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ไบเบสของกึ่งไฮเพอร์กรุป
On Bi-Bases of Semihypergroups

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บทคัดย่อ

จุดมุ่งหมายของบทความนี้คือการศึกษาแนวความคิดของไบเบสของกึ่งไฮเพอร์กรุป โดยแนะนำและอธิบายพัฒนาการของไบเบสของกึ่งไฮเพอร์กรุป ผลการวิจัยได้จากการขยายแนวคิดบนกึ่งกรุป
คำสำคัญ: กึ่งไฮเพอร์กรุป ไบไฮเพอร์ไอดีล ไบเบส ควอซีออเดอร์

ABSTRACT

The aim of this paper is to study the concept of bi-bases of a semihypergroup. The notions of bi-base of semihypergroups are introduced and described. The results obtained extend the results on semigroup.
Keywords: Semihypergroup, Bi-hyperideal, Bi-base, Quasi-order

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1. Introduction and Preliminaries

Hyperstructure theory was born in 1934 by a French mathematician, Marty [9]. He defined hypergroups, began to analyze their properties and applied them to groups and rational algebraic function. Many mathematicians have studied hypergroups from a theoretical perspective due to the applicability to many subjects of pure and applied mathematics. Fabrici [4] introduced the concepts of a two-sided base semigroup and Fabrici's results extended to ordered semigroups by Changpas and Summaprab [2]. In 2017, Changpas and Kummoon studied the notion of bi-base of a semigroup and bi-base of a $\Gamma-$semigroup [7 - 8]. The purpose of this paper is to introduce the concept of bi-base of a semihypergroup and extend the results in [7] to semihypergroups. Let H be a nonempty set. A mapping $\circ : H \times H \to P^*(H)$ where $P^*(H)$ denotes the family of all nonempty subsets of H. If A and B are two nonempty subsets of H, then, we denote

1. $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ for all $x \in H$,
2. $A^m = \underbrace{A \circ A \circ \cdots \circ A}_{m-1 \text{ times}}$ for all $m \in \mathbb{N}$,
3. $a^n = \underbrace{a \circ a \circ \cdots \circ a}_{n-1 \text{ times}}$ for all $n \in \mathbb{N}$ and $a \in A$.

A system $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$. A nonempty subset A of a semihypergroup H is called a *subsemihypergroup* of H if $A \circ A \subseteq A$. A subsemihypergroup A of a semihypergroup H is called a *bi-hyperideal* of H if $A \circ H \circ A \subseteq A$.

Proposition 1.1 Let H be a semihypergroup and A, B, C, D be nonempty subsets of H.

(1) If $A \subseteq B$ and $C \subseteq D$, then $A \circ C \subseteq B \circ D$.

(2) $A \circ (B \cup C) \subseteq A \circ B \cup A \circ C$ and $(B \cup C) \circ A \subseteq B \circ A \cup C \circ A$.

Proof. (1) Assume that $A \subseteq B$ and $C \subseteq D$. Let $x \in A \circ C$. Hence, $x \in a \circ c$ for some $a \in A$ and $c \in C$. Since $A \subseteq B$ and $C \subseteq D$, $x \in a \circ c$ for some $a \in B$ and $c \in D$. Hence, $x \in B \circ D$. Therefore, $A \circ C \subseteq B \circ D$.

(2) Let $x \in A \circ (B \cup C)$. Hence, $x \in s \circ t$ for some $s \in A$ and $t \in B \cup C$. There are three cases to be considered.

Case 1 $t \in B$ and $t \notin C$.

Hence, $x \in s \circ t$ for some $s \in A$ and $t \in B$. Thus, $x \in A \circ B \subseteq A \circ B \cup A \circ C$.

Case 2 $t \notin B$ and $t \in C$.

Hence, $x \in s \circ t$ for some $s \in A$ and $t \in C$. Thus, $x \in A \circ C \subseteq A \circ B \cup A \circ C$.

Case 3 $t \in B$ and $t \in C$.

Hence, $x \in s \circ t$ for some $s \in A$, $t \in B$ and $t \in C$. Thus, $x \in A \circ B \cup A \circ C$.

This implies that $A \circ (B \cup C) \subseteq A \circ B \cup A \circ C$. Similarly, $(B \cup C) \circ A \subseteq B \circ A \cup C \circ A$. □

From Proposition 1.1, if $a \in A$ and $b \in A$, then $a \circ b \subseteq A \circ A = A^2$.

Proposition 1.2 Let H be a semihypergroup and B_i be a bi-hyperideal of H for each i in an indexed set I. If $\bigcap_{i \in I} B_i \neq \emptyset$, then $\bigcap_{i \in I} B_i$ is a bi-hyperideal of H.

