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On the (s,t)-Pell and (s,t)-Pell-Lucas Polynomials

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Abstract

In this paper, we introduced the generalizations of Pell and Pell-Lucas polynomials, which are called (s,t)-Pell and (s,t)- Pell-Lucas polynomials. We also give the Binet formula and the generating function for these polynomials. Finally, we obtain some identities by using the Binet formulas.

Keywords: (s,t)-Pell Number, (s,t)-Pell-Lucas Number, (s,t)-Pell Polynomial, (s,t)-Pell-Lucas Polynomial.

1. Introduction

For over years, many recursive sequences have been studied in the literature. The famous examples of these sequences are Fibonacci, Lucas, Pell, and Pell-Lucas. They are used in many research areas such as Engineering, Architecture, Nature, and Art (for examples, see: (1-6)). The classical Fibonacci $\{F_n\}_{n=0}^{+\infty}$, Lucas $\{L_n\}_{n=0}^{+\infty}$, Pell $\{P_n\}_{n=0}^{+\infty}$, and Pell-Lucas $\{Q_n\}_{n=0}^{+\infty}$ sequences are defined by

$$F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2},$$
$$L_0 = 2, L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2},$$
$$P_0 = 0, P_1 = 1 \text{ and } P_n = 2P_{n-1} + P_{n-2},$$
$$Q_0 = 2, Q_1 = 2 \text{ and } Q_n = 2Q_{n-1} + Q_{n-2},$$

for $n \geq 2$, respectively. For more detailed information on the Fibonacci, Lucas, Pell, and Pell-Lucas sequences can be found in (1-2).

Recently, Fibonacci, Lucas, Pell, and Pell-Lucas sequences were generalized and studied by many authors in different ways to derive many identities. In 2012, Gulec and Taskara (7) introduced new generalizations of Pell and Pell-Lucas sequences, which are called (s,t)-Pell and (s,t)-Pell-Lucas sequences as in the following definition:

Definition 1.1 Let s,t be any real numbers with $s^2 + t > 0, s > 0$, and $t \neq 0$. Then the (s,t)-Pell sequence $\{P_n(s,t)\}_{n=0}^{+\infty}$ and the (s,t)-Pell-Lucas sequence $\{Q_n(s,t)\}_{n=0}^{+\infty}$ are defined respectively by

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t),$$
$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t)$$

for $n \geq 2$, with initial conditions $P_0(s,t) = 0$, $P_1(s,t) = 1$ and $Q_0(s,t) = 2$, $Q_1(s,t) = 2s$.

The terms of these sequences are called (s,t)-Pell and (s,t)-Pell-Lucas numbers, respectively. Also, they introduced the matrices sequences, which have elements of (s,t)-Pell and (s,t)-Pell-Lucas sequences. Further, they obtained some identities of (s,t)-Pell and (s,t)-Pell-Lucas matrices sequences by using elementary matrix algebra. After that, the (s,t)-Pell and (s,t)-Pell-Lucas numbers were studied in different ways to obtain many identities of these numbers. (See: (8-10))

On the other hand, the theory of the second-order recursive sequence of the polynomials has been studied in the literature. In 1883 E.C. Catalan and E. Jacobsthal introduced and studied the polynomials, which are defined by Fibonacci-like recurrence relations. Such polynomials, called the Fibonacci polynomials.

The Fibonacci polynomials studied by Catalan are defined by the recurrence relation.

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), n \geq 3$$

with initial conditions $F_1(x) = 1, F_2(x) = x$.

The Fibonacci polynomials studied by Jacobsthal are defined by the recurrence relation.

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), n \geq 3$$

with initial conditions $J_1(x) = J_2(x) = 1$.

In 1965, V.E. Hoggatt (11) introduced Lucas polynomials defined by the recurrence relation.

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 2$$

with initial conditions $L_0(x) = 2$, $L_1(x) = x$.

Pell polynomials sequence $\{P_n(x)\}_{n=0}^{+\infty}$ and Pell-Lucas polynomials sequence $\{P_n(x)\}_{n=0}^{+\infty}$ were studied in 1985 by A.F. Horadam and J.M. Mahon (12), and these polynomials are defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), n \geq 2$$
$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)\ , n \geq 2$$

with initial conditions $P_0(x) = 0$, $P_1(x) = 1$ and $Q_0(x) = 2$, $Q_1(x) = 2x$.

