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Some New Results of Weakly Contractive Mappings in G_b-Metric Space

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ABSTRACT

In our present work, we are proving some generalized weakly contractive mappings results in G_b-metric space. These outcomes extend, unify and develop the available results in literature. We also provide some corollaries and example to show the novelty and usefulness of outcomes.

Keywords: Fixed-point; G_b-metric space; Weakly contractive mapping

1. Introduction

The Banach's contraction principle, famous result of fixed point theory was developed in 1922. Since then, it has turn out to be a very accepted and a crucial way in solving existence problems because of its simplicity, convenience. Combining the idea of b-metric space and G-metric space Aghajani et al. [1] has initialized the notion of G_b-metric space. Afterward, a number of fascinating consequences in the framework of G_b-metric spaces have been reported [3, 2-5]. Definition of weakly contractive mappings in Hilbert Spaces, firstly given by Alber and Guerre-Delabriere [6]. They established various results of existence of fixed points in it. Rhoades [7] generalized this thought to Banach spaces. Dutta and Choudhury [8] developed (ψ,ϕ)-weakly contractive mappings in complete metric spaces. Since then a number of weakly contractive maps have been considered [7, 9-14]. We are proving some results of weakly contractive mappings in G_b-metric space.

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2. Preliminaries

Definition 2.1 ([1]). In a non void set Υ with $s \geq 1,$ suppose that a mapping $G_b : \Upsilon \times \Upsilon \times \Upsilon \to \Re^+$ satisfies the next conditions for every $\varpi, \gamma, \mu \in \Upsilon$:

(Gb1) $G_b(\varpi,\gamma,\mu) = 0$ if $\varpi = \gamma = \mu,$

(Gb2) $0 < G_b(\varpi,\gamma,\mu)$ with $\varpi \neq \gamma,$

(Gb3) $G_b(\varpi,\varpi,\mu) \leq G_b(\varpi,\gamma,\mu)$ with $\mu \neq \gamma,$

(Gb4) $G_b(\varpi,\gamma,\mu) = G_b(P\{\varpi,\gamma,\mu\})$ where P is a permutation of $\varpi,\gamma,\mu \in \Upsilon.$

(Gb5) $G_b(\varpi,\gamma,\mu) \leq s\{G_b(\varpi,x,x)G_b(x,\gamma,\mu)\}.$

The pair (Υ, G_b) is known to be G_b-metric space.

Remark. For $s = 1,$ G_b-metric space coincide with G-metric space.

Example 2.2 ([1]). let $\Upsilon = \Re$ and G-metric be

$$G_b(\varpi,\gamma,\mu) = \frac{1}{3}(|\varpi-\gamma|+|\gamma-\mu|+|\mu-\varpi|), \varpi,\gamma,\mu \in \Re.$$

$$G_b(\varpi,\gamma,\mu) = G_b(\varpi,\gamma,\mu)^2 = \frac{1}{9}(|\varpi-\gamma|+|\gamma-\mu|+|\mu-\varpi|)^2$$

is a G_b-metric space having $s = 2,$ but not a G-metric space in $\Re.$ Take $\varpi = 3, \gamma = 5, \mu = 7$ and $x = \frac{7}{2},$ we get

$$G_*(3,5,7) = \frac{64}{9}, G_*\left(3,\frac{7}{2},\frac{7}{2}\right) = \frac{1}{9}, G_*\left(\frac{7}{2},5,7\right) = \frac{49}{9}.$$

So,

$$G_*(3,5,7) = \frac{64}{9} \geq \frac{50}{9} = G_*\left(3,\frac{7}{2},\frac{7}{2}\right) + \left(\frac{7}{2},5,7\right).$$

Definition 2.3 ([9]). A G_b-metric is known to be symmetric if $G_b(\varpi,\gamma,\gamma) \leq G_b(\gamma,\varpi,\varpi),$ for every $\varpi,\gamma,\mu \in \Upsilon.$

Proposition 2.4 ([1]). In a G_b-metric space, following properties hold:

(1) If $G_b(\varpi,\gamma,\gamma) = 0,$ then $\varpi = \gamma = \mu,$

(2) $G_b(\varpi,\gamma,\mu) \leq s\{G_b(\varpi,\varpi,\gamma)G_b(\varpi,\varpi,\mu)\},$

(3) $G_b(\varpi,\gamma,\gamma) \leq 2sG_b(\gamma,\varpi,\varpi),$

(4) $G_b(\varpi,\gamma,\mu) \leq s\{G_b(\varpi,x,\mu)G_b(x,\gamma,\mu)\}.$

Definition 2.5 ([1]). A sequence $\{\varpi_i\}$ is said to be:

(1) G_b-Cauchy if for every $\varepsilon > 0,$ there always be a positive integer $i_0,$ such that for all $i,j \geq i_0, G_b(\varpi_i,\varpi_j,\varpi_j) < \varepsilon.$

