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*Original Article*

# New analytic solutions of some fourth-order nonlinear space-time fractional partial differential equations by $G'/G$-expansion method

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## Abstract

The exact traveling wave solutions of the space-time fractional Estevez-Mansfield-Clarkson (EMC) equation and the space-time fractional Ablowitz-Kaup-Newell-Segur (AKNS) water equation are investigated. Transforming the space-time fractional EMC and the space-time fractional AKNS water equations by the Jumarie's Riemann-Liouville derivative, nonlinear ordinary differential equations (ODEs) are achived. It is observed that the $G'/G$-expansion method is valid and reliable for solving some nonlinear fractional PDEs. Applying the $G'/G$-expansion method, the solutions of EMC and AKNS water equations are obtained in terms of hyperbolic functions, trigonometric functions and rational functions.

**Keywords:** $G'/G$-expansion method, fractional partial differential equations, fractional Estevez-Mansfield-Clarkson equation, fractional Ablowitz-Kaup-Newell-Segur water equation, analytic solution

## 1. Introduction

One of the fascinating problems in mathematical physics is solving nonlinear fractional PDEs. Recently, researchers have found many powerful methods to solve the nonlinear fractional PDEs such as first integral method (Eslami, Vajargah, Mirzazadeh, & Biswas, 2014; Ilie, Biazar, & Ayati, 2018; Lu, 2012), modified Kudryashov method (Ege, & Misirli, 2014; Kumar, Seadawy, & Joardar, 2018), extended Kudryashov method (Ege, & Misirli, 2018), generalized Kudryashov method (Demiray, Pandir, & Bulut, 2014; Gaber, Aljohani, Ebaid, & Machado, 2019), fractional sub-equation method (Alzaidy, 2013; Mohyud-Din, Nawaz, Azhar, & Akbar, 2017; Zhang, 2011) and $G'/G$-expansion method (Bekir, & Güner, 2013, 2017; Bin, 2012).

In this work, we applied the transformation using the Jumarie's Riemann-Liouville derivative and the $G'/G$-expansion method to illustrate the analytic solutions of the space-time fractional Estevez-Mansfield-Clarkson (EMC) equation and the space-time fractional Ablowitz-Kaup-Newell-Segur (AKNS) water equation. Some properties of Jumarie's Riemann-Liouville derivatives are stated as the following.

The Jumarie's Riemann-Liouville derivative of order $\alpha$ with respect to $t$ is defined as follows (Jumarie, 2006),

$$D_t^\alpha f(t) = \begin{cases} f(t) & ,\alpha = 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\zeta)^{-\alpha} (f(\zeta) - f(0)) d\zeta & ,0 < \alpha < 1, \\ \dfrac{d^n}{dt^n} D_t^{\alpha-n} f(t) & ,n \le \alpha < n+1, n \ge 1. \end{cases} \tag{1}$$

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Some important properties of fractional Riemann-Liouville derivatives are the following (Jumarie, 2009),

$$D_t^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} \qquad ,k \geq 0, \tag{2}$$

$$D_t^{\alpha}(f(t)g(t)) = f(t)D_t^{\alpha}g(t) + g(t)D_t^{\alpha}f(t), \tag{3}$$

$$D_t^{\alpha} f[g(t)] = D_g^{\alpha} f[g(t)](g'(t))^{\alpha} = f_g'[g(t)]D_t^{\alpha}g(t). \tag{4}$$

The procedure of solving space-time fraction EMC equation with $G'/G$-expansion method will be explained in the next section.

## 2. Space-Time Fractional EMC Equation

Consider the nonlinear space-time fractional EMC equation (Mansfield, & Clarkson, 1997),

$$D_y^{3\alpha} D_t^{\alpha} u + \beta D_y^{\alpha} u D_y^{\alpha} D_t^{\alpha} u + \beta D_y^{2\alpha} u D_t^{\alpha} u + D_t^{2\alpha} u = 0, t > 0, 0 < \alpha \leq 1, \tag{5}$$

where $u = u(x,y,t)$, $\beta$ is constant and $\alpha$ is a parameter explaining the order of the fractional space-time derivative. Using the transformation

$$u(x,y,t) = U(\zeta), \zeta = \frac{kx^{\alpha}}{\Gamma(\alpha+1)} + \frac{ly^{\alpha}}{\Gamma(\alpha+1)} - \frac{ct^{\alpha}}{\Gamma(\alpha+1)}, \tag{6}$$

