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Research Article

COMPLEXITY SCALING OF FINITE SURFACE METHOD IN HIGH REYNOLDS NUMBER FLOWS

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ABSTRACT:
This paper investigates the complexity scaling of the finite surface (FS) discretization applied to Navier-Stokes equations. The previous work on the turbulent channel flow suggests that the grids resolutions needed to achieve 1%-error in the first- and the second-order statistics is 30 and 11 wall-unit in the streamwise- and spanwise-directions. This grid size is about 10-times larger than the traditional recommendation and requires much lower computational resources. In this work, we investigate the application of this new resolution to the Reynolds number cases: Ret = 180, 390, 590, and 950. The mean and the RMS of the fluctuations are presented and compared with results from spectral codes. The result of the investigation indicates that the finite surface method can deliver a 1%-accurate prediction in the mean profile of turbulent channel flows using this resolution. Effectively, the number of grid points for 1%-accurate prediction scales with $(Re_\tau/22)^3+(Re_\tau/5.4)^2$.

Keywords: *Finite surface method, Turbulent flow, Partial differential equation, Higher-order methods, Direct numerical simulation*

1. INTRODUCTION

Direct Numerical Simulation (DNS) has always been a useful tool in the investigation of physical flow processes, turbulent flow phenomena, chemical mixing, and other fundamental flow physics. The underlying mechanism determining the accuracy and the reliability of the simulation result is the numerical algorithm solving Navier-Stokes Equations (NSE). Traditionally, three discretizations are dominant in the solution process of the NSE namely: (i) finite difference discretization: FDD, (ii) finite volume discretization: FVD and (iii) finite element discretization: FED. The first two discretization are closely related as they are solving the standard form of PDE, one is in the differential form and the other is the integral form. FED can be considered as an extension to the FVD. It discretizes the spatial domain in the same way FVD does. However, it multiplies the PDE with a trial function and solve the weak form. Thus it can also be considered as a different family. Recently, Hokpunna et. al. [1]. Proposes a new type of discretization called Finite Surface Discretization (FSD) which situates between the two discretizations. FDD and FVD discretize the three-dimensional continuous domain into a set of pointwise data (1D) and volume-averaged data (3D), respectively. However, FSD define the variable as a surface-averaged data (2D) enclosing ***the volume*** of the FVD. The two main advantages of this discretization are *(i).the discrete mass-conservation equation is exact* and *(ii).the discrete Laplacian for the pressure is most compact*. The standard implementation of the method is found to be 27-times faster than the second-order finite volume method [2] in turbulent channel flow at $Re_\tau=180$. In laminar smooth flow, this method is very efficient but it needs dealiasing procedure when the flow is not smooth,

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similar to other high order methods [3, 4]. On a relatively rough grid, it is more beneficial to apply a stronger dealiasing than the standard method proposed in [1]. When such dealiasing procedure is used, FSM can be 73-times faster than the second-order method [5].

In general, the most accurate class of numerical simulation of turbulent flow is the direct numerical simulation (DNS). This class assumes no turbulence modeling and solve the NSE directly. The DNS should resolve all relevant motions within the flow accurately, in order to capture correct interactions. Thus, a *fully resolved* DNS must capture all the relevant flow structures. This is usually translated to using the grid size in the range of Kolmogorov's scale. Unfortunately, it is very costly to follow this strict interpretation. Still, a prediction using such fine grid can deviate significantly from the true solution, when NSE are solved with lower-order approximations. For example, the smallest persistent structures in the turbulent channel flow are about $15y^+$ and $5y^+$ in the streamwise and the spanwise directions, respectively. In the wall-normal direction, they are about $1y^+$ at the wall and $10y^+$ at the center of the channel. This translates to the computational grid $N \approx (Re_\tau/9.1)^3$ per cubic volume of the channel half-width (H^3). When a second-order scheme is used, a very accurate prediction would be obtained only when the grid size is 50% smaller than the above-mentioned size, leading to $N \approx (Re_\tau/2.7)^3$.

