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Original Article

A new approach to fuzzy group theory using (α, β) -Pythagorean fuzzy sets

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Abstract

A Pythagorean fuzzy set (PFS) is a very efficient and powerful tool for handling uncertainty and vagueness. In this paper, we present the notion of (α, β) -Pythagorean fuzzy set *(*PFS*)* as a generalisation of Pythagorean fuzzy set *(*PFS*)*. We propose a new fuzzy subgroup, called (α, β) -Pythagorean fuzzy subgroup *(*PFSG*)*. We investigate properties of our proposed fuzzy subgroup. We present the concept of (α, β) -Pythagorean fuzzy coset *(*PFC*)* and (α, β) -Pythagorean fuzzy normal subgroup (PFNSG). Furthermore, we define (α, β) -Pythagorean fuzzy level subgroup (PFLSG) and establish properties of it.

Keywords: Pythagorean fuzzy set, (α, β) - