PROGRESS IN APPLIED SCIENCE AND TECHNOLOGY
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Vol.11 No.2 (2021): xx–xx
ISSN (Print): 2730-3012
ISSN (Online): 2730-3020
https://ph02.tci-thaijo.org/index.php/past

Received 12th May 2021,
Revised 14th June 2021,
Accepted 08th July 2021

DOI: 10.14456/x0xx00000x

The Common Fixed Point Theorems for Asymptotic Regularity on b-Metric Spaces

Areerat Arunchai[1], Methini Muangchai[1] and Kewalee Thala[1*]
[1] *Mathematics and Statistics Program, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, 398 Nakhon Sawantok sub-district Mueang district Nakhon Sawan, Thailand*
*E-mail: kewalee.t@nsru.ac.th

Abstract

In this paper, we propose the common fixed point theorems for asymptotic regularity on b-metric spaces. The results presented in the paper improve and extend some previous results.

Keywords: Common fixed point, Asymptotic regularity, b-metric space.

1. Introduction

In many branches of science, engineering, economics, computer science and the development of nonlinear analysis, the fixed point theory is one of the most important tool. Since the work of Banach (1) known as the Banach Contraction Principle, many mathematicians have extended and generalized the results in (1). The Banach Contraction Principle states that:

Theorem 1.1 (1) (The Banach Contraction Principle) If a self-mapping f of a complete metric space (X, d) satisfies the condition;

$$d(fx, fy) \leq Md(x, y),$$

$M \in [0,1)$, for each $x, y \in X$, then f has a unique fixed point.

In recent years, several authors have obtained fixed point theorem and common fixed point theorem of mappings in the setting of many generalized metric spaces. In 1966, Browder and Petryshyn (2) introduced the notion of asymptotic regularity. For example of asymptotic regularity and its relevance to metric fixed point theorem can be seen in (3-5).

In 2006, Proinov (6) was initiated study of fixed point of the class of mappings (1.1).

Theorem 1.2 (6) Let T be a continuous and asymptotically regular self-mapping on a complete metric space (X, d) satisfying the following conditions:

(i) There exists $\varphi \in \phi$ such that
$$d(Tx, Ty) \leq \varphi\big(D(x, y)\big)$$
for all $x, y \in X$;

(ii) $d(Tx, Ty) < \big(D(x, y)\big)$ for all $x, y \in X$ with $x = y$.

Then T has a contractive fixed point. Moreover, if $D(x, y) = d(x, y) + d(x, Tx) + d(y, Ty)$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then the continuity of T can be dropped.

In 2019, Górnicki (7) proved the fixed point theorem for a complete metric space and f is a continuous asymptotically regular mapping and if there exists $M \in [0,1)$ and $K \in [0, \infty)$ satisfying

$$d(fx, fy) \leq Md(x, y) + K\{d(x, fx) + d(y, fy)\}, \tag{1.1}$$

for all x, y . Then f has a unique fixed point.

In 2019, Bisht and Singh (8) obtain the existence of common fixed point theorems for mappings satisfying Lipschitz–Kannan type condition.

Theorem 1.3 (8) If (X, d) is a complete metric space and $f, g: X \to X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0,1)$ and $K \in [0, \infty)$ satisfying

$$d(fx, fy) \leq Md(gx, gy) + K\{d(fx, gx) + d(fy, gy)\}, \tag{1.2}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

The idea of b-metric was initiated from the works of Bourbaki (9) and Bakhtin (10).

In 1993, Czerwik (11) gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem.

Theorem 1.4 (11) Let (X, d) be e a complete b-metric space and let $T : X \to X$ satisfy

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big), x, y \in X$$

where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is an increasing function such that

$$\lim_{n\to\infty} \varphi^n(t) = 0,$$

for each $t > 0$. Then T has exactly one fixed point u and

$$\lim_{n\to\infty} d(T^n x, u) = 0,$$

for each $x \in X$.

In this paper, we establish new common fixed points theorems for asymptotic regularity on b-metric spaces. Our work extends and improves some important results due to many others.

2. Preliminaries

In this section we introduce the following definitions for this paper.

