EXPANSIONS OF REAL NUMBERS IN NON-INTEGER BASES

INTRODUCTION

Let $q \in (1, 2]$. By a *q-expansion* of 1, we mean a sequence $(e_i)_{i \geq 1}$ of integers in $\{0, 1\}$ satisfying the equality

$$1 = \sum_{i=1}^{\infty} \frac{e_i}{q^i}.$$

Such an expansion is not unique in general. There exist two particular expansions, known as the greedy and the lazy algorithms. In the greedy algorithm, we always choose the biggest possible value for e_i, while in the lazy algorithm, we always choose the smallest possible value for e_i.

In 1990, Erdös, Joo and Komornik began the work about characterizing the unique q-expansion of 1 for non-integer base q.

In 1991, Erdös, Horváth and Joo investigated the uniqueness of the q-expansions of 1. They showed that for almost all $q \in (1, 2]$, there are uncountably many different q-expansions, and surprisingly, there exist as well uncountably many exceptional $q \in (0, 1)$ for which there is only one q-expansion.

In 1998, Komornik and Loreti determined the smallest base number $q \in (1, 2]$ for which the q-expansion of 1 is unique.

In 1999, Komornik and Loreti gave a sufficient condition for which the number 1 has exactly two different q-expansions as well as using this information to construct the smallest number q for which the number 1 has exactly two different q-expansions.

In 2002, Dajani and Kraaikamp studied the ergodic properties of non-greedy series expansions to non-integer bases $\beta > 1$. It was shown that the so-called lazy expansion is isomorphic to the greedy expansion. Furthemore, a class of expansions to bases $\beta > 1$, $\beta \in \mathbb{Z}$, in between the lazy and the greedy expansions are introduced and studied. It was shown that these expansions of the form $Tx = \beta x + \alpha (mod 1)$.

In this thesis, our overall objective is to investigate how far the results mentioned above, excluding the cardinality and the ergodicity ones, continue to hold for the positive number x replacing the number 1.

In Chapter 1, general results about greedy q-expansions are derived. Chapter 2 does the same for lazy q-expansions. It is found that most results about q-expansions for real numbers greater than or equal to 1 are in somewhat opposite direction to those for real numbers less than or equal to 1, which illustrate the remarkable standing of the number 1 in this regard.

In Chapter 3, through the concept of U-sequences, we investigate the situation when a real number has unique q-expansion, and when it has exactly two q-expansions. In the last chapter, the smallest base number q yielding unique q-expansion of certain positive number is determined and a particular sequence is constructed which becomes the smallest sequence for certain positive number with corresponding base number q yielding exactly two q-expansions.

Objectives

1. To establish those results about q-expansions of a positive number x.

2. To find conditions for a q-expansion of a positive number x to be unique.

3. To find the smallest base number q for certain positive number x to have unique q-expansion.

4. To find the smallest sequence for certain positive number x to have exactly two q-expansions.

LITERATURE REVIEW

1. The work of Erdös *et al*. (1990)

Fix $q \in (1,2)$ and let

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = 1, \qquad \varepsilon_i \in \{0,1\}, \tag{1}$$

be a q-expansion of 1.

Erdös *et al* studied the greedy and lazy q-expansion of 1 and found conditions for the q-expansion of 1 to be unique. The main results are:

Theorem 1. *The q-expansion* (1) *is the greedy q-expansion of* 1 *if and only if*

$$(\varepsilon_{k+i}) \prec (\varepsilon_i) \quad \textit{whenever} \quad \varepsilon_k = 0. \tag{2}$$

The q-expansion (1) *is the unique expansion of* 1 *if and only if* (2) *and*

$$(1-\varepsilon_{k+i}) \prec (\varepsilon_i) \quad \textit{whenever} \quad \varepsilon_k = 1, \tag{3}$$

are satisfied.(For the definition of $\prec$*, see the section materials and methhod.)*

The proof of Theorem 1 is based on some lemmas concerning the more general expansions

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = x, \quad \varepsilon_i \in \{0,1\}, \tag{4}$$

for $x \in [0, \frac{1}{q-1}]$.

Lemma 1. *The q-expansion* (4) *is the greedy q-expansion of* x *if and only if*

$$\sum_{i=1}^{\infty} \frac{\varepsilon_{k+i}}{q^i} < 1 \quad \textit{whenever} \quad \varepsilon_k = 0. \tag{5}$$

The q-expansion (4) *is the lazy q-expansion of* x *if and only if*

$$\sum_{i=1}^{\infty} \frac{(1-\varepsilon_{k+i})}{q^i} < 1 \quad \textit{whenever} \quad \varepsilon_k = 1. \tag{6}$$

Lemma 2. (*a*) *If* $x \geq 1$ *and if its q-expansion* (4) *is greedy, then* (2) *is satisfied.*
(*b*) *If* $x \geq 1$ *and if its q-expansion* (4) *is unique, then* (2) *and* (3) *are satisfied.*

Lemma 3. *Assume that $x \le 1$ and that there is another q-expansion*

$$y = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}, \qquad \delta_i \in \{0,1\} \tag{7}$$

for some $0 \le y \le \frac{1}{q-1}$ such that

$$(\delta_{k+i}) \prec (\varepsilon_i) \quad \text{whenever} \quad \delta_k = 0. \tag{8}$$

Assume that either the q-expansion (4) of x is infinite (i.e. it contains infinitely many digits 1) or the q-expansion (7) is finite (i.e. it contains only finitely many digits 1). Then (4) is the greedy expansion of y.

Lemma 4. *(a) If $x < 1$ and if (2) is satisfied, then (4) is its greedy q-expansion.*
(b) If $x < 1$ and if (3) is satisfied, then (4) is its lazy q-expansion.

2. The work of Komornik and Loreti (1998)

Komornik and Loreti were interested in finding the smallest q for which the number 1 has unique q-expansion. Their main results are:

Theorem 2. *There is a smallest number $q \in (1,2)$ for which there is only one q-expansion of 1. This q is the unique positive solution of the equation*

$$1 = \sum_{i=1}^{\infty} \delta_i q^{-i},$$

where the sequence (δ_i) of zeros and ones is defined recursively as follows: first set $\delta_1 = 1$. If $n \ge 0$ and if $\delta_1, \dots, \delta_{2^n}$ are already defined, then set $\delta_{2^n+k} = 1 - \delta_k$ for $1 \le k < 2^n$ and $\delta_{2^{n+1}} = 1$.

Theorem 3. *The sequence (δ_i) given in Theorem 2 is the smallest admissible sequence.*

3. The work of Komornik and Loreti (1999)

Komornik and Loreti made a study about greedy and lazy q-expansions of 1 and established sufficient conditions for the number 1 to have exactly two different q-expansions. Their main results are:

Proposition 1. *A real number x has a q-expansion if and only if* $0 \le x \le \frac{1}{q-1}$.

Remark 1. The greedy algolithm also provides q-expansions for $x = 0$ with $q = 1$ given by all digits $a_i = 0$ $(i \ge 1)$ and for $x = 1$ given by the first digit $a_1 = x$ and all remaining digits $a_i = 0$ $(i \ge 2)$.

Definition 1. A sequence (a_i) of integers in $\{0, 1, \dots, m\}$ is *distinguished* if

$$a_{n+1}a_{n+2}\dots \prec a_1a_2\dots \quad \textit{whenever} \quad a_n = 0. \tag{9}$$

Theorem 4. *Denote by (ε_i) the greedy q-expansion of* 1 *for* $1 \le q \le 2$. *Then the map $q \mapsto (\varepsilon_i)$ is a strictly increasing bijection of the closed interval* $[1, 2]$ *onto the set of distinguished sequences.*

Theorem 5. *Fix $q \in (1, 2]$ and denote by (ε_i) the coresponding greedy q-expansion of* 1. *Furthemore, denote by (a_i) the greedy q-expansion of some x.*
(a) Assume that the sequence (ε_i) is infinite. Then the map $x \mapsto (a_i)$ is a strictly increasing bijection of the closed interval $[0, \frac{1}{q-1}]$ *onto the set of all sequences (a_i) satisfying*

$$a_{n+1}a_{n+2}\dots \prec \varepsilon_1\varepsilon_2\dots \qquad \textit{whenever} \quad a_n = 0. \tag{10}$$

(b) Assume that the sequence (ε_i) is finite. Let ε_k be its last nonzero element. Then the map $x \mapsto (a_i)$ is a strictly increasing bijection of the closed interval $[0, \frac{1}{q-1}]$ *onto the set of all sequences (a_i) satisfying* (10) *and which are not eventually periodic with period* $\varepsilon_1 \dots \varepsilon_{k-1}(\varepsilon_k - 1)$.

Lemma 5. *Fix $q \in (1, 2]$ and denote by (ε_i) the coresponding greedy q-expansion of* 1.
(a) The greedy q-expansion (a_i) of any $x \in [0, \frac{1}{q-1}]$ *satisfies the condition* (10).
(b) If the sequence (ε_i) is finite with a last nonzero digit ε_k, then no greedy q-expansion is eventually periodic with period $\varepsilon_1 \dots \varepsilon_{k-1}(\varepsilon_k - 1)$.

Lemma 6. *Fix $q \in (1, 2]$ and denote by (ε_i) the coresponding greedy q-expansion of* 1.
(a) Let (e_i) be an infinite q-expansion of 1. *Let (a_i) be a q-expansion of some real number x satisfying*

$$a_{n+1}a_{n+2}\dots \prec e_1e_2\dots \quad \textit{whenever} \quad a_n = 0. \tag{11}$$

Then (a_i) *is the greedy* q*-expansion of* x*.*
(b) Let (e_i) *be a finite* q*-expansion of* 1 *and denote by* e_k *its last nonzero element. Let* (a_i) *be a* q*-expansion of some real number* x *satisfying* (11)*. Assume that* (a_i) *is not eventually periodic with period* $\varepsilon_1 \dots \varepsilon_{k-1}(\varepsilon_k - 1)$*. Then* (a_i) *is the greedy* q*-expansion of* x*.*

Proposition 2. *Let* (e_i) *be an infinite* q*-expansion of* 1*. If a* q*-expansion* (b_i) *of some real number* y *satisfies the condition*

$$(1 - b_{n+i}) \prec (e_i) \quad \textit{whenever} \quad b_n > 0, \tag{12}$$

then (b_i) *is the lazy* q*-expansion of* y*.*

Remark 2. The condition in Proposition 2 is sufficient but not necessary. For example, the greedy q-expansion of 0 is unique and hence lazy, but the corresponding sequence $b_i \equiv 0$ for all i does not satisfy (12).

