

*Original Article*

## Quadratic parameter homotopy function for solving polynomial equations

Hafizudin Mohamad Nor and Siti Ainor Mohd Yatim\*

*School of Distance Education, Universiti Sains Malaysia, Minden, Penang, 11800 Malaysia*

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**Abstract**

Homotopy function is used in conjunction with homotopy continuation methods (HCM) and a classical numerical method to approximate the roots of nonlinear algebraic equations, particularly in solving polynomial equations. In this paper, we develop a new function, known as quadratic parameter homotopy function, and apply it to solve several non-application equations and several application equations. This homotopy function is used in two homotopy methods i.e., Newton-HCM and Ostrowski-HCM. The results obtained indicate that the proposed homotopy function performs better than the existing homotopy functions.

**Keywords:** numerical method, polynomial equations, homotopy function, quadratic parameter homotopy function, homotopy method

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**1. Introduction**

Consider a polynomial equation

$$f(x) = 0. \quad (1.1)$$

The solution of (1.1) has been widely studied by mathematicians for many years. The function  $f$  of  $x$  can be varied into linear and nonlinear. To solve (1.1), there are four categories of methods, namely graphical technique, closed method, open method, and global method. The graphical method is used to obtain rough approximates of roots as a guide for selecting good starting values or initial guesses. The closed method refers to using two initial guesses that bracket the true solution, whereas the open method does not require that the initial guesses that bracket the root, as stated in Chapra (2012). As mentioned by Gritton, Seader, and Lin (2001), a global method is defined as tracking a solution from an arbitrary initial guess. Bisection and false position methods are bracketing methods. Fixed point method, secant method, Newton's method, and Ostrowski's method are categorised as open

methods. An example of a global method is the homotopy continuation method (HCM).

There are numerous studies regarding finding the zeroes. A study by Torres-Munoz, Hernandez-Martinez, and Vazquez-Leal (2016) used homotopy path and spherical method. The purposes of the aforementioned techniques were to trace the homotopy trajectory and to resize the radius of sphere at each iteration, respectively. Besides that, Argyros and George (2018) extended the applicability of a homotopy method for locating an approximate zero using Newton's method. The results demonstrated a more precise location for the Newton's iterates, leading to at least as tight Lipschitz-type functions.

In this paper, we do not focus on a comparison between the aforementioned methods, but we prefer to focus on comparing homotopy functions when any HCM is applied. The homotopy functions include the standard homotopy function, the quadratic Bezier homotopy function introduced by Nor, Md Ismail, and Majid (2013, 2014), the cubic Bezier homotopy function introduced by (Ramli, Saharizan, & Salim Nasir, 2016) and our new proposed function named "quadratic parameter homotopy function". The reason for selecting the improved homotopy function is to improve the accuracy of solutions for zeroes.

Our research was inspired by other researchers, such as Milad, Aghil, Sahba, and Amir (2016), Mirmohammad,

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\*Corresponding author  
Email address: ainor@usm.my

Mohtasebi, and Yousefi-Koma (2016) and so on. Our contribution of Super Ostrowski homotopy method as cited in Nor, Rahman, Md Ismail, and Majid (2016) then was applied in Wu, Qin, Ma, Fu, and Xu (2017). Besides that, our idea of Quadratic Bezier Homotopy Function was developed for geodetic applications in Palancz and Awange (2017). Furthermore, Awange, Palancz, Lewis, and Volgyesi (2018) again cited our work (Nor *et al.*, 2014) in a chapter on Homotopy Solution of Nonlinear Systems.

Acknowledging the limitations of the homotopy function, we decided to improve the existing alternatives. Our new function was discussed roughly in Nor, Rahman, Md. Ismail, and Majid (2015) but not in a detailed manner. Therefore, in this paper, we investigate the performance of the proposed function as compared to the other available functions.

## 2. Quadratic and Cubic Bezier Homotopy Function

Quadratic Bezier Homotopy Function was introduced by Nor *et al.* (2013, 2014) first for the scalar nonlinear equations, and later for a system of polynomial equations. Then, this idea was extended to the Cubic Bezier Homotopy Function by Ramli *et al.* (2016). Both functions were developed from a linear homotopy function, also known as the standard homotopy function,  $H(x, t)$ .

