



# New Exponential Passivity Analysis of Neutral System with Time-Varying Delays and Nonlinear Perturbations

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## ABSTRACT

In this work, we study the delay-dependent exponential passivity and stability analysis of neutral system with time-varying delays and nonlinear perturbations. We derive new delay-dependent exponential passivity and stability conditions for the system in terms of linear matrix inequalities (LMIs) by constructing new class of augmented Lyapunov-Krasovskii functional, utilization of zero equation, descriptor model transformation, Leibniz-Newton formula and various inequalities. We also demonstrate the effectiveness and improvement of the results by giving some numerical examples.

**Keywords:** Exponential passivity analysis; Linear matrix inequality; Model transformation; Neutral system

## 1. Introduction

During the last decade, there has been interesting in the time-delay of neutral system due to their extensive applications, for example, in industrial systems, in communication, in engineering and so on. Stability of time-delay systems of neutral type has been divided into two classes depending on the size of the time-delay, delay-independent and delay-dependent. In [1–4], convex optimization algorithms has

been treated. Moreover, in [5–7], free-weighting method within convex optimization approach has been presented. Some results on robust stability for uncertain neutral system have been derived in [8–11].

The passivity is an essential tool in stability analysis. The stability of the feedback can be established by using the passivity theorem. Then, there has been an increasing of researchers, who are interesting in applying passivity to time-delay system

[12].

The results mentioned in the literatures are studied with the asymptotic stability ([2, 10, 11, 13, 14]) or only concerned with the exponential passivity of neural networks ([12, 15, 16]). However, the exponential passivity of neutral problem is also essential since it can estimate the convergence rate.

Motivated by the statement above, in this paper, the exponential passivity and stability analysis of neutral system with time-varying delays and nonlinear perturbations are investigated using new class of augmented Lyapunov-Krasovskii functional, utilization of zero equation, Leibniz-Newton formula and various inequalities. Two numerical examples are given to demonstrate the method.

## 2. Problem Formula and Preliminaries

We introduce some definitions and lemmas which we used throughout the paper and we also introduce the neutral system with time-varying delays and nonlinear perturbations of the form

$$\begin{aligned} \dot{y}(t) &= Ay(t) + By(t - r(t)) \\ &\quad + C\dot{y}(t - k(t)) + g_1(t, y(t)) \\ &\quad + g_2(t, y(t - r(t))) \\ &\quad + g_3(t, \dot{y}(t - k(t))) + D_1\omega(t), \\ t &\geq 0, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \tilde{z}(t) &= \tilde{C}_1y(t) + \tilde{C}_2y(t - r(t)) \\ &\quad + D_2\omega(t), \quad t \geq 0, \end{aligned} \quad (2.2)$$

$$\begin{aligned} y(t) &= \phi(t), \\ \forall t &\in [-\max\{r_2, k_2\}, 0], \end{aligned} \quad (2.3)$$

where  $y(t) \in \mathbb{R}^n$  is the state vector,  $\omega(t)$  is the disturbance input,  $\tilde{z}(t)$  is the output of the system and  $\phi(t)$  is a vector valued initial function.  $A, B, C, D_1, \tilde{C}_1, \tilde{C}_2$  and  $D_2 \in \mathbb{R}^{n \times n}$  are real constant matrices with appropriate

dimensions. The delays are time-varying delays satisfying

$$0 \leq r(t) \leq r_2, \quad (2.4)$$

$$0 \leq k(t) \leq k_2, \quad \dot{k}(t) \leq k_d, \quad (2.5)$$

where  $r_2, k_2$  and  $k_d$  are real constants. The uncertainties  $g_1(t, y(t)), g_2(t, y(t - r(t))), g_3(t, \dot{y}(t - k(t)))$  are nonlinear disturbances satisfying

$$g_1^T(t, y(t))g_1(t, y(t)) \leq l_1^2 y^T(t)y(t), \quad (2.6)$$

$$\begin{aligned} &g_2^T(t, y(t - r(t)))g_2(t, y(t - r(t))) \\ &\leq l_2^2 y^T(t - r(t))y(t - r(t)), \end{aligned} \quad (2.7)$$

$$\begin{aligned} &g_3^T(t, \dot{y}(t - k(t)))g_3(t, \dot{y}(t - k(t))) \\ &\leq l_3^2 \dot{y}^T(t - k(t))\dot{y}(t - k(t)), \end{aligned} \quad (2.8)$$

where  $l_1, l_2, l_3$  are given nonnegative constants.

**Definition 2.1** ([1]). If there exist positive real constants  $\alpha, M$  such that, for all  $\phi(t)$ , the solution  $y(t, \phi)$  of the system satisfies

$$\|y(t, \phi)\| \leq M\|\phi\|e^{-\alpha t}, \quad t \geq 0,$$

then the system Eq. (2.1) – Eq. (2.3) is called exponentially stable.

**Definition 2.2** ([12]). If there exists an exponential Lyapunov function  $V(t)$  defined on  $\mathbb{R}$ , and a constant  $\alpha > 0$  such that for all  $\omega(t)$ , all initial condition  $y(0)$  satisfy:

$$\dot{V}(t) + 2\alpha V(t) \leq 2\tilde{z}^T(t)\omega(t), \quad t \geq 0, \quad (2.9)$$

where  $\dot{V}(t)$  denotes the total derivative of  $V(t)$  along the state, then Eq. (2.1) – Eq. (2.3) is said to be exponentially passive from input  $\omega(t)$  to output  $\tilde{z}(t)$ , trajectories  $y(t)$  of Eq. (2.1) – Eq. (2.3).

