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Original Article

The length-biased power Garima distribution and its application to model lifetime data

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Abstract

In this paper, a new two-parameter distribution namely the length-biased power Garima (LBPG) distribution is proposed. It contains the length-biased Garima (LBG) distribution as a special case. Properties, such as moments, survival function, hazard rate function, and order statistics are included. The parameter estimation of the proposed distribution is illustrated by using the maximum likelihood estimation. Finally, the application study to show the flexibility of the proposed distribution in modeling six real data sets is illustrated in terms of data fitting when compared among the LBG, power Garima, and Garima distributions. The results presented to show the LBPG distribution fits better than other distributions. Applications to these practical data sets are given to demonstrate the usefulness of the proposed distribution.

Keywords: length biased power Garima distribution, length biased Garima distribution, lifetime data, hazard rate function

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1. Introduction

There numerous continuous distributions, such as exponential, gamma, Weibull, for modelling lifetime data, and their generalizations. One important lifetime distribution for modelling data from behavioral science was proposed by Shanker (2016): the Garima distribution. It can be obtained by mixing the exponential (β) and gamma $(2,\beta)$ distributions with the mixing proportion $(\beta+1)/(\beta+2)$. Let Y be a Garima random variable then the probability density function (pdf) and cumulative distribution function (cdf) of Y are given by

$$g(y)=\frac{\beta}{2+\beta}(1+\beta+\beta y)\exp(-\beta y), \tag{1}$$

$$G(y)=1-\left(1+\frac{\beta y}{2+\beta}\right)\exp(-\beta y); y>0, \beta>0. \tag{2}$$

Next, Abebe, Tesfay, Eyob, and Shanker (2019) proposed the power transformation $X=Y^{1/\lambda}$ for $\lambda>0$, where Y has the pdf with equation (1). The distribution of a random variable of X is called the power Garima (PG) distribution. The pdf and cdf of X respectively

$$g(x)=\frac{\lambda\beta}{2+\beta}(1+\beta+\beta x^{\lambda})x^{\lambda-1}\exp(-\beta x^{\lambda}), \tag{3}$$

$$G(x)=1-\frac{(2+\beta+\beta x^{\lambda})\exp(-\beta x^{\lambda})}{2+\beta}, x>0. \tag{4}$$

When $\lambda=1$ the PG distribution reduces to the Garima distribution with the parameter β.

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Weighted distributions are required when the recorded observation from an event cannot randomly sample from the actual distribution. This happens when the original observation damaged as well as an event that occurs in a non-observability manner. Due to these inappropriate situations, resulting in values are reduced, and units or events do not have the same chances of occurrences as if they follow the exact distribution. The weighted distributions are applied in various research areas related to biomedicine, reliability, ecology, etc. (Rather & Subramanian, 2018)

The weighted distribution reduces to length-biased distribution when the weight function considers only the length of the units. When the probability of selecting an individual in a population is proportional to its magnitude, it is called length biased sampling. However, when observations are selected with probability proportional to their length, the resulting distribution is called length-biased. The length-biased distribution was first introduced by Fisher (1934) to model ascertainment bias and formalized in a unifying theory by Rao (1965). Length-biased distributions have been found to be useful in probability sampling designs for forestry and other related studies (Al-Khadim & Hussein, 2014; Oluwafemi & Olalekan, 2017). Therefore, the development of distributions that could better describe certain phenomena and make them more flexible than the existing distribution is of great importance (Maxwell, Friday, Chukwudike, Runyi, & Bright, 2019). Thus, the choice of the length-biased model is also an important issue for reliable model parameter estimation (Maxwell, Oyamakin, & Th, 2018).

The concept of length-biased distribution is found in various applications in biomedical areas, such as family history and disease, survival analysis, intermediate events, and latency period of AIDS due to blood transfusion (Gupta & Akman, 1995). Much work has been done to characterize the relationships between the existing distributions and their length-biased versions. Patil and Rao (1978) expressed some basic distributions and their length-biased forms, such as the log-normal, gamma, Pareto, and beta distributions.

This article derives a new lifetime distribution to model lifetime data that is the length-biased version of the PG distribution. The theorem of the proposed distribution functions are presented. Some properties are established, such as moments, survival function, hazard rate function, and order statistics. The maximum likelihood estimation is determined to estimate the parameters of the proposed distribution. Moreover, application studies to illustrate the distribution used with different real-life data are shown. Finally, conclusions are presented.

2. The Length-Biased Power Garima Distribution

We first provide a definition of the proposed distribution which will subsequently reveal its pdf (Oluwafemi & Olalekan, 2017; Patil & Rao, 1978; Seenoi, Supapakorn, & Bodhisuwan, 2014).

