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Certain Unified Integrals Involving the Product of S-Function, V-Function and the Incomplete Aleph Functions

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ABSTRACT

In this article, two new integral findings have been established related to the product of the S-function, V-function and the incomplete Aleph function. The results so obtained are represented in the form of incomplete Aleph functions. By specializing the parameters of the S function, the V-function and incomplete Aleph functions, some special cases are listed in the paper. The conclusions attained in this article are broad in scope and extremely valuable in science and engineering.

Keywords: Incomplete aleph function; Mellin-Barnes contour integral; Mittag-Leffler function; Oberhettinger's integral formula; S-function; V-function

1. Introduction and Definitions

Fractional calculus is a generalization of conventional calculus that is applied to a wide range of scientific and technological domains. Many researchers have looked at the applications of fractional calculus.

In the 18th century, Lengendre [1] and Schlomilch invented the incomplete gamma function (IGFs). The series representation of IGFs is given by the Tannery, F.E. Prym et al. Since then, a number of researchers have been studying incomplete gamma functions deeply.

Recently, Srivastava et al. [2] studied and examined the incomplete Pochammer symbol in depth. In addition to this study, he proposed the generalized incomplete hyper-geometric functions. They also explores the Mellin-Barnes contour integral (GIHF), derivative expressions, and generalised incomplete hyper-geometric functions, which have significance in communication theory, groundwater extraction models, and probability theory.

Südland et al. [3-4] introduced and studied the Aleph function in the 19th century. Several researchers established

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interesting results and provided potential applications in the fields of physics, applied mathematics and various engineering disciplines. Srivastava et al. [5] recently studied and examined in depth the incomplete H-functions and incomplete $\overline{\text{H}}$-functions, which are a generalization of Fox's H-function. They defined the decomposition formula, derivative formula, classical integral transformations, and fractional calculus, as well as various incomplete H-function applications.

Bansal et al. [6] explored and investigated the incomplete Aleph (ℵ) functions. The Aleph (ℵ) functions and familiar I-function like special cases are among the incomplete Aleph (ℵ) functions. Following that, the authors uncovered several fascinating classical integral transformations of incomplete Aleph (ℵ) functions.

Recently, researchers [7-8] studied the S-function, which is a generalisation of the k-Mittag-Leffler function, K-function, M-series, Mittag-Leffler function, and many other special functions, as well as its relationships with other special functions. These specialised functions have recently proven important in the fields of applied sciences, biology, physics, and engineering.

The S-function is a generalization of generalized Mittag-Leffler function due to Prabhakar [9]. The S-function investigated by Saxena et al. [10]. The definition of S-function is given below

$$S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:x\right)$$
$$\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n...\left(a_u\right)_n\left(\gamma\right)_{\pi\tau\Re}}{\left(b_1\right)_n...\left(b_V\right)_n\Gamma_{\Re}\left(n\alpha+\beta\right)}\frac{x^n}{n!}. \tag{1.1}$$

Here $\Re\in R,\mathrm{Re}(\alpha)>0,\alpha,B,\gamma,\tau\in C;$
$a_1\left(i=1,...,U\right),b_j\left(j=1,...,V\right),$
$\mathrm{Re}(\alpha)K\,\mathrm{Re}(\tau)>V+1$
and $U<V+1$.

The V-function was introduced by Kumar [11] and defined as given below-

$$\mathrm{V}(z)=V_n^{a_u,h,b_v}\left(1,\mu',\zeta,\delta,m,k_u,A_v,B_w,\eta,v,\rho:z\right)$$
$$\varepsilon\sum_{n'=0}^{\infty}\frac{(-1)^{n'}\prod_{u=1}^{p}\left(h+\eta n'+v\right)^{-\mu'}}{\prod_{v=1}^{q}\left[\left(b_v\right)_{n'+A_v}\right]}\times$$
$$\frac{\left[\left(a_u\right)_{n'+k_u}\right]}{\prod_{w=1}^{r}\left[\left(h\right)_{\eta n'\rho+B_w}\right]}\left(\frac{z}{2}\right)^{n'\zeta+h\delta+m}, \tag{1.2}$$

where

1. $1,\mu',\zeta,\delta,m,v,\rho,k_u\left(u=1,...,p\right),$ $\mathrm{A}_V\left(v=1,...,q\right),B_w\left(w=1,...,r\right)$ are real numbers,
2. p,q and r are natural numbers,
3. $a_u,b_v\geq 1\left(u=1,...,p;v=1,...,q\right),$
4. $\eta>0,R(\mu')>0,R(h)>0,z$ is a complex variable and ε is arbitrary constants.

