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Original Article

Super edge-magic labeling for k-uniform, complete k-uniform and complete k-uniform k-partite hypergraphs

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Abstract

Let H be a hypergraph with a vertex set V and a hyperedge set E. Generalized from the super edge-magic in a graph, we say that a hypergraph H is super edge-magic if there is a bijection $f: V \cup E \rightarrow \{1,2,3,\ldots,|V|+|E|\}$ which satisfies: (i) there exists a constant Λ such that for all $e \in E$, $f(e) + \sum_{v\in e} f(v) = \Lambda$ and (ii) $f(V) = \{1,2,3,\ldots,|V|\}$. In this paper, we give a necessary condition for a k-uniform hypergraph to be super edge-magic. We show that the complete k-uniform hypergraph of n vertices is super edge-magic if and only if $k \in \{0,1,n-1,n\}$. Finally, we also prove that the complete k-uniform k-partite hypergraph with the same number of vertices in each partite, namely n, is super edge-magic if and only if $(n,k) = (1,k)$ for all $k \geq 2$ and $(n,k) = (2,3)$.

Keywords: super edge-magic, complete k-uniform hypergraph, complete k-uniform k-partite hypergraph, hypergraph, labeling

1. Introduction

For convenience, throughout this article, if a and b are integers such that $a < b$, then we use $[a,b]$ to represent the set of integers $\{a, a+1, a+2, \ldots, b\}$. Super edge-magic graph was first defined by Enomoto, Llado, Nakamigawaa and Ringel (1998), i.e., a graph $G = (V(G), E(G))$ is super edge-magic if there is a bijection $f: V(G) \cup E(G) \rightarrow [1, |V(G)| + |E(G)|]$ which satisfies (i) there exists a constant λ such that for all $xy \in E(G)$, $f(x) + f(y) + f(xy) = \lambda$ and (ii) $f\big(V(G)\big) = [1, |V(G)|]$. Moreover, Enomoto, Llado, Nakamigawaa and Ringel (1998) also showed that every super edge-magic graph having at least one edge satisfies $|E(G)| \leq 2|V(G)| - 3$. According to Gallian (2018), several classes of graphs were studied to determine whether they are super edge-magic. For example, the complete graph is super edge-magic if and only if it has at most three vertices. Labeling of graphs can be applied to several branches of mathematics and science such as coding theory, X-ray crystallography, radar, astronomy and circuit design.

Hypergraphs are the generalization of graphs since each edge or hyperedge of them may be incident to no or many vertices. Furthermore, if every hyperedge of a hypergraph contains exactly k vertices, then the hypergraph is said to be k-uniform. Actually, only a few researchers have explored hypergraph labeling. For examples, Sonntag (2002) gave the existence of antimagic labelings for hypergraphs, namely cacti, cycle and wheels. Recently, Boonklurb, Narissayaporn and Singhun (2016) generalized the notion of super edge-magic labelings of graphs to the one on hypergraphs. They showed that under some conditions the m-node k-uniform hyperpath and hypercycle are super edge-magic.

In this article, we give a necessary condition for a k-uniform hypergraph being super edge-magic and we consider the classes of hypergraphs, namely complete k-uniform hypergraphs and complete k-uniform k-partite hypergraphs

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with the same number of vertices in each partite, whether they are super edge-magic.

2. Necessary Condition

Definition 2.1 (Voloshin, 2009) A hypergraph H is the pair $(V(H), E(H))$ where $V(H)$ is a finite set and $E(H)$ is a subset of power set of $V(H)$, i.e., $E(H) \subseteq P\big(V(H)\big)$. The sets $V(H)$ and $E(H)$ are called *vertex set* and *hyperedge set*, respectively. Moreover, if every element of $E(H)$ has cardinality k, then H is k*-uniform* and denoted by $H^{(k)}$.

Note that by Definition 2.1, the vertex set of a hypergraph can be empty, however, in this article, we consider only hypergraphs having at least one vertex. When there is no ambiguity, we denote $V(H)$ and $E(H)$ by V and E, respectively.

