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A Class of Continuous Solutions of a Fourth Order Polynomial-like Iterative Equation

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Abstract

Using the so-called characteristic method, continuous solutions of the fourth order polynomial-like iterative equation

$$f^4(x) + a_3 f^3(x) + a_2 f^2(x) + a_1 f(x) + a_0 x = 0$$

were determined subject to certain natural conditions on its characteristic roots. The result so obtained complements earlier work in the cases of second and third order equations.

Keywords: polynomial-like iterative equation; continuous solution; characteristic root
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1. Introduction

For $n \in N$, the n^{th} iterate of a function f is defined by

$$f^n(x) = f(f^{n-1}(x)), \quad f^0(x) = x \ .$$

A polynomial-like iterative equation is a functional equation of the form

$$f^n(x) + a_{n-1} f^{n-1}(x) + ... + a_0 x = F(x) \ , \tag{1.1}$$

where $a_i \in R$ $(i = 0,1,2,...,n-1)$, F is a given function, and $f : R \to R$ is an unknown function. The homogeneous case of (1.1) (i.e., when $F(x) = 0$)

$$f^n(x) + a_{n-1} f^{n-1}(x) + ... + a_0 x = 0 \tag{1.2}$$

is of interest here. There have appeared a number of recent works [1-4] attempting to solve (1.2) using a technique mimicking that of Euler for solving linear differential equations with constant coefficients, which proceeds by assuming a solution of the form e^{rx}. Substituting this into the differential equation and simplifying, an algebraic equation in r, called its characteristic equation

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is obtained. Solving the characteristic equation yields linearly independent solutions of the differential equation, and the general solution follows by taking a linear combination of these independent solutions. In the case of iterative equation (1.2), we consider, instead, a continuous solution of the form $f(x) = rx$. Substituting into (1.2), we get an algebraic equation (in r)

$$r^n - a_{n-1}r^{n-1} - \ldots - a_1 r - a_0 = 0,$$

which, by abuse of language, is also called the *characteristic equation of (1.2)* and its roots are called its *characteristic roots*. Let $r_1, \ldots, r_n$ be all the characteristic roots. Using symmetric functions relations involving roots and coefficients, the iterative equation (1.2) is equivalent to

$$f^n(x) - (\sum_{i=1}^{n} r_i) f^{n-1}(x) + (\sum_{i<j}^{n} r_i r_j) f^{n-2}(x) + \ldots + (-1)^n r_1 r_2 \ldots r_n x = 0 . \qquad (1.3)$$

At the outset, we make several preliminary observations and define conventions to be adopted throughout the entire investigation here.

- The coefficient a_0 is always nonzero.

  (For otherwise, one of the characteristic roots is zero yielding the trivial solution function $f \equiv 0$ which must always be ruled out.)
- Any function solution $f : R \to R$ of (1.2) is injective.

  (If $x, y \in R$ are such that $f(x) = f(y)$, then $f^k(x) = f^k(y)$ for all $k \in N$ and (1.2) implies that $a_0 x = a_0 y$. Since $a_0 \neq 0$, we get $x = y$.)
- Any continuous function solution $f : R \to R$ of (1.2) is strictly monotone and surjective.

  (Since f is continuous and 1-1, clearly, it is strictly monotone. To show that f is onto, we consider only the case f is strictly increasing as the other case is similar. Suppose f is not onto on R . By its continuity, the range must be of the form $f(R) = I$, where $I \neq R$ is an interval, which can take one of the three shapes : $(-\infty, a)$, $(-b, a)$ or $(-b, \infty)$, for finite values $-b < a$. From $f : R \to f(R) := I$, we see that $f^2 : R \to f(I) \subseteq I ,\ldots, f^n : R \to f(I) \subseteq I$.

  From (1.2), we get

  $$|a_0 x| = |f^n(x) + a_{n-1} f^{n-1}(x) + \ldots + a_1 f(x)| .$$

  For the cases $I := (-\infty, a)$ or $I := (-b, a)$, letting $x \to \infty$, we see that the left-hand side $\to \infty$, while the right-hand expression (since the range of the function is bounded) is bounded, which is contradiction.

  For the case $I := (-b, \infty)$, letting $x \to -\infty$, the left-hand side $\to \infty$, the right-hand side is bounded, which is again a contradiction.)
- Since $a_0 \neq 0$ and f is bijective, the inverse function $f^{-1} : R \to R$ exists and the original iterative equation (1.3) is equivalent to the dual equation

  $$f^{-n}(x) - (\sum_{i=1}^{n} \frac{1}{r_i}) f^{-(n-1)}(x) + (\sum_{i<j}^{n} \frac{1}{r_i r_j}) f^{-(n-2)}(x) + \ldots + (-1)^n \frac{1}{r_1 r_2 \ldots r_n} x = 0 \qquad (1.4)$$

  where f^{-j} denotes the j^{th} iterate of f^{-1} .

In 2004, Yang and Zhang [3] constructed all continuous solutions of the equation (1.2) when $n \geq 2$ for the hyperbolic cases subject to the condition that characteristic roots belong to following ranges:

- all characteristic roots are in the interval $(1,\infty)$: $1 < r_1 < r_2 < .... < r_n$;
- all characteristic roots are in the interval $(0,1)$: $0 < r_1 < r_2 < .... < r_n < 1$;
- all characteristic roots are in the interval $(-\infty,-1)$: $r_1 < r_2 < .... < r_n < -1$;
- all characteristic roots are in the interval $(-1,0)$: $-1 < r_1 < r_2 < .... < r_n < 0$.

- no real characteristic roots (in this case n must be even and characteristic roots form pairs of conjugate complex numbers), it is shown that (1.2) has no continuous solution.
- all characteristic roots are equal, i.e., $r_1 = ... = r_n = r$.

(i) If $0 < r \neq 1,$ then f is strictly increasing, the function

$$F^{n-1}[r](f^0) := f^{n-1}(x) + \sum_{m=1}^{n-1} (-1)^m \binom{n-1}{m} r^m f^{n-1-m}(x)$$

is nondecreasing, $f(0) = 0,$ and $F^{n-1}[r](f^0) = 0$ for even n.

(ii) If $-1 \neq r < 0,$ then f is strictly decreasing, the function $F^{n-1}[r](f^0)$ is nondecreasing (respectively, nonincreasing) for odd (respectively, even) n, $f(0) = 0,$ and $F^{n-1}[r](f^0) = 0$ for even n.

(iii) If $r = 1,$ then f is strictly increasing. Additionally, $f(x) \equiv x$ if f has fixed point, otherwise, $F^{n-1}[r](f^0) \equiv a$ for all $x \in R,$ where a is a real constant which equals 0 for odd n. In particular, for $n = 3$, continuous solutions of equation (1.2) are of the form $f(x) = x + c,$ where c is a real constant.

(iv) If $r = -1,$ then $f(x) = -x$ for all $x \in R.$

The subcases (i), (ii) and (iii) provide a method to reduce the order of iteration, giving equivalent equations of lower order, although it does not give the construction of the general solution in all cases.

In their work, there remain un-resolved cases when the existing characteristic roots of (1.2)

- are both positive and negative, or
- have absolute values both greater and less than 1.

In the second order case (i.e., $n = 2$), Matkowski and Weinian [1] established continuous solutions by subdividing into the following cases.

- Noncritical cases: $r_1 r_2 > 0$, $|r_1| \neq 1$ and $|r_2| \neq 1$ with all possibilities
  $1 < r_1 < r_2$, $0 < r_1 < 1 < r_2$, $0 < r_1 < r_2 < 1$,
  $r_1 < r_2 < -1$, $r_1 < -1 < r_2 < 0$, $-1 < r_1 < r_2 < 0$.
- Noncritical cases: $r_1 r_2 < 0$, $|r_1| \neq 1$, $|r_2| \neq 1$, $r_2 \neq -r_1$ with all possibilities
  $0 < -r_1 < r_2 < 1$, $0 < r_2 < -r_1 < 1$, $1 < r_2 < -r_1$, $0 < r_2 < 1 < -r_1$, $0 < -r_1 < 1 < r_2$.

- Case $|r_1| = |r_2|$: $r_1 = r_2 = r \neq 0$.
- Critical cases: there is a characteristic root with absolute value 1 with all possibilities
  $0 < r_1 < r_2 = 1$, $1 \neq r_1 < 0 < r_2 = 1$, $-1 = r_1 < 0 < r_2 \neq 1$, $-1 = r_1 < r_2 < 0$, $r_1 < r_2 = -1$.
- Case with no real roots: in this case (1.2) has no continuous solution on R.

From their work, the un-resolved case is when $r_1 = -r = -r_2$ ($r > 0$). The governing equation of this case is

$$f^2(x) = r^2 x. \tag{1.5}$$

When $r = 1$, Kuczma's Theorem 15.2 [6] shows that (1.5) has a decreasing solution depending on an arbitrary function, but $f(x) = x$ is its unique increasing solution.

When $r \neq 1$, Kuczma's Theorems 15.7 [6] and 15.9 [6] indicate that (1.5) has not only increasing continuous solutions but also decreasing ones, all of which depend on arbitrarily given functions.

In the third order case (i.e., $n = 3$), Zhang and Gong [4] in 2014 solved (1.2) for continuous solutions in the hyperbolic cases not treated in the work of Yang and Zhang. They completed the following cases:

I. The three characteristic roots have different signs. There are two possibilities.

I.1 Two positive characteristic roots $0 < r_2 < r_3$, and one negative characteristic root $r_1 < 0$. Treated cases are

- $0 < -r_1 < 1$: $0 < -r_1 < r_2 < r_3 < 1$, $0 < r_2 < -r_1 < r_3 < 1$, $0 < r_2 < r_3 < -r_1 < 1$, $0 < -r_1 < 1 < r_2 < r_3$, $0 < r_2 < -r_1 < 1 < r_3$, $0 < -r_1 < r_2 < 1 < r_3$.
- $1 < -r_1$: $1 < -r_1 < r_2 < r_3$, $1 < r_2 < -r_1 < r_3$, $1 < r_2 < r_3 < -r_1$, $0 < r_2 < 1 < -r_1 < r_3$, $0 < r_2 < 1 < r_3 < -r_1$, $0 < r_2 < r_3 < 1 < -r_1$.

I.2 One positive characteristic root $r_3 > 0$, and two negative characteristic roots $0 > r_1 > r_2$.

- $0 < r_3 < 1$: $0 < -r_1 < -r_2 < r_3 < 1$, $0 < -r_1 < r_3 < -r_2 < 1$, $0 < r_3 < -r_1 < -r_2 < 1$, $0 < r_3 < 1 < -r_1 < -r_2$, $0 < r_3 < -r_1 < 1 < -r_2$, $0 < -r_1 < r_3 < 1 < -r_2$.
- $1 < r_3$: $1 < -r_1 < -r_2 < r_3$, $1 < -r_1 < r_3 < -r_2$, $1 < r_3 < -r_1 < -r_2$, $0 < -r_1 < 1 < -r_2 < r_3$, $0 < -r_1 < 1 < r_3 < -r_2$, $0 < -r_1 < -r_2 < 1 < r_3$.

