



วารสารคณิตศาสตร์ **Mathematical Journal** 66(703) มกราคม – เมษายน 2564

โดย สมาคมคณิตศาสตร์แห่งประเทศไทย ในพระบรมราชูปถัมภ์

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ลำดับโมเบียส

Mobius Sequences

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วันที่รับบทความ : 10 มกราคม 2563

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วันที่ตอบรับบทความ : 21 สิงหาคม 2563

บทคัดย่อ

ในงานวิจัยเรื่องนี้ผู้วิจัยหารูปแบบปิดของลำดับโมเบียส ซึ่งนิยามโดย

$$z_{n+1} = \frac{az_n + b}{cz_n + d}$$

เมื่อ a, b, c และ d เป็นจำนวนจริง โดยที่ $ad - bc \neq 0$

คำสำคัญ: โมเบียส ลำดับ

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ABSTRACT

In this research, we obtain a new derivation of the closed-form solution of a Mobius sequence defined by

$$z_{n+1} = \frac{az_n + b}{cz_n + d}$$

where a, b, c and d are real numbers with $ad - bc \neq 0$.

Keywords: Mobius, Sequence

1. Introduction

A Mobius sequence is a sequence defined by using iterated applications of a single Mobius transformation to an initial point on the real line (or more generally, on the extended complex plane). It is a common sequence appeared in many literatures yet a complete set of solutions are still rare to find. Many authors studied Mobius sequences in terms of their dynamical properties [1, 2, 5]. It is also applied in the study of continued fractions [3]. The complete set of solutions, in nontrivial cases, can be found in [4]. Liebeck [4] obtained the solution to the Mobius sequence by assuming that the sequence must converge to some complex number and constructing an *auxiliary equation* from this recurrence relation. He also explained the behavior of the solution in each case. See Theorem 2R in [4] for more details.

The aim of this paper is to collect, with proof different from [4], the closed-form solution of Mobius sequences defined on the real line. The novelty of our method is to define another *auxiliary sequence* which later becomes a homogeneous linear recurrence relation of degree 2 and can be easily solved. Our main result is summarized in Theorem 1 below.

2. Mobius Sequences and Their Closed-Form

A Mobius sequence is a sequence of real numbers $\{z_n\}$, with initial value z_1 , defined for any $n \in \mathbb{N}$ by the recurrence relation

$$z_{n+1} = \frac{az_n + b}{cz_n + d} \quad (1)$$

where a, b, c and d are real numbers with $ad - bc \neq 0$. The closed-form solution of a Mobius sequence is summarized in the following theorem.

Theorem 1 Let $\{z_n\}$ be a Mobius sequence defined recursively by (1).

(1) If $c = 0$ and $a = d$, then $z_n = z_1 + (n-1)\frac{b}{d}$ for all $n \in \mathbb{N}$.

(2) If $c = 0$ and $a \neq d$, then

$$z_n = \left(\frac{a}{d}\right)^{n-1} z_1 + \frac{b}{d^{n-1}} \left(\frac{a^{n-1} - d^{n-1}}{a-d}\right)$$

for all $n \in \mathbb{N}$.

(3) If $c \neq 0$ and $(a-d)^2 + 4bc > 0$, then

$$z_n = \frac{\left(\frac{\alpha-d}{c}\right)(\beta-d-cz_1)\alpha^{n-1} + \left(\frac{\beta-d}{c}\right)(cz_1+d-\alpha)\beta^{n-1}}{(\beta-d-cz_1)\alpha^{n-1} + (cz_1+d-\alpha)\beta^{n-1}}$$

for all $n \in \mathbb{N}$, where

$$\alpha = \frac{a+d - \sqrt{(a-d)^2 + 4bc}}{2} \quad \text{and} \quad \beta = \frac{a+d + \sqrt{(a-d)^2 + 4bc}}{2}.$$

(4) If $c \neq 0$ and $(a-d)^2 + 4bc = 0$, then

$$z_n = \frac{1}{2c} \left[\frac{(a-d)(2cz_1+d-a)n + (a-d)^2 + 4cdz_1}{(2cz_1+d-a)n + 2(a-cz_1)} \right]$$

for all $n \in \mathbb{N}$.

(5) If $c \neq 0$ and $(a-d)^2 + 4bc < 0$, then

$$z_n = \frac{r}{c} \left[\frac{(cz_1+d)\sin n\theta - r\sin(n-1)\theta}{(cz_1+d)\sin(n-1)\theta - r\sin(n-2)\theta} \right] - \frac{d}{c}$$

for all $n \in \mathbb{N}$, where $r = \sqrt{ad-bc}$ and $\theta = \cos^{-1}\left(\frac{a+d}{2\sqrt{ad-bc}}\right)$.