Thorough details of V-function are given by Chandak et al. [12] and Kumar [11]. The standard definition of gamma function $\Gamma(v)$ is defined in the below manner:

$$\Gamma(v)=\begin{Bmatrix}\int_0^{\infty}e^{-u}u^{v-1}du & \Re(v)>0,\\ \dfrac{\Gamma(v+k)}{(v)_k} & \left(v\in C\setminus z_0^-;k\in N_0\right)\end{Bmatrix}. \tag{1.3}$$

Here $(v)_k$ represent the Pochhammer symbol. The familiar incomplete gamma function $\gamma(v,y)$ and $\Gamma(v,y)$ are stated as:

$$\gamma(v,y)=\int_0^{y}e^{-u}u^{v-1}du,\left\{y\geq 0;\Re(v)>0\right\}, \tag{1.4}$$

and

$$\gamma(v,y)=\int_0^y e^{-u}u^{v-1}du,\{y\geq 0;\Re(v)>0\},$$

when $y=0$. (1.5)

Each satisfies the following decomposition formula:

$$\gamma(v,y)+\Gamma(v,y)=\Gamma(v)(\Re(v)>0). \quad (1.6)$$

The incomplete Aleph function ${}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)$ and ${}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)$ involving the incomplete gamma function $\gamma(v,y)$ and $\Gamma(v,y)$ as shown below:

$$ {}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)={}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z) \left[z\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N}\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}\\ (h_j,\hbar_j)_{1,M}\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1Q_i}\end{matrix}\right.\right] =\frac{1}{2\pi i}\int_\ell K(\xi,y)z^{-\xi}d\xi, \quad (1.7)$$

where $z\neq 0$, and

$$K(\xi,y)=\frac{\Gamma(1-g_1-H_1\xi,y)\prod_{j=1}^{M}\Gamma(h_1-\hbar_1\xi)\prod_{j=2}^{N}\Gamma(1-g_1-H_1\xi)}{\sum_{i=1}^{R}\varsigma_i\left[\prod_{j=M+1}^{Q_i}\Gamma(1-h_1-\hbar_1\xi)\prod_{j=N1}^{P_i}\Gamma(g_{ji}-\hbar_{ji}\xi)\right]} \quad (1.8)$$

and

$$ {}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)={}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z) \left[z\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N}\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}\\ (h_j,\hbar_j)_{1,M}\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1Q_i}\end{matrix}\right.\right] =\frac{1}{2\pi i}\int_\ell L(\xi,y)z^{-\xi}d\xi, \quad (1.9)$$

$$L(\xi,y)=\frac{\gamma(1-g_1-H_1\xi,y)\prod_{j=1}^{M}\Gamma(h_1-\hbar_1\xi)\prod_{j=2}^{N}\Gamma(1-g_1-H_1\xi)}{\sum_{i=1}^{R}\varsigma_i\left[\prod_{j=M+1}^{Q_i}\Gamma(1-h_1-\hbar_1\xi)\prod_{j=N1}^{P_i}\Gamma(g_{ji}-\hbar_{ji}\xi)\right]} \quad (1.10)$$

For all $y\geq 0$ the incomplete Aleph $(\aleph)$-function ${}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)$ and ${}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)$ exist in (1.7) and (1.9) under the set of valid conditions as shown below.

The parameters $H_j,\hbar_j,H_{ji},\hbar_{ji}$ are positive numbers, and g_j,h_j,g_{ji},h_{ji} are complex. Assuming that all poles of $K(\xi,y)$ and $L(\xi,y)$ is simple, and the null product is regarded as unity. The contour ℓ in the complex ξ-plane extends from $\gamma-i\infty$ to $\gamma+i\infty,\gamma\in R$, and poles of the gamma functions $\Gamma(h_1+\hbar_1\xi), j=\overline{\iota,M}$ do not exactly match with the poles of gamma function $\Gamma(1-g_j-H_1\xi), j=\overline{\iota,N}$. The parameters P_i,Q_i are non-negative integers satisfying $0\leq N\leq P_i, 0\leq M\leq Q_i$ for $i=\overline{1,R}$. (5)