II. The three characteristic roots have the same signs. The treated possibilities are

$0 < r_1 < 1 < r_2 < r_3$, $0 < r_3 < r_2 < 1 < r_1$, $r_1 < r_2 < -1 < r_3 < 0$, $r_3 < -1 < r_2 < r_1 < 0$.

The still un-resolved cases are:

- there is a characteristic root with absolute value 1.
- there is a characteristic root with multiplicity ≥ 2.

Zhang and Gong [4] also considered the 4-th order equation when the four characteristic roots have different signs lying in the following ranges:

$1 < -r_1 < r_2 < r_3 < r_4$, $1 < -r_1 < r_2 < r_4 < r_3$, $1 < -r_1 < r_4 < r_2 < r_3$, $1 < -r_1 < r_2 < -r_4 < r_3$, $1 < -r_1 < -r_4 < r_2 < r_3$, $1 < -r_4 < -r_1 < r_2 < r_3$.

2. Methodology

Following ideas from the work of Zhang and Gong [4], we determine here continuous solutions of the homogeneous equation (1.2) of order 4, i.e. the iterative equation

$$f^4(x) - (\sum_{i=1}^{4} r_i) f^3(x) + (\sum_{i<j}^{4} r_i r_j) f^2(x) - (\sum_{i<j<k}^{4} r_i r_j r_k) f(x) + r_1 r_2 r_3 r_4 x = 0, \tag{1.6}$$

when the characteristic roots r_1, r_2, r_3, r_4 are subject to the restrictions:

1) $|r_i| \neq 0,1$ $(i = 1,2,3,4)$, and

2) the absolute values of the chrarcteristic roots $|r_i|$ $(i = 1,2,3,4)$ are all distinct.

The dual equation of (1.6) is

$$f^{-4}(x) - (\sum_{i=1}^{4} \frac{1}{r_i}) f^{-3}(x) + (\sum_{i<j}^{4} \frac{1}{r_i r_j}) f^{-2}(x) - (\sum_{i<j<k}^{4} \frac{1}{r_i r_j r_k}) f^{-1}(x) + \frac{1}{r_1 r_2 r_3 r_4} x = 0. \tag{1.7}$$

To simplify our analysis, we leave out certain special cases consisting of

$$r_1 < r_2 < r_3 < r_4 < -1,\ -1 < r_1 < r_2 < r_3 < r_4 < 0,\ 0 < r_1 < r_2 < r_3 < r_4 < 1,\ 1 < r_1 < r_2 < r_3 < r_4$$

that have already been treated in Yang and Zhang [3]. In addition, the special case where all characteristic roots are not real, which has also been shown to have no continuous solution by them, is also left out.

Specifically, we solve (1.6) when the characteristic equation has

- one negative and three positive characteristic roots or
- three negative and one positive characteristic roots or
- two negative and two positive characteristic roots.

The solutions so obtained are displayed in Figures 1, 2 and 3, respectively. Though there are totally seventy subcases solved in this work, there remain two cases that are yet to be resolved for which the methods and techniques used here do not seem to work. These subcases are when

(i) all characteristic roots are positive, some being in $(0,1)$ and the others in $(1,\infty)$,

(ii) all characteristic roots are negative, some being in $(-1,0)$ and the others in $(-\infty,-1)$.

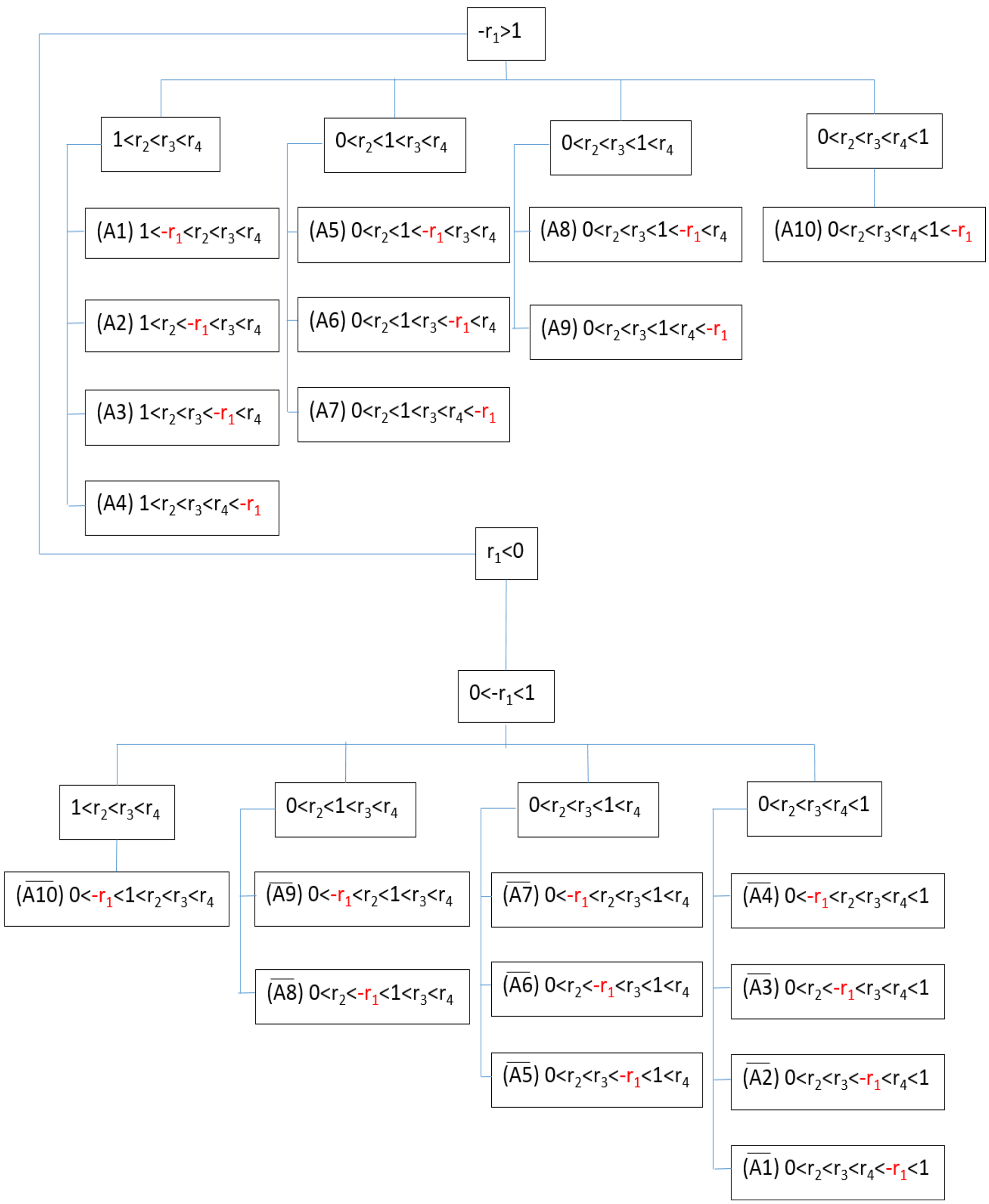

Figure 1. One negative characteristic root, r_1, and three positive characteristic roots, $0 < r_2 < r_3 < r_4$

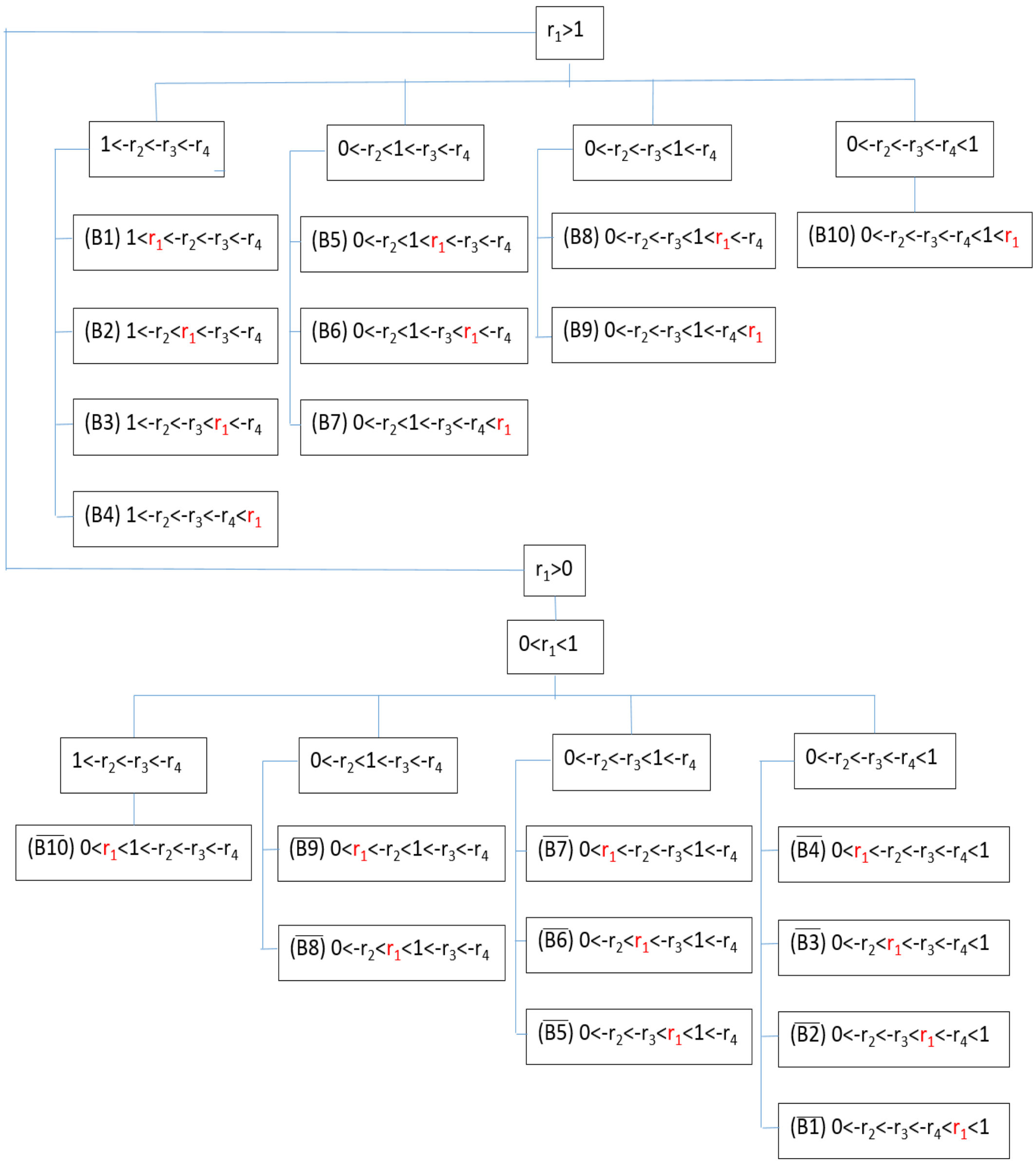

Figure 2. Three negative characteristic roots, $0 > r_2 > r_3 > r_4$, and one positive characteristic root, r_1