Proof. We consider the following cases:

Case 1 $c = 0$. Then $ad \neq 0$ and (1) reduces to

$$z_{n+1} = \frac{a}{d} z_n + \frac{b}{d}. \quad (2)$$

From (2), we get

$$z_n = \frac{a}{d} z_{n-1} + \frac{b}{d}. \quad (3)$$

Substituting (3) in (2) and continuing in the same manner, we have

$$\begin{aligned} z_{n+1} &= \frac{a}{d} \left(\frac{a}{d} z_{n-1} + \frac{b}{d} \right) + \frac{b}{d} \\ &= \left(\frac{a}{d} \right)^2 z_{n-1} + \frac{b}{d} \left(\frac{a}{d} + 1 \right) \\ &= \left(\frac{a}{d} \right)^2 \left(\frac{a}{d} z_{n-2} + \frac{b}{d} \right) + \frac{b}{d} \left(\frac{a}{d} + 1 \right) \\ &= \left(\frac{a}{d} \right)^3 z_{n-2} + \frac{b}{d} \left(\left(\frac{a}{d} \right)^2 + \left(\frac{a}{d} \right) + 1 \right) \\ &\quad \vdots \\ &= \left(\frac{a}{d} \right)^n z_1 + \frac{b}{d} \left(\left(\frac{a}{d} \right)^{n-1} + \left(\frac{a}{d} \right)^{n-2} + \dots + \left(\frac{a}{d} \right) + 1 \right). \end{aligned}$$

Thus, we have the following results.

(1) If $c = 0$ and $a = d$ then $z_{n+1} = z_1 + \frac{bn}{d}$. Hence $z_n = z_1 + \frac{b}{d}(n-1)$ for all

$n \in \mathbb{N}$.

(2) If $c = 0$ and $a \neq d$ then $z_{n+1} = \left(\frac{a}{d} \right)^n z_1 + \frac{b}{d} \left(\frac{\left(\frac{a}{d} \right)^n - 1}{\frac{a}{d} - 1} \right)$. Hence

$$z_n = \left(\frac{a}{d} \right)^{n-1} z_1 + \frac{b}{d^{n-1}} \left(\frac{a^{n-1} - d^{n-1}}{a - d} \right) \text{ for all } n \in \mathbb{N}.$$

Case 2 $c \neq 0$. Define a sequence $\{x_n\}$ by

$$x_{n+1} = (cz_n + d)x_n. \quad (4)$$

Since $cz_n + d \neq 0$, $x_n \neq 0$ for all $n \in \mathbb{N}$. Then we have

$$z_n = \frac{x_{n+1}}{cx_n} - \frac{d}{c}. \quad (5)$$

Substituting (5) in (1), we get

$$\begin{aligned} \frac{x_{n+2}}{cx_{n+1}} - \frac{d}{c} &= \frac{a \left(\frac{x_{n+1}}{cx_n} - \frac{d}{c} \right) + b}{c \left(\frac{x_{n+1}}{cx_n} - \frac{d}{c} \right) + d} \\ \frac{x_{n+2} - dx_{n+1}}{cx_{n+1}} &= \frac{ax_{n+1} - adx_n + bcx_n}{cx_{n+1}} \\ x_{n+2} - (a+d)x_{n+1} + (ad-bc)x_n &= 0. \end{aligned} \quad (6)$$

The auxiliary equation of (6) is

$$x^2 - (a+d)x + (ad-bc) = 0.$$

Solving for x , we get

$$x = \frac{a+d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

$$\text{Let } \alpha = \frac{a+d - \sqrt{(a-d)^2 + 4bc}}{2} \text{ and } \beta = \frac{a+d + \sqrt{(a-d)^2 + 4bc}}{2}.$$

We consider 3 subcases.

Case 2.1 $(a-d)^2 + 4bc > 0$. Since $ad-bc \neq 0$, we have $\alpha, \beta \neq 0$. Also, $\alpha \neq \beta$ because $(a-d)^2 + 4bc > 0$. Thus, the solution of (6) is

$$x_n = k_1 \alpha^n + k_2 \beta^n \quad (7)$$

where k_1 and k_2 are constants.

