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Original Article

A generalization of UP-algebras: Weak UP-algebras

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Abstract

Many logical algebras such as BE-, KU- and BCC-algebras have their weak version as generalization. It is known that a CI-algebra is a weak version of a BE-algebra, a JU-algebra is a weak version of a KU-algebra, and a BZ-algebra is a weak version of a BCC-algebra. In this paper, we introduce the concept of weak UP-algebras as a generalization of UP-algebras, and investigate its basic properties. Additionally, some types of ideals in this type of algebras are introduced and analyzed.

Keywords: UP-algebra, weak UP-algebra, closed wUP-ideal, (★)-ideal, ag-ideal, strong ideal, regular ideal, associative ideal, t-ideal

1. Introduction

The concept of UP-algebras as a generalization of KU-algebras was introduced by Iampan in 2017 (Iampan, 2017). Ansari, Koam, and Haider (2019) introduced the concept of roughness in UP-algebras and studied the lower and upper approximations of UP-subalgebras and UP-ideals. Senapati, Muhiuddin, and Shum (2017) studied interval-valued intuitionistic fuzzy structures in UP-algebras. Senapati, Jun, and Shum (2018) studied cubic sets in UP-algebras. Senapati and Shum (2016) and Senapati and Shum (2017) introduced and studied intuitionistic fuzzy bi-normed KU-ideals and intuitionistic fuzzy bi-normed KU-subalgebras of KU-algebras. Senapati (2018) introduced the concept of (imaginable) T-fuzzy KU-ideals of KU-algebras. Images and preimages of KU-ideals under homomorphism were investigated. Senapati, Jun, and Shum (2020) introduced the concepts of cubic intuitionistic KU-subalgebras and cubic intuitionistic KU-ideals of KU-algebras. Characterizations of cubic intuitionistic KU-subalgebras and cubic intuitionistic KU-ideals of KU-algebras were considered.

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In this paper, we introduce the concept of weak UP-algebras as a generalization of UP-algebras, and investigate its basic properties. This generalization is designed by weakening the axiomatic system of UP-algebras in the manner in which the aforementioned generalizations of BCC-algebras, BE-algebras and KU-algebras are designed by weakening the corresponding axiom system.

2. Preliminaries

In this section, we use the definitions of UP-algebras, UP-subalgebras, UP-ideals and other important terminologies and some related results from the literature (Iampan, 2017; Iampan, 2019).

Definition 2.1. ((Iampan, 2017), Definition 1.3) An algebra $A=(A,\cdot,0)$ of type $(2,0)$ is called a *UP-algebra* if it satisfies the following axioms:

(UP-1) $(\forall x,y,z\in A)((y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z))=0)$,

(UP-2) $(\forall x\in A)(0\cdot x=x)$,

(UP-3) $(\forall x\in A)(x\cdot 0=0)$, and

(UP-4) $(\forall x,y\in A)((x\cdot y=0\wedge y\cdot x=0)\Rightarrow x=y)$.

We denote this axiom system by [UP].

By weakening this axiomatic system by omitting the axiom (UP-3), a weak axiomatic system [wUP] is obtained:

Definition 2.2. An algebra $A=(A,\cdot,0)$ of type $(2,0)$ is called a *weak UP-algebra* if it satisfies the following axioms:
(wUP-1) $(\forall x,y,z\in A)((y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z))=0)$,
(wUP-2) $(\forall x\in A)(0\cdot x=x)$, and
(wUP-4) $(\forall x,y\in A)((x\cdot y=0\wedge y\cdot x=0)\Rightarrow x=y)$.
We denote this axiom system by [wUP] and we denote this type of algebraic structure by wUP-algebra.

Comment 2.1. If a wUP-algebra $(A,\cdot,0)$ satisfies also (UP-3), then it is a UP-algebra. Therefore, the concept of wUP-algebras is a generalization of the concept of UP-algebras since the axiom (UP-3) is independent of the other axioms in [wUP].

