



Certain Integral Formulae Involving Incomplete *I*-Functions

Sanjay Bhattar¹, Kamlesh Jangid², Sapna Meena¹, Sunil Dutt Purohit^{2,*}

¹Department of Mathematics, Malaviya National Institute of Technology, Jaipur 302017, India

²Department of HEAS (Mathematics), Rajasthan Technical University, Kota 324010, India

Received 25 August 2021; Received in revised form 25 November 2021

Accepted 28 November 2021; Available online 30 December 2021

ABSTRACT

In this paper, we have determined some integral formulae of incomplete *I*-functions involving the exponential function, the Legendre polynomials and generalized Laguerre polynomials.

Keywords: Exponential function; Generalized Laguerre polynomials; Incomplete *I*-functions; Legendre polynomials.

1. Introduction and Definitions

Along with its relevance in engineering and mathematical physics, special functions and their extensions have piqued the interest of numerous researchers, see [1–6]. Certainly, incomplete Gamma functions and their extensions have been observed to play an important role in a variety of situations in the fields of heat conduction and astrophysics. As a result, a great deal of work has been done in this area. See the latest work in [7–13] for further information on potential extensions and applications. Our goal is to look into some new integrals associated with incomplete *I*-functions that involve a variety of other special functions.

In the year 1997, Rathie [14] discov-

ered the *I*-function which is defined as follows:

$$\begin{aligned} I_{p,q}^{m,n}[z] &= I_{p,q}^{m,n} \left[z \left| \begin{matrix} (\mathbf{a}_1, \nu_1; E_1), \dots, (\mathbf{a}_p, \nu_p; E_p) \\ (\mathbf{b}_1, \omega_1; F_1), \dots, (\mathbf{b}_p, \omega_p; F_p) \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \psi(s) z^s ds, \end{aligned} \tag{1.1}$$

where,

$$\psi(s) = \frac{\prod_{j=1}^m [\Gamma(\mathbf{b}_j - \omega_j s)]^{F_j} \prod_{j=1}^n [\Gamma(1 - \mathbf{a}_j + \nu_j s)]^{E_j}}{\prod_{j=m+1}^q [\Gamma(1 - \mathbf{b}_j + \omega_j s)]^{F_j} \prod_{j=n+1}^p [\Gamma(\mathbf{a}_j - \nu_j s)]^{E_j}}, \tag{1.2}$$

and $m, n, p, q \in N_0$ with $0 \leq n \leq p$, $0 \leq m \leq q$, $\nu_j, E_j (j = 1, \dots, p)$, $\omega_j, F_j (j =$

$1, \dots, q) \in \mathbb{R}^+, \mathbf{a}_j, \mathbf{b}_j \in \mathbb{C}$. The appropriate conditions for the convergence of the \mathcal{L} contour appeared in Eq. (1.1) and the description of the I -function can be found in [14].

We next defined the well-known lower and upper incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ as follows:

$$\gamma(s, x) := \int_0^x y^{s-1} e^{-y} dy \quad (1.3)$$

$$(\Re(s) > 0; x \geq 0)$$

and

$$\Gamma(s, x) := \int_x^\infty y^{s-1} e^{-y} dy \quad (1.4)$$

$$(x \geq 0; \Re(s) > 0)$$

where $\Re(s)$ stands for real part of s . In particular if $x = 0$, then $\Gamma(s, x)$ reduces to the standard gamma function $\Gamma(s)$. These functions satisfy the following decomposition relation:

$$\gamma(s, x) + \Gamma(s, x) = \Gamma(s) \quad (\Re(s) > 0). \quad (1.5)$$

By making use of the incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ represented by Eq. (1.3) and Eq. (1.4), Srivastava et al. [15] discovered and defined the incomplete H -functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ as follows:

$$\begin{aligned} \gamma_{p,q}^{m,n}(z) &= \gamma_{p,q}^{m,n} \left[z \left| \begin{matrix} (\mathbf{a}_1, \nu_1, x), (\mathbf{a}_j, \nu_j)_{2,p} \\ (\mathbf{b}_j, \omega_j)_{1,q} \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} g(s, x) z^{-s} ds \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} \Gamma_{p,q}^{m,n}(z) &= \Gamma_{p,q}^{m,n} \left[z \left| \begin{matrix} (\mathbf{a}_1, \nu_1, x), (\mathbf{a}_j, \nu_j)_{2,p} \\ (\mathbf{b}_j, \omega_j)_{1,q} \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} G(s, x) z^{-s} ds, \end{aligned} \quad (1.7)$$

where,

$$g(s, x) = \frac{\gamma(1 - \mathbf{a}_1 - \nu_1, s, x) \prod_{j=1}^m \Gamma(\mathbf{b}_j + \omega_j s) \prod_{j=2}^n \Gamma(1 - \mathbf{a}_j - \nu_j s)}{\prod_{j=m+1}^q \Gamma(1 - \mathbf{b}_j - \omega_j s) \prod_{j=n+1}^p \Gamma(\mathbf{a}_j + \nu_j s)}, \quad (1.8)$$

and

$$G(s, x) = \frac{\Gamma(1 - \mathbf{a}_1 - \nu_1, s, x) \prod_{j=1}^m \Gamma(\mathbf{b}_j + \omega_j s) \prod_{j=2}^n \Gamma(1 - \mathbf{a}_j - \nu_j s)}{\prod_{j=m+1}^q \Gamma(1 - \mathbf{b}_j - \omega_j s) \prod_{j=n+1}^p \Gamma(\mathbf{a}_j + \nu_j s)}, \quad (1.9)$$

with the arrangement of conditions established in [15].

