P-ISSN 2586-9000
E-ISSN 2586-9027
Homepage : https://tci-thaijo.org/index.php/SciTechAsia
Science & Technology Asia
Vol. 26 No.4 October - December 2021
Page: [182-196]
Original research article

Semi-Analytic Solutions of Electrohydrodynamic Flow in a Circular Cylinder Conduit

Vipul K. Baranwal[1], Ram K. Pandey[2,*], Manoj P. Tripathi[3], Harendra Singh[4], Sushila[5]

[1]*Department of Mathematics, Deochand College, Hajipur 844101, India*
[2]*Department of Mathematics, Dr. Hari Singh Gaur University, Sagar 07582, India*
[3]*Department of Mathematics, Udai Pratap Autonomous College, Varanasi 221003, India*
[4]*Department of Mathematics, Post-Graduate College, Ghazipur 233001, India*
[5]*Department of Physics, Vivekananda Global University, Jaipur 303012, India*

Received 09 August 2021; Received in revised form 14 November 2021
Accepted 05 December 2021; Available online 30 December 2021

ABSTRACT

This paper is intended to construct a new modification of the optimal homotopy asymptotic method that can be used to solve various nonlinear boundary value problem (BVP). This modification is called the modified optimal homotopy asymptotic method (MOHAM). The modification is based on the unique way of representation of nonlinear terms in different powers of embedding parameter q. We have tested the proposed method-MOHAM to the nonlinear BVP that reveals the electrohydrodynamic (EHD) flow of a fluid in an ion drag configuration in a circular cylindrical conduit. This is a singular second-order ordinary differential equation. We have also given the solution of the EHD flow equation using optimal homotopy asymptotic method (OHAM) by taking the linear operator $L\left(=L_2\right)=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$ different from the previous study. Also, we have made the comparison of solution obtained by our proposed method and the existing results.

Keywords: Electrohydrodynamic flow; Modified optimal homotopy asymptotic method; Nonlinear boundary value problem; Optimal homotopy asymptotic method; Square residual error

***Corresponding author:** pandeywavelet@gmail.com

1. Introduction

Mckee et al. [1] proposed the following nonlinear boundary value problem (BVP)

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+H^2\left(1-\frac{u}{1-\alpha u}\right)=0, \quad (1.1)$$

$0<r<1,$

subject to initial conditions

$$u'(0)=0, u(1)=0, \quad (1.2)$$

that reveals the electrohydrodynamic (EHD) flow of a fluid in an ion drag configuration in a circular cylindrical conduit. Where, $u(r)$ is the fluid velocity and r the radial distance from the center of the cylindrical conduit. H is the Hartman electric number. The parameter α defines the pressure gradient, ion mobility, and current density at the inlet of the conduit and also measures the nonlinearity in the equation.

For $\alpha \ll 1$ and $\alpha \gg 1$, the authors of [1] are obtained the perturbation solution of Eq. (1.1) as

$$\begin{aligned} u(r) &= 1-\frac{I_0(Hr)}{I_0(H)}+\alpha[(u_1(Hr) \\ &+C_1)I_0(Hr)+v_1(Hr)K_0(Hr)] \\ &+\alpha^2[(u_2(Hr)+C_2)I_0(Hr) \\ &+v_2(Hr)K_0(Hr)] \end{aligned} \quad (1.3)$$

and

$$\begin{aligned} u(r) &= \frac{H^2}{4}\left(1+\frac{1}{\alpha}\right)(1-r^2) \\ &+\frac{1}{\alpha^2}\left(2\int_0^r \frac{\log(1-s^2)}{s}ds+\frac{\pi^2}{6}\right) \end{aligned} \quad (1.4)$$

respectively.

In 1999, Paullet [2] obtained the solution of Eqs. (1.1-1.2) which was different in nature for $\alpha \gg 1$. In 2011, Mastroberardino [3] applied homotopy analysis method (HAM) to find solution of Eqs. (1.1-1.2). Later in 2012, Pandey et al. [4] construct two semi-analytical algorithms based on OHAM (with the linear operator $L(=L_1)=\frac{d^2}{dr^2}$) and optimal homotopy analysis method to solve Eqs. (1.-1.2) and support Paullet solutions for $\alpha \gg 1$. Other contributory articles for the solution of EHD flow equations can be seen in [5-6].

In this paper, we are going to:

(1) to propose a modification of OHAM for solving nonlinear differential equations.

(2) to construct two semi-analytical algorithms, one based on MOHAM and other already discussed by Pandey et al. [4] based on well established OHAM but with a different linear operator ($L(=L_2)=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$) to solve Eqs. (1.1-1.2), and

(3) to show the efficiency of OHAM that also depends on the choice of a linear operator and to shot that to apply OHAM, we have to select linear operator carefully.

2. Optimal Homotopy Asymptotic Method

Marinca et al. [7-11] construct this method to solve different nonlinear differential equations.

Consider the following general nonlinear differential equation with B as a boundary operator:

$$A(u(r))+f(r)=0,\ B(u)=0,\ r\in\Omega \Leftrightarrow$$
$$L(u(r))+f(r)+N(u(r))=0,\ B(u)=0, \quad (2.1)$$
$$r\in\Omega$$

where, $A=L+N$, L and N are linear and nonlinear operators respectively, r denotes the independent variable. $u(r)$ and $f(r)$ are unknown and known functions respectively.

We construct a homotopy $h(\varphi(r,q),q): R\times[0,1]\rightarrow R$ satisfying:

$$(1-q)[L(\varphi(r,q))+f(r)] = H(q)[L(\varphi(r,q))+f(r)+N(\varphi(r,q))], \quad B(\varphi(r,q))=0 \quad (2.2)$$

where, $q\in[0,1]$ is an embedding parameter and as q tends from 0 to 1, the function $\varphi(r,q)$ tends from the initial approximation $u_0(r)$ to the solution $u(r)$. The auxiliary function $H(q)\neq 0$ as $q\neq 0$ and $H(0)=0$. Here, we choose $H(q)$ as

$$H(q)=qC_1+q^2C_2+q^3C_3+\cdots \quad (2.3)$$

where, $C_1, C_2, C_3, \cdots$ are constants to be determined later.

We expand $\varphi(r,q)$ in powers of q as

$$\varphi(r,q,C_1,C_2,\cdots)=u_0(r)+u_1(r,C_1)q+u_2(r,C_1,C_2)q^2+\cdots. \quad (2.4)$$

Substituting the values of $\varphi(r,q)$ and $H(q)$ from Eqs. (2.3) - (2.4) into Eq. (2.2) and equating the coefficients of like powers of q, we obtain the following zero, first, second, and the mth order equations.

$$L(u_0(r))+f(r)=0, \; B(u_0)=0, \quad (2.5)$$

$$L(u_1(r))=L(u_0(r))+f(r)+C_1[L(u_0(r))+N_0(u_0(r))+f(r)], \quad B(u_1)=0, \quad (2.6)$$

$$L(u_2(r))=L(u_1(r))+C_1[L(u_1(r))+N_1(u_0(r),u_1(r))]+C_2[L(u_0(r))+N_0(u_0(r))+f(r)], \quad B(u_2)=0, \quad (2.7)$$

$$L(u_3(r))=L(u_2(r))+C_1[L(u_2(r))+N_2(u_0(r),u_1(r),u_2(r))]+C_2[L(u_1(r))+N_1(u_0(r),u_1(r))]+C_3[L(u_0(r))+N_0(u_0(r))+f(r)], \; B(u_3)=0, \quad (2.8)$$

$$\vdots$$

$$\begin{aligned}L(u_m(r))=L(u_{m-1}(r))&+C_1[L(u_{m-1})+N_{m-1}(u_0,u_1,\cdots,u_{m-1})]\\&+C_2[L(u_{m-2})+N_{m-2}(u_0,u_1,\cdots,u_{m-2})]\\&+\cdots+C_{m-1}[L(u_1(r))+N_1(u_0(r),u_1(r))]\\&+C_m[L(u_0(r))+N_0(u_0(r))+f(r)], \; B(u_m)=0,\end{aligned} \quad (2.9)$$

$$\vdots.$$

where $N_m(u_0,u_1,\cdots,u_m)$ is the coefficient of q^m in the expansion of $N(\phi(r,q))$ about the embedding parameter q.

