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http://www.mathassociation.net Email: MathThaiOrg@gmail.com

ผลเฉลยของสมการไดโอแฟนไทน์เลขชี้กำลัง $7^x - 5^y = z^2$

The Solution of The Exponential Diophantine Equation $7^x - 5^y = z^2$

สุทธิวัฒน์ ทองนาค[1,*] วาเรียม ช่วยจันทร์[2] และ ธีรเดช เกื้อวงค์[3]
[1, 2, 3] สาขาวิชาคณิตศาสตร์และสถิติ คณะวิทยาศาสตร์ มหาวิทยาลัยทักษิณ พัทลุง 93210

Sutthiwat Thongnak[1,*] Wariam Chuayjan[2] and Theeradach Kaewong[3]
[1, 2, 3] Department of Mathematics and Statistics, Faculty of Science,
Thaksin University, Phatthalung 93210

Email: [1]tsutthiwat@tsu.ac.th [2]wariam_chuayjan@hotmail.com [3]theeradachkaewong@gmail.com

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บทคัดย่อ

ในบทความนี้ นำเสนอการพิสูจน์ว่า สมการไดโอแฟนไทน์เลขชี้กำลัง $7^x - 5^y = z^2$ มีผลเฉลยเพียงผลเฉลยเดียว เมื่อ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ ในการพิสูจน์ เราได้ใช้ข้อคาดการณ์ของคาตาลานและทฤษฎีต่าง ๆ ที่เกี่ยวข้องกับสมภาคมาช่วยในการพิสูจน์ ซึ่งพบว่า ผลเฉลยมีเพียงชุดเดียว คือ $(x, y, z) = (0, 0, 0)$

คำสำคัญ: สมการไดโอแฟนไทน์เลขชี้กำลัง ผลเฉลยจำนวนเต็ม ข้อคาดการณ์ของคาตาลาน

[*] ผู้เขียนหลัก

ABSTRACT

In this paper, we prove that the exponential Diophantine equation $7^x - 5^y = z^2$ has one solution where $x,\ y$ and z are non-negative integers. In the proof, we apply reasonably Catalan's conjecture and various theories concerning the congruence to obtain the solution. The result reveals that the unique solution is $(x, y, z) = (0, 0, 0)$.

Keywords: Exponential Diophantine equation, Integer solution, Catalan's conjecture

1. Introduction

Diophantine equations have been interesting to many mathe- maticians for a long time. Over the last decade, a number of the exponential Diophantine equations have been studied since Catalan presented Catalan's conjecture [3] and Mihailescu proved this conjecture [11]. In 2007, Acu studied the equation $2^x + 5^y = z^2$ [1]. He proved that $(3, 0, 3)$ and $(2, 1, 3)$ are the only two solutions (x, y, z) where $x,\ y$ and z are non-negative integers. In 2011, Suvarnamani et al. suggested that two equations which are $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no integer solutions [21]. In 2012, the Diophantine equation $4^x + p^y = z^2$ where $x,\ y$ and z are non-negative integers and p is any positive prime number was studied by Chotchaisthit. The study revealed that the equation does not have solutions [5]. Meanwhile, Sroysang [15 - 16] proved two Diophantine equations including $3^x + 5^y = z^2$ and $31^x + 32^y = z^2$. In the same year, Peker and Cenberci presented that the Diophantine equation $8^x + 19^y = z^2$ has no solution [12]. After that several Diophantine equations have been studied by different mathematical researchers [4, 7, 13, 14, 17, 18, 19, 20]. In 2017, Jayakumar and Shankarakalidoss proved that the Diophantine equation $47^x + 2^y = z^2$ has a unique non-negative solution $(x, y, z) = (0, 3, 3)$ [6] while Asthana and Singh showed that $(1, 0, 2)$, $(1,1,4)$, $(3, 2, 14)$ and $(5, 1, 16)$ are solutions (x, y, z) to the Diophantine equation $3^x + 13^y = z^2$ [2]. After that, Kumar et al. [8] studied two equations including $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They could show that the two equations have no solution. Recently, Laipaporn et al. proved the solution of $3^x + p5^y = z^2$ where p is a prime number. They found that the equations has infinitely many solutions [9]. While Makate et al. proved that $8^x + 61^y = z^2$ and $8^x + 67^y = z^2$ have same unique solution $(x, y, z) = (1, 0, 3)$ [10].

