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Original Article

An explicit solution of a recurrence differential equation and its application in determining the conditional moments of quadratic variance diffusion processes

Kiattisak Prathom[1] and Sanae Rujivan[1, 2*]

[1] *Division of Mathematics and Statistics, School of Science, Walailak University, Tha Sala, Nakhon Si Thammarat, 80161 Thailand*

[2] *Center of Excellence in Health System and Medical Research Walailak University, Tha Sala, Nakhon Si Thammarat, 80161 Thailand*

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Abstract

This paper investigates solutions of a recurrence differential equation (RDE) of the form:

$$A_1'(t) = a_1 A_1(t)$$

$$A_2'(t) = a_2 A_2(t) + b_2 A_1(t)$$

$$A_{k+2}'(t) = a_{k+2} A_{k+2}(t) + b_{k+2} A_{k+1}(t) + c_{k+2} A_k(t),$$

for $k = 1, 2, ..., K-2$ and any positive integer $K \ge 3$ subject to the initial conditions $A_i(0) = R_i \in \mathbf{R}$ for $i = 1, 2, ..., K$ where $b_i, c_i \in \mathbf{C}$, $a_i \in \mathbf{R}$ and $a_i \ne a_j$ for $i \ne j$. Firstly, we apply Laplace transform to the RDE to obtain a difference equation in Laplace space. Our success in performing Laplace inverse transform leads to an explicit solution of the RDE. Finally, we present an application of our results by deriving closed-form formulas for the conditional moment, variance, covariance, and correlation of quadratic variance diffusion processes which are commonly used for studying model variance or interest rate processes in financial engineering.

Keywords: recurrence differential equation, explicit solution, conditional moments, quadratic variance diffusion processes

1. Introduction

In this paper we investigate solutions of the recurrence differential equation (RDE) of the form:

$$A_1'(t) = a_1 A_1(t)$$

$$A_2'(t) = a_2 A_2(t) + b_2 A_1(t)$$

$$A_{k+2}'(t) = a_{k+2} A_{k+2}(t) + b_{k+2} A_{k+1}(t) + c_{k+2} A_k(t), \tag{1.1}$$

for $k = 1, 2, ..., K-2$ and any positive integer $K \ge 3$ subject to the initial condition

$$A_i(0) = R_i \in \mathbf{R} \tag{1.2}$$

*Corresponding author
Email address: rsanae@wu.ac.th

for $i = 1, 2, \ldots, K$ where $b_i, c_i \in \mathbf{C}$, $a_i \in \mathbf{R}$ and $a_i \neq a_j$ for $i \neq j$ and we denote $\mathbf{R}$ is the set of real numbers and $\mathbf{C}$ is the set of complex numbers.

Solving RDE is a key to understand complex systems. RDE appears in many problems in engineering and finance. Multidimensional modeling of diesel engine is explained as a system of ordinary differential equation which is in an RDE form (Belardini, Bertoli, Corsaro, & D'Ambra, 2005). Mathematical models of chemical kinetics are in the form of RDE (Gobbert, 1970; Edsberg, 1974; Nyengeri, Ndenzako, & Nizigiyimana, 2019; Rehman, 2018). A closed-form formula for the conditional moments of the extended CIR process have been derived by solving an RDE (Rujivan, 2016).

Solutions of a sequence of recurrence relations for first-order ordinary differential equations, which we have call an RDE, have been investigated by Batukhtin *et al.* (2017). They achieved solutions of a slightly different form of RDE compared to RDE (1.1) with a method of successive integration. Rujivan (2016) and Thamrongrat and Rujivan *(*2020*)* adopted a direct integration method to solve RDE (1.1) for a special case that the coefficients $c_{k+2}, j = 1, 2, \ldots, K-2$, are zero. Utilizing a solution of RDE (1.1) subject to the initial condition (1.2) as we shall present later on in this paper, one can derive closed-form formulas for the n^{th} conditional moment for any non-negative integer *n*, variance, covariance, and correlation of quadratic variance diffusion processes (QVD processes) introduced by Filipovic, Gourier, and Mancini (2016), which are usually found in modelling variance or interest rate processes appeared in financial engineering. These formulas can be used in pricing financial derivatives when the variance and interest rate processes are assumed to follow the QVD process.

There are two possible analytical approaches to solve RDE (1.1), investigating the eigenvalues of the matrix corresponding to (1.1) (see Chapter1 in Waltman (2004)) and using Laplace transformation (see Chapter 2 in Schiff (1999)). The former one is hard to handle due to the large size of *K* x *K* matrix for a fixed positive integer $K \geq 3$. Consequently, finding a closed-form of the eigenvalues of the matrix *K* x *K* corresponding to (1.1) which are used to represent the solution is a very difficult task. Alternatively, our proposed method is started by applying Laplace transform to RDE (1.1) under the initial condition (1.2) to obtain a difference equation in Laplace space. Then we apply the solution of linear difference equations with variable coefficients presented in Mallik (1998) to derive an explicit solution of the obtained difference equation. We next simplify the solution and then employing Laplace inversion to obtain an explicit solution of RDE (1.1).

The paper is organized as follows. In Section 2, we derive an explicit solution of RDE (1.1) subject to the initial condition (1.2). Our results in this section can be used to construct Mathematica codes which help us to double check the accuracy of the theorems. Examples shown in Section 2 illustrate how the theorems can be applied. Solution in each example can be achieved by our Mathematica programming codes or by manual calculation. Applications are given in Section 3 including closed-form formulas for the conditional moment, variance, covariance, and correlation of QVD processes. The conclusions of the paper are stated in Section4.

2. Main Results

We first introduce the index sets used in Mallik (1998) to present an explicit solution of RDE (1.1) subject to the initial condition (1.2).

