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Generalized Elliptic Integrals Involving Incomplete Fox Wright Function

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ABSTRACT

Elliptic integrals are used in various diverse physical fields (like some exclusive radiation field problems, analysis of crystallographic minimum surface problem, study of electromagnetic waves through an elliptic disk, elliptic crack problems etc.). Due to these applications a remarkably large number of elliptic integrals are defined and studied earlier. Motivated by these works we have derived some new elliptic integrals involving incomplete wright function. The results established in this paper are basic in nature and can be reduced in many special cases. Some of the special cases are mentioned here as corollaries.

Keywords: Beta and Gamma functions; Fox wright function; Incomplete elliptic integral; Riemann-Liouville fractional differential operator

1. Introduction

Nowadays, Research is centered on integral presentation of different type of special functions due to their use of various fields. Several integral formulas having generalized special function, plays an important role in physical and technical problems. The generalized special function with contrasting line of research have inspired the researchers to look into the area of integrals and linked generalized special function [1]. With the use of Riemann-Liouville fractional integral we describe many different hypergeometric notations and then we can use these representations for the derivation of many definite integrals [2-4]. Incomplete elliptic integrals and related definite integrals help to solve many physical problems and may find applications in definite engineering problems. Inspired by this idea, in this paper we estimated certain incomplete integrals and related general definite integrals. Here our target is to provide the theory of generalized incomplete elliptic integrals in an exclusive and generalized way.

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Incomplete Fox-Wright Ψ-functions $_r\psi_s^{(\Gamma)}(x)$ and $_r\psi_s^{(\gamma)}(x)$:

Here we define the family of the incomplete Fox-Wright function $_r\psi_s^{(\Gamma)}(x)$ and $_r\psi_s^{(\gamma)}(x)$, in both series and contour integral form which is a generalization of incomplete hyper geometric functions defined by Srivastava et al. [9, 13].

$$_r\psi_s^{(\Gamma)}\left[\begin{matrix}(a_1,\alpha_1,x),(a_j,\alpha_j)_{2,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\middle| x\right] = \frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)} \frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\frac{x^n}{n!}, \tag{1.1}$$

$$= \frac{1}{2\pi i}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\int_C \frac{\Gamma(a_1+v,y)\prod_{j=2}^{r}\Gamma(a_j+v)}{\prod_{j=1}^{s}\Gamma(b_j+v)}, \Gamma(-v)(-x)^v\, dv \ \left(|\arg(-x)|<\pi\right), \tag{1.2}$$

and

$$(\alpha_j>0\ (j=1,...,r);\ \beta_j>0\,(j=1,...,s);$$
$$1+\sum_{j=1}^{s}\beta_j-\sum_{j=1}^{r}\alpha_j\geq 0).$$

Similarly, Eq. (1.3) represents the series representation and Eq. (1.4) represents the contour integral of incomplete Fox-Wright function $_r\psi_s^{(\gamma)}(x)$:

$$_r\psi_s^{(\gamma)}\left[\begin{matrix}(a_1,\alpha_1,x),(a_j,\alpha_j)_{2,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\middle| x\right] = \frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)} \frac{\sum_{n=0}^{\infty}\gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\frac{x^n}{n!}, \tag{1.3}$$

$$= \frac{1}{2\pi i}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\int_C \frac{\gamma(a_1+v,y)\prod_{j=2}^{r}\Gamma(a_j+v)}{\prod_{j=1}^{s}\Gamma(b_j+v)}, \Gamma(-v)(-x)^v\, dv \ \left(|\arg(-x)|<\pi\right) \tag{1.4}$$

where

$$(\alpha_j>0\ (j=1,...,r);\ \beta_j>0\,(j=1,...,s);$$
$$1+\sum_{j=1}^{s}\beta_j-\sum_{j=1}^{r}\alpha_j\geq 0).$$
$$\alpha_j>0\ (j=1,...,r);\ \beta_j>0\,(j=1,...,s);$$
$$1+\sum_{j=1}^{s}\beta_j-\sum_{j=1}^{r}\alpha_j\geq 0).$$