where $k,l$ and $c$ are nonzero constants. Therefore, equation (5) can be transformed into an ODE,

$$-cl^3 \frac{d^4U}{d\zeta^4} - cl^2\beta \frac{dU}{d\zeta} \cdot \frac{d^2U}{d\zeta^2} - cl^2\beta \frac{d^2U}{d\zeta^2} \cdot \frac{dU}{d\zeta} + c^2 \frac{d^2U}{d\zeta^2} = 0. \tag{7}$$

$$-l^3 \frac{d^4U}{d\zeta^4} - 2l^2\beta \frac{dU}{d\zeta} \cdot \frac{d^2U}{d\zeta^2} + c\frac{d^2U}{d\zeta^2} = 0. \tag{8}$$

Integrating equation (8) and setting the constant to zero, we get

$$-l^3 \frac{d^3U}{d\zeta^3} - l^2\beta \left(\frac{dU}{d\zeta}\right)^2 + c\frac{dU}{d\zeta} = 0. \tag{9}$$

Applying the $G'/G$-expansion method, the solution may be exhibited in the form

$$U(\zeta) = \sum_{i=0}^{N} a_i \left(\frac{G'(\zeta)}{G(\zeta)}\right)^i, \tag{10}$$

where $a_i$ are real constants with $a_N$ is nonzero constant. The function *G* satisfied the ODE,

$$\frac{d^2G}{d\zeta^2} + \lambda \frac{dG}{d\zeta} + \mu G = 0, \tag{11}$$

where $\lambda$ and $\mu$ are nonzero constants. Solving equation (11), the solutions can be identified into three cases with arbitrary constants $c_1$ and $c_2$:

**Case I**: $\lambda^2 - 4\mu > 0$, the hyperbolic function solutions,

$$\frac{G'(\zeta)}{G(\zeta)}=\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}+c_2\cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}}{c_1\cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}+c_2\sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}}\right). \tag{12}$$

**Case II**: $\lambda^2-4\mu<0$, the trigonometric function solutions,

$$\frac{G'(\zeta)}{G(\zeta)}=\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}+c_2\cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}}{c_1\cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}+c_2\sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}}\right). \tag{13}$$

**Case III**: $\lambda^2-4\mu=0$, the rational function solutions,

$$\frac{G'(\zeta)}{G(\zeta)}=-\frac{\lambda}{2}+\left(\frac{c_2}{c_1+c_2\zeta}\right). \tag{14}$$

Balancing the highest order derivative and nonlinear terms of equation (9), we get $N=1$. Thus, equation (10) become

$$U(\zeta)=a_0+a_1\left(\frac{G'(\zeta)}{G(\zeta)}\right). \tag{15}$$

Using equation (15), we replace all $U(\zeta)$ in equation (9) and collect all terms which have the same power of $\frac{G'(\zeta)}{G(\zeta)}$. Setting each coefficient of them to zero, this gives

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^0: \qquad 2a_1l^3\mu^2+a_1l^3\lambda^2\mu-a_1^2l^2\beta\mu^2-ca_1\mu=0, \tag{16}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^1: \qquad a_1l^3\lambda^3+8a_1l^3\lambda\mu-2a_1^2l^2\beta\lambda\mu-ca_1\lambda=0, \tag{17}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^2: \qquad 8a_1l^3\mu+7a_1l^3\lambda^2-a_1^2l^2\beta\lambda^2-2a_1^2l^2\beta\mu-ca_1=0, \tag{18}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^3: \qquad 12a_1l^3\lambda-2a_1^2l^2\beta\lambda=0, \tag{19}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^4: \qquad 6a_1l^3-a_1^2l^2\beta=0. \tag{20}$$

Solving the system of equations (16)-(20) yields

$$a_1=\frac{6l}{\beta}, \lambda^2-4\mu=\frac{c}{l^3}. \tag{21}$$

Substituting equation (21) into equations (12)-(15), the exact solutions of the space-time fractional EMC equation may be described in the following three cases with arbitrary constants $c_1$ and $c_2$ and $\zeta$ :

