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Original Article

An analytical formula for pricing interest rate swaps in terms of bond prices under the extended Cox-Ingersoll Ross model

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Abstract

This paper presents an analytical formula for pricing interest rate swaps (IRSs) in terms of bond prices in which the interest rates are assumed to follow the extended Cox-Ingersoll-Ross model. Furthermore, we analytically investigate some asymptotic properties of the fair price of IRSs. Numerical tests are provided to demonstrate the accuracy and efficiency of our current approach compared with the Monte-Carlo simulations.

Keywords: interest rate swaps, bond prices**,** analytical pricing formula, extended CIR model

1. Introduction

An interest rate swap (IRS) is essentially a forward contract in which a company agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a number of years. At the maturity time of the forward contract, the company receives interest at a floating rate on the same notional principal for the same period of time. The floating rate in many interest rate swap agreements is the London Interbank Offer Rate (LIBOR).

There are various types of IRSs depending on the agreement between two companies to exchange cash flows in the future (see for example in Fat & Pop, 2015; Hull, 2002; Mallier & Alobaidi, 2004; Xiaofeng, Jinping, Shenghong, Cristoforo, & Xiaohu, 2010). This paper focuses the one described by Hull (2002), such that an IRS can be characterized as the difference between two bonds. In other words, the value of the IRS to a company receiving floating and paying fixed, denoted by V_{swap}, satisfies

$$V_{\text{swap}} = B_{\text{fl}} - B_{\text{fix}} \tag{1.1}$$

where B_{fl} and B_{fix} are the values of floating-rate and fixed-rate bond underlying the swap, respectively.

We can reasonably assume that the value of this swap is zero, i.e., $V_{\text{swap}} = 0$ when it is first initiated. After it has been in existence for some time, its value may become positive or negative. To calculate the value, we can regard the swap either as a long position in one bond combined with a short position in another bond or as a portfolio of forward rate agreements, whereas banks and other financial institutions usually discount cash flow in the over-the-counter market at LIBOR rates of interest. In either case, we use LIBOR zero rates for discounting. Therefore, B_{fix} can be written as

$$B_{\text{fix}} = \sum_{i=1}^{N} k_i e^{-(t_i - t_0)r_{t_i}} + Le^{-(t_N - t_0)r_{t_N}} \tag{1.2}$$

for some positive integer $N \geq 2$ where t_0 is the initiated time of the IRS, t_i is the time until i^{th} payments are exchanged for $i = 1, 2, \ldots, N$, L is the notional principal in swap

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agreement, k_i is the fixed payment made on payment time t_i, and r_{t_i} is the LIBOR zero rate corresponding to maturity time t_i.

As for the floating-rate bond, we have immediately after a payment date that $B_{\text{fl}} = L$. On the other hand, for the other payment dates, we can use the fact that B_{fl} will equal to L immediately after the next payment date. That is, $B_{\text{fl}} = L + k_0$, where k_0 is the floating-rate payment (already known) that will be made on the next payment date. In our setting, the time until the next payment date is t_1. This implies that the value of the swap today is its value just before the next payment discounted at rate r_{t_1} for time t_1:

$$B_{\text{fl}} = (L + k_0)e^{-(t_1 - t_0)r_{t_1}}. \tag{1.3}$$

By applying (1.2) and (1.3) to (1.1), the value of the swap at time t_N can be written as

$$V_{\text{swap}}(N) := (L + k_0)e^{-(t_1-t_0)r_{t_1}} - \left(\sum_{i=1}^{N} k_i e^{-(t_i-t_0)r_{t_i}} + Le^{-(t_N-t_0)r_{t_N}}\right) = \sum_{i=1}^{N} K_i e^{-(t_i-t_0)r_{t_i}} \tag{1.4}$$

for $N = 2, 3, \ldots$, where $K_1 = L + k_0 - k_1$, $K_N = -(k_N + L)$, and $K_i = -k_i$ for $i = 2, 3, \ldots, N-1$.

In this paper, we assume that the LIBOR zero rate at time t, denoted by r_t, is a random variable described by the extended Cox-Ingersoll-Ross (ECIR) process:

$$dr_t = \kappa(t)\big(\theta(t) - r_t\big)dt + \sigma(t)\sqrt{r_t}\,dW_t \tag{1.5}$$

for $t \in [t_0, T]$, $T = t_N$, and $0 \le t_0 < t_1$ where W_t is a standard Brownian motion under a probability space (Ω, F, P) with a filtration $(F_t)_{t \ge t_0}$.