The multivariable hyper geometric function defined by Srivastava & Daoust [5-7] is given by

$$F^{r:s_1,s_2,.......s_k}_{n:l_1,l_2,.......l_k}\left[\begin{matrix}(a_j;\alpha_j',...,\alpha_j^{(k)})_{1,r}:(c_j',t_j')_{1,s_1};...;\\(b_j;\beta_j',...,\beta_j^{(k)})_{1,n}:(d_j',\delta_j')_{1,l_1};...;\end{matrix}\ \begin{matrix}(c_j^{(k)},t_j^{(k)})_{1,s_k}\\(d_j^{(r)},\delta_j^{(r)})_{1,l_k}\end{matrix}\middle| x_1,x_2,...,x_k\right]$$

$$= \sum_{m_1,...,m_k}^{\infty}\frac{\prod_{j=1}^{r}(a_j)_{m_1\alpha_j'+...+m_k\alpha_j^{(k)}}}{\prod_{j=1}^{n}(b_j)_{m_1\beta_j'+...+m_k\beta_j^{(k)}}}\ \frac{\prod_{j=1}^{s_1}(c_j')_{m_1t_j'}...\prod_{j=1}^{s_k}(c_j^{(k)})_{m_1t_j^{(k)}}}{\prod_{j=1}^{m_1}(d_j')_{m_1\delta_j'}...\prod_{j=1}^{m_r}(d_j^{(k)})_{m_1\delta_j^{(k)}}} \tag{5}$$

with variable and parametric constraints, the above-mentioned series is absolutely convergent. The Pochhammer symbol denoted by $(c)_p$ is defined as:

$$(c)_p = \frac{\Gamma(c+p)}{\Gamma(c)} = \begin{cases} 1, & \{p=0;\lambda \in C/\{0\}\} \\ c(c+1)...(c+n-1), & \{p=n \in N = N_0; c \in C\} \end{cases} \quad (1.6)$$

where $c, p \in C$.

Riemann-Liouville operator $D_z^{\delta} f(z)$ of fractional calculus [8]:

$$D_Z^{\gamma-\delta}\left\{z^{\gamma-1}\prod_{j=1}^{k}\left\{\left(1-a_j z^{\delta_j}\right)^{-\alpha_j}\right\}\right\} = \frac{\Gamma(\gamma)}{\Gamma(\delta)} z^{\delta-1} F^{1:1;...1}_{1:0;...;0}\left[\begin{matrix}(\gamma;\delta_{1....}\delta_k):(\alpha_1,1); \\ (\delta;\delta_{1....}\delta_k);...;\end{matrix}\begin{matrix}...;(\alpha_k,1) \\ ...;......;\end{matrix}\middle| a_1 z^{\delta_1},...,a_k z^{\delta_k}\right] \quad (1.7)$$

where $R(\gamma) > 0; \delta_j > 0 (j=1,...,k);$

$$\max\left\{\left|a_1 z^{\delta_1}\right|,...,\left|a_k z^{\delta_k}\right|\right\} < 1$$

$$D_z^{\upsilon}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\upsilon)}\int_0^z (z-\xi)^{\upsilon-1} f(\xi) d\xi \ [R(\upsilon)<0] \\ \frac{d^n}{dz^n} D_z^{\upsilon-n}\{f(z)\} \quad [0 \le R(\upsilon) < n, n \in N_0] \end{cases} \quad (1.8)$$

Generalized elliptic integral of third kind [9]: The generalized form of elliptic integrals is defined by

$$R(\theta,h,\xi;\alpha,t) = \int_0^{\theta} \frac{1}{(1+\xi\sin^2\phi)^{\alpha}(1-h^2\sin^2\phi)^{1/2-t}} d\phi \quad (1.9)$$

$$R(\theta,h,\xi;\alpha,t) = \int_0^{\sin\theta} \frac{1}{(1+\xi e^2)^{\alpha}\sqrt{(1-e^2)}\,(1-h^2e^2)^{1/2-t}} de,$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}; t \in C, \alpha \ge 0\right) \quad (1.10)$$

where ξ is elliptic characteristic and $\xi > -1$. Also, by putting $\alpha = 1$ in (1.9) and (1.10), we have

$$I(\theta,h,\xi;t) = \int_0^{\theta} \frac{1}{(1+\xi\sin^2\phi)(1-h^2\sin^2\phi)^{1/2-t}} d\phi$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}; t \ge 0\right) \quad (1.11)$$

$$I(\theta,h,\xi;t) = \int_0^{\sin\theta} \frac{1}{(1+\xi e^2)\sqrt{(1-e^2)}\,(1-h^2e^2)^{1/2-t}} de,$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}; t \ge 0\right) \quad (1.12)$$

Similarly, on putting $t = 0$ and $\alpha = 1$ in equation (1.9) we get

$$\Pi(\theta,h,\xi) = \int_0^{\theta} \frac{1}{(1+\xi\sin^2\phi)(1-h^2\sin^2\phi)^{1/2}} d\phi,$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}\right) \quad (1.13)$$

On substituting $\xi = 0$ in equation (1.9), we get elliptic integral of third kind

$$H(\theta,k;\gamma) = \int_0^{\theta} \frac{1}{(1-h^2\sin^2\phi)^{1/2-t}} d\phi,$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}\right) \quad (1.14)$$

If we substitute $t = 0, \xi = 0$, then equation (1.9) reduces to $F(\phi,k)$

$$F(\theta,h) = \int_0^{\theta} \frac{1}{(1-h^2\sin^2\phi)^{1/2}} d\phi,$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}\right) \quad (1.15)$$

and if $t = 1, \xi = 0$, then our function (1.9) reduces to $E(\theta,h)$

$$E(\theta,h) = \int_0^{\theta} \sqrt{(1-h^2\sin^2\phi)} d\phi,$$

$$\left(\left|h^2\right| < 1; 0 \le \theta \le \frac{\pi}{2}\right). \quad (1.16)$$

Similarly, if we put $t = 0$ and $\theta = \pi/2$,then Eq. (1.14) reduces to K (h) =F (π/2, h)

$$K(h) = \int_0^{\pi/2} \frac{1}{\sqrt{(1-h^2\sin^2\phi)}} d\phi, \left(\left|h^2\right|<1\right), \quad (1.17)$$

Similarly,

$$E(h) = \int_0^{\pi/2} \sqrt{(1-h^2\sin^2\phi)} d\phi, \quad \left(\left|h^2\right|<1\right), \quad (1.18)$$

On applying k = 3 in Riemann-Liouville operator defined in (1.7) to the integral in the equation (1.9) with condition $t = \delta - 1 = 1$ and $z = -\sin\theta$, we get

$$R(\theta,h,\xi;\alpha,t) = \sin\theta F^{1:1;1;1}_{1:0;0;0}\left[\begin{array}{l}(1:2;2;2)(1/2-t,1); \\ (2:2;2;2); \quad ;\end{array}\right.$$
$$\left.\begin{array}{l}(1/2,1);(\alpha,1) \\ ; \quad ;\end{array}\middle| h^2\sin^2\theta, \sin^2\theta, -\xi\sin^2\theta\right]. \quad (1.19)$$

Similarly,

$$I(\theta,h,\xi;t) = \sin\theta F^{1:1;1;1}_{1:0;0;0}\left[\begin{array}{l}(1:2;2;2)(1/2-t,1); \\ (2:2;2;2); \quad ;\end{array}\right.$$
$$\left.\begin{array}{l}(1/2,1);(1,1) \\ ; \quad ;\end{array}\middle| h^2\sin^2\theta, \sin^2\theta, -\xi\sin^2\theta\right]. \quad (1.20)$$

With the help of definition of Pochhammer symbol

$$\frac{(1)_{2l+2m+2n}}{(2)_{2l+2m+2n}} = \frac{\Gamma(2l+2m+2n+1)}{\Gamma(2l+2m+2n+2)}$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)_{l+m+n}}{2\Gamma\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)_{l+m+n}} = \frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}} \quad (1.21)$$

where we used the duplication formula defined as follows:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad (1.22)$$

with the help of above relation defined by equation (1.19), we can write as follows:

$$R(\theta,h,\xi;\alpha,t)$$
$$= \sin\theta . F_1\left[\frac{1}{2}:\frac{1}{2}-t,\frac{1}{2},\alpha;\frac{3}{2};h^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right],$$
$$\left(\left|h^2\right|<1; 0\le\theta\le\frac{\pi}{2}; t\in C, \alpha\ge 0\right). \quad (1.23)$$

2. Main Results

Theorem1. If Incomplete Fox-Wright Ψ-functions ${}_r\psi_s^{(\Gamma)}(z)$, is given by

$${}_r\psi_s^{(\Gamma)}\left[\begin{array}{c}(a_1,\alpha_1,x),(a_j,\alpha_j)_{2,r} \\ (b_j,\beta_j)_{1,s}\end{array}\middle| x\right]$$
$$= \frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)} \frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)} \frac{x^n}{n!}.$$

Under the existence conditions

$$(\alpha_j > 0 \ (j=1,...,r);\ \beta_j > 0\,(j=1,...,s);$$
$$1+\sum_{j=1}^{s}\beta_j - \sum_{j=1}^{r}\alpha_j \ge 0).$$

Also let

$$\{\tau,\chi,\omega,\vartheta\}\ge 0, (\tau+\chi>0;\omega+\vartheta>0),$$

and

$$\sum_{m=1}^{\infty}\left|\frac{a_m}{m^{1/2(1+\lambda)}}\right|<\infty, \sum_{m=1}^{\infty}\left|\frac{a_m}{m^{1+\mu/2}}\right|<\infty;$$