Definition 2.2 (Boonklurb, Narissayapron, & Singhun, 2016) A hypergraph $H = (V, E)$ is said to be super edge-magic if there is a bijection labeling $f: V \cup E \to [1, |V| + |E|]$ which satisfies: (i) there exists a constant Λ such that for all $e \in E$, $f(e) + \sum_{v\in e} f(v) = \Lambda$ and (ii) $f(V) = [1, |V|]$.

Note that if $e = \emptyset$ is an edge of H, we let $\sum_{v\in e} f(v) = 0$. Observe that, from Definition 2.2, a hypergraph having no hyperedges ($E = \emptyset$) or having exactly one hyperedge is obviously super edge-magic.

Lemma 2.1 A hypergraph having at most one hyperedge is always super edge-magic.

Proof. If $E = \emptyset$, then the result follows immediately. If $E = \{e\}$, then by using $f: V \cup \{e\} \to [1, |V| + 1]$ by $f(V) = [1, |V|]$ and $f(e) = |V| + 1$, the result follows immediately.

The next result is a necessary condition for a k-uniform hypergraph to be super edge-magic.

Theorem 2.1 Let $H^{(k)} = (V, E)$ be a k-uniform hypergraph and $|E| \geq 1$. If $H^{(k)}$ is super edge-magic, then $|E| \leq k|V| - k^2 + 1$.

Proof. If $k = 0$, then $|E| = |\{\emptyset\}| = 1$ and the inequality holds. Suppose that $k > 0$. Assume that $H^{(k)}$ is super edge-magic with the labeling f. Let $e_1, e_2 \in E$ be such that $f(e_1) = |V| + |E|$ and $f(e_2) = |V| + 1$. Then, we obtain

$$\Lambda = f(e_1) + \sum_{v\in e_1} f(v) \geq (|V| + |E|) + (1 + 2 + 3 + \cdots + k)$$

and $\Lambda = f(e_2) + \sum_{v\in e_2} f(v) \leq (|V| + 1) + (|V| + (|V| - 1) + (|V| - 2) + \cdots + (|V| - k + 1))$.

Thus, $|V| + |E| + \frac{k(k+1)}{2} \leq (k+1)|V| - \frac{k(k-1)}{2} + 1$ and the desired inequality follows.

Note that if $k = 2$, $H^{(2)}$ is actually a graph and the result in Theorem 2.1 coincides with the result given by Enomoto, Llado, Nakamigawaa, and Ringel (1998).

3. Super Edge-Magic Complete k-Uniform Hypergraphs

Definition 3.1 Let n and k be integers such that $0 \leq k \leq n$. A complete k-uniform hypergraph, $K_n^{(k)}$, is a hypergraph that consists of $V\left(K_n^{(k)}\right) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E\left(K_n^{(k)}\right)$ is the family of all k-subsets of $V\left(K_n^{(k)}\right)$.

By Definition 3.1, it is easy to see that $K_n^{(k)}$ has n vertices and $\binom{n}{k}$ hyperedges. For example, consider a complete 3-uniform hypergraph $K_4^{(3)}$ with $V\left(K_4^{(3)}\right) = \{v_1, v_2, v_3, v_4\}$. Then,

$$E\left(K_4^{(3)}\right) = \big\{\{v_1, v_2, v_3\},\ \{v_1, v_2, v_4\},\ \{v_1, v_3, v_4\},\ \{v_2, v_3, v_4\}\big\}.$$

Lemma 3.1 (i) $K_n^{(0)}$ and $K_n^{(n)}$ are super edge-magic.
(ii) $K_n^{(1)}$ is super edge-magic.
(iii) $K_n^{(n-1)}$ is super edge-magic.

Proof. (i) Since $K_n^{(0)}$ and $K_n^{(n)}$ have only one hyperedge, the conclusion follows immediately by Lemma 2.1.