Similarly as in UP-algebras, in any wUP-algebra A we can introduce a natural relation "$\leq$" putting

$$(\forall x,y\in A)(x\leq y\Leftrightarrow x\cdot y=0).$$

It is not difficult to see that in wUP-algebras the following claims are valid:

Lemma 2.1. In a wUP-algebra A, the following properties hold:
(1) $(\forall x\in A)(x\leq x)$,
(2) $(\forall x,y,z\in A)((x\leq y\wedge y\leq z)\Rightarrow x\leq z)$,
(3) $(\forall x,y,z\in A)(x\leq y\Rightarrow z\cdot x\leq z\cdot y)$, and
(4) $(\forall x,y,z\in A)(x\leq y\Rightarrow y\cdot z\leq x\cdot z)$.

Proof. By examining the evidence for assertions (1), (2), (3), and (4) in Proposition 1.7 in article (Iampan, 2017), we can see that the demonstrations of these claims do not contain the axiom (UP-3). Therefore, these claims are valid claims in the axiom system [wUP], too.

Corollary 2.1. The relation $\leq$ is an order relation in a wUP-algebra A left compatible and right inverse compatible with the internal operation in A.

The map $\varphi: A\rightarrow A$, defined by
(5) $(\forall x\in A)(\varphi(x)=x\cdot 0)$, was formally introduced in Dudek and Thomys, 1990 for BCH-algebras, but, in fact, different properties of this map were used in Dudek and Thomys (1990), Dudek and Thomys (2012), and Romano (2018) to characterizations of special subclasses of weak BCC-algebras, and Romano (2020) in describing the properties of JU-algebras. Some important features of this mapping in wUP-algebras are given in the following proposition.

Proposition 2.1. Let A be a JU-algebra. Then
(6) $(\forall x,y\in A)(y\cdot x\leq\varphi(x\cdot y))$,
(7) $(\forall x\in A)(x\leq\varphi^2(x))$,
(8) $(\forall x,y\in A)(\varphi(y)\leq(x\cdot y)\cdot\varphi(x))$,
(9) $(\forall x,y\in A)(x\leq y\Rightarrow\varphi(y)\leq\varphi(x))$,
(10) $(\forall x,y\in A)(y\leq\varphi(x)\cdot(x\cdot y))$,
(11) $(\forall x\in A)(\varphi^3(x)=\varphi(x))$,
(12) $(\forall x,y\in A)(\varphi(x)=(y\cdot x)\cdot\varphi(y))$,
(13) $(\forall x,y\in A)(\varphi^2(y)=\varphi(x)\cdot\varphi(y\cdot x))$,
(14) $(\forall x,y\in A)(\varphi^2(x\cdot y)=\varphi^2(x)\cdot\varphi^2(y))$, and
(15) $(\forall x,y\in A)(\varphi^2(x\cdot y)=\varphi(y\cdot x))$.

Proof. If we put $z=x$ in (wUP-1), we get (6).
If we put $y=0$ in (6), we get (7).
If we put $z=0$ in (wUP-1), we get (8).
Let $x,y\in A$ be such that $x\leq y$. Then $x\cdot y=0$. If we put $x\cdot y=0$ in (8), we get (9). So the mapping φ is inversely monotone.

If we put $y=0$ and $z=y$ in (wUP-1), we get (10).

Applying procedure (9) to inequality (7), we obtain $\varphi^3(x)\leq\varphi(x)$. On the other hand, if we put $x=\varphi(x)$ into (7), we get $\varphi(x)\leq\varphi^3(x)$. These two inequalities give $\varphi(x)=\varphi^3(x)$, according to (wUP-4).