Definition 1. If X has a lifetime distribution with pdf $g(x)$ and expected value, $E_g(X)<\infty,$ the pdf of length-biased distribution of X can be defined as follows:

$$f(x)=\frac{x\cdot g(x)}{E_g(X)}. \tag{5}$$

Theorem 1. Let X be a random variable that has distributed the length-biased power Garima (LBPG) distribution with the positive parameters λ and $\beta,$ it will be denoted by X ~LBPG$(\lambda,\beta).$ Then the pdf and cdf of X are given by

$$f(x)=\frac{\lambda\beta^{1+1/\lambda}(1+\beta+\beta x^{\lambda})x^{\lambda}\exp(-\beta x^{\lambda})}{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)},\, x>0, \tag{6}$$

$$F(x)=\frac{(1+\beta)\Gamma\left(1+1/\lambda,\beta x^{\lambda}\right)+\Gamma\left(2+1/\lambda,\beta x^{\lambda}\right)}{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)}, \tag{7}$$

where $\Gamma(t)=\int_0^{\infty}s^{t-1}\exp(-s)ds$ and $\Gamma(t,x)=\int_x^{\infty}s^{t-1}\exp(-s)ds$ are the gamma function and upper incomplete gamma function, respectively, for $t>0.$

Proof. By replacing the pdf and expected value of the PG distribution (Abebe *et al.*, 2019), i.e., $E_g(X)=\dfrac{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)}{\beta^{1/\lambda}(2+\beta)}$ as in equation (5), we have the pdf

$$\begin{aligned} f(x)&=\frac{\lambda\beta(1+\beta+\beta x^{\lambda})x^{\lambda-1}\exp(-\beta x^{\lambda})\cdot\beta^{1/\lambda}(2+\beta)x}{(2+\beta)\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)} \\ &=\frac{\lambda\beta^{1+1/\lambda}(1+\beta+\beta x^{\lambda})x^{\lambda}\exp(-\beta x^{\lambda})}{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)}. \end{aligned}$$

This pdf satisfies the following properties: (i) $f(x) \geq 0$ for all x, and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$, i.e.,

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\lambda\beta^{1/\lambda}\left[(1+\beta)\int_0^{\infty}(\beta x^{\lambda})\exp(-\beta x^{\lambda})dx + \int_0^{\infty}(\beta x^{\lambda})^2\exp(-\beta x^{\lambda})dx\right]}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

Let $u = \beta x^{\lambda}$ then $dx = \frac{1}{\lambda\beta^{1/\lambda}}u^{1/\lambda-1}du$, and we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{(1+\beta)\int_0^{\infty}u^{1/\lambda}\exp(-u)du + \int_0^{\infty}u^{1/\lambda+1}\exp(-u)du}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}$$
$$= \frac{(1+\beta)\Gamma(1+1/\lambda)+\Gamma(2+1/\lambda)}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)} = 1.$$

The corresponding cdf of X is

$$F(x) = \int_{-\infty}^{x} f(s)ds = \frac{\lambda\beta^{1/\lambda}\left[(1+\beta)\int_0^{x}(\beta s^{\lambda})\exp(-\beta s^{\lambda})ds + \int_0^{x}(\beta s^{\lambda})^2\exp(-\beta s^{\lambda})ds\right]}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

Let $u = \beta s^{\lambda}$ then $ds = \frac{1}{\lambda\beta^{1/\lambda}}u^{1/\lambda-1}du$, and we have

$$F(x) = \frac{(1+\beta)\int_{\beta x^{\lambda}}^{\infty}u^{1/\lambda}\exp(-u)du + \int_{\beta x^{\lambda}}^{\infty}u^{1/\lambda+1}\exp(-u)du}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}$$
$$= \frac{(1+\beta)\Gamma(1+1/\lambda,\beta x^{\lambda})+\Gamma(2+1/\lambda,\beta x^{\lambda})}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

For $t > 0$ we have the gamma function, $\Gamma(t) = \Gamma(t,x) + \gamma(t,x)$ where $\Gamma(t,x)$ is the upper incomplete gamma function and $\gamma(t,x) = \int_0^{x} s^{t-1}\exp(-s)ds$ is the lower incomplete gamma function. Then the LBPG cdf can written in the form

$$F(x) = 1 - \frac{(1+\beta)\gamma(1+1/\lambda,\beta x^{\lambda})+\gamma(2+1/\lambda,\beta x^{\lambda})}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}. \quad (8)$$

The pdf behavior of the LBPG distribution is shown in Figure 1, which has a unimodal distribution. The LBPG pdf has various shape, i.e., the left-skewed shape, right-skewed shape, and close to symmetric shape. Figure 2 shows the LBPG cdf curves as a non-decreasing function.

