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Existence theorems and iterative schemes for nonconvex variational inequalities with different nonlinear operators

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ABSTRACTS

In this research, we introduce the variational inequalities with nonlinear bi-mapping on nonempty closed subset of Hilbert space with is uniformly prox-regular. We define the algorithm for solving the solution of nonconvex variational inequalities and show that this algorithm convergence to the solution of nonconvex variational problems. The particular case some known results in this field.

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CHAPTER I
Introduction

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [1, 7, 8] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [17], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In recent years, Bounkhel *et al.* [5], Moudafi [10], Wen [18], Kazmi *et al.* [9], Noor [12, 13, 15] and the relevant references cited therein, Alimohammady *et al.* [3], Balooee *et al.* [4], suggested and analyzed iterative algorithms for solving some classes (systems) of nonconvex variational inequality problems in the setting of uniformly prox-regular sets.

The existence and iterative scheme of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator are convex sets. Recently, the concept of convex sets has been generalized in many differnt ways. It is known that the uniformly prox-regular sets are an immediate consequence of the generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

In 2003, Bounkhel [6], 2004 Noor [11], Moudafi [10] and 2007 Pang et al. [16], considered the variational inequality probelm over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique.

Recently, in 2009, Noor [14] introduced and studied some new classes of variational and the Wiener-Hiof equations and established the equivalent between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equation, by using the projection technique. Noor also present some new projection methods for solving the nonconvex variational inequalities and prove the convergence of iterative method under suitable conditions.

In the same year, Moudafi [10], introduce the convergence of two-step projection methods for a system of nonconvex variational inequalities problems for a mapping T is γ-strongly monotone and L-Lipschitz continuous.

Very recently, in 2013, Al-Shemas [2], introduced the strongly nonlinear general nonconvex variational inequalities who prove the convergence of the predictor-corrector method only requires pseudomonotonicity, which is weaker condition than monotonicity.

The purpose of this paper is to show that the modified two-step projection method converge to thesolution of theproblem (3.2) and show some existence os solution.

CHAPTER II

Preliminaries

In this chapter, we give some defnitions, notations, and some useful results that will be used in the later chapters.

2.1 Basic results.

Definition 2.1.1. Let X be a linear space over the field $\mathbb{K}$, denote either $\mathbb{R}$ or $\mathbb{C}$ A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a norm on X if it satisfies the following conditions:

(i) $\|x\| \geq 0$, $\forall x \in X$

(ii) $\|x\| = 0 \Leftrightarrow x = 0$

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$

(iv) $\|\alpha x\| = |\alpha|\|x\|$, $\forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.1.2. Let X be a linear space over the field $\mathbb{K}$. A function $< \cdot, \cdot >: X \times X \to \mathbb{K}$ that assigns to each ordered pair (x, y) of vectors in X a scalar $< x, y >$ is said to be an *inner product* on X if it satisfies the following conditions:

(i) $\langle x, x\rangle \geq 0, \quad \forall\ x \in X$ and $< x, x >= 0 \Leftrightarrow x = 0$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}, \quad \forall\ x, y \in X$

(iii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \quad \forall\ x, y \in X$ and $\forall \alpha \in \mathbb{K}$

(iv) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \quad \forall\ x, y, z \in X$.

Definition 2.1.3. A norm space X is said to be a *complete norm space* if every Cauchy sequence in X is a convergent sequence in X.

Definition 2.1.4. A complete norm linear space over the field $\mathbb{K}$ is called a *Banach space* over $\mathbb{K}$.

Definition 2.1.5. A subset C of a linear space X over the field $\mathbb{K}$ is *convex* if for any $x, y \in C$ implies

$$M = \{z \in X : z = \alpha x + (1 - \alpha)y, 0 < \alpha < 1\} \subset C.$$

(M is called *closed segment with boundary point* x, y) or a subset C of X is *convex* if every $x, y \in C$ the segment joining x and y is contained in C.

Definition 2.1.6. A sequence $\{x_n\}$ in a normed space X is said to be $strongly$ $convergence$ (or convergence in norm) if there exists $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0 \text{ denote by } x_n \to x.$$

2.2 Useful lemmas.

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let C be a nonempty closed set in H, not necessarily convex.

First, we recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis.

Definition 2.2.1. The *proximal normal cone* of C at $u \in H$ is given by

$$N_C^P(u) := \{\xi \in H : u \in P_C(u + \alpha\xi)\},$$

where $\alpha > 0$ is a constant and P_C is projection operator of H onto C, that is,

$$P_C(u) = \{u^* \in C : d_C(u) = \|u - u^*\|\},$$

where $d_C(u)$ is the usual distance function to the subset C, that is,

$$d_C(u) = \inf_{v\in C} \|v - u\|.$$

The proximal normal cone $N_C^P(u)$ has the following characterization.

Lemma 2.2.2. *Let* C *be a nonempty closed subset of H. Then* $\xi \in N_C^P(u)$ *if and only if there exists a constant* $\alpha > 0$ *such that*

$$\langle \xi, v - u\rangle \le \alpha\|v - u\|^2, \forall v \in C.$$

Definition 2.2.3. The *Clarke normal cone,* denoted by $N^C(K;u)$, is defined as

$$N_K^C(C;u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}A$ means the closure of the convex hull of A.

