Songklanakarin J. Sci. Technol.
43 (2), 492-495, Mar. - Apr. 2021

http://www.sjst.psu.ac.th

Original Article

Total domination game on ladder graphs

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Received: 1 August 2019; Revised: 16 February 2020; Accepted: 21 March 2020

Abstract

The total domination game is played on a simple graph G by two players, named Dominator and Staller. They alternately select a vertex of G; each chosen vertex totally dominates its neighbors. In this game, each chosen vertex must totally dominate at least one new vertex not totally dominated before. The game ends when all vertices in G are totally dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game total domination number is the number of chosen vertices when both players play optimally. There are two types of such number, one where Dominator starts the game and another where Staller starts the game. In this paper, we determine the game total domination numbers of the ladders, the circular ladders and the Möbius ladders.

Keywords: total domination game, Game total domination number, ladder, circular ladder, Möbius ladder

1. Introduction

The domination game in graphs was introduced by Brešar, Klavžar, and Rall (2010), where the original idea is attributed to Henning in 2003 and the game has been extensively studied afterwards in (Brešar, Dorbec, Klavžar, & Košmrlj, 2014; Dorbec, Košmrlj, & Renault, 2015; Kinnersley, West, & Zamani, 2013; Košmrlj, 2017) and elsewhere. A subset S of vertices in a graph G is called a *dominating set* if every vertex not in S is adjacent to some vertex in S. If a vertex $u \in S$ is adjacent to a vertex $v \in V(G)$, we say that *u dominates v*. *Domination game* is played on a graph G by two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is dominated. Playing a vertex will make all vertices in its close neighborhood dominated. The game ends when the chosen vertices from two players form a dominating set i.e. all vertices are dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. Note that domination game is a game without winner or loser but the players want to play optimally according to their purposes. The *game domination number* is the size of the dominating set of chosen vertices when both players play optimally, denoted by $\gamma_g(G)$ when Dominator starts the game and denoted by $\gamma_g'(G)$ when Staller starts the game.

There are many results about game domination numbers on various families of graphs. Brešar *et al.* (2010) showed a lower bound on the game domination number of an arbitrary Cartesian product of two graphs. Košmrlj (2017) determined the game domination numbers for paths and cycles. Dorbec *et al.* (2015) showed how the game domination number of the union of two graphs from a certain family corresponds to the game domination numbers of the initial graphs. Ruksasakchai *et al.* (2019) showed the game domination numbers of a disjoint union of paths and cycles.

In this paper, we study the total version of domination game which is called the total domination game. A set S of vertices of a graph G is a *total dominating set*, abbreviated TD-set, if every vertex of G is adjacent to some vertex in S If a vertex $u \in S$ is adjacent to another vertex $v \in V(G)$, we say that *u totally dominates v*. Note that in total domination a vertex does not totally dominate itself so it is required that there is no isolated vertex in the graph. The *total domination number* of a graph G is the minimum cardinality of a total dominating set of G, denoted by $\gamma_t(G)$.

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For any graph G which has no isolated vertex, $\gamma_t(G)$ exists and $\gamma_t(G)\geq 2.$

The *total domination game* was introduced by Henning, Klavžar, and Rall (2015). This game is played on a graph G by two players, *Dominator and Staller*, who alternate taking turns to choose a vertex from a graph G Each chosen vertex must totally dominate at least one new vertex not totally dominated before. The game ends when the set of vertices chosen by the two players is a total dominating set in G Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game total domination number* is the size of the total dominating set of chosen vertices when both players play optimally, denoted by $\gamma_{tg}(G)$ when Dominator starts the game and denoted by $\gamma'_{tg}(G)$ when Staller starts the game.

Henning *et al.* (2015) showed a bound of game total domination numbers of a graph in terms of total domination number. More precisely, if G is a graph on at least two vertices, then $\gamma_t(G)\leq\gamma_{tg}(G)\leq 2\gamma_t(G)-1.$ They also show that the values of the two types of game total domination number of a graph differ by at most 1.

