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Original Article

Approximations of normal distribution by its *q*-generalizations

Mongkhon Tuntapthai*

Department of Mathematics, Faculty of Science,
Khon Kaen University, Mueang, Khon Kaen, 40002 Thailand

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Abstract

A concept of *q*-generalization of normal distribution arises in the context of statistical mechanics. In this article, we introduce a *q*-generalization of normal approximation. By using Stein's method, zero and square bias approaches are applied to derive the bound in *q*-normal approximation. Our results also cover the bound of normal approximation for Student's *t*-distribution.

Keywords: coupling approach, non-normal approximation, *q*-Gaussian distribution, size bias distribution, square bias distribution

1. Introduction

A *q*-generalization of normal distribution was proposed by Tsallis (1988), who opened the door to nonextensive statistical mechanics. The Tsallis' distribution is now known as the *q*-normal distribution, and arises from the Boltzmann-Gibbs' theory which plays a fundamental law in thermodynamic systems. However, the classical thermo dynamic law breaks down in the systems with long-range interactions. During the last two decades, a *q*-generalization of central limit theorem was established by Umarov, Tsallis, and Steinberg (2008), and Hilhorst (2010). It turns out that the *q*-normal distribution can be applied to the sum of *q*-independent random variables, which is a kind of strong correlation. (The definition of *q-independent random variables* was introduced by Umarov, Tsallis, & Steinberg, 2008. In this context, the values of *q* are real numbers.) Over the years, the *q*-normal distribution has appeared in several fields of applications, including statistical mechanics, machine learning, economics and finance. The interested reader can refer to Borland (2002), Juniper (2007), Stavroyiannis, Makris, and Nikolaidis (2009), Tsallis (2009), Tsallis (2011), Katz, and Tian (2013), Domingo, Onofrio, and Flandoli (2017).

With three parameters $-\infty < \mu < \infty, \sigma^2 > 0, q < 3$ but $q \neq 1$, a *q*-normal density function is given by

$$p_q(w;\mu,\sigma^2) = \frac{C_q}{\sqrt{(3-q)\sigma^2}}\left(1+(q-1)\frac{(w-\mu)^2}{(3-q)\sigma^2}\right)^{1/(1-q)},$$

where C_q is a normalization factor. A random variable *W* with *q*-normal density function is called a *q*-normal random variable and denoted by $W \sim Normal_q(\mu,\sigma^2)$. It is straightforward to verify that

$$C_q = \begin{cases} \dfrac{\Gamma\left(\frac{5-3q}{2(1-q)}\right)}{\Gamma\left(\frac{2-q}{1-q}\right)}\sqrt{\dfrac{1-q}{\pi}}, & \text{for } q<1, \\ \dfrac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{3-q}{2(q-1)}\right)}\sqrt{\dfrac{q-1}{\pi}}, & \text{for } 1<q<3, \end{cases}$$

*Corresponding author
Email address: mongkhon@kku.ac.th

and that $E[W]=\mu$ for $q<2$, $Var(W)=(3-q)\sigma^2/(5-3q)$ for $q<5/3$. In the special case of parameters $\mu=0$ and $\sigma^2=1$, the standard *q*-normal random variable denoted by $W\sim Normal_q(0,1)$ has probability density function

$$\phi_q(w):=p_q(w;0,1)=\frac{C_q}{\sqrt{3-q}}\left(1+\frac{q-1}{3-q}w^2\right)^{1/(1-q)},$$

and distribution function

$$\Phi_q(z):=\int_{-\infty}^{z}\phi_q(w)\,dw.$$

Observe that the classical normal distribution can be recovered by Φ_q in the limit of $q\to 1$. When $q<1$, the density ϕ_q is defined for $|w|\le\sqrt{3-q}/\sqrt{1-q}$, so that *W* is a bounded random variable. When $1<q<3$, the distributions Φ_q cover the Student's *t*-distributions. For further properties of these distributions, the reader can refer to Hilhorst, and Schehr (2007), Diaz, and Pariguan (2009).