(2) G_b-convergent to $\kappa \in \Upsilon$ if for every $\varepsilon > 0,$ there always be a positive integer $i_0,$ such that for every $i,j \geq i_0, G_b(\varpi_i,\varpi_j,\kappa) < \varepsilon.$

Lemma 2.6 ([5]). Let Υ be a G_b-metric space. Assume that $\{\varpi_i\} \to \kappa.$ Then, we have

$$\frac{1}{s}G_b(\kappa,\gamma,\gamma) \leq \liminf_{i\to\infty} G_b(\varpi_i,\gamma,\gamma)$$
$$\leq \limsup_{i\to\infty} G_b(\varpi_i,\gamma,\gamma)$$
$$\leq G_b(\kappa,\gamma,\gamma).$$

In particular, if $\varpi = \gamma,$ then we have $\lim_{i\to\infty} G_b(\varpi_i,\gamma,\gamma) = 0.$ Throughout the paper we will denote Ψ as the collection of decreasing and continuous mappings from $[0,\infty) \to [0,\infty)$ and such that $\Psi(y) = 0$ if and only if $y = 0.$ And Φ as the collection of continuous functions from $[0,\infty) \to [0,\infty)$ and such that $\Phi(y) = 0$ if and only if $y = 0.$

Definition 2.7 ([12]). A mapping $\Im : \Upsilon \to \Upsilon$ is known to be a weakly contractive on metric space (Υ, d), if there always be $\psi, \phi \in \Psi$ such that

$$\psi\left(d\left(\Im\varpi, \Im\gamma\right)\right) \leq \psi\left(d\left(\varpi, \gamma\right)\right) - \phi\left(d\left(\varpi, \gamma\right)\right),$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are non decreasing and continuous mappings such that $\psi(y) = 0$ if and only if $y = 0$.

3. Results and Discussion

Theorem 3.1. In a complete G_b-metric space (Υ, G_b) with $s \geq 1$, let $\Im$ be (α, Ψ, Φ) generalized weakly contractive self map on Υ and there are $\alpha \in (0,1), \psi \in \Psi$ and $\phi \in \Phi$ such that:

$$\psi\left(4s^4 G_b\left(\Im\varpi, \Im\gamma, \Im\mu\right)\right) \leq \psi\left(\mathcal{A}_a\left(\varpi, \gamma, \mu\right)\right) - \phi\left(\mathcal{A}_a\left(\varpi, \gamma, \mu\right)\right), \tag{3.1}$$

where

$$\mathcal{A}_a(\varpi, \gamma, \mu) = \max\left\{\begin{array}{l} G_b(\varpi, \gamma, \mu), \\ G_b(\varpi, \Im\varpi, \Im\varpi), \\ G_b(\gamma, \Im\gamma, \Im\gamma), \\ G_b(\mu, \Im\gamma\mu\Im\mu), \\ \frac{1}{2s^2}\left[\begin{array}{l} \alpha G_b(\Im\varpi, \Im\gamma, \Im\mu) + \\ (1-\alpha) G_b(\Im\varpi, \gamma, \gamma) \\ G_b(\Im\gamma, \mu, \mu) G_b(\Im\mu, \varpi, \varpi) \end{array}\right] \end{array}\right\},$$

for every $\varpi, \gamma, \mu \in \Upsilon$. Then, $\Im$ has a unique fixed point.

Proof. Let $\varpi_0 \in \Upsilon$ be an initial point and $\{\varpi_i\}$ be a sequence in Υ such that $\varpi_{i+1} = \Im\varpi_i$. Then, ϖ_i is fixed point and we get the result. Now, assume that $\varpi_i \neq \varpi_{i+1}$

$$\psi\left(G_b\left(\varpi_i, \varpi_{i+1}, \varpi_{i+1}\right)\right) \leq \psi\left(2s^4 G_b\left(\Im\varpi_{i-1}, \Im\varpi_i, \Im\varpi_i\right)\right) \leq \psi\left(\mathcal{A}_a\left(\varpi_{i-1}, \varpi_i, \varpi_i\right)\right) - \phi\left(\mathcal{A}_a\left(\varpi_{i-1}, \varpi_i, \varpi_i\right)\right), \tag{3.2}$$

where

$$\mathcal{A}_a(\varpi_{i-1}, \varpi_i, \varpi_i) = \max\left\{\begin{array}{l} G_b(\varpi_{i-1}, \varpi_i, \varpi_i), \\ G_b(\varpi_{i-1}, \Im\varpi_{i-1}, \Im\varpi_{i-1}), \\ G_b(\varpi_i, \Im\varpi_i, \Im\varpi_i), \\ G_b(\varpi_{i+1}, \Im\varpi_{i+1}, \Im\varpi_{i+1}), \\ \frac{1}{2s^2}\left[\begin{array}{l} \alpha G_b(\Im\varpi_{i-1}, \Im\varpi_i, \Im\varpi_i) + \\ (1-\alpha) G_b(\Im\varpi_{i-1}, \varpi_i, \varpi_i) \\ G_b(\Im\varpi_i, \varpi_i, \varpi_i) G_b(\Im\varpi_i, \varpi_{i-1}, \varpi_{i-1}) \end{array}\right] \end{array}\right\},$$

$$= \max\left\{\begin{array}{c} G_b(\varpi_{i-1}, \varpi_i, \varpi_i), \\ G_b(\varpi_i, \Im\varpi_{i+1}, \Im\varpi_{i+1}), \end{array}\right\}.$$