If we apply (9) to inequality (10), we get $\varphi(\varphi(x)\cdot(x\cdot y))\leq\varphi(y)$. If we extend the resulting inequality on the left side with respect (6) and on the right side with respect (8), we obtain

$$(x\cdot y)\cdot\varphi(x)\leq\varphi(\varphi(x)\cdot(x\cdot y))\leq\varphi(y)\leq(x\cdot y)\cdot\varphi(x).$$

Therefore, (12) is a valid formula in [wUP].
If we put $x=\varphi(y)$ and $y=y\cdot x$ into (12), we get
$\varphi^2(y)=\varphi(\varphi(y))=((y\cdot x)\cdot\varphi(y))\cdot\varphi(y\cdot x)=\varphi(x)\cdot\varphi(y\cdot x)$.
If we put $x=\varphi(x)$ and $y=x\cdot y$ into (13), we get
$\varphi^2(x\cdot y)=\varphi(\varphi(x))\cdot\varphi((x\cdot y)\cdot\varphi(x)=\varphi^2(x)\cdot\varphi(\varphi(y))=$
$\varphi^2(x)\cdot\varphi^2(y)$.

3. Ideals in wUP-algebras

A non-empty subset S of a wUP-algebra A is called a *wUP-subalgebra* of A if

$$(\forall x,y\in A)((x\in S\wedge y\in S)\Rightarrow x\cdot y\in S)$$ holds.

In this section, we introduce the concept of ideals in wUP-algebras and some types of ideals such as closed ideal, (★)-ideal, ag-ideal, strong ideal, regular ideal, associative ideal and t-ideal of a wUP-algebra.

Definition 3.1. A subset J of a wUP-algebra A is called a *wUP-ideal* of A if
(J1) $0\in J$, and
(J2) $(\forall x,y,z\in A)((x\cdot(y\cdot z)\in J\wedge y\in J)\Rightarrow x\cdot z\in J)$.

The following results are important properties of wUP-ideals of wUP-algebras.

Proposition 3.1. Let *J* be a wUP-ideal of a wUP-algebra *A*. Then

(J3) $(\forall y,z\in A)((y\cdot z\in J\wedge y\in J)\Rightarrow z\in J)$,

(J4) $(\forall x,y\in A)((\varphi(x)\in J\wedge y\in J)\Rightarrow x\cdot y\in J)$, and

(J5) $(\forall x,y\in A)((x\cdot\varphi(y)\in J\wedge y\in J)\Rightarrow\varphi(x)\in J)$.

Proof. If we put $x=0$ in (J2), we get (J3) with respect to (wUP-2).

(J4) is obtained from (J2) if we put $z=y$ in (J2).

(J5) is obtained from (J2) if we put $z=0$ in (J2).

Corollary 3.1. Let *J* be a wUP-ideal of a wUP-algebra *A*. Then

(J6) $(\forall y,z\in A)(y\leq z\wedge y\in J)\Rightarrow z\in J)$.

Proof. Let $y,z\in A$ be such that $y\leq z$ and $y\in J$. Then $y\cdot z=0\in J$ and $y\in J$. Thus $z\in J$ by (J3).

Example 3.1. $\mathrm{Ker}\varphi=\{x\in A:\varphi(x)=0\}$ is a wUP-ideal of a wUP-algebra A. First, we have $0\in\mathrm{Ker}\varphi$ because $0\cdot 0=0$ by (1). Let $x,y,z\in A$ be arbitrary elements such that $x\cdot(y\cdot z)\in\mathrm{Ker}\varphi$ and $y\in\mathrm{Ker}\varphi$.

Then $\varphi(x\cdot(y\cdot z))=0$ and $\varphi(y)=0$. Thus, the following holds

$$0=\varphi(x\cdot(y\cdot z))=\varphi^2(y\cdot z)\cdot x)=\varphi^2(y\cdot z)\cdot\varphi^2(x)=(\varphi^2(y)\cdot\varphi^2(z))\cdot\varphi^2(x)$$
$$=(0\cdot\varphi^2(z))\cdot\varphi^2(x)=\varphi^2(z)\cdot\varphi^2(x)=\varphi^2(z\cdot x)=\varphi(x\cdot z).$$

As shown, the set $\mathrm{Ker}\varphi$ satisfies both conditions (J1) and (J2). So, $\mathrm{Ker}\varphi$ is a wUP-ideal of A.