These incomplete H -functions satisfy the following decomposition relation:

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z). \quad (1.10)$$

For $x \geq 0$, the incomplete H -functions presented in Eq. (1.6) and Eq. (1.7) exist.

Jangid et al. [16] described a class of the incomplete I -functions $\gamma I_{p,q}^{m,n}(z)$ and $\Gamma I_{p,q}^{m,n}(z)$ which are defined as follow:

$$\begin{aligned} \gamma I_{p,q}^{m,n}[z] &= \gamma I_{p,q}^{m,n} \left[z \left| \begin{matrix} (\mathbf{a}_1, \nu_1; E_1 : x), (\mathbf{a}_j, \nu_j; E_j)_{(2,p)} \\ (\mathbf{b}_j, \omega_j; F_j)_{(1,q)} \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \phi(s, x) z^s ds, \end{aligned} \quad (1.11)$$

and

$$\begin{aligned} \Gamma I_{p,q}^{m,n}[z] &= \Gamma I_{p,q}^{m,n} \left[z \left| \begin{matrix} (\mathbf{a}_1, \nu_1; E_1 : x), (\mathbf{a}_j, \nu_j; E_j)_{(2,p)} \\ (\mathbf{b}_j, \omega_j; F_j)_{(1,q)} \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s, x) z^s ds, \end{aligned} \quad (1.12)$$

for all $z \neq 0$, where,

$$\begin{aligned} \phi(s, x) &= [\gamma(1 - \mathbf{a}_1 + \nu_1, s, x)]^{E_1} \\ &\times \frac{\prod_{j=1}^m [\Gamma(\mathbf{b}_j - \omega_j s)]^{F_j} \prod_{j=2}^n [\Gamma(1 - \mathbf{a}_j + \nu_j s)]^{E_j}}{\prod_{j=m+1}^q [\Gamma(1 - \mathbf{b}_j + \omega_j s)]^{F_j} \prod_{j=n+1}^p [\Gamma(\mathbf{a}_j - \nu_j s)]^{E_j}}, \end{aligned} \quad (1.13)$$

and

$$\Phi(s, x) = [\Gamma(1 - \alpha_1 + \nu_1 s, x)]^{E_1} \times \frac{\prod_{j=1}^m [\Gamma(\mathfrak{b}_j - \omega_j s)]^{F_j} \prod_{j=2}^n [\Gamma(1 - \alpha_j + \nu_j s)]^{E_j}}{\prod_{j=m+1}^q [\Gamma(1 - \mathfrak{b}_j + \omega_j s)]^{F_j} \prod_{j=n+1}^p [\Gamma(\alpha_j - \nu_j s)]^{E_j}} \quad (1.14)$$

The definitions Eq. (1.11) and Eq. (1.12) at once yield the following decomposition relation:

$${}^\gamma I_{p,q}^{m,n}[z] + \Gamma I_{p,q}^{m,n}[z] = I_{p,q}^{m,n}[z] \quad (1.15)$$

for the familiar *I*-function (see [14]).

The incomplete *I*-functions ${}^\gamma I_{p,q}^{m,n}[z]$ and $\Gamma I_{p,q}^{m,n}[z]$ defined in Eq. (1.11) and Eq. (1.12) exist for all $x \geq 0$, under the set of conditions given by Rathie [14], with

$$\Delta > 0, |\arg z| < \Delta \frac{\pi}{2},$$

where

$$\Delta = \sum_{j=1}^m F_j \omega_j - \sum_{j=m+1}^q F_j \omega_j + \sum_{j=1}^n E_j \nu_j - \sum_{j=n+1}^p E_j \nu_j.$$

For more recent development in the theory of incomplete *H* and *I*-functions, one can see the papers [21–23] and references therein.

2. Main Results

In the following segment, we shall derive certain integrals including the incomplete *I*-functions with exponential function, Legendre polynomials and generalized Laguerre polynomials, respectively.

2.1 Exponential function and Incomplete *I*-function

Here, we shall derive the integrals for the product of the incomplete *I*-function and exponential function with the help of the following Integrals formulae:

$$(i) \int_{-\pi}^{\pi} (\cos \theta)^{\alpha-1} \exp(i\beta\theta) d\theta = \frac{\pi \Gamma(\alpha)}{2^{\alpha-1} \Gamma\left(\frac{\alpha+\beta+1}{2}\right) \Gamma\left(\frac{\alpha-\beta+1}{2}\right)}, \quad (2.1)$$

where $\Re(\alpha) > 0$.

(ii) The following integral was given by Nielson [17],

$$\int_0^{\pi} (\sin \theta)^{\alpha} e^{-\beta \theta} d\theta = \frac{\pi e^{-\frac{\pi\beta}{2}} \Gamma(\alpha + 1)}{2^{\alpha} \Gamma\left(1 + \frac{\alpha+i\beta}{2}\right) \Gamma\left(1 + \frac{\alpha-i\beta}{2}\right)}, \quad (2.2)$$

where $\Re(\alpha) > 1$.