Definition 2.1

Let i and r be positive integers and k be a non-negative integer such that $i \leq k+1$ and $r \leq k+2-i$. We define the following:

$$\begin{aligned} &Case\ k \geq 1: L_{i,k,r} = \{(l_1, \ldots, l_r) \mid l_1 + \cdots + l_r = k+2-i,\ 1 \leq l_1, \ldots, l_r \leq 2,\ l_r \geq 3-i\} \\ &Case\ k = 0: i = r = 1,\ L_{1,0,1} = \{(1)\}. \end{aligned} \tag{2.1}$$

The index $L_{i,k,r}$ is the set of *r* partitions of $k+2-i$ where each entry of the partition is an integer 1 or 2. To illustrate the definition, we provide the following examples:

$L_{3,4,1} = f = L_{1,1,2}$
$L_{3,4,2} = \{(1,2),(2,1)\}$
$L_{3,4,3} = \{(1,1,1)\}$.

In terms of programming, each index set can be computed by a computer programme. For fixed positive integers $i, k \geq 0$, and *r*, $L_{i,k,r}$ can be found by using Code 2.1 written by using Mathematica provided in Appendix.

The important result in Mallik (1998) which is used in this paper is stated as follows.

Proposition 2.1 (Mallik, 1998)

The solution of the difference equation

$$y_{k+N}=\sum_{j=1}^{N}a_{k,j}y_{k+N-j}+x_{k+N},\quad k\geq 1$$

of $N\geq 2$ with complex coefficients $a_{k,j}$, $j=1,\ldots,N$, complex forcing term x_{k+N}, and complex initial values $y_1,\ldots,y_N$ is given by

$$y_{k+N}=\sum_{j=1}^{N}d_{k,j}y_{N+1-j}+\sum_{j=2-k}^{0}d_{k,j}x_{N+1-j}+x_{k+N},\quad k\geq 1,$$

where

$$d_{k,j}=\sum_{r=1}^{k+j-1}\sum_{\substack{(l_1,\ldots,l_r)\\ 1\leq l_1,\ldots,l_r\leq N\\ l_r\geq j\\ l_1+l_2+\cdots+l_r=k+j-1}}\left[\prod_{m=1}^{r}d_{k+l_m-\sum_{n=1}^{m}l_n,l_m}\right]$$

for $j=2-k,\ldots,N,\ k\geq 1.$

The next theorem provides a solution of RDE (1.1) under the initial condition (1.2).

Theorem 2.1

A solution of RDE (1.1) subject to the initial condition (1.2) can be expressed as

$$A_1(t)=R_1e^{a_1t}$$
$$A_{k+2}(t)=R_{k+2}e^{a_{k+2}t}+\sum_{r=1}^{k}\sum_{L_{2,k,r}}\left[\sum_{m=1}^{r}\beta_{k+2+l_m-\sum_{n=1}^{m}l_n}e^{a_{k+2+l_m-\sum_{n=1}^{m}l_n}t}+\beta_1e^{a_1t}+\beta_2e^{a_2t}\right]$$
$$+\sum_{i=1}^{k+1}\sum_{r=1}^{k+2-i}\sum_{L_{i,k,r}}\left[\sum_{m=1}^{r}\alpha_{k+2+l_m-\sum_{n=1}^{m}l_n,i}e^{a_{k+2+l_m-\sum_{n=1}^{m}l_n}t}+\alpha_{i,i}e^{a_it}\right],\qquad (2.2)$$

for $k=0,1,\ldots,K-2$ and any positive integer $K\geq 3$ where

$$\alpha_{j,i}=\lim_{s\to a_j}\left[(s-a_j)\prod_{m=1}^{r}d_{k+l_m-\sum_{n=1}^{m}l_n,l_m}x_i\right]\quad for\ (l_1,\ldots,l_r)\in L_{i,k,r},$$

$$\beta_j=\lim_{s\to a_j}\left[(s-a_j)\prod_{m=1}^{r}d_{k+l_m-\sum_{n=1}^{m}l_n,l_m}x_1d_{0,1}\right]\quad for\ (l_1,\ldots,l_r)\in L_{2,k,r},$$

and

$$d_{k,1}=\frac{b_{k+2}}{s-a_{k+2}},\quad d_{k,2}=\frac{c_{k+2}}{s-a_{k+2}},\quad x_k=\frac{R_k}{s-a_k}.$$

Proof. Let $K\geq 3$ be a positive integer. For $k=1,2,\ldots,K-2,$ we let

$$d_{k,1}=\frac{b_{k+2}}{s-a_{k+2}},\quad d_{k,2}=\frac{c_{k+2}}{s-a_{k+2}},\quad x_k=\frac{R_k}{s-a_k}$$

where we set $x_0=0$.

Let $y_i(s)$ be Laplace transform of the unknown function $A_i(t)$; i.e.,

$$y_i(s)=\int_0^{\infty}e^{-st}A_i(t)dt$$

where s is a complex number frequency parameter. Hence, solving RDE (1.1) subject to the initial condition (1.2) is equivalent to solving the difference equation with variable coefficients in Laplace space such that

$$y_1(s) = x_1$$
$$y_{k+2}(s) = d_{k,1}y_{k+1}(s) + d_{k,2}y_k(s) + x_{k+2}$$

for $k = 0,1,...,K-2$. The above linear difference equation with variable coefficients can be solved by using Proposition 2.1 with $N = 2$ to obtain an explicit solution as

$$y_{k+2}(s) = \sum_{j=1}^{2} D_{k,j}y_{3-j}(s) + \sum_{j=2-k}^{0} D_{k,j}x_{3-j} + x_{k+2} \tag{2.3}$$

where

$$D_{k,j} = \sum_{r=1}^{k+j-1} \sum_{L_{3-j,k,r}} \left[\prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m} \right].$$

Equation (2.3) can be rewritten as follows:

$$y_{k+2}(s) = x_{k+2} + d_{0,1}D_{k,1}x_1 + \sum_{i=1}^{k+1} D_{k,3-i}x_i \tag{2.4}$$

for $k = 0,1,...,K-2$.