Following the ideas presented in Dudek and Thomys (1990), Dudek, Zhang, and Wang (2009), Romano (2020), and Thomys and Zhang, (2013) in this section we will introduce several types of ideals in this class of weak algebras and show the connection between them. Names and designations of newly constructed ideals in wUP-algebra are retrieved from Dudek *et al.* (2009) by analogy.

Definition 3.2. A wUP-ideal J of a wUP-algebra A is called a *closed wUP-ideal* of A if the following holds

(C) $(\forall x\in A)(x\in J\Rightarrow\varphi(x)\in J)$.

The following theorem shows that closed wUP-ideals and wUP-subalgebras coincide in wUP-algebras.

Theorem 3.1. A wUP-ideal J of a wUP-algebra A is a closed wUP-ideal of A if and only if J is a wUP-subalgebra of A.

Proof. Let J be a wUP-subalgebra of A. Then $0\in J$ and $x\in J$ implies $\varphi(x)=x\cdot 0\in J$. So, the wUP-ideal J is a closed wUP-ideal of A.

Conversely, assume that J is a closed wUP-ideal of A. Then for any $x,y\in A$, we have $\varphi(x)\in J$ and $y\in J$. Thus $x\cdot y\in J$ by (J4). This means that J is a wUP-subalgebra of A.

Example 3.2. $\mathrm{Ker}\varphi$ is a closed wUP-ideal of a wUP-algebra A. For any elements $x,y\in A$ such that $\varphi(x)=0$ and $\varphi(y)=0$, the following holds

$$\varphi(x\cdot y)=\varphi^2(y\cdot x)=\varphi^2(y)\cdot\varphi^2(x)=0\cdot 0=0.$$

Hence, $x\cdot y\in\mathrm{Ker}\varphi$. Since $\mathrm{Ker}\varphi$ is a wUP-subalgebra of A, the wUP-ideal $\mathrm{Ker}\varphi$ is closed according to Theorem 3.1.

Definition 3.3. A wUP-ideal J of a wUP-algebra A is called (★)-*wUP-ideal* of A if the following holds

(★) $(\forall x,y\in A)((x\in J\wedge\neg(y\in J))\Rightarrow y\cdot x\in J)$.

The following theorem shows the necessary and sufficient conditions of closed (★)-ideals in wUP-algebras.

Theorem 3.2. A wUP-ideal J of a wUP-algebra is a closed (★)-ideal of A if and only $\varphi(A)\subseteq J$.

Proof. Let J be a wUP-ideal of A. If $\varphi(A)\subseteq J$, then obviously $\varphi(J)\subseteq J$, i.e. the wUP-ideal J is a closed wUP-ideal of A. Let $x,y\in A$ be elements such that $x\in J$ and $\neg(y\in J)$. Then $x\in J$ and $\varphi(y)\in J$ by hypothesis. At the other hand, from from (10), in the form $x\leq\varphi(y)\cdot(y\cdot x)$, and $x\in J$ follows $\varphi(y)\cdot(y\cdot x)$ by (J6). From here it follows $x\cdot y$ according to (J3) because $\varphi(y)\in J$.

For the converse, suppose that a wUP-ideal J of A is a closed (★)-ideal of A. Then for any $y \in A$, we have $y \in J \vee \neg(y \in J)$. If $y \in J$, then $\varphi(y) \in J$ because J is closed. If $\neg(y \in J)$, then from $0 \in J$ and $\neg(y \in J)$ it follows $\varphi(y) = y \cdot 0 \in J$ by (★). So, $\varphi(A) \subseteq J$.

For any wUP-ideal J of a wUP-algebra A we can define a binary relation θ_J on A putting:

$$(\forall x, y \in A)((x, y) \in \theta_J \Leftrightarrow (x \cdot y \in J \wedge y \cdot x \in J)).$$

Such defined relation is an equivalence relation. It is a congruence on A because J is a wUP-ideal of A. Then the family A/θ_J is a wUP-algebra with respect to the operation "•" defined by

$$(\forall x, y \in A)([x] \bullet [y] = [x \cdot y]).$$

Clearly $[0] = \{x \in A : \varphi(x) = x \cdot 0 \in J \wedge x = 0 \cdot x \in J\} = \{x \in A : \varphi(x) \in J\} \subseteq J$. For closed wUP-ideal J of A we have $[0] = J$.