Our attention in the *q*-normal distribution was captured by the fact that

$$\lim_{q\to 1}\left(1+\frac{q-1}{3-q}w^2\right)^{1/(1-q)}=e^{-w^2/2}.$$

In this article, we concentrate on the case of $q\to 1^+$. Our main result gives an absolute error bound between normal distribution and its *q*-generalization with speed of convergence $\sqrt{q-1}$.

Theorem 1.1. For $1<q<5/3$, we have

$$\sup_{z\in R}\left|\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}\,dw-\Phi_q(z)\right|\le A_q\sqrt{q-1},$$

where

$$A_q=\frac{\sqrt{(q-1)}(11-5q)}{\sqrt{2\pi(3-q)}(5-3q)}+\frac{2\sqrt{2}(5-3q)^{3/2}}{\sqrt{\pi}(3-q)^{3/2}}\left\{\frac{\sqrt{(q-1)}(4-2q)}{2C_q(3-q)}+\frac{2}{3-q}\right\}.$$

It is recognized that for large ν degree of freedom, the Student's *t*-distribution is so close to the standard normal distribution. Also, notice that the distribution Φ_q is equal to Student's *t*-distribution with ν degree of freedom where $q=(\nu+3)/(\nu+1)$. From these facts, we can apply Theorem 1.1 to derive the bound of normal approximation for Student's *t*-distributions with the speed of order $1/\sqrt{\nu}$.

Our result can be extended to the *q*-normal approximation for any random variables with zero mean and variance $(3-q)/(5-3q)$ for $1<q<5/3$. We are going to derive the bound of *q*-normal approximation by using zero and square bias approaches which were introduced by Goldstein and Rinott (1996).

Theorem 1.2. For $1<q<5/3$, let $\alpha_q=(q-1)/(3-q)$ and $\gamma_q=(4-2q)/(3-q)$. If *X* is a random variable with zero mean and variance $\sigma^2=1/\left(\gamma_q-\alpha_q\right)$, then

$$\sup_{z\in R}|P(X\le z)-\Phi_q(z)|\le\left(\sqrt{(3-q)}\gamma_q/2C_q+\left(\gamma_q-2\alpha_q\right)/\sqrt{\alpha_q}\right)\left\{E|X-X^*|+\alpha_q\sigma^2E|X^{\Box}-X^*|\right\},$$

where X^* and $X^{\Box}$ have the *X*-zero bias and *X*-square bias distribution, respectively.

The behavior of *X* can be described by *q*-normal distribution, if two of these differences $|X-X^*|$, $|X-X^{\Box}|$ and $|X^{\Box}-X^*|$ are close to zero. Note that $\alpha_q\to 0$, $\gamma_q\to 1$ and $C_q\to 1/\sqrt{\pi}$ in the limit of $q\to 1$. Also we note that there are four expressions in the bound of Theorem 1.2,

$$\sqrt{(3-q)}\gamma_q/2C_q\cdot E|X-X^\star|\to\sqrt{\pi}\cdot E|X-X^\star|$$

$$(\gamma_q 2\alpha_q)/\sqrt{\alpha_q}\cdot E|X-X^\star|\to(1/0)\cdot E|X-X^\star|$$

$$\sqrt{(3-q)}\,\gamma_q/2C_q \cdot \alpha_q\sigma^2 E|X^\square - X^\star| \to 0$$

$$(\gamma_q 2\alpha_q)/\sqrt{\alpha_q} \cdot \alpha_q\sigma^2 E|X^\square - X^\star| \to 0$$

the first and second expressions are close to zero if $E|X - X^\star| \le A\alpha_q$. So we can apply Theorem 1.2 when $E|X - X^*| \le A\alpha_q$ for some constant *A*. In view of the above statement, our result can compare with the bound of normal approximation for Student's *t*-distribution in Shao (2005), Ley, Reinert, and Swan (2017). Shao (2015) obtained a non-uniform Berry-Esseen bound for the *t*-statistic with order $1/\sqrt{\nu}$. Ley, Reinert, and Swan (2017) obtained the bounds in total variation distance $d_{TV}(Z, W_\nu) \le \frac{4}{\nu-2}$ where *Z* is a standard normal, and W_ν is a Student's *t*-random variable with $\nu > 2$ degrees of freedom. In this work, we can apply Theorem 1.1 to derive the bound of order $1/\sqrt{(\nu+1)}$.