If $G_b(\varpi_i, \varpi_{i+1}, \varpi_{i+1})$ is maximum, then, $\phi\left(G_b\left(\varpi_i, \varpi_{i+1}, \varpi_{i+1}\right)\right) = 0$ and $\varpi_i = \varpi_{i+1}$, contrary to our assumption.
So,

$$\psi\left(G_b\left(\varpi_i, \varpi_{i+1}, \varpi_{i+1}\right)\right) < \psi\left(G_b\left(\varpi_{i-1}, \varpi_i, \varpi_i\right)\right).$$

Hence, $\psi\left(G_b\left(\varpi_i, \varpi_{i+1}, \varpi_{i+1}\right)\right)$ is a decreasing sequence of positive real numbers and there always be a $p \geq 0$ such that

$$\lim_{i\to\infty} G_b\left(\varpi_i, \varpi_{i+1}, \varpi_{i+1}\right) = p. \tag{3.3}$$

Using Eq. (3.3) in Eq. (3.2), $\psi(p) \leq \psi(p) - \phi(p)$, and this means that $\phi(p) = 0$. Hence, $p = 0$ and

$$\lim_{i\to\infty} G_b\left(\varpi_i, \varpi_{i+1}, \varpi_{i+1}\right) = 0. \tag{3.4}$$

Again from Eq. (3.1)

$$\psi\left(G_b\left(\varpi_{i+1}, \varpi_i, \varpi_i\right)\right) \leq \psi\left(4s^4 G_b\left(\Im\varpi_i, \Im\varpi_{i-1}, \Im\varpi_{i-1}\right)\right) \leq \psi\left(\mathcal{A}_a\left(\varpi_i, \varpi_{i-1}, \varpi_{i-1}\right)\right) - \phi\left(\mathcal{A}_a\left(\varpi_i, \varpi_{i-1}, \varpi_{i-1}\right)\right), \tag{3.5}$$

where

$$\mathcal{A}_a\left(\varpi_i, \varpi_{i-1}, \varpi_{i-1}\right) = \max\left\{\begin{array}{c} G_b\left(\varpi_i, \varpi_{i-1}, \varpi_{i-1}\right) \\ \frac{\alpha}{2s^2} G_b\left(\varpi_{i+1}, \varpi_i, \varpi_i\right) \end{array}\right\}.$$

If $\frac{\alpha}{2s^2}G_b\left(\varpi_{i+1},\varpi_i,\varpi_i\right)$ is maximum, then Eq. (3.5) will be approached to

$$G_b\left(\varpi_{i+1},\varpi_i,\varpi_i\right)\leq\frac{\alpha}{2s^2}G_b\left(\varpi_{i+1},\varpi_i,\varpi_i\right),$$

a contradiction to our assumption.
Therefore,

$$\psi\left(G_b\left(\varpi_{i+1},\varpi_i,\varpi_i\right)\right)\leq\psi\left(G_b\left(\varpi_i,\varpi_{i-1},\varpi_{i-1}\right)\right), \tag{3.6}$$

and by the same steps used in Eq. (3.6) and Eq. (3.4), we can show that

$$\lim_{i\to\infty}G_b\left(\varpi_{i+1},\varpi_i,\varpi_i\right)=0. \tag{3.7}$$

Assume that $\{\varpi_i\}$ is not a G_b-Cauchy sequence. Then there will be $\varepsilon>0$ and monotone increasing sequences of real numbers $i(\kappa),j(\kappa)\in N$ with $i(\kappa)>j(\kappa)>\kappa,$ such that

$$G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\geq\varepsilon$$

and

$$G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)<\varepsilon. \tag{3.8}$$