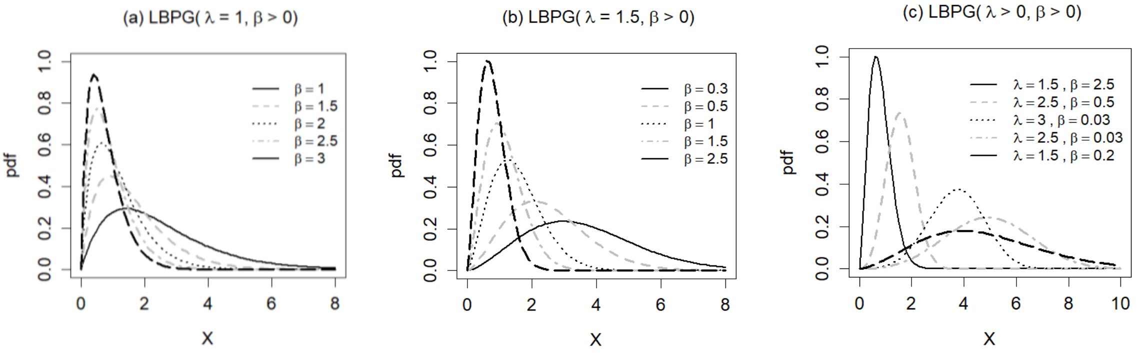

Figure 1. The pdf plots of X ~LBPG (λ, β) with specified parameters λ and β

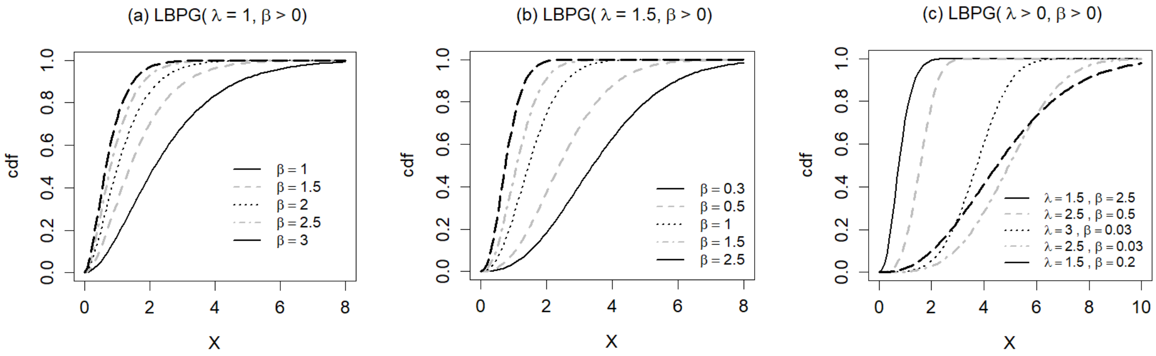

Figure 2. The cdf plots of $X \sim \text{LBPG}(\lambda, \beta)$ with specified parameters λ and β

Corollary 2.1 If $X \sim \text{LBPG}(\lambda, \beta)$ for $\lambda = 1,$ the LBPG distribution reduces to the length-biased Garima (LBG) distribution, which will be denoted by $X \sim \text{LBG}(\beta)$. Then the pdf and cdf of X are

$$f_{LBG}(x) = \frac{\beta^2 x(1+\beta+\beta x)\exp(-\beta x)}{(3+\beta)}; x > 0, \beta > 0, \tag{9}$$

$$F_{LBG}(x) = \frac{(1+\beta)\Gamma(2, \beta x) + \Gamma(3, \beta x)}{(3+\beta)}. \tag{10}$$

Proof. By replacing equations (6) and (7) with $\lambda = 1,$ we have the pdf and cdf of X as in equations (9) and (10) respectively.

3. Statistical Properties

In this section, we introduce some properties of the proposed distribution including moments, survival and hazard rate functions, and order statistics.

3.1 Moments and related measures

First, we provide explicit formulas for the r th moments of the LBPG distribution. Next, its related measures, such as mean and variance are introduced.