We have established the following results:

Theorem 2.1. *If $h > 0, k, \lambda \in \mathbb{C}, \mu \geq 0, \Delta > 0, |\arg z| < \frac{1}{2}\pi\Delta$ and $\Re\left(k + \lambda - h \frac{\alpha_i}{\nu_i}\right) > 1 - \frac{h}{\nu_i} \quad i = 1, \dots, n$, then the following integral holds;*

$$\int_{-\pi}^{\pi} (\cos \theta)^{k+\lambda-2} \exp[i(k-\lambda)\theta] \times \Gamma I_{p,q}^{m,n} \left[z (e^{i\theta} \cos \theta)^{-h} \middle| \begin{matrix} (\alpha_1, \nu_1; E_1; x), \\ (\mathfrak{b}_1, \omega_1; F_1) \\ (\alpha_j, \nu_j; E_j)_{2,p} \\ (\mathfrak{b}_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta = \frac{\pi}{2^{k+\lambda-2} \Gamma(\lambda)} \times \Gamma I_{p+1,q+1}^{m+1,n} \left[2^h z \middle| \begin{matrix} (\alpha_1, \nu_1; E_1; x), \\ (k+\lambda-1, h; 1), \\ (\alpha_j, \nu_j; E_j)_{2,p}, (k, h; 1) \\ (\mathfrak{b}_j, \omega_j; F_j)_{1,q} \end{matrix} \right]. \quad (2.3)$$

Proof. To demonstrate the outcome Eq. (2.3), take the left-hand side assertion of Eq. (2.1). Express the incomplete I -function in terms of Mellin-Barnes type integral defined in Eq. (1.12) and then on changing the order of the integrations, we have

$$\begin{aligned} & \int_{-\pi}^{\pi} (\cos \theta)^{k+\lambda-2} \exp [i(k-\lambda)\theta] \\ & \times \frac{1}{2\pi i} \int z^s e^{-i\theta hs} (\cos \theta)^{-hs} \Phi(s, x) ds d\theta \\ & = \frac{1}{2\pi i} \int z^s \Phi(s, x) \\ & \times \int_{-\pi}^{\pi} (\cos \theta)^{(k+\lambda-hs-1)-1} e^{i(k-\lambda-hs)\theta} d\theta ds \end{aligned}$$

using the integral formula Eq. (2.1), we obtain

$$\begin{aligned} & = \frac{1}{2\pi i} \int z^s \Phi(s, x) \\ & \times \frac{\pi \Gamma(k+\lambda-hs-1)}{2^{(k+\lambda-hs-2)} \Gamma(k-hs) \Gamma(\lambda)} ds \\ & = \frac{\pi}{2^{k+\lambda-2} \Gamma(\lambda)} \\ & \times \frac{1}{2\pi i} \int \frac{z^s 2^{hs} \Gamma(k+\lambda-1-hs) \Phi(s, x)}{\Gamma(k-hs)} ds \end{aligned} \tag{2.4}$$

using Eq. (1.14), we obtain the required R.H.S of Eq. (2.3).

Theorem 2.2. If $h > 0, k, \lambda \in C, \mu \geq 0, \Delta > 0, |argz| < \frac{1}{2}\pi\Delta$ and $\Re\left(k+\lambda-h\frac{\alpha_i}{\nu_i}\right) > 1 - \frac{h}{\nu_i}, i = 1, \dots, n,$ then the following integral holds:

$$\begin{aligned} & \int_{-\pi}^{\pi} (\cos \theta)^{k+\lambda-2} \exp [i(k-\lambda)\theta] \times \\ & \Gamma_{I_{p,q}^{m,n}} \left[z e^{i\theta} (\sec \theta)^h \mid \begin{matrix} (a_1, \nu_1; E_1; x), \\ (b_1, \omega_1; F_1) \\ \\ (a_j, \nu_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta \\ & = \frac{\pi}{2^{k+\lambda-2} \Gamma(k)} \times \\ & \Gamma_{I_{p+1,q+1}^{m+1,n}} \left[2^h z \mid \begin{matrix} (a_1, \nu_1; E_1; x), \\ (k+\lambda-1, h; 1), \\ \\ (a_j, \nu_j; E_j)_{2,p}, (\lambda, h; 1) \\ (b_j, \omega_j; F_j)_{1,q} \end{matrix} \right]. \end{aligned} \tag{2.5}$$

Proof. To demonstrate the outcome Eq. (2.5), take the left-hand side assertion of Eq. (2.2). Express the incomplete I -function in terms of Mellin-Barnes type integral defined in Eq. (1.12), we get

$$\begin{aligned} & \int_{-\pi}^{\pi} (\cos \theta)^{k+\lambda-2} \exp [i(k-\lambda)\theta] \times \\ & \frac{1}{2\pi i} \int z^s e^{i\theta hs} (\sec \theta)^{hs} \Phi(s, x) ds d\theta \end{aligned}$$

on changing the order of integrations,

$$\begin{aligned} & = \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \\ & \int_{-\pi}^{\pi} (\cos \theta)^{(k+\lambda-hs-1)-1} e^{i(k-\lambda+hs)\theta} d\theta ds \end{aligned}$$

using the integral formula Eq. (2.1), we obtain

$$\begin{aligned} & = \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \\ & \frac{\pi \Gamma(k+\lambda-hs-1)}{2^{(k+\lambda-hs-2)} \Gamma(\lambda-hs) \Gamma(k)} ds \\ & = \frac{\pi}{2^{k+\lambda-2} \Gamma(k)} \times \\ & \frac{1}{2\pi i} \int \frac{z^s 2^{hs} \Gamma(k+\lambda-1-hs) \Phi(s, x)}{\Gamma(\lambda-hs)} ds \end{aligned} \tag{2.6}$$

using Eq. (1.14), we obtain the required R.H.S of Eq. (2.5).