$$g_j>0,|\arg(z)|<\frac{\pi}{2}g_j, i=\overline{1,R}, \quad (1.11)$$

$$g_j\geq 0,|\arg(z)|<\frac{\pi}{2}g_j, i=\overline{1,R} \text{ and } R(\varphi_i)+1<0, \quad (1.12)$$

where

$$g_j=\sum_{j=1}^{N}H_j+\sum_{j=1}^{M}\hbar_j-\varsigma_i\left(\sum_{j=1}^{P_i}H_{ji}+\sum_{j=1}^{Q_i}\hbar_{ji}\right), \quad (1.13)$$

$$\varphi_i=\sum_{j=1}^{M}h_j-\sum_{j=1}^{N}g_j-\varsigma_i\left(\sum_{j=M+1}^{Q_i}H_{ji}+\sum_{j=1}^{P_i}\hbar_{ji}\right)+\frac{1}{2}(P_i-Q_i), i=\overline{1,R}. \quad (1.14)$$

The incomplete Aleph($\aleph$)-function ${}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)$ and ${}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\varsigma_i,R}(z)$ defined in (1.7) and (1.9) respectively, has been simplified to several special functions like Aleph function, incomplete I-function [13],I-function, incomplete H-function [14], and Fox's H-function.

More details of Incomplete Aleph function is given in the paper of Bansal et al. [6].

2. Oberhettinger's Integral Formula

Many authors established unified integral formulas containing a number of special functions. The integral formula of the incomplete Aleph-functions (1.7) and (1.9) containing S and V-function is established in this section.

The Oberhettinger's integral expression [15], is given as:

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\rho}\frac{\Gamma(2\rho)\Gamma(\lambda-\rho)}{\Gamma(1+\rho+\mu)} \tag{2.1}$$

provided that $0<\mathcal{R}(\rho)<R(\lambda)$.

Theorem 2.1. If $\rho,\lambda\in\mathcal{C}$ with $R(h)>0$, $bp,q,r\in N; 0<\mathcal{R}(\rho)<R(\lambda); a_u,b_v\geq 1,\eta>0$, and $x>0,\varepsilon>0$ be arbitrary constants then the following integral formula holds:

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}dx\times S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:\frac{z_1}{x+a+\sqrt{x+2ax}}\right)\times V_{n'}^{a_u,h,b_v}\left(1,\mu',\zeta,\delta,m,k_u,A_v,B_w,\eta,v,\rho:\frac{z_3}{x+a+\sqrt{x+2ax}}\right)\times {}^{(\Gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}(z)\left[\frac{z_2}{x+a+\sqrt{x+2ax}}\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N}\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}\\(h_j,\hbar_j)_{1,M}\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1Q_i}\end{matrix}\right.\right]dx$$

$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\varepsilon\sum_{K=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}\frac{(\gamma)_{\pi\tau,\Re}\left(\frac{z_1}{a}\right)^K}{\Gamma_\Re(Ka+B)K!}\times\frac{(-1)^{n'}\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\times\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}{}^{(\Gamma)}\aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left(\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right), \tag{2.2}$$

where

$A^*=(g_1,H_1,y)$,
$(g_j,H_j)_{2,N'}(-\lambda-K-n'\zeta-h\delta-m,1)$,
$(1-\lambda-K-n'\zeta-h\delta-m+\rho,1),\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}$,
$B^*=(h_j,\hbar_j)_{1,M}(1-\lambda-K-n'\zeta-h\delta-m,1)$,
$(-\lambda-K-n'\zeta-h\delta-m-\rho,1),\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1,Q_i}$,

provided that the conditions of incomplete Aleph function ${}^{(\Gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}(z)$ in (1.6) are satisfied.

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Proof. For suitability, let us represent the left hand side of assertion (2.2) by $\mathcal{T}$. Using (1.1-1.2) and (1.7) in the right hand side of (2.2)

$$T=\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}dx\times\frac{\left(\frac{z_1}{x+a+\sqrt{x+2ax}}\right)^K(-1)^{n'}}{K!}\times\sum_{n'=0}^{\infty}\varepsilon\frac{\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu'}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\times\left(\frac{z_3}{2\left(x+a+\sqrt{x+2ax}\right)}\right)^{n'\zeta+h\delta+m}\times$$

$$=\frac{1}{2\pi i}\int_\ell K(\xi,y)\left(\frac{z_2}{x+a+\sqrt{x+2ax}}\right)^{-\xi}d\xi dx,$$

where $K(\xi,y)$ is defined in (1.7).