Total domination game played on various families of graphs have been studied. Dorbec and Henning (2016) determined the game total domination numbers for cycles and paths. Henning and Rall (2017) showed that if G is a forest with no isolated vertex, then $\gamma_{tg}(G)\leq\gamma'_{tg}(G).$ They also characterized the trees with equal total domination and game total domination number. In order to prove that certain circular ladders are total domination game critical, Henning and Klavžar (2018) determined the game total domination numbers of the circular ladder of order 8*k* and the game total domination numbers of the MÖbius ladder of order 4*k* for $k\geq 1.$ In this paper, we complement this last result by determining the game total domination numbers of all ladders, circular ladders and MÖbius ladders.

2. Preliminaries

A *path* P_n is a graph whose vertices can be listed in the order $v_1,v_2,...,v_n$ such that v_i and v_{i+1} are adjacent where $i=1,2,...,n-1.$ A *cycle* C_n is a graph whose vertices can be listed in the order $v_1,v_2,...,v_n$ such that v_1 and v_n are adjacent, and v_i and v_{i+1} are adjacent where $i=1,2,...,n-1.$ The *Cartesian product* of graphs G and H, denoted by $G\square H$, is the graph whose vertex set is the Cartesian product $V(G)\times V(H)$, two vertices (u,u') and (v,v') are adjacent in $G\square H$, if and only if either $u=v$ and u' is adjacent to v' in H, or $u'=v'$ and u is adjacent to V in G. The *disjoint union* of m copies of a graph G is denoted by mG.

To determine the game total domination numbers of the three families of ladder graphs, we will show that playing the total domination game on each such graph is equivalent to playing the domination game on a certain disjoint union of paths or cycles. Here we recall some useful results of domination game of the related graphs.

Lemma 1. (Košmrlj, 2017) The game domination numbers of P_n and C_n are given by

$$\gamma_g(P_n)=\gamma_g(C_n)=\begin{cases}\left\lceil\frac{n}{2}\right\rceil-1; & n\equiv 3\pmod 4\\ \left\lceil\frac{n}{2}\right\rceil; & otherwise\end{cases},$$

$$\gamma'_g(P_n)=\left\lceil\frac{n}{2}\right\rceil$$

and

$$\gamma'_g(C_n)=\begin{cases}\left\lceil\frac{n-1}{2}\right\rceil-1; & n\equiv 2\pmod 4\\ \left\lceil\frac{n-1}{2}\right\rceil; & otherwise\end{cases}.$$

Lemma 2. (Ruksasakchai *et al.*, 2019) The game domination numbers of $2P_n$ are given by

$$\gamma_g(2P_n)=\gamma'_g(2P_n)=\begin{cases}n+1; & n\equiv 1\pmod 4\\ n; & otherwise\end{cases}.$$

Lemma 3. (Ruksasakchai *et al.*, 2019) The game domination numbers of $2C_n$ are given by

$$\gamma_g(2C_n)=\begin{cases}n-1; & n\equiv 2,3\pmod 4\\ n; & otherwise\end{cases}$$

and

$$\gamma'_g(2C_n)=\begin{cases}n-2; & n\equiv 2(\mathrm{mod}\,4)\\ n-1; & n\equiv 1,3(\mathrm{mod}\,4)\\ n; & n\equiv 0(\mathrm{mod}\,4)\end{cases}.$$

3. Ladder Graphs

For $n\geq 1,$ the ladder L_n is the Cartesian product $P_n\square P_2.$ Throughout the paper, we let $V(L_n)=\{x_1,x_2,\ldots,x_n,$ $y_1,y_2,\ldots,y_n\}$ and $E(L_n)=\{x_iy_j\mid 1\leq i,j\leq n$ and $|i-j|\leq 1\}.$ The ladders L_4 and L_5 are shown in Figure 1(a)(b). In this section, we determine the game total domination numbers of the ladder $L_n.$ To do so, we show that playing the total domination game on L_n is equivalent to playing the domination game on $2P_n$.