Solving Eqs. (2.5) - (2.9), we get the mth iterate u_m, $m=0,1,2,\cdots$.

Substituting these values of u_m, $m=0,1,2,\cdots$ in Eq. (2.4) and taking $q\to 1$, we get

$$u(r,C_1,C_2,C_3\cdots)=u_0(r)+\sum_{k=1}^{\infty}u_k(r,C_1,C_2,\cdots,C_k) \quad (2.10)$$

Truncating Eq. (2.10) at $k=m$, the m[th] order solution is given by

$$\tilde{u}_m(r,C_1,C_2,\cdots,C_m)=u_0(r)+\sum_{i=1}^{m}u_i(r,C_1,C_2,\cdots,C_i) \quad (2.11)$$

In order to find the optimal values of C_i, $i=1,2,3,\cdots$, we first use Eq. (2.1) and Eq. (2.11) and construct the functional (-called the square residual error) as,

$$E_m(C_1,C_2,\cdots,C_m)=\int_{\Omega}R_m^2(r,C_1,C_2,\cdots,C_m)dr, \quad (2.12)$$

where,

$$R_m(r,C_1,C_2,\cdots,C_m)=L(\tilde{u}_m(r,C_1,C_2,\cdots,C_m))+f(r)+N(\tilde{u}_m(r,C_1,C_2,\cdots,C_m)). \quad (2.13)$$

and then minimizing it using

$$\frac{\partial E_m}{\partial C_1} = \frac{\partial E_m}{\partial C_2} = \cdots = \frac{\partial E_m}{\partial C_m} = 0. \qquad (2.14)$$

The m[th] order approximate solution $\tilde{u}_m$ is obtained by substituting these optimal values of C_i's into Eq. (2.11).

3. Modified Optimal Homotopy Asymptotic Method (MOHAM)

In modified OHAM, we expand $N(\varphi(r,q))$ in a power series with respect to the parameter q as

$$N(\varphi(r,q)) = N(u_0(r)) + \sum_{n=1}^{\infty} \begin{matrix}[N(u_0+u_1+\cdots+u_n) \\ -N(u_0+u_1+\cdots+u_{n-1})]q^n\end{matrix}. \qquad (3.1)$$

As the embedding parameter q increases from 0 to 1, $N(\varphi(r,q))$ varies from $N(u_0(r))$ to $N(u(r))$. Substituting Eq. (2.3), Eq. (2.4) and Eq. (3.1) into Eq. (2.2) and equating the coefficients of like powers of q, the zero, first, second, and the mth order problems are given as

$$L(u_0(r)) + f(r) = 0, \; B(u_0) = 0, \qquad (3.2)$$

$$\begin{aligned} L(u_1(r)) = {} & L(u_0(r)) + f(r) + C_1[L(u_0(r)) \\ & + N(u_0(r)) + f(r)], \; B(u_1) = 0, \end{aligned} \qquad (3.3)$$

$$\begin{aligned} L(u_2(r)) = {} & L(u_1(r)) + C_1[L(u_1(r)) \\ & + \{N(u_0(r) + u_1(r)) \\ & - N(u_0(r))\}] + C_2[L(u_0(r)) \\ & + N(u_0(r)) + f(r)], \; B(u_2) = 0, \end{aligned} \qquad (3.4)$$

$$\begin{aligned} L(u_3(r)) = {} & L(u_2(r)) + C_1[L(u_2(r)) \\ & + \{N(u_0(r) + u_1(r) + u_2(r)) \\ & - N(u_0(r) + u_1(r))\}] \\ & + C_2[L(u_1(r)) + \{N(u_0(r) + u_1(r)) \\ & - N(u_0(r))\}] + C_3[L(u_0(r)) \\ & + N(u_0(r)) + f(r)], \; B(u_3) = 0, \end{aligned} \qquad (3.5)$$

$\vdots$

$$\begin{aligned} L(u_m(r)) = {} & L(u_{m-1}(r)) + C_1[L(u_{m-1}) \\ & + \{N(u_0 + u_1 + \cdots + u_{m-1}) \\ & - N(u_0 + u_1 + \cdots + u_{m-2})\}] \\ & + C_2[L(u_{m-2}) \\ & + \{N(u_0 + u_1 + \cdots + u_{m-2}) \\ & - N(u_0 + u_1 + \cdots + u_{m-3})\}] \\ & + \cdots + C_{m-1}[L(u_1(r)) \\ & + \{N(u_0(r) + u_1(r)) \\ & - N(u_0(r))\}] + C_m[L(u_0(r)) \\ & + N(u_0(r)) + f(r)], \; B(u_m) = 0, \end{aligned} \qquad (3.6)$$

$\vdots$

Remark: The MOHAM mth order problem (3.6) is different from the corresponding OHAM mth order problem (2.9).

Solving Eqs. (3.2)-(3.6) we get the various iterates $u_m(r)$ and the mth order solution is given by

$$\tilde{u}_m(r, C_1, C_2, \cdots, C_m) = u_0(r) + \sum_{i=1}^{m} u_i(r, C_1, C_2, \cdots, C_i). \qquad (3.7)$$

Using Eq. (3.7) and Eqs. (2.12)-(2.14), the optimal values of C_i's are obtained. Substituting these optimal values of C_i's in Eq. (3.7), we get the mth order approximate solution $\tilde{u}_m$.