Definition 2. A sequence $a_1, a_2, \dots$ of integers in $0, 1, \dots, m$ is called *1-admissible* if

$$(a_{n+i}) \prec (a_i) \quad \text{whenever} \quad a_n = 0. \tag{13}$$

and

$$(1 - a_{n+i}) \prec (a_i) \quad \text{whenever} \quad a_n = 1. \tag{14}$$

Theorem 6. *The number* 1 *has a unique* q*-expansion if and only if its greedy* q*-expansion* (ε_i) *is a 1-admissible sequence. In other words, the bijection* $q \mapsto (\varepsilon_i)$ *of Theorem* 4 *establishes a bijection between the set of numbers* q *having the uniqueness property and the set of 1-admissible sequences.*

Let (ε_i) be a sequence of zeroes and ones, satisfying the condition

$$\varepsilon_{n+1}\varepsilon_{n+2} \dots \prec \varepsilon_1\varepsilon_2 \dots \quad \text{whenever} \quad \varepsilon_n = 0. \tag{15}$$

Hence there exists a unique $q \in (1, 2]$ satisfying

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = 1. \tag{16}$$

Assume that there exists a positive integer m satisfying

$$\varepsilon_m = 1. \tag{17}$$

and

$$\varepsilon_i + \varepsilon_{i+m} \in \{0,1\} \quad \text{for all} \quad i \geq 1. \tag{18}$$

Then we can define another q-expansion of 1,

$$\sum_{i=1}^{\infty} \frac{\delta_i}{q^i} = 1. \tag{19}$$

by setting

$$\delta_i = \begin{cases} \varepsilon_i & \text{if } i < m \\ 0 & \text{if } i = m \\ \varepsilon_i + \varepsilon_{i-m} & \text{if } i > m. \end{cases} \tag{20}$$

Theorem 7. *Assume* (15),(17),(18) *and define* q *by* (16). *Furthermore, define the sequence* (δ_i) *by* (20), *and assume that*

$$\overline{\delta_{n+1}\delta_{n+2}\ldots} \prec \varepsilon_1\varepsilon_2\ldots \quad \textit{whenever} \quad \delta_n = 1, \tag{21}$$

$$\overline{\varepsilon_{n+1}\varepsilon_{n+2}\ldots} \prec \varepsilon_1\varepsilon_2\ldots \quad \textit{whenever} \quad \varepsilon_n = 1 \quad \textit{and} \quad n > m, \tag{22}$$

$$\delta_{n+1}\delta_{n+2}\ldots \prec \varepsilon_1\varepsilon_2\ldots \quad \textit{whenever} \quad \delta_n = 0 \quad \textit{and} \quad n > m. \tag{23}$$

Then for this q *there are exactly two different* q*-expansions, those given by* (16) *and* (19).

Consider the sequence (e_i') given by

$$111\ 001\ 001\ \ldots = 111\ \underline{\underline{001}}.$$

Denote the unique positive solution of the equation

$$\sum_{i=1}^{\infty} \frac{e_i'}{q_i'} = 1$$

by $q' \approx 1.871349313$.

Theorem 8. *All possible base numbers are greater than* q'. *On the other hand,* q' *is an accumulation point of the set of possible base numbers.*

Let the sequence of 0 and 1, (ε_i), satisfy the condition

$$\varepsilon_1 \ldots \varepsilon_5 \ \underline{\prec}\ 11100. \tag{24}$$

Proposition 3. *Every admissible sequence* (ε_i) *satisfying* (24) *begins with* 11100 *and is eventually periodic with period* 100*. Moreover, denoting by* ε_k *the first one digit in the sequence* $\varepsilon_{m+1}\varepsilon_{m+2}\ldots$ *and by* δ_l *the first zero digit in the subsequence* $\delta_{m+1}\delta_{m+2}\ldots$*, the subsequence*

$$\varepsilon_N\varepsilon_{N+1}\ldots, \qquad N = max\,\{m+k+1, l+1\}$$

is periodic with one of the three periods 100*,* 010 *or* 001*.*

Remark 3. 1) There are only countably many admissible sequences satisfying (24). 2) If (ε_i) is an admissible sequence beginning with 11100, then m cannot be a multiple of 3. Indeed, otherwise (20) would lead to an eventually periodic sequence (δ_i) with period 200.

Theorem 9. *If* (ε_i) *is an eventually periodic sequence, then* $q \in (1,2)$ *given by* (16) *is irrational.*

MATERIALS AND METHODS

Let $q \in (1, 2]$. By an expansion with respect to q, or *q-expansion*, of a positive real number x we mean a sequence $(e_1, e_2, \ldots) \subseteq \{0, 1\}$ satisfying

$$\sum_{i=1}^{\infty} \frac{e_i}{q^i} = x.$$

It is known that x has an expansion if and only if $0 \leq x \leq \frac{1}{q-1}$.

The *lexicographical order* $\prec$ is defined as follows: given two real sequences (a_i) and (b_i), we write $(a_i) \prec (b_i)$ or $a_1 a_2 \ldots \prec b_1 b_2 \ldots$ if there exists a positive integer n such that $a_i = b_i$ for all $i < n$ but $a_n < b_n$. Note that this is a complete ordering.

A sequence $(a_i)_{i \geq 1}$ of integers in $\{0, 1\}$ is said to be *distinguished* if

$$(a_{n+i}) \prec (a_i) \quad \text{whenever} \quad a_n = 0. \tag{25}$$

A sequence $(a_i)_{i \geq 1}$ of integers in $\{0, 1\}$ is said to be *U-sequence* if

$$(a_{n+i}) \prec (a_i) \quad \text{whenever} \quad a_n = 0$$

and

$$(1 - a_{n+i}) \prec (a_i) \quad \text{whenever} \quad a_n = 1.$$

We write for brevity $\overline{\varepsilon_i}$ instead of $1 - \varepsilon_i$ and also $\overline{s}$ instead of $\overline{\varepsilon_1 \varepsilon_2 \ldots}$ if $s = (\varepsilon_i)$ is a finite or infinte sequence of zeros and ones. Thus the condition of U-sequence may be rewritten in the form

$$(a_{n+i}) \prec (a_i) \quad \text{whenever} \quad a_n = 0$$

and

$$\overline{a_{n+i}} \prec (a_i) \quad \text{whenever} \quad a_n = 1.$$

If (a_i) begins with N (≥ 2) consecutive 1 digits and if there are neither N consecutive 1 digits, nor N consecutive 0 digits later, then the sequence (a_i) is U-sequence.

A sequence $(e_i)_{i \geq 1}$ of integers in $\{0, 1\}$ is said to be *T-sequence* if

i) $(e_{n+i}) \prec (e_i)$ whenever $e_n = 0$ (On the other hand, (e_i) is distinguished sequence.)

ii) There exists a positive integer m satisfying

$$e_m = 1$$

Let sequence $(\varepsilon_i)_{i\geq 1}$ be the sequence of zeros and ones satisfying the condition

$$e_{i=m} + \varepsilon_i \in \{0,1\} \quad \text{for all} \quad i \geq 1.$$

Let sequence $(\delta_i)_{i\geq 1}$ be the sequence of zeros and ones define by

$$\delta_i = \begin{cases} e_i & \text{if } i < m \\ 0 & \text{if } i = m \\ e_i + \varepsilon_{i-m} & \text{if } i > m. \end{cases}$$

iii) A sequence (δ_i) and (e_i) satisfying

$$\begin{aligned} &\overline{\delta_{n+1}\delta_{n+2}\dots} \prec e_1e_2\dots \quad \text{whenever} \quad \delta_n = 1, \\ &\overline{e_{n+1}e_{n+2}\dots} \prec e_1e_2\dots \quad \text{whenever} \quad e_n = 1 \quad \textit{and} \quad n > m, \\ &\delta_{n+1}\delta_{n+2}\dots \prec e_1e_2\dots \quad \text{whenever} \quad \delta_n = 0 \quad \textit{and} \quad n > m. \end{aligned}$$

A real number $q \in (1,2]$ is called a *T-base number* if for some a positive real number x with exactly two different q-expansion.

From the definition of distinguished sequence we have

Theorem 10. *Let (e_i) be distinguished sequence. Then the map $q \mapsto \sum_{i=1}^{\infty} \frac{e_i}{q^i}$ is continuous and strictly decreasing of the interval $(1,2]$ onto the set of positive real numbers.*

Proof. Let $q \in (1,2]$ and $F(q) = \sum_{i=1}^{\infty} \frac{e_i}{q^i}$. That this map is strictly decreasing is clear. Let $q_1 < q_2$. Then

$$|F(q_1) - F(q_2)| = \sum_{i=1}^{\infty} \left| \frac{e_i(q_1^i - q_2^i)}{q_1^i q_2^i} \right| \leq \frac{|q_1 - q_2|}{q_1 q_2} \sum_{i=1}^{\infty} \frac{i}{q_1^{i-1}},$$

showing that F is continuous. □

Location and Duration of Research

Location Department of Mathematics, Kasetsart University.

Duration of Research May 2006-September 2006.

RESULTS AND DISCUSSION

Chapter I
Greedy expansion

Let $q \in (1,2]$ and $x \in [0, \frac{1}{q-1}]$. We define the *greedy q-expansion* (a_i) of $x \in [0, \frac{1}{q-1}]$ as follows: if for some positive integer n the numbers a_i are defined for all $i < n$ (no assumption if $n = 1$), then set $a_n = 1$ if

$$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} \leq x,$$

and set $a_n = 0$ otherwise.