In the next section, we will introduce the Stein's equation for standard *q*-normal distribution. In order to apply Stein's technique, we need to estimate the bounds of solution to the Stein's equation, and its derivatives. These results can be easily obtained by density identities, Mills' ratio inequalities, and L'Hopital's rule. The proof of Theorem 1.1 and 1.2 will be shown in the last section. A crucial step is zero and square bias approaches.

2. Stein's Method

An effective method for establishing the bound of normal approximation in some weak dependent conditions was introduced by Stein (1972). The basic idea of this method is based on a characterization of normal distribution. A random variable *X* has normal distribution with zero mean and variance σ^2 if and only if

$$\sigma^2 E\left[f'(X)\right] = E\left[Xf(X)\right],$$

for all absolutely continuous functions f such that $E\left|f'(Z)\right|$ exists, where *Z* is a normal random variable with zero mean and variance σ^2. This characterization leads to a first order linear differential equation called the Stein's equation for normal distribution,

$$\sigma^2 f'(x) - xf(x) = \mathbf{1}_{(-\infty,z]}(x) - P(Z \le z).$$

It is known that for a fixed real number z, there exists a unique absolutely continuous function f_z solving the Stein's equation and satisfying that

$$\sigma^2 E\left[f_z'(X)\right] - E\left[Xf_z(X)\right] = P(X \le z) - P(Z \le z),$$

for any random variable *X* with zero mean and variance σ^2. This equation may be a subtle way in the context of normal approximation, since the bound of left-hand side of the above equation can be easily obtained by using the following three approaches: the concentration inequality presented in the original paper of Stein (1972), the inductive approach discussed in the works of Barbour and Hall (1984), and the coupling approach introduced by Goldstein and Rinott (1996).

Over the years, Stein's method has been applied to a wide range of applications, including random graph theory, random matrix theory, and statistical physics. The interested reader can refer to Pekoz, and Ross (2013), Mackey, Jordan, Chen, Farrell, and Tropp (2014), Eichelsbacher, and Martschink (2014). In addition, Stein's method has recently been developed to several approximations by non-normal distributions, such as the exponential, gamma, beta, and Laplace distributions. For an overview of literature, the reader can refer to Chatterjee, Fulman, and Röllin (2011), Chatterjee, and Shao (2011), Gaunt (2014), Pike, and Ren (2014), Dobler (2015), Gaunt, Pickett, and Reinert (2017), Ley *et al.* (2017). In this section, we will develop the Stein's method to the *q*-normal distributions. We begin by introducing a characterization of the *q*-normal distribution.

2.1. A characterization of *q*-normal distribution

For a parameter $1 < q < 5/3$, we denote

$$\alpha_q = \frac{q-1}{3-q} \text{ and } \gamma_q = \frac{4-2q}{3-q}.$$

Let $W \sim Normal_q(0,1)$. We first observe that the density function

$$\phi_q(w) = \frac{C_q}{\sqrt{3-q}}\left(1 + \alpha_q w^2\right)^{1/(1-q)},$$

satisfies the identities

$$\left\{(1+\alpha_q w^2)\phi_q(w)\right\}' = -\gamma_q w\phi_q(w), \quad \text{and} \tag{2.1}$$

$$\left\{(1+\alpha_q w^2)^2\phi_q(w)\right\}' = -(\gamma_q - 2\alpha_q)w(1+\alpha_q w^2)\phi_q(w). \tag{2.2}$$

The *q*-normal distribution satisfies the symmetric property around its mean, which implies that

$$\min\left\{1-\Phi_q(z), \Phi_q(z)\right\} \le \Phi_q(0) = \frac{1}{2}, \tag{2.3}$$

and it also satisfies the Mills' ratio inequality that for $w > 0$,

$$1-\Phi_q(w) \le \frac{C_q}{\sqrt{3-q}}\int_w^\infty \frac{x}{w}(1+\alpha_q x^2)^{1/(1-q)}dx = \frac{1}{\gamma_q w}(1+\alpha_q w^2)\phi_q(w). \tag{2.4}$$