Let us consider the standard homotopy function before we show the quadratic and the cubic Bezier homotopy functions. The standard homotopy function, denoted by  $H(x, t)$  is as follows

$$H(x, t) = (1 - t)g(x) + tf(x) \quad (2.1)$$

where  $g(x)$  and  $f(x)$  are the initial system and the target system, respectively.

Parameter  $t$  is the homotopy parameter that varies from zero to one. We have

$$H(x, 0) = g(x) \quad (2.2)$$

$$H(x, 1) = f(x) \quad (2.3)$$

for these two endpoints. In other words, the homotopy function will start from curve  $g(x)$  and will finish at curve  $f(x)$ . The solution is then approximated and tracked when  $f(x) = 0$ . The concept of a homotopy function was earlier discussed in detail by Palancz, Awange, Zaletnyik, and Lewis (2010), and then discussed again clearly in Nor *et al.* (2013).

Then, Nor *et al.* (2013) made a simple analogy between a linear homotopy function and the linear De Casteljau algorithm, such that

$$P(t) = (1 - t)P_0 + tP_1 \quad (2.4)$$

where  $P_0, P_1$  are control points and  $t \in [0, 1]$ . The De Casteljau algorithm describes the movement of point on a curve, so that homotopy is a movement of a curve that contains a set of points. There is also a second order De Casteljau algorithm, in which

$$P(t) = (1 - t)^2P_0 + 2t(1 - t)P_1 + t^2P_2, \quad (2.5)$$

where  $P_0, P_1, P_2$  are control points and  $t \in [0, 1]$ . Thus, Nor *et al.* (2013) took this chance to introduce a new homotopy function of second order as follows.

$$H_2(x, t) = (1 - t)^2g(x) + 2t(1 - t)H(x, t) + t^2f(x), \quad (2.6)$$

This new homotopy function started from an investigation of solving scalar nonlinear equations. Then, the investigation was extended to the system of polynomial equations, named Quadratic Bezier Homotopy function

$$H_2(x, t) = (1 - t)^2G(x) + 2t(1 - t)H(x, t) + t^2F(x) \quad (2.7)$$

Then, quadratic Bezier homotopy function was extrapolated and expanded by Ramli *et al.* (2016) to the third order as follows.

$$H_3(x, t) = (1 - t)^3g(x) + 3t(1 - t)^2H(x, t) + 3t^2(1 - t)H_2(x, t) + t^3f(x) \quad (2.8)$$

Quadratic Bezier homotopy function requires three functions  $g(x)$ ,  $H(x, t)$  and  $f(x)$  to perform the movement of function as the parameter  $t$  increases from zero to one. Meanwhile, Cubic Bezier requires four functions  $g(x)$ ,  $H(x, t)$ ,  $H_2(x, t)$  and  $f(x)$ . Someone probably will introduce Bezier homotopy function of fourth and fifth orders, named Quartic Bezier and Quintic Bezier homotopy functions, respectively. The aforementioned Bezier homotopy functions need five terms of functions and six terms of functions. Generally, the  $n$ th order of Bezier homotopy function will need  $n+1$  terms of functions, making it more complicated. Motivated by this, we developed an easier and shorter function, named the Quadratic Parameter Homotopy Function.

## 3. Development of Quadratic Parameter Homotopy Function

Before we introduce the aforementioned function, let us show a function that is related to our function, used by Vershelde (1996). The function is as follows

$$H(x, t) = \gamma(1 - t)^kG(x) + t^kF(x), \quad (3.1)$$

where  $\gamma \in \mathbb{C}_0$ ,  $k \in \mathbb{N}_0$ ,  $t \in [0, 1]$ ,  $x = \{x_1, x_2, \dots, x_n\}$ ,  $G(x) = (g_1(x), g_2(x), \dots, g_n(x))^T$  and  $F(x) = (f_1(x), f_2(x), \dots, f_n(x))^T$  to solve a sparse polynomial system. Since we are now dealing with the solution of a single equation and of second degree in the parameter  $t$ , therefore  $k = 2$  and  $n = 1$ , and the homotopy function (3.1) can be written as

$$H(x, t) = \gamma(1 - t)^2g(x) + t^2f(x), \gamma \in \mathbb{C} \quad (3.2)$$

A new homotopy function was introduced by Nor *et al.* (2013) and named Quadratic Bezier homotopy function in Nor *et al.* (2014), such that is used to solve scalar nonlinear equations as well as systems of polynomial equations. The Quadratic Bezier homotopy function was developed from a widely-used standard homotopy function. The standard homotopy function is  $H(x, t)$  as mentioned by Gritton *et al.* (2001), and Jalali-Farahani and Seader (2000), a collection of

Wu’s works (Wu, 2005a, 2005b, 2006, 2007) and (Rahimian, Jalali, & Seader, 2010) as (2.1).