We note that $P_n(1) = P_n$, $Q_n(1) = Q_n$, $P_n(\frac{1}{2}) = F_n$, and $Q_n(\frac{1}{2}) = L_n$ where P_n, Q_n, F_n, and L_n are the n^{th} Pell, Pell-Lucas, Fibonacci, and Lucas numbers, respectively. Moreover, $P_n(\frac{1}{2}x) = F_n(x)$ and $Q_n(\frac{1}{2}x) = L_n(x)$ where $F_n(x)$ and $L_n(x)$ are the n^{th} Fibonacci and Lucas polynomials, respectively (see (12)). For more detailed information on Fibonacci, Lucas, Pell, and Pell-Lucas polynomials can be found in (1-2). In the last decade, Fibonacci, Lucas, Pell, and Pell-Lucas polynomials have been generalized and studied by many authors (see (13-14)).

In this paper, we introduced the generalizations of Pell and Pell-Lucas polynomials, which are called (s,t)-Pell and (s,t)- Pell-Lucas polynomials. We also give the Binet formula and the generating function for these polynomials and then some identities are obtained by using the Binet formulas.

2. Main results

We begin this section with the following definition.

Definition 2.1 Let s,t be any real numbers with $s^2 + t > 0, s > 0$, and $t \neq 0$. Then the (s,t)-Pell polynomial sequence $\{P_n(s,t)(x)\}_{n=0}^{+\infty}$ and the (s,t)-Pell-Lucas polynomial sequence $\{Q_n(s,t)(x)\}_{n=0}^{+\infty}$ are defined respectively by

$$P_n(s,t)(x) = 2sxP_{n-1}(s,t)(x) + tP_{n-2}(s,t)(x),$$
$$Q_n(s,t)(x) = 2sxQ_{n-1}(s,t)(x) + tQ_{n-2}(s,t)(x)$$

for $n \geq 2$, with initial conditions $P_0(s,t)(x) = 0$, $P_1(s,t)(x) = 1$ and $Q_0(s,t)(x) = 2$, $Q_1(s,t)(x) = 2sx$.

The first few terms of the (s,t)-Pell polynomial sequence are $0, 1, 2sx, 4s^2x^2 + t, 8s^3x^3 + 4stx$. Also, the first few terms of the (s,t)-Pell-Lucas polynomial sequence are $2, 2sx$, $4s^2x^2 + 2t$, $8s^3x^3 + 6stx, 16s^4x^4 + 16s^2tx^2 + 2t^2$. The terms of the (s,t)-Pell and the (s,t)-Pell-Lucas polynomial sequences are called (s,t)-Pell and (s,t)-Pell-Lucas polynomials, respectively.

In particular, if $s = \frac{1}{2}$, $t = 1$, then the classical Fibonacci and Lucas polynomial sequences are obtained, if $s = t = 1$, then the classical Pell and Pell-Lucas polynomial sequences are obtained, and if $s = 1$, then the k-Pell and k-Pell-Lucas polynomial sequences are obtained.

Throughout this paper, for convenience, we will use the symbol $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$ instead of $P_n(s,t)(x)$ and $Q_n(s,t)(x)$, respectively.

First, we give the explicit formula for the n^{th} (s,t)-Pell and (s,t)-Pell-Lucas polynomials.

Theorem 2.2 (Binet's formulas) *The n^{th} (s,t)-Pell and n^{th} (s,t)-Pell-Lucas polynomials are given by*

$$\mathcal{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 0 \quad (2.1)$$
$$\mathcal{Q}_n(x) = \alpha^n + \beta^n, n \geq 0 \quad (2.2)$$

respectively, where α, β are the roots of the characteristic equation $r^2 - 2sxr - t = 0$ and $\alpha > \beta$.

Proof. The characteristic equation for the recurrence relations in Definition 2.1 is

$$r^2 - 2sxr - t = 0 \quad (2.3)$$

Let α, β be the roots of equation (2.1) and $\alpha > \beta$, we have $\alpha = sx + \sqrt{s^2x^2 + t}$ and $\beta = sx - \sqrt{s^2x^2 + t}$. Note that

$$\alpha + \beta = 2sx, \alpha - \beta = 2\sqrt{s^2x^2 + t} \text{ and } \alpha\beta = -t.$$

Since equation (2.3) has two distinct roots, then

$$\mathcal{P}_n(x) = a_1\alpha^n + a_2\beta^n$$

and

$$\mathcal{Q}_n(x) = b_1\alpha^n + b_2\beta^n$$

are the solutions for the recurrence relations in Definition 2.1, respectively, for some real numbers a_1, a_2, b_1, b_2. Giving to n the values $n = 0$ and $n = 1$, then solving system of linear equations, we obtain a unique solution $a_1 = \frac{1}{\alpha - \beta}, a_2 = -\frac{1}{\alpha - \beta}$ and $b_1 = b_2 = 1$. So,

$$\mathcal{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$\mathcal{Q}_n(x) = \alpha^n + \beta^n$$ □

Theorem 2.3 (Generating Functions) *The generating functions for the (s,t)-Pell and (s,t)-Pell-Lucas polynomials are given by*

$$f(y) = \frac{y}{1 - 2sxy - ty^2},$$
$$g(y) = \frac{2 - 2sxy}{1 - 2sxy - ty^2}$$

respectively.