**Lemma 2.3** ([2]). For all constant matrix  $M \in \mathbb{R}^{n \times n}$ ,  $M = M^T > 0$ , positive real constant  $r_2$  and vector-valued function  $\dot{y} : [-r_2, 0] \rightarrow \mathbb{R}^n$  such that the integrations concerned are well defined,

$$-r_2 \int_{t-r_2}^t \dot{y}^T(s)M\dot{y}(s)ds \leq$$

$$-\left(\int_{t-r_2}^t \dot{y}(s)ds\right)^T M \left(\int_{t-r_2}^t \dot{y}(s)ds\right).$$

Replacing the term  $\int_{t-r_2}^t \dot{y}(s)ds$  with  $y(t) - y(t - r_2)$ , we can obtain the following inequality

$$-r_2 \int_{t-r_2}^t \dot{y}^T(s)M\dot{y}(s)ds \leq \begin{bmatrix} y(t) \\ y(t-r_2) \end{bmatrix}^T \begin{bmatrix} -M & M \\ * & -M \end{bmatrix} \begin{bmatrix} y(t) \\ y(t-r_2) \end{bmatrix}.$$

**Lemma 2.4** ([10]). For all constant matrix  $M \in \mathbb{R}^{n \times n}$ ,  $M = M^T > 0$ ,  $r(t)$  is time-varying delays with  $0 \leq r(t) \leq r_2$ ,  $r_2 \in \mathbb{R}$  and a vector-valued function  $y : [-r_2, 0] \rightarrow \mathbb{R}^n$  such that the integrations concerned are well defined,

$$-r_2 \int_{t-r_2}^t y^T(s)My(s)ds \leq -\int_{t-r(t)}^t y^T(s)dsM \int_{t-r(t)}^t y(s)ds - \int_{t-r_2}^{t-r(t)} y^T(s)dsM \int_{t-r_2}^{t-r(t)} y(s)ds.$$

**Lemma 2.5** ([10]). For all constant matrices  $M_1, M_2, M_3 \in \mathbb{R}^{n \times n}$ ,  $M_1 \geq 0, M_3 > 0$ ,  $\begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix} \geq 0$ ,  $r(t)$  is time-varying delays with  $0 \leq r(t) \leq r_2$ ,  $r_2 \in \mathbb{R}$ , vector-valued functions  $y$  and  $\dot{y} : [-r_2, 0] \rightarrow \mathbb{R}^n$  such that the following integration is well defined,

$$-r_2 \int_{t-r_2}^t \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix} \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix} ds \leq \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \\ \int_{t-r(t)}^t y(s)ds \\ \int_{t-r_2}^{t-r(t)} y(s)ds \end{bmatrix}^T \times \begin{bmatrix} -M_3 & M_3 & 0 & -M_2^T & 0 \\ * & -M_3 - M_3^T & M_3 & M_2^T & -M_2^T \\ * & * & -M_3 & 0 & M_2^T \\ * & * & * & -M_1 & 0 \\ * & * & * & * & -M_1 \end{bmatrix} \times \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \\ \int_{t-r(t)}^t y(s)ds \\ \int_{t-r_2}^{t-r(t)} y(s)ds \end{bmatrix}.$$

**Lemma 2.6** ([1]). The following inequality holds, for a positive matrix  $M$ :

$$-\frac{(\alpha - \beta)^2}{2} \int_{\beta}^{\alpha} \int_s^{\alpha} y^T(u)My(u)duds \leq -\left(\int_{\beta}^{\alpha} \int_s^{\alpha} y(u)duds\right)^T M \left(\int_{\beta}^{\alpha} \int_s^{\alpha} y(u)duds\right).$$

**Lemma 2.7** ([10]). Let  $y(t) \in \mathbb{R}^n$  be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any constant matrices  $N, M_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, 2, \dots, 5$  and  $r(t)$  is time-varying delays with  $0 \leq r(t) \leq r_2$ ,  $r_2 \in \mathbb{R}^+$ ,

$$-\int_{t-r_2}^t \dot{y}^T(s)Q\dot{y}(s)ds \leq \Phi^T \omega \Phi + r_2 \Phi^T \Omega \Phi,$$

where

$$\Phi = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \end{bmatrix}, \quad \Omega = \begin{bmatrix} M_3 & M_4 & 0 \\ * & M_3 + M_5 & M_4 \\ * & * & M_5 \end{bmatrix}, \quad \omega = \begin{bmatrix} M_1 + M_1^T & -M_1^T + M_2 & 0 \\ * & \varrho & -M_1^T + M_2 \\ * & * & -M_2 - M_2^T \end{bmatrix}, \quad \varrho = M_1 + M_1^T - M_2 - M_2^T$$

and  $\begin{bmatrix} N & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0$ .

**Lemma 2.8** ([14]). For all constant matrix  $M \in \mathbb{R}^{n \times n}$ ,  $M = M^T > 0$ , nonnegative real constants  $r_2$  and a vector-valued function  $\dot{y} : [-r_2, 0] \rightarrow \mathbb{R}^n$  such that the following integrals are well defined,

$$-r_2 \int_{t-r_2}^t \dot{y}^T(s)M\dot{y}(s)ds \leq \omega^T(t) \Theta \omega(t),$$

where

$$\omega(t) = \left[ y^T(t), y^T(t-r_2), \frac{1}{r_2} \int_{t-r_2}^t y^T(s)ds \right]^T$$

and  $\Theta = \begin{bmatrix} -4M & -2M & 6M \\ * & -4M & 6M \\ * & * & -12M \end{bmatrix}$ .

**Lemma 2.9** ([13, 17]). For any constant matrices  $M, N \in \mathbb{R}^{n \times n}$ ,  $M \geq 0$ ,  $\begin{bmatrix} M & N \\ * & M \end{bmatrix} \geq 0$ ,  $r(t)$  is time-varying delay with Eq. (2.5), a vector-valued function  $\dot{y} : [-r_2, 0] \rightarrow \mathbb{R}^n$  such that the concerned integrations are well defined,

$$-r_2 \int_{t-r_2}^t \dot{y}^T(s)M\dot{y}(s)ds \leq \tilde{\omega}^T(t)\tilde{\Theta}\tilde{\omega}(t),$$

where

$$\tilde{\omega}(t) = [y^T(t), y^T(t-r(t)), y^T(t-r_2)]^T$$

and  $\tilde{\Theta} = \begin{bmatrix} -M & M-N & N \\ * & -2M+N+N^T & M-N \\ * & * & -M \end{bmatrix}$ .