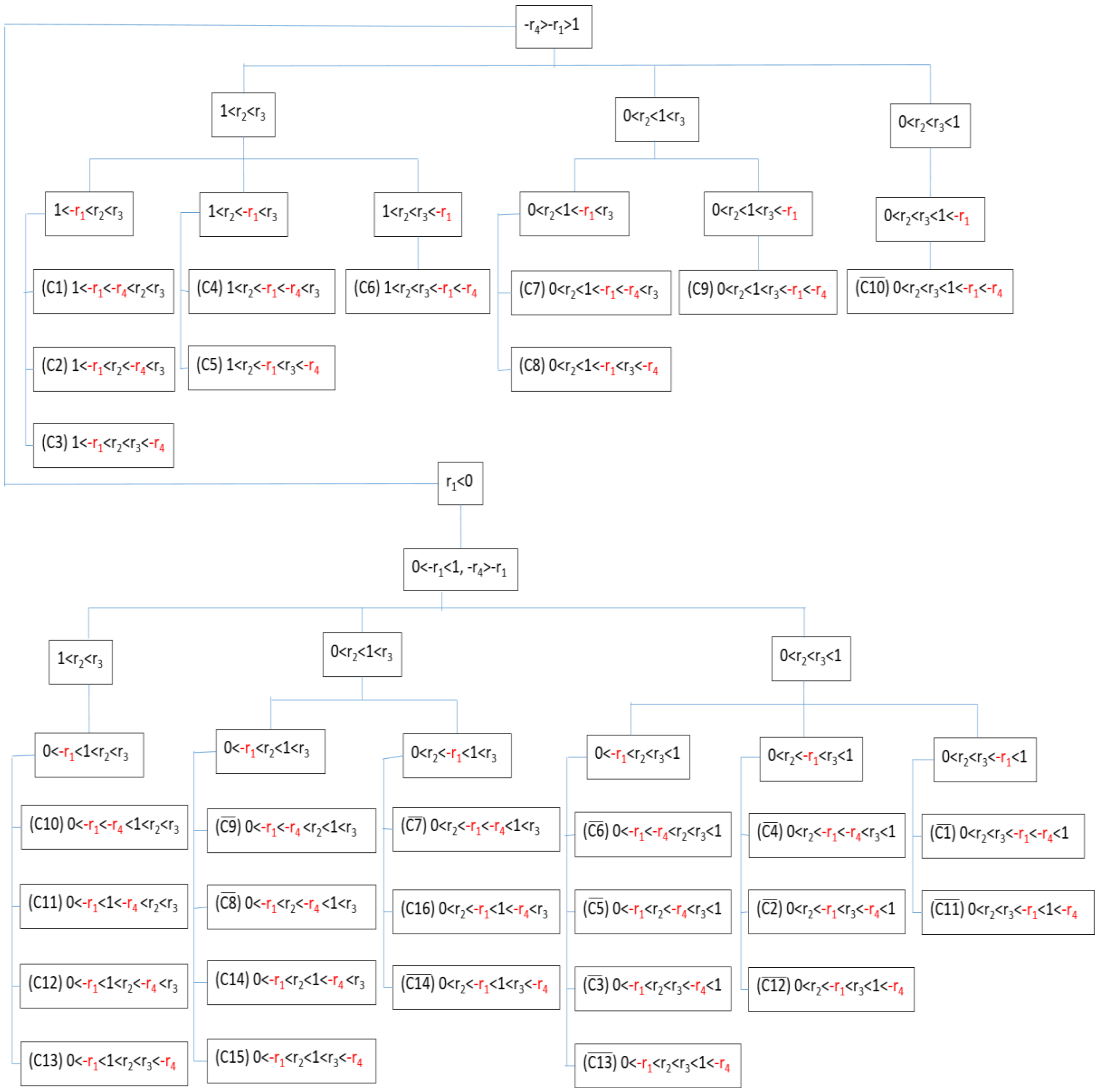

Figure 3. Two negative characteristic roots, $0 > r_1 > r_4$, and two positive characteristic roots, $0 < r_2 < r_3$

Detailed proofs of our results are given in the next section. Preliminary results needed throughout are displayed in the above two boxes.

To simplify the presentation, define

$$CS(x) = \{ f : R \to R;\ f \text{ is a continuous solution of the iterative equation } (x)\},$$
$$CSI(x) = \{ f : R \to R;\ f \text{ is a continuous strictly increasing solution of the iterative equation } (x)\},$$
$$CSD(x) = \{ f : R \to R;\ f \text{ is a continuous strictly decreasing solution of the iterative equation } (x)\}.$$

To consider our continuous solutions, we need the continuous solutions from the works of Yang and Zhang [3], Matkowski and Weinian [1], and Zhang and Gong [4] as shown below.

Works of Yang and Zhang [3]

Theorem 1.1 (Theorem 2 [3]) Suppose that $1 < r_1 < ... < r_n$. If $f \in CS(1.2)$, then

(i) f is strictly increasing,

(ii) $f(0) = 0$, and

(iii) $f^{n-1}(x) - (\sum_{i=1}^{n-1} r_i) f^{n-2}(x) + (\sum_{i<j}^{n-1} r_i r_j) f^{n-3}(x) + ... + (-1)^{n-1} r_1 r_2 ... r_{n-1} x$ is nondecreasing

and $f^{n-1}(x) - (\sum_{i=2}^{n} r_i) f^{n-2}(x) + (\sum_{i<j\neq 1}^{n} r_i r_j) f^{n-3}(x) + ... + (-1)^{n-1} r_2 r_3 ... r_n x$ is non-decreasing (resp. non-increasing) for odd (resp. even) n.

Conversely, given positive numbers $x_0, ..., x_{n-1}$ such that

$$x_{n-1} - (\sum_{i\neq k}^{n} r_i) x_{n-2} + (\sum_{i<j, i, j\neq k}^{n} r_i r_j) x_{n-3} + ... + (-1)^{n-1} r_1 ... r_{k-1} r_{k+1} ... r_n \geq 0 \text{ (resp. } \leq 0\text{)},$$

if $n-k$ is even (resp. odd) and given a continuous function $f_* : [x_0, x_{n-1}] \to [x_1, x_n]$, where

$$x_n := (\sum_{i=1}^{n} r_i) x_{n-1} - (\sum_{i<j}^{n} r_i r_j) x_{n-2} + ... + (-1)^{n-1} r_1 ... r_n x_0, \text{ satisfying}$$

(I) $f_*(x_j) = x_{j+1}\ (0 \leq j \leq n-1)$ and

(II) Each $f_*^{n-1}(x) - (\sum_{i=1\neq k}^{n} r_i) f_*^{n-2}(x) + (\sum_{i<j\neq k}^{n} r_i r_j) f_*^{n-3}(x) + ... + (-1)^{n-1} r_1 ... r_{k-1} r_{k+1} ... r_n x,$

$k = 1, 2, ..., n$, is nondecreasing (resp. nonincreasing) on $[x_0, x_1]$ if $n-i$ is even (resp. odd), then equation (1.2) has a unique continuous solution $\phi : (o, \infty) \to (o, \infty)$ satisfying $\phi|_{[x_0, x_{n-1}]} = f_*$.

Furthermore, given f_{*1} and f_{*2} arbitrarily like f_*, the function

$$f(x) := \begin{cases} \phi_1(x), & x > 0, \\ 0, & x = 0, \\ -\phi_2(-x), & x < 0 \end{cases}$$

is a continuous solution of equation (1.2) on $\mathbb{R}$, where ϕ_1 and ϕ_2 are functions determined correspondingly by f_{*1} and f_{*2}.

Corollary 1.2 (Remark 4 [3]) The case that $0 < r_1 < ... < r_n < 1$ can be reduced to the case of above Theorem 1 by considering the dual equation (1.4).
Theorem 1.3 (Theorem 3 [3]) The case that $r_1 < ... < r_n < -1$. Suppose $f \in CS(1.2)$. Then
(i) f is strictly decreasing and has a unique fixed point 0;
(ii) $f^{n-1}(x) - (\sum_{i=1}^{n-1} r_i) f^{n-2}(x) + (\sum_{i<j}^{n-1} r_i r_j) f^{n-3}(x) + ... + (-1)^{n-1} r_1 r_2 ... r_{n-1} x$ is nondecreasing and $f^{n-1}(x) - (\sum_{i=2}^{n} r_i) f^{n-2}(x) + (\sum_{i<j\neq 1}^{n} r_i r_j) f^{n-3}(x) + ... + (-1)^{n-1} r_2 r_3 ... r_n x$ is nondecreasing (resp. nonincreasing) for odd (resp. even) n.
Moreover, (1.2) has symmetric continuous solutions in the form

$$f(x) := \begin{cases} -\varphi(x), & x \geq 0, \\ \varphi(-x), & x < 0, \end{cases}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is an arbitrarily given function satisfying

$$\varphi^n(x) = (\sum_{i=1}^{n} r_i)\varphi^{n-1}(x) - (\sum_{i<j}^{n} r_i r_j)\varphi^{n-1}(x) + ... + (-1)^{n-1} r_1 ... r_n x, \ \ x \in (0,\infty).$$

Theorem 1.4 (Theorem 4 [3]) The case that $r_1 < ... < r_n < -1$. Given real $x_0, ..., x_{n-1}$ arbitrarily, which are not all zero such that $x_{n-1} - (\sum_{i\neq k}^{n} r_i) x_{n-2} + (\sum_{i<j,i,j\neq k}^{n} r_i r_j) x_{n-3} + ... + (-1)^{n-1} r_1 ... r_{k-1} r_{k+1} ... r_n \geq 0$ (resp. ≤ 0), if n-k is even (resp. odd). Given a continuous function $f_* : \bigcup_{j=0}^{n-2} [x_j ; x_{j+2}] \to \bigcup_{j=0}^{n-2} [x_{j+1} ; x_{j+3}]$, where $x_k := (\sum_{i=1}^{k} r_i) x_{k-1} - (\sum_{i<j}^{k} r_i r_j) x_{k-2} + ... + (-1)^{k-1} r_1 ... r_k x_{k-n}$, for $k = n, n+1$ such that

(I) $f_*(x_j) = x_{j+1}$ $j = 0, ..., n$ and

(II) each $f_*^{n-1}(x) - (\sum_{i=1\neq k}^{n} r_i) f_*^{n-2}(x) + (\sum_{i<j\neq k}^{n} r_i r_j) f_*^{n-3}(x) + ... + (-1)^{n-1} r_1 ... r_{k-1} r_{k+1} ... r_n x,$

$k = 1, 2, ..., n$, is nondecreasing (resp. nonincreasing) on $[x_0, x_2]$ if $n-i$ is even (resp. odd), then equation (1.2) has a unique continuous solution f such that $f|_{\bigcup_{j=0}^{n-2}[x_j, x_{j+2}]} = f_*$.

Corollary 1.5 (Remark 6 [3]) The case that $-1 < r_1 < ... < r_n < 0$ can be reduced to the case of Theorem 4 by considering the dual equation (1.4).