In this paper, we present a different Diophantine equation $7^x - 5^y = z^2$ and prove that the equation has only one trivial solution.

2. Preliminaries

Proposition 2.1 [11] (Catalan's conjecture) $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where $a,\ b,\ x$ and y are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2.2 The Diophantine equation $7^x - z^2 = 1$ has no solution (x, z) where x and z are integers and $\min\{x, z\} > 1$.

Proof. Suppose that x and z are integers where $\min\{x, z\} > 1$ such that $7^x - z^2 = 1$. By Proposition 2.1, it is sufficient to conclude that the equation has no solution. ☐

3. Main Result

Theorem 3.1 The Diophantine equation $7^x - 5^y = z^2$ has a unique solution $(x, y, z) = (0,0,0)$ where $x,\ y$ and z are non-negative integers.

Proof. Let x, y and z be non-negative integers such that

$$7^x - 5^y = z^2 \tag{1}$$

First, we consider in two cases including $x = 0$ and $x \geq 1$.

<u>Case 1</u> If $x = 0$, then from (1) we have $z^2 + 5^y = 1$. Obviously, the equation has only one solution when z and $y = 0$.

<u>Case 2</u> If $x \geq 1$, then we separate y in two subcases including $y = 0$ and $y \geq 1$.

Subcase 1 If $y = 0$, then $7^x - z^2 = 1$. By Lemma 2.2, it is sufficient to consider only $x = 1$ or $z \leq 1$. Hence, we consider x and z which are as following. For $x = 1$, we have

$z^2 = 6$. This is impossible. For $z = 0$, it follows that $7^x = 1$. It has no solution. For $z = 1$, we have 7^x =2. This is impossible.

Subcase 2 If $y \geq 1$, then z^2 is even which implies that $z^2 \equiv 0 \pmod 4$. From (1), it follows that $z^2 \equiv (-1)^x - 1 \pmod 4$. Then we have $(-1)^x - 1 \equiv 0 \pmod 4$. This implies that x is positive even, so we let $x = 2m$ where $m \in Z^+$. From (1), we have $5^y = 7^{2m} - z^2$. It is written as

$$5^y = (7^m + z)(7^m - z). \tag{2}$$

From (2), it yields (3) and (4):

$$5^\alpha = 7^m - z \tag{3}$$

and

$$5^\beta = 7^m + z \tag{4}$$

where $0 \leq \alpha < \beta \leq y$ and $\alpha + \beta = y$. From (3) and (4), we have

$$2 \cdot 7^m = 5^\alpha\left(1 + 5^{\beta-\alpha}\right). \tag{5}$$

We separate α into $\alpha = 0$ and α ≥1.

If $\alpha = 0$, then (5) becomes $2 \cdot 7^m = 1 + 5^\beta$. Thus we have $2 \equiv 1 + (-1)^\beta \pmod 3$. This implies that β is even. Let $\beta = 2t$ where $t \in Z^+$, we have

$$2 \cdot 7^m = 1 + 25^t. \tag{6}$$

From (6), we have $2 \cdot (-1)^m \equiv 1 + 1 \pmod 8$. This implies that m is even. Suppose $m = 2l$ where $l \in Z^+$. Then we have

$$2 \cdot 49^l = 1 + 25^t. \tag{7}$$

From (7), we have $2 \cdot (-1)^l \equiv 1 \pmod 5$ which is impossible.

If $\alpha \geq 1$, (5) implies that $5 \mid 2 \cdot 7^m$ which contradicts the fact that 2, 5 and 7 are relatively primes. Hence, the theorem is proved. □

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