Definition 2.2.4. For a given $r \in (0,\infty]$, a subset C of H is said to be *normalized uniformly r-prox-regular* if and only if every nonzero proximal to K can be realized by any r-ball, that is, $\forall u \in C$ and $0 \neq \xi \in N_C^P(u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, v-u\rangle \leq \tfrac{1}{2r}\|v-u\|^2, \forall v \in C$$

Lemma 2.2.5. Let C be a nonempty closed subset of Hilbert space H, $r \in (0, +\infty]$ and set $C_r = \{x \in H : d(x, C) < r\}$. If C is uniformly r-prox-regular, then the following hold:

1. for all $x \in C_r, P_C(x) \neq \emptyset$,

2. for all $s \in (0, r)$, P_C is Lipschitz continuous with constant $\frac{r}{r-s}$ on C_s

Throughout the paper unless otherwise stated, we assume that C is a prox-regular subset of H. Let $T : H \times H \to H$. For any constant $\rho > 0$,we consider for a fixed $u \in C$ the problem of finding $(x_1, x_2) \in H \times H$ such that

$$\langle \rho T(x_1, x_2), y-u\rangle + \frac{1}{2r}\|y-u\|^2 \geq 0, \forall y \in C. \tag{2.1}$$

Lemma 2.2.6. Let H be a real Hilbert space. Then for any $x, y \in H$ we have

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$

(ii) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$

(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0, 1]$.

CHAPTER III
Main Results

3.1 Nonconvex Variational inequality Problems with different nonlinear mapping

Let H be real Hilbert space and C a nonempty closed subset of H. For a mapping $T : C_r \times C_r \to C_r$ we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} y^* - x^* - \rho T(y^*, x^*) &\in N_C^P(x^*) \\ x^* - y^* - \eta T(x^*, y^*) &\in N_C^P(y^*) \end{aligned} \tag{3.2}$$

where ρ, η are fixed positive real numbers.

Let $T : H \times H \to H$ be a nonlinear mapping. Then T is said to be δ-strongly monotone in the first argument if there exists a constant $\delta > 0$ such that

$$\langle T(x, u) - T(y, u), x - y \rangle \geq \delta \|x - y\|^2, \forall x, y, u \in H.$$

A mapping T is said to be (u_1, u_2) mixed Lipschitz continuous if there exist constants $\mu_1, \mu_2 > 0$ such that

$$\|T(x_1, x_2) - T(y_1, y_2)\| \leq u_1\|x_1 - y_1\| + u_2\|x_2 - y_2\|$$

for all $x_1, x_2, y_1, y_2 \in H$. First me prove the following technical Lemmas.

Lemma 3.1.1. Let $(x_1, x_2) \in C_r \times C_r$ be a solution of (3.2) if and only if it satisfies the relation

$$u = P_{C_r}[u - \rho T(x_1, x_2)], \tag{3.3}$$

for each $u \in H, \rho > 0$ and P_{C_r} is the projection operator of H onto the prox-regular set C_r.

พิสูจน์. From Lemma 2.2.2, we have the solution (3.2) equivalent to

$$\begin{aligned} 0 \in \rho T(x_1, x_2) + N_C^P(u) &\Leftrightarrow 0 \in \rho T(x_1, x_2) - u + u + N_C^P(u) \\ &\Leftrightarrow 0 \in \rho T(x_1, x_2) - u + (I + N_C^P)(u) \\ &\Leftrightarrow u \in (I + N_C^P)^{-1}(u - \rho T(x_1, x_2)) \\ &\Leftrightarrow u \in proj_C(u - \rho T(x_1, x_2)) \end{aligned}$$

where $(I + N_C^P)^{-1} = P_C$. hence $u = P_C[u - \rho T(x_1, x_2)]$. □

Now, in this paper we introduce a nonconvex variational inequality problems define on the uniformly prox-regular set in Hilbert space. By using the properties of projection operator over uniformly prox-regular set, we suggest some iterative algorithms for finding the approximate solution of nonconvex variational inequalities problems.

Algorithm 3.1.2. Let C be an r-uniformly prox-regular subset of H. Assume that $T : C_r \times C_r \to H$ is nonlinear mapping. Let $x_0 \in C_r$, we consider the following two-step projection method:

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T(x_n, x_n)] \\ x_{n+1} &= P_{C_r}[y_n - \rho T(y_n, y_n)] \end{aligned} \tag{3.4}$$

where ρ, η are positive real number, which appeared in problem (3.2).

Now we will prove the existence theorem of problem (3.1), when C_r is a closed uniformly r-uniformly prox-regular.

Theorem 3.1.3. Let C be a uniformly r-uniformly prox-regular closed subset of a Hilbert space H, and let $T : C_r \times C_r \to H$ be a nonlinear mapping such that T is a γ-strongly monotone and (μ_1, μ_2) mined Lipschitz continuous mapping and satisfy the following condition, there exists $s \in (0, r)$ such that

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \epsilon < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \epsilon, \frac{1}{t_s \mu_2}\}, \tag{3.5}$$

where $t_s = \frac{r}{(r-s)}$ and $\epsilon = \frac{\sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$.

Then the problem (3.2)) has solution. Moreover, the sequence (x_n, y_n) which is generated by (4.17) strongly converges to a solution $(x^*, y^*) \in C_r \times C_r$ of the problem (3.2).