Note that

$$D_{k,3-i}x_i = \sum_{r=1}^{k+2-i} \sum_{L_{i,k,r}} \left[\prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m} \frac{R_i}{s-a_i} \right].$$

Since $a_q \neq a_r$ for all $q \neq r$, the denominators of $d_{k+l_m-\sum_{n=1}^{m} l_n, l_m}$ are distinct. Moreover, each denominator of $d_{k+l_m-\sum_{n=1}^{m} l_n, l_m}$ cannot be $s - a_i$ because $k + l_m - \sum_{n=1}^{m} l_n > i - 2$. Then

$$D_{k,3-i}x_i = \sum_{r=1}^{k+2-i} \sum_{L_{i,k,r}} \left[\sum_{m=1}^{r} \frac{\alpha_{k+2+l_m-\sum_{n=1}^{m} l_n, i}}{s - a_{k+2+l_m-\sum_{n=1}^{m} l_n}} + \frac{\alpha_{i,i}}{s-a_i} \right] \tag{2.5}$$

where

$$\alpha_{j,i} = \lim_{s \to a_j} \left[(s-a_j) \prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m} x_i \right] \text{ for } (l_1,\dots,l_r) \in L_{i,k,r}.$$

Using the similar idea of getting (2.5), we have

$$D_{k,1}x_1d_{0,1} = \sum_{r=1}^{k} \sum_{L_{2,k,r}} \left[\sum_{m=1}^{r} \frac{\beta_{k+2+l_m-\sum_{n=1}^{m} l_n}}{s - a_{k+2+l_m-\sum_{n=1}^{m} l_n}} + \frac{\beta_1}{s-a_1} + \frac{\beta_2}{s-a_2} \right] \tag{2.6}$$

where

$$\beta_j = \lim_{s \to a_j} \left[(s-a_j) \prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m} x_1 d_{0,1} \right] \text{ for } (l_1,\dots,l_r) \in L_{2,k,r}.$$

By substituting (2.5) and (2.6) in (2.4) and then applying Laplace inverse transformation, we then obtain

$$A_{k+2}(t) = R_{k+2}e^{a_{k+2}t} + \sum_{r=1}^{k}\sum_{L_{2,k,r}}\left[\sum_{m=1}^{r}\beta_{k+2+l_m-\sum_{n=1}^{m}l_n} e^{a_{k+2+l_m-\sum_{n=1}^{m}l_n}t} + \beta_1 e^{a_1 t} + \beta_2 e^{a_2 t}\right]$$
$$+\sum_{i=1}^{k+1}\sum_{r=1}^{k+2-i}\sum_{L_{i,k,r}}\left[\sum_{m=1}^{r}\alpha_{k+2+l_m-\sum_{n=1}^{m}l_n,i} e^{a_{k+2+l_m-\sum_{n=1}^{m}l_n}t} + \alpha_i e^{a_i t}\right],$$

for $k = 0,1,...,K-2$. It easy to check that $A_1(t) = R_1e^{a_1t}$. Therefore, RDE (1.1) subject to the initial condition (1.2) has an explicit solution as shown in (2.2).

In terms of programming with Mathematica, Code 2.2 provided in Appendix can be used to derive $A_k(t), k = 1,2,\dots,K$ as written in (2.2) where we compute $A_k(t)$ by using the module A[k_,t_,La_,Lb_,Lc_,LR_] in Code 2.2 such that

La $\to \{a_1,\dots,a_K\}$, Lb$\to \{b_1,\dots,b_K\}$, Lc$\to \{c_1,\dots,c_K\}$, and LR$\to \{R_1,\dots,R_K\}$.

Example 2.1

Consider an RDE with complex coefficients as follows:

$$A_1'(t) = 2A_1(t)$$
$$A_2'(t) = 3A_2(t) + \mathbf{i}A_1(t)$$
$$A_3'(t) = 4A_3(t) + (1+\mathbf{i})A_2(t) + A_1(t)$$

subject to the initial conditions

$$A_1(0) = 1, A_2(0) = 2, A_3(0) = 3$$

where $\mathbf{i} = \sqrt{-1}$. It is obvious that $A_1(t) = e^{2t}$. Under index set $L_{1,0,1}$, $\alpha_{2,1}$ and $\alpha_{1,1}$ can be computed by using Theorem 2.1 as follows:

$$\alpha_{2,1} = \lim_{s\to a_2}\left[(s-a_2)d_{0,1}x_1\right] = \frac{b_2 r_1}{s-a_1} = \mathbf{i}, \quad \alpha_{1,1} = \lim_{s\to a_1}\left[(s-a_1)d_{0,1}x_1\right] = \frac{b_2 r_1}{s-a_2} = -\mathbf{i}.$$

We then have

$$A_2(t) = 2e^{3t} + \sum_{L_{1,0,1}}\left[\alpha_{2,1}e^{3t} + \alpha_{1,1}e^{2t}\right] = 2e^{3t} + \mathbf{i}e^{3t} - \mathbf{i}e^{2t} = (2+\mathbf{i})e^{3t} - \mathbf{i}e^{2t}.$$

As shown in (2.2) of Theorem 2.1, $A_3(t)$ depends upon the index sets $L_{2,1,1}, L_{1,1,1}$, and $L_{1,1,2}$. Since $L_{1,1,2}$ is the empty set, we get

$$A_3(t) = 3e^{4t} + \sum_{L_{2,1,1}}\left[\beta_3 e^{4t} + \beta_2 e^{3t} + \beta_1 e^{2t}\right] + \sum_{L_{1,1,1}}\left[\alpha_{3,1}e^{4t} + \alpha_{1,1}e^{2t}\right] + \sum_{L_{2,1,1}}\left[\alpha_{3,2}e^{4t} + \alpha_{2,2}e^{3t}\right].$$

By Theorem 2.1, we have the following:

- Under index set $L_{2,1,1}$, we have $\beta_3 = \frac{\mathbf{i}-1}{2}$, $\beta_2 = 1-\mathbf{i}$, $\beta_3 = \frac{\mathbf{i}-1}{2}$.
- Under index set $L_{1,1,1}$, we have $\alpha_{3,1} = \frac{1}{2}$, $\alpha_{1,1} = -\frac{1}{2}$.
- Under index set $L_{2,1,1}$, we have $\alpha_{3,2} = 2(1+\mathbf{i})$, $\alpha_{2,2} = -2(1+\mathbf{i})$.