From the identity (2.1) and Fubini's theorem, it is easy to verify that

$$\begin{aligned} E[Wf(W)] &= \int_{-\infty}^{\infty} wf(w)\phi_q(w)\,dw \\ &= \int_{-\infty}^{\infty} w\int_{-\infty}^{w} f'(x)\,dx\phi_q(w)\,dw \\ &= \int_{-\infty}^{\infty}\int_{x}^{\infty} wf'(x)\phi_q(w)\,dw\,dx \\ &= \frac{1}{\gamma_q}\int_{-\infty}^{\infty}(1+\alpha_q x^2)f'(x)\phi_q(x)\,dx \\ &= \frac{1}{\gamma_q}E[(1+\alpha_q W^2)f'(W)] \end{aligned}.$$

So, we have that

$$E\left[(1+\alpha_q W^2)f'(W)\right] = \gamma_q E[Wf(W)], \tag{2.5}$$

for every absolutely continuous function f such that $E|f(w)|$ and $E|f'(W)|$ exist.

We now introduce a first order linear differential equation called the Stein's equation for *q*-normal distribution, for a fixed real number z,

$$(1+\alpha_q w^2)f'(w) - \gamma_q wf(w) = \mathbf{1}_{(-\infty,z]}(w) - \Phi_q(z). \tag{2.6}$$

Multiplying both sides of (2.6) with the integrating factor $(1+\alpha_q w^2)^{1/(1-q)}$ yields the solution f_z given by

$$\begin{aligned} f_z(w) &= (1+\alpha_q w^2)^{(q-2)/(1-q)}\int_{-\infty}^{w}\left\{\mathbf{1}_{(-\infty,z]}(w) - \Phi_q(z)\right\}(1+\alpha_q x^2)^{1/(1-q)}\,dx \\ &= \left\{(1+\alpha_q w^2)\phi_q(w)\right\}^{-1}\int_{-\infty}^{w}\left\{\mathbf{1}_{(-\infty,z]}(w) - \Phi_q(z)\right\}\phi_q(x)\,dx, \end{aligned}$$

or equivalently

$$f_z(w) = \left\{(1+\alpha_q w^2)\phi_q(w)\right\}^{-1}\int_{w}^{\infty}\left\{\mathbf{1}_{(-\infty,z]}(w) - \Phi_q(z)\right\}\phi_q(x)\,dx.$$

Therefore, the solution to Stein's equation (2.6) is given by

$$f_z(w) = \begin{cases} \left\{(1+\alpha_q w^2)\phi_q(w)\right\}^{-1}\left\{1-\Phi_q(z)\right\}\Phi_q(w), & \text{for } w \le z, \\ \left\{(1+\alpha_q w^2)\phi_q(w)\right\}^{-1}\Phi_q(z)\left\{1-\Phi_q(w)\right\}, & \text{for } w > z, \end{cases} \tag{2.7}$$

which is absolutely continuous and infinitely differentiable at $w \neq z$.

By the identity (2.5) and the solution given by (2.7), we obtain a characterization of *q*-normal distributions on the class of absolutely continuous functions f_z.

Lemma 2.1. A random variable *X* satisfies the equation

$$E\left[(1+\alpha_q X^2)f'(X)\right] = \gamma_q E\left[X f(X)\right],$$

for every absolutely continuous function f such that $E|f'(X)|$ exists, if and only if $X \sim Normal_q(0,1)$.

2.2. Properties of the solution to Stein's equation

We derive some properties of the solution to Stein's equation (2.6). From the density identity (2.1), the first derivative of solution (2.7) is given by

$$f_z'(w) = \begin{cases} \left\{(1+\alpha_q w^2)^2\phi_q(w)\right\}^{-1}\left\{1-\Phi_q(z)\right\}G(w), & \text{for } w<z,\\ -\left\{(1+\alpha_q w^2)^2\phi_q(w)\right\}^{-1}\Phi_q(z)H(w), & \text{for } w>z,\end{cases}$$

where

$$G(w) = (1+\alpha_q w^2)\phi_q(w) + \gamma_q w\Phi_q(w), \tag{2.8}$$

$$H(w) = (1+\alpha_q w^2)\phi_q(w) - \gamma_q w\left\{1-\Phi_q(w)\right\}. \tag{2.9}$$