4. Analysis of the Methods for Electrohydrodynamic Flow Equation (1.1) with (1.2)

4.1 Optimal homotopy asymptotic method

We consider a linear operator $L\left(=L_2\right) = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$ different from linear operator $L\left(=L_1\right) = \frac{d^2}{dr^2}$ used by Pandey et al. [4] to obtain the corresponding OHAM

solutions of the BVP (1.1-1.2) which shows that our operator gives better result.
We take

$$L = L_2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

and

$$f(x) = H^2$$

and applying the OHAM as developed in section 2 to the BVP (1.1-1.2), the various order problems obtained from Eqs. (2.5-2.9) are as follows:
Zero order problem:

$$\frac{d^2u_0(r)}{dr^2} + \frac{1}{r}\frac{du_0(r)}{dr} + H^2 = 0, \quad . \qquad (4.1)$$
$$u_0'(1) = 0, u_0(0) = 0$$

First order problem:

$$\begin{aligned}&\frac{d^2u_1(r)}{dr^2} + \frac{1}{r}\frac{du_1(r)}{dr} = \frac{d^2u_0(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_0(r)}{dr} + H^2 + C_1[\frac{d^2u_0(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_0(r)}{dr} - \alpha u_0(r)\frac{d^2u_0(r)}{dr^2} \qquad (4.2)\\ &-\frac{\alpha}{r}u_0(r)\frac{du_0(r)}{dr} + H^2(1-(1+\alpha)u_0(r))],\\ &u_1'(1) = 0, u_1(0) = 0.\end{aligned}$$

Second order problem:

$$\begin{aligned}&\frac{d^2u_2(r)}{dr^2} + \frac{1}{r}\frac{du_2(r)}{dr} = \frac{d^2u_1(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_1(r)}{dr} + C_1[\frac{d^2u_1(r)}{dr^2} + \frac{1}{r}\frac{du_1(r)}{dr}\\ &-\alpha u_0(r)\frac{d^2u_1(r)}{dr^2} - \alpha u_1(r)\frac{d^2u_0(r)}{dr^2} \qquad (4.3\\ &-\frac{\alpha}{r}u_0(r)\frac{du_1(r)}{dr} - \frac{\alpha}{r}u_1(r)\frac{du_0(r)}{dr}\\ &-H^2(1+\alpha)u_0(r)] + C_2[Au_0 + H^2],\\ &u_2'(1) = 0, u_2(0) = 0.\end{aligned}$$

)

Third order problem:

$$\begin{aligned}&\frac{d^2u_3(r)}{dr^2} + \frac{1}{r}\frac{du_3(r)}{dr} = \frac{d^2u_2(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_2(r)}{dr} + C_1[\frac{d^2u_2(r)}{dr^2} + \frac{1}{r}\frac{du_2(r)}{dr}\\ &-\alpha u_0(r)\frac{d^2u_2(r)}{dr^2} - \alpha u_1(r)\frac{d^2u_1(r)}{dr^2}\\ &-\frac{\alpha}{r}u_0(r) - \alpha u_2(r)\frac{d^2u_0(r)}{dr^2}\frac{du_2(r)}{dr}\\ &-\frac{\alpha}{r}u_1(r)\frac{du_1(r)}{dr} - \frac{\alpha}{r}u_2(r)\frac{du_0(r)}{dr}\\ &-H^2(1+\alpha)u_2(r)] + C_2[\frac{d^2u_1(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_1(r)}{dr} - \alpha u_0(r)\frac{d^2u_1(r)}{dr^2}\\ &-\alpha u_1(r)\frac{d^2u_0(r)}{dr^2} - \frac{\alpha}{r}u_0(r)\frac{du_1(r)}{dr}\\ &-\frac{\alpha}{r}u_1(r)\frac{du_0(r)}{dr} - H^2(1+\alpha)u_1(r)] \qquad (4.4)\\ &+C_3[Au_0 + H^2], \quad u_3'(1) = 0, u_3(0) = 0\\ &\vdots\end{aligned}$$

The mth order problem:

$$\begin{aligned}&\frac{d^2u_m(r)}{dr^2} + \frac{1}{r}\frac{du_m(r)}{dr} = \frac{d^2u_{m-1}(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_{m-1}(r)}{dr} + C_1[\frac{d^2u_{m-1}(r)}{dr^2}\\ &+\frac{1}{r}\frac{du_{m-1}(r)}{dr} - \alpha u_0(r)\frac{d^2u_{m-1}(r)}{dr^2}\\ &-\alpha u_1(r)\frac{d^2u_{m-2}(r)}{dr^2} - \cdots\\ &-\alpha u_{m-1}(r)\frac{d^2u_0(r)}{dr^2} - \frac{\alpha}{r}u_0(r)\frac{du_{m-1}(r)}{dr}\\ &-\frac{\alpha}{r}u_1(r)\frac{du_{m-2}(r)}{dr} - \quad \cdots - \frac{\alpha}{r}u_{m-1}(r)\frac{du_0(r)}{dr}\end{aligned}$$

$$-H^2(1+\alpha)u_{m-1}(r)]+C_2[\frac{d^2u_{m-2}(r)}{dr^2}$$
$$+\frac{1}{r}\frac{du_{m-2}(r)}{dr}-\alpha u_0(r)\frac{d^2u_{m-2}(r)}{dr^2}$$
$$-\alpha u_1(r)\frac{d^2u_{m-3}(r)}{dr^2}-\cdots$$
$$-\alpha u_{m-2}(r)\frac{d^2u_0(r)}{dr^2}-\frac{\alpha}{r}u_0(r)\frac{du_{m-2}(r)}{dr}$$
$$-\frac{\alpha}{r}u_1(r)\frac{du_{m-3}(r)}{dr}-\cdots-\frac{\alpha}{r}u_{m-2}(r)\frac{du_0(r)}{dr}$$
$$-H^2(1+\alpha)u_{m-2}(r)]+\cdots$$
$$+C_{m-1}[\frac{d^2u_1(r)}{dr^2}+\frac{1}{r}\frac{du_1(r)}{dr}-\alpha u_0(r)\frac{d^2u_1(r)}{dr^2}$$
$$-\alpha u_1(r)\frac{d^2u_0(r)}{dr^2}-\frac{\alpha}{r}u_0(r)\frac{du_1(r)}{dr}$$
$$-\frac{\alpha}{r}u_1(r)\frac{du_0(r)}{dr}-H^2(1+\alpha)u_1(r)]$$
$$+C_m[Au_0+H^2],\ \ u_m'(1)=0, u_m(0)=0, \tag{4.5}$$

$\vdots$.

Solving these equations (4.1) - (4.5), we get the various iterates as:

$$u_0(r)=\frac{1}{4}(H^2-H^2r^2), \tag{4.6}$$

$$u_1(r)=\frac{1}{64}H^4C_1(3-4r^2+r^4), \tag{4.7}$$

$$u_2(r)=\frac{1}{2304}H^4(-1+r^2)(36C_1(-3+r^2)$$
$$+36C_2(-3+r^2)+C_1^2(36(-3+r^2)$$
$$+H^2(-19+r^2(8-14\alpha)$$
$$+22\alpha+r^4(-1+4\alpha)))), \tag{4.8}$$

$$u_3(r)=\frac{1}{147456}H^4(-1+r^2)(2304(C_2+C_3)$$
$$(-3+r^2)+128C_1^2(36(-3+r^2)+H^2(-19$$
$$+r^2(8-14\alpha)+22\alpha+r^4(-1+4\alpha)))$$
$$+C_1^3(2304(-3+r^2)+128H^2(-19$$
$$+r^2(8-14\alpha)+22\alpha+r^4(-1+4\alpha))$$
$$+H^4(-211+738\alpha-300\alpha^2+r^4(-15+170\alpha$$
$$-156\alpha^2)+r^6(1-22\alpha+36\alpha^2)$$
$$+r^2(93-478\alpha+276\alpha^2)))$$
$$+128C_1(18(-3+r^2)+C_2(36(-3+r^2)$$
$$+H^2(-19+r^2(8-14\alpha)+22\alpha$$
$$+r^4(-1+4\alpha))))), \tag{4.9}$$

$\vdots$.