Works of Matkowski and Weinian [1]

Theorem 1.6 (Theorem 1 [1]) Suppose $1 < r_1 < r_2$.
(i) If $f \in CS(1.2)$ for $n = 2$, then $f(0) = 0$ and f, strictly increasing, satisfies $r_1 \leq (f(x) - f(x'))/(x - x') \leq r_2$ for $x \neq x' \in R$.
(ii) Conversely, equation (1.2) for $n = 2$ has a continuous solution depending on an arbitrary function. More precisely, for every $x_0 > 0, x_1 > 0$ and $f_0 : [x_0, x_1] \to R$ such that

$$r_1x_0 \le x_1 \le r_2x_0,$$
$$f_0(x_0) = x_1, \quad f_0(x_1) = (r_1 + r_2)x_1 - r_1r_2x_0,$$
$$r_1 \le \frac{f_0(x) - f_0(x')}{x - x'} \le r_2; \quad \forall x, x' \in [x_0, x_1],$$

there is a unique continuous function $p:(0,\infty) \to (0,\infty)$ satisfying equation (1.2) for $n = 2$ on $(0,\infty)$ and $p = f_0$ on $[x_0, x_1]$; for two arbitrary initial functions f_{01} and f_{02} like f_0, the function

$$f(x) := \begin{cases} p_1(x), & x > 0, \\ 0, & x = 0, \\ -p_2(-x), & x < 0. \end{cases} \tag{3.1}$$

is a continuous function of equation (1.2) for $n = 2$ on R, where p_1 and p_2 are functions like p determined as above by f_{01} and f_{02}. (3.1) gives all continuous solutions of equation (1.2) for $n = 2$ in R.

Corollary 1.7 (Page 427 [1]) The case where $0 < r_1 < r_2 < 1$ can be obviously reduced to the case of Theorem 6 by considering the dual equation (1.4) for $n = 2$.

Theorem 1.8 (Theorem 2 [1]) Suppose $0 < r_1 < 1 < r_2$.
(i) If $f \in CS(1.2)$ for $n = 2$ then f is strictly increasing. If, additionally, f has a fixed point then

$$f(x) = \begin{cases} r_i x, x \ge 0 \\ r_j x, x < 0 \end{cases} \qquad \exists i, j = 1, 2.$$

(ii) Conversely, every $f \in CS(1.2)$ for $n = 2$ without fixed points depends on an arbitrary initial function. More precisely, for $x_0 = 0$, for every $x_1 > 0$ $(resp. < 0)$ and for every function $f_0 : [x_0, x_1] \to R$ $(resp. f_0 : [x_1, x_0] \to R)$ such that

$$f_0(x_0) = f_0(0) = x_1, \quad f_0(x_1) = (r_1 + r_2)x_1,$$
$$r_1 \le (f_0(x) - f_0(x'))/(x - x') \le r_2, \quad \forall x, x' \ne 0.$$

There exists a unique continuous function $f : R \to R$ satisfying equation (1.2) for $n = 2$ and $f(x) = f_0(x)$ on $[x_0, x_1]$ ($resp.$ on $[x_1, x_0]$).

Theorem 1.9 (Theorem 3 [1]) Suppose $r_1 < r_2 < -1$.
(i) If $f \in CS(1.2)$ for $n = 2$ then f is strictly decreasing with a unique fixed point 0 and satisfies the condition $r_1 \le (f(x) - f(x'))/(x - x') \le r_2$, for $x \ne x' \in R$.
(ii) Conversely, equation (1.2) for $n = 2$ has a continuous solution depending on an arbitrary function, given by $f(x) = -p(x)$ when $x \ge 0$ and $f(x) = p(-x)$ when $x < 0$ where $p:[0,\infty) \to [0,\infty)$ has been constructed in Theorem 6 as an arbitrary solution of the functional equation

$$p^2(x) = ((-r_1) + (-r_2))\,p(x) - (-r_1)(-r_2)x, \quad x \in [0,\infty).$$

Theorem 1.10 (Theorem 4 [1]) Suppose $r_1 < -1 < r_2 < 0$. Then every $f \in CS(1.2)$ for $n = 2$ is strictly decreasing and 0 is its unique fixed point, and $r_1 \le (f(x) - f(x'))/(x - x') \le r_2, \forall x \ne x'$.

Corollary 1.11 (Page 427 [1]) The case where $-1 < r_1 < r_2 < 0$ can be obviously reduce to the case of Theorem 9 by considering the dual equation (1.4) for $n = 2$.

Theorem 1.12 (Theorem 5 [1]) Suppose that $r_1 < 0, r_1 \ne -1, r_2 > 0, r_2 \ne 1$ and $r_2 \ne -r_1$. If $f \in CS(1.2)$ for $n = 2$ then $f(x) = r_1 x$ or $f(x) = r_2 x$ for $x \in R$.

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Lemma 1.13 (Lemma 2.5 [4]) Suppose that the characteristic equation (1.3) has n distinct roots $r_1, ..., < r_n \in C$. If $f \in CS(1.2)$, then

$$f^m = \sum_{j=1}^{n} \frac{r_j^m}{\prod_{i=1, i\ne j}^{n} (r_j - r_i)} [f^{n-1}(x) - (\sum_{k \ne j}^{n-1} r_k) f^{n-2}(x) + (\sum_{k>t\ne j}^{n-1} r_k r_t) f^{n-3}(x) + ... + (-1)^{n-1} \prod_{i\ne j}^{n-1} r_i x]$$

$$f^{-m} = \sum_{j=1}^{n} \frac{r_j^{-m}}{\prod_{i=1, i\ne j}^{n} (\frac{1}{r_j} - \frac{1}{r_i})} [f^{-(n-1)}(x) - (\sum_{k \ne j}^{n-1} \frac{1}{r_k}) f^{-(n-2)}(x) + (\sum_{k>t\ne j}^{n-1} \frac{1}{r_k r_t}) f^{-(n-3)}(x) + ... + (-1)^{n-1} \prod_{i\ne j}^{n-1} \frac{1}{r_i} x]$$

for all integers $m \ge 1$ and $i = 1, 2, ..., n$.

Lemma 1.14 (Lemma 3.1 [4]) Suppose that $0 < r_1 < 1 < r_2 < r_3$. Then for $x_0 = 0$ and arbitrarily given x_1, x_2 such that

$$x_1 > 0 \text{ and } (r_1 + r_2)x_1 < x_2 < (r_1 + r_3)x_1,$$

the sequence $(..., x_{-2}, x_{-1}; x_0, x_1, x_2, ...)$ defined by

$$x_{n+2} = (r_1 + r_2 + r_3)x_{n+1} - (r_1 r_2 + r_1 r_3 + r_2 r_3)x_n + r_1 r_2 r_3 x_{n-1}, \tag{3.2}$$

$$x_{-n} = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) x_{-n+1} - \left(\frac{1}{r_1 r_2} + \frac{1}{r_1 r_3} + \frac{1}{r_2 r_3}\right) x_{-n+2} + \frac{1}{r_1 r_2 r_3} x_{-n+3}, \tag{3.3}$$

is strictly increasing and satisfies

$$\lim_{n\to+\infty} x_n = +\infty, \qquad \lim_{n\to-\infty} x_n = -\infty.$$

Lemma 1.15 (Lemma 3.2 [4]) Suppose that $0 < r_1 < 1 < r_2 < r_3$. Then for $x_0 = 0$ and arbitrarily given x_1, x_2 such that

$$x_1 < 0 \text{ and } (r_1 + r_3)x_1 < x_2 < (r_1 + r_2)x_1,$$

the sequence $(..., x_2, x_1; x_0, x_{-1}, x_{-2}, ...)$ defined by (3.2) and (3.3) is strictly decreasing and satisfies

$$\lim_{n\to+\infty} x_n = -\infty, \qquad \lim_{n\to-\infty} x_n = +\infty.$$

Theorem 1.16 (Theorem 3.1 [4]) Suppose that $0 < r_1 < 1 < r_2 < r_3$. Then all $f \in CS(1.2)$ for $n = 3$ are strictly increasing. Additionally:

(i) If f has fixed points, then 0 is the unique fixed point and

$$f(x)=\begin{cases} f_i(x), x\geq 0 \\ f_j(x), x<0 \end{cases} \qquad i,j=1,2$$

where $f_1(x)=r_1x$ and $f_2(x)$ is a solution given in Theorem 1 of Nabeya [5].

(ii) If $f(x)>x$ for all $x\in R,$ then the set of f contains both $f(x)>max(r_1x,r_ix)(i=2,3)$ constructed by Theorem 2 of [5] and

$$f(x):=\begin{cases} f_n(x), & x\in[x_n,x_{n+1}], n=0,1,..., \\ f_{-n}^{-1}(x), & x\in[x_{-n},x_{-n+1}], n=1,2,..., \end{cases}$$

where the bilateral sequence (x_i) is given in Lemma 14, and $f_n:[x_n,x_{n+1}]\to[x_{n+1},x_{n+2}]$ and $f_{-n}:[x_{-n+1},x_{-n+2}]\to[x_{-n},x_{-n+1}], n=1,2,...$ are orientation-preserving homeomorphisms defined inductively as

$$f_{n+2}(x)=(r_1+r_2+r_3)x-(r_1r_2+r_1r_3+r_2r_3)f_{n+1}^{-1}(x)+r_1r_2r_3f_n^{-1}(f_{n+1}^{-1}(x)),$$

$$f_{-n}(x)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)x-\left(\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_2r_3}\right)f_{-n+1}^{-1}(x)+\frac{1}{r_1r_2r_3}f_{-n+2}^{-1}(f_{-n+1}^{-1}(x))$$

which are uniquely determined by two given function $f_0:[x_0,x_1]\to[x_1,x_2]$ and $f_1:[x_1,x_2]\to[x_2,x_3]$.

(iii) If $f(x)<x$ for all $x\in R,$, then the set of f contains both $f(x)<max(r_1x,r_ix)(i=2,3)$ constructed by Theorem 2 of Nabeya [5] and

$$f(x):=\begin{cases} f_n(x), & x\in[x_{n+1},x_n], n=0,1,..., \\ f_{-n}^{-1}(x), & x\in[x_{-n+1},x_{-n}], n=1,2,..., \end{cases}$$

where the bilateral sequence (x_i) is given in Lemma 15, and $f_n:[x_{n+1},x_n]\to[x_{n+2},x_{n+1}]$ and $f_{-n}:[x_{-n+2},x_{-n+1}]\to[x_{-n+1},x_{-n}]$ are orientation-preserving homeomorphisms defined inductively as

$$f_{n+2}(x)=(r_1+r_2+r_3)x-(r_1r_2+r_1r_3+r_2r_3)f_{n+1}^{-1}(x)+r_1r_2r_3f_n^{-1}(f_{n+1}^{-1}(x)),$$

$$f_{-n}(x)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)x-\left(\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_2r_3}\right)f_{-n+1}^{-1}(x)+\frac{1}{r_1r_2r_3}f_{-n+2}^{-1}(f_{-n+1}^{-1}(x))$$

which are uniquely determined by two given function $f_0:[x_1,x_0]\to[x_2,x_1]$ and $f_1:[x_2,x_1]\to[x_3,x_2]$.

Corollary 1.17 (Corollary 3.1 [4]) Suppose that $0<r_3<r_2<1<r_1$. Then every $f\in CS(1.3)$ for $n=3$ is strictly increasing and f^{-1} is a solution given in Theorem 16.

Theorem 1.18 (Theorem 3.2 [4]) Suppose that $r_1<r_2<-1<r_3<0.$ Then all $f\in CS(1.3)$ for $n=3$ are strictly decreasing and $x=0$ is the unique fixed point of every f. Moreover,

(i) If $x=0$ is attractive fixed point of f^2, then $f(x)=r_3x.$

(ii) If $x=0$ is repelling fixed point of f^2, then f is a solution in the class given in Theorem 3 of Nabeya [5].