Proof. The generating function for the (s,t)-Pell polynomial sequence is defined by

$$f(y) = \sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n.$$

Since
$$\sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n = y + 2sx\sum_{n=2}^{+\infty} \mathcal{P}_{n-1}(x)y^n + t\sum_{n=2}^{+\infty} \mathcal{P}_{n-2}(x)y^n,$$
$$= y + 2sxy\sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n + ty^2\sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n,$$

we get that

$$f(y) = \sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n = \frac{y}{1 - 2sxy - ty^2}.$$

Similarly, we obtain the generating function for the (s,t)-Pell-Lucas polynomial sequence as follows

$$g(y) = \frac{2 - 2sxy}{1 - 2sxy - ty^2}$$ □

Next, using the Binet's formulas (2.1) and (2.2), we obtain Catalan's identities, Cassini's identities, and d'Ocagne's identities for the (s,t)-Pell and (s,t)-Pell-Lucas polynomials, which are stated in the following theorems.

Theorem 2.4 (Catalan's Identities) *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*

(i) $\mathcal{P}_{n+r}(x)\mathcal{P}_{n-r}(x) - \mathcal{P}_n^2(x) = -(-t)^{n-r}\mathcal{P}_r^2(x)$,
(ii) $\mathcal{Q}_{n+r}(x)\mathcal{Q}_{n-r}(x) - \mathcal{Q}_n^2(x) = (-t)^{n-r}(\mathcal{Q}_r^2(x) - 4(-t)^r)$,
for all $n \geq 0$, $r \geq 0$, *and* $n \geq r$.

Proof. Using Binet's formula (2.1), we have

$$\mathcal{P}_{n+r}(x)\mathcal{P}_{n-r}(x) - \mathcal{P}_n^2(x) = \frac{-\alpha^{n+r}\beta^{n-r} - \alpha^{n-r}\beta^{n+r} + 2\alpha^n\beta^n}{(\alpha-\beta)^2} = \frac{-(\alpha\beta)^{n-r}(\alpha^r-\beta^r)^2}{(\alpha-\beta)^2} = -(-t)^{n-r}\mathcal{P}_r^2(x),$$

and using Binet's formula (2.2), we obtain

$$\begin{aligned}\mathcal{Q}_{n+r}(x)\mathcal{Q}_{n-r}(x) - \mathcal{Q}_n^2(x) &= \alpha^{n-r}\beta^{n-r}(\alpha^{2r} + \beta^{2r} - 2\alpha^r\beta^r)\\ &= (\alpha\beta)^{n-r}((\alpha^r+\beta^r)^2 - 4(\alpha\beta)^r)\\ &= (-t)^{n-r}(\mathcal{Q}_r^2(x) - 4(-t)^r).\end{aligned}$$

□

Theorem 2.5 (Cassini's Identities) *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*

(i) $\mathcal{P}_{n+1}(x)\mathcal{P}_{n-1}(x) - \mathcal{P}_n^2(x) = -(-t)^{n-1}$
(ii) $\mathcal{Q}_{n+1}(x)\mathcal{Q}_{n-1}(x) - \mathcal{Q}_n^2(x) = 4(-t)^{n-1}(s^2x^2+t)$,
for all $n \geq 1$.

Proof. Take $r = 1$ in Theorem 2.4 and using the initial conditions $\mathcal{P}_1(x) = 1$, $\mathcal{Q}_1(x) = 2sx$. Then we get the results. □

Theorem 2.6 (d'Ocagne's Identities) *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*
(i) $\mathcal{P}_m(x)\mathcal{P}_{n+1}(x) - \mathcal{P}_{m+1}(x)\mathcal{P}_n(x) = (-t)^n\mathcal{P}_{m-n}(x)$,
(ii) $\mathcal{Q}_m(x)\mathcal{Q}_{n+1}(x) - \mathcal{Q}_{m+1}(x)\mathcal{Q}_n(x) = 2(-t)^n\sqrt{s^2x^2+t}(\mathcal{Q}_{m-n}(x) - 2(sx+\sqrt{s^2x^2+t})^{m-n})$,
for all $m \geq 1$, $n \geq 1$, *and* $m \geq n$.