Using Eq. (3.2), Eq. (3.5) and Eq. (3.8)

$$\psi\left(4s^4\varepsilon\right)\leq\psi\left(4s^4G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\right)$$
$$\leq\psi\left(\mathcal{A}_a\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)\right)-$$
$$\phi\left(\mathcal{A}_a\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)\right),$$

where

$$\mathcal{A}_a\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)$$

$$=\max\left\{\begin{array}{c}G_b\left(\varpi_{j(\kappa)-1},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right),\\ G_b\left(\varpi_{j(\kappa)-1},\varpi_{j(\kappa)},\varpi_{j(\kappa)}\right),\\ G_b\left(\varpi_{i(\kappa)-1},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right),\\ G_b\left(\varpi_{i(\kappa)-1},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right),\\ \frac{1}{2s^2}\left[\begin{array}{l}\alpha G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)+\\ (1-\alpha)G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)\\ G_b\left(\varpi_{i(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)\\ G_b\left(\varpi_{i(\kappa)},\varpi_{j(\kappa)-1},\varpi_{j(\kappa)-1}\right)\end{array}\right]\end{array}\right\},$$

taking limit $\kappa\to\infty,$

$$\mathcal{A}_a\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)$$
$$=\max\left\{\begin{array}{c}G_b\left(\varpi_{j(\kappa)-1},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right),\\ \frac{\alpha}{2s^2}G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\end{array}\right\},$$

if $\frac{\alpha}{2s^2}G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)$ is maximum, then

$$\psi\left(4s^4G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\right)$$
$$\leq\psi\left(\frac{\alpha}{2s^2}G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\right),$$

this gives rise to $8s^6\leq\alpha,$ which is not possible as $\alpha\in(0,1)$ and $s\geq1.$ So

$$\psi\left(4s^4G_b\left(\varpi_{j(\kappa)},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\right)$$
$$\leq\psi\left(G_b\left(\varpi_{j(\kappa)-1},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right)\right)$$
$$\leq\psi\left(s\left\{G_b\left(\varpi_{j(\kappa)-1},\varpi_{i(\kappa)},\varpi_{i(\kappa)}\right)\right\}\right)+ \tag{3.9}$$
$$G_b\left(\varpi_{i(\kappa)},\varpi_{i(\kappa)-1},\varpi_{i(\kappa)-1}\right),$$

taking $\kappa\to\infty,$ in Eq. (3.9), we get

$\frac{4s^3\varepsilon}{\alpha} \le \varepsilon$ and $4s^3 < \varepsilon$, again not possible, hence $\{\varpi_i\}$ is a G_b-Cauchy sequence in Υ. Completeness property of (Υ, G_b), enable us to find a sequence which is G_b convergent to a point $\kappa \in \Upsilon$.
Assume that $\Im\kappa \ne \kappa$ and from lemma 2.6, we have

$$\frac{1}{s}G_b(\Im\kappa,\kappa,\kappa) \le 4s^4 \liminf_{i\to\infty} G_b(\Im\kappa,\Im\varpi_i,\Im\varpi_i).$$

This implies that

$$\begin{aligned}\frac{4s^4}{s}G_b(\Im\kappa,\kappa,\kappa) &\le \liminf_{i\to\infty} G_b(\Im\kappa,\Im\varpi_i,\Im\varpi_i)\\ &\le \psi\left(4s^4 \limsup_{i\to\infty} G_b(\Im\kappa,\Im\varpi_i,\Im\varpi_i)\right)\\ &\le \psi\left(\limsup_{i\to\infty}\mathcal{A}_a(\kappa,\varpi_i,\varpi_i)\right) - \\ &\phi\left(\limsup_{i\to\infty}\mathcal{A}_a(\kappa,\varpi_i,\varpi_i)\right),\end{aligned}$$

where

$$\mathcal{A}_a(\kappa,\varpi_i,\varpi_i) = \max\left\{\begin{array}{c} G_b(\kappa,\varpi_i,\varpi_i), \\ G_b(\kappa,\Im\kappa,\Im\kappa), \\ G_b(\varpi_i,\Im\varpi_i,\Im\varpi_i), \\ G_b(\varpi_i,\Im\varpi_i,\Im\varpi_i), \\ \frac{1}{2s^2}\left[\begin{array}{l}\alpha G_b(\Im\kappa,\Im\varpi_i,\Im\varpi_i),+ \\ (1-\alpha)G_b(\Im\kappa,\varpi_i,\varpi_i) \\ G_b(\Im\varpi_i,\varpi_i,\varpi_i) \\ G_b(\Im\varpi_i,\kappa,\kappa)\end{array}\right]\end{array}\right\},$$

$$= \max\left\{\begin{array}{l} G_b(\kappa,\Im\kappa,\Im\kappa), \\ \frac{\alpha}{2s^2}G_b(\Im\kappa,\kappa,\kappa)\end{array}\right\}.$$