Then, by altering the sequence of summation, integration, and contour integral involved (allowed under the specified conditions), we get

$$T=\sum_{K=0}^{\infty}\sum_{n'=0}^{\infty}\varepsilon\frac{(a_1)_K...(a_u)_K(\gamma)_{\pi\tau,\Re}}{(b_1)_K...(b_v)_K\Gamma_\Re(Ka+B)}\times\frac{(z_1)^K}{K!}\frac{(-1)^{n'}\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu'}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}\times$$

$$\frac{1}{2\pi i}\int_{\ell} K(\xi,y)(z_2)^{-\xi} \times$$
$$\int_0^{\infty} x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-(\lambda+K+n'\zeta+h\delta+m-\xi)} d\xi dx.$$

Finally, apply (2.1) to the aforementioned integral and re-explain it as an Aleph function. Wearrive at the conclusion (2.2).

Theorem 2.2. If $\rho,\lambda \in \mathcal{C}$ with $R(h)>0$, $bp,q,r \in N; 0<\mathcal{R}(\rho)<R(\lambda); a_u, b_v \geq 1, \eta>0$, and $x>0, \varepsilon>0$ be arbitrary constants then the following integral formula holds:

$$\int_0^{\infty} x^{\rho-1}\left(x+a+\sqrt{x+2a\mathcal{X}}\right)^{-\lambda} \times$$
$$S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\mathfrak{R})}\left(a_1...a_n;b_1...b_V : \frac{z_1}{x+a+\sqrt{x+2ax}}\right)\times$$
$$V_{n'}^{a_u,h,b_v}\left(1,\mu',\zeta,\delta,m,k_u,A_v,B_w,\eta,v,\rho : \frac{z_3}{x+a+\sqrt{x^2}}\right)\times$$
$$^{(\gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}\left[\frac{z_2}{x+a+\sqrt{x+2ax}}\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N},\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}\\ (h_j,\hbar_j)_{1,M}\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1Q_i}\end{matrix}\right.\right]dx$$
$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\varepsilon\sum_{K=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}$$
$$\frac{(\gamma)_{\pi\tau,\mathfrak{R}}\left(\frac{z_1}{a}\right)^K}{\Gamma_{\mathfrak{R}}(Ka+B)K!}\times$$
$$\frac{(-1)^{n'}\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\times$$
$$\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}{}^{(\Gamma)}\aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left(\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right), \tag{2.3}$$

where

$$A^*=(g_1,H_1,y),(g_j,H_j)_{2,N},(-\lambda-K-n'\zeta-h\delta-m,1),$$
$$(1-\lambda-K-n'\zeta-h\delta-m+\rho,1),\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i},$$
$$B^*=(h_j,\hbar_j)_{1,M}(1-\lambda-K-n'\zeta-h\delta-m,1),$$
$$(-\lambda-K-n'\zeta-h\delta-m-\rho,1),\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1,Q_i}.$$

Proof. Let $\mathcal{T}$ represent the left hand side of assertion (2.3) for suitability. Also using (1.1-1.2), and (1.9) in the right hand side of assertion (2.3)

$$T=\int_0^{\infty}x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}dx\times$$
$$\sum_{K=0}^{\infty}\frac{(a_1)_K...(a_u)_K(\gamma)_{\pi\tau,\mathfrak{R}}}{(b_1)_K...(b_v)_K\Gamma_{\mathfrak{R}}(Ka+B)}\times\frac{\left(\frac{z_1}{x+a+\sqrt{x+2ax}}\right)^K}{K!}\times$$
$$\sum_{n'=0}^{\infty}\varepsilon\frac{(-1)^{n'}\prod_{u=1}^{P}(h+\eta n'+v)^{-\mu'}\left[(a_u)_{n'+ku}\right]}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\times$$
$$\left(\frac{z_3}{2\left(x+a+\sqrt{x+2ax}\right)}\right)^{n'\zeta+h\delta+m}\times$$
$$=\frac{1}{2\pi i}\int_{\ell}L(\xi,y)\left(\frac{z_2}{x+a+\sqrt{x+2ax}}\right)^{-\xi}d\xi dx,$$

where $L(\xi,y)$ is defined in (1.10).