This canonical interpretation is well suited for scientific investigation where we want to obtain a timeless and error-free solution. *In actual engineering applications, a prediction does not have to be that accurate*. Simplifications and additional modeling have always been used to achieve a reasonable approximation for the job at hands. Computational Fluid Dynamics solving Reynolds Averaged Navier Stokes Equations (RANS) is a prime example of such approach. CFD experts and practitioners use them in daily working life knowingly that the predictions are coarse and they would be significantly deviate from the actual values. Nevertheless, those predictions are sufficient for engineering applications and RANS has become the contemporary industry standard for the past twenty years. As the computational power is increased, large companies started to shift the simulation technique to the Large-eddy simulation, and sometimes to the coarse DNS. These two approaches are very accurate, but they are still too expensive at present. However, in the coming decades they would become the norm. Most of the fluid dynamics problem in industry will be solvable by LES in 2050, and perhaps even by DNS.

When a DNS is performed with the grid resolutions coarser than the theoretical ones, we call it a *coarse DNS*. In the engineering applications mentioned previously, the prediction error of 3% is an acceptable threshold. Using a lower order method with theoretical grid resolutions usually produces errors larger than this range. This is due to the severe dispersion error of low-order methods. In practice, the grid in each direction must be 1.5-times finer than the theoretical resolutions. In contrast, higher-order methods can deliver a solution very close to that of the spectral methods on the same grid [2]. Therefore, if one aims at 1% or 3%-accurate prediction, it is possible to use the resolutions coarser than the theoretical sizes. For example, the fourth-order FVM can deliver a 1%-accurate solution using the total grid point equals to $(Re_\tau/17.6)^3$ per H^3 and the new sixth-order FSM needs $(Re_\tau/22)^3$ [1]. The FSM can deliver the 1%-error prediction using 2.4-times less grid points than the theoretical requirement. This low complexity requirement coupled with the lower computational complexity per grid point allows the FSM to deliver the result 100 to 1,000-times faster than the spectral scheme on the theoretical grid. This improvement in the capability of the solver allows us to study the turbulent flow faster and at a lower cost. It can shorten the transition process from RANS to LES or coarse DNS. However, the scaling of the total number of grid point $(Re_\tau/22)^3$ of the finite surface method is an asymptotic value ($Re_\tau \to \infty$) and this scaling is derived based on a single correlation from Re_τ=180. Therefore, it is necessary to validate it over a wider range of Reynolds number and develop a tighter scaling. These are the two objectives of this work.

This paper is organized as follows. First, the setting of the turbulent channel flow and the numerical grids are described. The 1% criteria is explained next, and then we present the first- and the second-order statistics from Re_τ =180 to 950 in comparison with spectral simulations. The flow structure is inspected. The second order statistics is presented and the behavior of the energy cascade is checked. Finally, the conclusion and outlook are presented in the last section.

2. SETUP OF THE TURBULENT CHANNEL FLOW SIMULATION

Turbulent channel flow is one of the most studied cases. Numerical results from the Stanford's Center of Turbulence Research have been highly regarded as the canonical reference. This group continuously releases new results over the past four decades. Started from the ground breaking paper in 1987 [6] where the simulated friction Reynolds number is 180. In 1999, Re_τ=395 and 590 are presented [7]. Just five years later, the group collaborates with Spanish researcher and releases the result up to Re_τ=2,000 [8]. Then, after ten years, the result up to $Re_\tau \approx$5,200 [9] is released. The highest Re_τ in the present work is 950.

In this work, we conduct a series of direct numerical simulations of the turbulent channel flow. The bulk flow Reynolds number ranges from ~5,600 to ~37,000. The discretization used in this work is the finite surface method with sixth-order approximation as described in [1]. The sixth-order FSM method is implemented in FORTRAN on the MGLET code developed originally at the Universität der Bundeswehr München and continue at Technical University of Munich. It should be emphasized that, in every simulation presented here, there are no turbulence modeling, artificial diffusion, nor other stabilization methods. The aliasing error is dealiased by applying a low-pass filter to the nonlinear term which is employed in every high-order and spectral codes.