$$[\tau=0; R(\lambda)>-1, R(\mu)>-2].$$

Then,

$$\int_0^1 h^\lambda\left(\sqrt{1-h^2}\right)^\mu {}_r\psi_s^{(\Gamma)}\left[\begin{array}{c}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,r} \\ (b_j,\beta_j)_{1,s}\end{array}\middle|\right.$$
$$\left. xh^\tau\left(\sqrt{(1-h^2)}\right)^\chi\right] R\left(\theta,\zeta h^\omega\left(\sqrt{1-h^2}\right)^\vartheta,\xi;\alpha,t\right)dh$$
$$= \frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)} \frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}$$
$$\frac{\sin\theta}{2} B\left(\frac{\lambda+1}{2},\frac{\mu+1}{2}\right) F^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{array}{l}A:(a_1,\alpha_1)...(a_n,\alpha_n); \\ B:(b_1,\beta_1)...(b_n,\beta_n);\end{array}\right.$$

$$\left.\begin{matrix}\left(\frac{1}{2},1\right);(\alpha,1) \\ -;-;\end{matrix}\right| x,\zeta^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\Bigg] \tag{2.1}$$

$$A=\left(\frac{1}{2};0,1,1,1\right),\left(\frac{\lambda+1}{2},\frac{\tau}{2},\omega,0,0\right),$$

$$\left(\frac{\mu+2}{2},\frac{\chi}{2},\vartheta,0,0\right);$$

$$B=\left(\frac{3}{2};0,1,1,1\right),\left(\frac{\mu+\lambda+3}{2}:\frac{\chi+\tau}{2},\omega+\vartheta,0,0\right),$$

where $R(\lambda)>-1$ and

$|\zeta|<1\left[or|\zeta|=1 and\ R(\lambda+2\omega)>-1\right].$

Proof. To prove Theorem 1, we use the series form of Incomplete Fox-Wright Ψ-function given by equation (1.1) and formula of generalized elliptic function of third kind from equation (1.19) in L.H.S. part of theorem, we get

$$\int_0^1 h^\lambda\left(\sqrt{1-h^2}\right)^\mu {}_r\psi_s^{(\Gamma)}\left[\left.\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,r} \\ (b_j,\beta_j)_{1,s}\end{matrix}\right| xh^\tau\left(\sqrt{(1-h^2)}\right)^\chi\right]R\left(\theta,\zeta h^\omega\left(\sqrt{1-h^2}\right)^\vartheta,\xi;\alpha,t\right)dh$$

$$=\int_0^1 h^\lambda\left(\sqrt{1-h^2}\right)^\mu \frac{\prod_{j=1}^s\Gamma(b_j)\sum_{n=0}^\infty\Gamma(a_1+\alpha_1 n,y)}{\prod_{j=1}^r\Gamma(a_j)\prod_{j=1}^s\Gamma(b_j+\beta_j n)}$$

$$\prod_{j=2}^r\Gamma(a_j+\alpha_j n)\frac{x^n}{n!}h^{\tau n}(1-h^2)^{\frac{n\chi}{2}}\sin\theta$$

$$F_{1;0;0;0}^{1;1;1;1}\left[\left.\begin{matrix}(1;2,2,2) & \left(\frac{1}{2}-t,1\right)\left(\frac{1}{2},1\right) & (\alpha,1) \\ (2;2,2,2) & ;\ ;\ ; & \end{matrix}\right| \zeta^2h^{2\omega}(1-h^2)^\vartheta\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right]dh.$$

Now expanding the multivariable hypergeometric function with the help of (1.5), we get

$$=\int_0^1 h^\lambda\left(\sqrt{1-h^2}\right)^\mu\frac{\prod_{j=1}^s\Gamma(b_j)}{\prod_{j=1}^r\Gamma(a_j)}\sum_{n=0}^\infty\frac{\Gamma(a_1+\alpha_1 n,y)}{\prod_{j=1}^r\Gamma(b_j+\beta_j n)}$$

$$\prod_{j=2}^r\Gamma(a_j+\alpha_j n)\frac{x^n}{n!}h^{\tau n}(1-h^2)^{\frac{n\chi}{2}}\sin\theta$$

$$\frac{(1)_{2n_1+2n_2+2n_3}}{(2)_{2n_1+2n_2+2n_3}}\frac{\left(\frac{1}{2}-t\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(\alpha)_{n_3}}{n_1!\ n_2!\ n_3!}\zeta^{2n_1}$$

$$\zeta^{2n_1}(\sin^2\theta)^{n_1+n_2+n_3}(-\xi)^{n_3}h^{2\omega n_1}(1-h^2)^{\vartheta n_1}dh.$$

On simplification we get

$$=\frac{1}{2}\frac{\prod_{j=1}^s\Gamma(b_j)}{\prod_{j=1}^r\Gamma(a_j)}\sum_{n=0}^\infty\frac{\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^r\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^r\Gamma(b_j+\beta_j n)}\frac{x^n}{n!}$$

$$(-\xi)^{n_3}\zeta^{2n_1}(\sin\theta)(\sin^2\theta)^{n_1+n_2+n_3}\frac{\left(\frac{1}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}$$

$$\frac{\left(\frac{1}{2}-t\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(\alpha)_{n_3}}{n_1!\ n_2!\ n_3!}\int_0^1 h^{\lambda+\tau n+2\omega n_1}(1-h^2)^{\frac{\mu}{2}+\frac{n\chi}{2}+\vartheta n_1}dh. \tag{2.2}$$