It is known that *G* and *H* are positive, since

$$G(w) = -\int_{-\infty}^{w}\gamma_q x\phi_q(x)\,dx + \gamma_q w\Phi_q(w) = \gamma_q\int_{-\infty}^{w}\Phi_q(x)\,dx,$$

$$H(w) = \int_{w}^{\infty}\gamma_q x\phi_q(x)\,dx - \gamma_q w\left\{1-\Phi_q(w)\right\} = \gamma_q\int_{w}^{\infty}\left\{1-\Phi_q(x)\right\}dx.$$

Thus f_z is increasing on $(-\infty, z]$ and decreasing on (z,∞). By the continuity of f_z, we can conclude that z is the extreme point, so that $\|f_z\| = f_z(z)$, where we use the notation $\|f_z\| = \sup_w |f(w)|$. By the symmetry and (2.3), we obtain the bound

$$\|f_z\| = f_z(z) \le f_0(0) = \frac{\sqrt{3-q}}{4C_q}. \tag{2.10}$$

To obtain the bound of $\|f_z'\|$, we have to rearrange the Stein's equation (2.6) as

$$f_z'(w) = (1+\alpha_q w^2)^{-1}\left\{1_{(-\infty,z]}(w) - \Phi_q(w) + \gamma_q w f_z(w)\right\}.$$

Applying the Mills' ratio inequality (2.4) gives

$$|f_z(w)| \le \left\{(1+\alpha_q w^2)\phi_q(w)\right\}^{-1}\min\left\{1-\Phi_q(w),\Phi_q(w)\right\} \le \frac{1}{\gamma_q|w|},$$

which implies that $|f_z'(w)| \le 2/(1+\alpha_q w^2)$. Thus

$$\|f_z'\| \le 2. \tag{2.11}$$

The second derivative of the solution (2.7) is given by

$$f_z''(w) = \begin{cases} \left\{(1+\alpha_q w^2)^3\phi_q(w)\right\}^{-1}\left\{1-\Phi_q(z)\right\}G_1(w), & \text{for } w<z,\\ -\left\{(1+\alpha_q w^2)^3\phi_q(w)\right\}^{-1}\Phi_q(z)H_1(w), & \text{for } w>z,\end{cases}$$

where

$$G_1(w) = (1+\alpha_q w^2)\gamma_q\Phi_q(w) + (\gamma_q - 2\alpha_q)wG(w),$$

$$H_1(w) = (1+\alpha_q w^2)\gamma_q\left\{1-\Phi_q(w)\right\} - (\gamma_q - 2\alpha_q)wH(w).$$

For $w<0$, we define a function G_2 from the first term of G_1 by

$$G_2(w) = \left\{(1+\alpha_q w^2)^2\phi_q(w)\right\}^{-1}\gamma_q\Phi_q(w).$$

By L'Hopital's rule and identity (2.2), it is easy to see that

$$\lim_{w\to-\infty} G_2(w) = \lim_{w\to-\infty} \frac{-\gamma_q\phi_q(w)}{(\gamma_q-2\alpha_q)w(1+\alpha_q w^2)\phi_q(w)} = 0,$$

which implies that G_2 possess the limit of $w \to -\infty$. Its derivative is given by

$$G_2'(w) = \left\{(1+\alpha_q w^2)^3\phi_q(w)\right\}^{-1}\gamma_q\left\{G(w) - 2\alpha_q w\phi_q(w)\right\},$$

where G given by (2.8), is positive on $(-\infty, 0]$.

Thus $G_2(w) \le G_2(0) = \sqrt{3-q}\,\gamma_q/2C_q$.