### 3. Main Results

In this section, we will indicate the exponential passivity criteria dependent on time-varying delays. We define a new parameter

$$\sum = [\Xi_{i,j}]_{16 \times 16}, \quad (3.1)$$

where  $\Xi_{i,j} = \Xi_{j,i}^T$ ,  $i, j = 1, 2, 3, \dots, 16$ ,

$$\begin{aligned} S_{16} &= P_1G, \\ \Xi_{1,1} &= -S_5 - S_5^T + r_2^2S_{10} + P_2 + r_2^2P_7 \\ &\quad - M_2 - M_2^T, \\ \Xi_{1,2} &= P_1^T - S_4^T + S_5A + r_2^2S_9^T, \\ \Xi_{1,3} &= -M_1^T + M_2, \\ \Xi_{1,4} &= S_5B - S_6^T, \\ \Xi_{1,6} &= S_5C, \\ \Xi_{1,11} &= S_5, \\ \Xi_{1,12} &= S_5, \\ \Xi_{1,13} &= S_5, \\ \Xi_{2,2} &= S_1A + S_2 + A^T S_1 + S_2^T + 2\alpha S_1 \\ &\quad + S_{16} + S_{16}^T + S_4A + A^T S_4^T + r_2^2S_8 \\ &\quad + r_2^2P_4 - e^{-2\alpha r_2}S_{10} + S_{11} + S_{11}^T \\ &\quad + r_2S_{13} + r_2C_1 + r_2C_1^T + r_2C_4 \\ &\quad + \epsilon_1 l_1^2 I + 2\alpha P_1 + P_3 + r_2^2P_5 \\ &\quad - e^{-4\alpha r_2}r_2^2P_9 + r_2^2P_{10} + r_2C_4^T \\ &\quad - 4e^{-2\alpha r_2}P_{11} - e^{-2\alpha r_2}P_{12}, \end{aligned}$$

$$\begin{aligned} \Xi_{2,3} &= A^T J_1^T, \\ \Xi_{2,4} &= S_1BS_{16} - S_4B + A^T S_6^T - r_2C_4 \\ &\quad - S_{11}^T + S_{12} + r_2S_{14} + r_2C_5^T \\ &\quad + e^{-2\alpha r_2}P_{12} - e^{-2\alpha r_2}P_{13} \\ &\quad + e^{-2\alpha r_2}S_{10}, \\ \Xi_{2,5} &= -S_2 + r_2C_2^T - r_2C_1 - 2e^{-2\alpha r_2}P_{11} \\ &\quad + e^{-2\alpha r_2}P_{13}, \\ \Xi_{2,6} &= S_1C + S_4C + A^T J_2^T, \\ \Xi_{2,7} &= A^T S_2 + S_3 + 2\alpha S_2 - e^{-2\alpha r_2}S_9^T, \\ \Xi_{2,8} &= A^T S_2 + S_3 + 2\alpha S_2, \\ \Xi_{2,9} &= -S_{16} + S_7^T + r_2C_6^T - r_2C_4, \\ \Xi_{2,10} &= r_2C_3^T - r_2C_1, \\ \Xi_{2,11} &= S_1 + S_4, \\ \Xi_{2,12} &= S_1 + S_4, \\ \Xi_{2,13} &= S_1 + S_4, \\ \Xi_{2,14} &= e^{-4\alpha r_2}r_2^2P_9 + 6e^{-2\alpha r_2}P_{11}, \\ \Xi_{2,16} &= S_1D_1 - \tilde{C}_1^T, \\ \Xi_{3,1} &= -M_1 + M_2^T, \\ \Xi_{3,3} &= M_1 + M_1^T - J_1 - J_1^T + \frac{r_2^4}{2}P_8 \\ &\quad + \frac{r_2^4}{2}P_9 - e^{-2\alpha r_2}L_2 + r_2^2P_{11} \\ &\quad + r_2^2P_{12}, \\ \Xi_{3,4} &= J_1B, \\ \Xi_{3,6} &= J_1C - J_2^T, \\ \Xi_{3,11} &= J_1, \\ \Xi_{3,12} &= J_1, \\ \Xi_{3,13} &= J_1, \\ \Xi_{4,4} &= S_6B + B^T S_6^T - e^{-2\alpha r_2}S_{10} + S_{11} \\ &\quad - e^{-2\alpha r_2}S_{10}^T + S_{11}^T - S_{12} - S_{12}^T \\ &\quad + r_2S_{13} + r_2S_{15} - r_2C_5 - r_2C_5^T \\ &\quad - 2e^{-2\alpha r_2}P_{12} + e^{-2\alpha r_2}P_{13} \\ &\quad + e^{-2\alpha r_2}P_{13}^T + \epsilon_2 l_2^2 I, \\ \Xi_{4,5} &= e^{-2\alpha r_2}S_{10} - S_{11}^T + S_{12} + r_2S_{14} \\ &\quad + e^{-2\alpha r_2}P_{12} - e^{-2\alpha r_2}P_{13}, \\ \Xi_{4,6} &= S_6C + B^T J_2^T, \\ \Xi_{4,7} &= B^T S_2 + e^{-2\alpha r_2}S_9^T, \end{aligned}$$

$$\begin{aligned}
 \bar{\Xi}_{4,8} &= B^T S_2 - e^{-2\alpha r_2} S_9^T, \\
 \bar{\Xi}_{4,9} &= -S_7^T - r_2 C_6^T - r_2 C_5, \\
 \bar{\Xi}_{4,11} &= S_6, \\
 \bar{\Xi}_{4,12} &= S_6, \\
 \bar{\Xi}_{4,13} &= S_6, \\
 \bar{\Xi}_{5,5} &= -e^{-2\alpha r_2} S_{10} - S_{12} - S_{12}^T + r_2 S_{15} \\
 &\quad - r_2 C_2 - r_2 C_2^T - e^{-2\alpha r_2} P_3 \\
 &\quad - 4e^{-2\alpha r_2} P_{11} - e^{-2\alpha r_2} P_{12}, \\
 \bar{\Xi}_{5,7} &= -S_3, \\
 \bar{\Xi}_{5,8} &= -S_3 + e^{-2\alpha r_2} S_9^T, \\
 \bar{\Xi}_{5,10} &= -r_2 C_3^T - r_2 C_2, \\
 \bar{\Xi}_{5,14} &= 6e^{-2\alpha r_2} P_{11}, \\
 \bar{\Xi}_{6,6} &= -e^{-2\alpha k_2} W + k_d W + \epsilon_3 l_3^2 I + J_2 C \\
 &\quad + C^T J_2^T, \\
 \bar{\Xi}_{6,7} &= C^T K_{12}, \\
 \bar{\Xi}_{6,8} &= C^T K_{12}, \\
 \bar{\Xi}_{6,11} &= J_2, \\
 \bar{\Xi}_{6,12} &= J_2, \\
 \bar{\Xi}_{6,13} &= J_2, \\
 \bar{\Xi}_{7,7} &= 2\alpha S_3 - e^{-2\alpha r_2} S_8 - e^{-2\alpha r_2} P_4 \\
 &\quad - e^{-2\alpha r_2} P_5, \\
 \bar{\Xi}_{7,8} &= 2\alpha S_3 - e^{-2\alpha r_2} P_4, \\
 \bar{\Xi}_{7,11} &= S_2^T, \\
 \bar{\Xi}_{7,12} &= S_2^T, \\
 \bar{\Xi}_{7,13} &= S_2^T, \\
 \bar{\Xi}_{7,16} &= S_2^T D_1, \\
 \bar{\Xi}_{8,7} &= 2\alpha S_3 - e^{-2\alpha r_2} P_4, \\
 \bar{\Xi}_{8,8} &= 2\alpha S_3 - e^{-2\alpha r_2} S_8 - e^{-2\alpha r_2} P_4 \\
 &\quad - e^{-2\alpha r_2} P_5, \\
 \bar{\Xi}_{8,11} &= S_2^T, \\
 \bar{\Xi}_{8,12} &= S_2^T, \\
 \bar{\Xi}_{8,13} &= S_2^T, \\
 \bar{\Xi}_{8,16} &= S_2^T D_1, \\
 \bar{\Xi}_{9,9} &= -S_7 S_7^T - r_2 C_6 - r_2 C_6^T \\
 &\quad - e^{-2\alpha r_2} P_6, \\
 \bar{\Xi}_{10,10} &= r_2 C_3 - r_2 C_3^T - e^{-2\alpha r_2} P_6,
 \end{aligned}$$