We now improve the Quadratic Bezier homotopy function (2.6) to a simpler alternative i.e.

$$H_2^*(x, t) = \frac{\gamma}{d!} (1 - t)^2 g(x) + (2t - t^2) f(x), \tag{3.3}$$

where  $\gamma$  is a complex constant and  $d$  is the highest degree of polynomial equations. Function (3.3) is formulated from

$$H_2^*(x, t) = \frac{\gamma}{d!} (1 - t)^2 g(x) + (1 - (1 - t)^2) f(x). \tag{3.4}$$

The complement of parameter  $t$  is  $1 - t$  and vice versa. Therefore, the function  $(2t - t^2)$  becomes the coefficient of target function since it is the complement of the parameter  $(1 - t)^2$ . The function looks simpler than the existing functions when we eliminate the middle term of Quadratic Bezier homotopy function. The equation (2.6) and (3.3) can also be expressed using matrix representation as

$$H_2(x, t) = \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} g(x) \\ H(x, t) \\ f(x) \end{pmatrix}, \tag{3.5}$$

and

$$H_2^*(x, t) = \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & -1 \\ -2\alpha & 0 & 2 \\ \alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} g(x) \\ H(x, t) \\ f(x) \end{pmatrix}, \tag{3.6}$$

where  $\alpha = \frac{\gamma}{d!}$ . All the equations (3.2), (3.5), and (3.6) also fulfill the following two boundary conditions at the endpoints

$$H(x, 0) = H_2(x, 0) = H_2^*(x, 0) \equiv g(x) = 0, \tag{3.7}$$

$$H(x, 1) = H_2(x, 1) = H_2^*(x, 1) \equiv f(x) = 0, \tag{3.8}$$

when  $t = 0$  and  $t = 1$  respectively.

#### 4. Numerical Experiments and Discussion

We test three equations of non-applications and three equations of applications involving polynomial equations. The Quadratic Parameter Homotopy Function is demonstrated using Newton-HCM and Ostrowski-HCM as described in detail in Wu (2005b) and Nor *et al.* (2014) respectively. For better understanding of the basic concepts of Newton-HCM, we refer to W Ismail, Nor, and Ishak (2015). Then we compared with other homotopy functions namely the standard homotopy, the quadratic Bezier homotopy, and the cubic Bezier homotopy functions. We use the value  $\gamma = i$  and the stopping criterion used is

$$|f(x)| < \varepsilon, \tag{4.1}$$

where  $\varepsilon = 10^{-6}$ . To increase the accuracy of approximate solutions, we use a technique from Palancz *et al.* (2010) as follows

$$x_{i+1} = \text{NewtonRaphson}(H(x, t_{i+1}), \{x, x_i\}) \tag{4.2}$$

and

$$x_{i+1} = \text{OstrowskiMethod}(H(x, t_{i+1}), \{x, x_i\}) \tag{4.3}$$

where  $x_i$  is the initial value for calculating next  $x_{i+1}$ . However, we only iterate two times for each  $t_{i+1}$ . The best function is then selected based on the lowest number of iterations required and the least time of computation to converge to the aforementioned stopping criterion.

#### 4.1 Equations of non-application

**Example 4.1.** Consider the following equation discussed in Palancz *et al.* (2010)

$$f(x) = x^2 + 8x - 9 = 0, \tag{4.4}$$

for which the exact solutions are  $x_1 = -9$  and  $x_2 = 1$ . The results are shown in Table 1 and Table 2 for the homotopy functions from the selected homotopy methods.