Theorem 3.3. A closed wUP-ideal J of a wUP-algebra A is an (★)-ideal of A if and only if A/J is a UP-algebra.

Proof. Let J be a closed (★)-ideal of a wUP-algebra A. For any $x \in A$ holds $\varphi(x) \in J$ by Theorem 3.2. Thus $\varphi(x) \cdot 0 \in J$ and $0 \cdot \varphi(x) \in J$ by Theorem 3.1 because J is a wUP-subalgebra of A. This means $x \cdot 0 \in J$ by (J4). Hence, $[x] \bullet [0] = [0]$. Finally, we have that A/J is a UP-algebra.

Conversely, if A/J is a UP-algebra, then $[x] \bullet [0] = [x \cdot 0] = [0]$ for any $x \in A$. Hence, $\varphi(x) = x \cdot 0 \in J$. So, the wUP-ideal J is a closed (★)-ideal of A by Theorem 3.2.

Definition 3.4. A wUP-ideal J of a wUP-algebra A is called an *ag-ideal* of A if the following holds

(AG) $(\forall x \in A)(\varphi^2(x) \in J \Rightarrow x \in J)$.

Example 3.3. The wUP-ideal $\mathrm{Ker}\varphi$ is an ag-ideal of a wUP-algebra A. Indeed, let $x \in A$ be an element such that $\varphi^2(x) \in \mathrm{Ker}\varphi$. Then $\varphi(\varphi^2(x)) = 0$. Thus $\varphi(x) = \varphi^3(x) = \varphi(\varphi^2(x)) = 0$. Hence, $x \in \mathrm{Ker}\varphi$. So, the wUP-ideal $\mathrm{Ker}\varphi$ is an ag-ideal of A.

The following theorem shows the equivalent conditions with ag-ideals in wUP-algebras.

Theorem 3.4. For a wUP-ideal J of a wUP-algebra A, the following conditions are equivalent:

(a) J is an ag-ideal of A,

(b) $(\forall x, y \in A)((x \leq y \wedge y \in J) \Rightarrow x \in J)$,

(c) $(\forall x, y, z \in A)(((z \cdot x) \cdot (z \cdot y) \in J \wedge x \in J) \Rightarrow y \in J)$, and

(d) $(\forall x, z \in A)(\varphi(z) \cdot (z \cdot y) \Rightarrow y \in J)$.

Proof. (a)$\Longrightarrow$(b). Let $x, y \in A$ be elements such that $x \leq y$ and $y \in J$. Then $x \cdot y = 0$ and $\varphi^2(y \cdot x) = \varphi(x \cdot y) = \varphi(0) = 0 \in J$ by (15). Thus $y \cdot x \in J$ by (AG). Now, from $y \cdot x \in J$ and $y \in J$ it follows $x \in J$ by (J3). This means that the condition (b) is a valid formula in a wUP-algebra A.

(b)$\Longrightarrow$(c). Let $x, y, z \in A$ be arbitrary elements such that $(z \cdot x) \cdot (z \cdot y) \in J$ and $x \in J$. If we put $y = x$, $z = y$ and $x = z$ in (wUP-1), we get $x \cdot y \leq (z \cdot x) \cdot (z \cdot y)$. From this and $(z \cdot x) \cdot (z \cdot y) \in J$ it follows $x \cdot y \in J$ by (J6). Now, from $x \cdot y \in J$ and $x \in J$ it follows $y \in J$ by (J3).

(c)$\Longrightarrow$(d). Putting $x = 0$, we get (d).

(d)$\Longrightarrow$(a). Putting $y = z$, we get (a).

Theorem 3.5. A wUP-ideal J of a wUP-algebra A is a closed ag-ideal of A if and only if for every $x \in A$ both x and $\varphi(x)$ belong or not belong to J.