**Case I**: $\lambda^2 - 4\mu > 0$, the hyperbolic function solutions,

$$u(x,y,t) = a_0 + \frac{6l}{\beta}\left(\frac{-\lambda}{2} + \frac{\sqrt{c/l^3}}{2}\left(\frac{c_1 \sinh\left\{\frac{\sqrt{c/l^3}}{2}\zeta\right\} + c_2 \cosh\left\{\frac{\sqrt{c/l^3}}{2}\zeta\right\}}{c_1 \cosh\left\{\frac{\sqrt{c/l^3}}{2}\zeta\right\} + c_2 \sinh\left\{\frac{\sqrt{c/l^3}}{2}\zeta\right\}}\right)\right). \tag{22}$$

**Case II**: $\lambda^2 - 4\mu < 0$, the trigonometric function solutions,

$$u(x,y,t) = a_0 + \frac{6l}{\beta}\left(\frac{-\lambda}{2} + \frac{\sqrt{-c/l^3}}{2}\left(\frac{-c_1 \sin\left\{\frac{\sqrt{-c/l^3}}{2}\zeta\right\} + c_2 \cos\left\{\frac{\sqrt{-c/l^3}}{2}\zeta\right\}}{c_1 \cos\left\{\frac{\sqrt{-c/l^3}}{2}\zeta\right\} + c_2 \sin\left\{\frac{\sqrt{-c/l^3}}{2}\zeta\right\}}\right)\right). \tag{23}$$

**Case III**: $\lambda^2 - 4\mu = c = 0$, the rational function solutions,

$$u(x,y,t) = a_0 + \frac{6l}{\beta}\left(\frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\zeta}\right). \tag{24}$$

The hyperbolic function solutions, equation (22), of the space-time fractional EMC equation forms a kink wave with $a_0 = 0, \beta = 1, \lambda = 1, c = 9, k = 1, l = 1, t = 1, \alpha = 0.5,$ $c_1 = 1, c_2 = 1, 50 \le x \le 70$ and $50 \le y \le 70$ as shown in Figure 1. Setting $a_0 = 0, \beta = 1, \lambda = 1,\ c = -9, k = 1, l = 1, t = 1, \alpha = 0.5, c_1 = 1, c_2 = 1, 50 \le x \le 70$ and $50 \le y \le 70$ in equation (23), the periodic wave is displayed in Figure 2. Next, Figure 3 shows the kink wave of equation (24) with parameters $a_0 = 0, \beta = 1, \lambda = 1, k = 1, l = 1, t = 1, \alpha = 0.5, c_1 = 1, c_2 = 1,$ $50 \le x \le 70$ and $50 \le y \le 70$.

Next, another application of $G'/G$-expansion method is the search for a solution of space-time fractional AKNS water equation. This procedure is described in Section 3.

## 3. Space-Time Fractional AKNS Water Equation

Consider the following nonlinear space-time fractional AKNS water equation (Helal, Seadawy, & Zekry, 2013).

$$4D_x^\alpha D_t^\alpha u + D_x^{3\alpha} D_t^\alpha u + 8D_x^\alpha u D_x^\alpha D_y^\alpha u + 4D_x^{2\alpha} u D_y^\alpha u - \gamma D_x^{2\alpha} u = 0, t > 0, 0 < \alpha \le 1, \tag{25}$$

where $u = u(x,y,t),$ $\gamma$ is constant and $\alpha$ is a parameter explaining the order of the fractional space-time derivative. Apply the equation (6) into equation (25). We get the ODE,

$$-(4c + \gamma k)\frac{d^2U}{d\zeta^2} - ck^2\frac{d^4U}{d\zeta^4} + 12kl\frac{dU}{d\zeta}\cdot\frac{d^2U}{d\zeta^2} = 0. \tag{26}$$

Integrating equation (26) and setting constant to zero, we obtain

$$-(4c + \gamma k)\frac{dU}{d\zeta} - ck^2\frac{d^3U}{d\zeta^3} + 6kl\left(\frac{dU}{d\zeta}\right)^2 = 0. \tag{27}$$

Similar to the previous section, balancing the equation (26), we have $N = 1.$ Therefore, equation (10) modifies

$$U(\zeta) = a_0 + a_1\left(\frac{G'(\zeta)}{G(\zeta)}\right). \tag{28}$$

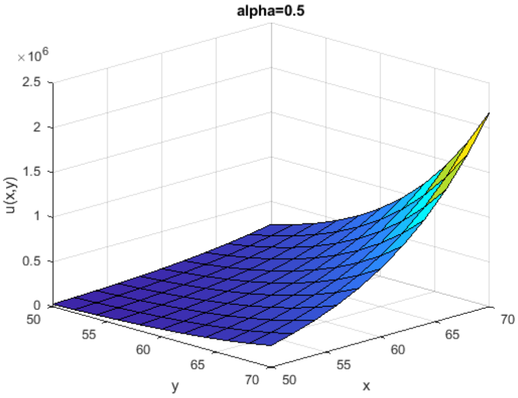


Figure 1. The hyperbolic function solution of the space-time fractional EMC equation forms the kink wave.