Substituting the above iterations in (2.11), the mth order approximate solution for (1.1) using L_2 is obtained as

$$\tilde{u}_m(r,C_1,C_2,C_3,\cdots,C_m)=u_0(r)+u_1(r,C_1)$$
$$+u_2(r,C_1,C_2)+\cdots+u_m(r,C_1,C_2,\cdots,C_m). \tag{4.10}$$

From (2.12) the mth order residual is

$$R_m(r,C_1,C_2,C_3,\cdots,C_m)=\frac{d^2\tilde{u}_m}{dr^2}+\frac{1}{r}\frac{d\tilde{u}_m}{dr}$$
$$+H^2(1-\frac{\tilde{u}_m}{1-\alpha\tilde{u}_m}). \tag{4.11}$$

Using (2.13) and (4.11), we solve (2.14) to get the optimal values of convergence control parameters $C_1,C_2,C_3,\cdots,C_m$ that minimize the exact square residual error E_m.

4.2 Modified optimal homotopy asymptotic method

Taking

$$L=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr},\ f(r)=H^2$$

and applying the MOHAM as developed in section 3 to the BVP (1.1-1.2), the various order problems (3.2-3.5) for Eq. (1.1) with the boundary conditions (1.2) becomes

Zeroth order problem:

$$\frac{d^2u_0(r)}{dr^2}+\frac{1}{r}\frac{du_0(r)}{dr}+H^2=0,$$
$$u_0'(1)=0, u_0(0)=0$$

that gives $u_0(r)=\frac{1}{4}(H^2-H^2r^2)$. (4.12)

First order problem:

$$\frac{d^2u_1(r)}{dr^2}+\frac{1}{r}\frac{du_1(r)}{dr}=\frac{d^2u_0(r)}{dr^2}+\frac{1}{r}\frac{du_0(r)}{dr}+H^2+C_1[\frac{d^2u_0(r)}{dr^2}+\frac{1}{r}\frac{du_0(r)}{dr}-\alpha u_0(r)\frac{d^2u_0(r)}{dr^2}-\frac{\alpha}{r}u_0(r)\frac{du_0(r)}{dr}+H^2(1-(1+\alpha)u_0(r))], \quad (4.13)$$
$$u_1'(1)=0, u_1(0)=0.$$

Second order problem:

$$\frac{d^2u_2(r)}{dr^2}+\frac{1}{r}\frac{du_2(r)}{dr}=\frac{d^2u_1(r)}{dr^2}+\frac{1}{r}\frac{du_1(r)}{dr}+C_1[\frac{d^2u_1(r)}{dr^2}+\frac{1}{r}\frac{du_1(r)}{dr}-\alpha\{(u_0(r)+u_1(r))\frac{d^2}{dr^2}(u_0(r)+u_1(r))-u_0(r)\frac{d^2u_0(r)}{dr^2}\}-\frac{\alpha}{r}\{(u_0(r)+u_1(r))\frac{d}{dr}(u_0(r)+u_1(r))-u_0(r)\frac{du_0(r)}{dr}\}-H^2(1+\alpha)u_1(r)]+C_2[Au_0+H^2], u_2'(1)=0, u_2(0)=0. \quad (4.14)$$

Third order problem:

$$\frac{d^2u_3(r)}{dr^2}+\frac{1}{r}\frac{du_3(r)}{dr}=\frac{d^2u_2(r)}{dr^2}+\frac{1}{r}\frac{du_2(r)}{dr}+C_1[\frac{d^2u_2(r)}{dr^2}+\frac{1}{r}\frac{du_2(r)}{dr}-\alpha\{(u_0(r)+u_1(r)+u_2(r))\frac{d^2}{dr^2}(u_0(r)+u_1(r)+u_2(r))-(u_0(r)+u_1(r))\frac{d^2}{dr^2}(u_0(r)+u_1(r))\}-\frac{\alpha}{r}\{(u_0(r)+u_1(r)+u_2(r))\frac{d}{dr}(u_0(r)+u_1(r)+u_2(r))-(u_0(r)+u_1(r))\frac{d}{dr}(u_0(r)+u_1(r))\}-H^2(1+\alpha)u_2(r)]+C_2[\frac{d^2u_1(r)}{dr^2}+\frac{1}{r}\frac{du_1(r)}{dr}-\alpha\{(u_0(r)+u_1(r))\frac{d^2}{dr^2}(u_0(r)+u_1(r))-u_0(r)\frac{d^2u_0(r)}{dr^2}\}-\frac{\alpha}{r}\{(u_0(r)+u_1(r))\frac{d}{dr}(u_0(r)+u_1(r))-u_0(r)\frac{du_0(r)}{dr}\}-H^2(1+\alpha)u_1(r)]+C_3[Au_0+H^2],\ u_3'(1)=0, u_3(0)=0, \quad (4.15)$$

$\vdots$

The m[th] order problem:

$$\frac{d^2u_m(r)}{dr^2}+\frac{1}{r}\frac{du_m(r)}{dr}=\frac{d^2u_{m-1}(r)}{dr^2}+\frac{1}{r}\frac{du_{m-1}(r)}{dr}+C_1[\frac{d^2u_{m-1}(r)}{dr^2}+\frac{1}{r}\frac{du_{m-1}(r)}{dr}-\alpha\{(u_0+u_1+u_2+\cdots u_{m-1})\frac{d^2}{dr^2}(u_0+u_1+u_2+\cdots u_{m-1})-(u_0+u_1+u_2+\cdots u_{m-2})\frac{d^2}{dr^2}(u_0+u_1+u_2+\cdots u_{m-2})\}-\frac{\alpha}{r}\{(u_0+u_1+u_2+\cdots u_{m-1})$$

$$
\begin{aligned}
&\frac{d}{dr}(u_0+u_1+u_2+\cdots u_{m-1})-(u_0+u_1+u_2+\cdots u_{m-2})\\
&\frac{d}{dr}(u_0+u_1+u_2+\cdots u_{m-2})\}-H^2(1+\alpha)u_{m-1}(r)]\\
&+C_2[\frac{d^2u_{m-2}(r)}{dr^2}+\frac{1}{r}\frac{du_{m-2}(r)}{dr}\\
&-\alpha\{(u_0+u_1+u_2+\cdots u_{m-2})\frac{d^2}{dr^2}(u_0+u_1+u_2+\cdots u_{m-2})\\
&-(u_0+u_1+u_2+\cdots u_{m-3})\frac{d^2}{dr^2}(u_0+u_1+u_2+\cdots u_{m-3})\}\\
&-\frac{\alpha}{r}\{(u_0+u_1+u_2+\cdots u_{m-2})\frac{d}{dr}(u_0+u_1+u_2+\cdots u_{m-2})\\
&-(u_0+u_1+u_2+\cdots u_{m-3})\frac{d}{dr}(u_0+u_1+u_2+\cdots u_{m-3})\}\\
&-H^2(1+\alpha)u_{m-2}(r)]+\cdots+C_{m-1}[\frac{d^2u_1(r)}{dr^2}+\frac{1}{r}\frac{du_1(r)}{dr}\\
&-\alpha\{(u_0(r)+u_1(r))\frac{d^2}{dr^2}(u_0(r)+u_1(r))-u_0(r)\frac{d^2u_0(r)}{dr^2}\}\\
&-\frac{\alpha}{r}\{(u_0(r)+u_1(r))\frac{d}{dr}(u_0(r)+u_1(r))\\
&-u_0(r)\frac{du_0(r)}{dr}\}-H^2(1+\alpha)u_1(r)]\\
&+C_m[Au_0+H^2],\quad u_m{}'(1)=0,u_m(0)=0.
\end{aligned}
\tag{4.16}
$$

$\vdots$.