Proposition 1. Let $X \sim \text{LBPG}(\lambda, \beta)$ then the r th moment expression of X is

$$\mu_r' = E(X^r) = \frac{(1+\beta)\Gamma\left(1+\dfrac{r+1}{\lambda}\right) + \Gamma\left(2+\dfrac{r+1}{\lambda}\right)}{\beta^{r/\lambda}(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}, r = 1, 2, 3, ... \tag{11}$$

where parameters λ and $\beta > 0.$

Proof. The r th raw moments (or moment about origin) of the LBPG is derived as follows:

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx$$

$$= \frac{\lambda\beta^{1+1/\lambda}\left[(1+\beta)\int_0^{\infty} x^{\lambda+r}\exp(-\beta x^{\lambda})dx + \beta\int_0^{\infty} x^{2\lambda+r}\exp(-\beta x^{\lambda})dx\right]}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

Let $u = \beta x^{\lambda}$, then $dx = \dfrac{1}{\lambda\beta^{1/\lambda}} u^{1/\lambda - 1} du,$ and we have

$$\mu_r' = \frac{(1+\beta)\Gamma(1+1/\lambda+r/\lambda)+\Gamma(2+1/\lambda+r/\lambda)}{\beta^{r/\lambda}(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}$$
$$= \frac{(1+\beta)\Gamma[1+(r+1)/\lambda]+\Gamma[2+(r+1)/\lambda]}{\beta^{r/\lambda}(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

From the moments of X, we have the mean and variance of the LBPG distribution given by:

$$E(X) = \frac{(1+\beta)\Gamma(1+2/\lambda)+\Gamma(2+2/\lambda)}{\beta^{1/\lambda}(2+1/\lambda+\beta)\Gamma(1+1/\lambda)},$$

$$V(X) = \frac{1}{\beta^{2/\lambda}}\left\{\frac{(1+\beta)\Gamma(1+3/\lambda)+\Gamma(2+3/\lambda)}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)} - \frac{[(1+\beta)\Gamma(1+2/\lambda)+\Gamma(2+2/\lambda)]^2}{(2+1/\lambda+\beta)^2\Gamma^2(1+1/\lambda)}\right\}.$$

3.2 Survival and hazard rate functions

Proposition 3.2 Let X ~LBPG (λ,β) then the survival function of X can be written:

$$S(x) = \frac{(1+\beta)\gamma(1+1/\lambda,\beta x^\lambda)+\gamma(2+1/\lambda,\beta x^\lambda)}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}, x>0, \lambda>0, \beta>0. \tag{12}$$

Proof. From the pdf in equation (6), the survival function of X is given by:

$$S(x) = \int_x^\infty f(s)ds$$
$$= \frac{\lambda\beta^{1/\lambda}\left[(1+\beta)\int_x^\infty(\beta s^\lambda)\exp(-\beta s^\lambda)ds + \int_x^\infty(\beta s^\lambda)^2\exp(-\beta s^\lambda)ds\right]}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

Let $u=\beta s^\lambda$, then $ds = \frac{1}{\lambda\beta^{1/\lambda}}u^{1/\lambda-1}du$, and we have

$$S(x) = \frac{(1+\beta)\int_0^{\beta x^\lambda}u^{1/\lambda}\exp(-u)du + \int_0^{\beta x^\lambda}u^{1/\lambda+1}\exp(-u)du}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}$$
$$= \frac{(1+\beta)\gamma(1+1/\lambda,\beta x^\lambda)+\gamma(2+1/\lambda,\beta x^\lambda)}{(2+1/\lambda+\beta)\Gamma(1+1/\lambda)}.$$

Alternatively, the survival function of X can be obtained by replacing the cdf in equation (7) as in form, i.e., $S(x)=1-F(x)$.

Proposition 3. Let X be a LBPG random variable with positive parameters λ and β. Then the hazard rate function of X is

$$h(x) = \frac{\lambda\beta^{1+1/\lambda}(1+\beta+\beta x^\lambda)x^\lambda\exp(-\beta x^\lambda)}{(1+\beta)\gamma(1+1/\lambda,\beta x^\lambda)+\gamma(2+1/\lambda,\beta x^\lambda)}, x>0. \tag{13}$$

Proof. The LBPG hazard rate function is $h(x)=f(x)/S(x)$, where $f(x)$ and $S(x)$ are as in equations (6) and (12) respectively. Finally, we obtain the hazard rate function of X as in equation (13).

Figure 3, shows the LBPG survival function is a monotonically decreasing. In addition, plots of the LBPG hazard rate function are presented in Figure 4, it can be shown that; (i) for $0<\lambda\le 1$, hazard rate function is a monotonically increasing and (ii) for $\lambda>1$, hazard rate function is unimodal or reversed bathtub shapes.