Hence,

$$A_3(t) = 3e^{4t} + \frac{\mathbf{i}-1}{2}e^{4t} + (1-\mathbf{i})e^{3t} + \frac{\mathbf{i}-1}{2}e^{2t} + \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} + 2(1+\mathbf{i})e^{4t} - 2(1+\mathbf{i})e^{3t}$$
$$= 5\left(1+\frac{\mathbf{i}}{2}\right)e^{4t} + \frac{\mathbf{i}-2}{2}e^{2t} - (1+3\mathbf{i})e^{3t}.$$

By applying Code 2.2, we have

$$A_1(t) = \text{A[1,t,\{2,3,4\},\{0,Sqrt[-1],1+Sqrt[-1]\},\{0,0,1\},\{1,2,3\}]}$$
$$= e^{2t},$$

$$A_2(t) = \text{A[2,t,\{2,3,4\},\{0,Sqrt[-1],1+Sqrt[-1]\},\{0,0,1\},\{1,2,3\}]}$$
$$= (2+\mathbf{i})e^{3t} - \mathbf{i}e^{2t},$$

$$A_3(t) = \text{A[3,t,\{2,3,4\},\{0,Sqrt[-1],1+Sqrt[-1]\},\{0,0,1\},\{1,2,3\}]}$$
$$= 5\left(1+\frac{\mathbf{i}}{2}\right)e^{4t} + \frac{\mathbf{i}-2}{2}e^{2t} - (1+3\mathbf{i})e^{3t}.$$

The results from Code 2.2 are the same as computed directly from Theorem 2.1.

We note here that the condition $a_i \neq a_j$ for $i \neq j$ in Theorem 2.1 is sufficient to get Equation (2.5). Without the condition, (2.5) will not have a closed simple form because the expression $\prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m}$ in the proof of Theorem 2.1 can be written as $\prod_j \frac{l_j}{(s-a_j)^{k_j}}$ if we have k_j numbers of a_j with constant l_j. Although we can write the product of $\frac{l_j}{(s-a_j)^{k_j}}$ as the summation we obtained in (2.5), the procedure produces a complicated formula, which is not suitable to be written in a limited space, due to the partial fraction having to be determined carefully case by case. Therefore, we let the case when some a_j are equal to be our future work.

For some special cases of initial conditions (1.2) that shall be applied to derive a closed-form formula for the conditional moments of QVD process in the next section, we deduce the following corollary.

Corollary 2.1

The RDE

$$\begin{aligned} A_1'(t) &= a_1 A_1(t) \\ A_2'(t) &= a_2 A_2(t) + b_2 A_1(t) \\ A_{k+2}'(t) &= a_{k+2} A_{k+2}(t) + b_{k+2} A_{k+1}(t) + c_{k+2} A_k(t) \end{aligned} \tag{2.7}$$

for $k = 1, 2, ..., K-2$ and any positive integer $K \geq 3$ subject to the initial conditions

$$A_1(0) = 1,\ A_i(0) = 0,\ i = 2, \ldots, K \tag{2.8}$$

where $b_i, c_i \in \mathbf{C},\ a_i \in \mathbf{R}\ and\ a_i \neq a_j\ for\ i \neq j,$ has a solution in the form

$$A_1(t) = e^{a_1 t}$$
$$A_{k+2}(t) = \sum_{r=1}^{k} \sum_{L_{2,k,r}} \left[\sum_{m=1}^{r} \beta_{k+2+l_m-\sum_{n=1}^{m} l_n} e^{a_{k+2+l_m-\sum_{n=1}^{m} l_n} t} + \beta_1 e^{a_1 t} + \beta_2 e^{a_2 t} \right]$$
$$+ \sum_{r=1}^{k+1} \sum_{L_{1,k,r}} \left[\sum_{m=1}^{r} \alpha_{k+2+l_m-\sum_{n=1}^{m} l_n} e^{a_{k+2+l_m-\sum_{n=1}^{m} l_n} t} + \alpha_1 e^{a_1 t} \right] \tag{2.9}$$

$$for\ k = 0, 1, ..., K-2\ where\ \alpha_j = \lim_{s \to a_j} \left[\frac{s-a_j}{s-a_1} \prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m} \right] for\ (l_1, \ldots, l_r) \in L_{1,k,r},$$

$$\beta_j = \lim_{s\to a_j}\left[d_{0,1}\left(\frac{s-a_j}{s-a_1}\right)\prod_{m=1}^{r} d_{k+l_m-\sum_{n=1}^{m} l_n, l_m}\right] for\ (l_1,\ldots,l_r)\in L_{2,k,r},\ d_{k,1} = \frac{b_{k+2}}{s-a_{k+2}},\ d_{k,2} = \frac{c_{k+2}}{s-a_{k+2}}.$$

Proof. The proof is rather trivial by using Theorem 2.1, thus omitted here.