In addition, in order to ensure that the stochastic differential equation (SDE) (1.5) has a path-wise unique strong solution, in which r_t avoids zero a.s. P for all $t \in (t_0, T]$, we further assume that $\theta(t)$, $\kappa(t)$, and $\sigma(t)$ are strictly positive smooth and bounded time dependent parameter functions on $[0, T]$ such that the inequality

$$\frac{\theta(t)\kappa(t)}{\sigma^2(t)} \ge \frac{1}{2} \tag{1.6}$$

holds for all $t \in [0, T]$.

In the context of forward contract pricing when the underlying process is assumed to follow the ECIR process (1.5), the fair price of an IRS corresponding to (1.4) at date t_0 is defined by the conditional expectation of V_{swap} with respect to the probability measure P as

$$\text{FIRS}_{\text{swap}}(r_0, t_0; N) := E^P[V_{\text{swap}}(n) \mid F_{t_0}] = \sum_{i=1}^{N} K_i E^P[e^{-(t_i-t_0)r_{t_i}} \mid r_{t_0} = r_0] \tag{1.7}$$

for the predetermined dates $0 \le t_0 < t_1 < t_2 < t_3 < \cdots < t_N = T$ and LIBOR zero rate $r_0 > 0$ at date t_0 where we denote by $E^P[X \mid F_t]$, the conditional expectation of a random variable X with respect to the probability measure P and $\sigma-$field F_t.

It should be noted from (1.7) that the problem of IRS pricing is in fact reduced to computing the conditional expectations of the interest rate process r_t of the form:

$$E^P[e^{-(t_i-t_0)r_{t_i}} \mid r_{t_0} = r_0] \tag{1.8}$$

for $i = 1, \ldots, N$.

The computation of the conditional expectation (1.8) is non-trivial because it contains a nonlinear function (exponential function) of the random variable r_{t_i} with the result that the distribution of $y_{t_i} = e^{-(t_i - t_0)r_{t_i}}$ might be unavailable in closed form. As explained in Thamrongrat and Rujivan (2019), we need to solve the forward Kolmogorov equation associated with the process y_{t_i} in order to determine its transition probability density. The approaches proposed in Rujivan (2009) and Rujivan (2010) can be adopted to derive a closed-form expansion for the transition probability density. However, this is a difficult and complicated task in general for arbitrary real-valued functions $\theta(t)$, $\kappa(t)$, and $\sigma(t)$. Fortunately, we thank for the work of Sutthimat, Mekchay, and Rujivan (2019) who extended the results proposed in Rujivan (2016) for computing the conditional moments of ECIR processes, to obtain a closed-form formula for the conditional expectations of product of polynomial and exponential function of the ECIR process (1.5).

There are two major contributions of the present paper. First, we provide an analytical formula for the fair price of an IRS in which the interest rates are assumed to follow the ECIR model (1.5). More specifically, our analytical approach produces the exact values of the conditional expectations in the RHS of (1.7) and avoids the utilization of Monte Carlo (MC) simulations. As shown in our numerical results, this can substantially reduce the computational burden which is a major drawback of MC methods. Second, our current analytical formula has a simple form, which can be easily used by practitioners. With these contributions, our formula presented in the paper should be valuable in both theoretical and practical senses.

The paper is organized as follows. In Section 2, we present an analytical formula for pricing IRSs under the ECIR model (1.5) and derive some asymptotic properties of the fair prices of IRSs. In Section 3, we demonstrate the accuracy and efficiency of our current approach compared with the MC simulations. The conclusion is provided in Section 4.

2. An Analytical Formula for Pricing IRSs under the ECIR Model

According to (1.7), we denote $\mathrm{FIRS}_{\mathrm{swap}}(r_0,t_0;N)$ by $\mathrm{FIRS}^{E}_{\mathrm{swap}}(r_0,t_0;N)$ if r_t follows the ECIR model. On the other hand, if r_t follows the CIR process that is $\theta(t)=\theta$, $\kappa(t)=\kappa$, and $\sigma(t)=\sigma$ for some positive values q, k, and s , then we write $\mathrm{FIRS}^{C}_{\mathrm{swap}}(r_0,t_0;N)$ instead of $\mathrm{FIRS}^{E}_{\mathrm{swap}}(r_0,t_0;N)$.