Proof. Let J be a closed ag-ideal of A. Then $x \in J$ implies $\varphi(x) \in J$. Similarly, $\varphi(x) \in J$ implies $\varphi^2(x) \in J$. Since J is an ag-ideal of A, it follows $x \in J$. Hence, both x and $\varphi(x)$ belong or not belong to J.

Conversely, any wUP-ideal J with the property that both x and $\varphi(x)$ belong or not belong to J, is obviously closed wUP-ideal of A. Moreover, if $\varphi^2(x) \in J$, then also $\varphi(x) \in J$. Thus $x \in J$ by hypothesis. So, the wUP-ideal J is a closed ag-ideal of A.

In what follows, the concept of strong ideal of a wUP-algebra is introduced and analyzed.

Definition 3.5. A wUP-ideal J of a wUP-algebra A is called a *strong ideal* of A if the following holds

(S) $(\forall x, y \in A)((x \in J \wedge \neg(y \in J)) \Rightarrow \neg(x \cdot y \in J))$.

Lemma 3.1. The condition (S) is equivalent to the condition

(S1) $(\forall x, y \in A)(x \cdot y \in J \wedge \neg(x \in J)) \Rightarrow y \in J)$.

Proof. The condition (S1) is obtained from the condition (S) by the logical contraposition and vice versa, (S) is obtained from (S1) by the logical contraposition.

The following theorem shows that strong ideals and closed ag-ideals coincide in wUP-algebras.

Theorem 3.6. A wUP-ideal J of a wUP-algebra A is a strong ideal of A if and only if it is closed ag-ideal of A.

Proof. Let J be a closed ag-ideal of A. Suppose $x \in J$, $\neg(y \in J)$ and $y \cdot x \in J$. Then $\varphi(y \cdot x) \in J$ by (C). Also $\varphi(x) \in J$ by (C). At the other hand, we have $\varphi^2(y) = \varphi(x) \cdot \varphi(y \cdot x) \in J$ by (12) and because J is a wUP-subalgebra of A by Theorem 3.1. Thus $\varphi^2(y) \in J$. Hence $y \in J$ by (AG). We got a contradiction. So, it must be $\neg(yx \in J)$. Therefore, J is a strong ideal of A.

Let A be a strong ideal of a wUP-algebra A. Suppose $\varphi^2(x) \in J$ and $\neg(x \in J)$. Then $\neg(x \cdot \varphi^2(x) \in J)$ by (S). At the other hand, from (7): $x \leq \varphi^2(x)$ it follows $x \cdot \varphi^2(x) \geq x \cdot x = 0 \in J$ by (3). Then $x \cdot \varphi^2(x) \in J$ by (J6). The obtained contradiction overthrows the assumption $\neg(x \in J)$. So, it must be $x \in J$ and we conclude that J is an ag-ideal of A. To prove that J is closed we select arbitrary $x \in J$. Them $\varphi^2(x) \in J$ by (7) and (J6). Suppose $\neg(\varphi(x) \in J)$. Now, from this and $0 \in J$ it follows $\neg(\varphi(x) \cdot 0 = \varphi^2(x) \in J)$. The resulting contradiction overthrows the assumption $\neg(\varphi(x) \in J)$. So, it must be $\varphi(x) \in J$. This proves that J is a closed ag-ideal of A.

The next type of ideals in wUP-algebras is the type of the regular ideals.

Definition 3.6. A wUP-ideal J of a wUP-algebra A is called a *regular ideal* of A if the following holds

(R) $(\forall x, y \in A)((x \cdot y \in J \wedge y \in J) \Rightarrow x \in J)$.

The following theorem shows that regular ideals and closed ag-ideals coincide in wUP-algebras.

Theorem 3.7. A wUP-ideal J of a wUP-algebra A is regular if and only if it is closed ag-ideal of A.