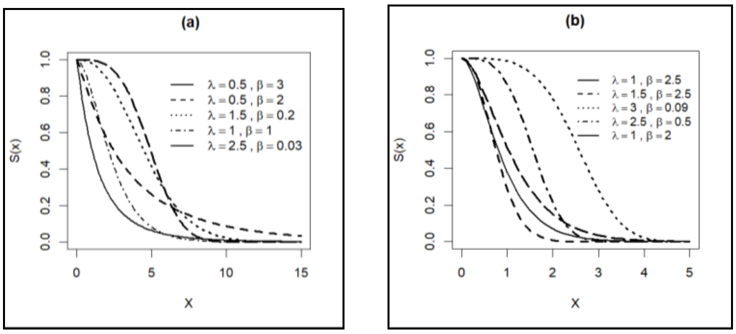

Figure 3. Plots of survival function of X ~LBPG(λ, β)

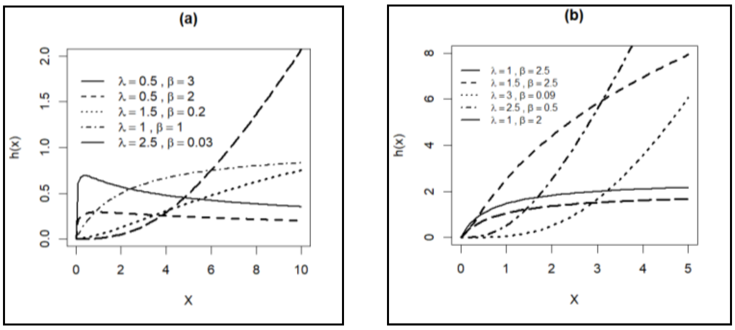

Figure 4. Plots of hazard rate function of X ~LBPG(λ, β)

3.3 Order statistics

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics of a random variable X_i, $i = 1, 2, \dots, n$ from the LBPG distribution with the pdf in equation (6) and cdf in equation (7); then the pdf of $X_{(j)}, j = 1, 2, \dots, n$ is

$$
\begin{aligned}
f_{X_{(j)}}(x) &= \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j} \\
&= \frac{(1+\beta+\beta x^{\lambda})\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)}\left[1-\frac{(1+\beta)\Gamma\left(1+1/\lambda, \beta x^{\lambda}\right)+\Gamma\left(2+1/\lambda, \beta x^{\lambda}\right)}{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)}\right]^{n-j} \\
&\times \frac{\lambda \beta^{1+1/\lambda} x^{\lambda} \exp(-\beta x^{\lambda})}{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)}\left[\frac{(1+\beta)\Gamma\left(1+1/\lambda, \beta x^{\lambda}\right)+\Gamma\left(2+1/\lambda, \beta x^{\lambda}\right)}{\left(2+1/\lambda+\beta\right)\Gamma\left(1+1/\lambda\right)}\right]^{j-1}.
\end{aligned}
$$

4. The Parameter Estimation

Estimation of the parameters of the LBPG model by the method of maximum likelihood is investigated. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be a random variable that has distributed the LBPG distribution with the pdf in equation (6). The likelihood function of X_i~LBPG (λ, β) is

$$
L(\lambda,\beta) = \frac{\lambda^{n} \beta^{n(1+1/\lambda)} \prod_{i=1}^{n}(1+\beta+\beta x_i^{\lambda}) x_i^{\lambda} \exp(-\beta x_i^{\lambda})}{\left(2+1/\lambda+\beta\right)^{n}\left[\Gamma\left(1+1/\lambda\right)\right]^{n}}
$$

The corresponding log-likelihood function is given by

$$\log \mathrm{L} = n\log(\lambda) + n(1+1/\lambda)\log(\beta) - \sum_{i=1}^{n}\beta \mathrm{x}_i^{\lambda} - n\log\left(2+1/\lambda+\beta\right)$$
$$-n\log\left[\Gamma\left(1+1/\lambda\right)\right] + \sum_{i=1}^{n}\log(1+\beta+\beta \mathrm{x}_i^{\lambda}) + \lambda\sum_{i=1}^{n}\log(\mathrm{x}_i). \quad (15)$$

The maximum likelihood estimates (MLEs) such as $\hat{\lambda}$ and $\hat{\beta}$ of parameters λ and β respectively are calculated as follows:

$$\frac{\partial \mathrm{L}}{\partial \lambda} = \frac{n}{\lambda} + n\log(\beta)\log(\lambda) - \frac{n\log\left(\lambda\right)}{2+\beta+1/\lambda} - \frac{n\Gamma'\left(1+1/\lambda\right)}{\Gamma\left(1+1/\lambda\right)}$$
$$+\sum_{i=1}^{n}\frac{\beta \mathrm{x}_i^{\lambda}}{(1+\beta+\beta \mathrm{x}_i^{\lambda})\log(\mathrm{x}_i)} + \sum_{i=1}^{n}\log(\mathrm{x}_i) = 0, \quad (16)$$

$$\frac{\partial \mathrm{L}}{\partial \beta} = \frac{n}{\beta}(1+\frac{1}{\lambda}) - \frac{n}{2+1/\lambda+\beta} + \sum_{i=1}^{n}\frac{1+\mathrm{x}_i^{\lambda}}{1+\beta+\beta \mathrm{x}_i^{\lambda}} - \sum_{i=1}^{n}\mathrm{x}_i^{\lambda} = 0. \quad (17)$$