(ii) Let $V\left(K_n^{(1)}\right) = \{v_1, v_2, v_3, \ldots, v_n\}$. Then, $E\left(K_n^{(1)}\right) = \{\{v_1\}, \{v_2\}, \{v_3\}, \ldots, \{v_n\}\}$. Define a bijection $f: V\left(K_n^{(1)}\right) \cup E\left(K_n^{(1)}\right) \to [1, 2n]$ by $f(v_i) = i$ and $f(\{v_i\}) = 2n + 1 - i$ for $1 \leq i \leq n$. Thus, $\Lambda = f(\{v_i\}) + f(v_i) = 2n + 1$ for $1 \leq i \leq n$ and $f\left(V\left(K_n^{(1)}\right)\right) = [1, n]$, that is $K_n^{(1)}$ is super edge-magic.

(iii) Let $V = V\left(K_n^{(n-1)}\right) = \{v_1, v_2, v_3, \ldots, v_n\}$. Then, $E\left(K_n^{(n-1)}\right) = \{V - \{v_1\}, V - \{v_2\}, V - \{v_3\}, \ldots, V - \{v_n\}\}$. Define a bijection $f: V\left(K_n^{(n-1)}\right) \cup E\left(K_n^{(n-1)}\right) \to [1, 2n]$ by $f(v_i) = i$ and $f(V - \{v_i\}) = n + i$ for $1 \leq i \leq n$. Thus, $\Lambda = f(V - \{v_i\}) + \sum_{j\neq i} f(v_j) = (n + i) + \left(\frac{n(n+1)}{2} - i\right) = \frac{n(n+3)}{2}$ for $1 \leq i \leq n$ and $f\left(V\left(K_n^{(n-1)}\right)\right) = [1, n]$, that is $K_n^{(n-1)}$ is super edge-magic.

Now, the general question would be asked is "are there any other super edge-magic complete k-uniform hypergraphs?". The answer of this question is "no". We use the necessary condition together with the following lemma to verify our answer.

Lemma 3.2 Let n and k be integers such that $0 \leq k \leq n$. Then, $\binom{n}{k} \leq kn - k^2 + 1$ if and only if $k \in \{0,1\}$ or $n \in \{k, k+1\}$.

Proof. It is easy to see that if $k \in \{0,1\}$ or $n \in \{k, k+1\}$, then the inequality holds. Now, assume that $k \geq 2$. We will show that if $n \geq k + 2$, then $\binom{n}{k} > kn - k^2 + 1$ by using mathematical induction on n. First, let $n = k + 2$. Then,

$$\binom{n}{k} = \frac{k^2 + 3k}{2} + 1 > \frac{k + 3k}{2} + 1 = 2k + 1 = kn - k^2 + 1.$$

Next, let n be an integer such that $n \geq k + 2$ and $\binom{n}{k} > kn - k^2 + 1$. Since $n - k \geq 2$ and $k > 1$, we obtain

$\frac{kn-k^2+1}{n+1-k} > 1$. Observe that

$$\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k} > \left(1+\frac{k}{n+1-k}\right)(kn-k^2+1)$$

$$> kn-k^2+1+k = k(n+1)-k^2+1.$$

Therefore, by mathematical induction, $\binom{n}{k} > kn - k^2 + 1$ for every n such that $n \geq k+2$. Consequently, since $0 \leq k \leq n$, $\binom{n}{k} \leq kn - k^2 + 1$ if and only if $k \in \{0,1\}$ or $n \in \{k, k+1\}$.

Since $\left|V\left(K_n^{(k)}\right)\right| = n$ and $\left|E\left(K_n^{(k)}\right)\right| = \binom{n}{k}$, we can use Theorem 2.1 to conclude the following.

Theorem 3.3 A complete k-uniform hypergraph $K_n^{(k)}$ is super edge-magic if and only if $k \in \{0,1\}$ or $n \in \{k, k+1\}$.