Definition 2.1 (12) Let X be a non-empty set and $s \geq 1$ be a given real number. A function $b: X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties for each $x, y, z \in X$:

(b1): $b(x, y) = 0 \Leftrightarrow x = y$;

(b2): $b(x, y) = b(y, x)$;

(b3): $b(x, z) \leq s[b(x, y) + b(y, z)]$.

The pair (X, b) is called a b-metric space.

Example 2.2 Let $X = \{2,3,4\}$ and a function $b: X \times X \to \mathbb{R}^+$ be defined by

$b(2,2) = b(3,3) = b(4,4) = 0$,

$b(2,3) = b(3,2) = b(3,4) = b(4,3) = 1$,

$b(2,4) = b(4,2) = m$,

where s is a given real number such that $m \geq 2$.

It is easy to see

$$b(x, z) \leq \frac{m}{2}[b(x, y) + b(y, z)]$$

for all $x, y, z \in X$. Therefore, (X, b) is a b-metric space with constant $s = \frac{m}{2}$. However, if $m > 2$, the ordinary triangle inequality does not hold and thus (X, b) is not a metric space.

Definition 2.3 (12) Let (X, b) be a b-metric space. A sequence $\{x_n\}$ in X is said to be:

(i) Cauchy if and only if $b(x_n, x_m) \to 0$ $as\ n, m \to \infty$;

(ii) convergent if and only if there exist $x \in X$ such that $b(x_n, x) \to 0$ $as\ n \to \infty$; and we write $\lim_{n\to\infty} x_n = x$;

(iii) the b-metric space (X, b) is complete if every Cauchy sequence is convergent.

Definition 2.4 (13) Let f and g be self-maps of a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

The set of coincidence points and point of coincidences of f and g are denoted by $C(f, g)$ and $PC(f, g)$, respectively. If $w = x$ then x is a common fixed point of f and g and the set of common fixed points is denoted by $F(f, g)$.

Definition 2.5 Let f and g be two self-mappings of a b-metric space (X, b). Then

(i) f is asymptotically regular with respect to g at $x_0 \in X$, if there exists a sequence $\{x_n\}$ in X such that $gx_{n+1} = fx_n$, for$n = 0,1,2,...$, and $\lim_{n\to\infty} b(gx_{n+1}, gx_{n+2}) = 0$.

(ii) f and g are called compatible iff $\lim_{n\to\infty} b(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some t in X.

Definition 2.6 Let f and g be two self-mappings of a b-metric space (X, b) and let $\{x_n\}$ be a sequence in X such that $fx_n = gx_{n+1}$. Then the set

$$O(x_0, f, g) = \{fx_n : n = 0,1,2, ...\}$$

is called the (f, g)-orbit at x_0 and

g is called (f, g)-orbitally continuous if $\lim_{n\to\infty} fx_n = z$ implies $\lim_{n\to\infty} gfx_n = gz$ or

f is called (f, g)-orbitally continuous if $\lim_{n\to\infty} fx_n = z$ implies $\lim_{n\to\infty} ffx_n = fz$.

We say f and g are orbitally continuous if f is (f, g)-orbitally continuous and g is (f, g)-orbitally continuous.

3. Main results

We start this section by the following result:

Theorem 3.1 If (X, b) is a complete b-metric space and $f, g: X \to X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0,1)$ and $K \in [0, \infty)$ satisfying

$$b(fx, fy) \leq Mb(gx, gy) + K\{b(fx, gx) + b(fy, gy)\}, \tag{3.1}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in X$, there exists a sequence $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n\to\infty} b(gx_{n+1}, gx_{n+2}) = \lim_{n\to\infty} b(y_n, y_{n+1}) = 0.$$

We show that $\{y_n\}$ is a Cauchy sequence. Since (3.1), for any n and any $p > 0$,

$$\begin{aligned} b(fx_{n+p}, fx_n) &= b(y_{n+p}, y_n) \\ &\le s[b(y_{n+p}, y_{n+p+1}) + b(y_{n+p+1}, y_n)] \\ &\le sb(y_{n+p}, y_{n+p+1}) + sb(y_{n+p+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) + s^2[b(y_{n+p+1}, y_{n+1}) + b(y_{n+1}, y_n)] \\ &\le sb(y_{n+p}, y_{n+p+1}) + s^2 b(y_{n+p+1}, y_{n+1}) + s^2 b(y_{n+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) + s^2[Mb(y_{n+p}, y_n) + K\{b(y_{n+p+1}, y_{n+p}) + b(y_{n+1}, y_n)\}] + s^2 b(y_{n+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) + s^2 Mb(y_{n+p}, y_n) + s^2 Kb(y_{n+p+1}, y_{n+p}) + s^2 Kb(y_{n+1}, y_n) + s^2 b(y_{n+1}, y_n). \end{aligned}$$