The MLEs of $\hat{\lambda}$ and $\hat{\beta}$ can be obtained numerically from the non-linear equations (16) and (17). Because these equations are non-linear equations, we solve these equations simultaneously using a numerical procedure with the Newton-Raphson method. The **nlm** function in the **stats** package, contribution package in R (R Core Team, 2018) is used to find the MLEs.

5. Application

In this section, we demonstrate the flexibility and the potentiality of the LBPG model through six applications on practical data sets having different natures. Six real datasets are as follow; Data 1 contains 64 observations of time, in seconds, between consecutive eruptions of the geyser Kiama (Smyth, 2012), and data 2 is times (in minutes) between failures for repairable items (Murthy, Xie, & Jiang, 2003), see Souza, Junior, de Brito, Ferreira, and Soares (2019). The data set of data 3 consists of the waiting time (in minutes), which were given in Abebe *et al.* (2019). Data sets 4-5 are the breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006) and the time to failure (10^3h) of turbocharger of one type of engine given in Xu, Xie, Tang, and Ho (2003), both data sets which appeared in Handique and Chakraborty (2016). Finally, data set 6 consists of 63 observations of the strength of 1.5cm glass fibers taken from Smith and Naylor (1987), see Abebe *et et al.* (2019). Some descriptive statistics of all data sets are presented in Table 1.

All the model parameters are estimated by the maximum likelihood method, as presented in Section 4 for the proposed model. These applications will be used to determine the estimated parameters of each distribution. The value of MLEs of each distributions are obtained by using the **nlm** function in the **stats** package, contribution package in R (R Core Team, 2018). In each application, we first compare the LBPG model with other useful competitive models, such as the LBG, PG, and Garima distributions via the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), i.e., $\mathrm{AIC} = 2k - 2\log\hat{L}$ and $\mathrm{BIC} = k\log(n) - 2\log\hat{L},$ where k is the number of parameters estimated by the model, n is the sample size, and $\log\hat{L}$ is the maximum value of the log-likelihood function of the model. Moreover, a goodness of fit test of distance between the empirical distribution function $\mathrm{F}_n(\mathrm{x})$ of the sample and the cdf of the reference distribution $\mathrm{F}_0(\mathrm{x})$ are considered by using test statistics such as the Kolmogorov-Smirnov (K-S), Anderson Darling (AD), and Cramer-von Mises (W). The model gives the smallest values of AIC, BIC, K-S, AD, and W statistics; therefore it is the best model for fitting data.

The results of the MLEs, AIC, BIC, K-S, AD, and W for fitted distributions of each data sets are illustrated in Table 2. The LBPG distribution gives smaller statistics than the LBG, PG, and Garima distributions. These results show that the LBPG distribution is the best model to fit these data sets when compared to other distributions. Moreover, the estimated pdf plots in Figure 5 and the probability plot of the proposed distribution in Figure 6 indicate that the LBPG distribution provides a closer fit to these data.

Table 1. Descriptive statistics of real datasets.

Data	n	Min	Max	Median	Mean	Variance
1: Kiama Blowhole	64	7.00	169.00	28.00	39.83	1139.10
2: Times between failures	30	0.11	0.72	1.24	1.54	1.27
3: Waiting time	65	2.00	90.00	23.00	26.48	300.91
4: Breaking stress of carbon fibers	100	0.39	5.56	2.70	2.62	1.03
5: Time to failure of the turbocharger	40	1.60	9.00	6.50	6.25	3.82
6: Strength of glass fibers	63	0.55	2.24	1.59	1.51	0.11

Table 2. Values of MLEs, AIC, BIC, K-S, AD, and W of each distribution for fitting various real data sets