Our first lemma reveals some intrinsic relations between greedy q-expansion of $\sigma \geq 1$ and greedy q-expansion of any real number.

Lemma 7. *Let $1 < q \leq 2$, $\sigma \geq 1$ and let (e_i) be the greedy q-expansion of σ.*
(a) The greedy q-expansion, (a_i), of any number $x \in [0, 1/(q-1)]$ satisfies the condition $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$.
(b) If the sequence (e_i) is finite with a last nonzero digit e_k, then no greedy q-expansion is eventually periodic with the period $e_1e_2\ldots e_{k-1}(e_k-1)$.

Proof. (a) Assume that $a_n = 0$. If $(a_{n+i}) \succ (e_i)$, then there exists an integer k such that $a_{n+i} = e_i$ for $i = 1, 2, \ldots, k-1$, but $a_{n+k} > e_k$. Thus $e_k = 0$ and $a_{n+k} = 1$ and so, by the definition of greedy q-expansion of σ,

$$\sum_{i=1}^{k-1} \frac{e_i}{q^i} + \frac{1}{q^k} > \sigma.$$

Thus

$$\sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} \geq \sum_{i=1}^{k-1} \frac{a_{n+i}}{q^i} + \frac{1}{q^k} = \sum_{i=1}^{k-1} \frac{e_i}{q^i} + \frac{1}{q^k} > \sigma.$$

Consequently,

$$x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n}\left(\frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots\right) > \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{\sigma}{q^n} \geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n},$$

contradicting the defintion of greedy q-expansion of x as $a_n = 0$.

If $(a_{n+i}) = (e_i)$, then $a_{n+i} = e_i$ for all $i \geq 1$. Thus

$$x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n}\left(\frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots\right) = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n}\left(\frac{e_1}{q} + \frac{e_2}{q^2} + \cdots\right)$$
$$= \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{\sigma}{q^n} \geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n},$$

again contradicting the defintion of the greedy q-expansion of x as $a_n = 0$.

(b) Assume on the contrary that the greedy q-expansion (a_i) of some $x \in [0, 1/(q-1)]$ is eventually periodic with period $e_1 e_2 \cdots e_{k-1}(e_k - 1)$. Since $\sigma - \frac{1}{q^k} = \frac{e_1}{q^1} + \cdots + \frac{e_k - 1}{q^k}$, we have

$$\begin{aligned} x = & \left(\frac{a_1}{q} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q} + \cdots + \frac{e_k - 1}{q^k}\right) \\ & + \left(\frac{e_1}{q} + \cdots + \frac{e_k - 1}{q^k}\right)\left(\frac{1}{q^{r+k}} + \frac{1}{q^{r+2k}} + \cdots\right) \\ & = \left(\frac{a_1}{q} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q} + \cdots + \frac{e_k - 1}{q^k}\right) + \left(\sigma - \frac{1}{q^k}\right)\left(\frac{\frac{1}{q^{r+k}}}{1 - \frac{1}{q^k}}\right) \\ & \geq \left(\frac{a_1}{q^1} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q^1} + \cdots + \frac{e_k - 1}{q^k}\right) + \left(1 - \frac{1}{q^k}\right)\left(\frac{\frac{1}{q^{r+k}}}{1 - \frac{1}{q^k}}\right) \\ & = \left(\frac{a_1}{q^1} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q^1} + \cdots + \frac{e_k}{q^k}\right) = \sum_{i=1}^{r+k-1} \frac{a_i}{q^i} + \frac{1}{q^{r+k}}, \end{aligned}$$

contradicting the definition of the q-greedy expansion of x since $a_{r+k} = 0$. □

Remark 4. 1) The case where $\sigma = 1$ is Lemmas 2(a) and 5.
2) The converse of Lemma 7(a) is not true, i.e., there exists an $x \in [0, \frac{1}{q-1}]$, whose q-expansion, (a_i), satisfies the condition $a_{n+1}a_{n+2}\ldots \prec e_1 e_2 \ldots$ whenever $a_n = 0$, but this expansion is not the q-greedy expansion of x, as seen in the following example.

Example. Take $q = \frac{9}{5}$ and $x = \sigma = \frac{27199387096045}{22876792454961} = \frac{27199387096045}{9^{14}} \geq 1$. We have

n	$1/q^n$	$\sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{1}{q^n}$	e_n
1	$\frac{5}{9} = \frac{12709329141645}{9^{14}}$	$\frac{12709329141645}{9^{14}}$	1
2	$(\frac{5}{9})^2 = \frac{7060738412025}{9^{14}}$	$\frac{19770067553670}{9^{14}}$	1
3	$(\frac{5}{9})^3 = \frac{3922632451125}{9^{14}}$	$\frac{23692700004795}{9^{14}}$	1
4	$(\frac{5}{9})^4 = \frac{2179240250625}{9^{14}}$	$\frac{25871940255420}{9^{14}}$	1
5	$(\frac{5}{9})^5 = \frac{1210689028125}{9^{14}}$	$\frac{27082629283545}{9^{14}}$	1
6	$(\frac{5}{9})^6 = \frac{672605015625}{9^{14}}$	$\frac{27755234299170}{9^{14}}$	0
7	$(\frac{5}{9})^7 = \frac{373669453125}{9^{14}}$	$\frac{27456298736670}{9^{14}}$	0
8	$(\frac{5}{9})^8 = \frac{207594140625}{9^{14}}$	$\frac{27290223424170}{9^{14}}$	0
9	$(\frac{5}{9})^9 = \frac{115330078125}{9^{14}}$	$\frac{27197959361670}{9^{14}}$	1
10	$(\frac{5}{9})^{10} = \frac{64072265625}{9^{14}}$	$\frac{27262031627295}{9^{14}}$	0
11	$(\frac{5}{9})^{11} = \frac{35595703125}{9^{14}}$	$\frac{27233555064795}{9^{14}}$	0
12	$(\frac{5}{9})^{12} = \frac{19775390625}{9^{14}}$	$\frac{27217734752295}{9^{14}}$	0
13	$(\frac{5}{9})^{13} = \frac{10986328125}{9^{14}}$	$\frac{27208945689795}{9^{14}}$	0
14	$(\frac{5}{9})^{14} = \frac{6103515625}{9^{14}}$	$\frac{27204062877295}{9^{14}}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Here, $(e_i) = (1,1,1,1,1,0,0,0,1,0,0,0,0,0,\ldots)$ is the greedy q-expansion of $x = \sigma$ and $(a_i) = (1,1,1,1,1,0,0,0,0,1,1,0,1,1)$ is a finite q-expansion of $x = \sigma$ which satisfies the condition $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$, but (a_i) is not q-greedy.

Next we derive further characterizations of greedy q-expansions.

Lemma 8. *Let $1 < q \leq 2$. A sequence (a_i) is the greedy q-expansion of x if and only if $\sum_{i=1}^{\infty} \frac{a_{k+i}}{q^i} < 1$ whenever $a_k = 0$.*

Proof. Let (a_i) be the greedy q-expansion of x and assume $a_k = 0$. By definition,

$$\sum_{i=1}^{k-1} \frac{a_i}{q^i} + \frac{1}{q^k} > x,$$

and so

$$\frac{1}{q^k} > \sum_{i=k}^{\infty} \frac{a_i}{q^i} = \sum_{i=k+1}^{\infty} \frac{a_i}{q^i} = \sum_{i=1}^{\infty} \frac{a_{k+i}}{q^{k+i}}.$$

The required inequality follows after multiplying by q^k.

Assume $\sum_{i=1}^{\infty} \frac{a_{k+i}}{q^i} < 1$ whenever $a_k = 0$. If $a_k = 1$, then

$$x = \sum_{i=1}^{k-1} \frac{a_i}{q^i} + \frac{1}{q^k} + \sum_{i>k} \frac{a_i}{q^i} \geq \sum_{i=1}^{k-1} \frac{a_i}{q^i} + \frac{1}{q^k}$$

If $a_k = 0$, then

$$\sum_{i=1}^{\infty} \frac{a_{k+i}}{q^i} < 1,$$

and so

$$\sum_{i=1}^{\infty} \frac{a_{k+i}}{q^{k+i}} < \frac{1}{q^k}.$$

Thus

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i} = \sum_{i \neq k} \frac{a_i}{q^i} = \sum_{i=1}^{k-1} \frac{a_i}{q^i} + \sum_{i=1}^{\infty} \frac{a_{k+i}}{q^{k+i}} < \sum_{i=1}^{k-1} \frac{a_i}{q^i} + \frac{1}{q^k},$$

i.e., (a_i) is the greedy q-expansion of x. $\square$

Remark 5. Lemma 8 is Lemma 1 but the proof here is different.

Lemma 9. *Let* $1 < q \leq 2$.