So,

$$\begin{aligned} & b(y_{n+p}, y_n) - s^2 Mb(y_{n+p}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) + s^2 Kb(y_{n+p+1}, y_{n+p}) + s^2 Kb(y_{n+1}, y_n) + s^2 b(y_{n+1}, y_n), \end{aligned}$$

and then

$$\begin{aligned} & b(y_{n+p}, y_n)(1 - s^2 M) \\ &\le b(y_{n+p}, y_{n+p+1})(s + s^2 K) + b(y_{n+1}, y_n)(s^2 + s^2 K). \end{aligned}$$

Hence,

$$b(y_{n+p}, y_n) \le \frac{(s+s^2K)}{(1-s^2M)}\, b(y_{n+p}, y_{n+p+1}) + \frac{(s^2+s^2K)}{(1-s^2M)}\, b(y_{n+1}, y_n) \tag{3.2}$$

By f is asymptotic regularity with respect to g, we have $b(y_{n+p}, y_n) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. By completeness of X there exists $t \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

Since orbital continuity of f and g implies that

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = ft,$$

and

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt.$$

Since compatibility of f and g, $\lim_{n\to\infty} b(fgx_n, gfx_n) = 0$. Letting $n \to \infty$, we get

$$ft = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ggx_n = gt.$$

Hence $C(f,g) \ne \emptyset$.
By compatibility of f and g, we have

$$gft = fgt = fft = ggt.$$

Using (3.1), we obtain

$$\begin{aligned} b(ft, fft) &\le Mb(gt, gft) + K\{b(ft, gt) + b(fft, gft)\} \\ &= Mb(gt, gft) + K\{b(ft, ft) + b(fft, fft)\} \\ &= Mb(gt, gft) \\ &= Mb(ft, fft). \end{aligned}$$

So,

$$b(ft, fft) - Mb(ft, fft) \le 0,$$

and then

$$(1 - M) b(ft, fft) \le 0.$$

Therefore,

$$b(ft, fft) = 0.$$

Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Suppose that st and ft are common fixed point of f and g implies that $fst = gst = st$ and $fft = gft = ft$. We show that $ft = st$. Using (3.1), we obtain

$$\begin{aligned} b(ft, st) &= b(fft, fst) \\ &\le Mb(gft, gst) + K\{b(fft, gft) + b(fst, gst)\} \\ &= Mb(gft, gst) \\ &= Mb(ft, st). \end{aligned}$$

So,

$$b(ft, st) - Mb(ft, st) \le 0.$$

Hence,

$$0 \le b(ft, st) \le 0.$$

Therefore,

$$b(ft, st) = 0.$$

Hence $ft = st$ and f and g have a unique common fixed point.

Corollary 3.2 If (X, b) is a complete b-metric space and $f, g: X \to X$. Suppose that f is asymptotically regular with respect to g and there exist $K \in [0, \infty)$ satisfying

$$b(fx, fy) \le K\{b(fx, gx) + b(fy, gy)\}, \tag{3.3}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \ne \emptyset$ and f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in X$, there exists a sequence $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n\to\infty} b(gx_{n+1}, gx_{n+2}) = \lim_{n\to\infty} b(y_n, y_{n+1}) = 0.$$