Data	Distribution	Estimators		$-\log \hat{L}$	AIC	BIC	KS	AD	W
		$\hat{\beta}$	$\hat{\lambda}$						
1	LBPG	0.2354	0.7141	295.46	594.92	599.24	0.1078	0.7897	0.1068
	LBG	0.0658	-	299.75	601.49	603.65	0.3459	25.1908	3.9710
	PG	0.0088	1.2581	340.99	685.98	690.30	0.2545	7.5483	1.4257
	Garima	0.0254	-	343.38	688.76	690.92	0.1995	5.1916	0.9833
2	LBPG	1.7147	0.9238	39.64	83.27	86.08	0.0659	0.1394	0.0180
	LBG	1.5783	-	39.73	81.47	82.87	0.3069	7.6507	1.3665
	PG	0.9121	1.0867	46.09	96.17	98.97	0.1630	1.0446	0.1862
	Garima	0.9737	-	46.27	94.54	95.94	0.1949	1.5513	0.2920
3	LBPG	0.1522	0.8916	271.77	547.54	551.89	0.0876	0.6712	0.0704
	LBG	0.0994	-	272.15	546.30	548.47	0.3568	17.8156	3.1117
	PG	0.0062	1.5095	315.35	634.70	639.05	0.1894	5.6942	0.9981
	Garima	0.0385	-	321.80	645.60	647.78	0.2096	4.2700	0.7611
4	LBPG	0.2534	2.0174	141.31	286.62	291.83	0.0628	0.3964	0.0662
	LBG	0.9625	-	162.19	326.38	328.99	0.3975	21.0319	4.5456
	PG	0.0620	2.6269	207.20	418.40	423.61	0.2065	7.5127	1.5174
	Garima	0.5123	-	239.02	480.05	482.65	0.2794	14.0128	2.6290
5	LBPG	0.0071	2.9294	82.86	169.72	173.10	0.1066	0.6904	0.0793
	LBG	0.4185	-	97.98	197.96	199.65	0.4866	9.5493	2.1082
	PG	0.0006	3.8657	110.19	224.38	227.76	0.1601	3.2886	0.4885
	Garima	0.1714	-	138.01	278.03	279.71	0.3658	6.6627	1.2490
6	LBPG	0.1804	4.7498	14.98	33.96	38.25	0.1461	1.1723	0.1951
	LBG	1.6254	-	63.61	129.22	131.37	0.4878	18.0592	3.9892
	PG	0.0761	5.4126	54.57	113.13	117.42	0.2228	4.4721	0.8189
	Garima	0.9965	-	94.93	191.86	194.00	0.4467	16.2278	3.3191