Now evaluate the integral part using equation (2.2) we arrive

$$=\frac{1}{2}\frac{\prod_{j=1}^s\Gamma(b_j)}{\prod_{j=1}^r\Gamma(a_j)}\sum_{l=0}^\infty\frac{\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^r\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^r\Gamma(b_j+\beta_j n)}\frac{x^n}{n!}$$

$$(-\xi)^{n_3}\zeta^{2n_1}(\sin\theta)(\sin^2\theta)^{n_1+n_2+n_3}\frac{\left(\frac{1}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}\frac{\left(\frac{1}{2}-t\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(\alpha)_{n_3}}{n_1!\ n_2!\ n_3!}\times$$

$$\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{\lambda+1}{2}\right)_{\frac{\tau n}{2}+\omega n_1}\Gamma\left(\frac{\mu+1}{2}\right)\left(\frac{\mu+1}{2}\right)_{\frac{\chi n}{2}+\vartheta n_1}}{\Gamma\left(\frac{\lambda+\mu+3}{2}\right)\left(\frac{\lambda+\mu+3}{2}\right)_{\frac{(\tau+\chi)n}{2}+(\omega+\vartheta)n_1}}.$$

Finally, interpreting the result in terms of multivariable hypergeometric function [10] we obtain the R.H.S. of (2.1).

3. Particular Cases

3.1 Case-1

By using the elliptic function given in equation (1.11), we obtained the following result:

$$\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu} {}_r\psi_s^{(\Gamma)}\left[\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,r}\\(b_j,B_j)_{1,s}\end{matrix}\middle| xh^{\tau}\left(\sqrt{(1-h^2)}\right)^{\chi}\right] I\left(\theta,\zeta h^{\omega}\left(\sqrt{1-h^2}\right)^{\vartheta},\xi;t\right)dh$$

$$=\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\times$$

$$\frac{\sin\theta}{2}\beta\left(\frac{\lambda+1}{2},\frac{\mu+1}{2}\right)\times$$

$$F^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{matrix}A:(a_1,\alpha_1)\ldots(a_n,\alpha_n);\left(\frac{1}{2}-t,1\right);\left(\frac{1}{2},1\right);(1,1)\\B:(b_1,\beta_1)\ldots(b_n,\beta_n);-;-;\end{matrix}\middle| x,\zeta^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right], \tag{3.1}$$

where

$$A=\left(\frac{1}{2};0,1,1,1\right),\left(\frac{\lambda+1}{2},\frac{\tau}{2},\omega,0,0\right),\left(\frac{\mu+2}{2},\frac{\chi}{2},\vartheta,0,0\right);$$

$$B=\left(\frac{3}{2};0,1,1,1\right),\left(\frac{\mu+\lambda+3}{2}:\frac{\chi+\tau}{2},\omega+\vartheta,0,0\right)$$

and $R(\lambda)>-1$ and

$$|\zeta|<1\left[or|\zeta|=1 and\ R(\lambda+2\omega)>-1\right].$$

Proof.

$$\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu} {}_r\psi_s^{(\Gamma)}\left[\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,r}\\(b_j,B_j)_{1,s}\end{matrix}\middle| xh^{\tau}\left(\sqrt{(1-h^2)}\right)^{\chi}\right] I\left(\theta,\zeta h^{\omega}\left(\sqrt{1-h^2}\right)^{\vartheta},\xi;t\right)dh.$$

By using equation (1.1) and (1.20), we get

$$\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}$$

$$\frac{x^l}{n!}h^{\tau n}(1-h^2)^{\frac{n\chi}{2}}\sin\theta\times$$

$$F^{1;1;1;1}_{1;0;0;0}\left[\begin{matrix}(1;2,2,2) & \left(\frac{1}{2}-t,1\right)\left(\frac{1}{2},1\right) & (1,1)\\(2;2,2,2) & ;\ ;\ ; & \end{matrix}\middle| \zeta^2 h^{2\omega}(1-h^2)^{\vartheta}\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right]dh.$$