Proof. Let J be a regular ideal of a wUP-algebra A. For $x, y \in A$ such that $x \leq y$ and $y \in J$, it follows $x \cdot y = 0 \in J$ and $y \in J$. Thus $x \in J$ by (R). So, J is an ag-ideal of A by (b). Moreover, for $x \in J$, by (7) and (J6), we have $\varphi^2(x) \in J$. But $\varphi(x) \cdot 0 = \varphi^2(x) \in J$ and $0 \in J$ give $\varphi(x) \in J$ by regularity of J. Thus J is a closed ag-ideal of A.

Opposite, suppose that J is a closed ag-ideal of A. For $x, y \in A$ such that $x \cdot y \in J$ and $y \in J$ have to be $\varphi(x \cdot y) \in J$ and $\varphi(y) \in J$ by (X). Then $\varphi^2(x) = \varphi(y) \cdot \varphi(x \cdot y) \in J$ because J is a wUP-subalgebra of A. Thus, from $\varphi^2(x) \in J$, it follows $x \in J$ by (AG). This proves that J is a regular ideal of A.

The concept of associative ideals of wUP-algebras was introduced by the following definition.

Definition 3.7. A wUP-ideal J of a wUP-algebra A is called an *associative ideal* of A if the following holds

(A) $(\forall x, z \in A)(\varphi(z) \cdot z \in J \Rightarrow x \cdot z \in J)$.

Theorem 3.8. An associate ideal of a wUP-algebra A is a closed ag-ideal of A.

Proof. Suppose J is an associate ideal of a wUP-algebra A. Putting $z = 0$ in (A), we get (C). Putting $x = 0$ in (A), we get (AG).

The family of possible ideals in wUP-algebras is supplemented by the concept of t-ideals, introduced by the following definition.

Definition 3.8. A wUP-ideal J of a wUP-algebra A is called a *t-ideal* of A if the following holds

(T) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \wedge y \in J) \Rightarrow x \cdot z \in J)$.

The following results are important properties of t-ideals of wUP-algebras.

Proposition 3.2. Let J be a t-ideal of a wUP-algebra A. Then

(e) $(\forall x, y \in A)((\varphi(x \cdot y) \in J \wedge y \in J) \Rightarrow \varphi(x) \in J)$,

(f) $(\forall x, z \in A)(\varphi(x) \cdot z \in J \Rightarrow x \cdot z \in J)$, and

(g) $(\forall x \in A)(\varphi^2(x) \in J \Rightarrow \varphi(x) \in J)$.

Proof. If we put $z = 0$ in (T), we get (e). Putting $y = 0$ in (T), we get (f). Putting $z = 0$ and $y = 0$ in (T), we get (g).

Theorem 3.9. Let J be a wUP-ideal of a wUP-algebra A. Then the condition (T) is equivalent to the condition (f).

Proof. The implication (T)$\Longrightarrow$(f) was shown in the previous proposition.

If we put $y = 0$ and $z = y$ in (wUP-1), we get $0 \cdot y \leq (x \cdot 0) \cdot (x \cdot y)$, i.e. $y \leq \varphi(x) \cdot (x \cdot y)$. Hence $\varphi(x) \cdot (x \cdot y) \in J$ by (J3) because $y \in J$. On the other hand, if we put $x = \varphi(x)$, $y = x \cdot y$ in (wUP-1), we get $(x \cdot y) \cdot z \leq (\varphi(x) \cdot (x \cdot y)) \cdot (\varphi(x) \cdot z)$. From here and from $(x \cdot y) \cdot z \in J$ it follows $(\varphi(x) \cdot (x \cdot y)) \cdot (\varphi(x) \cdot z) \in J$ according to (J6). Now, we have

$$(\varphi(x) \cdot (x \cdot y) \in J \wedge (\varphi(x) \cdot (x \cdot y)) \cdot (\varphi(x) \cdot z) \in J) \Rightarrow \varphi(x) \cdot z \in J$$

by (J3). Thus $x \cdot z \in J$ according (f). So, J is a t-ideal of A.

The following theorem shows the necessary and sufficient conditions of t-ideals in wUP-algebras.

Theorem 3.10. A wUP-ideal J of a wUP-algebra A is t-ideal of A if and only if the following holds

(h) $(\forall x, z \in A)(\varphi(x) \cdot z \Rightarrow \varphi^2(x) \cdot z)$.