**Example 4.2.** Consider the following scalar polynomial equation in (Wu, 2005b)

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 1 = 0, \tag{4.5}$$

for which the ‘exact’ solutions are  $x_1 = -3.6527047588514683$ ,  $x_2 = 0.16465538573865882$ , and  $x_3 = 4.988049373112809$ . The results are shown in Table 1 and Table 2 for homotopy functions of the selected homotopy methods.

**Example 4.3.** Consider the following fourth-order of polynomial equation in Abd. Rahman, Ibrahim, and Jayes (2011)

$$f(x) = x^4 + 6x - 40 = 0. \tag{4.6}$$

The number of solutions of (4.6) should be not more than  $d = 4$ . The ‘exact’ solutions are  $x_1 = -2.7409949444$ ,  $x_2 = 2.2667282725$  and the one pair of complex conjugates  $x_3, x_4 = 0.237133336 \pm 2.526219981i$ . The results are shown in Table 1 and Table 2 for the homotopy functions of the selected homotopy methods.

Performance of the different functions was then measured by using Newton-HCM as in Table 1.

Table 1 shows that the Quadratic Parameter Homotopy Function performs the fastest among the three alternatives. The number of iterations using this homotopy function is less than 15 even when the location of initial value is far from any roots.

However, one homotopy method is not sufficient to arrive at a comprehensive conclusion. Therefore, we test the same functions by using a second method, named Ostrowski-HCM. The results can be summarized as in Table 2.

Table 2 also shows that the Quadratic Parameter Homotopy Function performs the fastest among the candidate homotopy functions in the three examples. For Example 4.3 in Table 2, the results for  $x_0 = 257$  were inconclusive in deciding for the best function. However, the results give advantage to the Quadratic Parameter Homotopy Function over the other candidate functions, when the initial value is bigger  $x_0 = 2579$ ,

Table 1. Demonstration of different homotopy functions using newton-HCM for equations of non-application

Equation	Initial value $x_0$	Standard homotopy function $H(x, t)$	Quadratic Bezier homotopy function $H_2(x, t)$	Cubic Bezier homotopy function $H_3(x, t)$	Quadratic parameter homotopy function $H_2^*(x, t)$
Example 4.1	100	98	17	11	8
	CPU time, s	0.2028	0.0156	0.0156	0.0156
Example 4.2	-100	239	27	14	7
	CPU time,s	1.638	0.0624	0.0312	0.0156
Example 4.3	257	274	29	15	10
	CPU time,s	1.794	0.1092	0.0312	0.0156

Table 2. Demonstration of different homotopy functions using Ostrowski-HCM for equations of non-application

Equation	Initial value $x_0$	Standard homotopy function $H(x, t)$	Quadratic Bezier homotopy function $H_2(x, t)$	Cubic Bezier homotopy function $H_3(x, t)$	Quadratic parameter homotopy function $H_2^*(x, t)$
Example 4.1	100	3	3	3	2
	CPU time, s	0.0156	0.0156	0.0936	0.0156
	1000	16	7	6	4
Example 4.2	CPU time, s	0.1092	0.156	0.1248	0.078
	-100	6	4	4	3
	CPU time, s	0.0156	0.0312	0.078	0.0156
Example 4.3	257	5	5	5	5
	CPU time, s	0.0156	0.0468	0.0936	0.0312
	2579	40	11	8	7
	CPU time, s	0.2652	0.1248	0.2184	0.0312

based on the criteria specified. The number of iterations using this homotopy function is less than 10 even when the location of the initial value is far from the zeroes. Besides that, the CPU time to converge gives advantage to the proposed homotopy function.

4.2 Equations of application

**Example 4.4.** Consider the following azeotropic-point calculation discussed in Gritton *et al.* (2001)

$$f(x) = x^2 - 6.5886408x + 4.0777367 = 0, \tag{4.7}$$

where the ‘exact’ solutions are  $x_1 = 0.691473843274813$  and  $x_2 = 5.89716695672519$ . The results are shown in Table 3 and Table 4 for homotopy functions using the selected homotopy methods.

**Example 4.5.** Consider the following scalar polynomial equation in Gritton *et al.* (2001), which relates to the calculation of the specific volume of a gas using a virial equation of state:

$$f(x) = x^3 - 471.34x^2 + 74944.6341x - 4242149.1 = 0, \tag{4.8}$$

where the ‘exact’ solutions are  $x_1 = 212.95801506571$  and the one pair of complex roots  $x_2, x_3 = 129.190992467147 \pm 56.831390313885 i$ . The results are shown in Table 3 and

Table 4 for homotopy functions using the selected homotopy methods.