$$\begin{aligned}
 \bar{\Xi}_{11,11} &= -\epsilon_1 I, \\
 \bar{\Xi}_{12,12} &= -\epsilon_2 I, \\
 \bar{\Xi}_{13,13} &= -\epsilon_3 I, \\
 \bar{\Xi}_{14,14} &= -e^{-4\alpha r_2} r_2^2 P_9 - r_2^2 e^{-2\alpha r_2} P_{10} \\
 &\quad - 12e^{-2\alpha r_2} P_{11}, \\
 \bar{\Xi}_{15,15} &= -e^{-4\alpha r_2} P_8, \\
 \bar{\Xi}_{16,16} &= -D_2 - D_2^T
 \end{aligned}$$

and others are equal to zero.

**Theorem 3.1.** For  $\|C\| + l_3 < 1$ , the system Eq. (2.1) – Eq. (2.3) is exponentially passive with a decay rate  $\alpha > 0$  if there exist positive definite symmetric matrices  $P_m, m = 1, 2, \dots, 13$ , any appropriate dimensional matrices  $S_l, l = 1, 2, \dots, 16$ , and positive real constants  $\epsilon_n, n = 1, 2, 3$ , such that the following symmetric linear matrix inequalities hold

$$\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0, \quad (3.2)$$

$$\begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} > 0, \quad (3.3)$$

$$\begin{bmatrix} e^{-2\alpha r_2} P_7 & S_{11} & S_{11} \\ * & S_{13} & S_{13} \\ * & * & S_{15} \end{bmatrix} \geq 0, \quad (3.4)$$

$$\begin{bmatrix} P_{12} & P_{13} \\ * & P_{12} \end{bmatrix} > 0, \quad (3.5)$$

$$\sum < 0. \quad (3.6)$$

*Proof.* Initially, we use descriptor system to rewrite the system Eq. (2.1)

$$\dot{y}(t) = z(t), \quad (3.7)$$

$$\begin{aligned}
 0 &= -z(t) + Ay(t) + By(t - r(t)) \\
 &\quad + C\dot{y}(t - k(t)) + g_1(t, y(t)) \\
 &\quad + g_2(t, y(t - r(t))) \\
 &\quad + g_3(t, \dot{y}(t - k(t))) \\
 &\quad + D_1\omega(t).
 \end{aligned} \quad (3.8)$$

Apply the following utilization of zero equation, we have

$$0 = Gy(t) - Gy(t - r(t))$$

$$-G \int_{t-r(t)}^t z(s)ds, \quad (3.9)$$

where  $G \in \mathbb{R}^{n \times n}$  will be chosen to assure the exponential passivity of system Eq. (2.1) – Eq. (2.3). By Eq. (3.9), the system Eq. (3.7) and Eq. (3.8) can be represented in the form of the descriptor delayed system

$$\begin{aligned} \dot{y}(t) &= z(t) + Gy(t) - Gy(t - r(t)) \\ &\quad -G \int_{t-r(t)}^t z(s)ds, \quad (3.10) \\ 0 &= -z(t) + Ay(t) + By(t - r(t)) \\ &\quad + C\dot{y}(t - k(t)) + g_1(t, y(t)) \\ &\quad + g_2(t, y(t - r(t))) \\ &\quad + g_3(t, \dot{y}(t - k(t))) \\ &\quad + D_1\omega(t). \quad (3.11) \end{aligned}$$

Construct a Lyapunov-Krasovskii functional candidate for the system Eq. (2.1) – Eq. (2.3), Eq. (3.10) – Eq. (3.11) of the form