Now using equation (1.5), we have

$$=\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\times$$

$$\frac{x^l}{n!}h^{\tau n}(1-h^2)^{\frac{n\chi}{2}}\sin\theta\times\frac{(1)_{2n_1+2n_2+2n_3}}{(2)_{2n_1+2n_2+2n_3}}\times$$

$$\frac{\left(\frac{1}{2}-t\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(1)_{n_3}}{n_1!\ n_2!\ n_3!}\times$$

$$\zeta^{2n_1}(\sin^2\theta)^{n_1+n_2+n_3}(-\xi)^{n_3}h^{2\omega n_1}(1-h^2)^{\vartheta n_1}\,dh,$$

$$=\frac{1}{2}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{r}\Gamma(b_j+B_j n)}\times$$

$$\frac{x^n}{n!}(-\xi)^{n_3}\zeta^{2n_1}(\sin\theta)(\sin^2\theta)^{n_1+n_2+n_3}\times$$

$$\frac{\left(\frac{1}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}\int_0^1 h^{\lambda+\tau n+2\omega n_1}(1-h^2)^{\frac{\mu}{2}+\frac{n\chi}{2}+\vartheta n_1}dh.$$

To solve

$$=\int_0^1 h^{\lambda+\tau n+2\omega n_1}(1-h^2)^{\frac{\mu}{2}+\frac{n\chi}{2}+\vartheta n_1}dh,$$

let $h^2=u\Rightarrow h=u^{1/2}\Rightarrow dh=\frac{1}{2}u^{-1/2}du$

$$=\int_0^1 u^{\frac{\lambda+\tau n+2\omega n_1}{2}}(1-u)^{\frac{\mu}{2}+\frac{\chi n}{2}+\vartheta n_1}\frac{1}{2}u^{-1/2}du$$

$$=\frac{1}{2}\int_0^1 u^{\frac{\lambda+\tau n+2\omega n_1-1}{2}+1-1}(1-u)^{\frac{\mu}{2}+\frac{\chi n}{2}+\vartheta n_1+1-1}du$$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{\lambda+\tau n+1}{2}+\omega n_1\right)\Gamma\left(\frac{\mu+\chi n+\vartheta n_1}{2}+1\right)}{\Gamma\left\{\left(\frac{\lambda+\tau n+\mu+\chi n+\vartheta n_1+1}{2}\right)+\omega n_1+1\right\}}$$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{\lambda+1}{2}\right)_{\frac{\tau n}{2}+\omega n_1}\Gamma\left(\frac{\mu+1}{2}\right)\left(\frac{\mu+1}{2}\right)_{\frac{\chi n}{2}+\vartheta n_1}}{\Gamma\left(\frac{\lambda+\mu+3}{2}\right)\left(\frac{\lambda+\mu+3}{2}\right)_{\frac{(\tau+\chi)n}{2}+(\vartheta+\omega)n_1}}$$

$$=\frac{1}{2}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\sum_{n=0}^{\infty}\frac{\gamma(a_1+n,y)\prod_{j=2}^{r}\Gamma(a_j+n)}{\prod_{j=1}^{r}\Gamma(b_j+l)}\times$$

$$\frac{x^n}{n!}(-\xi)^{n_3}\zeta^{2n_1}(\sin\theta)(\sin^2\theta)^{n_1+n_2+n_3}\times$$

$$\frac{\left(\frac{1}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}\frac{\left(\frac{1}{2}-t\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(1)_{n_3}}{n_1!\ n_2!\ n_3!}\times$$

$$\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{\lambda+1}{2}\right)_{\frac{\tau n}{2}+\omega n_1}\Gamma\left(\frac{\mu+1}{2}\right)\left(\frac{\mu+1}{2}\right)_{\frac{\chi n}{2}+\vartheta n_1}}{\Gamma\left(\frac{\lambda+\mu+3}{2}\right)\left(\frac{\lambda+\mu+3}{2}\right)_{\frac{(\tau+\chi)n}{2}+(\vartheta+\omega)n_1}}$$

$$=\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\times$$

$$\frac{\sin\theta}{2}\beta\left(\frac{\lambda+1}{2},\frac{\mu+1}{2}\right)\times$$

$$F^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{array}{l}A:(a_1,\alpha_1)\ldots(a_n,\alpha_n);\left(\frac{1}{2}-t,1\right);\left(\frac{1}{2},1\right);(1,1)\\ B:(b_1,\beta_1)\ldots(b_n,\beta_n);-;-;\end{array}\right|$$

$$x,\zeta^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\Big].$$

3.2 Case-2

By using the elliptic function given in equation (1.13) we obtained the following result:

$$\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu}{}_r\psi_s^{(\Gamma)}\left[\begin{array}{c}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,r}\\(b_j,B_j)_{1,s}\end{array}\right|$$

$$xh^{\tau}\left(\sqrt{(1-h^2)}\right)^{\chi}\Big]\prod\left(\theta,\zeta h^{\omega}\left(\sqrt{1-h^2}\right)^{\vartheta},\xi;t\right)dh$$