**Example 4.6.** Consider the following fourth-order of polynomial equation in Gritton *et al.* (2001) associated with a chemical equilibrium problem:

$$f(x) = x^4 - 1.3x^3 + 0.699096x^2 - 0.1816915x + 0.00850239 = 0. \tag{4.9}$$

The number of solutions of (4.9) should be no more than  $d = 4$ . The ‘exact’ solutions are  $x_1 = 0.0586545664906674$ ,  $x_2 = 0.600323276889981$  and the one pair of complex conjugates  $x_3, x_4 = 0.320511078309676 \pm 0.372474896861919 i$ . The results are shown in Table 3 and Table 4 for the homotopy functions using the selected homotopy methods.

Table 3 shows that the Quadratic Parameter Homotopy Function performs the fastest among the three alternatives. The number of iterations using this homotopy function is less than 20 even when the location of the initial value is far from any roots.

As one homotopy method is not sufficient to derive a comprehensive conclusion, we test the same functions using a second method, named Ostrowski-HCM. The result are summarized in Table 4.

Table 4 also shows that the Quadratic Parameter Homotopy Function performs the fastest among the candidate functions in the three examples. For Example 4.5 in Table 4,

Table 3. Demonstration of different homotopy functions using Newton-HCM for equations of application

Equation	Initial value $x_0$	Standard homotopy function $H(x, t)$	Quadratic Bezier homotopy function $H_2(x, t)$	Cubic Bezier homotopy function $H_3(x, t)$	Quadratic parameter homotopy function $H_2^*(x, t)$
Example 4.4	100	245	27	14	12
	CPU time, s	1.170	0.0468	0.0312	0.1872
Example 4.5	-100 000	144	21	14	14
	CPU time, s	0.5304	0.0468	0.0312	0.0156
	-1 000 000	1417	66	26	16
Example 4.6	CPU time, s	49.14	0.2652	0.0936	0.0312
	1	163	22	13	4
	CPU time, s	0.7956	0.0468	0.0312	0.0156

Table 4. Demonstration of different homotopy functions using Ostrowski-HCM for equations of application

Equation	Initial value $x_0$	Standard homotopy function $H(x, t)$	Quadratic Bezier homotopy function $H_2(x, t)$	Cubic Bezier homotopy function $H_3(x, t)$	Quadratic parameter homotopy function $H_2^*(x, t)$
Example 4.4	100	6	4	4	3
	CPU time, s	0.0156	0.0156	0.0468	0.1872
Example 4.5	-100	3	3	3	3
	CPU time, s	0.0156	0.0156	0.0624	0.0156
	-100 000	7	7	7	7
	CPU time, s	0.0156	0.0624	0.1872	0.0468
Example 4.6	-100 000 000	410	40	22	11
	CPU time, s	32.276	1.4664	1.747	0.078
	1	4	4	3	2
	CPU time, s	0.0156	0.0312	0.0468	0.0156

the results for small negative values are inconclusive. However, the results give advantage to the Quadratic Parameter Homotopy Function over the other functions when the initial value is near the smallest negative value. The difference in time of computation between standard and proposed homotopy functions, required for converged solution, is very significant.

The superior performance of the quadratic parameter homotopy function may be due to the function itself that is simpler than the quadratic and cubic Bezier homotopy functions. In other words, the proposed function only needs two functions,  $g(x)$  and  $f(x)$ , which accelerates the convergence.

**4. Conclusions**

The results from Tables 1-4 indicate that quadratic parameter homotopy function performs better than the other candidates tested. This improved performance has been demonstrated by testing in six examples; with three example equations of non-applications and another three from applications. The superior performance has also been demonstrated by using the Newton-HCM and Ostrowski-HCM methods. The effectiveness and efficiencies of the proposed function were measured by the ability of the function to track the zeroes, either real or complex roots, with fewer iterations and lesser time to compute. The results show that only the quadratic parameter homotopy function could track the complex root even when the initial guess was located far from any roots.

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