พิสูจน์. From the algorithm (4.17), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \big\|P_C[y_n - \rho T(y_n, y_n)] - P_C[y_{n-1} - \rho T(y_{n-1}, y_{n-1})]\big\| \\ &\le t_s\big\|y_n - y_{n-1} - \rho(T(y_n, y_n) - T(y_{n-1}, y_{n-1}))\big\| \\ &\le t_s\big[\|y_n - y_{n-1}\| - \rho(T(y_n, y_n) - T(y_{n-1}, y_n) - \rho(T(y_{n-1}, y_n) - T(y_{n-1}, y_{n-1}))\big] \\ &\le t_s\big[\|y_n - y_{n-1} - \rho(T(y_n, y_n) - T(y_{n-1}, y_n))\| + \rho\|T(y_{n-1}, y_n) - T(y_{n-1}, y_{n-1})\|\big] \end{aligned} \tag{3.6}$$

Since the mapping T is γ-strongly monotone and (μ_1, μ_2) Lipschitz continuous, we obtain

$$\begin{aligned} \|y_n - y_{n-1} - \rho(T(y_n, y_n) - T(y_{n-1}, y_n))\|^2 &\le \|y_n - y_{n-1}\|^2 - 2\rho\langle y_n - y_{n-1}, T(y_n, y_n) - T(y_{n-1}, y_n)\rangle \\ &\quad + \rho^2\|T(y_n, y_n) - T(y_{n-1}, y_n)\|^2 \\ &\le \|y_n - y_n\|^2 - 2\gamma\rho\|y_n - y_{n-1}\|^2 + \rho^2\mu_1\|y_n - y_{n-1}\|^2 \\ &= [1 - 2\gamma\rho + \rho^2\mu_1]\|y_n - y_{n-1}\|^2. \end{aligned}$$

Hence,

$$\|y_n - y_{n-1} - \rho(T(y_n, y_n) - T(y_{n-1}, y_n))\| \le \sqrt{1 - 2\gamma\rho + \rho^2\mu_1^2}\|y_n - y_{n-1}\|, \tag{3.7}$$

and then we have

$$\|T(y_{n-1}, y_n) - T(y_{n-1}, y_{n-1})\| \leq \mu_2 \|y_n - y_{n-1}\|. \tag{3.8}$$

Replace (3.7) and (3.8) in (3.6), it follows that

$$\|x_{n+1} - x_n\| \;\; \leq \;\; t_s[\mu_2 \rho + \sqrt{1 - 2\gamma\rho r + \rho^2 \mu^2}]\|y_n - y_{n-1}\|. \tag{3.9}$$

Next, we consider

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \big\|P_C[x_n - \eta T(x_n, x_n)] - P_C[x_{n-1} - \eta T(x_{n-1}, x_{n-1})]\big\| \\
&\leq t_s \big\|x_n - x_{n-1} - \eta(T(x_n, x_n) - T(x_{n-1}, x_{n-1}))\big\| \\
&\leq t_s \big[\|x_n - x_{n-1}\| - \eta(T(x_n, x_n) - T(x_{n-1}, x_n) - \eta(T(x_{n-1}, x_n) - T(x_{n-1}, x_{n-1}))\big] \\
&\leq t_s \big[\|x_n - x_{n-1} - \eta(T(x_n, x_n) - T(x_{n-1}, x_n))\| + \eta\|T(x_{n-1}, y_n) - T(x_{n-1}, x_{n-1})\|\big]
\end{aligned} \tag{3.10}$$

From T is γ-strongly monotone and (μ_1, μ_2) Lipschitz continuous, we obtain

$$\begin{aligned}
\|x_n - x_{n-1} - \rho(T(x_n, x_n) - T(x_{n-1}, x_n))\|^2 &\leq \|x_n - x_{n-1}\|^2 - 2\eta\langle x_n - x_{n-1}, T(x_n, x_n) - T(x_{n-1}, x_n)\rangle \\
&\quad + \eta^2\|T(x_n, x_n) - T(x_{n-1}, x_n)\|^2 \\
&\leq \|x_n - x_n\|^2 - 2\gamma\eta\|x_n - x_{n-1}\|^2 + \eta^2\mu_1\|x_n - x_{n-1}\|^2 \\
&= [1 - 2\gamma\eta + \eta^2\mu_1]\|x_n - x_{n-1}\|^2.
\end{aligned}$$

Hence,

$$\|x_n - x_{n-1} - \rho(T(x_n, x_n) - T(x_{n-1}, x_n))\| \leq \sqrt{1 - 2\gamma\eta + \eta^2\mu_1^2}\|x_n - x_{n-1}\|, \tag{3.11}$$

and then we have

$$\|T(x_{n-1}, x_n) - T(x_{n-1}, x_{n-1})\| \leq \mu_2\|x_n - x_{n-1}\|. \tag{3.12}$$

Combining (3.11) and (3.12) in (3.10), it follows that

$$\|y_{n+1} - y_n\| \leq t_s[\mu_2\eta + \sqrt{1 - 2\gamma\eta r + \eta^2\mu^2}]\|x_n - x_{n-1}\|. \tag{3.13}$$

Replacing (3.13) into (3.9), it implies that

$$\|x_{n+1} - x_n\| \leq t_s^2 \triangle_\rho \triangle_\eta \|x_n - x_{n-1}\|, \tag{3.14}$$

where $\triangle_\rho = \mu_2\rho + \sqrt{1 - 2\gamma\rho + \rho^2\mu_1^2}$ and $\triangle_\eta = \mu_2\eta + \sqrt{1 - 2\gamma\eta + \eta^2\mu_1^2}$. From condition (4.18), we have $t_s\triangle_\rho$ and $t_s\triangle_\eta$ are element in $(0, 1)$. From (3.14), it implies that

$$\|x_{n+1} - x_n\| \leq \kappa^n\|x_n - x_{n-1}\|, \text{ for all } n = 1, 2, 3, ...,$$

where $\kappa = t_s^2\triangle_\rho\triangle_\eta$. Thus, for any $m > n > 1$, it follows that

$$\|x_m - x_n\| \leq \sum_{i=1}^{m-1}\|x_{i+1} - x_i\| \leq \sum_{i=1}^{m-1}\kappa^i\|x_1 - x_0\| \leq \frac{\kappa^n}{1-\kappa}\|x_1 - x_0\|.$$

Since $\kappa < 1$, it follows that $\lim_{n\to\infty}\kappa^n = 0$ and then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in C_r and from (3.13), we have $\{y_n\}^{\infty}$ is a Cauchy sequence in C_r. By Lemma 3.1.1, there exists $(x^*, y^*) \in C_r \times C_r$ such that $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$.

Now, we claim that $(x^*, y^*) \in C_r \times C_r$ is a solution of problem (3.2). By the definition of the proximal normal cone, from (4.17), we have

$$\begin{aligned}(x_n - y_n) - \eta T(x_n, x_n) &\in N_C^P(y_n)\\ (y_n - x_{n+1}) - \rho T(y_n, y_n) &\in N_C^P(x_{n+1})\end{aligned}$$

By letting $n \to \infty$, using the closedness property of the proximal cone together with the continuity of T, it follows that

$$\begin{aligned}(x^* - y^*) - \eta T(x^*, x^*) &\in N_C^P(y^*)\\ (y^* - x^*) - \rho T(y^*, y^*) &\in N_C^P(x^*).\end{aligned}$$

This complete the proof. □

3.2 Existence result

In this section we can using the Theorem 3.1.3 obtain an existence theorem of the following problem: find $x^* \in C$ such that

$$-T(x^*, x^*) \in N_{C_r}^P(x^*). \tag{3.15}$$

Theorem 3.2.1. A nonconvex variational inequality problem (3.15) is equivalent to the following nonconvex variational inclusions of finding $(x^*, x^*) \in C_r \times C_r$ such that

$$0 \in T(x^*, x^*) + N_{C_r}^P(x^*),$$

where $N_{C_r}^P(x^*)$ denotes the proximal normal cone of C_r at $x^* \in H$ in the sense of nonconvex analysis.

พิสูจน์. Let $(x^*, x^*) \in C \times C$ be a solution of (3.15), we have

$$\langle T(x^*, x^*), y - x^*\rangle + \frac{1}{2r}\|y - x^*\|^2, \forall y \in C_r,$$

it implies that

$$\langle -T(x_1, x_2), y - u \rangle \leq \frac{1}{2r}\|y - x^*\|^2, \forall y \in C_r,$$

If $T(x^*, x^*) = 0$ it clear. Assume that $T(x^*, x^*) \neq 0$, by Lemma 2.2.2, taking $\alpha = \frac{1}{2r}$, we have

$$-T(x^*, x^*) \in N^P_{C_r}(x^*)$$

it follows that

$$0 \in T(x_1, x_2) + N^P_{C_r}(x^*)$$

□

CHAPTER IV

CONCLUSIONS

From chapter 3 we have 2 theorems for submitted to thai journals of mathematics.

4.1 Outputs Results.

Lemma 4.1.1. Let $(x_1, x_2) \in C_r \times C_r$ be a solution of (3.2) if and only if it satisfies the relation

$$u = P_{C_r}[u - \rho T(x_1, x_2)], \tag{4.16}$$

for each $u \in H, \rho > 0$ and P_{C_r} is the projection operator of H onto the prox-regular set C_r.

Algorithm 4.1.2. Let C be an r-uniformly prox-regular subset of H. Assume that $T : C_r \times C_r \to H$ is nonlinear mapping. Let $x_0 \in C_r$, we consider the following two-step projection method:

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T(x_n, x_n)] \\ x_{n+1} &= P_{C_r}[y_n - \rho T(y_n, y_n)] \end{aligned} \tag{4.17}$$

where ρ, η are positive real number, which appeared in problem (3.2).

Now we will prove the existence theorem of problem (3.1), when C_r is a closed uniformly r-uniformly prox-regular.

Theorem 4.1.3. Let C be a uniformly r-uniformly prox-regular closed subset of a Hilbert space H, and let $T : C_r \times C_r \to H$ be a nonlinear mapping such that T is a γ-strongly monotone and (μ_1, μ_2) mined Lipschitz continuous mapping and satisfy the following condition, there exists $s \in (0, r)$ such that

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \epsilon < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \epsilon, \frac{1}{t_s \mu_2}\}, \tag{4.18}$$

where $t_s = \frac{r}{(r-s)}$ and $\epsilon = \frac{\sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$.

Then the problem (3.2)) has solution. Moreover, the sequence (x_n, y_n) which is generated by (4.17) strongly converges to a solution $(x^*, y^*) \in C_r \times C_r$ of the problem (3.2).

Theorem 4.1.4. A nonconvex variational inequality problem (3.15) is equivalent to the following nonconvex variational inclusions of finding $(x^*, x^*) \in C_r \times C_r$ such that

$$0 \in T(x^*, x^*) + N_{C_r}^P(x^*),$$

where $N_{C_r}^P(x^*)$ denotes the proximal normal cone of C_r at $x^* \in H$ in the sense of nonconvex analysis.

4.2 Outputs 1 paper.

1. Iterative Algorithm for Solving Nonconvex Variational inequality Problems with nonlinear mapping. Submitted to Thai Journal of Mathematics, and oral presentation on the Tenth International Conference on Nonlinear Analysis and Convex Analysis. July 4th - 9 th 2017, in Chitose, Hokkaido, Japan.

หนังสืออ้างอิง

[1] A. Bensoussan and J. L. Lions, *Application des Inequations Variationelles en Control et en Stochastiques*, Dunod, Paris 1978.

[2] E. Al-Shemas, *An Implicit Method For Solving Nonconvex Variational Inequalities*, Italian Journal of Pure and Applied Mathematics, 31(2013), 333-342.

[3] M. Alimohammady, J.Baloone, Y.J. Cho and M. Roohi, *Iterative algorythms for a new class of extended general nonconvex set-valued variational inequalities*, Nonlinear Anal. TMA sereies A 73(12)(2010) 3907-3923.