Then, by altering the sequence of summation, integration, and contour integral involved (allowedunder the specified conditions), we get

$$T=\sum_{K=0}^{\infty}\sum_{n'=0}^{\infty}\varepsilon\frac{(a_1)_K...(a_u)_K(\gamma)_{\pi\tau,\mathfrak{R}}}{(b_1)_K...(b_v)_K\Gamma_{\mathfrak{R}}(Ka+B)}\times$$
$$\frac{(z_1)^K}{K!}\frac{(-1)^{n'}\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu'}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}\times$$
$$\frac{1}{2\pi i}\int_{\ell}L(\xi,y)(z_2)^{-\xi}\times$$
$$\int_0^{\infty}x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-(\lambda+K+n'\zeta+h\delta+m-\xi)}d\xi dx.$$

Finally, apply (2.1) to the aforementioned integral and rewrite it as an Aleph function. We arrive atthe conclusion (2.3).

3. Special Cases

We found several exciting special cases of major results (Theorem 2.1 and Theorem 2.2) in this sectionas follows:

Corollary 3.1. If $\rho,\lambda \in \mathcal{C}$ with $R(h)>0,$ $bp,q,r \in N; 0<\mathcal{R}(\rho)<R(\lambda); a_u, b_v \geq 1, \eta>0,$ and $x>0, \varepsilon>0$ the following improper integral formula holds:

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}\times S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:\frac{z_1}{x+a+\sqrt{x+2ax}}\right)\times V_{n'}^{a_u,h,b_v}\left(1,\mu',\zeta,\delta,m,k_u,A_v,B_w,\eta,v,\rho:\frac{z_3}{x+a+\sqrt{x^2}}\right)\times {}^{(\gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}\left[\frac{z_2}{x+a+\sqrt{x+2ax}}\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N},\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}\\(h_j,\hbar_j)_{1,M}\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1Q_i}\end{matrix}\right.\right]dx$$
$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\varepsilon\sum_{K=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}\frac{(\gamma)_{\pi\tau,\Re}\left(\frac{z_1}{a}\right)^K}{\Gamma_\Re(Ka+B)K!}\times\frac{(-1)^{n'}\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\times\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}{}^{(\Gamma)}\aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left[\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right],\tag{3.1}$$

and

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}\times S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:\frac{z_1}{x+a+\sqrt{x^2+2ax}}\right)\times V_{n'}^{a_u,h,b_v}\left(1,\mu',\zeta,\delta,m,k_u,A_v,B_w,\eta,v,\rho:\frac{z_3}{x+a+\sqrt{x^2+2ax}}\right)\times {}^{(\gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}\left[\frac{z_2}{x+a+\sqrt{x^2+2ax}}\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N},\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i}\\(h_j,\hbar_j)_{1,M}\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1Q_i}\end{matrix}\right.\right]dx$$
$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\varepsilon\sum_{K=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}\frac{(\gamma)_{\pi\tau,\Re}\left(\frac{z_1}{a}\right)^K}{\Gamma_\Re(Ka+B)K!}\times\frac{(-1)^{n'}\prod_{u=1}^{P}\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu}}{\prod_{v=1}^{q}\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^{r}\left[(h)_{\eta n'\rho+B_w}\right]}\times\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}{}^{(\Gamma)}I_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left[\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right],\tag{3.2}$$

where

$$A^*=(g_1,H_1,y),(g_j,H_j)_{2,N},(-\lambda-K-n'\zeta-h\delta-m,1),(1-\lambda-K-n'\zeta-h\delta-m+\rho,1),\left[\varsigma_i(g_{ji},H_{ji})\right]_{N+1,P_i},$$
$$B^*=(h_j,\hbar_j)_{1,M}(1-\lambda-K-n'\zeta-h\delta-m,1),(-\lambda-K-n'\zeta-h\delta-m-\rho,1),\left[\varsigma_i(h_{ji},\hbar_{ji})\right]_{M+1,Q_i}.$$

Proof. Substituting $\varsigma_i=1$ in the results (2.2) and (2.3) the incomplete Aleph function reduces into incomplete I-function [16], the desired result (3.1) and (3.2) will be obtained.