The dealiasing method is sixth-order except for RE395 where the fourth-order dealiasing method is applied. The domain sizes are documented in Table 1 and the channel opening is set to $2H$. The streamwise and the spanwise direction is set to x and y, respectively. The grid convergence study of the FSM is already investigated in [1] and it is reported that the resolution $[dx^+, dz^+, dy_w^+, dy_c^+]$ =[30,11,4,30] is sufficient for a 1%-accurate solution. Therefore, in this work we set the numerical grid as close as possible to this resolutions. The effective grid resolutions in term of friction length scale are shown together with the relative bulk flow velocity velocity (U_b/u_τ), the total number of grid point and the corresponding relative velocity from the reference.

The fourth-order dealiasing is reported to handle the coarse resolution better than the sixth order one [5]. Therefore, the grid resolutions in the streamwise-spanwise plane are set to be coarser in RE395 than the rest. This is done to check whether the grid resolution requirement of the sixth-order FSM with a fourth-order dealiasing behaves as suggested in [5], at a higher Reynolds number. Note that the overall resolutions are slightly coarser than the suggestion. The domain size in RE180 case is the largest because the flow needs longer time to "*forget*" its previous state, and thereby the longer traveling distance is necessary. The RE950 case requires the smallest domain and this small domain size has been shown to be sufficient to deliver the 1%-accurate result [1, 10]. The periodic conditions are imposed in the streamwise-spanwise plane (*xz*-plane) and the no-slip conditions are set at the top and bottom walls (*y*-axis). The flow is driven by a constant pressure gradient which yields the friction velocity (u_τ) and the bulk flow velocity (U_b) as shown in the table. The initial conditions are taken from a snapshot of RE180 and transfer to the respective cases. A transition is performed in each case for 100 H/U_b to allow the flow structures to develop for the respective Reynolds number. Afterwards, the flow statistics are sampled over 600 H/U_b.

Table 1: Domain size and the numerical grid use in the study u_b/u_τ (ref) is taken from [6, 7].

Case	[Lx/H, Lz/H]	dx^+	dz^+	dy^+_w	dy^+_c	N (10^6)	U_b/u_τ	U_b/u_τ (ref)	Error of U_b/u_τ (%)
RE180	[4π, 4π/3]	35.0	11.7	5.3	18.4	0.131	15.54	15.68	0.89
RE395	[2π, π]	38.7	15.5	4.0	19.7	0.297	17.56	17.55	0.06
RE590	[2π, π]	29.0	12.9	4.7	29.4	1.475	18.61	18.65	0.21
RE950	[2π, π/2]	34.5	12.6	4.7	27.1	2.673	19.82	19.83	0.05

3. Result and discussion

The overall structure of the flow in the three highest Reynolds number is shown in Fig. 1 where we plot the streamwise velocity contour at y^+=5. The range of the contour is fixed at 0 to 0.5075 which is equal to $10u_\tau$(RE950) It can be seen that as the Reynolds number increases, the complexity of the flow structure is increased. The fine details of the flow is missing from the contour due to the grid is too coarse to capture them. Note that the standard grid resolutions for channel flow $[dx^+, dz^+, dy_w^+, dy_c^+]$ are [15, 5, 1, 10]. The volume of our grid in the table is more than 8-times larger, thus the very fine scale cannot be fully resolved by the grid, but the absence of the very fine scale does not apparent in the bulk flow rate. In turbulent channel flow, the bulk flow rate is a function of the driving pressure, density, viscosity and the channel opening. These factors form *friction Reynolds number (Re_τ)* for smooth walls. This is similar to the relationship between the pressure gradient and the flow rate in pipe flows. The *effective bulk flow*