Theorem 2.1 Suppose that r_t follows the ECIR process (1.5). Then,

$$\mathrm{FIRS}^{E}_{\mathrm{swap}}(r_0,t_0;N)=\sum_{i=1}^{N}K_iE^P[e^{-(t_i-t_0)r_{t_i}}\mid r_{t_0}=r_0]=\sum_{i=1}^{N}K_iA(\Delta t_i;t_i,-\Delta t_i)e^{B(\Delta t_i;t_i,-\Delta t_i)r_0} \tag{2.1}$$

where $\Delta t_i = t_i - t_0$ and

$$A(\tau;t,\alpha):=e^{\int_0^\tau \kappa(t-u)\theta(t-u)B(u;t,\alpha)du} \tag{2.2}$$

$$B(\tau;t,\alpha):=\frac{\alpha e^{-\int_0^\tau \kappa(t-u)du}}{1-\frac{\alpha}{2}\int_0^\tau \sigma^2(t-s)e^{-\int_0^s \kappa(t-u)du}ds} \tag{2.3}$$

for $\tau\geq 0, t>0, and\ \alpha\in\mathbb{R}$.

Proof. First, we compute the conditional expectations of the nonlinear function of the interest rate process as expressed in (1.8). By applying Theorem 2 proposed by (Sutthimat *et al.*, 2019) by setting $n=0,\alpha=-\Delta t_i,\beta=0,\tau=\Delta t_i$, , and $v = r_0$, we immediately yield

$$E^P[e^{-(t_i-t_0)r_{t_i}}\mid r_{t_0}=r_0]=A(\Delta t_i;t_i,-\Delta t_i)e^{B(\Delta t_i;t_i,-\Delta t_i)r_0}. \tag{2.4}$$

We replace $E^P[e^{-(t_i-t_0)r_{t_i}}\mid r_{t_0}=r_0]$ in (1.7) with the RHS of (2.4) to complete the proof.

Next, we derive some asymptotic properties of the fair price of an IRS as follows.

Corollary 2.1 Suppose that r_t follows the ECIR process (1.5). Then,

1) $$\lim_{r_0\to 0^+}\mathrm{FIRS}^{E}_{\mathrm{swap}}(r_0,t_0;N)=\sum_{i=1}^{N}K_iA(\Delta t_i;t_i,-\Delta t_i). \tag{2.5}$$

2) $\lim_{r_0 \to \infty} \mathrm{FIRS}^{E}_{\mathrm{swap}}(r_0, t_0; N) = 0.$ (2.6)

Proof. We directly get (2.5) by taking $r_0 \to 0^+$ in (2.4). Moreover, from (2.3), we have $B(\Delta t_i; t_i, -\Delta t_i)$ is always be negative and this implies (2.6) holds since

$$\lim_{r_0 \to \infty} e^{B(\Delta t_i; t_i, -\Delta t_i) r_0} = 0 \tag{2.7}$$

for all $i = 1, 2, \ldots, N$.

In the case that the model parameters of the ECIR process (1.5) become constants, we simplify the formula (2.1) for the CIR process as shown in the following theorem.

Theorem 2.2 Suppose that r_t follows the ECIR process (1.5) such that $\theta(t) = \theta$, $\kappa(t) = \kappa$, and $\sigma(t) = \sigma$ for some positive values $q, k,$ and s satisfying the inequality $\kappa\theta \geq \sigma^2 / 2$. Then,

$$\mathrm{FIRS}^{C}_{\mathrm{swap}}(r_0, t_0; N) = \sum_{i=1}^{N} K_i \left(\frac{2\kappa}{-\Delta t_i \sigma^2 + e^{\kappa \Delta t_i}(2\kappa + \Delta t_i \sigma^2)} \right)^{\frac{2\kappa\theta}{\sigma^2}} e^{\left(\frac{-2\Delta t_i \kappa}{-\Delta t_i \sigma^2 + e^{\kappa \Delta t_i}(2\kappa + \Delta t_i \sigma^2)} \right) r_0 + \frac{2\kappa\theta\Delta t_i}{\sigma^2}} \tag{2.8}$$

where $\Delta t_i = t_i - t_0$.