4. Super Edge-Magic Complete *k*-Uniform *k*-Partite Hypergraphs

Definition 4.1 (Kuhl & Schroeder, 2013) Let n and k be positive integers such that $k \geq 2$. A complete k-uniform k-partite hypergraph with the same number of vertices in each partite, $K_{k\times n}^{(k)}$, is a hypergraph that consists of $V\left(K_{k\times n}^{(k)}\right) = V_1 \cup V_2 \cup V_3 \cup \cdots \cup V_k$, where $V_i = \{v_{i1}, v_{i2}, v_{i3}, \ldots, v_{in}\}$ and $E\left(K_{k\times n}^{(k)}\right) = \left\{\{v_{1j_1}, v_{2j_2}, v_{3j_3}, \ldots, v_{kj_k}\} \mid j_i \in \{1,2,3,\ldots,n\}\right\}$.

Notice that each partite of $K_{k\times n}^{(k)}$ has the same number of vertices. Thus, by Definition 4.1, it is easy to see that $K_{k\times n}^{(k)}$ has kn vertices and n^k hyperedges. For example, consider a complete 3-uniform 3-partite hypergraph $K_{3\times 2}^{(3)}$ with $V\left(K_{3\times 2}^{(3)}\right) = \{v_{11}, v_{12}, v_{21}, v_{22}, v_{31}, v_{32}\}$. Then,

$$E\left(K_{3\times 2}^{(3)}\right) = \{\{v_{11}, v_{21}, v_{31}\}, \{v_{11}, v_{21}, v_{32}\}, \{v_{11}, v_{22}, v_{31}\}, \{v_{11}, v_{22}, v_{32}\}, \{v_{12}, v_{21}, v_{31}\}, \{v_{12}, v_{21}, v_{32}\}, \{v_{12}, v_{22}, v_{31}\}, \{v_{12}, v_{22}, v_{32}\}\}.$$

Likewise, since the number of vertices and hyperedges of $K_{k\times n}^{(k)}$ are determined, we first find all solutions of n and k agreeing with the necessary condition. The result is proved in Lemma 4.1.

Lemma 4.1 $n^k > k^2(n-1)+1$ for $k \geq 2$ and $n \geq 4$.

Proof. Let $n = 4$. If $k = 2$, then $4^2 = 16 > 13 = 2^2(4-1)+1$. Let $l \geq 2$ be an integer and assume that $4^l > 3l^2 + 1$. Thus, $4^{l+1} > 3l^2 + 2(4^l) + 4^l > 3l^2 + 2(3l^2) + 4 > 3l^2 + 2(3l) + 4 = 3(l+1)^2 + 1$.

Next, let $m \geq 4$ be an integer and assume that $m^k > k^2(m-1)+1$ for all $k \geq 2$. Now, if $k = 2$, then $(m+1)^2 = m^2 + 2m + 1 > 4(m-1) + 1 + 2m + 1 > 4(m+1-1) + 1$. Let $s \geq 2$ be an integer and assume that $(m+1)^s > s^2m + 1$. Thus, $(m+1)^{s+1} = (m+1)(m+1)^s > (m+1)(s^2m+1) = s^2m^2 + s^2m + m + 1 > s^2m + 2sm + m + 1 = (s+1)^2m + 1$.

Therefore, by the mathematical induction, $n^k > k^2(n-1)+1$ for $k \geq 2$ and $n \geq 4$.

By Theorem 2.1 and Lemma 4.1, we obtain that $K_{k\times n}^{(k)}$ is super edge-magic implies that $k = 1$ or $n \leq 3$. Since we consider only k-uniform hypergraphs with $k \geq 2$, by direct computation, all (n, k) satisfying the necessary condition of super edge-magic are $(1, k)$, $(2, 2)$, $(2, 3)$, $(2, 4)$ and $(3, 2)$.

Lemma 4.2 (i) $K_{k\times 1}^{(k)}$ and $K_{3\times 2}^{(3)}$ are super edge-magic for all $k \geq 2$.

(ii) $K_{2\times 2}^{(2)}$, $K_{2\times 3}^{(2)}$ and $K_{4\times 2}^{(4)}$ are not super edge-magic.