Theorem 4. The game total domination numbers of the ladder L_n are given by

$$\gamma_{tg}(L_n)=\gamma'_{tg}(L_n)=\begin{cases}n+1; & n\equiv 1\pmod 4\\ n; & otherwise\end{cases}.$$

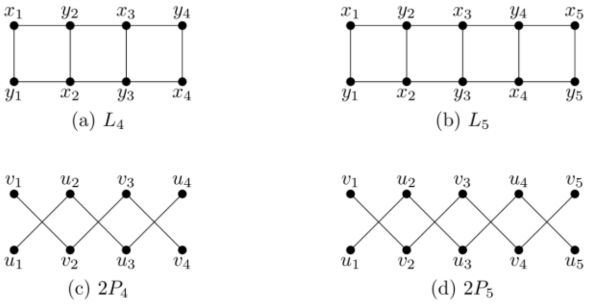

Figure 1. The ladders L_4 and L_5, and the graphs $2P_4$ and $2P_5$

Proof. Let $G = L_n$. Let $H = 2P_n$ be the disjoint union of the paths $u_1u_2\ldots u_n$ and $v_1v_2\ldots v_n$. The graphs $2P_4$ and $2P_5$ are shown in Figure 1(c)(d). We define a bijection *f* from $V(G)$ to $V(H)$ by

$$f(x_i) = u_i,\ f(y_i) = v_i$$

for $i \in \{1, 2, \ldots, n\}$. For convenience, we define the reflections $\overline{}$ on $V(G)$ and on $V(H)$ by

$$\overline{x_i} = y_i,\ \overline{y_i} = x_i,\ \overline{u_i} = v_i,\ \overline{v_i} = u_i$$

for $i \in \{1, 2, \ldots, n\}$. Consider the total domination game on *G* and the domination game on *H*. Recall that in total domination game, playing a vertex a will totally dominate its open neighborhood $N(a)$ while in domination game, playing a vertex a will dominate its closed neighborhood $N(a)$. Observe that f and f^{-1} preserve domination, that is $f(N_G(a)) = \overline{N_H[f(a)]}$ and $f^{-1}(N_H[b]) = \overline{N_G(f^{-1}(b))}$. Therefore, if $a_1, a_2, ..., a_t$ is a sequence of moves played in *G*, then $f(a_1), f(a_2), ..., f(a_t)$ is a sequence of moves played in *H*, and if $b_1, b_2, ..., b_t$ is a sequence of moves played in *H*, then $f^{-1}(b_1), f^{-1}(b_2), ..., f^{-1}(b_t)$ is a sequence of moves played in *G* In other words, the two games are essentially the same so $\gamma_{tg}(G) = \gamma_g(H)$ and $\gamma'_{tg}(G) = \gamma'_g(H)$. The result follows from Lemma 2.

4. Circular Ladder Graphs

For $n \ge 3$, the circular ladder CL_n is the Cartesian product $C_n \square P_2$. Here CL_n is the graph obtained from the ladder L_n by adding the edges x_1x_n, y_1y_n when *n* is odd, and the edges x_1x_n, y_1y_n when *n* is even. The circular ladders CL_4 and CL_5 are shown in Figure 2(a)(b). In this section, we determine the game total domination numbers of the circular ladders. To do so, we show that (i) playing the total domination game on CL_{2k+1} is equivalent to playing the domination game on C_{4k+2}; and (ii) playing the total domination game on CL_{2k} is equivalent to playing the domination game on $2C_{2k}$.

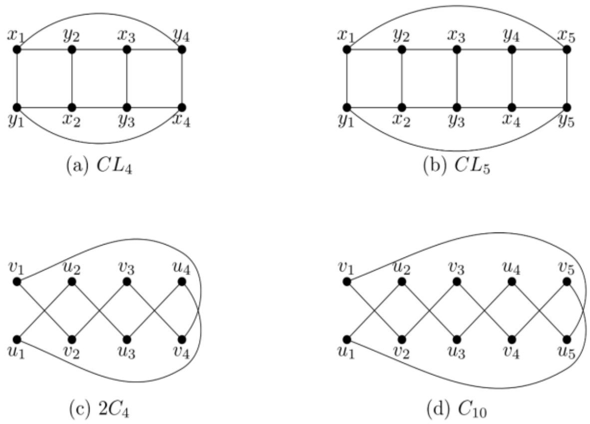

Figure 2. The circular ladders CL_4 and CL_5, and the graphs $2C_4$ and C_{10}

Theorem 5. The game total domination numbers of the circular ladder CL_n are given by

$$\gamma_{tg}(CL_n) = \begin{cases} n-1; & n \equiv 2 \pmod 4 \\ n; & otherwise \end{cases}$$

and

$$\gamma'_{tg}(CL_n) = \begin{cases} n-2; & n \equiv 2 \pmod 4 \\ n-1; & n \equiv 1,3 \pmod 4. \\ n; & n \equiv 0 \pmod 4 \end{cases}$$