Theorem 2.3. *If $h > 0, k, \lambda \in C, \mu \geq 0, \Delta > 0, |arg z| < \frac{1}{2}\pi\Delta$ and $\Re(k + \lambda - h\frac{a_i}{v_i}) > 1 - \frac{h}{v_i}, i = 1, \dots, n,$ then the following integral holds:*

$$\int_{-\pi}^{\pi} (\cos \theta)^{k+\lambda-2} \exp [i(k - \lambda)\theta] \times \Gamma I_{p, q}^{m, n} \left[z (\sec \theta)^{2h} \left| \begin{matrix} (a_1, v_1; E_1; x), \\ (b_1, \omega_1; F_1) \end{matrix} \right. \right. \\ \left. \left. \begin{matrix} (a_j, v_j; E_j)_{2,p}, \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta \right. \\ = \frac{\pi}{2^{k+\lambda-2}} \times \Gamma I_{p+2, q+1}^{m+1, n} \left[2^{2h} z \left| \begin{matrix} (a_1, v_1; E_1; x), (a_j, v_j; E_j)_{2,p}, \\ (k + \lambda - 1, 2h; 1), \\ (k, h; 1), (\lambda, h; 1) \\ (b_j, \omega_j; F_j)_{1,q} \end{matrix} \right. \right]. \tag{2.7}$$

Proof. To prove the result Eq. (2.7), we will take the L.H.S. of Eq. (2.3). Now, write the incomplete *I*-function in terms of the Mellin-Barnes type contour integral defined in Eq. (1.12) and then, on changing the order of the integrals, we get

$$\int_{-\pi}^{\pi} (\cos \theta)^{k+\lambda-2} \exp [i(k - \lambda)\theta] \times \frac{1}{2\pi i} \int z^s (\sec \theta)^{2hs} \Phi(s, x) ds d\theta \\ = \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \int_{-\pi}^{\pi} (\cos \theta)^{(k+\lambda-2hs-1)-1} e^{[i(k-\lambda)\theta]} d\theta ds$$

using the integral formula Eq. (2.1)

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \frac{\pi \Gamma(k + \lambda - 2hs - 1)}{2^{(k+\lambda-2hs-2)} \Gamma(\lambda - hs) \Gamma(k - hs)} ds \\ = \frac{\pi}{2^{k+\lambda-2}} \times \frac{1}{2\pi i} \int z^s \frac{2^{2hs} \Gamma(k + \lambda - 1 - 2hs) \Phi(s, x)}{\Gamma(\lambda - hs) \Gamma(k - hs)} ds \tag{2.8}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.7).

Theorem 2.4. *Let $\Delta > 0, |arg z| < \frac{1}{2}\pi\Delta$ or $\Delta = 0, \Re(\delta) < -1,$ then the following integral formula holds:*

$$\int_0^{\pi} (\sin \theta)^{\gamma-1} e^{-\eta \theta} \times \Gamma I_{p, q}^{m, n} \left[z (\sin \theta)^{2h} \left| \begin{matrix} (a_1, v_1; E_1; x), \\ (b_1, \omega_1; F_1) \end{matrix} \right. \right. \\ \left. \left. \begin{matrix} (a_j, v_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta \right. \\ = \frac{\pi}{2^{\gamma-1}} \exp \left(-\frac{\pi\eta}{2} \right) \times \Gamma I_{p+1, q+2}^{m, n+1} \left[\frac{z}{4h} \left| \begin{matrix} (1 - \gamma, 2h; 1), (a_1, v_1; E_1; x), \\ (b_j, \omega_j; F_j)_{1,q}, \\ (a_j, v_j; E_j)_{2,p} \\ \left(\frac{1-\gamma+i\eta}{2}, h; 1 \right), \left(\frac{1-\gamma-i\eta}{2}, h; 1 \right) \end{matrix} \right. \right], \tag{2.9}$$

where $\gamma, \eta \in C, h > 0$ are such that $\Re(\gamma) + 2h \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{\omega_j} \right] > 0$ for $\Delta > 0, |arg z| < \frac{1}{2}\pi\Delta$ or $\Delta = 0, \mu \geq 0,$ and $\Re(\gamma) + 2h \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{\omega_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] > 0$ for $\Delta = 0$ and $\mu < 0.$

Proof. To prove the result Eq. (2.9), take the left-hand side assertion of Eq. (2.4). Now, write the incomplete *I*-function in terms of the Mellin-Barnes contour type

integral defined in Eq. (1.12), we get

$$\int_0^\pi (\sin \theta)^{\gamma-1} e^{-\eta\theta} \times \frac{1}{2\pi i} \int z^s (\sin \theta)^{2hs} \Phi(s, x) ds d\theta$$

On changing the order of the integrations,

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \int_0^\pi (\sin \theta)^{\gamma+2hs-1} e^{-\eta\theta} d\theta ds$$

using the integral formula Eq. (2.2)

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \frac{\pi e^{-\frac{\pi}{2}\eta}}{2^{(\gamma+2hs-1)}} \frac{\Gamma(\gamma+2hs)}{\Gamma\left(1 + \frac{\gamma+i\eta-1}{2} + hs\right) \Gamma\left(1 + \frac{\gamma-i\eta-1}{2} + hs\right)} ds$$

$$= \frac{\pi}{2^{\gamma-1}} e^{-\frac{\pi}{2}\eta} \times \frac{1}{2\pi i} \int \Phi(s, x) \frac{z^s}{4^{hs}} \frac{\Gamma(\gamma+2hs)}{\Gamma\left(1 + \frac{\gamma+i\eta-1}{2} + hs\right) \Gamma\left(1 + \frac{\gamma-i\eta-1}{2} + hs\right)} ds$$

(2.10)

using definition Eq. (1.14), we will get the desired result Eq. (2.9).