Putting $n=1, 2$ in (7), we get the system of equations

$$\left. \begin{aligned} \alpha k_1 + \beta k_2 &= x_1 \\ \alpha^2 k_1 + \beta^2 k_2 &= x_2 \end{aligned} \right\} \quad (8)$$

Using Cramer's rule, the solutions for (8) are

$$k_1 = \frac{\beta x_1 - x_2}{\alpha(\beta - \alpha)} \text{ and } k_2 = \frac{x_2 - \alpha x_1}{\beta(\beta - \alpha)}.$$

Thus,

$$x_n = \left(\frac{\beta x_1 - x_2}{\beta - \alpha} \right) \alpha^{n-1} + \left(\frac{x_2 - \alpha x_1}{\beta - \alpha} \right) \beta^{n-1} \quad (9)$$

Putting (9) in (5) and simplifying, we get

$$z_n = \frac{\left(\frac{\alpha - d}{c} \right) (\beta - d - cz_1) \alpha^{n-1} + \left(\frac{\beta - d}{c} \right) (cz_1 + d - \alpha) \beta^{n-1}}{(\beta - d - cz_1) \alpha^{n-1} + (cz_1 + d - \alpha) \beta^{n-1}}$$

for all $n \in \mathbb{N}$.

Case 2.2 $(a-d)^2 + 4bc = 0$. Then $\alpha = \beta = \frac{a+d}{2}$. Thus, the solution of (6) is

$$x_n = (k_1 + k_2 n) \alpha^n \quad (10)$$

where k_1 and k_2 are constants.

Putting $n=1, 2$ in (10), we get the system of equations

$$\left. \begin{aligned} \alpha k_1 + \alpha k_2 &= x_1 \\ \alpha^2 k_1 + 2\alpha^2 k_2 &= x_2 \end{aligned} \right\} \quad (11)$$

Using Cramer's rule, the solutions for (11) are

$$k_1 = \frac{x_1}{\alpha^2} (2\alpha - cz_1 - d) \text{ and } k_2 = \frac{x_1}{\alpha^2} (cz_1 + d - \alpha).$$

Thus,

$$\begin{aligned} x_n &= \left[\frac{x_1}{\alpha^2} (2\alpha - cz_1 - d) + \frac{x_1}{\alpha^2} (ncz_1 + nd - n\alpha) \right] \alpha^n \\ &= x_1 \alpha^{n-2} \left[a - cz_1 + ncz_1 + \frac{n}{2} (d - a) \right] \end{aligned}$$

$$= x_1 \alpha^{n-2} \left[(n-1)cz_1 + \frac{(d-a)}{2}n + a \right] \quad (12)$$

Putting (12) in (5) and simplifying, we have

$$\begin{aligned} z_n &= \frac{x_1 \alpha^{n-1} \left[ncz_1 + \left(\frac{d-a}{2} \right) (n+1) + a \right]}{cx_1 \alpha^{n-2} \left[(n-1)cz_1 + \left(\frac{d-a}{2} \right) n + a \right]} - \frac{d}{c} \\ &= \left(\frac{a+d}{2c} \right) \left[\frac{ncz_1 + \left(\frac{d-a}{2} \right) (n+1) + a}{(n-1)cz_1 + \left(\frac{d-a}{2} \right) n + a} \right] - \frac{d}{c} \\ &= \frac{1}{2c} \left[\frac{(a-d)(2cz_1 + d - a)n + (a-d)^2 + 4cdz_1}{(2cz_1 + d - a)n + 2(a - cz_1)} \right] \end{aligned}$$

for all $n \in \mathbb{N}$.

Case 2.3 $(a-d)^2 + 4bc < 0$. Then α, β are nonreal complex numbers. Note that

$$ad - bc = \frac{1}{4}(a+d)^2 - \frac{1}{4}[(a-d)^2 + 4bc] > 0.$$

We also have

$$\alpha = \frac{a+d - i\sqrt{-(a-d)^2 - 4bc}}{2} \quad \text{and} \quad \beta = \frac{a+d + i\sqrt{-(a-d)^2 - 4bc}}{2}.$$

Let θ be the argument of β and $r = |\beta|$.

Then $r = \sqrt{ad - bc}$ and $\sin \theta = \sqrt{\frac{-(a-d)^2 - 4bc}{4(ad - bc)}} > 0$. Thus, $\theta \in (0, \pi)$ and hence

$\theta = \cos^{-1} \left(\frac{a+d}{2\sqrt{ad - bc}} \right)$. Then $\alpha = r(\cos \theta - i \sin \theta)$ and $\beta = r(\cos \theta + i \sin \theta)$.

Thus, the solution of (6) is

$$x_n = r^n (k_1 \cos n\theta + k_2 \sin n\theta) \quad (13)$$

where k_1 and k_2 are constants.