Solving these equations (4.13)-(4.16), we get the various iterates as:

$$u_1(r)=\frac{1}{64}H^4C_1(3-4r^2+r^4), \tag{4.17}$$

$$
\begin{aligned}
&u_2(r)=-\frac{1}{147456}H^4(-1+r^2)(-2304C_1\\
&(-3+r^2)-2304C_2(-3+r^2)+H^4C_1^3\\
&(-251+181r^2-71r^4+9r^6)\alpha-64C_1^2\\
&(36(-3+r^2)+H^2(-19+r^2(8-14\alpha)\\
&+22\alpha+r^4(-1+4\alpha)))),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&u_3(r)=-\frac{1}{106542032486400}H^4(-1+r^2)\\
&(-1664719257600(C_2+C_3)(-3+r^2)\\
&+H^{12}C_1^7(-244819371+286497429r^2\\
&-252051771r^4+142692229r^6-52009271r^8\\
&+12313225r^{10}-1700775r^{12}+99225r^{14})\alpha^3\\
&+722534400C_1^2(-4608(-3+r^2)\\
&+5H^4C_2(-251+181r^2-71r^4+9r^6)\alpha\\
&-128H^2(-19+r^2(8-14\alpha)+22\alpha\\
&+r^4(-1+4\alpha)))-451584H^4C_1^4\alpha(-8000(-251\\
&+181r^2-71r^4+9r^6)+H^4C_2(-29427\\
&+28073r^2-18702r^4+7298r^6-1527r^8\\
&+125r^{10})\alpha-16H^2(-21863+r^2(15812\\
&-23990\alpha)+r^6(1412-4590\alpha)+26010\alpha\\
&+r^8(-113+560\alpha)+6r^4(-1148+2435\alpha)))\\
&-256H^8C_1^6\alpha^2(1764(-29427+28073r^2\\
&-18702r^4+7298r^6-1527r^8+125r^{10})\\
&+H^2(-9165783+r^2(9190842-12002982\alpha)\\
&+r^6(2860042-5003332\alpha)+10708518\alpha\\
&+5625r^{12}(-1+4\alpha)-250r^{10}(-394+1135\alpha)\\
&+r^8(-723083+1545518\alpha)+r^4(-6585933\\
&+9848568\alpha)))+28901376C_1^3(-57600(-3+r^2)\\
&-3200H^2(-19+r^2(8-14\alpha)+22\alpha\\
&+r^4(-1+4\alpha))+H^6C_2\alpha(-2213+r^2(1712-2323\alpha)\\
&+2577\alpha+13r^8(-1+4\alpha)-9r^6(-18+47\alpha)\\
&+r^4(-788+1377\alpha))-25H^4(-211+1491\alpha\\
&+502C_2\alpha-300\alpha^2\\
&+r^4(-15+383\alpha+142C_2\alpha-156\alpha^2)
\end{aligned}
$$

$$+r^6(1-49\alpha-18C_2\alpha+36\alpha^2)+r^2(93-1021\alpha$$
$$-362C_2\alpha+276\alpha^2)))+100352H^4C_1^5\alpha(7200(-251$$
$$+181r^2-71r^4+9r^6)+288H^2(-2213+r^2(1712$$
$$-2323\alpha)+2577\alpha+13r^8(-1+4\alpha)-9r^6$$
$$(-18+47\alpha)+r^4(-788+1377\alpha))+H^4(-56211$$
$$+395859\alpha-76488\alpha^2+r^8(-886+18959\alpha$$
$$-6688\alpha^2)+25r^{10}(2-61\alpha+32\alpha^2)$$
$$-3r^4(8087-82978\alpha+20696\alpha^2)$$
$$+r^6(6539-94666\alpha+27512\alpha^2)$$
$$+r^2(46389-377241\alpha+81912\alpha^2)))$$
$$+722534400C_1(-2304(-3+r^2)$$
$$+H^4C_2^2(-251+181r^2-71r^4+9r^6)\alpha$$
$$-128C_2(36(-3+r^2)+H^2(-19$$
$$+r^2(8-14\alpha)+22\alpha+r^4(-1+4\alpha))))), \tag{4.19}$$

$\vdots$.

Substituting the above iterations in (3.7), we obtained the m[th] order approximate solution for (1.1) as

$$\tilde{u}_m(r,C_1,C_2,C_3,\cdots,C_m)=u_0(r)+u_1(r,C_1)$$
$$+u_2(r,C_1,C_2)+\cdots+u_m(r,C_1,C_2,\cdots,C_m). \tag{4.20}$$

The m[th] order residual for Eq. (1.1) is obtained by substituting Eq. (4.20) in Eq. (2.12) and is given by

$$R_m(r,C_1,C_2,C_3,\cdots,C_m)=\frac{d^2\tilde{u}_m}{dr^2}+\frac{1}{r}\frac{d\tilde{u}_m}{dr}$$
$$+H^2\left(1-\frac{\tilde{u}_m}{1-\alpha\tilde{u}_m}\right). \tag{4.21}$$

Using (2.13) and (4.21), we solve (2.14) to get the optimal values of convergence control parameters $C_1,C_2,C_3,\cdots,C_m$ that minimize the exact square residual error E_m.

5. Convergence of the Solutions

From Figs. 1-2, we see that the square residual error E_3 obtained from MOHAM is more stable even for larger deviations from the its optimal value of C_1 as compared to square residual errors E_3 to OHAM with operator L_2, E_4 to OHAM with operator L_1 (see Fig. 1 of [4]) corresponding to their optimal values of C_1 and E_{20} to HAM (see Fig. 2 of [3]) corresponding to the optimal value of $\hbar$. Thus, we conclude that the sensitivity of E_3 (OHAM corresponding to the operator L_2) is much less compare to the sensitivity of E_4 (OHAM corresponding to the operator L_1) with respect to the variation in C_1 about their optimal values and E_{20} (HAM) with respect to the variation in $\hbar$ about its optimal value but this is larger than the sensitivity of E_3 (MOHAM) with respect to the variation in C_1 about its optimal values. In Figs. 3-8 we plot the residual functions $R_3(r)$ for the optimal values of the convergence control parameters C_1,C_2,C_3 (given in Tables 1-3) and the different values of the parameters $\alpha=0.5,1$ and $H^2=4$. These plots demonstrate the accuracy of MOHAM solution over OHAM with operator L_2, OHAM with operator L_1 (see Figs. (5-6) of [4]) and HAM solution (see Fig. 6 of [3]).