[4] J. Balooee, Y.J. Cho and M.K. Kang, *Projection methods and a new system of extended general regularized nonconvex set-valued variational inequalities* , J. Appl. Math. Article ID(2012) 690-648.

[5] M. Bounkhel, L.Tadj and A. Hamdi, *Iterative schemes to solve nonconvex variational problems* , J. Inequal. pure Appl. Math. 28(4) (2003), 1-14.

[6] M. Bounkhel, L. Tadj, A. Hamdi, *Iterative schemes to solve nonconvex variational problems*, J. Inequal. Pure Appl. Math. 4(2003), 1-14.

[7] R. Glowinski and P. Letallec, *Augmented Kargrangin and Operator-splitting Methods in Control Theory*, Springer-Verlag, New York, 1989.

[8] P. T. Harker and J. S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithm and applications*, Math. Program. 48(1990) 161 - 220.

[9] K.R. Kazmi, F.A. Khan and M. Shahzad, *Two step algorithm for solving regularized generalized mixed variational inequality problem* , Bull. Korean Math. Soc. 47(2010) 675-685.

[10] A. Moudafi, *Projection methods for a system of nonconvex variational inequalities*, Nonlinear Anal. TMA. sereies A 71(2009) 517-520.

[11] M. A. Noor, *Iterative schemes for nonconvex variational inequalities*, Journal of Optimization Theory and Applications, vol. 121, no. 2(2004), 385-395.

[12] M. A Noor, *Projection methods for nonconvex variational inequalities*, Optim. Lett. 3(2009) 411-418.

[13] M.A. Noor and E. Al-said, *Iterative methods for general nonconvex variational inequalities*, Appl. Comput. Math 10(2)(2011) 309-320.

[14] M. A Noor, *Iterative methods for general nonconvex variational inequalities*,Albanian J. Math. 3(2009), 117-127.

[15] M.A. Noor and K.I. Noor, *New system of general nonconvex variational inequalities*, Appl. Math. E-Notes 10(2010) 76-85.

[16] L. P. Pang, J. Shen and H. S. Song, *A modified predictor-corrector Algorithm for solving Nonlinear Generalized Variational Inequalities*, Computers Math. Appl., 54(2007), 319-325.

[17] G. Stampacchia, *Formes bilinยคeaires coercitives sur les ensembles convexes*,Comptes Rendus de lย'Acadยคemie des Sciences, vol. 258(1964) 4413-4416.

[18] D.J. Wen *Projection methods for a generalized system of nonconvex variational inequalities with different nonlinear operators* , Nonlinear Analysis 73(2010) 2292-2297.

BIOGRAPHY

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APPENDIX

Iterative Algorithm for Solving Nonconvex Variational inequality Problems with nonlinear mapping

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Abstract

In this paper, we introduce the variational inequalities with nonlinear bi-mapping on nonempty closed subset of Hilbert space with is uniformly prox-regular. We define the algorithm for solving the solution of nonconvex variational inequalities and show that this algorithm convergence to the solution of nonconvex variational problems. The particular case some known results in this field.

1 Introduction

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [1, 7, 8] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [17], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In recent years, Bounkhel *et al.* [5], Moudafi [10], Wen [18], Kazmi *et al.* [9], Noor [12, 13, 15] and the relevant references cited therein, Alimohammady *et al.* [3], Balooee *et al.* [4], suggested and analyzed iterative algorithms for solving some classes (systems) of nonconvex variational inequality problems in the setting of uniformly prox-regular sets.

The existence and iterative scheme of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator are convex sets. Recently, the concept of convex sets has been generalized in many diffent ways. It is known that

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the uniformly prox-regular sets are an immediate consequence of the generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

In 2003, Bounkhel [6], 2004 Noor [11], Moudafi [10] and 2007 Pang et al. [16], considered the variational inequality probelm over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique.

Recently, in 2009, Noor [14] introduced and studied some new classes of variational and the Wiener-Hiof equations and established the equivalent between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equation, by using the projection technique. Noor also present some new projection methods for solving the nonconvex variational inequalities and prove the convergence of iterative method under suitable conditions.

In the same year, Moudafi [10], introduce the convergence of two-step projection methods for a system of nonconvex variational inequalities problems for a mapping T is γ-strongly monotone and L-Lipschitz continuous.

Very recently, in 2013, Al-Shemas [2], introduced the strongly nonlinear general nonconvex variational inequalities who prove the convergence of the predictor-corrector method only requires pseudomonotonicity, which is weaker condition than monotonicity.

The purpose of this paper is to show that the modified two-step projection method converge to thesolution of theproblem (3.1) and show some existence os solution.

2 Preliminary

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let C be a nonempty closed set in H, not necessarily convex.

First, we recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. The *proximal normal cone* of C at $u \in H$ is given by

$$N_C^P(u) := \{\xi \in H : u \in P_C(u + \alpha\xi)\},$$

where $\alpha > 0$ is a constant and P_C is projection operator of H onto C, that is,

$$P_C(u) = \{u^* \in C : d_C(u) = \|u - u^*\|\},$$

where $d_C(u)$ is the usual distance function to the subset C, that is,

$$d_C(u) = \inf_{v \in C} \|v - u\|.$$

The proximal normal cone $N_C^P(u)$ has the following characterization.

Lemma 2.2. *Let C be a nonempty closed subset of H. Then $\xi \in N_C^P(u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \xi, v-u\rangle \leq \alpha\|v-u\|^2, \forall v \in C.$$

Definition 2.3. The *Clarke normal cone,* denoted by $N^C(K;u)$, is defined as

$$N_K^C(C;u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}A$ means the closure of the convex hull of A.