For the second term of G_1, by (2.11), we note that

$$\begin{aligned}&\left\{(1+\alpha_q w^2)^3\phi_q(w)\right\}^{-1}\left\{1-\Phi_q(z)\right\}(\gamma_q-2\alpha_q)wG(w)\\ &= \frac{(\gamma_q-2\alpha_q)w}{(1+\alpha_q w^2)}f_z'(w) \le \frac{(\gamma_q-2\alpha_q)}{2\sqrt{\alpha_q}}\|f_z'\| \le \frac{(\gamma_q-2\alpha_q)}{\sqrt{\alpha_q}}.\end{aligned}$$

So, we obtain the bound of $|f_z''(w)|$ for $w < z$,

$$|f_z''(w)| \le \sqrt{3-q}\,\gamma_q/2C_q + (\gamma_q-2\alpha_q)/\sqrt{\alpha_q}. \tag{2.12}$$

Similar to the bound of $|f_z''(w)|$ when $z < w < 0$.
For $w > 0$, we also obtain the bound of $|f_z''(w)|$ by the symmetry of G_2.
The next statement shows some properties of the Stein's solution given by (2.7), and the bounds of its derivatives given by (2.10), (2.11), (2.12).

Proposition 2.2. For a real number z, the solution f_z given by (2.7) is absolutely continuous and infinitely differentiable at $w \neq z$ with the bounds

$$\|f_z\| \le \sqrt{3-q}/4C_q, \quad \|f_z'\| \le 2, \quad \text{and} \quad \|f_z''\| \le \sqrt{3-q}\,\gamma_q/2C_q + (\gamma_q-2\alpha_q)/\sqrt{\alpha_q}.$$

3. Proof of Results

In this section, we are going to derive the bounds of q-normal approximation by using the zero bias and square bias approaches which were introduced by Goldstein and Rinott (1996), Chen et al. (2010). Let X be a random variable with zero mean and finite variance; recall that a random variable X^* has the X-zero bias distribution if

$$E[X^2]E[f'(X^*)] = E[Xf(X)],$$

for all absolutely continuous functions f for which these expectations exist, and recall that a random variable X^* has the X-square bias distribution if

$$E[X^2]E[f(X^\square)] = E[X^2 f(X)],$$

for all functions f for which $E|Xf(X)| < \infty$.

Proof of Theorem 1.1. Let X have the classical normal distribution with zero mean and variance $\sigma^2 = 1/(\gamma_q - \alpha_q)$, where α_q and γ_q are given in Section 2.1. For each real number z, let f_z be the solution to Stein's equation (2.6). Then

$$\begin{aligned}P(X\le z) - \Phi_q(z) &= E[f_z'(X)] + \alpha_q E[X^2 f_z'(X)] - \gamma_q E[Xf_z(X)]\\ &= E[f_z'(X)] - (\gamma_q-\alpha_q)E[Xf_z(X)] + \alpha_q\left\{E[X^2f_z'(X)] - E[Xf_z(X)]\right\}\\ &= \alpha_q\left\{E[X^2f_z'(X)] - E[Xf_z(X)]\right\},\end{aligned}$$

where we used a characterization of X having the classical normal distribution. Let X^* have the X-square bias distribution, and U have the uniform distribution on (-1,1) which is independent of X^*. By Proposition 2.3 of Chen *et al.* (2010), the random variable $X^* = UX^*$ has the X-zero bias distribution. Thus

$$P(X\le z) - \Phi_q(z) = \alpha_q\left\{E[X^2f_z'(X)] - E[Xf_z(X)]\right\}$$

$$
\begin{aligned}
&= \alpha_q \sigma^2 \left\{ E[f_z'(X^\Box)] - E[f_z'(X^*)] \right\} \\
&= \alpha_q \sigma^2 \left\{ E[f_z'(X^\Box)] - E[f_z'(UX^\Box)] \right\} \\
&\le \alpha_q \sigma^2 \|f_z''\| E|X^\Box - UX^\Box| \\
&= \alpha_q \sigma^2 \|f_z''\| E|X^\Box| \\
&= \alpha_q E|X^3| \cdot \|f_z''\| \le \alpha_q \frac{2\sqrt{2}}{\sqrt{\pi}\sigma^3} \left(\sqrt{(3-q)}\gamma_q/2C_q + (\gamma_q - 2\alpha_q)/\sqrt{\alpha_q} \right) \\
&= \alpha_q \frac{2\sqrt{2}}{\sqrt{\pi}} (\gamma_q - \alpha_q)^{3/2} \left(\sqrt{(3-q)}\gamma_q/2C_q + (\gamma_q - 2\alpha_q)/\sqrt{\alpha_q} \right),
\end{aligned} \tag{3.1}
$$

where we used the bound of $\|f_z''\|$ in Proposition 2.2.
Let Φ be the standard normal distribution. Next, we claim that

$$
\sup_z |\Phi(z) - P(X \le z)| \le \alpha_q \frac{(2\sigma^2 + 1)}{\sqrt{2\pi}} = \frac{\alpha_q (2 + \gamma_q - \alpha_q)}{\sqrt{2\pi}\,(\gamma_q - \alpha_q)}. \tag{3.2}
$$