Corollary 3.2. If $\rho,\lambda \in \mathcal{C}$ with $R(h)>0,$ $bp,q,r \in N; 0<\mathcal{R}(\rho)<R(\lambda); a_u, b_v \geq 1, \eta>0,$ and $x>0, \varepsilon>0$, then the following improper integral formula holds:

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}\times S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:\frac{z_1}{x+a+\sqrt{x^2+2ax}}\right)\times$$
$$V_{n'}^{a_u,h,b_v}\left(1,\mu',\zeta,\delta,m,k_u,A_v,B_w,\eta,v,\rho:\frac{z_3}{x+a+\sqrt{x^2+2ax}}\right)$$
$$^{(\gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}\left[\frac{z_2}{x+a+\sqrt{x^2+2ax}}\left|\begin{matrix}(g_1,H_1,y),\left[\varsigma_i\left(g_{ji},H_{ji}\right)\right]_{N+1,P_i}\\ \left(h_j,\hbar_j\right)_{1,M}\left[\varsigma_i\left(h_{ji},\hbar_{ji}\right)\right]_{M+1Q_i}\end{matrix}\right.\right]dx$$
$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\varepsilon\sum_{K=0}^\infty\sum_{n'=0}^\infty\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}\frac{(\gamma)_{\pi\tau,\Re}\left(\frac{z_1}{a}\right)^K}{\Gamma_\Re(Ka+B)K!}\times$$
$$\frac{(-1)^{n'}\prod_{u=1}^P\left[(a_u)_{n'+ku}\right](h+\eta n'+v)^{-\mu}}{\prod_{v=1}^q\left[(b_v)_{n'+A_v}\right]\prod_{w=1}^r\left[(h)_{\eta n'\rho+B_w}\right]}\times$$
$$\left(\frac{z_3}{2}\right)^{n'\zeta+h\delta+m}\,^{(\Gamma)}I_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left[\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right],\qquad(3.3)$$

where

$$A^*=(g_1,H_1,y),(g_j,H_j)_{2,N},(-\lambda-K-n'\zeta-h\delta-m,1),$$
$$(1-\lambda-K-n'\zeta-h\delta-m+\rho,1),\left[\varsigma_i\left(g_{ji},H_{ji}\right)\right]_{N+1,P_i},$$
$$B^*=\left(h_j,\hbar_j\right)_{1,M}(1-\lambda-K-n'\zeta-h\delta-m,1),$$
$$(-\lambda-K-n'\zeta-h\delta-m-\rho,1),\left[\varsigma_i\left(h_{ji},\hbar_{ji}\right)\right]_{M+1,Q_i}.$$

Proof. By simplifying incomplete Aleph function to Aleph function [3-4], we can get the results (3.2) and (3.3) by taking y=0 in the result (2.2).

Corollary 3.3. If $\rho,\lambda\in\mathcal{C}$ with $R(h)>0$, $bp,q,r\in N;0<\mathcal{R}(\rho)<R(\lambda);a_u,b_v\geq1$, $R(\mu')>0,\eta>0$, and $x>0$, then the following improper integral formula holds:

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}\times S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:\frac{z_1}{x+a+\sqrt{x^2+2ax}}\right)\times$$
$$\mathcal{H}_{n'}\left(\frac{z_3}{x+a+\sqrt{x^2+2ax}}\right)\times^{(\gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}$$
$$\left[\frac{z_2}{x+a+\sqrt{x^2+2ax}}\left|\begin{matrix}(g_1,H_1,y),\left[\varsigma_i\left(g_{ji},H_{ji}\right)\right]_{N+1,P_i}\\ \left(h_j,\hbar_j\right)_{1,M}\left[\varsigma_i\left(h_{ji},\hbar_{ji}\right)\right]_{M+1Q_i}\end{matrix}\right.\right]dx$$
$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\sum_{K=0}^\infty\sum_{n'=0}^\infty\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}$$
$$\frac{(\gamma)_{\pi\tau,\Re}\left(\frac{z_1}{a}\right)^K}{\Gamma_\Re(Ka+B)K!}\frac{(-1)^{n'}\left(\frac{z_3}{2}\right)^{2n'+h+1}}{\Gamma\left(2n'+\frac{3}{2}\right)\Gamma\left(n'+h+\frac{3}{2}\right)}\times$$
$$^{(\Gamma)}\aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left[\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right],\qquad(3.4)$$