Corollary 1.19 (Corollary 3.2 [4]) Suppose that $r_3 < -1 < r_2 < r_1 < 0$. Then all $f \in CS(1.3)$ for $n = 3$ are strictly decreasing with the unique fixed point 0, and the inverse f^{-1} is a solution given in Theorem 18.

From now, Zhang and Gong [4] considered all continuous solutions of equation (1.3) for $n = 3$ for one negative root and two positive roots as in Table 1 and two negative roots and one positive root as in Table 2.

Table 1. Two positive and one negative characteristic roots

$(i)\ 1 < -r_1 < r_2 < r_3$	$(iii)\ 1 < r_2 < -r_1 < r_3$	$(v)\ 1 < r_2 < r_3 < -r_1$
$(\overline{v})\ 0 < -r_1 < r_2 < r_3 < 1$	$(\overline{iii})\ 0 < r_2 < -r_1 < r_3 < 1$	$(\overline{i})\ 0 < r_2 < r_3 < -r_1 < 1$
$(ii)\ 0 < -r_1 < 1 < r_2 < r_3$	$(iv)\ 0 < r_2 < 1 < -r_1 < r_3$	$(vi)\ 0 < r_2 < 1 < r_3 < -r_1$
$(\overline{ii})\ 0 < r_2 < r_3 < 1 < -r_1$	$(\overline{iv})\ 0 < r_2 < -r_1 < 1 < r_3$	$(\overline{vi})\ 0 < -r_1 < r_2 < 1 < r_3$

Table 2. One positive and two negative characteristic roots

$(a)\ 1 < -r_1 < -r_2 < r_3$	$(c)\ 1 < -r_1 < r_3 < -r_2$	$(e)\ 1 < r_3 < -r_1 < -r_2$
$(\overline{e})\ 0 < -r_1 < -r_2 < r_3 < 1$	$(\overline{c})\ 0 < -r_1 < r_3 < -r_2 < 1$	$(\overline{a})\ 0 < r_3 < -r_1 < -r_2 < 1$
$(b)\ 0 < -r_1 < 1 < -r_2 < r_3$	$(d)\ 0 < -r_1 < 1 < r_3 < -r_2$	$(f)\ 0 < r_3 < 1 < -r_1 < -r_2$
$(\overline{b})\ 0 < r_3 < -r_1 < 1 < -r_2$	$(\overline{d})\ 0 < -r_1 < r_3 < 1 < -r_2$	$(\overline{f})\ 0 < -r_1 < -r_2 < 1 < r_3$

Theorem 1.20 (Theorem 4.1 [4]) Cases $(i)\ 1 < -r_1 < r_2 < r_3$, $(ii)\ 0 < -r_1 < 1 < r_2 < r_3$, $(iii)\ 1 < r_2 < -r_1 < r_3$ and $(v)\ 1 < r_2 < r_3 < -r_1$ in Table 1.

1) If $f \in CSI(1.3)$ for $n = 3$, then f is a function in the class given in Theorem 1 of [5].

2) If $f \in CSD(1.3)$ for $n = 3$, then $f(x) = r_1 x$.

Corollary 1.21 (Corollary 4.1 [4]) Cases $(\overline{i})\ 0 < r_2 < r_3 < -r_1 < 1$, $(\overline{ii})\ 0 < r_2 < r_3 < 1 < -r_1$, $(\overline{iii})\ 0 < r_2 < -r_1 < r_3 < 1$ and $(\overline{v})\ 0 < -r_1 < r_2 < r_3 < 1$ in Table 1.

If $f \in CS(1.3)$ for $n = 3$, then f^{-1} is a function in the class given in Theorem 20.

Theorem 1.22 (Theorem 4.2 [4]) Cases $(\overline{iv})\ 0 < r_2 < -r_1 < 1 < r_3$ and $(\overline{vi})\ 0 < -r_1 < r_2 < 1 < r_3$ in Table 1.

1) If $f \in CSI(1.3)$ for $n = 3$, then f is a function in the class given in Theorem 2 of [5].

2) If $f \in CSD(1.3)$ for $n = 3$, then $f(x) = r_1 x$.

Corollary 1.23 (Corollary 4.2 [4]) Cases $(iv)\ 0 < r_2 < 1 < -r_1 < r_3$ and $(vi)\ 0 < r_2 < 1 < r_3 < -r_1$ in Table 1.

If $f \in CS(1.3)$ for $n = 3$, then f^{-1} is a function in the class given in Theorem 22.

Theorem 1.24 (Theorem 4.3 [4]) Cases $(a)\ 1 < -r_1 < -r_2 < r_3$, $(c)\ 1 < -r_1 < r_3 < -r_2$, $(e)\ 1 < r_3 < -r_1 < -r_2$ and $(f)\ 0 < r_3 < 1 < -r_1 < -r_2$ in Table 2.

1) If $f \in CSI(1.3)$ for $n=3$, then $f(x) = r_3 x$.

2) If $f \in CSD(1.3)$ for $n=3$, then f is a function in the class given in Theorem 3 of Nabeya [5].

Corollary 1.25 (Corollary 4.3 [4]) Cases $(\bar{a})\ 0 < r_3 < -r_1 < -r_2 < 1$, $(\bar{c})\ 0 < -r_1 < r_3 < -r_2 < 1$, $(\bar{e})\ 0 < -r_1 < -r_2 < r_3 < 1$ and $(\bar{f})\ 0 < -r_1 < -r_2 < 1 < r_3$ in Table 2.

If $f \in CS(1.3)$ for $n=3$, then f^{-1} is a function in the class given in Theorem 24.

Theorem 1.26 (Theorem 4.4 [4]) Cases $(b)\ 0 < -r_1 < 1 < -r_2 < r_3$ and $(d)\ 0 < -r_1 < 1 < r_3 < -r_2$ in Table 2.

1) If $f \in CSI(1.3)$ for $n=3$, then $f(x) = r_3 x$.

2) If $f \in CSD(1.3)$ for $n=3$, then $f(x) = r_1 x$ or $f(x) = r_2 x$.

Corollary 1.27 (Corollary 4.4 [4]) Cases $(\bar{b})\ 0 < r_3 < -r_1 < 1 < -r_2$ and $(\bar{d})\ 0 < -r_1 < r_3 < 1 < -r_2$ in Table 2.

If $f \in CS(1.3)$ for $n=3$, then f^{-1} is a function in the class given in Theorem 26.

We single out from Lemma 1.13, two useful formulas for the iterates of an element in $CS(1.6)$.

Lemma 1.28 Suppose that (1.6) has four different characteristic roots $r_1, r_2, r_3, r_4 \in C$. If $f \in CS(1.6)$, then

$$f^m(x) = \sum_{j=1}^{4} \frac{r_j^m}{\prod_{i=1, i\neq j}^{4} (r_j - r_i)} [f^3(x) - (\sum_{k \neq j}^{4} r_k) f^2(x) + (\sum_{k>t, \neq j}^{4} r_k r_t) f(x) - \prod_{i \neq j}^{4} r_i x]$$

$$f^{-m}(x) = \sum_{j=1}^{4} \frac{r_j^{-m}}{\prod_{i=1, i\neq j}^{4} (\frac{1}{r_j} - \frac{1}{r_i})} [f^{-3}(x) - (\sum_{k \neq j}^{4} \frac{1}{r_k}) f^{-2}(x) + (\sum_{k>t, \neq j}^{4} \frac{1}{r_k r_t}) f^{-1}(x) - \prod_{i \neq j}^{4} \frac{1}{r_i} x]$$

for any integer $m \geq 0$.

3. Results and Discussion

3.1 Results

Our results are derived in accordance with those listed in Cases A, B and C. Apart from adopting the approach Zhang and Gong [4], we introduce a novel technique of using a second limiting criterion for the cases starting from (A2),(A3) (i.e., from Theorem 2.23) onwards.

Theorem 2.1 Cases (A1):$1 < -r_1 < r_2 < r_3 < r_4$, (A4):$1 < r_2 < r_3 < r_4 < -r_1$ and $(\overline{\text{A10}})\ 0 < -r_1 < 1 < r_2 < r_3 < r_4$.

i) If $f \in CSI(1.6)$, then f is a function given in Theorem 1.1.
ii) If $f \in CSD(1.6)$, then $f(x) = r_1 x$, the form given by Theorem 1.20.

Proof. We know that if $f \in CSI(1.6)$, then $f^{-1} \in CSI(1.7)$. If $f \in CSD(1.6)$, then $f^{-1} \in CSD(1.7)$.

Case (A1) : $1 < -r_1 < r_2 < r_3 < r_4$.

i) Let $f \in CSI(1.6)$. From Lemma 1.28, we have

$$\lim_{m\to\infty} \frac{f^{-m}(x)}{r_1^{-m}} = \frac{1}{(\frac{1}{r_1}-\frac{1}{r_2})(\frac{1}{r_1}-\frac{1}{r_3})(\frac{1}{r_1}-\frac{1}{r_4})} \left(f^{-3}(x) - (\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}) f^{-2}(x) + (\frac{1}{r_2 r_3}+\frac{1}{r_2 r_4}+\frac{1}{r_3 r_4}) f^{-1}(x) - \frac{1}{r_2 r_3 r_4} x \right).$$

For a fixed $x \in R$, since $f^{-1}(x)$ is strictly increasing, the limiting function $\lim_{m\to\infty} \frac{f^{-m}(x)}{r_1^{-m}}$ is nondecreasing for even m and nonincreasing for odd m, which implies that it must be a constant, i.e.,

$$f^{-3}(x) - (\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}) f^{-2}(x) + (\frac{1}{r_2 r_3}+\frac{1}{r_2 r_4}+\frac{1}{r_3 r_4}) f^{-1}(x) - \frac{1}{r_2 r_3 r_4} x = c_1 \in R, \forall x \in R. \quad (2.1)$$

Substituting (2.1) into (1.7), we get $c_1 = r_1^{-1} c_1$ implying that $c_1 = 0$, and so

$$f^{-3}(x) - (\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}) f^{-2}(x) + (\frac{1}{r_2 r_3}+\frac{1}{r_2 r_4}+\frac{1}{r_3 r_4}) f^{-1}(x) - \frac{1}{r_2 r_3 r_4} x = 0,$$

equivalently,

$$f^3(x) - (r_2 + r_3 + r_4) f^2(x) + (r_2 r_3 + r_2 r_4 + r_3 r_4) f(x) - r_2 r_3 r_4 x = 0. \quad (2.2)$$

The equation (2.2) is of 3rd-order with three positive distinct characteristic roots > 1, and its solutions are as given in Theorem 1.1.

ii) Let $f \in CSD(1.6)$. From Lemma 1.28, we have

$$\lim_{m\to\infty} \frac{f^m}{r_4^m} = \frac{1}{(r_4 - r_1)(r_4 - r_2)(r_4 - r_3)} \left(f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x \right).$$

For a fixed $x \in R$, since $f(x)$ is strictly decreasing, the limiting function $\lim_{m\to\infty} \frac{f^m(x)}{r_4^m}$ is nondecreasing for even m and nonincreasing for odd m, which forces it to be a constant, i.e.,

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = c_1 \in R, \forall x \in R. \quad (2.3)$$

Substituting (2.3) into (1.6), we get $c_1 = r_4 c_1$ implying that $c_1 = 0$, and so

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0.$$

Since the three distinct characteristic roots satisfy $1 < -r_1 < r_2 < r_3$, Theorem 1.20 gives $f(x) = r_1 x$.