Proof. Using Binet's formula (2.1), we have

$$\mathcal{P}_m(x)\mathcal{P}_{n+1}(x) - \mathcal{P}_{m+1}(x)\mathcal{P}_n(x) = \frac{(\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n}-\beta^{m-n})}{(\alpha-\beta)^2} = (\alpha\beta)^n \cdot \frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta} = (-t)^n\mathcal{P}_{m-n}(x),$$

and using Binet's formula (2.2), we obtain

$$\begin{aligned}\mathcal{Q}_m(x)\mathcal{Q}_{n+1}(x) - \mathcal{Q}_{m+1}(x)\mathcal{Q}_n(x) &= \alpha^n\beta^n(\alpha^{m-n}\beta + \alpha\beta^{m-n} - \alpha^{m-n+1} - \beta^{m-n+1})\\ &= (\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n} + \beta^{m-n} - 2\alpha^{m-n})\\ &= 2(-t)^n\sqrt{s^2x^2+t}\left(\mathcal{Q}_{m-n}(x) - 2(sx + \sqrt{s^2x^2+t})^{m-n}\right).\end{aligned}$$

□

Again, in the following theorems, we obtain important elementary relationships involving $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$ by using Binet's formulas (2.1) and (2.2).

Theorem 2.7 *For all* $n \geq 1$, *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*

(i) $\mathcal{P}_{n+1}(x) + t\mathcal{P}_{n-1}(x) = \mathcal{Q}_n(x)$,
(ii) $\mathcal{Q}_{n+1}(x) + t\mathcal{Q}_{n-1}(x) = 4(s^2x^2+t)\mathcal{P}_n(x)$,
(iii) $2sx\mathcal{P}_n(x) + 2t\mathcal{P}_{n-1}(x) = \mathcal{Q}_n(x)$.

Proof. Using Binet's formulas (2.1) and (2.2), we have

$$\mathcal{P}_{n+1}(x) + t\mathcal{P}_{n-1}(x) = \frac{\alpha^{n+1}-\beta^{n+1}-\alpha\beta(\alpha^{n-1}-\beta^{n-1})}{\alpha-\beta} = \frac{(\alpha-\beta)(\alpha^n+\beta^n)}{\alpha-\beta} = \mathcal{Q}_n(x).$$

Similarly, we obtain the result (ii). By using (i) and $\mathcal{P}_{n+1}(x) = 2sx\mathcal{P}_n(x) + t\mathcal{P}_{n-1}(x)$, we obtain the result (iii). □

Finally, we obtain summations involving $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$, which are stated in the following theorem.

Theorem 2.8 *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then for all* $m \geq 1, n \geq 1, r \geq 0$ *and* $\mathcal{Q}_m(x) \neq (-t)^m + 1$, *the following statements hold.*
(i) $\sum_{k=1}^{n}\mathcal{P}_{mk+r}(x) = \frac{\mathcal{P}_{mn+m+r}(x) - \mathcal{P}_{m+r}(x) - (-t)^m(\mathcal{P}_{mn+r}(x) - \mathcal{P}_r(x))}{\mathcal{Q}_m(x) - (-t)^m - 1}$,
(ii) $\sum_{k=1}^{n}\mathcal{Q}_{mk+r}(x) = \frac{\mathcal{Q}_{mn+m+r}(x) - \mathcal{Q}_{m+r}(x) - (-t)^m(\mathcal{Q}_{mn+r}(x) - \mathcal{Q}_r(x))}{\mathcal{Q}_m(x) - (-t)^m - 1}$.

Proof. Using Binet's formulas (2.1) and (2.2), we have

$$\begin{aligned}\sum_{k=1}^{n}\mathcal{P}_{mk+r}(x) &= \sum_{k=1}^{n}\frac{1}{\alpha-\beta}(\alpha^{mk+r} - \beta^{mk+r})\\ &= \frac{1}{\alpha-\beta}\left(\frac{\alpha^{m+r}(1-\alpha^{mn})}{1-\alpha^m} - \frac{\beta^{m+r}(1-\beta^{mn})}{1-\beta^m}\right)\\ &= \frac{\alpha^{m+r}(1-\alpha^{mn})(1-\beta^m) - \beta^{m+r}(1-\beta^{mn})(1-\alpha^m)}{(\alpha-\beta)(1-\alpha^m)(1-\beta^m)}\\ &= \frac{\mathcal{P}_{mn+m+r}(x) - \mathcal{P}_{m+r}(x) - (-t)^m(\mathcal{P}_{mn+r}(x) - \mathcal{P}_r(x))}{\mathcal{Q}_m(x) - (-t)^m - 1}.\end{aligned}$$

Similarly, by the same argument as above, we obtain the result (ii). □

3. Conclusions

In this paper, the generalizations of Pell and Pell-Lucas polynomials, are introduced and the Binet formulas and the generating functions for these polynomials are obtained. Furthermore, some identities are given by using the Binet formulas.

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