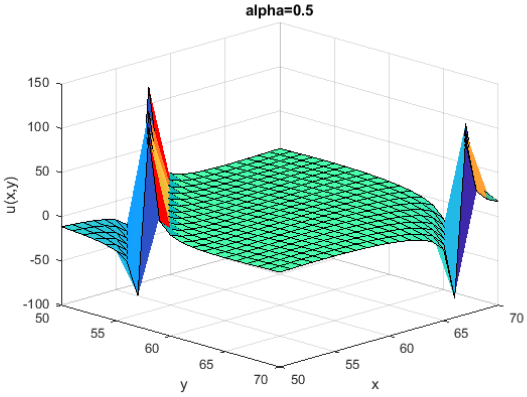


Figure 2. The trigonometric function solution of the space-time fractional EMC equation shows the periodic wave.

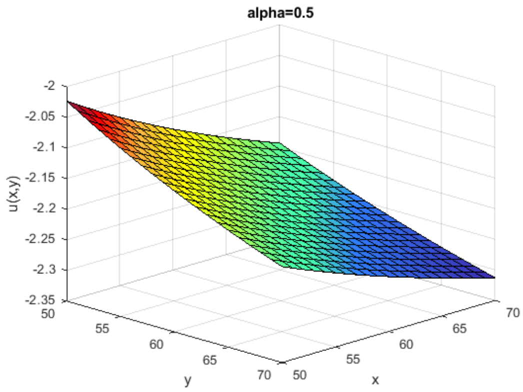


Figure 3. The rational function solution of the space-time fractional EMC equation presents the kink wave.

Replace $U(\zeta)$ in equation (27) using equation (28). Collect all terms which have the same power of $\frac{G'(\zeta)}{G(\zeta)}$. Setting each coefficient of them to zero, we gain

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^0: \qquad 4ca_1\mu+\gamma ka_1\mu+2ck^2a_1\mu^2+ck^2a_1\mu\lambda^2+6kla_1^2\mu^2=0, \tag{29}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^1: \qquad 4ca_1\lambda+\gamma ka_1\lambda+ck^2a_1\lambda^3+8ck^2a_1\mu\lambda+12kla_1^2\mu\lambda=0, \tag{30}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^2: \qquad 4ca_1+\gamma ka_1+8ck^2a_1\mu+7ck^2a_1\lambda^2+6kla_1^2\lambda^2+12kla_1^2\mu=0, \tag{31}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^3: \qquad 12ck^2a_1\lambda+12kla_1^2\lambda=0, \tag{32}$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^4: \qquad 6ck^2a_1+6kla_1^2=0. \tag{33}$$

Solving the system of equations (29)-(33), we get

$$a_1 = -\frac{ck}{l}, \lambda^2 - 4\mu = \frac{4c+\gamma k}{ck^2}. \tag{34}$$

By equations (12)-(15) and equation (34), the following exact solutions of the space-time fractional AKNS water equation are represented with arbitrary constants $c_1$ and $c_2$ and $\zeta$ :

**Case I**: $\lambda^2 - 4\mu > 0$, the hyperbolic function solutions,

$$u(x,y,t) = a_0 - \frac{ck}{l}\left(\frac{-\lambda}{2} + \sqrt{\frac{4c+\gamma k}{4ck^2}}\left(\frac{c_1 \sinh\left\{\frac{\zeta}{2}\sqrt{\frac{4c+\gamma k}{ck^2}}\right\} + c_2 \cosh\left\{\frac{\zeta}{2}\sqrt{\frac{4c+\gamma k}{ck^2}}\right\}}{c_1 \cosh\left\{\frac{\zeta}{2}\sqrt{\frac{4c+\gamma k}{ck^2}}\right\} + c_2 \sinh\left\{\frac{\zeta}{2}\sqrt{\frac{4c+\gamma k}{ck^2}}\right\}}\right)\right). \tag{35}$$