If $\frac{\alpha}{2s^2}G_b(\Im\kappa,\kappa,\kappa)$ is maximum, then

$$\psi\left(4s^3G_b(\Im\kappa,\kappa,\kappa)\right) \le \psi\left(\frac{\alpha}{2s^2}G_b(\Im\kappa,\kappa,\kappa)\right),$$

$$4s^3G_b(\Im\kappa,\kappa,\kappa) \le \frac{\alpha}{2s^2}G_b(\Im\kappa,\kappa,\kappa),$$

this shows that $8s^5 < \alpha$, contradiction to our assumption.
Hence

$$\begin{aligned}&\psi\left(4s^3G_b(\Im\kappa,\kappa,\kappa)\right)\\ &\le \psi\left(G_b(\kappa,\Im\kappa,\Im\kappa)\right) - \phi\left(G_b(\kappa,\Im\kappa,\Im\kappa)\right)\\ &< \psi\left(G_b(\kappa,\Im\kappa,\Im\kappa)\right).\end{aligned}$$

If $\Im\kappa \ne \kappa$, then

$$\begin{aligned}4s^3G_b(\Im\kappa,\kappa,\kappa) &< G_b(\kappa,\Im\kappa,\Im\kappa)\\ &< 2sG_b(\Im\kappa,\kappa,\kappa),\end{aligned}$$

this shows that $2s^2 < 1$, a contradiction, hence $\Im\kappa = \kappa$.

Uniqueness: If possible there is another point κ_1 such that $\Im\kappa_1 = \kappa_1$.

$$\begin{aligned}\left(4s^3G_b(\kappa,\kappa_1,\kappa_1)\right) &= \psi\left(4s^3G_b(\Im\kappa,\Im\kappa_1,\Im\kappa_1)\right)\\ &\le \psi\left(\mathcal{A}_a(\kappa,\kappa_1,\kappa_1)\right) - \phi\left(\mathcal{A}_a(\kappa,\kappa_1,\kappa_1)\right),\end{aligned}$$

where

$$\mathcal{A}_a(\kappa,\kappa_1,\kappa_1) = \max\left\{\begin{array}{c} G_b(\kappa,\kappa_1,\kappa_1)G_b(\kappa,\Im\kappa,\Im\kappa), \\ G_b(\kappa_1,\Im\kappa_1,\Im\kappa_1)G_b(\kappa_1,\Im\kappa_1,\Im\kappa_1), \\ \frac{1}{2s^2}\left[\begin{array}{l}\alpha G_b(\Im\kappa,\Im\kappa_1,\Im\kappa_1)+ \\ (1-\alpha)G_b(\Im\kappa,\kappa_1,\kappa_1) \\ G_b(\Im\kappa_1,\kappa_1,\kappa_1)G_b(\Im\kappa_1,\kappa,\kappa)\end{array}\right]\end{array}\right\}$$

$$= G_b(\kappa,\kappa_1,\kappa_1),$$

therefore

$\psi\left(4s^4 G_b\left(\kappa,\kappa_1,\kappa_1\right)\right) \le \psi\left(G_b\left(\kappa,\kappa_1,\kappa_1\right)\right),$

and we have $4s^4 < 1,$ a contradiction, hence $G_b\left(\kappa,\kappa_1,\kappa_1\right) = 0$ and $\kappa = \kappa_1.$

Example 3.2.

$\Upsilon = \left[0, \frac{12}{5}\right],$ take $G_b : \Upsilon \times \Upsilon \times \Upsilon \to \Re$ by

$$G_b\left(\varpi,\gamma,\mu\right) = \frac{1}{9}\left(\left|\varpi-\gamma\right|^2 + \left|\gamma-\mu\right|^2 + \left|\mu-\varpi\right|^2\right).$$

Then $\left(\Upsilon, G_b\right)$ is a complete G_b-metric space with $s = 2.$

We take

$$\Im\varpi = \left\{\begin{array}{l} \frac{1}{6}, \text{ if } \varpi \in [0,1] \\ \frac{\varpi}{9} - \frac{1}{18} \text{ if } \varpi \in \left(1, \frac{8}{5}\right] \end{array}\right\}.$$

And $\psi,\phi : [0,\infty) \to [0,\infty)$ by $\psi(y) = y$ and $\phi(y) = \frac{y}{3}$ for every $y > 0.$

Case 1: if $\varpi,\gamma,\mu \in [0,1],$ then

$$\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right) = 0,$$

and the inequality holds.

Case 2: if $\varpi,\gamma,\mu \in \left[1, \frac{8}{5}\right]$ and we assume that $\varpi > \gamma > \mu,$ then, we have

$$\begin{aligned} &\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right) \\ &= \frac{4.2^4}{9}\left\{\left|\frac{\varpi}{9} - \frac{\gamma}{9}\right| + \left|\frac{\gamma}{9} - \frac{\mu}{9}\right| + \left|\frac{\mu}{9} - \frac{\varpi}{9}\right|\right\}^2 \\ &\le \frac{64 \times 9 \times 9}{9 \times 9 \times 9 \times 25} = \frac{64}{225} \le \frac{578}{2187} \\ &= \frac{289}{729} - \frac{289}{2187} \\ &= \frac{2}{3} G_b\left(\varpi,\Im\varpi,\Im\varpi\right) = \frac{2}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right) \\ &= \mathcal{A}_a\left(\varpi,\gamma,\mu\right) - \frac{1}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right) \\ &= \psi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right) - \phi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right). \end{aligned}$$