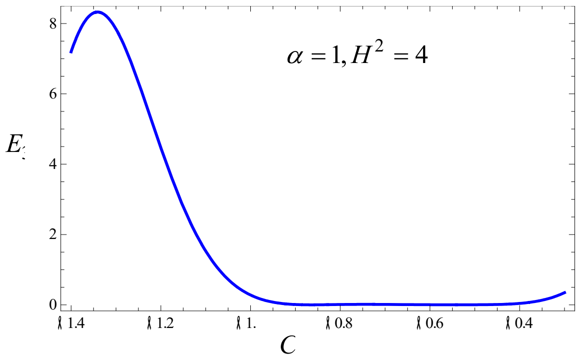

Fig. 1. E_3 for the 3[rd] order MOHAM $C_2=-0.0219947, C_3=0.0143855$.

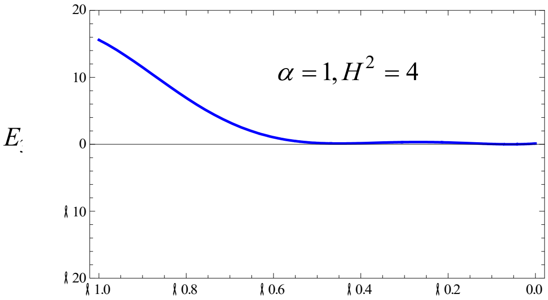

Fig. 2. E_3 for the 3rd order OHAM (corresponding to L_2) $C_2 = -0.943826$, $C_3 = 1.1979$.

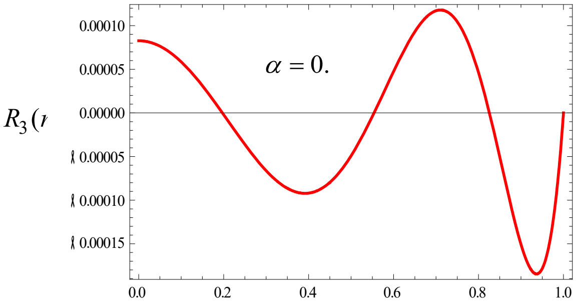

Fig. 3. The residual of the 3rd order MOHAM solution for $H^2 = 4, \alpha = 0.5$.

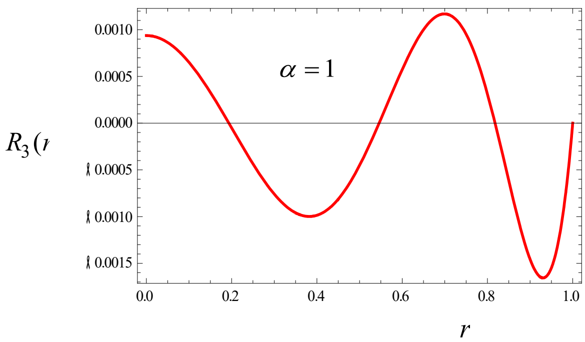

Fig. 4. The residual of the 3rd order MOHAM solution for $H^2 = 4, \alpha = 1$.

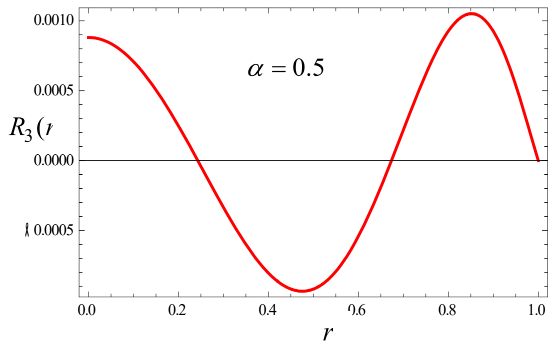

Fig. 5. The residual of the 3rd order OHAM corresponding to L_2 solution for $H^2 = 4, \alpha = 0.5$.

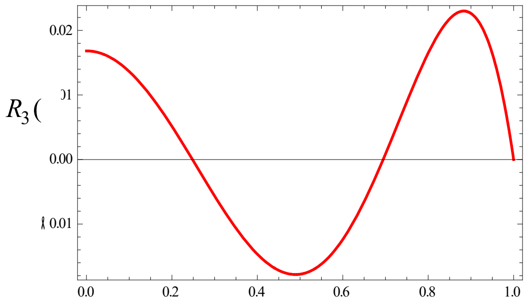

Fig. 6. The residual of the 3rd order OHAM corresponding to L_2 solution for $H^2 = 4, \alpha = 1$.

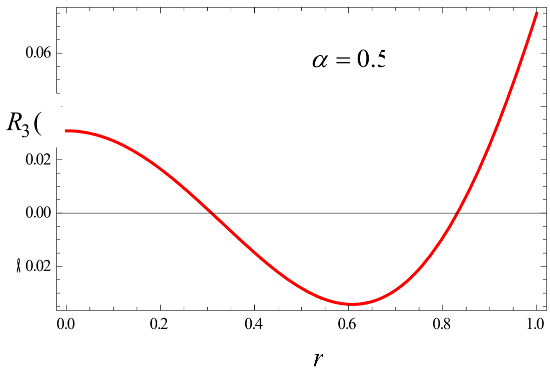

Fig. 7. The residual of the 3rd order OHAM corresponding to L_1 solution for $H^2 = 4, \alpha = 0.5$.

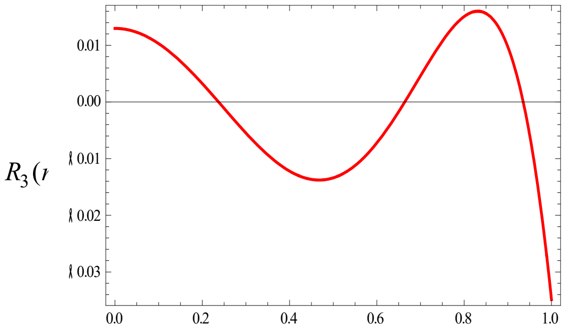

Fig. 8 The resid r the 3rd order OHAM corresponding to L_1 solution for $H^2 = 4, \alpha = 1$.

6. Discussion

Case (i) For $\alpha << 1$, the solution obtained by our proposed algorithms are similar to that of obtained by [1-4] as is evident from Figs (9-11). The third order MOHAM solution profiles are shown in Figs. 9-10 for $H^2 = 1$ and 10, respectively. Whereas, for $H^2 = 100$, fourth order MOHAM solution profile is plotted in Fig. 11. We also get same figures using OHAM corresponding to the operator L_2.

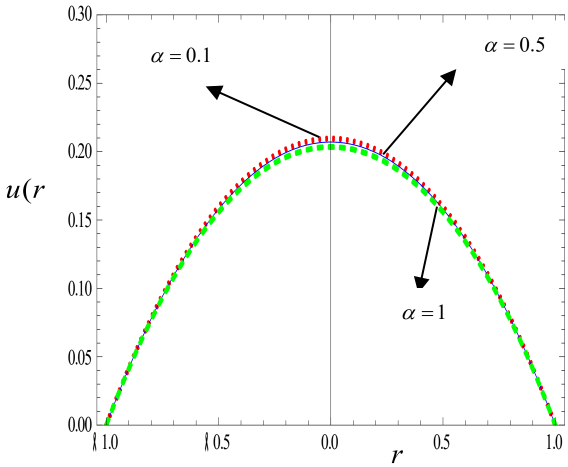

Fig. 9. 3rd order MOHAM Solution $u(r)$, for $H^2 = 1$ and $\alpha = 0.1, 0.5, 1$.