Proof. Assume that $A = \bigcap_{i \in I} B_i \neq \emptyset$. Let $a \in A \circ H \circ A$. We have $a \in b_1 \circ h \circ b_2$ for some $b_1, b_2 \in A$ and $h \in H$. From $b_1, b_2 \in A = \bigcap_{i \in I} B_i$, so $b_1, b_2 \in B_i$ for all $i \in I$. Since B_i is a bi-hyperideal for all $i \in I$, we have $a \in b_1 \circ h \circ b_2 \subseteq B_i$ for all $i \in I$. Thus, $a \in \bigcap_{i \in I} B_i = A$. Therefore, $A = \bigcap_{i \in I} B_i$ is a bi-hyperideal of H. □

Definition 1.3 Let A be a nonempty subset of a semihypergroup H. Then, the intersection of all bi-hyperideals of H containing A is the *smallest bi-hyperideal of* H *generated by* A and is denoted by $(A)_b$.

Proposition 1.4 Let A be a nonempty subset of a semihypergroup H. Then,

$$(A)_b = A \cup A \circ A \cup A \circ H \circ A.$$

Proof. Let $B = A \cup A \circ A \cup A \circ H \circ A.$ Consider,

$$\begin{aligned} B \circ B &= (A \cup A \circ A \cup A \circ H \circ A) \circ (A \cup A \circ A \cup A \circ H \circ A) \\ &\subseteq A \circ A \cup A \circ H \circ A \subseteq B. \end{aligned}$$

Hence, B is a subsemihypergroup of H. Consider,

$$\begin{aligned} B \circ H \circ B &= (A \cup A \circ A \cup A \circ H \circ A) \circ H \circ (A \cup A \circ A \cup A \circ H \circ A) \\ &\subseteq A \circ H \circ A \cup A \circ H \circ A^2 \cup A \circ H \circ A \circ H \circ A \cup A^2 \circ H \circ A \\ &\quad \cup A^2 \circ H \circ A^2 \cup A^2 \circ H \circ A \circ H \circ A \cup A \circ H \circ A \circ H \circ A \\ &\quad \cup A \circ H \circ A \circ H \circ A^2 \cup A \circ H \circ A \circ H \circ A \circ H \circ A \\ &\subseteq A \circ H \circ A \cup A \circ H \circ H \circ A \cup A \circ H \circ H \circ H \circ A \cup A \circ H \circ H \circ A \\ &\quad \cup A \circ H \circ H \circ H \circ A \cup A \circ H \circ H \circ H \circ H \circ A \cup A \circ H \circ H \circ H \circ A \\ &\quad \cup A \circ H \circ H \circ H \circ H \circ A \cup A \circ H \circ H \circ H \circ H \circ H \circ A \\ &\subseteq A \circ H \circ A \subseteq B. \end{aligned}$$

Therefore, B is a bi-hyperideal of H containing $A.$

Let C be a bi-hyperideal of H containing $A.$ Clearly, $A \subseteq C.$ Since C is a subsemihypergroup of $H,$ $A \circ A \subseteq C \circ C \subseteq C.$ Consider, $A \circ H \circ A \subseteq C \circ H \circ C \subseteq C.$ Thus, $B = A \cup A \circ A \cup A \circ H \circ A \subseteq C.$ Hence, B is a smallest bi-hyperideal of H containing $A.$ Therefore, $(A)_b = A \cup A \circ A \cup A \circ H \circ A$. □

Definition 1.5 Let H be a semihypergroup. A subset B of H is called a *bi-base* of H if it satisfies the following two conditions:

(1) $H = (B)_b$ (i.e., $H = B \cup B \circ B \cup B \circ H \circ B$).

(2) If A is a nonempty subset of B and $H = (A)_b,$ then, $A = B.$

Example 1.6 Let $H = \{a, b, c, d, e\}$. The hyperoperation is defined by

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
b	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
c	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a\}$	$\{a,b,c\}$
d	$\{a,b,d\}$	$\{a,b,d\}$	H	$\{a,b,d\}$	H
e	$\{a,b,d\}$	$\{a,b,d\}$	H	$\{a,b,d\}$	H

From [6], $(H,\circ)$ is a semihypergroup. Consider $B_1=\{e\}$ and $B_2=\{c,d\}$. Thus, B_1 and B_2 are bi-bases of H.

2. Main Results

In this section, we characterize bi-bases of semihypergroups and find a condition that a bi-base is a subsemihypergroup.

Lemma 2.1 Let B be a bi-base of a semihypergroup H and $a,b\in B.$
If $a\in b\circ b\cup b\circ H\circ b,$ then $a=b.$

Proof. Assume that $a\in b\circ b\cup b\circ H\circ b.$ Suppose that $a\neq b$. Consider $A=B\setminus\{a\}.$ Thus, $A\subset B.$ Since $A\subset B,$ we have $(A)_b\subseteq (B)_b=H.$ Hence, $(A)_b\subseteq H.$ From $(B)_b=H,$ so $x\in B\cup B\circ B\cup B\circ H\circ B$ for all $x\in H.$ Let $x\in H.$ There are three cases to be considered.