Putting $n=1, 2$ in (13), we get the system of equations

$$\left. \begin{aligned} (r \cos \theta)k_1 + (r \sin \theta)k_2 &= x_1 \\ (r^2 \cos 2\theta)k_1 + (r^2 \sin 2\theta)k_2 &= x_2 \end{aligned} \right\} \quad (14)$$

Using Cramer's rule and $x_2 = (cz_1 + d)x_1$, the solutions for (14) are

$$\begin{aligned} k_1 &= \frac{rx_1 \sin 2\theta - x_2 \sin \theta}{r^2 \sin \theta} \\ &= \frac{rx_1 \sin 2\theta - x_1 (cz_1 + d) \sin \theta}{r^2 \sin \theta} \\ &= \frac{x_1}{r^2 \sin \theta} [r \sin 2\theta - (cz_1 + d) \sin \theta] \end{aligned}$$

and

$$\begin{aligned} k_2 &= \frac{x_2 \cos \theta - rx_1 \cos 2\theta}{r^2 \sin \theta} \\ &= \frac{x_1 (cz_1 + d) \cos \theta - rx_1 \cos 2\theta}{r^2 \sin \theta} \\ &= \frac{x_1}{r^2 \sin \theta} [(cz_1 + d) \cos \theta - r \cos 2\theta]. \end{aligned}$$

Substituting k_1, k_2 and (13) in (5), we get

$$\begin{aligned} z_n &= \frac{r}{c} \left[\frac{(\cos(n+1)\theta)[r \sin 2\theta - (cz_1 + d) \sin \theta] + (\sin(n+1)\theta)[(cz_1 + d) \cos \theta - r \cos 2\theta]}{(\cos n\theta)[r \sin 2\theta - (cz_1 + d) \sin \theta] + (\sin n\theta)[(cz_1 + d) \cos \theta - r \cos 2\theta]} \right] - \frac{d}{c} \\ &= \frac{r}{c} \left[\frac{(cz_1 + d)[\sin(n+1)\theta \cos \theta - \cos(n+1)\theta \sin \theta]}{(cz_1 + d)[\sin n\theta \cos \theta - \cos n\theta \sin \theta] + r[\cos n\theta \sin 2\theta - \sin n\theta \cos 2\theta]} \right. \\ &\quad \left. + \frac{r[\cos(n+1)\theta \sin 2\theta - \sin(n+1)\theta \cos 2\theta]}{(cz_1 + d)[\sin n\theta \cos \theta - \cos n\theta \sin \theta] + r[\cos n\theta \sin 2\theta - \sin n\theta \cos 2\theta]} \right] - \frac{d}{c} \\ &= \frac{r}{c} \left[\frac{(cz_1 + d) \sin n\theta - r \sin(n-1)\theta}{(cz_1 + d) \sin(n-1)\theta - r \sin(n-2)\theta} \right] - \frac{d}{c}. \end{aligned}$$

3. Examples

We illustrate results in Theorem 1 by the following examples.

1. Let $a=1$, $b=1$, $c=0$, $d=1$ and $z_1=1$.

Then (1) becomes $z_{n+1} = z_n + 1$. By Theorem 1 (1), we get $z_n = n$ for all $n \in \mathbb{N}$.

2. Let $a=2$, $b=1$, $c=0$, $d=1$ and $z_1=2$.

Then (1) becomes $z_{n+1} = 2z_n + 1$. By Theorem 1 (2), we get $z_n = 3 \cdot 2^{n-1} - 1$ for all $n \in \mathbb{N}$.

3. Let $a=1$, $b=4$, $c=1$, $d=1$ and $z_1=1$.

Then (1) becomes $z_{n+1} = \frac{z_n + 4}{z_n + 1}$. By Theorem 1 (3), we get $z_n = \frac{2 \cdot 3^n + 2(-1)^n}{3^n - (-1)^n}$ for all $n \in \mathbb{N}$.

4. Let $a=3$, $b=-1$, $c=1$, $d=1$ and $z_1=2$.

Then (1) becomes $z_{n+1} = \frac{3z_n - 1}{z_n + 1}$. By Theorem 1 (4), we get $z_n = \frac{n+3}{n+1}$ for all $n \in \mathbb{N}$.

5. Let $a=0$, $b=1$, $c=-1$, $d=1$ and $z_1=2$.

Then (1) becomes $z_{n+1} = \frac{1}{-z_n + 1}$. By Theorem 1 (5), we get

$$z_n = \frac{\sin \frac{(n-2)\pi}{3} - \sin \frac{n\pi}{3}}{\sin \frac{(n-2)\pi}{3} + \sin \frac{(n-1)\pi}{3}} \text{ for all } n \in \mathbb{N}.$$

Acknowledgement

The second author is grateful to the Development and Promotion of Science and Technology Project (DPST).

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