Example 2.2

Consider an RDE

$$A_1'(t) = 2A_1(t)$$
$$A_2'(t) = 3A_2(t) + A_1(t)$$
$$A_3'(t) = 4A_3(t) + A_2(t) + A_1(t)$$

subject to the initial conditions

$$A_1(0) = 1,\ A_2(0) = A_3(0) = 0.$$

By Corollary 2.1,

$$\begin{aligned} A_1(t) &= e^{2t} \\ A_2(t) &= \sum_{L_{1,0,1}}\left[\alpha_2 e^{3t} + \alpha_1 e^{2t}\right] = e^{3t} - e^{2t} \\ A_3(t) &= \sum_{L_{2,1,1}}\left[\beta_3 e^{4t} + \beta_2 e^{3t} + \beta_1 e^{2t}\right] + \sum_{L_{1,1,1}}\left[\alpha_3 e^{4t} + \alpha_1 e^{2t}\right] \\ &= e^{4t} - e^{3t}. \end{aligned}$$

3. Application in Determining the Conditional Moments of QVD Processes

Firstly, we derive an explicit solution of a partial differential equation (PDE) by applying Corollary 2.1.

Theorem 3.1

Let n be a positive integer, $T > 0$ and $u^{(n)}(v,t)$ be a solution of the PDE:

$$-\frac{\partial u^{(n)}(v,\tau)}{\partial \tau} + \frac{1}{2}(a+\alpha v + cv^2)\frac{\partial^2 u^{(n)}(v,\tau)}{\partial v^2} + (b+\beta v)\frac{\partial u^{(n)}(v,\tau)}{\partial v} = 0. \tag{3.1}$$

for $(v,\tau)\in(0,\infty)\times[0,T]$ subject to an initial condition

$$u^{(n)}(v,0) = v^n \tag{3.2}$$

where β *and* c *are positive real numbers,* $a,\alpha,b\in\mathbf{R}$. *Then* $u^{(n)}(v,\tau)$ *can be expressed as*

$$u^{(n)}(v,\tau) = \sum_{k=0}^{n} A_{k+1}(\tau)v^{n-k} \tag{3.3}$$

in which the coefficient functions $A_{k+1}(t), k = 0,\ldots,n,$ satisfy the RDE (2.7) and the initial conditions (2.8) by setting

$$a_{k+1} = \left(\frac{1}{2}c(n-k-1)+\beta\right)(n-k) \tag{3.4}$$

$$b_{k+1} = \left(\frac{1}{2}\alpha(n-k)+b\right)(n-k+1) \tag{3.5}$$

$$c_{k+1} = \frac{1}{2}a(n-k+2)(n-k+1). \tag{3.6}$$

Proof. We first suppose that

$$u^{(n)}(v,\tau) = \sum_{k=-\infty}^{\infty} A_{k+1}(\tau)v^{n-k}. \tag{3.7}$$

Since $u^{(n)}(v,t)$ must satisfy the initial condition (3.2). This implies that the coefficient functions $A_{k+1}(t), k = 0,\ldots,n$, satisfy the initial conditions

$$A_1(0) = 1 \text{ and } A_k(0) = 0 \text{ for } k \neq 1. \tag{3.8}$$

By adopting the method proposed in Rujivan (2016), we compute the partial derivatives of $u^{(n)}(v,t)$ appeared in the PDE (3.1) based on the formula (3.7) as

$$\frac{\partial u^{(n)}(v,\tau)}{\partial \tau} = \sum_{k=-\infty}^{\infty} A'_{k+1}(\tau)v^{n-k} \tag{3.9}$$

$$\frac{\partial u^{(n)}(v,\tau)}{\partial v} = \sum_{k=-\infty}^{\infty} (n-k)A_{k+1}(\tau)v^{n-k-1} \tag{3.10}$$

$$\frac{\partial^2 u^{(n)}(v,\tau)}{\partial v^2} = \sum_{k=-\infty}^{\infty} (n-k)(n-k-1)A_{k+1}(\tau)v^{n-k-2}. \tag{3.11}$$

Next, we insert the partial derivatives (3.9)-(3.11) into the PDE (3.1) and then collecting the coefficients of $v^{n-k}, k = 0, \pm 1, \ldots$, gives us

$$\sum_{k=-\infty}^{\infty} \left[-A'_{k+1}(\tau) + a_{k+1}A_{k+1}(\tau) + b_{k+1}A_k(\tau) + c_{k+1}A_{k-1}(\tau)\right]v^{n-k} = 0 \tag{3.12}$$

where a_{k+1}, b_{k+1}, and c_{k+1} are given in (3.4)-(3.6), respectively.

Let consider $\sum_{k=-N}^{N} A_{k+1}(\tau)v^{n-k}$ where $N \in \mathbb{N}$ and $N > n+1$. From (3.12),

$$\begin{aligned}
A'_{N+1}(\tau) &= a_{N+1}A_{N+1}(\tau) + b_{N+1}A_N(\tau) + c_{N+1}A_{N-1}(\tau) \\
&\vdots \\
A'_{n+2}(\tau) &= a_{n+2}A_{n+2}(\tau) + b_{n+2}A_{n+1}(\tau) + c_{n+2}A_n(\tau) \\
A'_{n+1}(\tau) &= a_{n+1}A_{n+1}(\tau) + b_{n+1}A_n(\tau) + c_nA_{n-1}(\tau) \\
A'_n(\tau) &= a_nA_n(\tau) + b_nA_{n-1}(\tau) + c_nA_{n-2}(\tau) \\
&\vdots \\
A'_1(\tau) &= a_1A_1(\tau) + b_1A_0(\tau) + c_1A_{-1}(\tau) \\
&\vdots \\
A'_{1-N}(\tau) &= a_{1-N}A_{1-N}(\tau)
\end{aligned}$$

The right-hand side of the last equation has only one term because $\sum_{k=-N}^{N} A_{k+1}(\tau)v^{n-k}$ has neither $A_{-N}(\tau)$ nor $A_{-N-1}(\tau)$. From (3.5)-(3.6), we have that the coefficients $b_{n+2} = 0 = c_{n+2}$. From the initial conditions (3.8), it follows that

$A_{n+2}(t) = \dots = A_{N+1}(t) = 0 = A_{1-N}(t) = \dots = A_0(t)$.