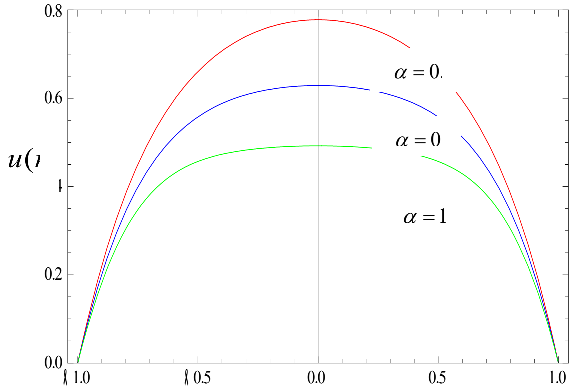

Fig. 10. 3rd order MOHAM Solutio r, for $H^2 = 10$ and $\alpha = 0.1, 0.5, 1$.

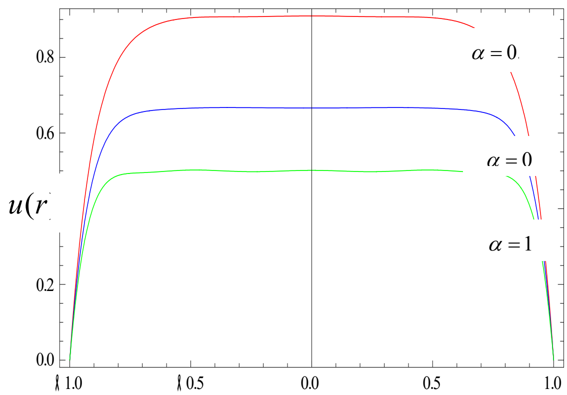

Fig. 11 4th order MOHAM Solution r for $H^2 = 100$ and $\alpha = 0.1, 0.5, 1$.

Case (ii) For $\alpha >> 1$, the solution of MOHAM are shown in Figs. 12-14 for $\alpha = 4, 10$ and $H^2 = 1, 10, 20$. For lower value of $H^2 = 1$, third order solution profiles are given in Fig. 12 whereas for higher values of $H^2 = 10$ and 20, sixth order solution profiles are drawn in Figs. 13 and 15 respectively. Fig. 13 shows that the solution profile obtained from MOHAM solution for large values of H^2 and α is better than OHAM solution (see Fig. 4 of [4]).

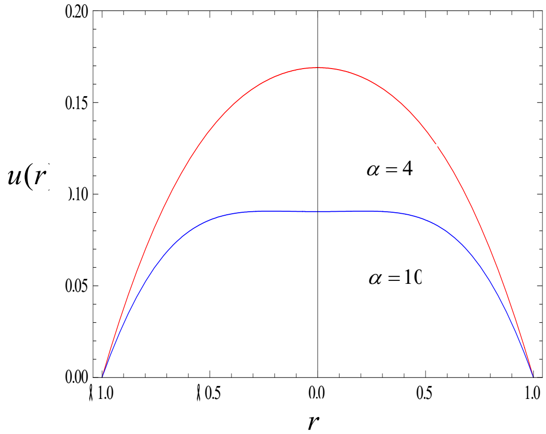

Fig. 12. 3[rd] order MOHAM Solution $u(r)$, for $H^2 = 1$ and $\alpha = 4$ (Red), $\alpha = 10$ (Blue).

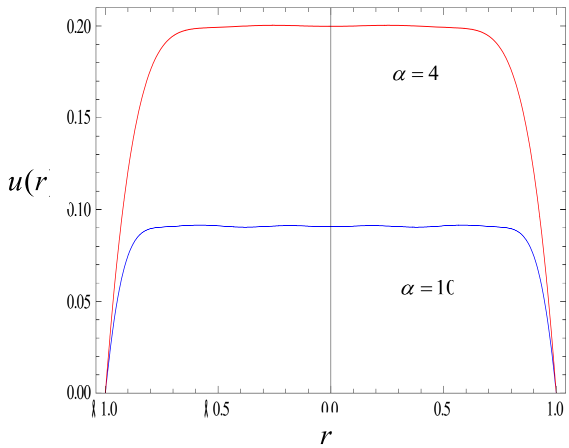

Fig. 13 6[th] order MOHAM Solution $u(r)$, for $H^2 = 10$ and $\alpha = 4$ (Red), $\alpha = 10$ (Blue)

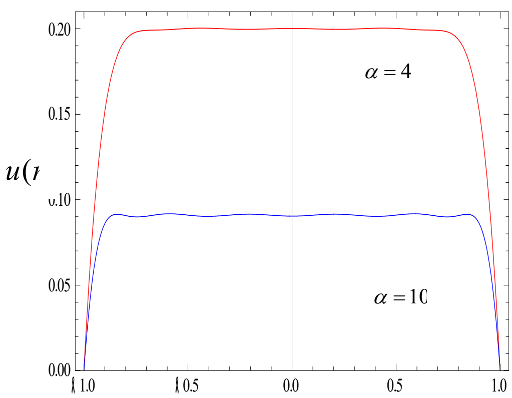

Fig. 14 6[th] order MOHAM Solution $u(r)$, for $H^2 = 20$ and $\alpha = 4$ (Red), $\alpha = 10$ (Blue).

7. Comparison of MOHAM Solution with OHAM and HAM Solutions

Tables 1-2 shows the square residual errors E_3 (MOHAM), E_3 (OHAM corresponding to the operators L_1 and L_2) and E_{20} (HAM) at various values of parameters H^2 and $\alpha = 0.5, 1$ respectively. The square residual errors E_3 (MOHAM), E_3 (OHAM corresponding to the operators L_1 and L_2) and E_9 (HAM) at various values of parameters H^2 and $\alpha = 1.5$ are given in Table 3. From these tables we observed that the square residual errors E_3 (obtained by MOHAM and OHAM corresponding to the operator L_2) are more accurate as comparison to square residual errors E_3 obtained by OHAM corresponding to the operator L_1 and E_{20} / E_9 (obtained by HAM) for all values of α and H^2, whereas for higher values of α and H^2 the square residual error E_3 obtained by MOHAM solution is better than the square residual error E_3 obtained by OHAM solutions corresponding to the both the linear operators L_1 and L_2.

Table 1. Comparison of square residual error at $\alpha = 0.5$ and different values of H^2 by MOHAM, OHAM, and HAM.