Case 3: if $\varpi,\gamma \in [0,2]$ and $\mu \in \left[1, \frac{8}{5}\right]$ we assume that $\varpi > \gamma$ then, we have

$$\begin{aligned} &\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right) \\ &= \frac{4.2^4}{9}\left\{\left|\frac{\varpi}{9} - \frac{\gamma}{9}\right| + \left|\frac{\gamma}{9} - \frac{\mu}{9}\right| + \left|\frac{\mu}{9} - \frac{\varpi}{9}\right|\right\}^2 \\ &\le \frac{64 \times 9 \times 9}{9 \times 9 \times 9 \times 25} = \frac{64}{225} \le \frac{578}{2187} \\ &= \frac{289}{729} - \frac{289}{2187} \\ &= \frac{2}{3} G_b\left(\varpi,\Im\varpi,\Im\varpi\right) = \frac{2}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right) \\ &= \mathcal{A}_a\left(\varpi,\gamma,\mu\right) - \frac{1}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right) \\ &= \psi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right) - \phi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right). \end{aligned}$$

Case 4: if $\varpi,\gamma \in [0,2]$ and we assume that $\gamma > \mu,$ then, we have

$$\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right)$$
$$=4.2^4 G_b\left(\frac{1}{6},\frac{1}{6},\frac{\varpi}{9}-\frac{1}{18}\right)$$
$$\le \frac{64\times 4\times|2-\mu|^2}{9\times 9\times 9\times 25}=\frac{1024}{18225}\le\frac{578}{2187}$$
$$=\frac{289}{729}-\frac{289}{2187}$$
$$=\frac{2}{3}G_b\left(\varpi,\Im\varpi,\Im\varpi\right)=\frac{2}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\mathcal{A}_a\left(\varpi,\gamma,\mu\right)-\frac{1}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\psi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right)-\phi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right).$$

Case 5: if $\mu\in[0,2]$ and $\varpi,\gamma\in\left[1,\frac{8}{5}\right]$ we assume that $\varpi>\gamma$ then, we have

$$\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right)$$
$$=\frac{4.2^4}{9}G_b\left(\frac{\varpi}{9}-\frac{1}{18},\frac{\gamma}{9}-\frac{1}{18},\frac{1}{6}\right)$$
$$\le \frac{64\times 4\times 4}{9\times 9\times 9\times 25}=\frac{1024}{225}\le\frac{578}{2187}$$
$$=\frac{289}{729}-\frac{289}{2187}$$
$$=\frac{2}{3}G_b\left(\varpi,\Im\varpi,\Im\varpi\right)=\frac{2}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\mathcal{A}_a\left(\varpi,\gamma,\mu\right)-\frac{1}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\psi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right)-\phi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right).$$

Case 6: if $\mu\in[0,2]$ and $\varpi,\gamma\in\left[1,\frac{8}{5}\right]$ we assume that $\varpi>\gamma$ then, we have

$$\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right)$$
$$=4.2^4 G_b\left(\frac{\varpi}{9}-\frac{1}{18},\frac{\gamma}{9}-\frac{1}{18},\frac{1}{6}\right)$$
$$\le \frac{64\times 4\times|\varpi-2|^2}{9\times 9\times 9}=\frac{1024}{18225}\le\frac{578}{2187}$$
$$=\frac{289}{729}-\frac{289}{2187}$$
$$=\frac{2}{3}G_b\left(\varpi,\Im\varpi,\Im\varpi\right)=\frac{2}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\mathcal{A}_a\left(\varpi,\gamma,\mu\right)-\frac{1}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\psi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right)-\phi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right).$$

Case 7: if $\varpi\in[0,2]$ and $\gamma,\mu\in\left[1,\frac{8}{5}\right]$ we assume that $\gamma>\mu,$ then, we have

$$\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right)$$
$$=4.2^4 G_b\left(\frac{1}{6},\frac{\gamma}{9}-\frac{1}{18},\frac{\gamma}{9}-\frac{1}{18}\right)$$
$$\le \frac{64\times 4\times|\gamma-2|^2}{9\times 9\times 9}=\frac{1024}{18225}\le\frac{578}{2187}$$
$$=\frac{289}{729}-\frac{289}{2187}$$
$$=\frac{2}{3}G_b\left(\varpi,\Im\varpi,\Im\varpi\right)=\frac{2}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\mathcal{A}_a\left(\varpi,\gamma,\mu\right)-\frac{1}{3}\mathcal{A}_a\left(\varpi,\gamma,\mu\right)$$
$$=\psi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right)-\phi\left(\mathcal{A}_a\left(\varpi,\gamma,\mu\right)\right).$$