Proof. We apply Theorem 4 proposed by Sutthimat *et al.* (2019) under the CIR model by setting $n = 0, \alpha = -\Delta t_i, \beta = 0, \tau = \Delta t_i$,and $v = r_0$. Thus, we obtain

$$E^P[e^{-(t_i - t_0) r_{t_i}} \mid r_{t_0} = r_0] = \left(\frac{2\kappa}{-\Delta t_i \sigma^2 + e^{\kappa \Delta t_i}(2\kappa + \Delta t_i \sigma^2)} \right)^{\frac{2\kappa\theta}{\sigma^2}} e^{\left(\frac{-2\Delta t_i \kappa}{-\Delta t_i \sigma^2 + e^{\kappa \Delta t_i}(2\kappa + \Delta t_i \sigma^2)} \right) r_0 + \frac{2\kappa\theta\Delta t_i}{\sigma^2}}. \tag{2.9}$$

To complete the proof, we replace $E^P[e^{-(t_i - t_0) r_{t_i}} \mid r_{t_0} = r_0]$ in (1.7) with the RHS of (2.9).

Next, we present some interesting properties of $\mathrm{FIRS}^{E}_{\mathrm{swap}}$.

Corollary 2.2 Suppose that r_t follows the ECIR process (1.5) such that $\theta(t) = \theta$, $\kappa(t) = \kappa$, and $\sigma(t) = \sigma$ for some positive values $q, k,$ and s satisfying the inequality $\kappa\theta \geq \sigma^2 / 2$. Assume $k_1 - k_0 \geq L$. Then $\mathrm{FIRS}^{C}_{\mathrm{swap}}(r_0, t_0; N)$ is always negative for all $r_0 > 0$. Moreover, $\mathrm{FIRS}^{C}_{\mathrm{swap}}(r_0, t_0; N)$ is a strictly increasing function with respect to r_0 on $(0, \infty)$.

Proof. Since $k_1 - k_0 \geq L$, this implies $K_1 \leq 0$. On the other hand, $K_i < 0$ for all $i = 2, \ldots, N$, and the inequality $\left(\frac{2\kappa}{-\Delta t_i \sigma^2 + e^{\kappa \Delta t_i}(2\kappa + \Delta t_i \sigma^2)} \right)^{\frac{2\kappa\theta}{\sigma^2}} \geq 1$ holds for $\Delta t_i \geq 0$. Therefore, we can conclude from (2.8) that $\mathrm{FIRS}^{C}_{\mathrm{swap}}(r_0, t_0; N) < 0$ for all $r_0 > 0$. Next, one can easily verify by applying the first-derivative test to show that $\mathrm{FIRS}^{C}_{\mathrm{swap}}(r_0, t_0; N)$ is a strictly increasing function with respect to r_0 on $(0, \infty)$.

3. Numerical Tests and Discussions

In this section, we investigate the accuracy of our analytical formula (2.1) through numerical experiments. Theoretically, there would be no need to discuss the accuracy of the formula and present numerical results. However, some comparisons with the MC simulations may give readers a sense of verification for the newly found solution. This is particularly for some practitioners who are familiar with the MC method and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison.

In our numerical tests, we consider the ECIR process:

$$dr_t = \kappa(\theta(t) - r_t)dt + \sigma(t)\sqrt{r_t}\,dW_t \tag{3.1}$$

where $\theta(t) = \frac{4}{\kappa}\sigma_0^2 d e^{2\sigma_1 t}$ and $\sigma(t) = \sigma_0 e^{\sigma_1 t}$ with κ, d, σ_0 and σ_1 are positive constants. In our MC simulations, we have employed the simple Euler–Maruyama discretization for the ECIR process (3.1)

$$r_{t_i}(\omega) = r_{t_{i-1}}(\omega) + \kappa(\theta(t_{i-1}) - r_{t_{i-1}}(\omega))\Delta t + \sigma(t_{i-1})\sqrt{r_{t_{i-1}}(\omega)}\sqrt{\Delta t}\, z_{t_i}(\omega) \tag{3.2}$$

for $i = 1, 2, \ldots, m$, a sample path $\omega \in \Omega$, and $\Delta t = T / m$ for some positive integer m, where z_{t_i} is a standard normal random variable. We generate M sample paths of r_t on the interval $[0, T]$ with $T = 1$ by setting $t_0 = 0$ and the parameters $\kappa = 0.03$, $d = 0.34$, $\sigma_0 = 0.01$, and $\sigma_1 = 0.02$. We remark here that the current parameter setting makes the inequality (1.6) holds for all $t \in (0, T]$. This implies that the SDE (3.1) has a path-wise unique strong solution, in which r_t avoids zero a.s. P for all $t \in (0, T]$.