Proof. (i) Let $k \geq 2$. Since $K_{k\times 1}^{(k)}$ has only one hyperedge, the conclusion follows immediately by Lemma 2.1.

Next, consider $K_{3\times 2}^{(3)}$. We define $f: V\left(K_{3\times 2}^{(3)}\right) \cup E\left(K_{3\times 2}^{(3)}\right) \to [1, 14]$ by
$f(v_{11}) = 1$, $f(v_{12}) = 5$, $f(v_{21}) = 2$,
$f(v_{22}) = 3$, $f(v_{31}) = 4$, $f(v_{32}) = 6$,
$f(\{v_{11}, v_{21}, v_{31}\}) = 14$, $f(\{v_{11}, v_{21}, v_{32}\}) = 12$,
$f(\{v_{11}, v_{22}, v_{31}\}) = 13$, $f(\{v_{11}, v_{22}, v_{32}\}) = 11$,
$f(\{v_{12}, v_{21}, v_{31}\}) = 10$, $f(\{v_{12}, v_{21}, v_{32}\}) = 8$,
$f(\{v_{12}, v_{22}, v_{31}\}) = 9$ and $f(\{v_{12}, v_{22}, v_{32}\}) = 7$.
Then, the result follows immediately with $\Lambda = 21$.

(ii) If $(n, k) \in \{(2, 2), (3, 2)\}$, then $K_{k\times n}^{(k)}$ is ordinary complete bipartite graphs $K_{2,2}$ or $K_{3,3}$. Enomoto, Llado, Nakamigawa and Ringel (1998) already justified that $K_{2,2}$ and $K_{3,3}$ are not super edge-magic.

Next, suppose that $K_{4\times 2}^{(4)}$ is super edge-magic with a labeling function f. Note that all vertex-labels are in $[1, 8]$ and all hyperedge-labels are in $[9, 24]$. Without loss of generality, we assume that $f(v_{i1}) < f(v_{i2})$ for all $i \in [1,4]$. Then, the sum of vertex-labels in hyperedges $e_1 = \{v_{11}, v_{21}, v_{31}, v_{41}\}$ and $e_2 = \{v_{12}, v_{22}, v_{32}, v_{42}\}$ attain the minimum and maximum, respectively. Thus, $K_{4\times 2}^{(4)}$ is super edge-magic implies that $f(e_1) = 24$ and $f(e_2) = 9$. Now, we have

$$2\Lambda = \left(f(e_1) + \sum_{v\in e_1} f(v)\right) + \left(f(e_2) + \sum_{v\in e_2} f(v)\right)$$
$$= (24+9) + (1+2+3+4+5+6+7+8)$$

which contradicts the fact that Λ is an integer. Therefore, $K_{4\times 2}^{(4)}$ is not super edge-magic.

Therefore, by Theorem 2.1, Lemma 4.1 and Lemma 4.2, we can conclude the following result.

Theorem 4.3 A complete k-uniform k-partite hypergraph with the same number of vertices in each partite $K_{k\times n}^{(k)}$ is super edge-magic if and only if $(n, k) = (1, k)$ for all $k \geq 2$ and $(n, k) = (2, 3)$.

5. Conclusions and Discussion

Since hypergraph is a new area, some of the definitions may not be the same in some of the literatures. Jirimutu and Wang (2001) and Boonklurb, Singhun and Termtanasombat (2015) also gave a definition of a complete k-uniform bipartite hypergraph and a complete k-uniform tripartite hypergraph and we modify their definition to be a complete k-uniform r-partite hypergraph as follows.

Definition 5.1 Let n, k and r be positive integers such that $2 \leq k \leq nr$. A complete k-uniform r-partite hypergraph with the same number of vertices in each partite, $\widehat{K}_{r\times n}^{(k)}$, is a hypergraph that consists of $V\left(\widehat{K}_{r\times n}^{(k)}\right) = V_1 \cup V_2 \cup V_3 \cup \cdots \cup V_r$, where $V_i = \{v_{i1}, v_{i2}, v_{i3}, \dots, v_{in}\}$ and $E\left(\widehat{K}_{r\times n}^{(k)}\right) = \{e | e \subseteq V\left(\widehat{K}_{r\times n}^{(k)}\right), |e| = k$ and $|e \cap V_i| < k$ for all $1 \leq i \leq r\}$.