Case 1 $x\in B.$

Subcase 1.1 $x\neq a.$ Thus, $x\in B\setminus\{a\}=A\subseteq (A)_b.$

Subcase 1.2 $x=a.$ By assumption,

$$x=a\in b\circ b\cup b\circ H\circ b\subseteq A\circ A\cup A\circ H\circ A\subseteq (A)_b.$$

Case 2 $x\in B\circ B$. Hence, $x\in b_1\circ b_2$ for some $b_1,b_2\in B.$ There are four subcases to be considered.

Subcase 2.1 $b_1=a$ and $b_2=a.$ We have

$$\begin{aligned}
x&\in b_1\circ b_2\\
&=a\circ a\\
&\subseteq (b\circ b\cup b\circ H\circ b)\circ(b\circ b\cup b\circ H\circ b)\\
&=b^4\cup b^3\circ H\circ b\cup b\circ H\circ b^3\cup b\circ H\circ b^2\circ H\circ b\\
&\subseteq A^4\cup A^3\circ H\circ A\cup A\circ H\circ A^3\cup A\circ H\circ A^2\circ H\circ A\\
&\subseteq A\circ H^2\circ A\cup A\circ H^3\circ A\cup A\circ H^3\circ A\cup A\circ H^4\circ A\\
&\subseteq A\circ H\circ A\\
&\subseteq (A)_b.
\end{aligned}$$

Subcase 2.2 $b_1\neq a$ and $b_2=a.$ We have

$$\begin{aligned}
x &\in b_1 \circ b_2 \\
&= b_1 \circ a \\
&\subseteq (B\setminus\{a\}) \circ (b \circ b \cup b \circ H \circ b) \\
&= (B\setminus\{a\}) \circ b \circ b \cup (B\setminus\{a\}) \circ b \circ H \circ b \\
&\subseteq A^3 \cup A^2 \circ H \circ A \\
&\subseteq A \circ H \circ A \cup A \circ H^2 \circ A \\
&\subseteq A \circ H \circ A \\
&\subseteq (A)_b.
\end{aligned}$$

Subcase 2.3 $b_1 = a$ and $b_2 \neq a.$ We have

$$\begin{aligned}
x &\in b_1 \circ b_2 \\
&= a \circ b_2 \\
&\subseteq (b \circ b \cup b \circ H \circ b) \circ (B\setminus\{a\}) \\
&= b \circ b \circ (B\setminus\{a\}) \cup b \circ H \circ b \circ (B\setminus\{a\}) \\
&\subseteq A^3 \cup A \circ H \circ A^2 \\
&\subseteq A \circ H \circ A \cup A \circ H^2 \circ A \\
&\subseteq A \circ H \circ A \\
&\subseteq (A)_b.
\end{aligned}$$

Subcase 2.4 $b_1 \neq a$ and $b_2 \neq a.$ By assumption, $A = B\setminus\{a\}.$ We have

$$\begin{aligned}
x &\in b_1 \circ b_2 \\
&\subseteq (B\setminus\{a\}) \circ (B\setminus\{a\}) \\
&= A \circ A \\
&\subseteq (A)_b.
\end{aligned}$$

<u>Case 3</u> $x \in B \circ H \circ B.$ Hence, $x \in b_3 \circ h \circ b_4$ for some $b_3, b_4 \in B$ and $h \in H.$ There are four subcases to be considered.

Subcase 3.1 $b_3 = a$ and $b_4 = a.$ We have

$$\begin{aligned}
x &\in b_3 \circ h \circ b_4 \\
&= a \circ h \circ a \\
&\subseteq (b \circ b \cup b \circ H \circ b) \circ H \circ (b \circ b \cup b \circ H \circ b) \\
&= b \circ b \circ H \circ b \circ b \cup b \circ b \circ H \circ b \circ H \circ b \cup b \circ H \circ b \circ H \circ b \circ b \\
&\quad \cup b \circ H \circ b \circ H \circ b \circ H \circ b
\end{aligned}$$

$$\subseteq A\circ A\circ H\circ A\circ A\cup A\circ A\circ H\circ A\circ H\circ A\cup A\circ H\circ A\circ H\circ A\circ A$$
$$\cup A\circ H\circ A\circ H\circ A\circ H\circ A$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 3.2 $b_3\neq a$ and $b_4=a.$ We have

$$x\in b_3\circ h\circ b_4$$
$$=b_3\circ h\circ a$$
$$\subseteq (B\setminus\{a\})\circ H\circ (b\circ b\cup b\circ H\circ b)$$
$$=(B\setminus\{a\})\circ H\circ b\circ b\cup (B\setminus\{a\})\circ H\circ b\circ H\circ b$$
$$\subseteq A\circ H\circ A\circ A\cup A\circ H\circ A\circ H\circ A$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 3.3 $b_3=a$ and $b_4\neq a.$ We have

$$x\in b_3\circ h\circ b_4$$
$$=a\circ h\circ b_4$$
$$\subseteq (b\circ b\cup b\circ H\circ b)\circ H\circ (B\setminus\{a\})$$
$$=b\circ b\circ H\circ (B\setminus\{a\})\cup b\circ H\circ b\circ H\circ (B\setminus\{a\})$$
$$\subseteq A\circ A\circ H\circ A\cup A\circ H\circ A\circ H\circ A$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 3.4 $b_3\neq a$ and $b_4\neq a.$ By assumption, $A=B\setminus\{a\}.$ We have

$$x\in b_3\circ h\circ b_4$$
$$\subseteq (B\setminus\{a\})\circ H\circ (B\setminus\{a\})$$
$$= A\circ H\circ A$$
$$\subseteq (A)_b.$$