Case (A4) : $1 < r_2 < r_3 < r_4 < -r_1$.

i) Assume that $f \in CSI(1.6)$. By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $1<r_2<r_3<r_4$, the solution function is given by Theorem 1.1.

ii) Assume that $f\in CSD(1.6)$. By the same proof as that of the case (A1), we obtain

$$f^{-3}(x)-(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_1r_3r_4}x=0,$$

equivalently,

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0.$$

Since $1<r_3<r_4<-r_1$, Theorem 1.20 yields $f(x)=r_1x$.

Case $(\overline{\text{A10}}):0<-r_1<1<r_2<r_3<r_4$.

The proof for this case is the same as that of the case (A1) and is omitted.

Corollary 2.2 Cases $(\overline{\text{A1}}):0<r_2<r_3<r_4<-r_1<1,$ $(\overline{\text{A4}}):0<-r_1<r_2<r_3<r_4<1$ and $(\text{A10}):0<r_2<r_3<r_4<1<-r_1$.

Proof. For the case $(\overline{\text{A1}})$ since $0<r_2<r_3<r_4<-r_1<1$, we get $\frac{1}{r_2}>\frac{1}{r_3}>\frac{1}{r_4}>-\frac{1}{r_1}>1$.

The reciprocals are characteristic roots of (1.7), which is the dual equation of (1.6). Thus, its solution f^{-1} is a function given in Theorem 2.1 depending on its behavior (increasing or decreasing).

The cases $(\overline{\text{A4}}):0<-r_1<r_2<r_3<r_4<1$ and $(\text{A10}):0<r_2<r_3<r_4<1<-r_1$ are reasoned similarly.

The solution functions in the forthcoming corollaries are derived via the same arguments as in Corollary 2.2.

Theorem 2.3 Case $(\text{A7}):0<r_2<1<r_3<r_4<-r_1$.

i) If $f\in CSI(1.6)$, then f is as given in Theorem 1.16.

ii) If $f\in CSD(1.6)$, then $f(x)=r_1x$, the form given by Theorem 1.20.

Proof. i) Assume that $f\in CSI(1.6)$. By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $0<r_2<1<r_3<r_4$, the solution function f is as given in Theorem 1.16.

ii) Assume that $f\in CSD(1.6)$. By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0.$$

Since $1<r_3<r_4<-r_1$, again Theorem 1.20 yields $f(x)=r_1x$.

Corollary 2.4 Case $(\overline{\text{A7}}):0<-r_1<r_2<r_3<1<r_4$.

If $f\in CS(1.6)$, then f^{-1} is a function given in Theorem 2.3.

Theorem 2.5 Case $(\overline{\text{A9}}):0<-r_1<r_2<1<r_3<r_4$.

i) If $f \in CSI(1.6)$, then f is a function given in Theorem 1.16.

ii) If $f \in CSD(1.6)$, then $f(x) = r_1 x$, the form given by Theorem 1.22.

Proof. i) Assume that $f \in CSI(1.6)$. By the same proof as the case (A1) by Lemma 1.28 (form of f^{-m}) and removing $\frac{1}{r_1}$ we obtain

$$f^3(x) - (r_2 + r_3 + r_4) f^2(x) + (r_2 r_3 + r_2 r_4 + r_3 r_4) f(x) - r_2 r_3 r_4 x = 0.$$

Since $0 < r_2 < 1 < r_3 < r_4$, then f is a function in the class given in Theorem 1.16.

ii) Assume that $f \in CSD(1.6)$. By the same proof as that of the case (A1), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0.$$

Since $0 < -r_1 < r_2 < 1 < r_3$, Theorem 1.22 yields $f(x) = r_1 x$.

Corollary 2.6 Case $(\text{A9}): 0 < r_2 < r_3 < 1 < r_4 < -r_1$.

If $f \in CS(1.6)$, then f^{-1} is a function given in Theorem 2.5.

Theorem 2.7 Cases $(\text{B1}): 1 < r_1 < -r_2 < -r_3 < -r_4$, $(\text{B4}): 1 < -r_2 < -r_3 < -r_4 < r_1$ and $(\overline{\text{B10}}): 0 < r_1 < 1 < -r_2 < -r_3 < -r_4$.

i) If $f \in CSI(1.6)$, then $f(x) = r_1 x$, the form given by Theorem 1.24.

ii) If $f \in CSD(1.6)$, then f is given by Theorem 1.4.

Proof. Case $(\text{B1}): 1 < r_1 < -r_2 < -r_3 < -r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0.$$

Since $1 < r_1 < -r_2 < -r_3$, Theorem 1.24 gives $f(x) = r_1 x$.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x) - (r_2 + r_3 + r_4) f^2(x) + (r_2 r_3 + r_2 r_4 + r_3 r_4) f(x) - r_2 r_3 r_4 x = 0.$$

Since $1 < -r_2 < -r_3 < -r_4$, the solution function f is given by Theorem 1.4.

Case $(\text{B4}): 1 < -r_2 < -r_3 < -r_4 < r_1$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x) - (r_1 + r_3 + r_4) f^2(x) + (r_1 r_3 + r_1 r_4 + r_3 r_4) f(x) - r_1 r_3 r_4 x = 0.$$

Since $1 < -r_3 < -r_4 < r_1$, Theorem 1.24 gives $f(x) = r_1 x$.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x) - (r_2 + r_3 + r_4) f^2(x) + (r_2 r_3 + r_2 r_4 + r_3 r_4) f(x) - r_2 r_3 r_4 x = 0.$$

Since $1 < -r_2 < -r_3 < -r_4$, the solution function is given by Theorem 1.4.

Case $(\overline{\text{B10}}): 0 < r_1 < 1 < -r_2 < -r_3 < -r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0.$$

Since $0 < r_1 < 1 < -r_2 < -r_3$, Theorem 1.24 then yields $f(x) = r_1 x$.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $1<-r_2<-r_3<-r_4$, the solution f is given by Theorem 1.4.

Corollary 2.8 Cases $(\overline{\mathrm{B1}}):0<-r_2<-r_3<-r_4<r_1<1$, $(\overline{\mathrm{B4}}):0<r_1<-r_2<-r_3<-r_4<1$ and $(\mathrm{B10}):0<-r_2<-r_3<-r_4<1<r_1$.

If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.7.

Theorem 2.9 Case $(\mathrm{B7}):0<-r_2<1<-r_3<-r_4<r_1$.

i) If $f\in CSI(1.6)$, then $f(x)=r_1x$, the form given by Theorem 1.24.

ii) If $f\in CSD(1.6)$, then the solution function f is as given in Theorem 1.18.

Proof. i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0.$$

Since $1<-r_3<-r_4<r_1$, Theorem 1.24 then gives $f(x)=r_1x$.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $0<-r_2<1<-r_3<-r_4$, the solution f is given by Theorem 1.18.

Corollary 2.10 Case $(\overline{\mathrm{B7}}):0<r_1<-r_2<-r_3<1<-r_4$.

If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.9.

Theorem 2.11 Case $(\overline{\mathrm{B9}}):0<r_1<-r_2<1<-r_3<-r_4$.

i) If $f\in CSI(1.6)$, then $f(x)=r_1x$, the form given in Corollary 1.27.

ii) If $f\in CSD(1.6)$, then f is given by Theorem 1.18.

Proof. i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $0<r_1<-r_2<1<-r_3$, Theorem 1.27 yields $f(x)=r_1x$.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $-r_2<1<-r_3<-r_4$, the solution function f is given by Theorem 1.18.

Corollary 2.12 Case $(\mathrm{B9}):0<-r_2<-r_3<1<-r_4<r_1$.

If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.11.

Theorem 2.13 Cases $(C1):1<-r_1<-r_4<r_2<r_3$, $(C2):1<-r_1<r_2<-r_4<r_3$, $(C5)\;1<r_2<-r_1<r_3<-r_4$, and $(C6):1<r_2<r_3<-r_1<-r_4$.

i) If $f\in CSI(1.6)$, then f is given by Theorem 1.20.

ii) If $f\in CSD(1.6)$, then f is given by Theorem 1.24.

Proof. Case $(C1):1<-r_1<-r_4<r_2<r_3$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $1<-r_4<r_2<r_3$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_4)f^2(x)+(r_1r_2+r_1r_4+r_3r_4)f(x)-r_1r_2r_4x=0\ .$$

Since $1<-r_1<-r_4<r_2$, the solution f is given by Theorem 1.24.

Case $(C2):1<-r_1<r_2<-r_4<r_3$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $1<r_2<-r_4<r_3$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_4)f^2(x)+(r_1r_2+r_1r_4+r_2r_4)f(x)-r_1r_2r_4x=0\ .$$

Since $1<-r_1<r_2<-r_4$, the solution f is given by Theorem 1.24.

Case $(C5):1<r_2<-r_1<r_3<-r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $1<r_2<-r_1<r_3$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0\ .$$

Since $1<-r_1<r_3<-r_4$, the solution f is given by Theorem 1.24.

Case $(C6):1<r_2<r_3<-r_1<-r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $1<r_2<r_3<-r_1$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0\ .$$

Since $1<r_3<-r_1<-r_4$, the solution f is given by Theorem 1.24**.**

Corollary 2.14 Cases $(\overline{C1}):0<r_2<r_3<-r_1<-r_4<1,$ $(\overline{C2}):0<r_2<-r_1<r_3<-r_4<1,$ $(\overline{C5}):0<-r_1<r_2<-r_4<r_3<1$ and $(\overline{C6}):0<-r_1<-r_4<r_2<r_3<1.$
If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.13.

Theorem 2.15 Cases $(C8):0<r_2<1<-r_1<r_3<-r_4$ and $(C9):0<r_2<1<r_3<-r_1<-r_4$.
i) If $f\in CSI(1.6)$, then f is given by Corollary 1.23.
ii) If $f\in CSD(1.6)$, then f is given by Theorem 1.24.
Proof. Case $(C8):0<r_2<1<-r_1<r_3<-r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $0<r_2<1<-r_1<r_3$, the solution f is given by Corollary 1.23.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0 .$$

Since $1<-r_1<r_3<-r_4$, the solution f is given by Theorem 1.24.

Case $(C9): 0<r_2<1<r_3<-r_1<-r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $0<r_2<1<r_3<-r_1$, the solution f is given by Corollary 1.23.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0 .$$

Since $1<r_3<-r_1<-r_4$, the solution f is given by Theorem 1.24.

Corollary 2.16 Cases $(\overline{C8}):0<-r_1<r_2<-r_4<1<r_3$ and $(\overline{C9}):0<-r_1<-r_4<r_2<1<r_3$.
If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.15.