Reynolds number (Re_b) will be determined by the turbulence interactions predicted by the numerical schemes used in the NSE solver. For channel flow, the relationship of the two Reynolds number is known to follow Dean's correlation: $Re_\tau=0.09Re_b^{0.88}$. Figure 2 shows that the prediction of the FSM using the proposed grid resolutions follows the correlation very well. The spectral simulation in the graph is from Moser et. al. [6] and Lee et al. [9]. At higher Reynolds number, the Dean's correlation seems to underestimate to correlation obtained from recent simulations and experiments. This is expected because it is known that at high Reynolds number, the domain size strongly affects the mean flow.

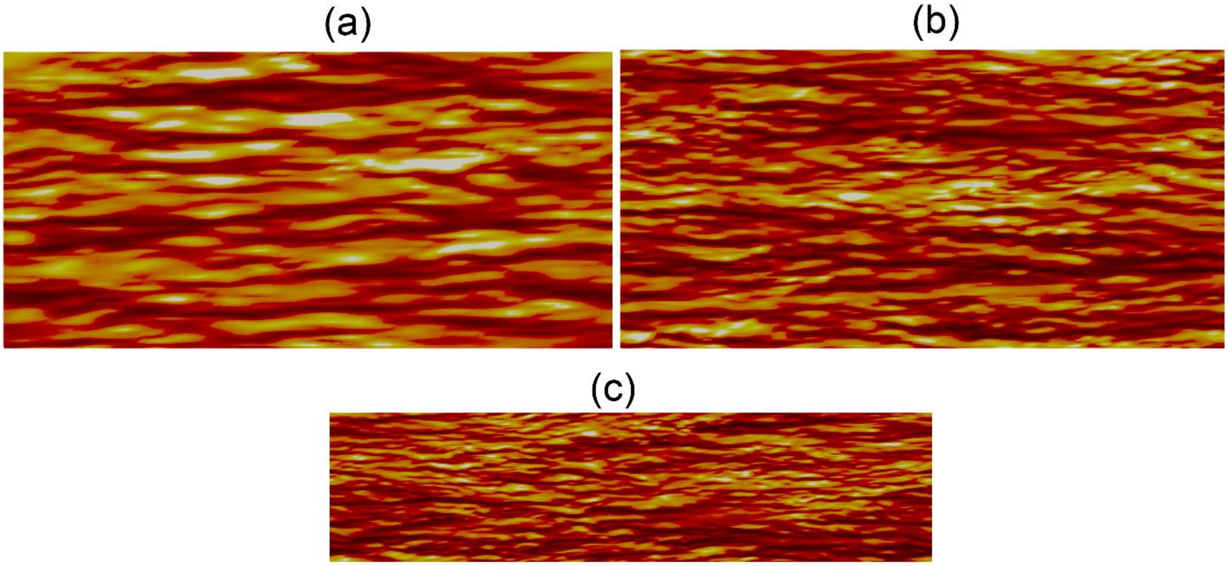

Fig. 1. Overview of the flow structure from the streamwise velocity contour at 5 wall-unit above the wall: a.) RE395, b.) RE590 and c.) RE950

Next, let us consider the accuracy of the mean streamwise profile in Fig. 3. The profiles from FSM lie on top of those from spectral scheme that has been simulated with a much higher grid density. This, however, does not imply that the FSM is more accurate than the spectral scheme. It just indicates that the FSM can capture the large and intermediate scale very well and the small scales do not affect the flow much. This implies that the dissipation predicted by the numerical scheme is adequate. At the lowest Reynolds number, the mean streamwise velocity is slightly underpredicted. This is because the streamwise grid resolution in this case is coarsest among the sixth-order dealiasing. Even though the grid resolution in the wall-normal plane is coarsest in RE395, but the fourth-order dealiasing offer a stronger suppression of the aliasing error and thus the mean profile is more accurate. The second most important quantity after the mean flow profile is the magnitude of the velocity fluctuations. This quantity is directly related to fatigue, vibration and noise in the system. Figure 4 shows that the peak of the streamwise RMS profiles are positioned relatively close.