This implies that $(A)_b=H.$ This is a contradiction. Therefore, $a=b.$ □

Lemma 2.2 Let B be a bi-base of a semihypergroup H and $a,b,c\in B.$
If $a\in c\circ b\cup c\circ H\circ b,$ then $a=b$ or $a=c.$

Proof. Assume that $a\in c\circ b\cup c\circ H\circ b.$ Suppose that $a\neq b$ and $a\neq c.$
Consider $A=B\setminus\{a\},$ we have $A\subset B.$ Since $a\neq b$ and $a\neq c,$ we have $b,c\in A.$

Since $A \subset B,$ we have $(A)_b \subseteq (B)_b = H.$ Hence, $(A)_b \subseteq H.$ Since $(B)_b = H,$ we have $x \in B \cup B \circ B \cup B \circ H \circ B$ for all $x \in H.$ Let $x \in H.$ There are three cases to be considered.

<u>Case 1</u> $x \in B.$

Subcase 1.1 $x \neq a.$ Thus, $x \in B \setminus \{a\} = A \subseteq (A)_b.$

Subcase 1.2 $x = a.$ By assumption,

$$x = a \in c \circ b \cup c \circ H \circ b \subseteq A \circ A \cup A \circ H \circ A \subseteq (A)_b.$$

<u>Case 2</u> $x \in B \circ B.$ Hence, $x \in b_1 \circ b_2$ for some $b_1, b_2 \in B.$ There are four subcases to be considered.

Subcase 2.1 $b_1 = a$ and $b_2 = a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= a \circ a \\ &\subseteq (c \circ b \cup c \circ H \circ b) \circ (c \circ b \cup c \circ H \circ b) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2 $b_1 \neq a$ and $b_2 = a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= b_1 \circ a \\ &\subseteq (B \setminus \{a\}) \circ (c \circ b \cup c \circ H \circ b) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.3 $b_1 = a$ and $b_2 \neq a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= a \circ b_2 \\ &\subseteq (c \circ b \cup c \circ H \circ b) \circ (B \setminus \{a\}) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.4 $b_1 \neq a$ and $b_2 \neq a.$ By assumption, $A = B \setminus \{a\}.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &\subseteq (B \setminus \{a\}) \circ (B \setminus \{a\}) \\ &= A \circ A \\ &\subseteq (A)_b. \end{aligned}$$

<u>Case 3</u> $x \in B \circ H \circ B$. Hence, $x \in b_3 \circ h \circ b_4$ for some $b_3, b_4 \in B$ and $h \in H.$ There are four subcases to be considered.

Subcase 3.1 $b_3 = a$ and $b_4 = a.$ We have

$$\begin{aligned} x &\in b_3 \circ h \circ b_4 \\ &= a \circ h \circ a \\ &\subseteq (c \circ b \cup c \circ H \circ b) \circ H \circ (c \circ b \cup c \circ H \circ b) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2 $b_3 \neq a$ and $b_4 = a.$ We have

$$\begin{aligned} x &\in b_3 \circ h \circ b_4 \\ &= b_3 \circ h \circ a \\ &\subseteq (B \setminus \{a\}) \circ H \circ (c \circ b \cup c \circ H \circ b) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.3 $b_3 = a$ and $b_4 \neq a.$ We have

$$\begin{aligned} x &\in b_3 \circ h \circ b_4 \\ &= a \circ h \circ b_4 \\ &\subseteq (c \circ b \cup c \circ H \circ b) \circ H \circ (B \setminus \{a\}) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.4 $b_3 \neq a$ and $b_4 \neq a.$ By assumption, $A = B \setminus \{a\}.$ We have

$$\begin{aligned} x &\in b_3 \circ h \circ b_4 \\ &\subseteq (B \setminus \{a\}) \circ H \circ (B \setminus \{a\}) \\ &= A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

This implies that $(A)_b = H.$ This is a contradiction. Therefore, $a = b$ or $a = c.$ ☐

Definition 2.3 Let H be a semihypergroup. For any $a,b \in H,$ define a *quasi-order* on H by

$$a \leq_b b \text{ if and only if } (a)_b \subseteq (b)_b.$$

From Definition 2.3, $a \not\leq_b b$ if and only if $(a)_b \not\subset (b)_b.$ The following example shows that the relation $\leq_b$ defined above is not a partial order.