$$V(t) = \sum_{i=1}^9 V_i(t), \quad (3.12)$$

where

$$\begin{aligned} V_1(t) &= \begin{bmatrix} y(t) \\ \int_{t-r_2}^t y(s)ds \end{bmatrix}^T \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} \\ &\quad \times \begin{bmatrix} y(t) \\ \int_{t-r_2}^t y(s)ds \end{bmatrix}, \\ V_2(t) &= y^T(t)P_1y(t) \\ &= \begin{bmatrix} y(t) \\ z(t) \\ y(t - r(t)) \\ \int_{t-r(t)}^t z(s)ds \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S_4 & S_5 & S_6 & S_7 \end{bmatrix} \\ &\quad \times \begin{bmatrix} y(t) \\ z(t) \\ y(t - r(t)) \\ \int_{t-r(t)}^t z(s)ds \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} V_3(t) &= r_2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \begin{bmatrix} y(\theta) \\ \dot{y}(\theta) \end{bmatrix}^T \\ &\quad \times \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(\theta) \\ \dot{y}(\theta) \end{bmatrix} d\theta ds, \\ V_4(t) &= \int_{t-k(t)}^t e^{2\alpha(s-t)} \dot{y}^T(s)P_2\dot{y}(s)ds \\ &\quad + \int_{t-r_2}^t e^{2\alpha(s-t)} y^T(s)P_3y(s), \\ V_5(t) &= r_2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} y^T(\theta)P_4 \\ &\quad \times y(\theta)d\theta ds + r_2^2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \\ &\quad y^T(\theta)P_5y(\theta)d\theta ds \\ &\quad + r_2^2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} z^T(\theta)P_6 \\ &\quad \times z(\theta)d\theta ds, \\ V_6(t) &= \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} z^T(\theta)P_7z(\theta)d\theta ds, \\ V_7(t) &= \left[ \frac{(r_2)^2}{2} \int_{-r_2}^0 \int_{\theta}^0 \int_{t+s}^t e^{2\alpha(\theta+s-t)} \right. \\ &\quad \left. z^T(\theta)P_8z(\theta)d\theta ds d\lambda \right] \\ &\quad + \left[ \frac{(r_2)^2}{2} \int_{-r_2}^0 \int_{\theta}^0 \int_{t+s}^t e^{2\alpha(\theta+s-t)} \right. \\ &\quad \left. z^T(\theta)P_9z(\theta)d\theta ds d\lambda \right], \\ V_8(t) &= r_2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} y^T(\theta)P_{10} \\ &\quad \times y(\theta)d\theta ds, \\ V_9(t) &= r_2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{y}^T(\theta)P_{11} \\ &\quad \times \dot{y}(\theta)d\theta ds + r_2 \int_{-r_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \\ &\quad \dot{y}^T(\theta)P_{12}\dot{y}(\theta)d\theta ds. \end{aligned}$$

The time derivative of  $V(t)$  along the trajectory of Eq. (2.1), Eq. (3.10) – Eq. (3.11) is given by

$$\dot{V}(t) = \sum_{i=1}^9 \dot{V}_i(t). \quad (3.13)$$

The time derivatives of  $V_1(t), V_2(t)$  are com-

puted as

$$\begin{aligned} \dot{V}_1(t) &= \left[ \int_{t-r(t)}^t y(s)ds + \int_{t-r_2}^{t-r(t)} y(s)ds \right]^T \\ &\times \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) - y(t-r_2) \end{bmatrix} \\ &+ \begin{bmatrix} \xi(t) \\ y(t) - y(t-r_2) \end{bmatrix}^T \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} \\ &\times \left[ \int_{t-r(t)}^t y(s)ds + \int_{t-r_2}^{t-r(t)} y(s)ds \right] \\ &+ 2\alpha V_1(t) - 2\alpha V_1(t), \\ \dot{V}_2(t) &= 2 \begin{bmatrix} y(t) \\ z(t) \\ y(t-r(t)) \\ \int_{t-r(t)}^t z(s)ds \end{bmatrix}^T \\ &\times \begin{bmatrix} P_1^T & 0 & 0 & S_4^T \\ 0 & 0 & 0 & S_5^T \\ 0 & 0 & 0 & S_6^T \\ 0 & 0 & 0 & S_7^T \end{bmatrix} \\ &\times \begin{bmatrix} I^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \\ \dot{y}(t-r(t)) \\ \int_{t-r(t)}^t \dot{z}(s)ds \end{bmatrix} \\ &= 2y^T R[\phi(t)] + 2y^T(t)S_4[\beta(t)] \\ &+ 2z^T(t)S_5[\beta(t)] + 2\alpha V_2(t) \\ &+ 2y^T(t-r(t))S_6[\beta(t)] \\ &+ 2 \int_{t-r(t)}^t z^T(s)ds S_7[\psi(t)] \\ &- 2\alpha V_2(t), \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= z(t) + Gy(t) - Gy(t-r(t)) \\ &- G \int_{t-r(t)}^t z(s)ds, \end{aligned}$$

$$\psi(t) = y(t) - y(t-r(t)) - \int_{t-r(t)}^t z(s)ds,$$

$$\begin{aligned} \xi(t) &= Ay(t) + By(t-r(t)) \\ &+ C\dot{y}(t-k(t)) + g_1(t, y(t)) \\ &+ g_2(t, y(t-r(t))) \\ &+ g_3(t, \dot{y}(t-k(t))) + D_1\omega(t) \end{aligned}$$

and

$$\beta(t) = -z(t) + Ay(t) + By(t-r(t))$$

$$\begin{aligned} &+ C\dot{y}(t-k(t)) + g_1(t, y(t)) \\ &+ g_2(t, y(t-r(t))) \\ &+ g_3(t, \dot{y}(t-k(t))) + D_1\omega(t). \end{aligned}$$

Obviously, we get  $e^{-2\alpha r_2} \leq e^{2\alpha(s-t)} \leq 1$ , for any a scalar  $s \in [t-r_2, t]$ , and apply with Lemma 2.5, we get  $\dot{V}_3(t)$  as follows

$$\begin{aligned} \dot{V}_3 &= r_2 \int_{-r_2}^0 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} ds \\ &- r_2 \int_{-r_2}^0 e^{2\alpha s} \begin{bmatrix} y(t+s) \\ \dot{y}(t+s) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \\ &\times \begin{bmatrix} y(t+s) \\ \dot{y}(t+s) \end{bmatrix} ds - 2\alpha V_3(t) \\ &= r_2^2 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\ &- r_2 \int_{t-r_2}^t e^{2\alpha(s-t)} \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \\ &\times \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix} ds - 2\alpha V_3(t) \\ &\leq r_2^2 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\ &- r_2 e^{-2\alpha r_2} \int_{t-r_2}^t \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \\ &\times \begin{bmatrix} y(s) \\ \dot{y}(s) \end{bmatrix} ds - 2\alpha V_3(t) \\ &\leq r_2^2 \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \\ &+ e^{-2\alpha r_2} \Upsilon^T \varrho \Upsilon - 2V_3(t) \\ &= r_2^2 \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \\ &+ e^{-2\alpha r_2} \Upsilon^T \varrho \Upsilon - 2V_3(t), \end{aligned}$$

where

$$\Upsilon = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \\ \int_{t-r(t)}^t y(s)ds \\ \int_{t-r_2}^{t-r(t)} y(s)ds \end{bmatrix}$$

and

$$\varrho = \begin{bmatrix} -S_{10} & S_{10} & 0 & -S_9^T & 0 \\ * & -S_{10} - S_{10}^T & S_{10} & S_9^T & -S_9^T \\ * & * & -S_{10} & 0 & S_9^T \\ * & * & * & -S_8 & 0 \\ * & * & * & * & -S_8 \end{bmatrix}.$$