To illustrate this point, we plot the one-dimensional energy spectra in Fig. 5, for RE950. The energy spectra are normalized by the energy of the wave number 1 and 2. In Kolmogorov energy cascade theory, the energy is dominantly transferred from the large scale to the small scale. The energy spectra from a similar flow should have the same. Thus, the energy cascade in RE950 is well captured by the finite surface method. The deviation from the spectral scheme occurs around wave number 70 which is about 78% of the resolvable wave number space. In second-order scheme, the deviation usually starts as early as 50%. Investigation of the one-dimensional energy spectra in the spanwise direction and the other Reynolds number show similar results. It should be emphasized that the 99% of the total energy of the streamwise momentum is contained in wavelength larger than 60+, at the top of the viscous sublayer and 74+ at the center of the channel, in RE180 case. Therefore, the 1%-accuracy grid size should be limited by these numbers. In the present work, the coarsest streamwise grid is 39+, with a fourth-order dealiasing. Thus, there are some room to improve in the future.

Consider that the spectral scheme solves RE180 with 2 million grid points and solves the RE590 with 37.9 million grid points. The finite surface method can deliver a similar prediction in the mean profiles using only 0.13 and 1.47 million grid points. Furthermore, the results here are computed using a single CPU. It is evident that the finite surface method is a very promising candidate for turbulent flow simulations.

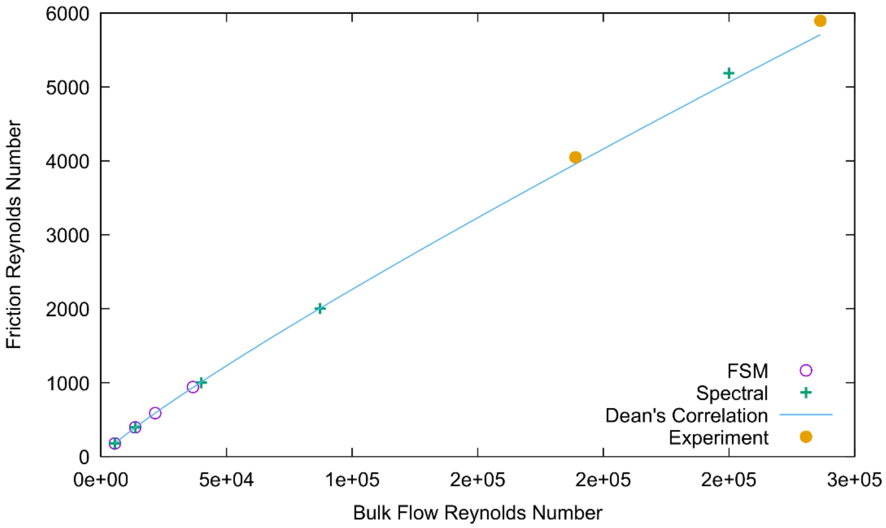

Fig. 2. Relationship between the Friction Reynolds number and the Bulk flow Reynolds number predicted by FSM comparing to spectral simulations and experiment [3].

3.1 The scaling of the sixth-order finite surface method

According the these results, we can conclude that the 1%-error grid resolutions $[dx^+,dz^+,dy_w^+,dy_c^+]$ =[30, 11, 4, 30] is applicable up to Re ≈ 1,000. In the next step, we present the scaling of the complexity as a function of Reynolds number. In wall bounded flow, the wall-normal grid is finest at the wall and it is stretched outwards to the center of the channel where it gets compressed towards the other wall. For 1%-accuracy, a 10% stretching factor is acceptable. The stretching region is dominant when Re_τ <256. Taking this part into account, the original scaling should be changed to

$$N = \frac{\mathrm{Re}^2}{360}\left(21 + \frac{\mathrm{Re}_\tau - 256}{30}\right) \tag{1}$$

$$N \approx \left(\frac{\mathrm{Re}_\tau}{22}\right)^3 + \left(\frac{\mathrm{Re}_\tau}{5.4}\right)^2 \tag{2}$$