(a) Let (e_i) *be an **infinite*** q*-expansion of* $y \in [0, 1]$ *and let* (a_i) *be a* q*-expansion of* $x \in [0, \frac{1}{q-1}]$*. If* $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ *whenever* $a_n = 0$*, then* (a_i) *is the greedy* q*-expansion of* x*.*

(b) Let (e_i) *be a* q*-expansion of* $y \in [0, 1]$ *and let* (a_i) *be a **finite*** q*-expansion of* $x \in [0, \frac{1}{q-1}]$*. If* $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ *whenever* $a_n = 0$*, then* (a_i) *is the greedy* q*-expansion of* x*.*

(c) Let (e_i) *be a **finite*** q*-expansion of* $y \in [0, 1]$ *and denote by* e_k *its last nonzero element. Let* (a_i) *be a* q*-expansion of* $x \in [0, \frac{1}{q-1}]$*. Assume* $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ *whenever* $a_n = 0$*.*

(c.1) If $y < 1$*, then* (a_i) *is the greedy* q*-expansion of* x*.*

(c.2) If $y = 1$ *and assume that* (a_i) *is not eventually periodic with period* $e_1 \ldots e_{k-1}(e_k - 1)$*, then* (a_i) *is the greedy* q*-expansion of* x*.*

Proof. There is nothing to prove if $a_n = 1$, while for those n with $a_n = 0$, the results follow from Lemma 8 if we can show that

$$\sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} < 1. \tag{26}$$

From $(a_{n+i}) \prec (e_i)$, we can construct a sequence of integers

$$n = k_0 < k_1 < \dots$$

satisfying the conditions: with $j \in \mathbb{N}$,

$$a_{k_{j-1}+i} = e_i \text{ for all } 1 \le i < k_j - k_{j-1}$$

and

$$a_{k_j} < e_{k_j - k_{j-1}}.$$

(a) If the sequence (e_i) is infinite, then

$$\begin{aligned}
\frac{1}{q^n}\sum_{i=1}^{\infty}\frac{a_{n+i}}{q^i} &= \frac{a_{k_0+1}}{q^{k_0+1}} + \frac{a_{k_0+2}}{q^{k_0+2}} + \dots + \frac{a_{k_1}}{q^{k_1}} + \frac{a_{k_1+1}}{q^{k_1+1}} + \cdots \\
&= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{k_j-k_{j-1}}\frac{a_{k_{j-1}+i}}{q^{k_{j-1}+i}}\right) \\
&= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{k_j-k_{j-1}}\frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}}\right) < \sum_{j=1}^{\infty}\left(\frac{y}{q^{k_{j-1}}} - \frac{1}{q^{k_j}}\right) \qquad (27) \\
&= \frac{1}{q^{k_0}},
\end{aligned}$$

proving (26).

(b) If the sequence (a_i) is finite, assume that there exits a positive integer m satisfying $a_i = 0$ for all $i > k_m$. Now

$$\begin{aligned}
\frac{1}{q^n}\sum_{i=1}^{\infty}\frac{a_{n+i}}{q^i} &= \frac{a_{k_0+1}}{q^{k_0+1}} + \frac{a_{k_0+2}}{q^{k_0+2}} + \dots + \frac{a_{k_1}}{q^{k_1}} + \frac{a_{k_1+1}}{q^{k_1+1}} + \cdots \\
&= \sum_{j=1}^{m}\left(\sum_{i=1}^{k_j-k_{j-1}}\frac{a_{k_{j-1}+i}}{q^{k_{j-1}+i}}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{k_j-k_{j-1}}\frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}}\right) \\
&\le \sum_{j=1}^{m}\left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}}\right) = \frac{1}{q^{k_0}} - \frac{1}{q^m} < \frac{1}{q^{k_0}},
\end{aligned}$$

proving (26).

(c) If the sequence (e_i) is finite, then proceeding as in the proof of (a) leads to (27) with strict inequality being now non-strict. Observe that $e_{k_j-k_{j-1}} = 1$ so $k_j - k_{j-1} \le k$.

A closer inspection of the proof reveals that we obtain equality exactly when $y = 1$ and $k_j - k_{j-1} = k$ for every j, i.e., when the sequence (a_{n+i}) is periodic with period $e_1 \dots e_{k-1}(e_k - 1)$. This contradicts the fact that (a_i) is not eventually periodic with period $e_1 \dots e_{k-1}(e_k - 1)$. Thus (a_i) is the greedy q-expansion of x. □

Remark 6. 1) Lemma 9 (a),(b) is Lemma 3 and the proofs given here are the same. Lemma 6 is a special case of Lemma 9 (c.2) above.
2) The converse of Lemma 9 is not true, i.e., there exist y with finite q-expansion (e_i), and x with greedy q-expansion (a_i), such that (a_i) does not satisfy the condition $a_{n+1}a_{n+2}\dots \prec e_1e_2\dots$ whenever $a_n = 0$ as seen in the following example.
Example. Take $q = \frac{4}{3}$. We have

n	$1/q^n$	$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n}$	a_n
1	$3/4 = 56668397794435742564352/4^{38}$	$56668397794435742564352/4^{38}$	1
2	$3^2/4^2 = 42501298345826806923264/4^{38}$	$99169696140262549487616/4^{38}$	0
3	$3^3/4^3 = 31875973759370105192448/4^{38}$	$88544371553805847756800/4^{38}$	0
4	$3^4/4^4 = 23906980319527578894336/4^{38}$	$80575378113963321458688/4^{38}$	0
5	$3^5/4^5 = 17930235239645684170752/4^{38}$	$74598633034081426735104/4^{38}$	0
6	$3^6/4^6 = 13447676429734263128064/4^{38}$	$70116074224170005692416/4^{38}$	0
7	$3^7/4^7 = 10085757322300697346048/4^{38}$	$66754155116736439910400/4^{38}$	1
8	$3^8/4^8 = 7564317991725523009536/4^{38}$	$74318473108461962919936/4^{38}$	0
9	$3^9/4^9 = 5673238493794142257152/4^{38}$	$72427393610530582167552/4^{38}$	0
10	$3^{10}/4^{10} = 4254928870345606692864/4^{38}$	$71009083987082046603264/4^{38}$	0
11	$3^{11}/4^{11} = 3191196652759205019648/4^{38}$	$69945351769495644930048/4^{38}$	0
12	$3^{12}/4^{12} = 2393397489569403764736/4^{38}$	$69147552606305843675136/4^{38}$	0
13	$3^{13}/4^{13} = 1795048117177052823552/4^{38}$	$68549203233913492733952/4^{38}$	0
14	$3^{14}/4^{14} = 1346286087882789617664/4^{38}$	$68100441204619229528064/4^{38}$	0
15	$3^{15}/4^{15} = 1009714565912092213248/4^{38}$	$67763869682648532123648/4^{38}$	1
16	$3^{16}/4^{16} = 757285924434069159936/4^{38}$	$68521155607082601283584/4^{38}$	0
17	$3^{17}/4^{17} = 567964443325551869952/4^{38}$	$68331834125974083993600/4^{38}$	0
18	$3^{18}/4^{18} = 425973332494163902464/4^{38}$	$68189843015142696026112/4^{38}$	0
19	$3^{19}/4^{19} = 319479999370622926848/4^{38}$	$68083349682019155050496/4^{38}$	0
20	$3^{20}/4^{20} = 239609999527967195136/4^{38}$	$68003479682176499318784/4^{38}$	0

n	$1/q^n$	$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n}$	a_n
21	$3^{21}/4^{21} = 1797074996459753963 52/4^{38}$	$679435771822945675200 00/4^{38}$	1
22	$3^{22}/4^{22} = 1347806247344815472 64/4^{38}$	$680783578070289890672 64/4^{38}$	0
23	$3^{23}/4^{23} = 1010854685508611604 48/4^{38}$	$680446626508453686804 48/4^{38}$	0
24	$3^{24}/4^{24} = 7581410141314587033 6/4^{38}$	$680193912837076533903 36/4^{38}$	0
25	$3^{25}/4^{25} = 5686057605985940275 2/4^{38}$	$680004377583543669227 52/4^{38}$	1
26	$3^{26}/4^{26} = 4264543204489455206 4/4^{38}$	$680430831903992614748 16/4^{38}$	0
27	$3^{27}/4^{27} = 3198407403367091404 8/4^{38}$	$680324218323880378368 00/4^{38}$	0
28	$3^{28}/4^{28} = 2398805552525318553 6/4^{38}$	$680244258138796201082 88/4^{38}$	0
29	$3^{29}/4^{29} = 1799104164393988915 2/4^{38}$	$680184287999983068119 04/4^{38}$	0
30	$3^{30}/4^{30} = 1349328123295491686 4/4^{38}$	$680139310395873218396 16/4^{38}$	0
31	$3^{31}/4^{31} = 1011996092471618764 8/4^{38}$	$680105577192790831104 00/4^{38}$	0
32	$3^{32}/4^{32} = 758997069353714073 6/4^{38}$	$680080277290479040634 88/4^{38}$	0
33	$3^{33}/4^{33} = 569247802015285555 2/4^{38}$	$680061302363745197783 04/4^{38}$	0
34	$3^{34}/4^{34} = 426935851511464166 4/4^{38}$	$680047071168694815644 16/4^{38}$	0
35	$3^{35}/4^{35} = 320201888633598124 8/4^{38}$	$680036397772407029040 00/4^{38}$	0
36	$3^{36}/4^{36} = 240151416475198593 6/4^{38}$	$680028392725191189086 88/4^{38}$	0
37	$3^{37}/4^{37} = 180113562356398945 2/4^{38}$	$680022388939779309122 04/4^{38}$	0
38	$3^{38}/4^{38} = 135085171767299208 9/4^{38}$	$680017886100720399148 41/4^{38}$	1

Here, $(a_i) = (1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ is the greedy expansion of $x = \frac{68001788610072039914841}{75557863725914323419136}$. Taking $y = x$, we get $(a_i) = (e_i)$. Note that (a_i) does not satisfy the condition $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$.

Lemma 10. *Let $q, q' \in (1, 2]$, $x \in [0, \frac{1}{q-1}] \cap [0, \frac{1}{q'-1}]$. Let (e_i), respectively, (e'_i) be the greedy q-expansion, respectively q'-expansion of x. If $q < q'$, then $(e_i) \prec (e'_i)$.*

Proof. Suppose that the conclusion is false. We have two possible cases.

Case 1: $(e_i) = (e'_i)$. Thus $x = \sum_{i=1}^{\infty} \frac{e'_i}{q'^i} < \sum_{i=1}^{\infty} \frac{e_i}{q^i} = x$, which is a contradiction.

Case 2: $(e_i) \succ (e'_i)$. Thus there exists an integer n such that $e_i = e'_i$ for all $1 \le i < n$ but $e_n > e'_n$. Now $e_n = 1$ and $e'_n = 0$. By the definition of the greedy q-expansion,

$$\sum_{i=1}^{n-1} \frac{e'_i}{q'^i} + \frac{1}{q'^n} < \sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{1}{q'^n} < \sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{1}{q^n} \le x,$$

contradicting the definition of greedy q'-expansion of x with respect to q' as $e'_n = 0$. $\square$

Chapter II
Lazy expansion

Let $q \in (1,2],\ y \in [0, \frac{1}{q-1}]$. The *lazy q-expansion* (b_i) of y is defined as follows: if for some positive integer n the numbers b_i are defined for all $i < n$ (no assumption when $n = 1$), then set $b_n = 0$ if

$$\sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i>n} \frac{1}{q^i} \geq y,$$

and set $b_n = 1$ otherwise.