Theorem 2.5. Let $\gamma, \eta \in \mathbb{C}, h > 0, |\arg z| < \frac{1}{2}\pi\Delta$ or $\Delta > 0, \Re(\gamma) > 0$, then

the following integral formula holds;

$$\int_0^\pi (\sin \theta)^{\gamma-1} e^{-\eta\theta} \times \Gamma I_{p,q}^{m,n} \left[z e^{i2h\theta} \middle| \begin{matrix} (a_1, \nu_1; E_1; x), \\ (b_1, \omega_1; F_1) \end{matrix} \right. \\ \left. \begin{matrix} (a_j, \nu_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta$$

$$= \frac{\pi \Gamma(\gamma)}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) \times \Gamma I_{p+1, q+1}^{m,n} \left[z e^{i\pi h} \middle| \begin{matrix} (a_1, \nu_1; E_1; x), (a_j, \nu_j; E_j)_{2,p}, \\ (b_j, \omega_j; F_j)_{1,q}, \end{matrix} \right. \\ \left. \begin{matrix} \left(\frac{1+\gamma-i\eta}{2}, h; 1\right) \\ \left(\frac{1-\gamma-i\eta}{2}, h; 1\right) \end{matrix} \right].$$

(2.11)

Proof. To demonstrate the result Eq. (2.11), take the left-hand side assertion of Eq. (2.5). Express incomplete I -function in terms of the Mellin-Barnes type contour integral defined in Eq. (1.12), we obtain

$$\int_0^\pi (\sin \theta)^{\gamma-1} e^{-\eta\theta} \times \frac{1}{2\pi i} \int z^s e^{i2h\theta s} \Phi(s, x) ds d\theta$$

On changing the order of the integrations,

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \int_0^\pi (\sin \theta)^{\gamma-1} e^{-[\eta-i2hs]\theta} d\theta ds$$

using the integral formula Eq. (2.2)

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \frac{\pi e^{-\frac{\pi}{2}[\eta-2ihs]}}{2^{(\gamma-1)}} \frac{\Gamma(\gamma)}{\Gamma\left(1 + \frac{\gamma+i\eta-1+2hs}{2}\right) \Gamma\left(1 + \frac{\gamma-i\eta-1-2hs}{2}\right)} ds$$

$$= \frac{\pi \Gamma(\gamma)}{2^{\gamma-1}} e^{-\frac{\pi}{2}\eta} \times \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \frac{e^{i\pi hs}}{\Gamma\left(1 + \frac{\gamma+i\eta-1+2hs}{2}\right) \Gamma\left(1 + \frac{\gamma-i\eta-1-2hs}{2}\right)} ds \tag{2.12}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.11).

Theorem 2.6. *If $\Delta > 0$, $\Re(\gamma) > 0$ or $\Delta = 0$, $\Re(\delta) < -1$, then the following integral formula;*

$$\int_0^\pi (\sin \theta)^{\gamma-1} e^{-\eta \theta} \times \Gamma I_{p,q}^{m,n} \left[z (\sin \theta)^{2\lambda} e^{i2h\theta} \left| \begin{matrix} (a_1, \nu_1; E_1; x), \\ (b_1, \omega_1; F_1) \end{matrix} \right. \right. \\ \left. \left. \begin{matrix} (a_j, \nu_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta \right. \\ = \frac{\pi}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) \times \Gamma I_{p+1,q+2}^{m,n+1} \left[\frac{z e^{i\pi h}}{4\lambda} \left| \begin{matrix} (1-\gamma, 2\lambda; 1), \\ (b_j, \omega_j; F_j)_{1,q}, \end{matrix} \right. \right. \\ \left. \left. \begin{matrix} (a_1, \nu_1; E_1; x), (a_j, \nu_j; E_j)_{2,p} \\ \left(\frac{1-\gamma-i\eta}{2}, \lambda+h; 1\right), \left(\frac{1-\gamma+i\eta}{2}, \lambda-h; 1\right) \end{matrix} \right] \right. \tag{2.13}$$

holds for $h > 0$, $\lambda > h$, $\gamma, \eta \in C$, such that $\Re(\gamma) + 2\lambda \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{\omega_j} \right] > 0$ for $\Delta > 0$ or $\Delta = 0$ and $\mu \geq 0$; and $\Re(\gamma) + 2\lambda \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{\omega_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] > 0$ for $\Delta = 0$ and $\mu < 0$.

Proof. To demonstrate the result Eq. (2.13), take the left-hand side assertion of Eq. (2.6). Express incomplete *I*-function in terms of the Mellin-Barnes type contour integral defined in Eq. (1.12), we obtain

$$\int_0^\pi (\sin \theta)^{\gamma-1} e^{-\eta \theta} \times \frac{1}{2\pi i} \int z^s (\sin \theta)^{2\lambda s} e^{2ih\theta s} \Phi(s, x) ds d\theta$$

On changing the order of the integrations,

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \int_0^\pi (\sin \theta)^{\gamma+2\lambda s-1} e^{-[\eta-2hsi]\theta} d\theta ds$$

Using the formula Eq. (2.2)

$$= \frac{1}{2\pi i} \int z^s \Phi(s, x) \times \frac{e^{-\frac{\pi}{2}[\eta-2hsi]}}{2^{\gamma+2\lambda s-1}} \times \frac{\Gamma(\gamma + 2\lambda s)}{\Gamma\left(1 + \frac{\gamma+i\eta-1}{2} + s(\lambda + h)\right)} \times \frac{1}{\Gamma\left(1 + \frac{\gamma-i\eta-1}{2} + s(\lambda + h)\right)} ds \\ = \frac{\pi}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) \times \frac{1}{2\pi i} \int \frac{z^s \Phi(s, x)}{4^{\lambda s}} \times \frac{e^{\frac{\pi}{2}[2hsi]}}{\Gamma\left(1 + \frac{\gamma+i\eta-1}{2} + s(\lambda + h)\right)} \times \frac{\Gamma(\gamma + 2\lambda s)}{\Gamma\left(1 + \frac{\gamma-i\eta-1}{2} + s(\lambda - h)\right)} ds \tag{2.14}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.13).