Example 2.4 Let $H = \{a,b,c,d\}.$ The hyperoperation is defined by

$\circ$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{a,c\}$	$\{b,c\}$	$\{d\}$
c	$\{c\}$	$\{b,c\}$	$\{a,b\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	H

From [5], $(H,\circ)$ is a semihypergroup. We have that the singleton sets consisting of an element of $H.$

Consider $(a)_b = a \cup a \circ a \cup a \circ H \circ a = H$ and $(b)_b = b \cup b \circ b \cup b \circ H \circ b = H.$ We have $(a)_b \subseteq (b)_b$ and $(b)_b \subseteq (a)_b.$ Hence, $a \leq_b b$ and $b \leq_b a.$ But $a \neq b.$ Therefore, $\leq_b$ is not a partial order on $H.$

Lemma 2.5 Let B be a bi-base of a semihypergroup $H.$ If $a,b \in B$ such that $b \neq a,$ then neither $a \leq_b b$ nor $b \leq_b a.$

Proof. Assume that $a,b \in B$ such that $a \neq b$.

Case 1 $a \leq_b b.$ Thus, $(a)_b \subseteq (b)_b.$ Consider $a \in (a)_b \subseteq (b)_b = \{b\} \cup b \circ b \cup b \circ H \circ b.$ Since $a \neq b,$ $a \in b \circ b \cup b \circ H \circ b.$ By Lemma 2.1, $a = b.$ This is a contradiction.

Case 2 $b \leq_b a.$ This can be proved similarly. □

Lemma 2.6 Let B be a bi-base of a semihypergroup $H.$ For all $a,b,c \in B$ and $h \in H,$

(1) if $a \in b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c,$ then $a = b$ or $a = c;$

(2) if $a \in b \circ h \circ c \cup (b \circ h \circ c)^2 \cup b \circ h \circ c \circ H \circ b \circ h \circ c,$ then $a = b$ or $a = c.$

Proof. (1) Assume that $a \in b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c.$ Suppose that $a \neq b$ and $a \neq c.$ Consider $A = B \setminus \{a\},$ we have $A \subset B.$ Since $a \neq b$ and $a \neq c,$ $b, c \in A.$ Since $A \subset B,$ we have $(A)_b \subseteq (B)_b = H.$ Hence, $(A)_b \subseteq H.$ Since $(B)_b = H,$ $x \in B \cup B \circ B \cup B \circ H \circ B$ for all $x \in H.$ Let $x \in H.$ There are three cases to be considered.

Case 1 $x \in B.$

Subcase 1.1 $x \neq a$. Thus, $x \in B \setminus \{a\} = A \subseteq (A)_b.$

Subcase 1.2 $x = a.$ By assumption,

$$\begin{aligned} x = a &\in b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c \\ &\subseteq A \circ A \cup A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Case 2 $x \in B \circ B.$ Hence, $x \in b_1 \circ b_2$ for some $b_1, b_2 \in B.$ There are four subcases to be considered.

Subcase 2.1 $b_1 = a$ and $b_2 = a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= a \circ a \\ &\subseteq \left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right) \circ \left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2 $b_1 \neq a$ and $b_2 = a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= b_1 \circ a \\ &\subseteq (B \setminus \{a\}) \circ (b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.3 $b_1 = a$ and $b_2 \neq a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= a \circ b_2 \\ &\subseteq \left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right) \circ (B \setminus \{a\}) \end{aligned}$$

$$\subseteq A \circ H \circ A$$
$$\subseteq (A)_b.$$

Subcase 2.4 $b_1 \neq a$ and $b_2 \neq a.$ By assumption, $A = B \setminus \{a\}.$ We have

$$x \in b_1 \circ b_2$$
$$\subseteq (B \setminus \{a\}) \circ (B \setminus \{a\})$$
$$\subseteq A \circ A$$
$$\subseteq (A)_b.$$

Case 3 $x \in B \circ H \circ B.$ Hence, $x \in b_3 \circ h \circ b_4$ for some $b_3, b_4 \in B$ and $k \in H.$ There are four subcases to be considered.

Subcase 3.1 $b_3 = a$ and $b_4 = a.$ We have

$$x \in b_3 \circ k \circ b_4$$
$$= a \circ k \circ a$$
$$\subseteq \left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right) \circ H \circ$$
$$\left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right)$$
$$\subseteq A \circ H \circ A$$
$$\subseteq (A)_b.$$

Subcase 3.2 $b_3 \neq a$ and $b_4 = a.$ We have

$$x \in b_3 \circ k \circ b_4$$
$$= b_3 \circ k \circ a$$
$$\subseteq (B \setminus \{a\}) \circ H \circ \left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right)$$
$$\subseteq A \circ H \circ A$$
$$\subseteq (A)_b.$$

Subcase 3.3 $b_3 = a$ and $b_4 \neq a.$ We have

$$x \in b_3 \circ k \circ b_4$$
$$= a \circ k \circ b_4$$
$$\subseteq \left(b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c\right) \circ H \circ (B \setminus \{a\})$$
$$\subseteq A \circ H \circ A$$
$$\subseteq (A)_b.$$