By symmetry of the normal distribution, we need only to show the claim in the case of $z > 0$. Observe that $\sigma^2 = (3-q)/(5-3q) > 1$, when $1 < q < 5/3$. For $0 < z \le 1$, by the mean value theorem, we have

$$
\begin{aligned}
\Phi(z) - P(X \le z) &= \frac{1}{\sqrt{2\pi}} \int_0^z e^{-w^2/2} - \frac{1}{\sigma} e^{-w^2/2\sigma^2}\, dw \\
&= \frac{z}{\sqrt{2\pi}} \left(e^{-w_o^2/2} - \frac{1}{\sigma} e^{-w_o^2/2\sigma^2} \right),
\end{aligned}
$$

where $0 < w_0 < z \le 1$. By using the facts that $1 - e^{-w} \le w$ and $we^{-w} \le 1$ for all $w > 0$, we obtain that for $0 < z \le 1$,

$$
\Phi(z) - P(X \le z) \le \frac{\alpha_q}{\sqrt{2\pi}} \left(2\sigma^2 + \frac{2}{1+\sigma} \right) \le \frac{\alpha_q}{\sqrt{2\pi}} (2\sigma^2 + 1).
$$

For $z > 1$, it is straightforward to check that

$$
\begin{aligned}
\Phi(z) - P(X \le z) &= \frac{1}{\sqrt{2\pi}} \int_z^\infty \left(e^{-w^2/2} - \frac{1}{\sigma} e^{-w^2/2\sigma^2} \right) dw \\
&\le \frac{1}{z\sqrt{2\pi}} \left(e^{-z^2/2} - \sigma e^{-z^2/2\sigma^2} \right) \le \frac{\alpha_q}{\sqrt{2\pi}} \left(\frac{\sqrt{2}\sigma}{e} + \frac{2e^{-1/2\sigma^2}}{(1+\sigma)} \right) \le \frac{\alpha_q}{\sqrt{2\pi}} (2\sigma^2 + 1).
\end{aligned}
$$

The proof is completed by combining (3.1) and (3.2).

Proof of Theorem 1.2. For each real number z, let f_z be the Stein's solution given by (2.7). Then

$$
\begin{aligned}
&P(X \le z) - \Phi_q(z) \\
&= E[f_z'(X)] + \alpha_q E[X^2 f_z'(X)] - \gamma_q E[X f_z(X)] \\
&= E[f_z'(X)] - (\gamma_q - \alpha_q) E[X f_z(X)] + \alpha_q \left\{ E[X^2 f_z'(X)] - E[X f_z(X)] \right\} \\
&= E[f_z'(X)] - E[f_z'(X^*)] + \alpha_q \sigma^2 \left\{ E[f_z'(X^\Box)] - E[f_z'(X^*)] \right\} \\
&\le \|f_z''\| \left\{ E|X - X^*| + \alpha_q \sigma^2 E|X^\Box - X^*| \right\}.
\end{aligned}
$$

The proof is complete by using Proposition 2.2, the bound of $\|f_z''\|$.

4. Conclusions

We have characterized the *q*-normal distribution *W* with density ϕ_q via Stein's method as the following identity $E[(1+\alpha_q W^2) f'(W)] = \gamma_q E[W f(W)]$. This identity is given by a nonhomogeneous linear differential equation which is called Stein's equation (2.6). The equation leads us to other possible methods for deriving the bound of a *q*-generalization of normal approximation. We have derived the bound of *q*-normal approximation for a standard normal distribution in Theorem 1.1. Our results can be applied to the Student's *t*-distribution, and also extended to the distribution of any random variable in Theorem 1.2.

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