Definition 2.4. For a given $r \in (0,\infty]$, a subset C of H is said to be *normalized uniformly r-prox-regular* if and only if every nonzero proximal to K can be realized by any r-ball, that is, $\forall u \in C$ and $0 \neq \xi \in N_C^P(u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, v-u\rangle \leq \tfrac{1}{2r}\|v-u\|^2, \forall v \in C$$

Lemma 2.5. *Let C be a nonempty closed subset of Hilbert space H, $r \in (0,+\infty]$ and set $C_r = \{x \in H : d(x,C) < r\}$. If C is uniformly r-prox-regular, then the following hold:*

1. *for all $x \in C_r, P_C(x) \neq \emptyset$,*
2. *for all $s \in (0,r)$, P_C is Lipschitz continuous with constant $\frac{r}{r-s}$ on C_s*

Throughout the paper unless otherwise stated, we assume that C is a prox-regular subset of H. Let $T : H \times H \to H$. For any constant $\rho > 0$,we consider for a fixed $u \in C$ the problem of finding $(x_1, x_2) \in H \times H$ such that

$$\langle \rho T(x_1,x_2), y-u\rangle + \frac{1}{2r}\|y-u\|^2 \geq 0, \forall y \in C. \tag{2.1}$$

3 Main results

Let H be real Hilbert space and C a nonempty closed subset of H. For a mapping $T : C_r \times C_r \to C_r$ we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &y^* - x^* - \rho T(y^*, x^*) \in N_C^P(x^*) \\ &x^* - y^* - \eta T(x^*, y^*) \in N_C^P(y^*) \end{aligned} \tag{3.1}$$

where ρ, η are fixed positive real numbers.

Let $T : H \times H \to H$ be a nonlinear mapping. Then T is said to be δ-strongly monotone in the first argument if there exists a constant $\delta > 0$ such that

$$\langle T(x,u) - T(y,u), x-y\rangle \geq \delta\|x-y\|^2, \forall x, y, u \in H.$$

A mapping T is said to be (u_1, u_2) mixed Lipschitz continuous if there exist constants $\mu_1, \mu_2 > 0$ such that

$$\|T(x_1,x_2) - T(y_1,y_2)\| \leq u_1\|x_1-y_1\| + u_2\|x_2-y_2\|$$

for all $x_1, x_2, y_1, y_2 \in H$. First me prove the following technical Lemmas.

Lemma 3.1. *Let $(x_1, x_2) \in C_r \times C_r$ be a solution of (3.1) if and only if it satisfies the relation*

$$u = P_{C_r}[u - \rho T(x_1, x_2)], \tag{3.2}$$

for each $u \in H, \rho > 0$ and P_{C_r} is the projection operator of H onto the prox-regular set C_r.

Proof. From Lemma 2.2, we have the solution (3.1) equivalent to

$$\begin{aligned} 0 \in \rho T(x_1, x_2) + N_C^P(u) \quad &\Leftrightarrow \quad 0 \in \rho T(x_1, x_2) - u + u + N_C^P(u) \\ &\Leftrightarrow \quad 0 \in \rho T(x_1, x_2) - u + (I + N_C^P)(u) \\ &\Leftrightarrow \quad u \in (I + N_C^P)^{-1}(u - \rho T(x_1, x_2)) \\ &\Leftrightarrow \quad u \in proj_C(u - \rho T(x_1, x_2)) \end{aligned}$$

where $(I + N_C^P)^{-1} = P_C$. hence $u = P_C[u - \rho T(x_1, x_2)]$. □

Now, in this paper we introduce a nonconvex variational inequality problems define on the uniformly prox-regular set in Hilbert space. By using the properties of projection operator over uniformly prox-regular set, we suggest some iterative algorithms for finding the approximate solution of nonconvex variational inequalities problems.

Algorithm 3.2. Let C be an r-uniformly prox-regular subset of H. Assume that $T : C_r \times C_r \to H$ is nonlinear mapping. Let $x_0 \in C_r$, we consider the following two-step projection method:

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T(x_n, x_n)] \\ x_{n+1} &= P_{C_r}[y_n - \rho T(y_n, y_n)] \end{aligned} \tag{3.3}$$

where ρ, η are positive real number, which appeared in problem (3.1).

Now we will prove the existence theorem of problem (3.1), when C_r is a closed uniformly r-uniformly prox-regular.

Theorem 3.3. *Let C be a uniformly r-uniformly prox-regular closed subset of a Hilbert space H, and let $T : C_r \times C_r \to H$ be a nonlinear mapping such that T is a γ-strongly monotone and (μ_1, μ_2) mined Lipschitz continuous mapping and satisfy the following condition, there exists $s \in (0, r)$ such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \epsilon < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \epsilon, \frac{1}{t_s \mu_2}\}, \tag{3.4}$$

where $t_s = \frac{r}{(r-s)}$ and $\epsilon = \frac{\sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$.

Then the problem (3.1)) has solution. Moreover, the sequence (x_n, y_n) which is generated by (3.3) strongly converges to a solution $(x^*, y^*) \in C_r \times C_r$ of the problem (3.1).