Theorem 2.17 Case $(C10): 0<-r_1<-r_4<1<r_2<r_3$.
i) If $f\in CSI(1.6)$, then f is given by Theorem 1.20.
ii) If $f\in CSD(1.6)$, then f is given by Corollary 1.25.
Proof. i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $0<-r_4<1<r_2<r_3$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_4)f^2(x)+(r_1r_2+r_1r_4+r_2r_4)f(x)-r_1r_2r_4x=0 .$$

Since $0<-r_1<-r_4<1<r_2$, the solution f is given by Corollary 1.25.

Corollary 2.18 Case $(\overline{C10}):0<r_2<r_3<1<-r_1<-r_4$.
If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.17.

Theorem 2.19 Cases $(C11):0<-r_1<1<-r_4<r_2<r_3$ and $(C12):0<-r_1<1<r_2<-r_4<r_3$.
i) If $f\in CSI(1.6)$, then f is given by Theorem 1.20.
ii) If $f\in CSD(1.6)$, then f is given by Theorem 1.26.
Proof. Case $(C11): 0<-r_1<1<-r_4<r_2<r_3$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $1<-r_4<r_2<r_3$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_4)f^2(x)+(r_1r_2+r_1r_4+r_2r_4)f(x)-r_1r_2r_4x=0 .$$

Since $0<-r_1<1<-r_4<r_2$, the solution f is given by Theorem 1.26.

Case $(C12): 0<-r_1<1<r_2<-r_4<r_3$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $1<r_2<-r_4<r_3$, the solution f is given by Theorem 1.20.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_4)f^2(x)+(r_1r_2+r_1r_4+r_2r_4)f(x)-r_1r_2r_4x=0\ .$$

Since $0<-r_1<1<r_2<-r_4$, the solution f is given by Theorem 1.26.

Corollary 2.20 Cases $(\overline{C11}):0<r_2<r_3<-r_1<1<-r_4$ and $(\overline{C12}):0<r_2<-r_1<r_3<1<-r_4$. If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.19.

Theorem 2.21 Case $(C14):0<-r_1<r_2<1<-r_4<r_3$.

i) If $f\in CSI(1.6)$, then f is given by Corollary 1.23.

ii) If $f\in CSD(1.6)$, then f is given by Corollary 1.27.

Proof. i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0.$$

Since $0<r_2<1<-r_4<r_3$, the solution f is given by Corollary 1.23.

ii) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_4)f^2(x)+(r_1r_2+r_1r_4+r_2r_4)f(x)-r_1r_2r_4x=0\ .$$

Since $0<-r_1<r_2<1<-r_4$, the solution f is given by Corollary 1.27.

Corollary 2.22 Case $(\overline{C14}):0<r_2<-r_1<1<r_3<-r_4$. If $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.21.

Theorem 2.23 Cases $(\mathrm{A2}):1<r_2<-r_1<r_3<r_4$ and $(\mathrm{A3}):1<r_2<r_3<-r_1<r_4$.

i) Assume that $f\in CSI(1.6)$. In the case (A2), further assume that $\lim_{m\to\infty}\frac{f^{-m}}{r_1^{-m}}$ exists. While in the case (A3) further assume that $\lim_{m\to\infty}\frac{f^{m}}{r_1^{m}}$ exists, then the solution function f is as given in Theorem 1.6.

ii) If $f\in CSD(1.6)$, then $f(x)=r_1x$, the form given by Theorem 1.20.

Proof. Case $(\mathrm{A2}):1<r_2<-r_1<r_3<r_4$.

Let $\lim_{m\to\infty}\frac{f^{-m}}{r_1^{-m}}=c$. From Lemma 1.28, we get

$$c=\lim_{m\to\infty}\frac{f^{-m}}{r_1^{-m}}=\frac{1}{(\frac{1}{r_1}-\frac{1}{r_2})(\frac{1}{r_1}-\frac{1}{r_3})(\frac{1}{r_1}-\frac{1}{r_4})}(f^{-3}(x)-(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_2r_3}+\frac{1}{r_2r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_2r_3r_4}x)$$

$$+\lim_{m\to\infty}\frac{(\frac{r_1}{r_2})^m}{(\frac{1}{r_2}-\frac{1}{r_1})(\frac{1}{r_2}-\frac{1}{r_3})(\frac{1}{r_2}-\frac{1}{r_4})}(f^{-3}(x)-(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_1r_3r_4}x)$$

showing that

$$\lim_{m\to\infty}\frac{(\frac{r_1}{r_2})^m}{(\frac{1}{r_2}-\frac{1}{r_1})(\frac{1}{r_2}-\frac{1}{r_3})(\frac{1}{r_2}-\frac{1}{r_4})}(f^{-3}(x)-(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_1r_3r_4}x)=0.$$

Thus

$$f^{-3}(x)-(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_1r_3r_4}x=0,$$

or equivalently,

$$f^3(x)-(r_1+r_3+r_4)f^2(x)+(r_1r_3+r_1r_4+r_3r_4)f(x)-r_1r_3r_4x=0\,. \qquad (2.4)$$

Returning to the limiting equation, we deduce that:

$$f^{-3}(x)-(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_2r_3}+\frac{1}{r_2r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_2r_3r_4}x=c_1 \qquad (2.5)$$

for some constant c_1. Substituting (2.5) in (1.7), we get $c_1=r_1^{-1}c_1$ yielding $c_1=0$.
Hence,

$$f^{-3}(x)-(\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4})f^{-2}(x)+(\frac{1}{r_2r_3}+\frac{1}{r_2r_4}+\frac{1}{r_3r_4})f^{-1}(x)-\frac{1}{r_2r_3r_4}x=0,$$

equivalently,

$$f^3(x)-(r_2+r_3+r_4)f^2(x)+(r_2r_3+r_2r_4+r_3r_4)f(x)-r_2r_3r_4x=0. \qquad (2.6)$$

Subtracting (2.4) from (2.6), we get

$$f^2(x)-(r_3+r_4)f(x)+r_3r_4x=0.$$

Since $1<r_3<r_4$, the solution function f is given in Theorem 1.6.

i) By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $1<r_2<-r_1<r_3$, the solution function is $f(x)=r_1x$, the form given by Theorem 1.20.

Case $(\text{A3}): 1<r_2<r_3<-r_1<r_4$.

i) By the same proof as that of the case (A2), we obtain

$$f^2(x)-(r_2+r_3)f(x)+r_2r_3x=0.$$

Since $1<r_2<r_3$, the solution function f is as given in Theorem 1.6.

ii) By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $1<r_2<r_3<-r_1$, the solution function is $f(x)=r_1x$, the form given by Theorem 1.20.

Corollary 2.24. Cases $(\overline{\text{A2}}){:}0<r_2<r_3<-r_1<r_4<1$ and $(\overline{\text{A3}}){:}0<r_2<-r_1<r_3<r_4<1$. Under the hypotheses of Theorem 2.23, if $f\in CS(1.6)$, then f^{-1} is given in Theorem 2.23.

Theorem 2.25. Cases $(\text{A5}){:}0<r_2<1<-r_1<r_3<r_4$ and $(\text{A6}){:}0<r_2<1<r_3<-r_1<r_4$.

i) Assume that $f\in CSI(1.6)$. In the case (A5), further assume that $\lim_{m\to\infty}\frac{f^{-m}}{r_1^{-m}}$ exists, while in the case (A6) further assume that $\lim_{m\to\infty}\frac{f^{m}}{r_1^{m}}$ exists. Then the solution function f is as given in Theorems 1.6 and 1.8., respectively.

ii) If $f\in CSD(1.6)$, then $f(x)=r_1x$, the form given by Corollary 1.23.

Proof. Case $(\text{A5}):0<r_2<1<-r_1<r_3<r_4$.

i) By the same proof as that of the case (A2), we get

$$f^2(x)-(r_3+r_4)f(x)+r_3r_4x=0.$$

Since $1<r_3<r_4$, the solution function f is as given in Theorem 1.6.

ii) By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $0<r_2<1<-r_1<r_3$, by Corollary 1.23 $f(x)=r_1x$.

Case $(\text{A6}):0<r_2<1<r_3<-r_1<r_4$.

i) By the same proof as that of the case (A2), we obtain

$$f^2(x)-(r_2+r_3)f(x)+r_2r_3x=0.$$

Since $0<r_2<1<r_3$, thesolution function f is as given in Theorem 1.8.

ii) By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $0<r_2<1<r_3<-r_1$, by Corollary 1.23 $f(x)=r_1x$.

Corollary 2.26. Cases $(\overline{\text{A5}}){:}0<r_2<r_3<-r_1<1<r_4$ and $(\overline{\text{A6}}){:}0<r_2<-r_1<r_3<1<r_4$. Under the hypotheses of Theorem 2.25, if $f\in CS(1.6)$, then f^{-1} is as given in Theorem 2.25.

Theorem 2.27. Case $(\overline{\text{A8}}):0<r_2<-r_1<1<r_3<r_4$.

i) If $f\in CSI(1.6)$ and it $\lim_{m\to\infty}\frac{f^{-m}}{r_1^{-m}}$ exists, then f is as given in Theorem 1.6.

ii) If $f\in CSD(1.6)$, then $f(x)=r_1x$, the form given by Theorem 1.22.

Proof. i) By the same proof as that of the case (A2), we obtain

$$f^2(x)-(r_3+r_4)f(x)+r_3r_4x=0.$$

Since $1<r_3<r_4$, the solution function f is given by Theorem 1.6.

ii) By the same proof as that of the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $0<r_2<-r_1<1<r_3$, Theorem 1.22 gives $f(x)=r_1x$.

Corollary 2.28. Case $(\mathrm{A8}):0<r_2<r_3<1<-r_1<r_4$.
Under the hypotheses of Theorem 2.27, if $f\in CS(1.6)$, then f^{-1} is as given by Theorem 2.27.

Theorem 2.29 Cases $(\mathrm{B2}):1<-r_2<r_1<-r_3<-r_4$ and $(\mathrm{B3}):1<-r_2<-r_3<r_1<-r_4$.
i) If $f\in CSI(1.6)$, then $f(x)=r_1x$, the form given by Theorem 1.24.

ii) Assume that $f\in CSD(1.6)$. In the case $(\mathrm{B2})$, assume further that $\lim_{m\to\infty}\frac{f^{-m}}{r_1^{-m}}$ exists, while in the case $(\mathrm{B3})$, assume further that $\lim_{m\to\infty}\frac{f^{m}}{r_1^{m}}$ exist. Then f is given by Theorem 1.9.

Proof. Case $(\mathrm{B2}):1<-r_2<r_1<-r_3<-r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $1<-r_2<r_1<-r_3$, Theorem 1.24 gives $f(x)=r_1x$.

ii) By the same proof as in the case (A2), we obtain

$$f^2(x)-(r_3+r_4)f(x)+r_3r_4x=0.$$

Since $1<-r_3<-r_4$, the solution f is given by Theorem 1.9.

Case $(\mathrm{B3}):1<-r_2<-r_3<r_1<-r_4$.

i) By the same proof as in the case (A1), we obtain

$$f^3(x)-(r_1+r_2+r_3)f^2(x)+(r_1r_2+r_1r_3+r_2r_3)f(x)-r_1r_2r_3x=0.$$

Since $1<-r_2<-r_3<r_1$, Theorem 1.24 gives $f(x)=r_1x$.

ii) By the same proof as in the case (A2), we obtain

$$f^2(x)-(r_2+r_3)f(x)+r_2r_3x=0\ .$$

Since $1<-r_2<-r_3$, the solution f is given by Theorem 1.9.