$$=\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\times$$

$$\frac{\sin\theta}{2}\beta\left(\frac{\lambda+1}{2},\frac{\mu+1}{2}\right)\times$$

$$F^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{array}{l}A:(a_1,\alpha_1)\ldots(a_n,\alpha_n);\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1)\\ B:(b_1,\beta_1)\ldots(b_n,\beta_n);-;-;\end{array}\right|$$

$$x,\zeta^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\Big],$$

$$A=\left(\frac{1}{2};0,1,1,1\right),\left(\frac{\lambda+1}{2},\frac{\tau}{2},\omega,0,0\right),$$

$$\left(\frac{\mu+2}{2},\frac{\chi}{2},\vartheta,0,0\right);$$

$$B=\left(\frac{3}{2};0,1,1,1\right),\left(\frac{\mu+\lambda+3}{2}:\frac{\chi+\tau}{2},\omega+\vartheta,0,0\right).\qquad(3.2)$$

Proof.

$$\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu}{}_r\psi_s^{(\Gamma)}\left[\begin{array}{c}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,r}\\(b_j,B_j)_{1,s}\end{array}\right|$$

$$\times xh^{\tau}\left(\sqrt{(1-h^2)}\right)^{\chi}\Big]\prod\left(\theta,\zeta h^{\omega}\left(\sqrt{1-h^2}\right)^{\vartheta},\xi;t\right)dh.$$

Now by equation (1.1) and (1.13), we get

$$=\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+B_j l)}\times$$

$$\frac{x^l}{n!}h^{\tau n}(1-h^2)^{\frac{n\chi}{2}}\sin\theta\times$$

$$F_{1;0;0;0}^{1;1;1;1}\left[\begin{matrix}(1;2,2,2) & \left(\frac{1}{2},1\right) & \left(\frac{1}{2},1\right) & (1,1)\\ (2;2,2,2) & ; & ; & ;\end{matrix}\right|$$

$$\left.\zeta^2 h^{2\omega}(1-h^2)^{\vartheta}\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right]dh.$$

Now by equation (1.5), we have

$$=\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\sum_{l=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+B_j n)}\times$$

$$\frac{x^n}{n!}h^{\tau n}(1-h^2)^{\frac{n\chi}{2}}\sin\theta\times\frac{(1)_{2n_1+2n_2+2n_3}}{(2)_{2n_1+2n_2+2n_3}}\frac{\left(\frac{1}{2}\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(1)_{n_3}}{n_1!\ n_2!\ n_3!}\times$$

$$\zeta^{2n_1}(\sin^2\theta)^{n_1+n_2+n_3}(-\xi)^{n_3}h^{2\omega n_1}(1-h^2)^{\vartheta n_1}dh$$

$$=\frac{1}{2}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{p}\Gamma(b_j+B_j l)}\times$$

$$\frac{z^n}{n!}(-\xi)^{n_3}\zeta^{2n_1}(\sin\theta)(\sin^2\theta)^{n_1+n_2+n_3}\frac{\left(\frac{1}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}\times$$

$$\int_0^1 h^{\lambda+\tau n+2\omega n_1}(1-h^2)^{\frac{\mu}{2}+\frac{n\chi}{2}+\vartheta n_1}dh.$$

To solve

$$=\int_0^1 h^{\lambda+\tau n+2\omega n_1}(1-h^2)^{\frac{\mu}{2}+\frac{n\chi}{2}+\vartheta n_1}dh.$$

Let $h^2=u\Rightarrow h=u^{1/2}\Rightarrow dh=\frac{1}{2}u^{-1/2}du$

$$=\int_0^1 u^{\frac{\lambda+\tau n+2\omega n_1}{2}}(1-u)^{\frac{\mu}{2}+\frac{\chi n}{2}+\vartheta n_1}\frac{1}{2}u^{-1/2}du$$

$$=\frac{1}{2}\int_0^1 u^{\frac{\lambda+\tau n+2\omega n_1-1}{2}+1-1}(1-u)^{\frac{\mu}{2}+\frac{\chi n}{2}+\omega n_1+1-1}du$$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{\lambda+\tau n+1}{2}+\omega n_1\right)\Gamma\left(\frac{\mu+\chi n+\vartheta n_1}{2}+1\right)}{\Gamma\left\{\left(\frac{\lambda+\tau n+\mu+\chi n+\vartheta n_1+1}{2}\right)+\omega n_1+1\right\}}$$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{\lambda+1}{2}\right)_{\frac{\tau n}{2}+\omega n_1}\Gamma\left(\frac{\mu+1}{2}\right)\left(\frac{\mu+1}{2}\right)_{\frac{\chi n}{2}+\vartheta n_1}}{\Gamma\left(\frac{\lambda+\mu+3}{2}\right)\left(\frac{\lambda+\mu+3}{2}\right)_{\frac{(\tau+\chi)n}{2}+(\vartheta+\omega)n_1}}$$