Proof. From the algorithm (3.3), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \big\|P_C[y_n-\rho T(y_n,y_n)]-P_C[y_{n-1}-\rho T(y_{n-1},y_{n-1})]\big\|\\ &\le t_s\big\|y_n-y_{n-1}-\rho(T(y_n,y_n)-T(y_{n-1},y_{n-1}))\big\|\\ &\le t_s\big[\|y_n-y_{n-1}\|-\rho(T(y_n,y_n)-T(y_{n-1},y_n)-\rho(T(y_{n-1},y_n)-T(y_{n-1},y_{n-1}))\big]\\ &\le t_s\big[\|y_n-y_{n-1}-\rho(T(y_n,y_n)-T(y_{n-1},y_n))\|+\rho\|T(y_{n-1},y_n)-T(y_{n-1},y_{n-1})\|\big].\end{aligned}\tag{3.5}$$

Since the mapping T is γ-strongly monotone and (μ_1,μ_2) Lipschitz continuous, we obtain

$$\begin{aligned}\|y_n-y_{n-1}-\rho(T(y_n,y_n)-T(y_{n-1},y_n))\|^2 &\le \|y_n-y_{n-1}\|^2-2\rho\langle y_n-y_{n-1},T(y_n,y_n)-T(y_{n-1},y_n)\rangle\\ &\quad+\rho^2\|T(y_n,y_n)-T(y_{n-1},y_n)\|^2\\ &\le \|y_n-y_n\|^2-2\gamma\rho\|y_n-y_{n-1}\|^2+\rho^2\mu_1\|y_n-y_{n-1}\|^2\\ &= [1-2\gamma\rho+\rho^2\mu_1]\|y_n-y_{n-1}\|^2.\end{aligned}$$

Hence,

$$\|y_n-y_{n-1}-\rho(T(y_n,y_n)-T(y_{n-1},y_n))\|\le\sqrt{1-2\gamma\rho+\rho^2\mu_1^2}\|y_n-y_{n-1}\|,\tag{3.6}$$

and then we have

$$\|T(y_{n-1},y_n)-T(y_{n-1},y_{n-1})\|\le\mu_2\|y_n-y_{n-1}\|.\tag{3.7}$$

Replace (3.6) and (3.7) in (3.5), it follows that

$$\|x_{n+1}-x_n\| \le t_s[\mu_2\rho+\sqrt{1-2\gamma\rho r+\rho^2\mu^2}]\|y_n-y_{n-1}\|.\tag{3.8}$$

Next, we consider

$$\begin{aligned}\|y_{n+1}-y_n\| &= \big\|P_C[x_n-\eta T(x_n,x_n)]-P_C[x_{n-1}-\eta T(x_{n-1},x_{n-1})]\big\|\\ &\le t_s\big\|x_n-x_{n-1}-\eta(T(x_n,x_n)-T(x_{n-1},x_{n-1}))\big\|\\ &\le t_s\big[\|x_n-x_{n-1}\|-\eta(T(x_n,x_n)-T(x_{n-1},x_n)-\eta(T(x_{n-1},x_n)-T(x_{n-1},x_{n-1}))\big]\\ &\le t_s\big[\|x_n-x_{n-1}-\eta(T(x_n,x_n)-T(x_{n-1},x_n))\|+\eta\|T(x_{n-1},y_n)-T(x_{n-1},x_{n-1})\|\big].\end{aligned}\tag{3.9}$$

From T is γ-strongly monotone and (μ_1,μ_2) Lipschitz continuous, we obtain

$$\begin{aligned}\|x_n-x_{n-1}-\rho(T(x_n,x_n)-T(x_{n-1},x_n))\|^2 &\le \|x_n-x_{n-1}\|^2-2\eta\langle x_n-x_{n-1},T(x_n,x_n)-T(x_{n-1},x_n)\rangle\\ &\quad+\eta^2\|T(x_n,x_n)-T(x_{n-1},x_n)\|^2\\ &\le \|x_n-x_n\|^2-2\gamma\eta\|x_n-x_{n-1}\|^2+\eta^2\mu_1\|x_n-x_{n-1}\|^2\\ &= [1-2\gamma\eta+\eta^2\mu_1]\|x_n-x_{n-1}\|^2.\end{aligned}$$

Hence,

$$\|x_n-x_{n-1}-\rho(T(x_n,x_n)-T(x_{n-1},x_n))\|\le\sqrt{1-2\gamma\eta+\eta^2\mu_1^2}\|x_n-x_{n-1}\|,\tag{3.10}$$

and then we have

$$\|T(x_{n-1},x_n)-T(x_{n-1},x_{n-1})\| \le \mu_2\|x_n-x_{n-1}\|. \tag{3.11}$$

Combining (3.10) and (3.11) in (3.9), it follows that

$$\|y_{n+1}-y_n\| \le t_s[\mu_2\eta+\sqrt{1-2\gamma\eta r+\eta^2\mu^2}]\|x_n-x_{n-1}\|. \tag{3.12}$$

Replacing (3.12) into (3.8), it implies that

$$\|x_{n+1}-x_n\| \le t_s^2\triangle_\rho\triangle_\eta\|x_n-x_{n-1}\|, \tag{3.13}$$

where $\triangle_\rho=\mu_2\rho+\sqrt{1-2\gamma\rho+\rho^2\mu_1^2}$ and $\triangle_\eta=\mu_2\eta+\sqrt{1-2\gamma\eta+\eta^2\mu_1^2}$. From condition (3.4), we have $t_s\triangle_\rho$ and $t_s\triangle_\eta$ are element in $(0,1)$. From (3.13), it implies that

$$\|x_{n+1}-x_n\| \le \kappa^n\|x_n-x_{n-1}\|, \text{ for all } n=1,2,3,...,$$

where $\kappa=t_s^2\triangle_\rho\triangle_\eta$. Thus, for any $m>n>1$, it follows that

$$\|x_m-x_n\| \le \sum_{i=1}^{m-1}\|x_{i+1}-x_i\| \le \sum_{i=1}^{m-1}\kappa^i\|x_1-x_0\| \le \frac{\kappa^n}{1-\kappa}\|x_1-x_0\|.$$

Since $\kappa<1$, it follows that $\lim_{n\to\infty}\kappa^n=0$ and then $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in C_r and from (3.12), we have $\{y_n\}^\infty$ is a Cauchy sequence in C_r. By Lemma 3.1, there exists $(x^*,y^*)\in C_r\times C_r$ such that $(x_n,y_n)\to(x^*,y^*)$ as $n\to\infty$.