Thus, instead of considering $\sum_{k=-N}^{N} A_{k+1}(\tau)v^{n-k}$ where $N \in \mathbb{N}$ and $N > n+1$, we can just consider $\sum_{k=0}^{n} A_{k+1}(\tau)v^{n-k}$. This implies that (3.7) can be rewritten as follows:

$$u^{(n)}(v,\tau) = \sum_{k=0}^{n} A_{k+1}(\tau)v^{n-k}.$$

Hence, the solution of the PDE (3.1) is in the form (3.3) where the coefficient functions $A_{k+1}(\tau), k = 0,\dots,n,$ solve the RDE (2.11) subject to the initial conditions (2.12). Since b and c are positive real numbers, we have $a_i \neq a_j$ for $i \neq j$. Hence, the coefficient functions $A_{k+1}(\tau), k = 0,\dots,n,$ can be derived by using (2.9) in Corollary 2.1.

In terms of financial modelling, QVD processes have been used to model the stochastic behaviours of interest rates as proposed in Filipovic *et al.* (2016). The QVD process $(v_t(w))_{t\geq 0}$ can be described by a stochastic differential equation (SDE).

$$dv_t = (b + \beta v_t)\,dt + \sqrt{a + \alpha v_t + c v_t^2}\,dW_t \tag{3.13}$$

for parameters β and c are positive real numbers and $a, \alpha, b \in \mathbf{R}$ where $(W_t(\omega))_{t\geq 0}$ is a standard Brownian motion under a probability space $(\Omega, F, P)$with a filtration $(F_t)_{t\geq 0}$. In some cases of pricing financial derivatives based on a stochastic interest rate model, we need to calculate an n^{th} conditional moment of the variance or interest rate process in the form of

$$E^P[v_T^n \mid F_t] = E^P[v_T^n \mid v_t = v]$$

for some $n \in \mathbb{N}$, $v > 0$ and $0 \leq t \leq T$. We set $\tau = T - t$ and define a real-valued function

$$u^{(n)}(v,\tau) := E^P[v_T^n \mid v_t = v] \tag{3.14}$$

for $(v,\tau) \in (0,\infty) \times [0,T]$.

Applying the Feynman–Kac theorem as presented in Rujivan (2016) but to the QVD process (3.13), we have that $u^{(n)}(v,t)$ satisfies the PDE (3.1) subject to the initial condition (3.2). It should be remarked here that the initial condition (3.2) is required for determining the conditional expectation (3.14) when $t = T$. Therefore, an n^{th} conditional moment of the QVD process (3.13) can be derived by using (2.9) in Corollary 2.1 in which the coefficient functions $A_{k+1}(\tau), k = 0,\dots,n,$ solve the RDE (2.7) subject to the initial conditions (2.8).

It should be noted that the QVD process (3.13) becomes the Cox-Ingersoll-Ross (CIR) process when the parameters a and c vanish. In this case, Theorem 2.4 in Rujivan (2016) provides a closed-form formula for the n^{th} conditional moment of CIR process for any non-negative integer *n*. On the other hand, Theorem 3.1 in this paper can be used to derive the conditional moment, variance, covariance, and correlation of QVD processes for any real number a and positive real number c as shown in the following examples.

Example 3.1

By applying Theorem 3.1, the 1st conditional moment of QVD process (3.13) can be expressed as

$$E^P[v_T \mid v_t = v] = u^{(1)}(v,\tau) = \sum_{k=0}^{1} A_{k+1}^{(1)}(\tau)v^{1-k} = A_1^{(1)}(\tau)v + A_2^{(1)}(\tau) \tag{3.15}$$

for $\tau = T - t$ and $0 \leq t \leq T$ where the coefficient functions can be computed by using Code 2.2 with setting (3.4)-(3.6) as

$$A_1^{(1)}(\tau) = \mathrm{e}^{\beta\tau} \tag{3.16}$$

$$A_2^{(1)}(\tau) = \frac{b\left(-1+\mathrm{e}^{\beta\tau}\right)}{\beta} \tag{3.17}$$

Similarly, the 2nd conditional moment of QVD process (3.13) can be expressed as

$$E^P[v_T^2 \mid v_t = v] = u^{(2)}(v,\tau) = \sum_{k=0}^{2} A_{k+1}^{(2)}(\tau)v^{2-k} = A_1^{(2)}(\tau)v^2 + A_2^{(2)}(\tau)v + A_3^{(2)}(\tau) \tag{3.18}$$

for $v > 0,\ \tau = T - t$ and $0 \le t \le T$ where the coefficient functions can be computed by using Code 2.2 with setting (3.4)-(3.6) as

$$A_1^{(2)}(\tau) = \mathrm{e}^{(c+2\beta)\tau} \tag{3.19}$$

$$A_2^{(2)}(\tau) = \frac{\mathrm{e}^{\beta\tau}\left(-1+\mathrm{e}^{(c+\beta)\tau}\right)\left(2b+\alpha\right)}{c+\beta} \tag{3.20}$$

$$A_3^{(2)}(\tau) = \frac{a\left(-1+\mathrm{e}^{(c+2\beta)\tau}\right)\beta\left(c+\beta\right)+2b^2\left(c-c\mathrm{e}^{\beta\tau}+\left(1-2\mathrm{e}^{\beta\tau}+\mathrm{e}^{(c+2\beta)\tau}\right)\beta\right)+b\alpha\left(c-c\mathrm{e}^{\beta\tau}+\left(1-2\mathrm{e}^{\beta\tau}+\mathrm{e}^{(c+2\beta)\tau}\right)\beta\right)}{\beta\left(c+\beta\right)\left(c+2\beta\right)}. \tag{3.21}$$