$$=\frac{1}{2}\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\sum_{n=0}^{\infty}\frac{\gamma(a_1+n,y)\prod_{j=2}^{r}\Gamma(a_j+n)}{\prod_{j=1}^{r}\Gamma(b_j+l)}\times$$

$$\frac{x^n}{n!}(-\xi)^{n_3}\zeta^{2n_1}(\sin\theta)(\sin^2\theta)^{n_1+n_2+n_3}\times$$

$$\frac{\left(\frac{1}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}\frac{\left(\frac{1}{2}\right)_{n_1}\left(\frac{1}{2}\right)_{n_2}(1)_{n_3}}{n_1!\ n_2!\ n_3!}\times$$

$$\times\frac{\Gamma\left(\frac{\lambda+1}{2}\right)\left(\frac{\lambda+1}{2}\right)_{\frac{\tau n}{2}+\omega n_1}\Gamma\left(\frac{\mu+1}{2}\right)\left(\frac{\mu+1}{2}\right)_{\frac{\chi n}{2}+\vartheta n_1}}{\Gamma\left(\frac{\lambda+\mu+3}{2}\right)\left(\frac{\lambda+\mu+3}{2}\right)_{\frac{(\tau+\chi)n}{2}+(\vartheta+\omega)n_1}}$$

$$=\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\times$$

$$\frac{\sin\theta}{2}\beta\left(\frac{\lambda+1}{2},\frac{\mu+1}{2}\right)\times$$

$$\times F_{2:n;0;0;0}^{3:n;1;1;1}\left[\begin{matrix}A:(a_1,\alpha_1)\ldots(a_n,\alpha_n);\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1)\\ B:(b_1,\beta_1)\ldots(b_n,\beta_n);-;-;\end{matrix}\right|$$

$$\left.x,\zeta^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right]$$

3.3 Case-3

On applying relation given by Srivastava et al. [11][Eq. 6.3] in Theorem 1 we get:

$$\int_0^1 h^{\lambda}\left(\sqrt{1-h^2}\right)^{\mu} \gamma_{r,s+1}^{1,r}\left[\begin{matrix}(1-a_1,\alpha_1,y),(1-a_j,\alpha_j)_{2,r}\\(0,1)\ ,(b_j,\beta_j)_{1,s}\end{matrix}\right|$$

$$\left.-xh^{\tau}\left(\sqrt{(1-h^2)}\right)^{\chi}\right]R\left(\theta,\zeta h^{\omega}\left(\sqrt{1-h^2}\right)^{\vartheta},\xi;\alpha,t\right)dh$$

$$=\frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{j=1}^{r}\Gamma(a_j)}\frac{\sum_{n=0}^{\infty}\Gamma(a_1+\alpha_1 n,y)\prod_{j=2}^{r}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{s}\Gamma(b_j+\beta_j n)}\times$$

$$\frac{\sin\theta}{2}B\left(\frac{\lambda+1}{2},\frac{\mu+1}{2}\right)\times$$

$$F_{2:n;0;0;0}^{3:n;1;1;1}\left[\begin{matrix}A:(a_1,\alpha_1)...(a_n,\alpha_n);\left(\frac{1}{2}-t,1\right);\left(\frac{1}{2},1\right);(\alpha,1)\\B:(b_1,\beta_1)...(b_n,\beta_n);-;-;\end{matrix}\right|$$

$$\left.x,\zeta^2\sin^2\theta,\sin^2\theta,-\xi\sin^2\theta\right]$$

$$A=\left(\frac{1}{2};0,1,1,1\right),\left(\frac{\lambda+1}{2},\frac{\tau}{2},\omega,0,0\right),$$

$$\left(\frac{\mu+2}{2},\frac{\chi}{2},\vartheta,0,0\right);$$

$$B=\left(\frac{3}{2};0,1,1,1\right),\left(\frac{\mu+\lambda+3}{2}:\frac{\chi+\tau}{2},\omega+\vartheta,0,0\right), \quad (28)$$

where $R(\lambda)>-1$ and

$$|\zeta|<1\left[or|\zeta|=1 and\ R(\lambda+2\omega)>-1\right].$$

4. Conclusion and Future Scope

In this work some elliptic integrals of incomplete hypergeometric functions have been established, which are inspired by various applications of different elliptic integrals of complete functions, studied earlier. We have also derived some other relations using the main result in respect of modulus as well as amplitude of the incomplete elliptic integrals. These elliptic integrals can be further used in finding several new integrals pertaining various general functions. These relations can also be employed in derivatives of other properties like analytic continuation and asymptotic behavior.

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