and

$$\int_0^\infty x^{\rho-1}\left(x+a+\sqrt{x+2ax}\right)^{-\lambda}\times S_{(U,V)}^{(\alpha,\beta,\gamma,\tau,\Re)}\left(a_1...a_n;b_1...b_V:\frac{z_1}{x+a+\sqrt{x^2+2ax}}\right)\times$$
$$\mathcal{H}_{n'}\left(\frac{z_3}{x+a+\sqrt{x^2+2ax}}\right)\times^{(\gamma)}\aleph_{P_i,Q_i,\varsigma_i,R}^{M,N}$$
$$\left[\frac{z_2}{x+a+\sqrt{x^2+2ax}}\left|\begin{matrix}(g_1,H_1,y),(g_j,H_j)_{2,N},\left[\varsigma_i\left(g_{ji},H_{ji}\right)\right]_{N+1,P_i}\\ \left(h_j,\hbar_j\right)_{1,M}\left[\varsigma_i\left(h_{ji},\hbar_{ji}\right)\right]_{M+1,Q_i}\end{matrix}\right.\right]dx$$
$$=2^{1-\rho}a^{\rho-\lambda}\Gamma(2\rho)\sum_{K=0}^\infty\sum_{n'=0}^\infty\frac{(a_1)_K...(a_u)_K}{(b_1)_K...(b_v)_K}$$
$$\frac{(\gamma)_{\pi\tau,\Re}\left(\frac{z_1}{a}\right)^K}{\Gamma_\Re(Ka+B)K!}\frac{(-1)^{n'}\left(\frac{z_3}{2}\right)^{2n'+h+1}}{\Gamma\left(2n'+\frac{3}{2}\right)\Gamma\left(n'+h+\frac{3}{2}\right)}\times$$
$$^{(\Gamma)}\aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2}\left[\frac{z_2}{a}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right],\qquad(3.5)$$

where

$$A^* = (g_1, H_1, y), (g_j, H_j)_{2,N}, (-\lambda - K - 2n' - h, 1),$$
$$(1 - \lambda - K - 2n' - h + \rho, 1), \left[\varsigma_i (g_{ji}, H_{ji})\right]_{N+1,P_i},$$
$$B^* = (h_j, \hbar_j)_{1,M} (-\lambda - K - 2n' - h, 1),$$
$$(-1 - \lambda - K - 2n' - h - \rho, 1), \left[\varsigma_i (h_{ji}, \hbar_{ji})\right]_{M+1,Q_i}.$$

Proof. Taking $u = 1, v = 2, w = 1, a_1 = 1, b_1 = \frac{3}{2},$ $b_2 = 1, l = 2, \mu' = 1, \zeta = 2, \varepsilon = \frac{1}{\Gamma(h)\Gamma\left(\frac{3}{2}\right)},$ $\delta = 1, m = 0, k_1 = 0, A_1 = A_2 = 0, B_1 = \frac{1}{2}, \rho = 1,$ in (2.3-2.4), the V-function reduces to Struve function [17], hence we get the desired outcomes (3.4-3.5).

Corollary 3.4. If $\rho, \lambda \in \mathcal{C}$ with $R(h) > 0,$ $bp, q, r \in N; 0 < \mathcal{R}(\rho) < R(\lambda); a_u, b_v \geq 1,$ $R(\mu') > 0, \eta > 0,$ and $x > 0,$ then the following improper integral formula holds:

$$\int_0^\infty x^{\rho-1} \left(x + a + \sqrt{x + 2ax}\right)^{-\lambda} \times E_{(0,0)}^{(\alpha,\beta,1,1,1)} \left(\frac{z_1}{x + a + \sqrt{x^2 + 2ax}}\right) \times E_{\eta,h} \left(\frac{z_3}{x + a + \sqrt{x^2 + 2ax}}\right) \times^{(\gamma)} \aleph_{P_i,Q_i,\varsigma_i,R}^{M,N} \left[\frac{z_2}{x + a + \sqrt{x^2 + 2ax}} \left| \begin{matrix} (g_1, H_1, y), (g_j, H_j)_{2,N}, \left[\varsigma_i (g_{ji}, H_{ji})\right]_{N+1,P_i} \\ (h_j, \hbar_j)_{1,M} \left[\varsigma_i (h_{ji}, \hbar_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right] dx$$
$$= 2^{1-\rho} a^{\rho-\lambda} \Gamma(2\rho) \sum_{K=0}^\infty \sum_{n'=0}^\infty \frac{\left(\frac{z_1}{a}\right)^K}{\Gamma(Ka + B)} \times \frac{\left(\frac{z_1}{a}\right)^{n'}}{\Gamma(n'\eta + h)} \times^{(\Gamma)} \aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2} \left[\frac{z_2}{a} \left| \begin{matrix} A^* \\ B^* \end{matrix} \right. \right], \tag{3.6}$$

and

$$\int_0^\infty x^{\rho-1} \left(x + a + \sqrt{x + 2ax}\right)^{-\lambda} \times E_{(0,0)}^{(\alpha,\beta,1,1,1)} \left(\frac{z_1}{x + a + \sqrt{x^2 + 2ax}}\right) \times E_{\eta,h} \left(\frac{z_3}{x + a + \sqrt{x^2 + 2ax}}\right) \times^{(\gamma)} \aleph_{P_i,Q_i,\varsigma_i,R}^{M,N} \left[\frac{z_2}{x + a + \sqrt{x^2 + 2ax}} \left| \begin{matrix} (g_1, H_1, y), (g_j, H_j)_{2,N}, \left[\varsigma_i (g_{ji}, H_{ji})\right]_{N+1,P_i} \\ (h_j, \hbar_j)_{1,M} \left[\varsigma_i (h_{ji}, \hbar_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right] dx$$
$$= 2^{1-\rho} a^{\rho-\lambda} \Gamma(2\rho) \sum_{K=0}^\infty \sum_{n'=0}^\infty \frac{\left(\frac{z_1}{a}\right)^K}{\Gamma(Ka + B)} \times \frac{\left(\frac{z_1}{a}\right)^{n'}}{\Gamma(n'\eta + h)} \times^{(\Gamma)} \aleph_{P_i+2,Q_i+2,\varsigma_i,R}^{M,N+2} \left[\frac{z_2}{a} \left| \begin{matrix} A^* \\ B^* \end{matrix} \right. \right], \tag{3.7}$$

where

$$A^* = (g_1, H_1, y), (g_j, H_j)_{2,N}, (-\lambda - K - n' - h, 1),$$
$$(1 - \lambda - K - n' - h + \rho, 1), \left[\varsigma_i (g_{ji}, H_{ji})\right]_{N+1,P_i},$$
$$B^* = (h_j, \hbar_j)_{1,M} (-\lambda - K - n' - h, 1),$$
$$(-1 - \lambda - K - n' - h - \rho, 1), \left[\varsigma_i (h_{ji}, \hbar_{ji})\right]_{M+1,Q_i}.$$

Proof. Taking $U, V = 0, \gamma, \tau, \Re = 1$ and $u = 1, v = 1, w = 1, a_1 = 1, b_1 = 1, l = 2, \mu' = 1, \zeta = 1,$ $\varepsilon = \frac{1}{\Gamma(h)}, \delta = 0, m = 0, k_1 = 0, A_1 = 0, B_1 = -1,$ $v = -1, \rho = 1,$ in (2.2-2.3) the S-function and V- function reduces in to Mittag-Leffler function [18-19]; we get the desired results (3.6-3.7).

4. Conclusion

The current study deals with the results of some interesting integrals including the product of S-function, generalized V-function and incomplete Aleph functions. The outcomes are represented in the form of incomplete Aleph function. Some special cases of S-functions, V-function and incomplete Aleph function (Mittag-Leffler function, Struve function, incomplete H function, Aleph function etc.) have been listed in the paper. An extensive variety of problems occurring in different fields of mathematical physics, such as neutron physics, radio physics, and plasma physics are related to V-function which provides solutions to a number of problems formulated in terms of fractional order operators. The Oberhettinger's integral formulas established in the recent article are very helpful to attain Mellin transform of a variety of simpler special functions. Hence the findings of this study are very useful in solving the problems arising in various fields of physics and engineering namely digital signals, image processing, sonar system etc.

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