Subcase 3.4 $b_3 \neq a$ and $b_4 \neq a.$ By assumption, $A = B \setminus \{a\}.$ We have

$$\begin{aligned} x &\in b_3 \circ k \circ b_4 \\ &\subseteq (B \setminus \{a\}) \circ H \circ (B \setminus \{a\}) \\ &= A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

This implies $(A)_b = H.$ This is a contradiction. Therefore, $a = b$ or $a = c.$

(2) Assume that $a \in b \circ h \circ c \cup (b \circ h \circ c)^2 \cup b \circ h \circ c \circ H \circ b \circ h \circ c.$ Suppose that $a \neq b$ and $a \neq c.$ Consider $A = B \setminus \{a\}.$ We have $A \subset B.$ Since $a \neq b$ and $a \neq c,$ $b, c \in A.$ Since $A \subset B,$ we have $(A)_b \subseteq (B)_b = H.$ Hence, $(A)_b \subseteq H.$ Since $(B)_b = H,$ $x \in B \cup B \circ B \cup B \circ H \circ B$ for all $x \in H.$ Let $x \in H.$ There are three cases to be considered.

Case 1 $x \in B$.

Subcase 1.1 $x \neq a.$ Thus, $x \in B \setminus \{a\} = A \subseteq (A)_b.$

Subcase 1.2 $x = a.$ By assumption,

$$\begin{aligned} x = a &\in b \circ h \circ c \cup (b \circ h \circ c)^2 \cup b \circ h \circ c \circ H \circ b \circ h \circ c \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Case 2 $x \in B \circ B.$ Hence, $x \in b_1 \circ b_2$ for some $b_1, b_2 \in B.$ There are four subcases to be considered.

Subcase 2.1 $b_1 = a$ and $b_2 = a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= a \circ a \\ &\subseteq \left(b \circ h \circ c \cup (b \circ h \circ c)^2 \cup b \circ h \circ c \circ H \circ b \circ h \circ c\right) \\ &\quad \circ \left(b \circ h \circ c \cup (b \circ h \circ c)^2 \cup b \circ h \circ c \circ H \circ b \circ h \circ c\right) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2 $b_1 \neq a$ and $b_2 = a.$ We have

$$\begin{aligned} x &\in b_1 \circ b_2 \\ &= b_1 \circ a \end{aligned}$$

$$\subseteq (B\setminus\{a\})\circ\big(b\circ h\circ c\cup(b\circ h\circ c)^2\cup b\circ h\circ c\circ H\circ b\circ h\circ c\big)$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 2.3 $b_1 = a$ and $b_2 \neq a.$ We have

$$x\in b_1\circ b_2$$
$$= a\circ b_2$$
$$\subseteq \big(b\circ h\circ c\cup(b\circ h\circ c)^2\cup b\circ h\circ c\circ H\circ b\circ h\circ c\big)\circ(B\setminus\{a\})$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 2.4 $b_1 \neq a$ and $b_2 \neq a.$ By assumption, $A = B\setminus\{a\}.$ We have

$$x\in b_1\circ b_2$$
$$\subseteq (B\setminus\{a\})\circ(B\setminus\{a\})$$
$$= A\circ A$$
$$\subseteq (A)_b.$$

Case 3 $x\in B\circ H\circ B.$ Hence, $x\in b_3\circ k\circ b_4$ for some $b_3, b_4\in B$ and $k\in H.$ There are four subcases to be considered.

Subcase 3.1 $b_3 = a$ and $b_4 = a.$ We have

$$x\in b_3\circ k\circ b_4$$
$$= a\circ k\circ a$$
$$\subseteq (b\circ h\circ c\cup(b\circ h\circ c)^2\cup b\circ h\circ c\circ H\circ b\circ h\circ c)$$
$$\circ H\circ(b\circ h\circ c\cup(b\circ h\circ c)^2\cup b\circ h\circ c\circ H\circ b\circ h\circ c)$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 3.2 $b_3 \neq a$ and $b_4 = a.$ We have

$$x\in b_3\circ k\circ b_4$$
$$= b_3\circ k\circ a$$
$$\subseteq (B\setminus\{a\})\circ H\circ\big(b\circ h\circ c\cup(b\circ h\circ c)^2\cup b\circ h\circ c\circ H\circ b\circ h\circ c\big)$$
$$\subseteq A\circ H\circ A$$
$$\subseteq (A)_b.$$

Subcase 3.3 $b_3 = a$ and $b_4 \neq a.$ We have

$$\begin{aligned} x &\in b_3 \circ k \circ b_4 \\ &= a \circ k \circ b_4 \\ &\subseteq \left(b \circ h \circ c \cup (b \circ h \circ c)^2 \cup b \circ h \circ c \circ H \circ b \circ h \circ c\right) \circ H \circ (B \setminus \{a\}) \\ &\subseteq A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.4 $b_3 \neq a$ and $b_4 \neq a.$ By assumption, $A = B \setminus \{a\}.$ We have

$$\begin{aligned} x &\in b_3 \circ k \circ b_4 \\ &\subseteq (B \setminus \{a\}) \circ H \circ (B \setminus \{a\}) \\ &= A \circ H \circ A \\ &\subseteq (A)_b. \end{aligned}$$