Lazy q-expansions enjoy two simple properties which we now decribe.

Property L1. A real number $y \in [0, \frac{1}{q-1}]$ has (b_i) as its lazy q-expansion if and only if the sequence $(a_i) := (1 - b_i)$ is the greedy q-expansion of $x := \frac{1}{q-1} - y$.
(This "duality "property implies that every $y \in [0, \frac{1}{q-1}]$ has a lazy q-expansion.)

Proof. First observe that (b_i) is a q-expansion of y
$\Leftrightarrow \sum_{i=1}^{\infty} \frac{b_i}{q^i} = y \ \Leftrightarrow \sum_{i=1}^{\infty} \frac{1-b_i}{q^i} = \frac{1}{q-1} - y \ \Leftrightarrow (1 - b_i)$ is a q-expansion of $\frac{1}{q-1} - y$.

Assume that (b_i) is the lazy q-expansion of y. If $1 - b_n = 0$, then

$$y > \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{1}{q^i},$$

and so

$$\frac{1}{q-1} - y < \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{b_i}{q^i} - \sum_{i=n+1}^{\infty} \frac{1}{q^i} = \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} + \frac{1}{q^n}.$$

If $1 - b_n = 1$, then $y \leq \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{1}{q^i}$, and so

$$\frac{1}{q-1} - y \geq \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{b_i}{q^i} - \sum_{i=n+1}^{\infty} \frac{1}{q^i} = \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} + \frac{1}{q^n}.$$

Thus $(1 - b_i)$ is the greedy q-expansion of $\frac{1}{q-1} - y$.

Assume that $(1 - b_i)$ is the greedy q-expansion of $\frac{1}{q-1} - y$. If $b_n = 0$, then $\frac{1}{q-1} - y \geq \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} + \frac{1}{q^n}$, and so

$$y \leq \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} - \frac{1}{q^n} = \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=1}^{n-1} \frac{1}{q^n}$$

If $b_n = 1$, then $\frac{1}{q-1} - y < \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} + \frac{1}{q^n}$, and so $y > \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{1}{q^i}$. Thus (b_i) is the lazy q-expansion of y. $\square$

Property L2. If (a_i) and (b_i) are the greedy and lazy q-expansions of x, and if there exists another q-expansion (c_i) of x, then

$$(b_i) \preceq (c_i) \preceq (a_i).$$

(In other words, the greedy q-expansion is the greatest and the lazy q-expansion is the smallest expansions of x with respect to the lexicographic order.)

Proof. Let (a_i) and (b_i) be the greedy, respectively lazy q-expansions of x and let (c_i) be another q-expansion of x.

To show that $(b_i) \preceq (c_i)$, assume $(b_i) \succ (c_i)$. Then there exists an integer n such that $b_i = c_i$ for all $1 \le i < n$ but $b_n > c_n$. Thus $b_n = 1$ and $c_n = 0$. By the definition of lazy q-expansion, we have

$$\sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{1}{q^i} < x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{n-1} \frac{c_i}{q^i} + \sum_{i=n}^{\infty} \frac{c_i}{q^i}.$$

Thus

$$\sum_{i=n+1}^{\infty} \frac{1}{q^i} < \sum_{i=n}^{\infty} \frac{c_i}{q^i} = \sum_{i=n+1}^{\infty} \frac{c_i}{q^i},$$

contradicting the definition of the sequence (c_i).

To show that $(c_i) \preceq (a_i)$, assume $(c_i) \succ (a_i)$. Then there exists an integer n such that $c_i = a_i$ for all $1 \le i < n$ but $c_n > a_n$. Thus $c_n = 1$ and $a_n = 0$. By the definition of greedy q-expansion, we have

$$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} > x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{n-1} \frac{c_i}{q^i} + \sum_{i=n}^{\infty} \frac{c_i}{q^i},$$

which implies $0 > \sum_{i=n+1}^{\infty} \frac{c_i}{q^i}$, contradicting the definition of the sequence (c_i). □

We next derive further characterizations of lazy q-expansions.

Lemma 11. *Let $q \in (1,2]$, $y \in [0, \frac{1}{q-1}]$. Then (b_i) is the lazy q-expansion of y if and only if $\sum_{i=1}^{\infty} \frac{1-b_{k+i}}{q^i} < 1$ whenever $b_k = 1$.*

Proof. Let (b_i) be the lazy q-expansion of y. Assuming $b_k = 1$, we get

$$\sum_{i=1}^{k} \frac{b_i}{q^i} + \sum_{i=k+1}^{\infty} \frac{1}{q^i} < y + \frac{1}{q^k} = \sum_{i=1}^{\infty} \frac{b_i}{q^i} + \frac{1}{q^k},$$

and so

$$\sum_{i=1}^{\infty} \frac{1-b_{k+i}}{q^i} < 1.$$

Conversely, assume $\sum_{i=1}^{\infty} \frac{1-b_{k+i}}{q^i} < 1$ whenever $b_k = 1$. If $b_k = 0$, then

$$y = \sum_{i=1}^{k-1} \frac{b_i}{q^i} + \sum_{i=k+1}^{\infty} \frac{b_i}{q^i} \leq \sum_{i=1}^{k-1} \frac{b_i}{q^i} + \sum_{i=k+1}^{\infty} \frac{1}{q^i}.$$

If $b_k = 1$, then from the assumption we have $\sum_{i=1}^{\infty} \frac{1-b_{k+i}}{q^{k+i}} < \frac{1}{q^k}$, and so $\sum_{i=1}^{\infty} \frac{1}{q^{k+i}} + \sum_{i=1}^{k} \frac{b_i}{q^i} < y + \frac{1}{q^k}$, i.e., $\sum_{i=1}^{k-1} \frac{b_i}{q^i} + \sum_{i=k+1}^{\infty} \frac{1}{q^i} < y$, showing that the q-expansion is lazy. □

Remark 7. Lemma 11 is Lemma 1, but the proof here is different.

Proposition 4. *Let (e_i) be an infinite q-expansion of $\sigma \leq 1$. If another q-expansion (b_i) of $y \in [0, \frac{1}{q-1}]$ satisfies the condition*

$$(1 - b_{n+i}) \prec (e_i) \quad \textit{whenever} \quad b_n > 0, \tag{28}$$

then (b_i) is the lazy q-expansion of y.

Proof. By Lemma 11, it suffices to show that if $b_k = 1$ then

$$\sum_{i=1}^{\infty} \frac{1-b_{k+i}}{q^i} < 1 \tag{29}$$

Let $b_k = 1$. Using the hypothesis, we can construct a sequence of integers $k = k_0 < k_1 < \ldots$ satisfying for each $j = 1, 2, \ldots$ the conditions

$$1 - b_{k_{j-1}+i} = e_i \qquad \text{when} \quad 1 \leq i < k_j - k_{j-1}$$

and

$$1 - b_{k_j} < e_{k_j - k_{j-1}}.$$

We have

$$\frac{1}{q^k} \sum_{i=1}^{\infty} \frac{1-b_{k+i}}{q^i} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{1-b_{k_{j-1}+i}}{q^{k_{j-1}+i}} \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right) \tag{30}$$
$$= \sum_{j=1}^{\infty} \left(\frac{1}{q^{k_{j-1}}} \sum_{i=1}^{k_j - k_{j-1}} \frac{e_i}{q^i} - \frac{1}{q^{k_j}} \right) < \sum_{j=1}^{\infty} (\frac{\sigma}{q^{k_{j-1}}} - \frac{1}{q^{k_j}}) \leq \frac{1}{q^{k_0}}.$$

Thus

$$\sum_{i=1}^{\infty} \frac{b_i}{q^i} + \sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^{k+i}} < \sum_{i=1}^{\infty} \frac{b_i}{q^i} + \frac{1}{q^k},$$

i.e.,

$$\sum_{i=1}^{\infty} \frac{1}{q^{k+i}} + \sum_{i=1}^{k} \frac{b_i}{q^i} < y + \frac{1}{q^k},$$

and so $\sum_{i=1}^{\infty} \frac{1}{q^{k+i}} + \sum_{i=1}^{k-1} \frac{b_i}{q^i} < y$, proving (29). □

Remark 8. Proposition 2 is a special case of Proposition 4 when $\sigma = 1$.

Proposition 5. *Let $q \in (1, 2]$, (e_i) be a finite q-expansion of $\sigma \le 1$ and denote by e_L its last nonzero digit. If a q-expansion (b_i) of $y \in [0, \frac{1}{q-1}]$ satisfies the condition*

$$(1 - b_{n+i}) \prec (e_i) \quad \textit{whenever } b_n > 0$$

and

$$L > \min\{k;\ \textit{for each } i \in \mathbb{N},\ \textit{if } b_i = 1,\ \textit{then } b_{i+j} \ne e_j \textit{ when } 1 \le j < k \\ \textit{and } b_{i+k} = e_k = 1\}, \tag{31}$$

then (b_i) is the lazy q-expansion of y.

Proof. Proceeding exactly as in the proof of Proposition 4, we end up at (30) but the strict inequality now becomes non-strict. If (30) is an equality, then $k_j - k_{j-1} = L$ for each j but the condition (31) prevents this from happening. □

Remark 9. Proposition 5 is new and complements Proposition 4. The condition (28) is not necessary when $y = 0$. For then y has only unique q-expansion which must then be (0) violating (28).

Chapter III

Numbers with unique q-expansion and numbers with exactly two q-expansions

1. Numbers with unique q-expansions

In this section, we aim to find conditions for which the greedy and lazy q-expansions of a fixed real number σ coincide. Equivalently, these are conditions for which the q-expansion of a fixed σ is unique.