It should be remarked here that when we set $b = k\theta, b = -k,$ and $a = s^2$ for $s > 0$, the QVD process (3.13) becomes a CIR process as presented in Rujivan (2016):

$$dv_t = \kappa(\theta - v_t)\,dt + \sigma\sqrt{v_t}\,dW_t.$$

With the parameter setting, the 1st and 2nd conditional moments of the CIR process can be immediately obtained by using (3.15) and (3.18), respectively, where

$$A_1^{(1)}(\tau) = \mathrm{e}^{-\kappa\tau},\ A_2^{(1)}(\tau) = \theta - \mathrm{e}^{-\kappa\tau}\theta$$

and

$$A_1^{(2)}(\tau) = \mathrm{e}^{-2\kappa\tau}$$

$$A_2^{(2)}(\tau) = \frac{\mathrm{e}^{-2\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)\left(2\theta\kappa+\sigma^2\right)}{\kappa}$$

$$A_3^{(2)}(\tau) = \frac{\mathrm{e}^{-2\kappa\tau}\left(-1+\mathrm{e}^{\kappa\tau}\right)^2\theta\left(2\theta\kappa+\sigma^2\right)}{2\kappa}.$$

One can verify that our results obtained above are the same as produced by using Theorem 2.4 in Rujivan (2016) for the 1st and 2nd conditional moments of the CIR process.

Example 3.2

The conditional variance of QVD process (3.13) is defined by

$$\mathrm{Var}^P[v_T \mid F_t] := E^P[(v_T - E^P[v_T \mid F_t])^2 \mid F_t] \tag{3.22}$$

for $0 \le t \le T$. By the definition of the conditional variance, one can show that

$$\mathrm{Var}^P[v_T \mid v_t = v] = u^{(2)}(v,\tau) - (u^{(1)}(v,\tau))^2 \tag{3.23}$$

for $v > 0$ and $\tau = T - t$ where $u^{(1)}(v,\tau)$ and $u^{(2)}(v,\tau)$ are given in (3.15) and (3.18), respectively.

Example 3.3

The conditional covariance of QVD process (3.13) is defined by

$$\mathrm{Cov}^P[v_{T_1}, v_{T_2} \mid F_t] := E^P[(v_{T_1} - E^P[v_{T_1} \mid F_t])(v_{T_2} - E^P[v_{T_2} \mid F_t]) \mid F_t] \tag{3.24}$$

for $0 \le t < T_1 \le T_2$. We first consider

$$E^P[(v_{T_1} - E^P[v_{T_1} \mid F_t])(v_{T_2} - E^P[v_{T_2} \mid F_t]) \mid F_t] = E^P[v_{T_1} v_{T_2} \mid F_t] - E^P[v_{T_1} \mid F_t] E^P[v_{T_2} \mid F_t]. \tag{3.25}$$

By using the tower property for the conditional expectation (see on page 29 in Brzezniak and Zastawniak (2000)), we have

$$E^P[v_{T_1} v_{T_2} \mid F_t] = E^P[v_{T_1} E^P[v_{T_2} \mid F_{T_1}] \mid F_t]. \tag{3.26}$$

Applying Theorem 3.1, we have

$$E^P[v_{T_2} \mid F_{T_1}] = E^P[v_{T_2} \mid v_{T_1}] = u^{(1)}(v_{T_1}, \tau_1) = \sum_{k=0}^{1} A_k^{(1)}(\tau_2) v_{T_1}^k \tag{3.27}$$

where $\tau_2 = T_2 - T_1$. Inserting (3.17) into (3.16) gives us

$$E^P[v_{T_1} v_{T_2} \mid v_t = v] = \sum_{k=0}^{1} A_k^{(1)}(\tau_2) E^P[v_{T_1}^{k+1} \mid F_t] = \sum_{k_2=0}^{1} \sum_{k_1=0}^{k_2+1} A_{k_2+1}^{(1)}(\tau_2) A_{k_1+1}^{(1)}(\tau_1) v^{k_1} \tag{3.28}$$

for $v > 0$ where $\tau_2 = T_2 - T_1$. By using (3.15)-(3.18), we now obtain

$$\mathrm{Cov}^P[v_{T_1}, v_{T_2} \mid v_t = v] = \sum_{k_2=0}^{1} \sum_{k_1=0}^{k_2+1} A_{k_2+1}^{(1)}(\tau_2) A_{k_1+1}^{(1)}(\tau_1) v^{k_1} - u_1^{(1)}(v,\tau_1) u_1^{(1)}(v,\tau_2) \tag{3.29}$$

where $u^{(1)}(v,t_1)$ and $u^{(1)}(v,t_2)$ can be computed by using (3.15).

Example 3.4

The conditional correlation of QVD process (3.13) is defined by

$$\mathrm{Corr}^P[v_{T_1}, v_{T_2} \mid F_t] := \frac{\mathrm{Cov}^P[v_{T_1}, v_{T_2} \mid F_t]}{\sqrt{\mathrm{Var}^P[v_{T_1} \mid F_t]}\sqrt{\mathrm{Var}^P[v_{T_2} \mid F_t]}} \tag{3.30}$$

or $0 \le t < T_1 \le T_2$. By applying (3.23) and (3.29), we immediately obtain that

$$\mathrm{Corr}^P[v_{T_1}, v_{T_2} \mid v_t = v] = \frac{\sum_{k_2=0}^{1} \sum_{k_1=0}^{k_2+1} A_{k_2+1}^{(1)}(\tau_2) A_{k_1+1}^{(1)}(\tau_1) v^{k_1} - u_1^{(1)}(v,\tau_1) u_1^{(1)}(v,\tau_2)}{\sqrt{u^{(2)}(v,\tau_1) - (u^{(1)}(v,\tau_1))^2}\sqrt{u^{(2)}(v,\hat{\tau}_2) - (u^{(1)}(v,\hat{\tau}_2))^2}} \tag{3.31}$$

for $v > 0$ where and $\tau_1 = T - T_1, \tau_2 = T_2 - T_1$, and $\hat{\tau}_2 = T_2 - t$.