Next, we use M sample paths to compute an approximate of $\mathrm{FIRS}^E_{\mathrm{swap}}(r_0, t_0; N)$ defined by

$$\mathrm{FIRS}^M_{\mathrm{swap}}(r_0, t_0; N, M) := \frac{1}{M}\sum_{k=1}^{M} V_{\mathrm{swap}}(N) = \frac{1}{M}\sum_{k=1}^{M}\left(\sum_{i=1}^{N} K_i e^{-(t_i - t_0) r_{t_i}(\omega_k)}\right) \tag{3.3}$$

for any sample path $\omega_k \in \Omega$. We set $L = 100$ (the notional principal in swap agreement) and $k_0 = 0$ and $k_i = 1, i = 1, \ldots, N$ (the fixed payment made on payment time t_i).

In our MC simulations with M = 10,000, N is varied from 2, 4, and 12 which refer to a half-year, quarterly, and monthly payment in one year, respectively. As displayed in Figure 1, we can clearly observe that the results obtained from our formula (2.1) perfectly match the results from the MC simulations for 20 sample points of $r_0 \in (0, 20]$ and $N = 2, 4, 12$.

Moreover, it should be pointed out from Figure 1 that $\mathrm{FIRS}^E_{\mathrm{swap}}(r_0, t_0; N)$ approaches zero as r_0 increases to infinity for $N = 2, 4, 12$. These results are confirmed by (2.6) in Corollary 2.1 such that an IRS is worth zero when the interest rate becomes very high.

In terms of efficiency of our approach, we compare the computational times of implementing (2.1) and the MC simulations (3.3) for computing averages of relative errors (%) from approximating $\mathrm{FIRS}^E_{\mathrm{swap}}(r_{0,i}, t_0; 12)$ by $\mathrm{FIRS}^M_{\mathrm{swap}}(r_{0,i}, 0; 12, M)$ with 100 sample points, $r_{0,i} = \frac{1}{5} i$, for $i = 1, 2, \ldots, 100$. The values of the model parameters are kept to be the same as in the previous setting except s_1 is increased by 4 in order to measure the efficiency. Table 1 illustrates the computational times for different values of sample paths (M) in MC simulations. In contrast to formidable computational time of 3131.03 seconds using the MC simulations with M = 100,000 to obtain 0.39 % for average of relative errors, our implementing (2.1) just consumed 0.12 seconds; a roughly 25,088 fold reduction in computational time for the sample points. Obviously, our approach can substantially reduce the computational burden by using the MC method and can be implemented efficiently.

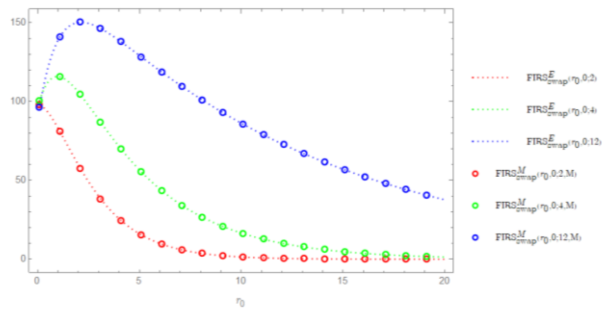

Figure 1. Comparisons of the fair prices of IRSs obtained by formula (2.1) and the MC simulations (3.3)

Table 1. Averages of percentage relative errors and computational times of MC simulations

No. of sample paths (M)	Averages of percentage relative errors (%)	Computational time (seconds)
10,000	0.62	327.76
50,000	0.43	1648.12
100,000	0.39	3131.03

4. Conclusions

The paper has provided an analytical formula for pricing IRSs in terms of bond prices in which the interest rates are assumed to follow the ECIR model. Utilizing our current analytical formula, we analytically investigate some asymptotic properties of the fair price of IRSs when the initial interest rate approaches zero and infinity. Furthermore, we have simplified our IRS pricing formula based on the CIR model and derived a monotonic property. The accuracy and efficiency of the current approach have been tested demonstrating its superiority over the MC method in terms of computational time and effort.

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