Taking the differential of  $\dot{V}_4(t)$ , we have

$$\begin{aligned} \dot{V}_4(t) &= \dot{y}^T(t)P_2\dot{y}(t) + y^T P_3 y(t) \\ &\quad - (1 - \dot{k}(t))e^{-2\alpha k(t)}\dot{y}^T(t - k(t)) \\ &\quad \times P_2\dot{y}(t - k(t)) - 2\alpha V_4(t) \\ &\quad - e^{-2\alpha r_2}y^T(t - r_2)P_3y(t - r_2) \\ &\leq \dot{y}^T(t)P_2\dot{y}(t) + y^T(t)P_3y(t) \\ &\quad - e^{-2\alpha k_2}\dot{y}(t - k(t))P_2\dot{y}(t - k(t)) \\ &\quad + k_d\dot{y}^T(t - k(t))P_2\dot{y}(t - k(t)) \\ &\quad - e^{-2\alpha r_2}y^T(t - r_2)P_3y(t - r_2) \\ &\quad - 2\alpha V_4(t) \\ &= z^T(t)P_2z(t) - e^{-2\alpha k_2}\dot{y}(t - k(t)) \\ &\quad \times P_2\dot{y}(t - k(t)) + k_d\dot{y}^T(t - k(t)) \\ &\quad \times P_2\dot{y}(t - k(t)) + y^T(t)P_3y(t) \\ &\quad - e^{-2\alpha r_2}y^T(t - r_2)P_3y(t - r_2) \\ &\quad - 2\alpha V_4(t). \end{aligned}$$

From Lemma 2.3 and Lemma 2.4, we obtain  $\dot{V}_5(t)$  as follows

$$\begin{aligned} \dot{V}_5(t) &= r_2 \int_{-r_2}^0 y^T(t)P_4y(t)ds \\ &\quad - r_2 \int_{-r_2}^0 e^{2\alpha s}y^T(t + s)P_4y^T(t + s)ds \\ &\quad + r_2 \int_{-r_2}^0 y^T(t)P_5y(t)ds \\ &\quad - r_2 \int_{-r_2}^0 e^{2\alpha s}y^T(t + s)P_5y^T(t + s)ds \\ &\quad + r_2 \int_{-r_2}^0 z^T(t)P_6z(t)ds - 2\alpha V_5(t) \\ &\quad - r_2 \int_{-r_2}^0 e^{2\alpha s}z^T(t + s)P_6z^T(t + s)ds \\ &= r_2^2 y^T(t)P_4y(t) \\ &\quad - r_2 \int_{t-r_2}^t e^{2\alpha(s-t)}y^T(s)P_4y(s)ds \\ &\quad + r_2^2 y^T(t)P_5y(t) - 2\alpha V_5(t) \\ &\quad - r_2 \int_{t-r_2}^t e^{2\alpha(s-t)}y^T(s)P_5y(s)ds \\ &\quad + r_2^2 z^T(t)P_6z(t) \\ &\quad - r_2 \int_{t-r_2}^t e^{2\alpha(s-t)}z^T(s)P_6z(s)ds \\ &\leq r_2^2 y^T(t)P_4y(t) + r_2^2 y^T(t)P_5y(t) \end{aligned}$$

$$\begin{aligned} &- r_2 e^{2\alpha r_2} \int_{t-r_2}^t y^T(s)P_4y(s)ds \\ &- r_2 e^{-2\alpha r_2} \int_{t-r_2}^t y^T(s)P_5y(s)ds \\ &\quad + r_2^2 z^T(t)P_6z(t) - 2\alpha V_5(t) \\ &- r_2 e^{-2\alpha r_2} \int_{t-r_2}^t z^T(s)P_6z(s)ds \\ &\leq r_2^2 y^T(t)P_4y(t) + r_2^2 y^T(t)P_5y(t) \\ &\quad - e^{-2\alpha r_2} \int_{t-r(t)}^t y^T(s)ds P_5 \\ &\quad \times \int_{t-r(t)}^t y^T(s)ds \\ &\quad - e^{2\alpha r_2} \left( \int_{t-r(t)}^t y^T(s)ds \right. \\ &\quad \left. + \int_{t-r_2}^{t-r(t)} y^T(s)ds \right) P_4 \\ &\quad \times \left[ \int_{t-r(t)}^t y^T(s)ds \right. \\ &\quad \left. + \int_{t-r_2}^{t-r(t)} y^T(s)ds \right] \\ &\quad - e^{-2\alpha r_2} \int_{t-r_2}^{t-r(t)} y^T(s)ds P_5 \\ &\quad \times \int_{t-r_2}^{t-r(t)} y^T(s)ds + r_2^2 z^T(t)P_6z(t) \\ &\quad - e^{-2\alpha r_2} \int_{t-r(t)}^t z^T(s)ds P_6 \\ &\quad \times \int_{t-r(t)}^t z^T(s)ds \\ &\quad - e^{-2\alpha r_2} \int_{t-r_2}^{t-r(t)} z^T(s)ds P_6 \\ &\quad \times \int_{t-r_2}^{t-r(t)} z^T(s)ds - 2\alpha V_5(t). \end{aligned}$$

Using Lemma 2.7,  $\dot{V}_6(t)$  will be obtained as

$$\begin{aligned} \dot{V}_6(t) &= \int_{-r_2}^0 z^T(t)P_7z(t)ds - 2\alpha V_6(t) \\ &\quad - \int_{-r_2}^0 e^{2\alpha s}z^T(t + s)P_7z(t + s)ds \\ &= r_2 z^T(t)P_7z(t) - 2\alpha V_6(t) \\ &\quad - \int_{t-r_2}^t e^{2\alpha(s-t)}z^T(s)P_7z(s)ds \end{aligned}$$

$$\begin{aligned} &\leq r_2 z^T(t) P_7 z(t) - 2\alpha V_6(t) \\ &\quad - e^{2\alpha r_2} \int_{t-r_2}^t z^T(s) P_7 z(s) ds \\ &\leq r_2 z^T(t) P_7 z(t) + \eta^T \Delta \eta \\ &\quad + r_2 \eta^T \Theta \eta - 2\alpha V_6(t), \end{aligned}$$

where

$$\eta = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \end{bmatrix},$$

$$\Delta = \begin{bmatrix} S_{11} + S_{11}^T & -S_{11}^T + S_{12} & 0 \\ * & \phi & -S_{11}^T + S_{12} \\ * & * & -S_{12} - S_{12}^T \end{bmatrix},$$

$$\phi = S_{11} + S_{11}^T - S_{12} - S_{12}^T$$

and

$$\Theta = \begin{bmatrix} S_{13} & S_{14} & 0 \\ * & S_{13} + S_{15} & S_{14} \\ * & * & S_{15} \end{bmatrix}.$$