Proof. (T)$\Longrightarrow$(h). If we put $x = \varphi^2(x)$ in (f), we get (h) by (11) since (f) is equivalent to (T).

Let J be a wUP-ideal of a wUP-algebra A satisfying the condition (h). Suppose $\varphi(x) \cdot z \in J$ for some $x, z \in A$. Then $\varphi^2(x) \cdot z \in J$ by (h). At the other hand, $\varphi^2(x) \cdot z \leq x \cdot z$ is obtained from $x \leq \varphi^2(x)$ by multiplying by right side with z with respect to (4). Now, from obtained inequality $\varphi^2(x) \cdot z \leq x \cdot z$ and $\varphi^2(x) \cdot z \in J$ it follows $x \cdot z \in J$ by (J6). Thus, the condition (f) is a consequence of (h). Hence, (h)$\Longrightarrow$(T) since (f)$\Longleftrightarrow$(T).

The following theorem shows another necessary and sufficient conditions of t-ideals in wUP-algebras.

Theorem 3.11. A wUP-ideal J of a wUP-algebra A is a t-ideal of A if and only if the following holds

(k) $(\forall x \in A)(x \cdot \varphi(x) \in J)$.

Proof. (T)$\Longrightarrow$(k). Let J be a t-ideal of a wUP-algebra A. To prove (k) observe that $(x \cdot 0) \cdot \varphi(x) = \varphi(x) \cdot \varphi(x) = 0 \in J$ for any $x \in A$ by (1). Now, from $(x \cdot 0) \cdot \varphi(x) \in J$ and $0 \in J$ it follows $x \cdot \varphi(x) \in J$ by (T). So, the condition (k) is a consequence of the condition (T).

(k)$\Longrightarrow$(T). Let $x\cdot\varphi(x)\in J$ be valid for any $x\in A$. Let us prove that the condition (f) is a valid formula in a wUP-algebra A. Suppose $\varphi(x)\cdot z\in J$ for some $x,z\in A$. Putting $y=\varphi(x)$ in (wUP-1), we get $\varphi(x)\cdot z\leq(x\cdot\varphi(x))\cdot(x\cdot z)$. Thus $(x\cdot\varphi(x))\cdot(x\cdot z)\in J$ by (J6) because $\varphi(x)\cdot z\in J$ by hypothesis. Now, from $(x\cdot\varphi(x))\cdot(x\cdot z)\in J$ and $x\cdot\varphi(x)\in J$ it follows $x\cdot z\in J$ by (J3). Therefore, condition (f) is satisfied, and therefore condition (T) is also satisfied.

4. Conclusions

In this paper, the concept of weak UP-algebras is introduced by omitting an axiom from the axiomatic system [UP] that determines UP-algebras. In the newly designed axiomatic system [wUP] several types of ideals were introduced and analyzed. The presented procedure for analyzing observed ideals in wUP-algebras can also be applied to some of the following conditions applied to ideals. Let J be a wUP-ideal of a wUP-algebra A. We may consider the following conditions imposed on the wUP-ideal J:

(X) $(\forall x,y,z\in A)(((x\cdot y)\cdot z\in J\wedge y\cdot z\in J)\Longrightarrow x\in J)$, and

(Y) $(\forall x,y,z\in A)((x\cdot(y\cdot z)\in J\wedge y\cdot z\in J)\Longrightarrow x\cdot z\in J)$.

It could, for example, prove to be valid.

Proposition 4.1. Let J be a wUP-ideal of a wUP-algebra A satisfying the condition (X). Then

(i) $(\forall x,y\in A)((\varphi(x\cdot y)\in J\wedge\varphi(y)\in J)\Longrightarrow x\in J)$,

(ii) $(\forall x,z\in A)((\varphi(x)\cdot z\in J\wedge z\in J)\Longrightarrow x\in J)$, and

(iii) $(\forall x,z\in A)((z\in J\wedge x\cdot z\in J)\Longrightarrow x\in J)$.

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