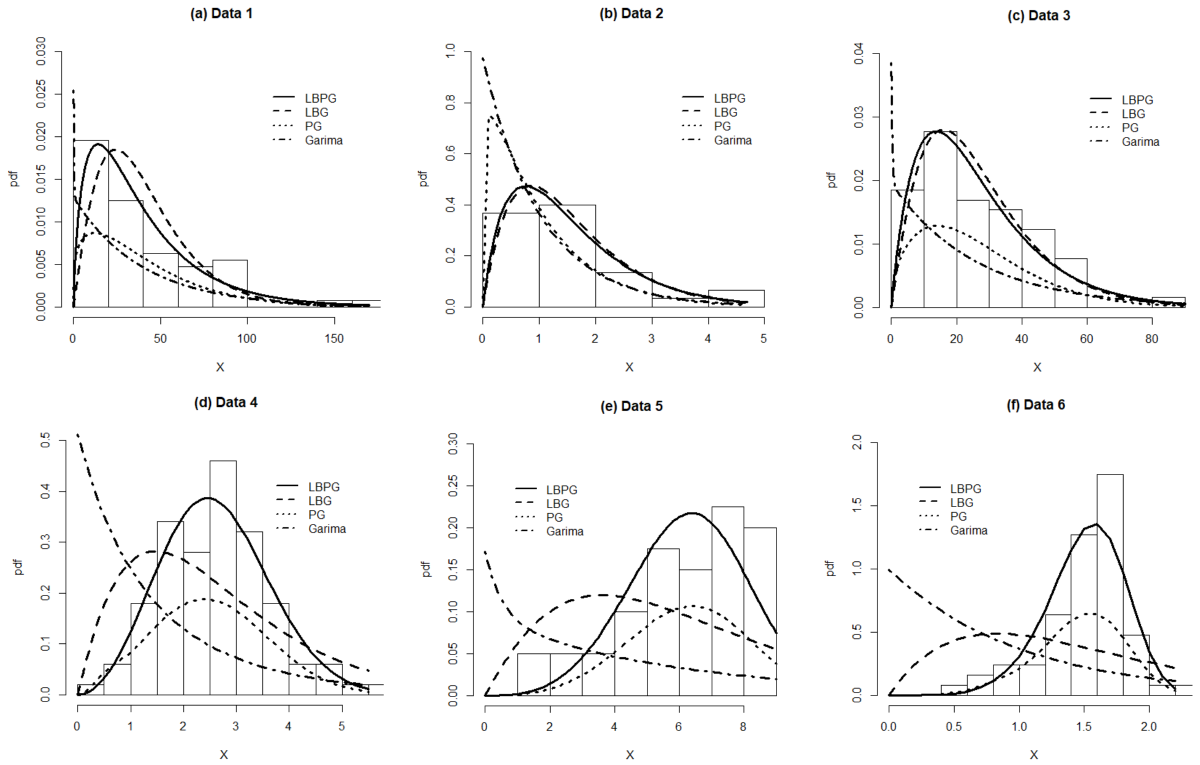

Figure 5. Plots of the observed histogram and estimated pdf of the fitted distributions for real data sets

6. Conclusions

We proposed the new lifetime distribution called the length-biased power Garima (LBPG) distribution. It has the length-biased Garima (LBG) distribution as a special case. The LBPG pdf behavior has a unimodal distribution of various shapes, i.e., the left-skewed shape, right-skewed shape, and close to symmetric shape. Statistical properties including moments, survival function, hazard rate function, and order statistics are derived. The parameters of the LBPG

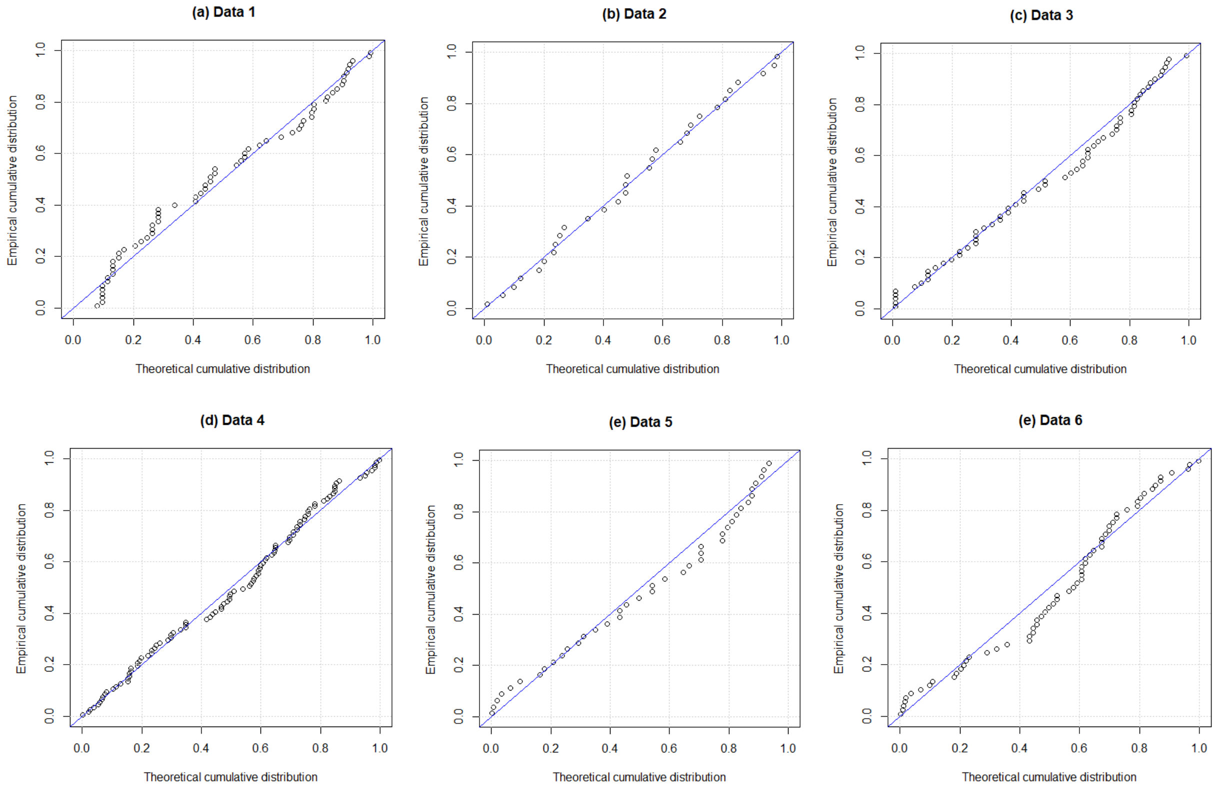

Figure 6. Probability plots of the fitted LBPG distribution for real data sets

distribution are estimated by using the maximum likelihood estimation. The result of data applications shows that the LBPG distribution provides better fits than the LBG, power Garima, and Garima distributions.

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