From Lemma 2.6,  $\dot{V}_7(t)$  can be estimated as

$$\begin{aligned} \dot{V}_7(t) &\leq \frac{(r_2)^4}{2} \dot{y}^T(t) P_8 \dot{y}(t) \\ &\quad + \frac{(r_2)^4}{2} \dot{y}^T(t) P_9 \dot{y}(t) \\ &\quad - \frac{r_2^2}{2} e^{-4\alpha r_2} \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}^T(s) P_8 \dot{y}(s) ds d\theta \\ &\quad - \frac{r_2^2}{2} e^{-4\alpha r_2} \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}^T(s) P_9 \dot{y}(s) ds d\theta \\ &\quad - 2\alpha V_7(t) \\ &\leq \frac{(r_2)^4}{2} \dot{y}^T(t) P_8 \dot{y}(t) \\ &\quad + \frac{(r_2)^4}{2} \dot{y}^T(t) P_9 \dot{y}(t) \\ &\quad - e^{-4\alpha r_2} \left( \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}^T(s) ds d\theta \right) P_8 \\ &\quad \times \left( \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}(s) ds d\theta \right) \\ &\quad - e^{-4\alpha r_2} \left( \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}^T(s) ds d\theta \right) P_9 \\ &\quad \times \left( \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}(s) ds d\theta \right) \\ &= \frac{(r_2)^4}{2} \dot{y}^T(t) P_8 \dot{y}(t) \\ &\quad + \frac{(r_2)^4}{2} \dot{y}^T(t) P_9 \dot{y}(t) \end{aligned}$$

$$\begin{aligned} &- e^{-4\alpha r_2} \left( \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}^T(s) ds d\theta \right) P_8 \\ &\quad \times \left( \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}(s) ds d\theta \right) \\ &\quad - e^{-4\alpha r_2} \left[ r_2 y^T(t) - \int_{t-r_2}^t y^T(\theta) d\theta \right] \\ &\quad \times P_9 \left[ r_2 y(t) - \int_{t-r_2}^t y(\theta) d\theta \right]. \end{aligned}$$

Using Lemma 2.3, we compute  $\dot{V}_8(t)$  as follows

$$\begin{aligned} \dot{V}_8(t) &= r_2^2 y^T(t) P_{10} y(t) - 2\alpha V_8(t) \\ &\quad - e^{-2\alpha r_2} \left( \frac{1}{r_2} \int_{t-r_2}^t y^T(s) ds \right) r_2^2 P_{10} \\ &\quad \times \left( \frac{1}{r_2} \int_{t-r_2}^t y(s) ds \right). \end{aligned}$$

By Lemma 2.8 and Lemma 2.9, we get  $\dot{V}_9(t)$  as

$$\begin{aligned} \dot{V}_9(t) &\leq r_2^2 \dot{y}(t) P_{11} \dot{y}(t) + e^{-2\alpha r_2} \Theta^T \Delta \Theta \\ &\quad + r_2^2 \dot{y}(t) P_{12} \dot{y}(t) \\ &\quad + e^{-2\alpha r_2} \Psi^T \Omega \Psi - 2\alpha V_9(t), \end{aligned}$$

where

$$\Theta = \begin{bmatrix} y(t) \\ y(t-r_2) \\ \frac{1}{r_2} \int_{t-r_2}^t y(s) ds \end{bmatrix},$$

$$\Delta = \begin{bmatrix} -4P_{11} & -2P_{11} & 6P_{11} \\ -2P_{11} & -4P_{11} & 6P_{11} \\ 6P_{11} & 6P_{11} & -12P_{11} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} y(t) \\ y(t-r(t)) \\ y(t-r_2) \end{bmatrix},$$

$$\Omega = \begin{bmatrix} -P_{12} & P_{12} - P_{13} & P_{13} \\ P_{12} - P_{13}^T & \phi & P_{12} - P_{13} \\ P_{13}^T & P_{12} - P_{13}^T & -P_{12} \end{bmatrix}$$

and

$$\phi = -2P_{12} + P_{13} + P_{13}^T.$$

From the Leibinz-Newton formula, for any real constant matrices  $C_i, i = 1, 2, \dots, 6$  with appropriate dimensions, the following equations are true

$$0 = 2r_2 [y^T(t) C_4 + y^T(t-r(t)) C_5]$$

$$\begin{aligned}
 & + \int_{t-r(t)}^t z^T(s)dsC_6] \\
 & [y(t) - y(t - r(t)) \\
 & - \int_{t-r(t)}^t z(s)ds], \quad (3.14)
 \end{aligned}$$

$$\begin{aligned}
 0 & = 2r_2[y^T(t)C_1 + y^T(t - r_2)C_2 \\
 & + \int_{t-r_2}^t z^T(s)dsC_3] \\
 & [y(t) - y(t - r_2) \\
 & - \int_{t-r_2}^t z(s)ds]. \quad (3.15)
 \end{aligned}$$

From Eq. (2.6) – Eq. (2.8), for any scalars  $\epsilon_1$ ,  $\epsilon_2$  and  $\epsilon_3$  are positive real constants, it can investigated that the following inequalities hold:

$$\begin{aligned}
 0 & \leq \epsilon_1 I_1^2 y^T(t)y(t) \\
 & - \epsilon_1 g_1^T(t, y(t))g_1(t, y(t)), \quad (3.16)
 \end{aligned}$$

$$\begin{aligned}
 0 & \leq \epsilon_2 I_2^2 y^T(t - r(t))y(t - r(t)) \\
 & - \epsilon_2 g_2^T(t, y(t - r(t))) \\
 & g_2(t, y(t - r(t))), \quad (3.17)
 \end{aligned}$$

$$\begin{aligned}
 0 & \leq \epsilon_3 I_3^2 \dot{y}^T(t - k(t))\dot{y}(t - k(t)) \\
 & - \epsilon_3 g_3^T(t, \dot{y}(t - k(t))) \\
 & g_3(t, \dot{y}(t - k(t))). \quad (3.18)
 \end{aligned}$$