2.2 Legendre Polynomials and Incomplete *I*-Function

Here, we shall establish certain integral formulae involving the product of the incomplete *I*-function and Legendre polynomials defined in [6].

The following known results will be use to prove our theorems.

(i) The following integral has obtained by Erdelyi [18, p. 316];

$$\int_{-1}^1 (1-x^2)^{\rho-1} P_\nu^m(x) dx$$

$$\begin{aligned}
 &= \frac{\pi 2^m \Gamma(\rho + \frac{1}{2}m)}{\Gamma(1 + \frac{1}{2}(v - m)) \Gamma(\frac{1}{2} - \frac{1}{2}(v + m))} \\
 &\times \frac{\Gamma(\rho - \frac{1}{2}m)}{\Gamma(\rho - \frac{1}{2}v) \Gamma(1 + \rho + \frac{1}{2}v)}
 \end{aligned} \tag{2.15}$$

provided $2 \Re(\rho) > |\Re(m)|$.

(ii) This integral is given by Milne-Thomson [19, p. 33];

$$E_a f(a) = f(a+1), E_a^n f(a) = E_a [E_a^{n-1} f(a)] \tag{2.16}$$

where E denotes the finite difference operator. And, we will use the following notation:

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)} = a(a+1) \cdots (a+r-1). \tag{2.17}$$

Theorem 2.7. Let $\rho, z \in C, \Delta > 0, |\arg z| < \frac{1}{2}\pi\Delta$ or $\Delta = 0, \Re(\mu) < -1$. Further, let $\rho \in C, k > 0$ satisfy the conditions

$$\Re(\rho) + k \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{\omega_j} \right] > \frac{1}{2} |\Re(\mu)|$$

for $\Delta > 0, |\arg z| < \frac{1}{2}\pi\Delta$ or $\Delta = 0, \mu \geq 0$ and

$$\Re(\rho) + k \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{\omega_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] > \frac{1}{2} |\Re(\mu)|,$$

For $\Delta = 0, \mu < 0$ then there holds the formula

$$\begin{aligned}
 &\int_{-1}^1 (1-x^2)^{\rho-1} P_v^\lambda(x) \times \\
 &\Gamma I_{p,q}^{m,n} \left[z(1-x^2)^k \mid \begin{matrix} (a_1, v_1; E_1; x), \\ (b_1, \omega_1; F_1) \\ \\ (a_j, v_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2^\lambda \pi}{\Gamma(\frac{2+v-\lambda}{2}) \Gamma(\frac{1-v-\lambda}{2})} \times \\
 &\Gamma I_{p+2, q+2}^{m, n+2} \left[z \mid \begin{matrix} (1-\rho \pm \frac{\lambda}{2}, k; 1), \\ (b_j, \omega_j; F_j)_{1,q}, \\ \\ (a_1, v_1; E_1; x), (a_j, v_j; E_j)_{2,p} \\ (1-\rho + \frac{v}{2}, k; 1), (-\rho - \frac{v}{2}, k; 1) \end{matrix} \right].
 \end{aligned} \tag{2.18}$$

Proof. To prove Eq. (2.18), we first write the incomplete I -function in terms of Mellin-Barnes contour integral form Eq. (1.12), we have

$$\begin{aligned}
 I_1(\rho) &= \int_{-1}^1 (1-x^2)^{\rho-1} P_v^\lambda(x) \\
 &\times \frac{1}{2\pi i} \int_{\mathcal{L}} z^s (1-x^2)^{ks} \Phi(s, x) ds dx
 \end{aligned}$$

on changing the order of the integration

$$\begin{aligned}
 &= \frac{1}{2\pi i} \int_{\mathcal{L}} z^s \Phi(s, x) \\
 &\times \int_{-1}^1 (1-x^2)^{\rho+ks-1} P_v^\lambda(x) dx ds
 \end{aligned}$$

Now, by making use of the formula Eq. (2.15) evaluate the internal integral

$$\begin{aligned}
 &= \frac{1}{2\pi i} \int_{\mathcal{L}} z^s \Phi(s, x) \times \frac{2^\lambda \pi}{\Gamma(\frac{2+v-\lambda}{2}) \Gamma(\frac{1-v-\lambda}{2})} \\
 &\times \frac{\Gamma(\rho + \frac{1}{2}\lambda + ks) \Gamma(\rho - \frac{1}{2}\lambda + ks)}{\Gamma(\rho - \frac{1}{2}v + ks) \Gamma(1 + \rho + \frac{1}{2}v + ks)} ds
 \end{aligned} \tag{2.19}$$

and finally reinterpreting the Mellin-Barnes contour integral thus involved by definition of incomplete I -function, we get the desired result.