**Case II**: $\lambda^2 - 4\mu < 0$, the trigonometric function solutions,

$$u(x,y,t) = a_0 - \frac{ck}{l}\left(\frac{-\lambda}{2} + \sqrt{-\frac{4c+\gamma k}{4ck^2}}\left(\frac{-c_1 \sin\left\{\frac{\zeta}{2}\sqrt{-\frac{4c+\gamma k}{ck^2}}\right\} + c_2 \cos\left\{\frac{\zeta}{2}\sqrt{-\frac{4c+\gamma k}{ck^2}}\right\}}{c_1 \cos\left\{\frac{\zeta}{2}\sqrt{-\frac{4c+\gamma k}{ck^2}}\right\} + c_2 \sin\left\{\frac{\zeta}{2}\sqrt{-\frac{4c+\gamma k}{ck^2}}\right\}}\right)\right). \tag{36}$$

**Case III**: $\lambda^2 - 4\mu = \frac{4c+\gamma k}{ck^2} = 0$, the rational function solutions,

$$u(x,y,t) = a_0 - \frac{ck}{l}\left(\frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\zeta}\right). \tag{37}$$

Equation (35) gives a kink wave when setting $a_0 = 0, \gamma = 1, \lambda = 1, c = 9, k = 1, l = 1, t = 1, \alpha = 0.5, c_1 = 1,$ $c_2 = 1, 50 \le x \le 60$ and $50 \le y \le 60,$ as shown in Figure 4. Using $a_0 = 0, \gamma = 1, \lambda = 1, c = 1, k = -5, l = 1, t = 1,$ $\alpha = 0.5, c_1 = 1, c_2 = 1, 50 \le x \le 70$ and $50 \le y \le 70$ in equation (36), the resulting periodic wave is presented in Figure 5. Next, we set $a_0 = 0, \beta = 1, \lambda = 1, c = 9, k = 1, l = 1, t = 1, \alpha = 0.5, c_1 = 1, c_2 = 1, 20 \le x \le 40$ and $20 \le y \le 40$ in equation (37), the rational function solution is a kink wave as demonstrated in Figure 6.

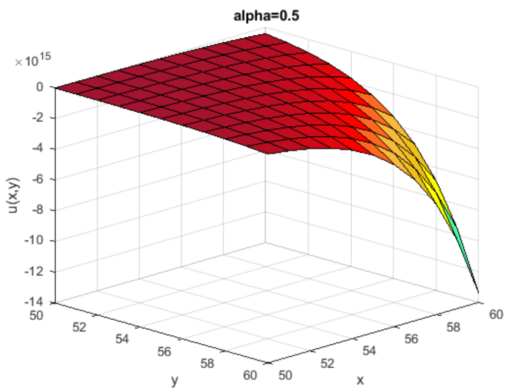


Figure 4. The hyperbolic function solution of the space-time fractional AKNS water equation demonstrates the kink wave.

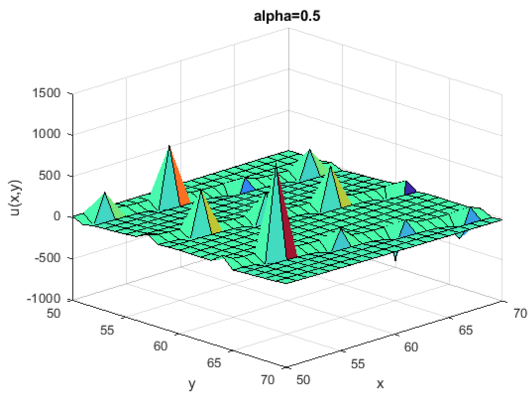


Figure 5. The trigonometric function solution of the space-time fractional AKNS water equation indicates the periodic wave.

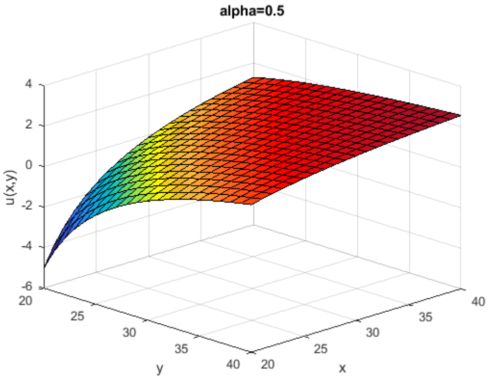


Figure 6. The rational function solution of the space-time fractional AKNS water equation represents the kink wave.