Now, we claim that $(x^*,y^*)\in C_r\times C_r$ is a solution of problem (3.1). By the definition of the proximal normal cone, from (3.3), we have

$$\begin{aligned}(x_n-y_n)-\eta T(x_n,x_n) &\in N_C^P(y_n)\\ (y_n-x_{n+1})-\rho T(y_n,y_n) &\in N_C^P(x_{n+1})\end{aligned}$$

By letting $n\to\infty$, using the closedness property of the proximal cone together with the continuity of T, it follows that

$$\begin{aligned}(x^*-y^*)-\eta T(x^*,x^*) &\in N_C^P(y^*)\\ (y^*-x^*)-\rho T(y^*,y^*) &\in N_C^P(x^*).\end{aligned}$$

This complete the proof. □

4 Existence result

In this section we can using the Theorem 3.3 obtain an existence theorem of the following problem: find $x^* \in C$ such that

$$-T(x^*, x^*) \in N^P_{C_r}(x^*). \tag{4.1}$$

Lemma 4.1. *A nonconvex variational inequality problem (4.1) is equivalent to the following nonconvex variational inclusions of finding* $(x^*, x^*) \in C_r \times C_r$ *such that*

$$0 \in T(x^*, x^*) + N^P_{C_r}(x^*),$$

where $N^P_{C_r}(x^*)$ *denotes the proximal normal cone of* C_r *at* $x^* \in H$ *in the sense of nonconvex analysis.*

Proof. Let $(x^*, x^*) \in C \times C$ be a solution of (4.1), we have

$$\langle T(x^*, x^*), y - x^*\rangle + \frac{1}{2r}\|y - x^*\|^2, \forall y \in C_r,$$

it implies that

$$\langle -T(x_1, x_2), y - u\rangle \leq \frac{1}{2r}\|y - x^*\|^2, \forall y \in C_r,$$

If $T(x^*, x^*) = 0$ it clear. Assume that $T(x^*, x^*) \neq 0$, by Lemma 2.2, taking $\alpha = \frac{1}{2r}$, we have

$$-T(x^*, x^*) \in N^P_{C_r}(x^*)$$

it follows that

$$0 \in T(x_1, x_2) + N^P_{C_r}(x^*)$$

□

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References

[1] A. Bensoussan and J. L. Lions, *Application des Inequations Variationelles en Control et en Stochastiques*, Dunod, Paris 1978.

[2] E. Al-Shemas, *An Implicit Method For Solving Nonconvex Variational Inequalities*, Italian Journal of Pure and Applied Mathematics, 31(2013), 333-342.

[3] M. Alimohammady, J.Baloone, Y.J. Cho and M. Roohi, *Iterative algorythms for a new class of extended general nonconvex set-valued variational inequalities*, Nonlinear Anal. TMA sereies A 73(12)(2010) 3907-3923.

[4] J. Balooee, Y.J. Cho and M.K. Kang, *Projection methods and a new system of extended general regularized nonconvex set-valued variational inequalities* , J. Appl. Math. Article ID(2012) 690-648.

[5] M. Bounkhel, L.Tadj and A. Hamdi, *Iterative schemes to solve nonconvex variational problems* , J. Inequal. pure Appl. Math. 28(4) (2003), 1-14.

[6] M. Bounkhel, L. Tadj, A. Hamdi, *Iterative schemes to solve nonconvex variational problems*, J. Inequal. Pure Appl. Math. 4(2003), 1-14.

[7] R. Glowinski and P. Letallec, *Augmented Kargrangin and Operator-splitting Methods in Control Theory*, Springer-Verlag, New York, 1989.

[8] P. T. Harker and J. S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithm and applications*, Math. Program. 48(1990) 161 - 220.

[9] K.R. Kazmi, F.A. Khan and M. Shahzad, *Two step algorithm for solving regularized generalized mixed variational inequality problem* , Bull. Korean Math. Soc. 47(2010) 675-685.

[10] A. Moudafi, *Projection methods for a system of nonconvex variational inequalities*, Nonlinear Anal. TMA. sereies A 71(2009) 517-520.

[11] M. A. Noor, *Iterative schemes for nonconvex variational inequalities*, Journal of Optimization Theory and Applications, vol. 121, no. 2(2004), 385-395.

[12] M. A Noor, *Projection methods for nonconvex variational inequalities*, Optim. Lett. 3(2009) 411-418.

[13] M.A. Noor and E. Al-said, *Iterative methods for general nonconvex variational inequalities*, Appl. Comput. Math 10(2)(2011) 309-320.

[14] M. A Noor, *Iterative methods for general nonconvex variational inequalities*,Albanian J. Math. 3(2009), 117-127.

[15] M.A. Noor and K.I. Noor, *New system of general nonconvex variational inequalities*, Appl. Math. E-Notes 10(2010) 76-85.

[16] L. P. Pang, J. Shen and H. S. Song, *A modified predictor-corrector Algorithm for solving Nonlinear Generalized Variational Inequalities*, Computers Math. Appl., 54(2007), 319-325.

[17] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*,Comptes Rendus de lAcademie des Sciences, vol. 258(1964) 4413-4416.

[18] D.J. Wen *Projection methods for a generalized system of nonconvex variational inequalities with different nonlinear operators* , Nonlinear Analysis 73(2010) 2292-2297.