Case 8: if $\varpi,\mu\in[0,2]$ and $\gamma\in\left[1,\frac{8}{5}\right]$ we assume that $\varpi>\mu,$ then, we have

$$\psi\left(4s^4 G_b\left(\Im\varpi,\Im\gamma,\Im\mu\right)\right)$$

$$= 4.2^4 G_b\left(\frac{1}{6}, \frac{\gamma}{9} - \frac{1}{18}, \frac{1}{6}\right)$$
$$\leq \frac{64\times 4\times|\gamma-2|^2}{9\times 9\times 9} = \frac{1024}{18225} \leq \frac{578}{2187}$$
$$= \frac{289}{729} - \frac{289}{2187}$$
$$= \frac{2}{3}G_b(\varpi, \Im\varpi, \Im\varpi) = \frac{2}{3}\mathcal{A}_a(\varpi,\gamma,\mu)$$
$$= \mathcal{A}_a(\varpi,\gamma,\mu) - \frac{1}{3}\mathcal{A}_a(\varpi,\gamma,\mu)$$
$$= \psi(\mathcal{A}_a(\varpi,\gamma,\mu)) - \phi(\mathcal{A}_a(\varpi,\gamma,\mu)).$$

Thus $\Im$ fulfills all the assumptions of the result 3.2 and $\frac{1}{6}$ is the unique fixed point of $\Im$.

Corollary 3.3. In a complete G_b-metric space (Υ, G_b) with $s \geq 1$, let $\Im$ be (α, Ψ, Φ) generalized weakly contractive self map on Υ and there are $\alpha \in (0,1), \psi \in \Psi$ and $\phi \in \Phi$ such that:

$$\psi\left(4s^4 G_b(\Im\varpi, \Im\gamma, \Im\mu)\right)$$
$$\leq \psi(\mathcal{A}_a(\varpi,\gamma,\mu)) - \phi(\mathcal{A}_a(\varpi,\gamma,\mu)).$$

where
$\mathcal{A}_a(\varpi,\gamma,\mu)$

$$= \max\left\{\begin{matrix} G_b(\varpi,\gamma,\mu) \\ G_b(\varpi,\Im\varpi,\Im\varpi) \\ G_b(\gamma,\Im\gamma,\Im\gamma), G_b(\mu,\Im\mu,\Im\mu) \\ \dfrac{G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\gamma,\Im\gamma,\Im\gamma)}{1+G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\gamma,\Im\gamma,\Im\gamma)} \\ \dfrac{G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\mu,\Im\mu,\Im\mu)}{1+G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\mu,\Im\mu,\Im\mu)} \\ \dfrac{1}{2s^2}\begin{bmatrix} \alpha G_b(\Im\varpi,\Im\gamma,\Im\mu) \\ (1-\alpha)G_b(\Im\varpi,\gamma,\gamma)+ \\ G_b(\Im\gamma,\mu,\mu)G_b(\Im\mu,\varpi,\varpi) \end{bmatrix} \end{matrix}\right\},$$

for every $\varpi,\gamma,\mu \in \Upsilon$. Then, $\Im$ has a unique fixed point.

Corollary 3.4. In a complete G_b-metric space (Υ, G_b) with $s \geq 1$, let $\Im$ be (α, Ψ, Φ) generalized weakly contractive self map on Υ and there are $\alpha \in (0,1), \psi \in \Psi$ such that:

$$\psi\left(4s^4 G_b(\Im\varpi, \Im\gamma, \Im\mu)\right) \leq \psi(\mathcal{A}_a(\varpi,\gamma,\mu)),$$

where
$\mathcal{A}_a(\varpi,\gamma,\mu)$

$$= \max\left\{\begin{matrix} G_b(\varpi,\gamma,\mu), G_b(\varpi,\Im\varpi,\Im\varpi) \\ G_b(\gamma,\Im\gamma,\Im\gamma), G_b(\mu,\Im\mu,\Im\mu) \\ \dfrac{G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\gamma,\Im\gamma,\Im\gamma)}{1+G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\gamma,\Im\gamma,\Im\gamma)} \\ \dfrac{G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\mu,\Im\mu,\Im\mu)}{1+G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\mu,\Im\mu,\Im\mu)} \\ \dfrac{1}{2s^2}\begin{bmatrix} \alpha G_b(\Im\varpi,\Im\gamma,\Im\mu) \\ (1-\alpha)G_b(\Im\varpi,\gamma,\gamma)+ \\ G_b(\Im\gamma,\mu,\mu)G_b(\Im\mu,\varpi,\varpi) \end{bmatrix} \end{matrix}\right\}.$$

For every $\varpi,\gamma,\mu \in \Upsilon$. Then, $\Im$ has a unique fixed point.