This implies $(A)_b = H.$ This is a contradiction. Therefore, $a = b$ or $a = c.$ □

Lemma 2.7 Let B be a bi-base of a semihypergroup $H.$

(1) For any $a, b, c \in B,$ if $a \neq b$ and $a \neq c,$ then $a \not\leq_b b \circ c.$

(2) For any $a, b, c \in B$ and $h \in H,$ if $a \neq b$ and $a \neq c,$ then $a \not\leq_b b \circ h \circ c.$

Proof. Let B be a bi-base of a semihypergroup H and $a, b, c \in B,$ $h \in H.$

(1) Suppose that $a \leq_b b \circ c.$ Thus, $(a)_b \subseteq (b \circ c)_b.$

We have $a \in (a)_b \subseteq (b \circ c)_b = b \circ c \cup (b \circ c)^2 \cup b \circ c \circ H \circ b \circ c.$

By Lemma 2.6 (1), we have $a = b$ or $a = c.$

(2) Suppose that $a \leq_b b \circ h \circ c.$ Thus, $(a)_b \subseteq (b \circ h \circ c)_b.$

We have $a \in (a)_b \subseteq (b \circ h \circ c)_b = b \circ h \circ c \cup b \circ h \circ c \circ b \circ h \circ c \cup b \circ h \circ c \circ H \circ b \circ h \circ c.$

By Lemma 2.6 (2), we have $a = b$ or $a = c.$ □

Theorem 2.8 A nonempty subset B of a semihypergroup H is a bi-base of H if and only if B satisfies the following conditions:

(1) For any $x \in H,$

(1.1) there exists $b \in B$ such that $x \leq_b b,$ or

(1.2) there exist $b_1, b_2 \in B$ such that $x \leq_b b_1 \circ b_2,$ or

(1.3) there exist $b_3, b_4 \in B$ and $h \in H$ such that $x \leq_b b_3 \circ h \circ b_4.$

(2) For any $a, b, c \in B,$ if $a \neq b$ and $a \neq c,$ then $a \not\leq_b b \circ c.$

(3) For any $a,b,c \in B$ and $h \in H,$ if $a \neq b$ and $a \neq c,$ then $a \not\leq_b b \circ h \circ c.$

Proof. Let B be a nonempty subset of a semihypergroup $H.$ Assume that B is a bi-base of $H.$ Therefore, $H = (B)_b.$ Suppose that $x \in H,$ so $x \in B \cup B \circ B \cup B \circ H \circ B.$ There are three cases to be considered.

<u>Case 1</u> $x \in B.$ Thus, $x = b$ for some $b \in B.$ This implies that $(x)_b \subseteq (b)_b.$ Hence, $x \leq_b b.$

<u>Case 2</u> $x \in B \circ B.$ Thus, $x \in b_1 \circ b_2$ for some $b_1, b_2 \in B.$ This implies that $(x)_b \subseteq (b_1 \circ b_2)_b.$ Hence, $x \leq_b b_1 \circ b_2.$

<u>Case 3</u> $x \in B \circ H \circ B.$ Thus, $x \in b_3 \circ h \circ b_4$ for some $b_3, b_4 \in B$ and $h \in H.$
This implies $(x)_b \subseteq (b_3 \circ h \circ b_4)_b.$ Hence, $x \leq_b b_3 \circ h \circ b_4.$
The validity of (2) and (3) follows from Lemma 2.7 (1) and Lemma 2.7 (2), respectively.

Conversely, assume that B satisfies (1), (2) and (3). We show that B is a bi-base of $H.$ Clearly, $(B)_b \subseteq H.$ Let $x \in H.$ From (1.1), it follows that $x \in (x)_b \subseteq (b)_b \subseteq (B)_b$ for some $b \in B.$ From (1.2), it follows that

$$\begin{aligned} x &\in (x)_b \\ &\subseteq (b_1 \circ b_2)_b \\ &= b_1 \circ b_2 \cup b_1 \circ b_2 \circ b_1 \circ b_2 \cup b_1 \circ b_2 \circ H \circ b_1 \circ b_2 \\ &\subseteq b_1 \circ b_2 \cup b_1 \circ H \circ b_2 \\ &\subseteq B \circ B \cup B \circ H \circ B \subseteq (B)_b \end{aligned}$$

for some $b_1, b_2 \in B.$ From (1.3), it follows that

$$\begin{aligned} x &\in (x)_b \\ &\subseteq (b_3 \circ h \circ b_4)_b \\ &= b_3 \circ h \circ b_4 \cup b_3 \circ h \circ b_4 \circ b_3 \circ h \circ b_4 \cup b_3 \circ h \circ b_4 \circ H \circ b_3 \circ h \circ b_4 \\ &\subseteq b_3 \circ H \circ b_4 \\ &\subseteq B \circ H \circ B \\ &\subseteq (B)_b \end{aligned}$$

for some $b_3, b_4 \in B$ and $h \in H.$ It remains to show that B is a minimal subset of H with the property $H = (B)_b.$ Assume that $H = (A)_b$ for some $A \subset B.$ There exists

$b \in B \setminus A.$ Since $b \in B \subseteq H = (A)_b,$ we have $b \in (A)_b.$ Thus, $b \in A \cup A \circ A \cup A \circ H \circ A.$ Since $b \notin A,$ $b \in A \circ A \cup A \circ H \circ A.$ There are two cases to be considered.