From Eq. (3.11), we have

$$\begin{aligned}
 0 & = 2\dot{y}(t)J_1[-\dot{y}(t) + Ay(t) + By(t - r(t)) \\
 & + C\dot{y}(t - k(t)) + g_1(t, y(t)) \\
 & + g_2(t, y(t - h(t))) \\
 & + g_3(t, \dot{y}(t - \tau(t)))]], \quad (3.19)
 \end{aligned}$$

$$\begin{aligned}
 0 & = 2\dot{y}(t - k(t))J_2[-\dot{y}(t) + Ay(t) \\
 & + By(t - r(t)) + C\dot{y}(t - k(t)) \\
 & + g_1(t, y(t)) + g_2(t, y(t - h(t))) \\
 & + g_3(t, \dot{y}(t - \tau(t)))]]. \quad (3.20)
 \end{aligned}$$

From Eq. (3.10), we obtain

$$0 = [\dot{y}^T M_1 + z^T(t)M_2][\dot{y}(t) - z(t)]. \quad (3.21)$$

According to Eq. (3.13) - Eq. (3.21), we can conclude that

$$\dot{V}(t) + 2\alpha V(t) - 2\tilde{z}^T(t)\omega(t) \leq \delta^T(t) \sum \delta(t),$$

where  $\delta(t) = [z(t), y(t), \dot{y}(t), y(t - r(t)), y(t - r_2), \dot{y}(t - k(t)), \int_{t-r(t)}^t y(s)ds, \int_{t-r_2}^{t-r(t)} y(s)ds, \int_{t-r(t)}^t z(s)ds, \int_{t-r_2}^{t-r(t)} z(s)ds, g_1(t, y(t)), g_2(t, y(t - r(t))), g_3(t, \dot{y}(t - k(t))), \frac{1}{r_2} \int_{t-r_2}^t y(\theta)d\theta, \int_{-r_2}^0 \int_{t+\theta}^t \dot{y}^T(s)dsd\theta, \omega(t)]$  and  $\sum$  is defined in Eq. (3.1). If conditions Eq. (3.2)-Eq. (3.6) hold, then

$$\dot{V}(t) + 2\alpha V(t) \leq 2\tilde{z}^T(t)\omega(t), \quad \forall t \in \mathbb{R}^+. \quad (3.22)$$

It means that systems Eq. (2.1)-Eq. (2.3) are exponentially passive.  $\square$

Now the exponential stability criteria of equation Eq. (2.1) when  $D_1 = 0$  is demonstrated. We define a new parameter

$$\tilde{\Sigma} = [\tilde{\Xi}_{i,j}]_{15 \times 15}, \quad (3.23)$$

where  $\tilde{\Xi}_{i,j} = \Xi_{j,i}^T, \quad i, j = 1, 2, 3, \dots, 15$ .

**Corollary 3.2.** For  $\|C\| + l_3 < 1$ , the system Eq. (2.1) where  $D_1 = 0$  is exponentially stable with a decay rate  $\alpha > 0$  if there exist positive definite symmetric matrices  $P_m, m = 1, 2, \dots, 13$ , any appropriate dimensional matrices  $S_l, l = 1, 2, \dots, 16$ , and positive real constants  $\epsilon_n, n = 1, 2, 3$ , such that the following symmetric linear matrix inequalities hold

$$\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0, \quad (3.24)$$

$$\begin{bmatrix} S_8 & S_9 \\ * & S_{10} \end{bmatrix} > 0, \quad (3.25)$$

$$\begin{bmatrix} e^{-2\alpha r_2} P_7 & S_{11} & S_{12} \\ * & S_{13} & S_{14} \\ * & * & S_{15} \end{bmatrix} \geq 0, \quad (3.26)$$

$$\begin{bmatrix} P_{12} & P_{13} \\ * & P_{12} \end{bmatrix} > 0, \quad (3.27)$$

$$\tilde{\Sigma} < 0. \quad (3.28)$$

#### 4. Numerical Examples

We give two numerical examples to present the improvement and performance of our stability criteria by comparing the least upper bounds of the parameter  $\lambda_2$  and considering the rate of convergence  $\alpha$  for guaranteeing exponential stability.

**Example 4.1.** Consider system Eq. (2.1) – Eq. (2.3) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix},$$

$$C = 0.1I, \quad \tilde{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

For  $l_1 = 0.1, l_2 = 0.05, l_3 = 0.05$ , and the different  $k_d$  and  $\alpha$ , the maximum value of  $r_2$  with convergence index  $\epsilon$  for exponential passivity of the system Eq. (2.1) is listed in Table 1. Moreover, we compute an upper bound  $\alpha$  with the different  $k_2$  and  $r_2$ , obtained from Theorem 1 are listed in Table 2 for different  $r_2$  and  $k_2$ .

**Table 1.** Maximum allowable bound  $r_2$

	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
$k_d = 0$	0.171	0.117	0.030
$k_d = 0.05$	0.168	0.114	0.026
$k_d = 0.1$	0.165	0.111	0.0220

**Table 2.** Maximum allowable bound  $\alpha$

	$k_2 = 0.5$	$k_2 = 0.9$
$r_2 = 0.01$	0.804	0.710
$r_2 = 0.05$	0.691	0.603
$r_2 = 0.1$	0.486	0.419

**Example 4.2.** Consider system Eq. (2.1) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix},$$

$$C = 0.1I, \quad D_1 = 0.$$

For  $l_1 = 0.1, l_2 = 0.05, l_3 = 0.05$ , the maximum value of  $r_2$  with a convergence index  $\epsilon$  for exponential stability of the system Eq. (2.1) is listed in Table 3.

**Table 3.** Maximum allowable bound  $r_2$

	$r_d = 0.5$	unknown
[18] $\alpha = 0.5$	0.731	-
[6] $\alpha = 0.5$	1.603	-
Corollary 11	-	1.151
[18] $\alpha = 0.7$	0.358	-
[6] $\alpha = 0.7$	1.169	-
Corollary 11	-	0.912
[18] $\alpha = 0.9$	0.103	-
[6] $\alpha = 0.9$	0.691	-
Corollary 11	-	0.711

## 5. Conclusion

From this paper, the problems of the delay-dependent exponential passivity and stability criteria for neutral system with time-varying delays and nonlinear perturbations have been investigated. Applying new class of augmented Lyapunov-Krasovskii functional, we acquire new exponential passivity criteria in terms of LMIs. The improvement and effectiveness of the proposed results have been guaranteed by some numerical examples.

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