It is easy to see that $\widehat{K}_{r\times n}^{(k)}$ has rn vertices and $\binom{rn}{k} - r\binom{n}{k}$ hyperedges. For example, consider a complete 3-uniform 2-partite hypergraph $\widehat{K}_{3\times 2}^{(3)}$ with

$$V\left(\widehat{K}_{3\times 2}^{(3)}\right) = \{v_{11}, v_{12}, v_{21}, v_{22}, v_{31}, v_{32}\}. \text{ Then,}$$

$$\begin{aligned} E\left(\widehat{K}_{3\times 2}^{(3)}\right) = \{&\{v_{11}, v_{21}, v_{31}\}, \{v_{11}, v_{21}, v_{32}\}, \\ &\{v_{11}, v_{22}, v_{31}\}, \{v_{11}, v_{22}, v_{32}\}, \\ &\{v_{12}, v_{21}, v_{31}\}, \{v_{12}, v_{21}, v_{32}\}, \\ &\{v_{12}, v_{22}, v_{31}\}, \{v_{12}, v_{22}, v_{32}\}, \\ &\{v_{11}, v_{12}, v_{21}\}, \{v_{11}, v_{12}, v_{22}\}, \\ &\{v_{11}, v_{12}, v_{31}\}, \{v_{11}, v_{12}, v_{32}\}, \\ &\{v_{21}, v_{22}, v_{11}\}, \{v_{21}, v_{22}, v_{12}\}, \\ &\{v_{21}, v_{22}, v_{31}\}, \{v_{21}, v_{22}, v_{32}\}, \\ &\{v_{31}, v_{32}, v_{11}\}, \{v_{31}, v_{32}, v_{12}\}, \\ &\{v_{31}, v_{32}, v_{21}\}, \{v_{31}, v_{32}, v_{22}\}\}. \end{aligned}$$

Thus, in the case that $r = k$, $\widehat{K}_{k\times n}^{(k)}$ defined in Definition 5.1 has more hyperedges than $K_{k\times n}^{(k)}$ defined in Definition 4.1. Therefore, in this case, the inequality in Theorem 2.1 most likely does not hold. In fact, from Lemma 4.1, we know that $k \geq 2$ and $n \geq 4$ implies $n^k > k^2(n-1) + 1$, i.e., $\widehat{K}_{k\times n}^{(k)}$ is super edge-magic implies $k = 1$ or $n \leq 3$. By direct computation, all (n, k) satisfying the necessary condition of super edge-magic and $k \geq 2$ are $(1, k)$, (2,2) and (3,2). Since $\widehat{K}_{2\times 2}^{(2)}$ and $\widehat{K}_{2\times 3}^{(2)}$ are ordinary complete bipartite graphs $K_{2,2}$ and $K_{3,3}$, respectively, we know from Enomoto, Llado, Nakamigawa and Ringel (1998) that $\widehat{K}_{2\times 2}^{(2)}$ and $\widehat{K}_{2\times 3}^{(2)}$ are not super edge-magic. However, if $n = 1$, then it is obvious that $\widehat{K}_{k\times 1}^{(k)}$ is super edge-magic. Therefore, we can conclude that $\widehat{K}_{k\times n}^{(k)}$ is super edge-magic if and only if $(n, k) = (1, k)$.

We can also notice that if the number of vertices in each partite of $K_{k\times n}^{(k)}$ or $\widehat{K}_{r\times n}^{(k)}$ are not the same, then it will be difficult to analyze the relation between the necessary condition and the number of vertices in each partite. However, one may think of some easy examples, e.g. $K_{1,k\times n}^{(k+1)}$ which has one vertex in the first partite and the rest have n vertices is super edge-magic by minor modification from the super edge-magic hypergraph $K_{k\times n}^{(k)}$.

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