Theorem 11. *If the number $\sigma \geq 1$ has a unique q-expansion for a given $q \in (1, 2]$, then this unique q-expansion (ε_i) is an U-sequence.*

Proof. Let $\sigma \geq 1$ and (ε_i) be unique, and so is a greedy q-expansion. We deduce from Lemma 7, using $x = \sigma$, that

$$(\varepsilon_{n+i}) \prec (\varepsilon_i) \quad \text{whenever} \quad \varepsilon_n = 0.$$

Since (ε_i) is also the lazy q-expansion of $\sigma \geq 1$, by Property 1, the q-expansion $(1-\varepsilon_i)$ is the greedy q-expansion of $\frac{1}{q-1} - \sigma$. By Lemma 7,

$$(1 - \varepsilon_{n+i}) \prec (\varepsilon_i) \quad \text{whenever} \quad 1 - \varepsilon_n = 0,$$

which shows that (ε_i) is U-sequence. □

Remark 10. Theorem 11 is Lemma 2, but the proof here is different.

Theorem 12. *If the greedy q-expansion (ε_i) of $\sigma \leq 1$ with $q \in (1, 2]$ is an U-sequence, then σ has a unique q-expansion for this given q .*

Proof. Assume the greedy q-expansion (ε_i) of $\sigma \leq 1$ is U-sequence. Then

$$(1 - \varepsilon_{n+i}) \prec (\varepsilon_i) \quad \text{whenever} \quad 1 - \varepsilon_n = 0.$$

Since (ε_i) is a q-expansion of σ, then as in the proof of Property 1, $(1 - \varepsilon_i)$ is a q-expansion of $\frac{1}{q-1} - \sigma$. By Lemma 9(a), $(1 - \varepsilon_i)$ is the greedy q-expansion of $\frac{1}{q-1} - \sigma$, taking $y = \sigma$, $x = \frac{1}{q-1} - \sigma$. By Property 1, (ε_i) is the lazy q-expansion of $\sigma \leq 1$. Thus the number $\sigma \leq 1$ has a unique q-expansion for this given q. □

Remark 11. Taking $\sigma = 1$ in Theorems 11 and 12, we get Theorem 6.

An example of Theorem 12 is given next.

Example Let $\sigma = \frac{559244998837862911002057902798057297233624110}{570658162108627174778971075491512021856922699}$ and $q = 1.9$. The greedy q-expansion of σ is

$$(\varepsilon_i) = (1,1,1,0,0,1,0,1,1,0,0,1,0,1,0,1,1,0,1,1,1,0,0,0,1,1,0,1,1,0,1,1,0,1,1),$$

which is an U-sequence. By Theorem 12, this is also a lazy q-expansion.

2.Numbers with exactly two expansions

In this section, we find conditions for which there are exactly two q-expansions, which must then be greedy and lazy, of a fixed σ.

Let $e = (e_i)$ be a sequence of zeros and ones, satisfying the condition

$$e_{n+1}e_{n+2}\ldots \prec e_1e_2\ldots \quad \text{whenever} \quad e_n = 0 \tag{32}$$

Then $e_1 = 1$. For otherwise applying (32) we would get $(e_i) \equiv (0)$, contradicting (32).

From Theorem 10, for $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right]$, there exists a unique $q \in (1, 2]$ satisfying

$$\sum_{i=1}^{\infty} \frac{e_i}{q^i} = y. \tag{33}$$

By Lemma 9, (e_i) is the greedy q-expansion of y. For this q, let (ε_i) be the corresponding greedy q-expansion of 1. Assume that there exists a positive integer m satisfying

$$e_m = 1 \tag{34}$$

and

$$e_{i+m} + \varepsilon_i \in \{0, 1\} \quad \text{for all} \quad i \geq 1. \tag{35}$$

Then we can define another q-expansion $\sum_{i=1}^{\infty} \frac{\delta_i}{q^i}$, which will shortly be shown to equal y, by setting

$$\delta_i = \begin{cases} e_i & \text{if } i < m \\ 0 & \text{if } i = m \\ e_i + \varepsilon_{i-m} & \text{if } i > m \end{cases} \tag{36}$$

By (35), $\delta = (\delta_i)$ is a sequence of zeros and ones. Furthermore, using (33) and (34) we have

$$\sum_{i=1}^{\infty} \delta_i q^{-i} = \sum_{i<m} e_i q^{-i} + \sum_{i>m} (e_i + \varepsilon_{i-m}) q^{-i}$$
$$= \sum_{i \neq m} e_i q^{-i} + q^{-m} \sum_{i=1}^{\infty} \varepsilon_i q^{-i}$$
$$= \sum_{i=1}^{\infty} e_i q^{-i} = y. \tag{37}$$

The q-expansions (e_i) and (δ_i) of y are different because $e_m = 1$ but $\delta_m = 0$.

Now we give a sufficient condition in order for a numbers to have exactly two q-expansions for y.

Theorem 13. *Let (e_i) be an infinite sequence of zeros and ones, satisfying (32). Let $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right]$ and $q \in (1,2]$ be such that $y = \sum_{i=1}^{\infty} \frac{e_i}{q^i}$. For this q, let (ε_i) be the greedy q-expansion of 1. Assume (34) and (35). Define the sequence (δ_i) by (36), and assume that*

$$\overline{\delta_{n+1}\delta_{n+2}\ldots} \prec e_1 e_2 \ldots \quad \textit{whenever} \quad \delta_n = 1, \tag{38}$$
$$\overline{e_{n+1}e_{n+2}\ldots} \prec e_1 e_2 \ldots \quad \textit{whenever} \quad e_n = 1 \quad \textit{and} \quad n > m, \tag{39}$$
$$\delta_{n+1}\delta_{n+2}\ldots \prec e_1 e_2 \ldots \quad \textit{whenever} \quad \delta_n = 0 \quad \textit{and} \quad n > m. \tag{40}$$

Then y has exactly two different q-expansions, given by (33) and (37).

Proof. Since the sequence (e_i) is infinite. We deduce from (38) and Proposition 4 that (33) and (37) are the greedy and lazy q-expansions of $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right]$.

It remains to verify that if a sequence (ρ_i) of zeros and ones satisfies the strict inequalities $(\delta_i) \prec (\rho_i) \prec (e_i)$, then

$$\sum_{i=1}^{\infty} \rho_i q^{-i} \neq y. \tag{41}$$

Fix such a sequence (ρ_i). Then $\rho_i = \delta_i = e_i$ for all $i < m$. Since $\delta_m = 0$ and $e_m = 1$, we have either $\rho_m = \delta_m$ or $\rho_m = e_m$. We distinguish two cases.

First case: $\rho_m = 0$. Then there is an integer $n > m$ such that $\rho_i = \delta_i$ for all $i < n$ and $\delta_n = 0 < 1 = \rho_n$. Using (40) and the same arguments as in the proof of Lemma 9 up to equation(40), we deduce $\sum_{i=1}^{\infty} \delta_{n+i} q^{-i} < 1$. Therefore

$$\sum_{i=1}^{\infty} \rho_i q^{-i} - y = q^{-n} + \sum_{i=n+1}^{\infty} (\rho_i - \delta_i) q^{-i} \geq q^{-n} - \sum_{i=n+1}^{\infty} \delta_i q^{-i} = q^{-n}(1 - \sum_{i=1}^{\infty} \delta_{n+i} q^{-i}) > 0,$$

proving (41).

Second case: $\rho_m = 1$. Then there is an integer $n > m$ such that $\rho_i = e_i$ for all $i < n$ and $\rho_n = 0 < 1 = e_n$. Using (39) and applying Lemma 7, we have

$$\sum_{i=1}^{\infty} \overline{e_{n+i}} q^{-i} < 1.$$

Hence

$$\begin{aligned} \sum_{i=1}^{\infty} \rho_i q^{-i} - y = -q^{-n} + \sum_{i=n+1}^{\infty} (\rho_i - e_i) q^{-i} &\leq -q^{-n} + \sum_{i=n+1}^{\infty} \overline{e_i} q^{-i} \\ &= -q^{-n}\Big(1 - \sum_{i=1}^{\infty} \overline{e_{n+i}} q^{-i}\Big) < 0, \end{aligned}$$

again implying (41). □

Remark 12. Theorem 7 is a special case of Theorem 13 above when $y = 1$.

Theorem 14. *Let* (e_i) *be a finite sequence of zeros and ones, with* e_L *its last nonzero digit, satisfying* (32). *Let* $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right]$ *and* $q \in (1, 2]$ *be such that* $y = \sum_{i=1}^{\infty} \frac{e_i}{q^i}$. *For this* q, *let* (ε_i) *be the greedy* q*-expansion of* 1. *Assume* (34) *and* (35). *Define the sequence* (δ_i) *by* (36), *and assume that* (38), (39), (40) *and* (31). *Then* y *has exactly two different* q*-expansions, given by* (33) *and* (37).

Proof. The proof proceeds exactly as in Theorem 13, except that now we appeal to Proposition 5 instead of Proposition 4 at the beginning of the proof. □

Remark 13. Theorem 14 is new and complements Theorem 13.

Next we give an example for Theorem 14.

Example Let $y = \frac{3902563888221395449817251061561905663982412670490}{3914144333903073791808962606796280957916632792441}$ and $q = 1.9$. We have $(e_i) = (1,1,1,0,1,0,0,1,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)$ is a q-expansion of y and $(\varepsilon_i) = (1,1,1,0,1,0,0,1,1,0,1,1,0,1,1,0,0,0,1,0,1,1,1,0,0,1,0,0,1,1,0,0,1,0,0,1,1,0,0,0,1,0,1,1,0,0,1,1,0,1,0,1,\ldots)$ is the greedy q-expansion of 1. Take $m = 10$. Thus $e_{i+m} + \varepsilon_i \in \{0, 1\}$ for all $i \geq 1$.

Chapter IV

Smallest base number for certain real numbers having unique q-expansion and smallest sequence of certain real number with exactly two q-expansions

3. Smallest base number with unique q-expansion

For certain real number y, among infinitely many base numbers q for which y has unique q-expansions, it is possible to determine the smallest such base number q, which we now show.