Theorem 2.8. Let k and d are positive integers, $\mathfrak{U} < \mathfrak{B}$ or $\mathfrak{U} = \mathfrak{B} + 1$ and $|c| < 1$ and none of $\beta_j, j = 1, \dots, \mathfrak{B}$ is a negative integer or zero, then the following integral

holds,

$$\int_{-1}^1 (1-x^2)^{\rho-1} P_v^\lambda(x) \times {}_U F_{\mathfrak{B}}(\alpha_1, \dots, \alpha_U; \beta_1, \dots, \beta_{\mathfrak{B}}; c(1-x^2)^d) \Gamma_{p,q}^{m,n} \left[z(1-x^2)^k \mid \begin{matrix} (a_1, \nu_1; E_1; x), \\ (b_1, \omega_1; F_1) \\ \\ (a_j, \nu_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] dx$$

$$= \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right) \Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \sum_{r=0}^\infty \frac{(\alpha_1)_r \cdots (\alpha_U)_r c^r}{(\beta_1)_r \cdots (\beta_{\mathfrak{B}})_r r!}$$

$$\times \Gamma_{p+2, q+2}^{m, n+2} \left[z \mid \begin{matrix} \left(1-\rho-rd \pm \frac{\lambda}{2}, k; 1\right), (a_1, \nu_1; E_1; x), \\ (b_j, \omega_j; F_j)_{1,q}, \left(1-\rho-rd + \frac{\nu}{2}, k; 1\right), \\ \\ (a_j, \nu_j; F_j)_{2,p} \\ \left(-\rho-rd - \frac{\nu}{2}, k; 1\right) \end{matrix} \right]. \tag{2.20}$$

Proof. On multiplying both sides of Eq. (2.18) by $\frac{\prod_{j=1}^U \Gamma(\alpha_j + \delta)(c)^\delta}{\prod_{j=1}^{\mathfrak{B}} \Gamma(\beta_j + \delta)}$ and applying the operator $\exp(E^d E_\delta)$ yields,

$$\exp(E^d E_\delta) \left[I_1(\rho) \frac{\prod_{j=1}^U \Gamma(\alpha_j + \delta)(c)^\delta}{\prod_{j=1}^{\mathfrak{B}} \Gamma(\beta_j + \delta)} \right]$$

$$= \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right) \Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \frac{\prod_{j=1}^U \Gamma(\alpha_j + \delta)(c)^\delta}{\prod_{j=1}^{\mathfrak{B}} \Gamma(\beta_j + \delta)}$$

$$\exp(E^d E_\delta) \left[z \mid \begin{matrix} \left(1-\rho \pm \frac{\lambda}{2}, k; 1\right), (a_1, \nu_1; E_1; x), \\ (b_j, \omega_j; F_j)_{1,q}, \left(1-\rho + \frac{\nu}{2}, k; 1\right), \\ \\ (a_j, \nu_j; F_j)_{2,p} \\ \left(-\rho - \frac{\nu}{2}, k; 1\right) \end{matrix} \right]. \tag{2.21}$$

Taking summation on both sides of Eq. (2.21) and using the definition of finite

difference operator Eq. (2.16), we get

$$\sum_{r=0}^\infty \left[\frac{\prod_{j=1}^U \Gamma(\alpha_j + \delta + r)(c)^{\delta+r}}{\prod_{j=1}^{\mathfrak{B}} \Gamma(\beta_j + \delta + r) r!} \times \int_{-1}^1 (1-x^2)^{\rho+rd-1} P_v^\lambda(x) \times \Gamma_{p,q}^{m,n} \left[z(1-x^2)^k \mid \begin{matrix} (a_1, \nu_1; E_1; x), (a_j, \nu_j; E_j)_{2,p} \\ ((f)_j, \omega_j; F_j)_{1,q} \end{matrix} \right] dx \right]$$

$$= \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right) \Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \times \sum_{r=0}^\infty \frac{\prod_{j=1}^U \Gamma(\alpha_j + \delta + r)(c)^{\delta+r}}{\prod_{j=1}^{\mathfrak{B}} \Gamma(\beta_j + \delta + r) r!}$$

$$\times \Gamma_{p+2, q+2}^{m, n+2} \left[z \mid \begin{matrix} \left(1-\rho \pm \frac{\lambda}{2}, k; 1\right), (a_1, \nu_1; E_1; x), \\ (b_j, \omega_j; F_j)_{1,q}, \left(1-\rho + \frac{\nu}{2}, k; 1\right), \\ \\ (a_j, \nu_j; F_j)_{2,p} \\ \left(-\rho - \frac{\nu}{2}, k; 1\right) \end{matrix} \right]. \tag{2.22}$$

Now, on the left hand side of Eq. (2.22), change the order of the integration and summation which is justified, using the result Eq. (2.17) and finally, replacing $\alpha_j + \delta$ by α_j and $\beta_j + \delta$ by β_j enable us to obtain the value of the integral Eq. (2.20).