Case 1 $b \in A \circ A.$ Thus, $b \in a_1 \circ a_2$ for some $a_1, a_2 \in A.$ Since $b \notin A,$ $b \neq a_1$ and $b \neq a_2.$ Thus, $(b)_b \subseteq (a_1 \circ a_2)_b.$ Hence, $b \leq a_1 \circ a_2.$ This contradicts (2).

Case 2 $b \in A \circ H \circ A.$ Thus, $b \in a_3 \circ h \circ a_4$ for some $a_3, a_4 \in A$ and $h \in H.$ Since $b \notin A,$ $b \neq a_3$ and $b \neq a_4.$ Thus, $(b)_b \subseteq (a_3 \circ h \circ a_4)_b.$ Hence, $b \leq a_3 \circ h \circ a_4.$ This contradicts (3).

Therefore, B is a bi-base of H and the proof is completed. □

In Example 1.6, we have that $\{e\}$ is a bi-base of $H.$ But $\{e\}$ is not a subsemihypergroup of $H.$ So, we find a condition that a bi-base is a subsemihypergroup.

Theorem 2.9 Let B be a bi-base of a semihypergroup $H.$

Then, B is a subsemihypergroup of H if and only if B satisfies the conditions $b \in b \circ c$ or $c \in b \circ c$ for any $b, c \in B.$

Proof. Assume that B is a subsemihypergroup of $H.$ Let $b, c \in B.$

Suppose that $b \notin b \circ c$ and $c \notin b \circ c.$ Let $a \in b \circ c.$ Thus, $a \neq b$ and $a \neq c.$

Since $a \in b \circ c \subseteq b \circ c \cup b \circ c \circ b \circ c \circ \cup b \circ c \circ H \circ b \circ c$ and by Lemma 2.6 (1), we have $a = b$ or $a = c.$ This is a contradiction.

Conversely, assume that $b \in b \circ c$ or $c \in b \circ c$ for any $b, c \in B.$ Let $a \in B \circ B.$ Thus, $a \in b \circ c$ for some $b, c \in B.$ Since $a \in b \circ c \cup b \circ c \circ b \circ c \cup b \circ c \circ H \circ b \circ c$ and by Lemma 2.6 (1), $a = b$ or $a = c.$ Hence, $a \in \{b, c\} \subseteq B.$ Therefore, B is a subsemihypergroup of $H.$ □

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References

[1] Anvariyeh, S. M., Mirvakili, S., and Davvaz, B. (2010). On $\Gamma-$hyperideals in $\Gamma-$Semihypergroups, *Carpathian Journal of Mathematics*, 26 (1), p. 11 – 23.

[2] Changphas, T., and Summaprab, P. (2014). On Two-Sided Bases of An Ordered Semigroup, *Quasigroups and Related Systems*, 22, p. 59 – 66.

[3] Fabrici, I. (1972). One-Sided Bases of Semigroups, *Matematický Casopis*, 22 (4), p. 286 – 290.

[4] Fabrici, I. (1975). Two-Sided Bases of Semigroups, *Matematický Casopis*, 25 (2), p. 173 – 178.

[5] Jafarabadi, H. M., Sarmin, H., and Molaei, M. R. (2012). Completely Simple and Regular Semi Hypergroups, *Bulletin of the Malaysian Mathematical Science Society*, 35 (2), p. 335 – 343.

[6] Kehayopulu, N. (2018). On Ordered Hypersemigroups Given by A Table of Multiplication and A Figure, *Turkish Journal of Mathematics*, 42, p. 2045 - 2060.

[7] Kummoon, P., and Changphas, T. (2017). On Bi-Bases of A Semigroup, *Quasigroups and Related Systems*, 25, p. 87 – 94.

[8] Kummoon, P., and Changphas, T. (2017). On Bi-Bases of $\Gamma-$Semigroup. *Thai Journal of Mathematics*, *Special issue: Annual Meeting in Mathematics 2017*, p. 75 – 86.

[9] Marty, F. (1934). Sur une Generalization de la Notion de Groupe, *the 8th Congress Math. Scandenave*. Stockholm.

[10] Tamura, T. (1955). One-Sided Bases and Translations of A Semigroup, *Mathematica Japonica,* 3, p. 137 – 141.