Corollary 2.30 Case $(\overline{\mathrm{B2}}):0<-r_2<-r_3<r_1<-r_4<1$ and $(\overline{\mathrm{B3}}):0<-r_2<r_1<-r_3<-r_4<1$.
Under the hypotheses of Theorem 2.29, if $f\in CS(1.6)$, then f^{-1} is given by Theorem 2.29.

Theorem 2.31. Cases $(\mathrm{B5}):0<-r_2<1<r_1<-r_3<-r_4$ and $(\mathrm{B6}):0<-r_2<1<-r_3<r_1<-r_4$.
i) If $f\in CSI(1.6)$, then $f(x)=r_1x$, the form given by Theorem 1.26.

ii) Assume that $f \in CSD(1.6)$. In the case (B5), assume further that $\lim_{m\to\infty} \frac{f^{-m}}{r_1^{-m}}$ exists, while in the case (B6), assume further that $\lim_{m\to\infty} \frac{f^{m}}{r_1^{m}}$ exists. Then the solution function f is given by Theorems 1.9 and 1.10, respectively.

Proof. Case $(B5): 0 < -r_2 < 1 < r_1 < -r_3 < -r_4$.

i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1r_2 + r_1r_3 + r_2r_3) f(x) - r_1r_2r_3x = 0 .$$

Since $0 < -r_2 < 1 < r_1 < -r_3$, Theorem 1.26 yield $f(x) = r_1x$.

ii) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_3 + r_4) f(x) + r_3r_4x = 0 .$$

Since $1 < -r_3 < -r_4$, the solution f is given by Theorem 1.9.

Case $(B6): 0 < -r_2 < 1 < -r_3 < r_1 < -r_4$.

i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1r_2 + r_1r_3 + r_2r_3) f(x) - r_1r_2r_3x = 0 .$$

Since $0 < -r_2 < 1 < -r_3 < r_1$, Theorem 1.26 yield $f(x) = r_1x$.

ii) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_2 + r_3) f(x) + r_2r_3x = 0 .$$

Since $0 < -r_2 < 1 < -r_3$, the solution f is given by Theorem 1.10.

Corollary 2.32. Cases $(\overline{B5}): 0 < -r_2 < -r_3 < r_1 < 1 < -r_4$ and $(\overline{B6}): 0 < -r_2 < r_1 < -r_3 < 1 < -r_4$. Under the hypotheses of Theorem 2.31, if $f \in CS(1.6)$, then f^{-1} is given by Theorem 2.31.

Theorem 2.33. Case $(\overline{B8}): 0 < -r_2 < r_1 < 1 < -r_3 < -r_4$.

i) If $f \in CSI(1.6)$, then $f(x) = r_1x$, the form given by Corollary 1.27.

ii) If $f \in CSD(1.6)$ and if $\lim_{m\to\infty} \frac{f^{-m}}{r_1^{-m}}$ exist, then f is given by Theorem 1.9.

Proof. i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1r_2 + r_1r_3 + r_2r_3) f(x) - r_1r_2r_3x = 0.$$

Since $0 < -r_2 < r_1 < 1 < -r_3$, Corollary 1.27 gives $f(x) = r_1x$.

ii) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_3 + r_4) f(x) + r_3r_4x = 0.$$

Since $1 < -r_3 < -r_4$, the solution f is given by Theorem 1.9.

Corollary 2.34. Case $(B8): 0 < -r_2 < -r_3 < 1 < r_1 < -r_4$.
Under the hypotheses of Theorem 2.33, if $f \in CS(1.6)$, then f^{-1} is given by Theorem 2.33.

Theorem 2.35. Case $(C3): 1 < -r_1 < r_2 < r_3 < -r_4$.

i) If $f \in CSI(1.6)$, then f is given by Theorem 1.20.

ii) If $f \in CSD(1.6)$ and if $\lim_{m\to\infty} \frac{f^m}{r_3^m}$ exists, then $f(x) = r_1 x$, the form given by Theorem 1.12.

Proof. i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0 .$$

Since $1 < -r_1 < r_2 < r_3$, the solution f is given by Theorem 1.20.

ii) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_1 + r_2) f(x) + r_1 r_2 x = 0 .$$

Since $1 < -r_1 < r_2$, Theorem 1.12 gives $f(x) = r_1 x$.

Corollary 2.36. Case $(\overline{C3}): 0 < -r_1 < r_2 < r_3 < -r_4 < 1$.

Under the hypotheses of Theorem 2.35, if $f \in CS(1.6)$, then f^{-1} is given by Theorem 2.35.

Theorem 2.37. Case $(C4): 1 < r_2 < -r_1 < -r_4 < r_3$.

i) If $f \in CSI(1.6)$ and if $\lim_{m\to\infty} \frac{f^m}{r_4^m}$ exists, then $f(x) = r_2 x$, the form given by Theorem 1.12.

ii) If $f \in CSD(1.6)$, then f is given by Theorem 1.24.

Proof. i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2) f(x) + r_1 r_2 x = 0 .$$

Since $1 < r_2 < -r_1$, Theorem 1.12 gives $f(x) = r_1 x$.

ii) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_4) f^2(x) + (r_1 r_2 + r_1 r_4 + r_2 r_4) f(x) - r_1 r_2 r_4 x = 0 .$$

Since $1 < r_2 < -r_1 < -r_4$, the solution f is given by Theorem 1.24.

Corollary 2.38. Case $(\overline{C4}): 0 < r_2 < -r_1 < -r_4 < r_3 < 1$.

Under the hypotheses of Theorem 2.37, if $f \in CS(1.6)$, then f^{-1} is given by Theorem 2.37.

Theorem 2.39. Case $(C7): 0 < r_2 < 1 < -r_1 < -r_4 < r_3$.

i) If $f \in CSI(1.6)$ and if $\lim_{m\to\infty} \frac{f^m}{r_4^m}$ exists, then $f(x) = r_2 x$, the form given in Theorem 1.12.

ii) If $f \in CSD(1.6)$, then f is given by Theorem 1.24.

Proof. i) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_1 + r_2) f(x) + r_1 r_2 x = 0 .$$

Since $0 < r_2 < 1 < -r_1$, Theorem 1.12 yields $f(x) = r_2 x$.

ii) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_4) f^2(x) + (r_1 r_2 + r_1 r_4 + r_2 r_4) f(x) - r_1 r_2 r_4 x = 0 .$$

Since $0 < r_2 < 1 < -r_1 < -r_4$, the solution f is given by Theorem 1.24.

Corollary 2.40. Case $(\overline{C7}): 0 < r_2 < -r_1 < -r_4 < 1 < r_3$.
Under the hypotheses of Theorem 2.39, if $f \in CS(1.6)$, then f^{-1} is given by Theorem 2.39.

Theorem 2.41. Case $(C13): 0 < -r_1 < 1 < r_2 < r_3 < -r_4$.
i) If $f \in CSI(1.6)$, then f is given by Theorem 1.20.
ii) If $f \in CSD(1.6)$ and if $\lim_{m\to\infty} \frac{f^m}{r_3^m}$ exists, then $f(x) = r_1 x$, the form given by Theorem 1.12.
Proof. i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0 .$$

Since $0 < -r_1 < 1 < r_2 < r_3$, the solution f is given by Theorem 1.20.
ii) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_1 + r_2) f(x) + r_1 r_2 x = 0 .$$

Since $0 < -r_1 < 1 < r_2$, Theorem 1.12 gives $f(x) = r_1 x$.

Corollary 2.42. Case $(\overline{C13}): 0 < -r_1 < r_2 < r_3 < 1 < -r_4$.
Under the hypotheses of Theorem 2.41, if $f \in CS(1.6)$, then f^{-1} is given by Theorem 2.41.

Theorem 2.43. Case $(C15): 0 < -r_1 < r_2 < 1 < r_3 < -r_4$.
i) If $f \in CSI(1.6)$, then f is given by Theorem 1.22.
ii) If $f \in CSD(1.6)$ and if $\lim_{m\to\infty} \frac{f^m}{r_3^m}$ exists, then , $f(x) = r_1 x$, the for given by Theorem 1.12.
Proof. i) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_3) f^2(x) + (r_1 r_2 + r_1 r_3 + r_2 r_3) f(x) - r_1 r_2 r_3 x = 0 .$$

Since $0 < -r_1 < r_2 < 1 < r_3$, the solution f is given by Theorem 1.22.
ii) By the same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_4) f^2(x) + (r_1 r_2 + r_1 r_4 + r_2 r_4) f(x) - r_1 r_2 r_4 x = 0 .$$

Since $0 < -r_1 < r_2 < 1$, Theorem 1.12 gives $f(x) = r_1 x$.

Theorem 2.44. Case $(C16): 0 < r_2 < -r_1 < 1 < -r_4 < r_3$.
i) If $f \in CSI(1.6)$ and if $\lim_{m\to\infty} \frac{f^m}{r_4^m}$ exists, then , $f(x) = r_2 x$, the form given by Theorem 1.12.
ii) If $f \in CSD(1.6)$, then f is given by Corollary 1.27.
Proof. i) By The same proof as in the case (A2), we obtain

$$f^2(x) - (r_1 + r_2) f(x) + r_1 r_2 x = 0 .$$

Since $0 < r_2 < -r_1 < 1$, Theorem 1.12 yields $f(x) = r_2 x$.
ii) By The same proof as in the case (A2), we obtain

$$f^3(x) - (r_1 + r_2 + r_4) f^2(x) + (r_1 r_2 + r_1 r_4 + r_2 r_4) f(x) - r_1 r_2 r_4 x = 0 .$$

Since $0 < r_2 < -r_1 < 1 < -r_4$, the solution f is given by Corollary 1.27.

3.2 General discussion

Based mainly on the ideas from the work of Zhang and Gong [4], all solution functions $f \in CSI(1.6)$ have been determined subject to the restrictions that

- the characteristic roots r_1, r_2, r_3, r_4 satisfy $|r_i| \neq 0,1 \ (i = 1,2,3,4)$,
- the absolute values of the charcteristic roots $|r_i| \quad (i = 1,2,3,4)$ are all distinct and
- the four characteristic roots have different signs.

In particular, the following cases have been completely solved.
A. One negative characteristic root $r_1 < 0$, and three positive characteristic roots $0 < r_2 < r_3 < r_4$.
B. Three negative characteristic roots $0 > r_2 > r_3 > r_4$, and one positive characteristic root $r_1 > 0$.
C. Two negative characteristic roots $0 > r_1 > r_4$, and two positive characteristic roots $r_3 > r_2 > 0$.

In certain cases an extra condition, namely, $\lim_{m\to\infty} \frac{f^m}{r_i^m}$ for some $i = 1,2,3,4$, is needed to obtain the solutions.

4. Conclusions

There are totally seventy subcases solved in this work, but there remain two cases that are yet to be resolved for which the methods and techniques used here do not seem to work. These subcases are when:
(i) all characteristic roots are positive, some being in $(0,1)$ and the others in $(1,\infty)$, and
(ii) all characteristic roots are negative, some being in $(-1,0)$ and the others in $(-\infty,-1)$.

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