Proof. This can be done by choosing $\psi(y) = \frac{y+\psi(y)}{2}$ and $\phi(y) = \frac{y+\phi(y)}{2}$.

Corollary 3.5. In a complete G_b-metric space (Υ, G_b) with $s \geq 1$, let $\Im$ be (α, Ψ, Φ) generalized weakly contractive self map on Υ and there are $\alpha \in (0,1), \psi \in \Psi$ and $\phi \in \Phi$ such that:

$$\psi\left(4s^4 G_b(\Im\varpi, \Im\gamma, \Im\mu)\right) \leq \frac{\rho}{4s^4}(\mathcal{A}_a(\varpi,\gamma,\mu)),$$

where

$$\mathcal{A}_a(\varpi,\gamma,\mu) = \max\left\{\begin{array}{c} G_b(\varpi,\gamma,\mu), G_b(\varpi,\Im\varpi,\Im\varpi) \\ G_b(\gamma,\Im\gamma,\Im\gamma), G_b(\mu,\Im\mu,\Im\mu) \\ \dfrac{G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\gamma,\Im\gamma,\Im\gamma)}{1+G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\gamma,\Im\gamma,\Im\gamma)} \\ \dfrac{G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\mu,\Im\mu,\Im\mu)}{1+G_b(\varpi,\Im\varpi,\Im\varpi)+G_b(\mu,\Im\mu,\Im\mu)} \\ \dfrac{1}{2s^2}\left[\begin{array}{c} \alpha G_b(\Im\varpi,\Im\gamma,\Im\mu) \\ (1-\alpha)G_b(\Im\varpi,\gamma,\gamma)+ \\ G_b(\Im\gamma,\mu,\mu)G_b(\Im\mu,\varpi,\varpi) \end{array}\right] \end{array}\right\}.$$

For every $\varpi,\gamma,\mu \in \Upsilon$. Then, $\Im$ has a unique fixed point.

4. Conclusion

From this paper, the results of weakly contractive mappings in etric space are proved. The importance and effectiveness of the proposed outcome have been shown by example and corollaries.

Reference

[1] Aghajani A, Abbas M, Roshan J R. Common fixed point of generalized weak contractive mappings in partially ordered Gb-metric spaces. Filomat 2014; 28(6):1087-101.

[2] Gupta V, Ege O, Saini R, Manuel D. Various fixed point results in complete Gb-metric spaces. Dynamic systems and applications 2021; 30(2): 277-93.

[3] Hamaizia T., Fixed point theorems involving C-class functions in Gb-metric spaces. Journal of applied mathematics and informatics 2021; 39(3_4): 529-39.

[4] Mustafa Z, Roshan J R, Parvaneh V. Existence of a tripled coincidence point in ordered Gb-metric spaces and applications to a system of integral equations. Journal of Inequalities and Applications 2013; 453.

[5] Sedghi S, Shobkolaei N, Rezaei Roshan J, Shatanawi W. Coupled fixed point theorems in Gb-metric spaces. Matematiki Vesnik 2014; 66(2): 190-01.

[6] Alber Ya I, Guerre-Delabriere S. Principle of weakly contractive maps in hilbert spaces. Operator theory Advances and Application 1997; 98: 7-22.

[7] Rhoades BE. Some theorems on weakly contractive maps. Nonlinear Analysis 2001; 47: 2683-693.

[8] Dutta PN, Choudhury BS. A generalization of contraction principle in metric spaces. Fixed Point Theory and Application 2008; Article ID 406368: 8 pages.

[9] Aydi H, Shatanawi W, Vetro C. On generalized weakly G-contraction mapping in G-metric spaces. Comput Math Application 2011; 62: 422-24.

[10] Babu GVR, Babu DR, Rao K N, Kumar BVS. Fixed points of (ψ, ϕ)-almost weakly contractive maps in G-metric spaces. Applied Mathematics E-Notes 2014; 14: 69-85.

[11] Choudhury BS, Konar P, Rhoades BE, Metiya N. Fixed point theorems for generalized weakly contractive mappings. Nonlinear Analysis 2011; 74: 2116-26.

[12] Doric D. Common fixed point for generalized (ψ,φ)-weak contractions. Applied Mathematics Letters 2009; 22:1896-900.

[13] Mustafa Z, Roshan J R, Parvaneh V. Coupled coincidence point results for

(ψ, ϕ) weakly contractive mappings in partially ordered Gb-metric spaces. Fixed Point Theory Application 2013; Article Number 206.

[14] Surender N, Reddy B K. Weakly compatible mapping satisfying generalized contraction principle in complete G-metric spaces. Annals of Pure and Applied Mathematics 2015; 10(2): 179-90.

[15] Aydi H, Rakic D, Aghajani A, Došenovic T, Noorani M, Qawaqneh H. On Fixed Point Results in Gb-Metric spaces. Mathematics 2019; 7(7): 617.