We show that $\{y_n\}$ is a Cauchy sequence. Since (3.3), for any n and any $p > 0$,

$$\begin{aligned} b(fx_{n+p}, fx_n) &= b(y_{n+p}, y_n) \\ &\le s[b(y_{n+p}, y_{n+p+1}) \\ &\quad + b(y_{n+p+1}, y_n)] \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + sb(y_{n+p+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2[b(y_{n+p+1}, y_{n+1}) \\ &\quad + b(y_{n+1}, y_n)] \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2 b(y_{n+p+1}, y_{n+1}) \\ &\quad + s^2 b(y_{n+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2[K\{ b(y_{n+p+1}, y_{n+p}) \\ &\quad + b(y_{n+1}, y_n)\}] \\ &\quad + s^2 b(y_{n+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2 K b(y_{n+p+1}, y_{n+p}) \\ &\quad + s^2 K b(y_{n+1}, y_n) + s^2 b(y_{n+1}, y_n) \\ &\le b(y_{n+p}, y_{n+p+1})(s + s^2 K) \\ &\quad + b(y_{n+1}, y_n)(s^2 K + s^2). \end{aligned}$$

By f is asymptotic regularity with respect to g, we have $b(y_{n+p}, y_n) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. By completeness of X there exists $t \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

Since orbital continuity of f and g implies that

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = ft,$$

and

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt.$$

Since compatibility of f and g, $\lim_{n\to\infty} b(fgx_n, gfx_n) = 0$. Letting $n \to \infty$, we get

$$ft = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ggx_n = gt.$$

Hence $C(f,g) \ne \emptyset$.
By compatibility of f and g implies

$$gft = fgt = fft = ggt.$$

Using (3.3), we have

$$b(ft, fft) \le K\{b(ft, gt) + b(fft, gft)\} = 0.$$

So,

$$0 \le b(ft, fft) \le 0.$$

Thus,

$$b(ft, fft) = 0.$$

Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Suppose that st and ft are common fixed point of f and g implies that $fst = gst = st$ and $fft = gft = ft$. We show that $ft = st$. Using (3.3), we obtain

$$\begin{aligned} b(ft, st) &= b(fft, fst) \\ &\le K\{b(fft, gft) + b(fst, gst)\} \\ &= 0. \end{aligned}$$

Then

$$b(ft, st) = 0.$$

Hence $ft = st$ and f and g have a unique common fixed point.

In the next theorem, we drop orbital continuity of a pair of mappings considered in Theorem 3.1 besides relaxing compatibility by the minimal non-commuting notion, i.e., non-trivially weak compatibility. We recall that f and g are weakly compatible (15) if $fgx = gfx$, whenever $fx = gx$ for some $x \in X$.

Theorem 3.3 If (X, b) is a b-metric space and f and g be self-mappings on an arbitrary non-empty set Y with values in a b-metric space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying

$$\begin{aligned} b(fx, fy) &\le Mb(gx, gy) \\ &\quad + K\{b(fx, gx) + b(fy, gy)\}, \end{aligned}$$

for all $x, y \in Y$. Then $C(f,g) \ne \emptyset$. Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in X$, there exists a sequence $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n\to\infty} b(gx_{n+1}, gx_{n+2}) = \lim_{n\to\infty} b(y_n, y_{n+1}) = 0.$$

We show that $\{y_n\}$ is a Cauchy sequence. Since (3.1), for any n and any $p > 0$,

$$\begin{aligned} b(fx_{n+p}, fx_n) &= b(y_{n+p}, y_n) \\ &\le s[b(y_{n+p}, y_{n+p+1}) \\ &\quad + b(y_{n+p+1}, y_n)] \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + sb(y_{n+p+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2[b(y_{n+p+1}, y_{n+1}) \\ &\quad + b(y_{n+1}, y_n)] \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2 b(y_{n+p+1}, y_{n+1}) \\ &\quad + s^2 b(y_{n+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2[Mb(y_{n+p}, y_n) \\ &\quad + K\{b(y_{n+p+1}, y_{n+p}) \\ &\quad + b(y_{n+1}, y_n)\}] + s^2 b(y_{n+1}, y_n) \\ &\le sb(y_{n+p}, y_{n+p+1}) \\ &\quad + s^2 M b(y_{n+p}, y_n) \\ &\quad + s^2 K b(y_{n+p+1}, y_{n+p}) \\ &\quad + s^2 K b(y_{n+1}, y_n) \\ &\quad + s^2 b(y_{n+1}, y_n). \end{aligned}$$