## 4. Conclusions

A combination of the $G'/G$-expansion method and the transformation of the Jumarie's Riemann-Liouville derivatives is a powerful method which gives the exact traveling wave solutions of the space-time fractional Estevez-Mansfield-Clarkson (EMC) equation and the space-time fractional Ablowitz-Kaup-Newell-Segur (AKNS) water equation. The solutions of EMC and AKNS equations can be expressed in the form of hyperbolic functions, trigonometric functions and rational functions.

## References


Alzaidy, J. F. (2013). Fractional sub-equation method and its applications to the space-time fractional differential equations in mathematical physics. *British Journal of Mathematics and Computer Science*, *3*(2), 153–163.

Bekir, A., & Güner, Ö. (2013). Exact solutions of nonlinear fractional differential equations by $G'/G$-expansion method. *Chinese Physics B*, *22*(11), 110202-1–110202-6.

Bekir, A., & Güner, Ö. (2017). The $G'/G$-expansion method using modified Riemann-Liouville derivative for some space-time fractional differential equations. *Ain Shams Engineering Journal*, *1798*(1), 959–965.

Bin, Z. (2012). $G'/G$-expansion method for solving fractional partial differential equations in the theory of mathematical physics. *Communications in Theoretical Physics*, *58*(5), 623–630.

Demiray, S. T., Pandir, Y., & Bulut, H. (2014). Generalized Kudryashov method for time-fractional differential equations. *Abstract and Applied Analysis*, *2014*, 1–13.

Ege, S. M., & Misirli, E. (2014). The modified Kudryashov method for solving some fractional-order nonlinear equations. *Advances in Difference Equations*, *2014*(1), 135-1–135-13.

Ege, S. M., & Misirli, E. (2018). Extended Kudryashov method for fractional nonlinear differential equations. *Mathematical Sciences and Applications E-Notes*, *6*(1), 19–28.

Eslami, M., Vajargah, B. F., Mirzazadeh, M., & Biswas, A. (2014). Application of first integral method to fractional partial differential equations. *Indian Journal of Physics*, *88*(2), 177–184.

Gaber, A. A., Aljohani, A. F., Ebaid, A., & Machado, J. T. (2019). The generalized Kudryashov method for nonlinear space-time fractional partial differential equations of Burgers type. *Nonlinear Dynamics*, *95*(1), 361–368.

Helal, M. A., Seadawy, A. R., & Zekry, M. H. (2013). Stability analysis solutions for the fourth-0rder nonlinear ablowitz-kaup-newell-segur water wave equation. *Applied Mathematical Sciences*, *7*(68), 3355–3365.

Ilie, M., Biazar, J., & Ayati, Z. (2018). The first integral method for solving some conformable fractional differential equations. *Optical and Quantum Electronics*, *50*(2), 55-1–55-11.

Jumarie, G. (2006). Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results. *Computers and Mathe matics with Applications*, *51*(9-10), 1367–1376.

Jumarie, G. (2009). Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. *Applied Mathematics Letters*, *22*(3), 378–385.

Kumar, D., Seadawy, A. R., & Joardar, A. K. (2018). Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology. *Chinese Journal of Physics*, *56*(1), 75–85.

Lu, B. (2012). The first integral method for some time fractional differential equations. *Journal of Mathematical Analysis and Applications*, *395*(2), 684–693.

Mansfield, E.L., & Clarkson, P.A. (1997). Symmetries and exact solutions for a 2+1-dimensional shallow water wave equation. *Mathematics and Computers in Simulation*, *43*(1), 39–55.

Mohyud-Din, S. T., Nawaz, T., Azhar, E., & Akbar, M. A. (2017). Fractional sub-equation method to space-time fractional Calogero-Degasperis and potential Kadomtsev-Petviashvili equations. *Journal of Taibah University for Science*, *11*(2), 258–263.

Zhang, S., & Zhang, H. (2011). Fractional sub-equation method and its applications to nonlinear fractional PDEs. *Physics Letters A*, *375*(7), 1069–1073.