Auxiliary Parameters		$\alpha = 0.5$, $H^2 = 0.5$	$\alpha = 0.5$, $H^2 = 1$	$\alpha = 0.5$, $H^2 = 2$	$\alpha = 0.5$, $H^2 = 4$
MOHAM	C_1	-0.783089	-0.762948	-0.939042	-0.808199
	C_2	-0.0697495	-0.0785508	0.000332704	-0.0155496
	C_3	0.0196492	0.0215928	0.00420497	0.00856689
	E_3	6.31664×10^{-13}	7.45912×10^{-11}	1.47635×10^{-10}	6.99385×10^{-9}
OHAM (L_1) [4]	C_1	-0.108084	-0.128187	-0.064627	-0.0236008
	C_2	-0.478377	-0.295243	-0.564313	-0.78968
	C_3	0.527761	0.270073	0.724121	1.19611
	E_3	7.527668×10^{-12}	1.51605×10^{-8}	5.41812×10^{-6}	7.36637×10^{-4}
OHAM (L_2)	C_1	-0.691588	-0.576155	-1.18931	-1.26823
	C_2	-0.057552	-0.0416597	-0.380533	-1.13223
	C_3	0.0132138	-0.00925248	-0.310485	-1.72174
	E_3	1.17153×10^{-12}	6.14706×10^{-10}	4.14274×10^{-8}	4.60801×10^{-7}
HAM [4]	$\hbar$	-0.375	-0.276	-0.275	-0.205
	E_{20}	7.772×10^{-12}	1.230×10^{-9}	5.319×10^{-8}	4.568×10^{-5}

Table 2. Comparison of square residual error at $\alpha = 1$ and different values of H^2 by MOHAM, OHAM, and HAM.

Auxiliary Parameters		$\alpha = 1$, $H^2 = 0.5$	$\alpha = 1$, $H^2 = 1$	$\alpha = 1$, $H^2 = 2$	$\alpha = 1$, $H^2 = 4$
MOHAM	C_1	-1.04278	-1.07716	-1.09016	-0.863868
	C_2	-0.00136297	-0.00604687	-0.00330252	-0.0219947
	C_3	4.92×10^{-6}	0.000342812	0.0169006	0.0143855
	E_3	1.22755×10^{-13}	6.00633×10^{-11}	6.17062×10^{-9}	6.95743×10^{-7}
OHAM (L_1)	C_1	-0.377725	-0.35981	-0.30792	-0.168637
	C_2	0.0420993	0.0623337	0.06334952	-0.1809055
	C_3	-0.01487	-0.0164235	-0.00274481	0.275378
	E_3	2.591989×10^{-11}	1.55047×10^{-9}	1.05204×10^{-7}	1.17518×10^{-4}
OHAM (L_2)	C_1	-0.77256	-0.709368	-0.172825	-0.0562493
	C_2	-0.0476014	-0.0269294	-1.23212	-0.943826
	C_3	0.0137832	0.00875074	1.56893	1.1979
	E_3	5.23876×10^{-12}	7.85814×10^{-12}	1.44811×10^{-7}	1.87544×10^{-4}
HAM	$\hbar$	-0.303	-0.292	-0.254	-0.198
	E_{20}	4.634×10^{-11}	4.996×10^{-9}	2.363×10^{-6}	3.461×10^{-4}

Table 3. Comparison of square residual error at $\alpha = 1.5$ and different values of H^2 by MOHAM, OHAM, and HAM.

Auxiliary Parameters		$\alpha = 1.5$ $H^2 = 0.5$	$\alpha = 1.5$ $H^2 = 1$	$\alpha = 1.5$ $H^2 = 2$	$\alpha = 1.5$ $H^2 = 4$
MOHAM	C_1	-0.935865	-1.1528	-1.18272	-0.920666
	C_2	-0.0295867	-0.00986575	-0.00291394	-0.047259
	C_3	0.0045882	0.000102707	0.0268436	0.0110118
	E_3	1.7622×10^{-10}	1.30481×10^{-9}	9.77178×10^{-8}	6.96376×10^{-6}
OHAM (L_1) [4]	C_1	-0.429884	-0.443299	-0.6898	-0.499435
	C_2	0.074246	0.139540	0.0298953	0.07564
	C_3	-0.0128337	-0.02021099	0.5407644	1.06037
	E_3	2.0763995×10^{-9}	3.8790×10^{-7}	2.6342×10^{-6}	1.5866×10^{-4}
OHAM (L_2)	C_1	-0.954502	-0.850206	-0.408103	-0.0590405
	C_2	0.0127759	0.0351817	-0.298564	-1.13711
	C_3	0.0000271039	-0.00107114	0.249194	1.54655
	E_3	3.06721×10^{-13}	1.52715×10^{-11}	1.12244×10^{-7}	6.06947×10^{-4}
HAM [4]	$\hbar$	-0.393372	-0.0164281	-0.6237173	-0.32296
	E_9	8.195638×10^{-8}	8.1234×10^{-4}	0.177825	0.551513

8. Conclusion

In this paper, we have successfully proposed a modification of OHAM (called MOHAM) to solve nonlinear differential equations. Two semi-analytical algorithms based on modified optimal homotopy asymptotic method (MOHAM) and optimal homotopy asymptotic method (OHAM) corresponding to linear operator L_2 are developed to solve EHD flow equation. Solution profile by MOHAM and OHAM are shown in figures. The norm of square residual error is also calculated and comparison is shown in various tables. From above discussions, it is observed that the semi-analytical solution obtained by MOHAM always give better results in comparison to the OHAM (corresponding to both the operators L_1 and L_2) and HAM. Thus, MOHAM is more effective semi-analytical method for solving nonlinear problems in comparison to OHAM and HAM. Also, the OHAM solution corresponding to the operator L_2 is better than the OHAM solution corresponding to the operator L_1. Therefore, the efficiency of OHAM depends upon the choice of the linear operators and before applying OHAM we have to select linear operators carefully.

References

[1] Mckee S. Calculation of electrohydrodynamic flow in a circular cylindrical conduit. Z Angew Math Mech 1997; 77: 457-65.

[2] Paullet JE. On the solution of electrohydrodynamic flow in a circular cylindrical conduit. Z Angew Math Mech 1999; 79: 357-60.

[3] Mastroberardino A. Homotopy analysis method applied to electrohydrodynamic

flow. Commun Nonlinear Sci Numer Simulat 2011; 16: 2730-6.

[4] Pandey RK, Baranwal VK, Singh CS, Singh OP. Semi-analytic algorithms for the electrohydrodynamic flow equation. Journal of Theoretical and Applied Physics 2012; 6:45.

[5] Tiwari S, Pandey RK, Solution of Electrohydrodynamic Flow Equation Using an Orthogonal Collocation Method Based on Mixed Interpolation, Journal of Applied Mathematics and Physics 6 (03), 2018, 573-87.

[6] Khan NA, Jamil M, Mahmood A, Ara1 A, Approximate Solution for the Electrohydrodynamic Flow in a Circular Cylindrical Conduit International Scholarly Research Network ISRN Computational Mathematics, 2012, 1-5.

[7] Herisanu N, Marinca V, Dordea T, Madescu G. A new analytical approach to nonlinear vibration of an electrical machine. Proceedings of the Romanian Academy, Series A 2008; 9(3): 229-36.

[8] Marinca V, Herisanu N, Nemes I. Optimal homotopy asymptotic method with application to thin film flow. Cent Eur J Phys 2008; 6(3):648-53.

[9] Marinca V, Herisanu N. An optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. I. Commun Heat Mass Transfer 2008; 35: 710-5.

[10] Marinca V, Herisanu N, Bota C, Marinca B. An optimal homotopy asymptotic method to the steady flow of a fourth grade fluid past a porous plate. Appl Math Lett 2009, 22, 245-51.

[11] Herisanu N, Marinca V. Accurate analytical solution to oscillators with discontinuities and fractional power restoring force by means of the optimal homotopy asymptotic method. Compt Math Appl 2010; 60: 1607-15.