Theorem 15. *Let (δ_i) be the sequence of zeros and ones defined recursively as follows:*

- *First set $\delta_1 = 1$.*
- *If $n \geq 0$ and if $\delta_1, \ldots, \delta_{2^n}$ are already defined, then set $\delta_{2^n+k} = 1 - \delta_k$ for $1 \leq k < 2^n$ and $\delta_{2^{n+1}} = 1$.*

If $y \in \left[\sum_{i=1}^{\infty} \frac{\delta_i}{2^i}, 1\right]$, then there is a smallest $q \in (1, 2]$ for which y has a unique q-expansion. This q is the unique positive solution of the equation

$$y = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

Proof. Fix $y \in \left[\sum_{i=1}^{\infty} \frac{\delta_i}{2^i}, 1\right]$. From Theorem 3, we have (δ_i) is the smallest U-sequence. By Theorem 10 using $(\delta_i) = (e_i)$, there exsits a unique $q \in (1, 2]$ satisfying

$$y = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

By Lemma 9(a) with $x = y$, $(e_i) = (a_i) = (\delta_i)$, we have (δ_i) is the greedy q-expansion of y.

If y has another (unique) U-sequence q'-expansion (e_i), then by exactly the same arguments, i.e., using Theorem 10 and Lemma 9(a), (e_i) is the only q'-expansion, hence greedy, of y. By Lemma 10, since $(e_i) \succ (\delta_i)$, we get $q' > q$. $\square$

4. Smallest sequence of numbers with exactly two expansions

In this section, we construct a sequence for certain real number y, and corresponding base number q for which y has exactly two q-expansions, with the constructed sequence being the smallest.

Let (e_i) be the sequence of zeros and ones, satisfying the conditions

$$e_{n+1}e_{n+2}\ldots \prec e_1e_2\ldots \quad \text{whenever} \quad e_n = 0 \tag{32}$$

Hence for $y \in [\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1]$, there exists a unique $q \in (1, 2]$ satisfying $\sum_{n=1}^{\infty} e_n q^{-n} = y$. For this q, let the greedy q-expansion of 1 be (ε_i). Assume that there exists a positive integer m satisfying

$$e_m = 1 \tag{34}$$

and

$$e_{i+m} + \varepsilon_i \in \{0, 1\} \quad \text{for all} \quad i \geq 1. \tag{35}$$

Hence we have another q-expansion of y defined by

$$\delta_i = \begin{cases} e_i & \text{if } i < m \\ 0 & \text{if } i = m \\ e_i + \varepsilon_{i-m} & \text{if } i > m \end{cases} \tag{36}$$

Recall that (e_i) is a T-sequence if it satisfies the condition (32),(34),(35) and

$$\overline{\delta_{n+1}\delta_{n+2}\dots} \prec e_1 e_2 \dots \quad \text{whenever} \quad \delta_n = 1, \tag{38}$$

$$\overline{e_{n+1}e_{n+2}\dots} \prec e_1 e_2 \dots \quad \text{whenever} \quad e_n = 1 \quad \text{and} \quad n > m, \tag{39}$$

$$\delta_{n+1}\delta_{n+2}\dots \prec e_1 e_2 \dots \quad \text{whenever} \quad \delta_n = 0 \quad \text{and} \quad n > m. \tag{40}$$

For a given real number y in an appropriate range, if y has an *T-sequence* as its q-expansion, then y has exactly two expansions with respect to q namely the greedy (e_i) and lazy (δ_i). Such q is then a T-base number. In this chapter we ask the quation: given the real number y in an appropriate range what is the smallet with respect to lexicographic order *T-sequence.*

Let the sequence (e'_i) be given by

$$111\ 001\ 001\ 001\ \dots = 111\ \underline{\underline{001}};$$

the symbol $\underline{\underline{s}}$ denotes the period s of a periodic sequence. For each y, there is unique q' satisfying $\sum_{n=1}^{\infty} e'_i q'^{-i} = y$.

Theorem 16. *If* (e_i) *is a* T-sequence *q-expansion of* $y \in [\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1] \cap [\sum_{i=1}^{\infty} \frac{e'_i}{2^i}, 1]$ *which begins with* 111 *and with* m *satisfying* (34) *and* (35) *not a multiple of* 3*, then* $q \geq q'$.

Proof. Let (e_n) be the T- sequence of $y \in [\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1]$ for q which begins with 111 and m satisfying (34) and (35) not a multiple of 3. We show that $q \geq q'$.

By Lemma10, it suffices to show that $(e_i) \succeq (e'_i)$. Assume

$$(e_i) \prec (e'_i) = 111\ \underline{\underline{001}}. \tag{42}$$

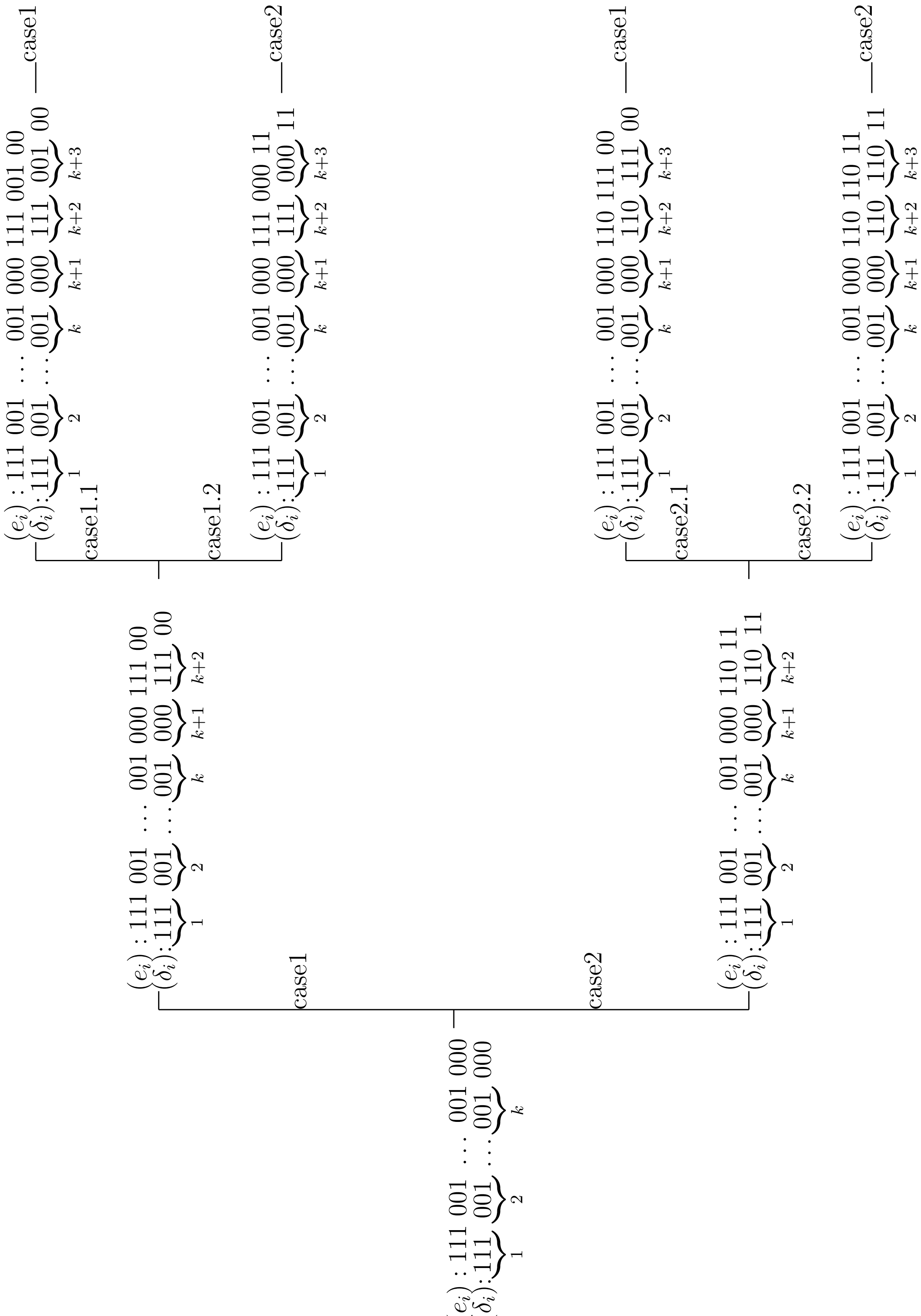
By the inequality (42), (e_i) begins with 11100. Thus (e_i) takes the form

$$\underbrace{111}_{1}\underbrace{001}_{2}\dots\underbrace{001}_{k}000 \quad \text{for some} \quad k \geq 2 \tag{43}$$

or 111000 which may be thought as (43) with $k = 1$. From (37) the sequence (e_i) also begins with (42), i.e., δ_i : $111001\dots001000$. Applying (38), we conclude that $\delta_{3k+4} = \delta_{3k+5} = 1$ (because $\delta_{3k} = 1$). Therefore, the sequence (e_i) also begins with

$$111\ 001\ \dots\ 001\ 000\ 11.$$

We distinguish two cases. The diagram below summarizes and provides a rapid check for some of the details following the arguments in the remaining proof of the theorem.


(e_i) : 111 001 ... 001 000
(δ_i) : 111 001 ... 001 000
case1
case2
(e_i) : 111 001 ... 001 000 111 00
(δ_i) : 111 001 ... 001 000 111 00
(e_i) : 111 001 ... 001 000 110 11
(δ_i) : 111 001 ... 001 000 110 11
case1.1
case1.2
case2.1
case2.2
(e_i) : 111 001 ... 001 000 111 001 00
(δ_i) : 111 001 ... 001 000 111 001 00
(e_i) : 111 001 ... 001 000 111 000 11
(δ_i) : 111 001 ... 001 000 111 000 11
(e_i) : 111 001 ... 001 000 110 111 00
(δ_i) : 111 001 ... 001 000 110 111 00
(e_i) : 111 001 ... 001 000 110 110 11
(δ_i) : 111 001 ... 001 000 110 110 11
case1
case2
case1
case2

Case 1: If $e_{3k+6} = 1$ then $\delta_{3k+6} = 1$ (since m cannot is a multiple of 3). From (32), $e_{3k+7} = e_{3k+8} = 0$ (because $e_{3k+3} = 0$).