Theorem 2.9. If $\lambda = 0$ and $\nu = \lambda$, where λ is a positive integer, then the theorem Eq. (2.8) reduces to the following result,

$$\int_{-1}^1 (1-x^2)^{\rho-1} P_\lambda(x) \times {}_U F_{\mathfrak{B}}(\alpha_1, \dots, \alpha_U; \beta_1, \dots, \beta_{\mathfrak{B}}; c(1-x^2)^d) \Gamma_{p,q}^{m,n} \left[z(1-x^2)^k \mid \begin{matrix} (a_1, \nu_1; E_1; x), (a_j, \nu_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{1,q} \end{matrix} \right] dx$$

$$= \frac{\pi}{\Gamma\left(\frac{2+\lambda}{2}\right) \Gamma\left(\frac{1-\lambda}{2}\right)} \sum_{r=0}^\infty \frac{(\alpha_1)_r \cdots (\alpha_U)_r c^r}{(\beta_1)_r \cdots (\beta_{\mathfrak{B}})_r r!} \times$$

$$\Gamma I_{p+2,q+2}^{m,n+2} \left[z \mid \begin{matrix} (1-\rho-rd, k; 1), (1-\rho-rd, k; 1), \\ (b_j, \omega_j; F_j)_{1,q}, (1-\rho-rd+\frac{\nu}{2}, k; 1), \\ (a_1, \nu_1; E_1; x), (a_j, \nu_j; F_j)_{2,p} \\ (-\rho-rd-\frac{\nu}{2}, k; 1) \end{matrix} \right]. \tag{2.23}$$

where $P_\lambda(x)$ is the Legendre polynomial defined in [6] and the conditions of the validity are the same as stated in Theorem Eq. (2.8) with $\lambda = 0$ and ν replaced by λ .

2.3 Generalized Laguerre Polynomial and Incomplete I-functions

Here, we shall derive an integral formula including the incomplete I-function and generalized Laguerre polynomial defined in [6].

Theorem 2.10. *The following formula holds true;*

$$\int_0^\infty x^\gamma e^{-x} L_k^\sigma(x) \times \Gamma I_{p,q}^{m,n} \left[z x^\eta \mid \begin{matrix} (a_1, \nu_1; E_1; x), \\ (b_1, \omega_1; F_1), \\ (a_j, \nu_j; E_j)_{2,p} \\ (b_j, \omega_j; F_j)_{2,q} \end{matrix} \right] dx = \frac{(-1)^k (2\pi)^{\frac{1}{2}(1-n)} \eta^{\gamma+k+\frac{1}{2}}}{k!} \times \Gamma I_{p+2\eta, q+\eta}^{m, n+2\eta} \left[z \eta^\eta \mid \begin{matrix} (\Delta(\eta, -\gamma), 1; 1), (\Delta(\eta, \sigma - \gamma), 1; 1), \\ (b_j, \omega_j; F_j)_{1,q}, \\ (a_1, \nu_1; E_1; x), (a_j, \nu_j; E_j)_{2,p} \\ (\Delta(\eta, \sigma - \gamma), 1; 1) \end{matrix} \right], \tag{2.24}$$

where, η is a positive integer, $\sum_{j=1}^p \nu_j - \sum_{j=1}^q \omega_j = \rho \leq 0$, $\sum_{i=1}^m \omega_i - \sum_{i=m+1}^q \omega_i + \sum_{i=1}^n \nu_i - \sum_{i=n+1}^p \nu_i = \Delta > 0$, $|\arg z| < \frac{1}{2}\Delta \pi$, and $\Re \left[\gamma + 1 + \eta \left(\frac{b_h}{\omega_h} \right) \right] > -1$ ($h = 1, 2, \dots, m$).

Proof. To prove Eq. (2.24), we express the incomplete I-function in the form of the Mellin-Barnes type of contour integral

Eq. (1.12) and change the order of the integrations, we have

$$\frac{1}{2\pi i} \int_{\mathcal{L}} z^s \Phi(s, x) \times \left\{ \int_0^\infty x^{\gamma+\eta s} e^{-x} L_k^\sigma(x) dx \right\} ds, \tag{2.25}$$

Now evaluating x -integral with the help of the result [18, p.292,(1)] :

$$\int_0^\infty x^{\beta-1} e^{-x} L_n^\alpha(x) dx = \frac{\Gamma(\alpha - \beta + n + 1)\Gamma(\beta)}{n! \Gamma(\alpha - \beta + 1)}, \quad (\Re(\beta) > 0),$$

and using the following relations

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad \frac{\Gamma(1 - a - n)}{\Gamma(1 - a)} = \frac{(-1)^n}{(a)_n},$$

and the Gauss's multiplication theorem for Gamma function [20, p. 26]:

$$\Gamma(kz) = (2\pi)^{\frac{1}{2}(1-k)} K^{kz-\frac{1}{2}} \prod_{s=1}^k \Gamma\left(z + \frac{s-1}{K}\right),$$

Eq. (2.25) reduces to

$$\frac{(-1)^k}{k!} (2\pi)^{\frac{1}{2}(1-n)} \eta^{\gamma+k+\frac{1}{2}} \times \frac{1}{2\pi i} \int_{\mathcal{L}} z^s \eta^{\eta s} \Phi(s, x) \times \frac{\prod_{i=0}^{\eta-1} \Gamma\left(\frac{1+\gamma+i}{\eta} - s\right) \prod_{i=0}^{\eta-1} \Gamma\left(\frac{1-\sigma+\gamma+i}{\eta} - s\right)}{\prod_{i=0}^{\eta-1} \Gamma\left(\frac{1-\sigma+\gamma-k+i}{\eta} - s\right)} ds. \tag{2.26}$$

Therefore, in accordance with the definition Eq. (1.14) of the incomplete I-function, Eq. (2.26) yields the value of the integral Eq. (2.24). \square

A family of integrals involving incomplete I -functions has been developed. The exponential function, Legendre polynomials, and modified Laguerre polynomials were used to analyse several integral formulas of incomplete I -functions. Because incomplete I -functions may be reduced to recognisable special functions (such as I -functions, incomplete H -functions, and Fox's H -functions), numerous special cases can be assessed based on our significant inventions by assigning appropriate values to the related parameters.

Acknowledgement

The authors thank the referees for their concrete suggestions which resulted in a better organization of this article.

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