Proof. Let $G = CL_n$. Let *H* be the disjoint union of the cycles $u_1u_2\ldots u_nu_1$ and $v_1v_2\ldots v_nv_1$ if *n* is even; otherwise let *H* be the cycle $u_1u_2\ldots u_nv_1v_2\ldots v_nu_1$. The graphs $2C_4$ and C_{10} are shown in Figure 2(c)(d). We define a bijection f from $V(G)$ to $V(H)$ by

$$f(x_i) = u_i,\ f(y_i) = v_i$$

for $i \in \{1, 2, \ldots, n\}$.

In the similar manner to the proof of Theorem 4, we see that the total domination game on *G* and the domination game on *H* are essentially the same so $\gamma_{tg}(G) = \gamma_g(H)$ and $\gamma'_{tg}(G) = \gamma'_g(H)$. The result follows from Lemma 1 and Lemma 3.

5. Möbius Ladder Graphs

For $n \geq 2,$ the Möbius ladder ML_n is the graph obtained from the ladder L_n by adding the edges x_1x_n, y_1y_n when n is even, and the edges x_1x_n, y_1y_n when n is odd. The Möbius ladders ML_4 and ML_5 are shown in Figure 3(a)(b). In this section, we determine the game total domination numbers of the Möbius ladder ML_n. To do so, we show that (i) playing the total domination game on ML_{2k+1} is equivalent to playing the domination game on $2C_{2k+1}$; and (ii) playing the total domination game on ML_{2k} is equivalent to playing the domination game on C_{4k}.

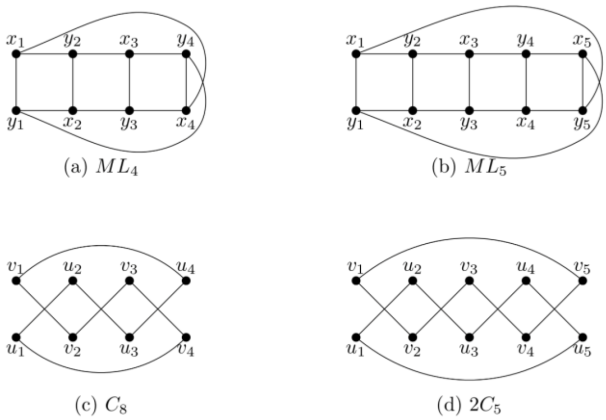

Figure 3. The Möbius ladders ML_4 and ML_5, and the graphs C_8 and $2C_5$

Theorem 6. The game total domination numbers of the Möbius ladder ML_n are given by

$$\gamma_{tg}(ML_n) = \begin{cases} n-1; & n \equiv 3 \pmod 4 \\ n; & otherwise \end{cases}$$

and

$$\gamma'_{tg}(ML_n) = \begin{cases} n-1; & n \equiv 1,3 \pmod 4 \\ n; & otherwise \end{cases}.$$

Proof. Let $G = ML_n$. Let H be the disjoint union of the cycles $u_1u_2\ldots u_nu_1$ and $v_1v_2\ldots v_nv_1$ if n is odd; otherwise let H be the cycle $u_1u_2\ldots u_nv_1v_2\ldots v_nu_1$. We define a bijection f from $V(G)$ to $V(H)$ by

$$f(x_i) = u_i,\ f(y_i) = v_i$$
for $i \in \{1, 2, \ldots, n\}$.

In the similar manner to the proof of Theorem 4, we see that the total domination game on G and the domination game on H are essentially the same so $\gamma_{tg}(G) = \gamma_g(H)$ and $\gamma'_{tg}(G) = \gamma'_g(H)$. The result follows from Lemma 1 and Lemma 3.

Acknowledgements

The authors would like to thank the anonymous reviewers for their useful comments and suggestions.

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