4. Conclusions

In this paper we have derived an explicit solution of RDE (1.1) subject to the initial condition (1.2) by using Laplace transform method and solution of difference equation proposed by Mallik (1998). Moreover, we have provided Mathematica codes for computing the solutions of RDE (1.1) with several examples. Finally, we have demonstrated an application of our results by deriving closed-form formulas for the conditional moments, variance, covariance, and correlation of QVD processes.

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References

Batukhtin, A., Batukhtina, I., Bass, M., Batukhtin, S., Safronov, P., & Baranovskaya, M. (2017). Solving a Sequence of Recurrence Relations for First-Order Differential Equations. *EURASIA Journal of Mathematics, Science and Technology Education*, *13*(11), 7179-7191. doi: 10.12973/ejmste/79043

Belardini, P., Bertoli, C., Corsaro, S., & D'Ambra, P. (2005). The Impact of Different Stiff ODE Solvers in Parallel Simulation of Diesel Combustion. *High Performance Computing and Communications Lecture Notes in Computer Science*, 958–968. doi: 10.1007/11557654_107

Brzezniak, Z., & Zastawniak, T. (2000). *Basic Stochastic Processes*. London, England: *Springer.*

Edsberg, L. (1974). Integration Package for Chemical Kinetics. *Stiff Differential Systems,* 81-94. doi: 10.1007/978-1-4684-2100-2_7

Filipovic, D., Gourier, E., & Mancini, L. (2016). Quadratic variance swap models. *Journal of Financial Economics*, *119*(1), 44–68. doi: 10.1016/j.jfineco.2015.08.015

Gobbert, K. (1970, January 1). Robertson's example for stiff differential equations. Retrieved from https://www.semanticscholar.org/paper/ROBERTSON'S-EXAMPLE-FOR-STIFF-DIFFERENTIAL-Gobbert/0fcdf1a2774a6c9d1f93eea341ba8a11f0f3c632

Mallik, R. K. (1998). Solutions of Linear Difference Equations with Variable Coefficients. *Journal of Mathematical Analysis and Applications*, *222*(1), 79–91. doi: 10.1006/jmaa.1997.5903

Nyengeri, H., Ndenzako, E., & Nizigiyimana, R. (2019). The Explicit Fatunla's Method for First-Order Stiff Systems of Scalar Ordinary Differential Equations: Application to Robertson Problem. *OALib*, *06*(04), 1–15. doi: 10.4236/oalib.1105291

Rehman, T. (2018). *Studies on plasma-chemical reduction*. Eindhoven: Technische Universiteit Eindhoven.

Rujivan, S. (2016). A closed-form formula for the conditional moments of the extended CIR process. *Journal of Computational and Applied Mathematics*, *297*, 75–84. doi: 10.1016/j.cam.2015.11.001

Schiff, J. L. (1999). *The Laplace transform: theory and applications*. New York: Springer.

Thamrongrat, N., & Rujivan, S. (2020). A closed-form formula for the conditional expectation of the extended CIR process. *Songklanakarin Journal of Science and Technology, 42* (2), 424-429.

Waltman, P. (2004). *A second course in elementary differential equations*. New York: Dover.

Appendix

Code 2.1

```
L[i_,k_,r_]:=If[k==0,Catenate[Permutations/@IntegerPartitions[k+2-i,{r},Range[1,2]]],Catenate[Permutations/@IntegerPartitions[k+2-i,{r},Range[Max[3-i,1],2]]]];
```

Code 2.2

```
L[i_,k_,r_]:=If[k==0,Catenate[Permutations/@IntegerPartitions[k+2-i,{r},Range[1,2]]],Catenate[Permutations/@IntegerPartitions[k+2-i,{r},Range[Max[3-i,1],2]]]];

G1[k_,t_,a_,b_,c_,R_]:=Sum[Sum[Sum[Limit[(R[[1]]/(S-a[[1]]))(b[[2]]/(S-a[[2]]))(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])Product[If[l[[m]]==1,(b[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])),(c[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]))],{m,r}]Exp[a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]t],S->a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]]+Limit[(S-a[[1]])(R[[1]]/(S-a[[1]]))(b[[2]]/(S-a[[2]]))Product[If[l[[m]]==1,(b[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])),(c[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]))],{m,r}]Exp[a[[1]]t],S->a[[1]]]+Limit[(S-a[[2]])(R[[1]]/(S-a[[1]]))(b[[2]]/(S-a[[2]]))Product[If[l[[m]]==1,(b[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])),(c[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]))],{m,r}]Exp[a[[2]]t],S->a[[2]]],{m,1,r}],{l,L[2,k,r]}],{r,1,k}];

G2[k_,t_,a_,b_,c_,R_]:=R[[k+2]]Exp[a[[k+2]]t]+Sum[Sum[Sum[Limit[(R[[i]]/(S-a[[i]]))(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])Product[If[l[[m]]==1,(b[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])),(c[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]))],{m,r}]Exp[a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]t],S->a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]]+Limit[(S-a[[i]])(R[[i]]/(S-a[[i]]))Product[If[l[[m]]==1,(b[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]])),(c[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]/(S-a[[k+2+l[[m]]-Sum[l[[n]],{n,1,m}]]]))],{m,r}]Exp[a[[i]]t],S->a[[i]]], {m,1,r}],{l,L[i,k,r]}],{i,1,k+1},{r,1,k+2-i}];

A[k_,t_,La_,Lb_,Lc_,LR_]:=If[k==1,LR[[1]]Exp[La[[1]]t],G1[k-2,t,La,Lb,Lc,LR]+G2[k-2,t,La,Lb,Lc,LR]];
```