Equation (1) and Eq. (2) are good approximations of the number of grid point in Tab. 1. Note that during the setup of the numerical simulation, we set the wall cell close to 4+, and adjust the stretching ratio such that the cell at the center is close to 30+. Thus, the stretching factors were not always 10%. The scaling of the complexity for 1%-accuracy prediction in Eq. (1) is specifically for the geometric grid with 10% stretching ratio. Other grid types would exhibit different scaling. As the Reynolds number increases, the third-order would outpace the second-order term and the complexity will return to the original scaling.

4. CONCLUSION AND OUTLOOK

We have demonstrated that the grid resolution of the sixth-order finite surface method proposed in [1] is valid for about half a decade of Reynolds numbers: Re = 5,600 to 37,000. A single CPU can use 2.7 million grid points and a period of two weeks to deliver a similar prediction obtained from 2.7 billion grid points (on a 24-times larger domain). The 50-times reduction in the grid density is made possible by the high resolving efficiency of the scheme and the discretization. The higher-order methods are made popular due the better resolving power. For a given set of flow structure (fix grid resolution), the higher-order schemes capture the fine scale structures better and this advantage in the reduction of the total grid point is a cubic function. However, decreasing the runtime using grid reduction would discard more and more physical interaction as the grid resolutions are reduced. Perhaps, it would be possible to identify the point of diminishing return, where increasing the grid point is more beneficial due to the "*the small physical interaction*" can be captured. Also, we will parallelize the finite surface method and try to match the current highest Reynolds number and verify whether the scaling and the proposed grid resolutions continue to hold.

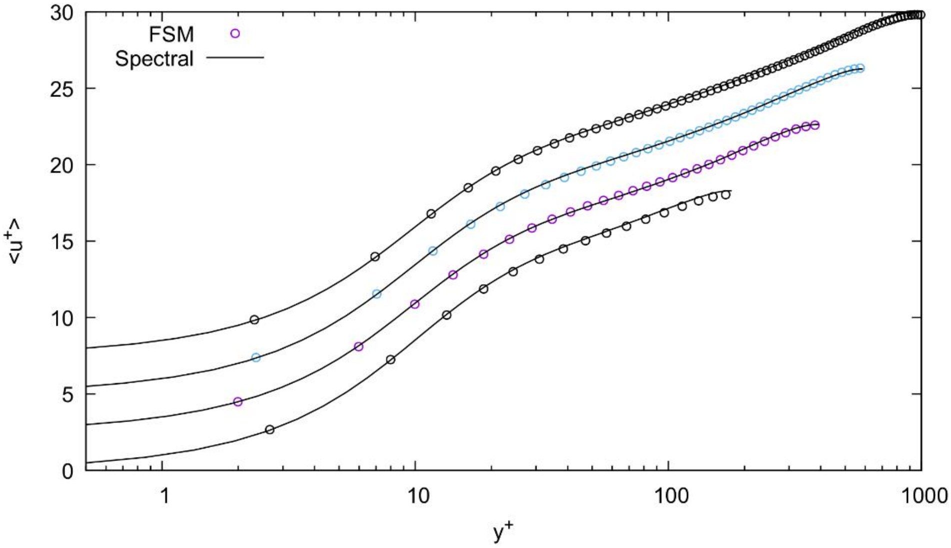

Fig. 3. Mean streamwise velocity profile of the FSM using the 1%-resolution compared against spectral simulation [6, 7]. The velocity of RE395, Re590 and RE950 are shifted 2.5u+ w.r.t. to the preceding Reynolds number