Subcase 1.1: If $e_{3k+9} = 1$ then $\delta_{3k+9} = 1$ (since m cannot is a multiple of 3). From (32), $e_{3k+10} = e_{3k+11} = 0$ (because $e_{3k+3} = 0$). Thus the step repeats as in Case 1.

Subcase 1.2: If $e_{3k+9} = 0$ then $\delta_{3k+9} = 0$ (because $3k + 9 < m$ and (37)) Thus the step repeats as in Case 2.

Case 2: If $e_{3k+6} = 0$ then $\delta_{3k+6} = 0$ (because $3k + 6 < m$ and (37)). From (38), $\delta_{3k+7} = \delta_{3k+8} = 1$ (because $\delta_{3k} = 1$). Thus $e_{3k+7} = e_{3k+8} = 1$ (because $3k + 8 < m$ and (37)). Thus the step repeats as in Case 1.

Subcase 2.1: If $e_{3k+9} = 1$ then $\delta_{3k+9} = 1$ (since m cannot is a multiple of 3).

Subcase 2.2: If $e_{3k+9} = 0$ then $\delta_{3k+9} = 0$ (because $3k+9 < m$ and (37)). From (38), $\delta_{3k+10} = \delta_{3k+11} = 1$ (because $\delta_{3k} = 1$). Thus the step repeats as in Case 2.

Continuing in the same manner, we deduce that m must be bigger than any integer, which is impossible. □

Remark 14. Theorem 8 is a special case of Theorem 16 when $y = 1$.

Next we give an example of Theorem 16.

Example Let $y = \frac{3902563888221395449817251061561905663982412670490}{3914144333903073791808962606796280957916632792441}$ and $q = 1.9$. Denote the unique positive solution of the equation

$$\sum_{i=1}^{\infty} \frac{e'_i}{q'_i} = y$$

by $q' \approx 1.874535175$. From Theorem 8, when $y = 1$ we have $q' = 1.871349313$.

Theorem 17. *Let* $(e'_i) = 111\ \underline{\underline{001}}$. *For each* $y \in [\sum_{i=1}^{\infty} \frac{e'_i}{2^i}, 1]$, *there is an unique* $q' \in (1, 2]$ *such that* (e'_i) *is a* q'*-expansion of* y *and this* q'*-expansion is always unique.*

Proof. By Lemma 9, (e'_i) is the greedy q-expansion of y.
Since (e'_i) satisfying the condition of U-sequence. Thus from 12, For this y has a unique q-expansion. □

Theorem 18. *Let* $(e'_i) = 111\ \underline{\underline{001}}, y \in [\sum_{i=1}^{\infty} \frac{e'_i}{2^i}, 1]$ *there exists an unique* $q' \in (1, 2]$ *satisfying* $y = \sum_{i=1}^{\infty} \frac{e'_i}{q'^i}$. *Assume there exists T-sequence* (e_i)

i) The sequence (ε_i) *in definition of T-sequence of* (e_i) *is greedy q-expansion of 1 for* *some* $q \in (1, 2]$.

ii) For all $k \in \mathbb{N}$*, the sequence* (ε_i) *satisfying*

$$\overline{(\varepsilon_i)} \prec (e_i) \quad \textit{whenevere} \;\; \varepsilon_n = 1 \;\; \textit{and} \;\; 1 \leq n \leq 3k+3 \tag{44}$$

$$(\varepsilon_i) \prec (e_i) \quad \textit{whenevere} \;\; \varepsilon_n = 0 \;\; \textit{and} \;\; 1 \leq n \leq 3k+4 \tag{45}$$

$$\varepsilon_{3k+4+3\mathbb{N}} = 0 \quad \textit{whenevere} \;\; 3\mathbb{N} = \{3t; t \in \mathbb{N}\} \tag{46}$$

$$\varepsilon_{3k+2+3\mathbb{N}}\varepsilon_{3k+3+3\mathbb{N}} \neq 11 \tag{47}$$

$$\varepsilon_{3k+2+3\mathbb{N}}\varepsilon_{3k+3+3\mathbb{N}}, \varepsilon_{3k+5+3\mathbb{N}}\varepsilon_{3k+6+3\mathbb{N}} \neq 01, 10. \tag{48}$$

Then q' *is an accumulation point of the set of T-base numbers.*

Proof. We choose an arbitrarily large positive k and insert between the kth and $(k+1)$th block 001 a block $100\ldots0$ formed by one followed by $3k+4$ zeros. And (ε_i) is greedy with respect to q of 1 satisfying $(44)-(48)$. Then we obtain an *T-sequence* (e'_i) with $m = 3k+4$. One can readily verify that the corresponding numbers q tend to q' as $k \to \infty$ □

CONCLUSION

The main results in this thesis are:

Lemma. Let $q \in (1,2]$, $\sigma \geq 1$ and let (e_i) be the greedy q-expansion of σ.

- The greedy q-expansion, (a_i) of $x \in [0, 1/(q-1)]$ satisfies $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$.
- If the sequence (e_i) is finite with a last nonzero digit e_k, then no greedy q-expansion is eventually periodic with the period $e_1e_2\ldots e_{k-1}(e_k-1)$.

Lemma Let $1 < q \leq 2$.
(a) Let (e_i) be an **infinite** q-expansion of $y \in [0,1]$ and let (a_i) be a q-expansion of $x \in [0, \frac{1}{q-1}]$. If $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$, then (a_i) is the greedy q-expansion of x.
(b) Let (e_i) be a q-expansion of $y \in [0,1]$ and let (a_i) be a **finite** q-expansion of $x \in [0, \frac{1}{q-1}]$. If $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$, then (a_i) is the greedy q-expansion of x.
(c) Let (e_i) be a **finite** q-expansion of $y \in [0,1]$ and denote by e_k its last nonzero element. Let (a_i) be a q-expansion of $x \in [0, \frac{1}{q-1}]$. Assume $a_{n+1}a_{n+2}\ldots \prec e_1e_2\ldots$ whenever $a_n = 0$.

(c.1) If $y < 1$, then (a_i) is the greedy q-expansion of x.

(c.2) If $y = 1$ and assume that (a_i) is not eventually periodic with period $e_1\ldots e_{k-1}(e_k-1)$, then (a_i) is the greedy q-expansion of x.

Lemma Let $q, q' \in (1,2]$, $x \in [0, \frac{1}{q-1}] \cap [0, \frac{1}{q'-1}]$. Let (e_i), respectively, (e_i') be the greedy q-expansion, respectively q'-expansion of x. If $q < q'$, then $(e_i) \prec (e_i')$.

Theorem. If the number $\sigma \geq 1$ has a unique q-expansion for a given $q \in (1,2]$, then this unique q-expansion (ε_i) is an U-sequence.

Theorem. If the greedy q-expansion (ε_i) of $\sigma \leq 1$ with $q \in (1,2]$ is an U-sequence, then this q-expansion of σ is unique.

Theorem. Let (e_i) be a infinite sequence of zeros and ones, satisfying (15). Let $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right]$ and $q \in (1,2]$ such that $y = \sum_{i=1}^{\infty} \frac{e_i}{q^i}$. Let (ε_i) be the greedy q-expansion of 1. Assume (17) and (18). Define the sequence (δ_i) by (20), and assume that

$$\overline{\delta_{n+1}\delta_{n+2}\dots} \prec e_1 e_2 \dots \quad \text{whenever} \quad \delta_n = 1, \tag{49}$$

$$\overline{e_{n+1}e_{n+2}\dots} \prec e_1 e_2 \dots \quad \text{whenever} \quad e_n = 1 \ \text{ and } \ n > m, \tag{50}$$

$$\delta_{n+1}\delta_{n+2}\dots \prec e_1 e_2 \dots \quad \text{whenever} \quad \delta_n = 0 \ \text{ and } \ n > m. \tag{51}$$

Then y has exactly two different q-expansions, given by (33) and (37).

Theorem. Let (e_i) be a finite sequence of zeros and ones, with e_L its last nonzero element, satisfying (32). Let $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right]$ and $q \in (1,2]$ be such that $y = \sum_{i=1}^{\infty} \frac{e_i}{q^i}$. Let (ε_i) be the greedy q-expansion of 1. Assume (34) and (35). Define the sequence (δ_i) by (36), and assume that (38),(39),(40) and (31). Then y has exactly two different q-expansions, given by (33) and (37).

Theorem. Let (δ_i) be a sequence of zeros and ones defined recursively by:
• first set $\delta_1 = 1$;
• if $n \geq 0$ and if $\delta_1, \dots, \delta_{2^n}$ are already defined, then set $\delta_{2^n+k} = 1 - \delta_k$ for $1 \leq k < 2^n$ and $\delta_{2^{n+1}} = 1$.

If $y \in \left[\sum_{i=1}^{\infty} \frac{\delta_i}{2^i}, 1\right]$, then there is a smallest $q \in (1,2]$ for which y has unique q-expansion. This q is the unique positive solution of the equation $y = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}$.

Theorem. Let the sequence (e_i') be given by 111 001 001 001 $\dots = 111\ \underline{\underline{001}}$; the symbol $\underline{\underline{s}}$ denotes the period s of a periodic sequence. For each y, there is unique q' satisfying $\sum_{n=1}^{\infty} e_i' q'^{-i} = y$. If (e_i) is an T-sequence q-expansion of $y \in \left[\sum_{i=1}^{\infty} \frac{e_i}{2^i}, 1\right] \cap \left[\sum_{i=1}^{\infty} \frac{e_i'}{2^i}, 1\right]$ which begins with 111 and with m satisfying (34), (35) not a multiple of 3, then $q \geq q'$.

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