So,

$$\begin{aligned} & b(y_{n+p}, y_n) - s^2 M b(y_{n+p}, y_n) \\ & \le sb(y_{n+p}, y_{n+p+1}) \\ & \quad + s^2 K b(y_{n+p+1}, y_{n+p}) \\ & \quad + s^2 K b(y_{n+1}, y_n) \\ & \quad + s^2 b(y_{n+1}, y_n), \end{aligned}$$

and then

$$b(y_{n+p}, y_n)(1 - s^2M) \leq b(y_{n+p}, y_{n+p+1})(s + s^2K) + b(y_{n+1}, y_n)(s^2 + s^2K).$$

Hence,

$$b(y_{n+p}, y_n) \leq \frac{(s + s^2K)}{(1 - s^2M)} b(y_{n+p}, y_{n+p+1}) + \frac{(s + s^2K)}{(1 - s^2M)} b(y_{n+1}, y_n).$$

Since asymptotic regularity of f with respect to g, $b(y_{n+p}, y_n) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence in gY. Since gY is a complete subset of X and $y_n = fx_n = gx_{n+1}$ is a Cauchy sequence in gY. Then there exists $t \in X$ such that

$$\lim_{n\to\infty} gx_{n+1} = gt.$$

Moreover,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = t.$$

By (3.1), we obtain

$$\begin{aligned} b(fx_n, ft) &\leq Mb(gx_n, gt) \\ &\quad + K\{b(fx_n, gx_n) + b(ft, gt)\} \\ &= Kb(ft, gt). \end{aligned}$$

So,

$$b(fx_n, ft) = b(gt, ft) \leq Kb(ft, gt).$$

Thus,

$$b(gt, ft) - Kb(ft, gt) \leq 0.$$

Therefore,

$$b(ft, gt) = 0.$$

Hence $C(f, g) \neq \emptyset$.
Since $Y = X$ and f and g non-trivially weakly compatible, then $gft = fgt$. This further implies that

$$fgt = gft = fft = ggt.$$

Using (3.1), we obtain

$$\begin{aligned} b(ft, fft) &\leq Mb(gt, gft) \\ &\quad + K\{b(ft, gt) + b(fft, gft)\} \\ &= Mb(gt, gft) \\ &\quad + K\{b(ft, ft) + b(fft, fft)\} \\ &= Mb(gt, gft) \\ &= Mb(ft, fft). \end{aligned}$$

So,

$$b(ft, fft) - Mb(ft, fft) \leq 0.$$

Then

$$(1 - M)b(ft, fft) \leq 0.$$

Therefore,

$$b(ft, fft) = 0.$$

Hence $ft = fft = gft$ and ft is a common fixed point of f and g.

Suppose that st and ft are common fixed point of f and g implies that $fst = gst = st$ and $fft = gft = ft$. We show that $ft = st$. Using (3.1), we obtain

$$\begin{aligned} b(ft, st) &= b(fft, fst) \\ &\leq Mb(gft, gst) \\ &\quad + K\{b(fft, gft) + b(fst, gst)\} \\ &= Mb(gft, gst) \\ &= Mb(ft, st). \end{aligned}$$

So,

$$b(ft, st) - Mb(ft, st) \leq 0,$$

and then

$$(1 - M)b(ft, st) \leq 0.$$

Hence,

$$0 \leq b(ft, st) \leq 0.$$

Therefore,

$$b(ft, st) = 0.$$

Hence $ft = st$ and f and g have a unique common fixed point.

Remark. Let $K = 0$ in Theorem 3.3, we obtain

$$\begin{aligned} b(fx, fy) &\leq Mb(gx, gy) \\ &\quad + K\{b(fx, gx) + b(fy, gy)\} \\ &= Mb(gx, gy) \\ &\leq Mmax\{ b(gx, gy) \\ &\quad , b(fx, gx), b(fy, gy) \\ &\quad , \frac{b(fx, gy) + b(fy, gx)}{2}\}. \end{aligned}$$

Hence satisfy the condition in Corollary 3.4.

Corollary 3.4 If (X, b) is a b-metric space and f and g be self-mappings on an arbitrary non-empty set Y with values in a b-metric space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying

$$b(fx, fy) \leq Mmax\{ b(gx, gy), b(fx, gx), b(fy, gy), \frac{b(fx, gy) + b(fy, gx)}{2}\},$$

for all $x, y \in Y$. Then $C(f, g) \neq \emptyset$. Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

The following example illustrates Theorem 3.3.
Example 3.5 Let $X = [2,20]$ and a function $b: X \times X \to \mathbb{R}$ be defined by $b(x, y) = |x - y|^2$ for all $x, y \in \mathbb{R}$ is a b-metric on X. Define self-mappings f and g on X as follows

$$fx = \begin{cases} 2, & x = 2 \\ 6, & x \in (2,5] \\ 2, & x \in (5, \infty) \end{cases}$$

and

$$gx = \begin{cases} 2, & x = 2 \\ 11, & x \in (2,5] \\ \dfrac{x+1}{3}, & x \in (5, \infty). \end{cases}$$

Then f and g satisfy the following conditions of Theorem 3.3 and have a unique common fixed point $x = 2$ which f and g are discontinuous:

(i) $gY = [2,7] \cup \{11\}$ is a complete subspace of $X = [2,20]$;
(ii) f and g satisfy the condition

$$b(fx, fy) \leq \frac{4}{5}\{b(fx, gx) + b(fy, gy)\},$$

for all $x, y \in Y$ and $M \in [0,1)$;

(iii) f and g are non-trivially weakly compatible as f and g commute at their only coincidence point $x = 2$.

Declaration of conflicting interests

The authors declared that they have no conflicts of interest in the research, authorship, and this article's publication.

References

1. Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund math. 1922;3(1):133-81.
2. Browder FE, Petryshyn W. The solution by iteration of nonlinear functional equations in Banach spaces. Bulletin of the american mathematical society. 1966;72(3):571-5.
3. Bailion J, Bruck RE, Reich S. On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces. 1978.
4. Ćirić L. Some recent results in metrical fixed point theory. Belgrade: University of Belgrade. 2003.
5. Todorčević V. Harmonic quasiconformal mappings and hyperbolic type metrics: Springer; 2019.
6. Proinov PD. Fixed point theorems in metric spaces. Nonlinear Analysis: Theory, Methods & Applications. 2006;64(3):546-57.
7. Górnicki J. Remarks on asymptotic regularity and fixed points. Journal of Fixed Point Theory and Applications. 2019;21(1):1-20.
8. Bisht RK, Singh NK. On asymptotic regularity and common fixed points. The Journal of Analysis. 2020;28(3):847-52.
9. Bourbaki N. Topologie Générale Hermann. Paris1961 Zbl0249. 1965;54001.
10. Bakhtin I. The contraction mapping principle in quasimetric spaces. Func An, Gos Ped Inst Unianowsk. 1989;30:26-37.
11. Czerwik S. Contraction mappings in b-metric spaces. Acta mathematic a et informatica universitatis ostraviensis. 1993;1(1):5-11.
12. Kadak U. On the Classical Sets of Sequences with Fuzzy b-Metric. General Mathematics Notes. 2014;23(1).
13. Pariya A, Pathak P, Badshah V, Gupta N. Common fixed point theorems for generalized contraction mappings in modular metric spaces. Adv Inequal Appl. 2017;2017:Article ID 9.
14. Sastry K, Naidu S, Rao I, Rao K. Common fixed points for asymptotically regular mappings. Indian J Pure Appl Math. 1984;15(8):849-54.
15. Jungck G. Common fixed points for noncontinuous nonself maps on nonmetric spaces. Far East Journal of Mathematical Sciences. 1996, 4(2):199–212.
16. Kannan R. Some results on fixed points—II. The American Mathematical Monthly. 1969;76(4):405-8.
17. Subrahmanyam P. Completeness and fixed-points. Monatshefte für Mathematik. 1975;80(4):325-30.
18. Reich S. Some remarks concerning contraction mappings. Canadian Mathematical Bulletin. 1971;14(1):121-4.
19. Bisht RK. A note on the fixed point theorem of Górnicki. Journal of Fixed Point Theory and Applications. 2019;21(2):1-3.
20. Górnicki J. Fixed point theorems for Kannan type mappings. Journal of fixed point theory and applications. 2017;19(3):2145-52.
21. Jungck G. Commuting mappings and fixed points. The American Mathematical Monthly. 1976;83(4):261-3.