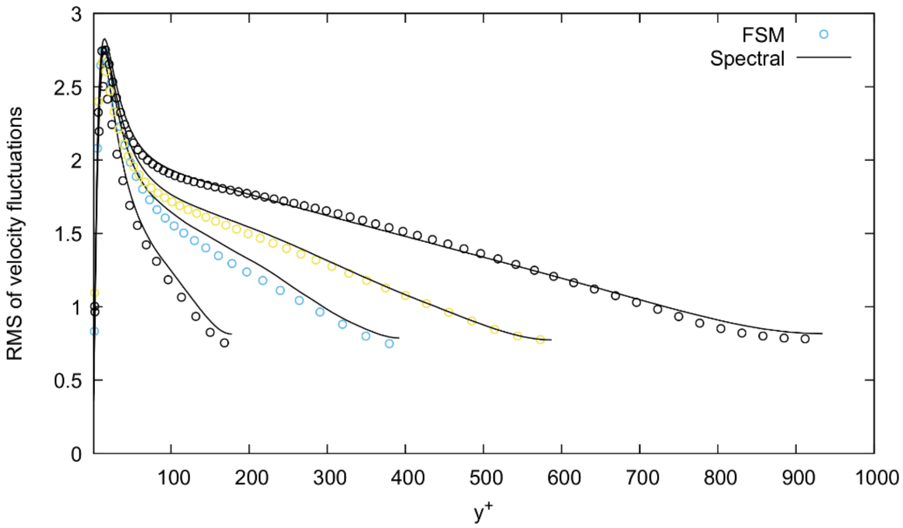

Fig. 4. Root mean square of the streamwise velocity fluctuations. Reynolds number is increased from the left to the right.

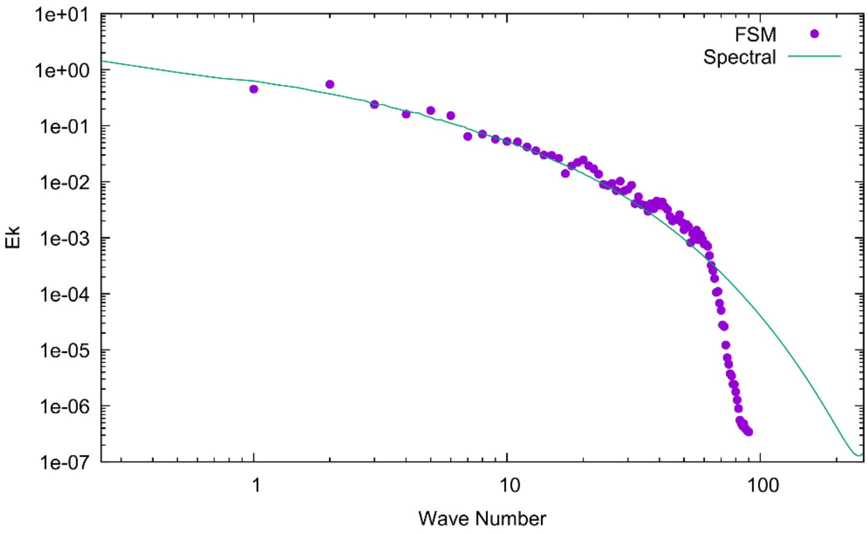

Fig. 5. One-dimensional energy spectra of the streamwise velocity at the center of the channel (RE950).

NOMENCLATURE

Re	Reynolds number
L	Domain length
N	Total number of grid points
U	streamwise velocity (outer scale)
u	streamwise velocity (inner scale)
dx	grid spacing in x-direction
dy	grid spacing in y-direction
dz	grid spacing in z-direction
x	x-direction (use as the streamwise direction)
y	y-direction (use as the spanwise direction)
z	z-direction (use as the wall-normal direction)

Subscripts

τ	friction
b	bulk flow
w	at the wall
c	at the center

Superscripts

+	normalized by friction

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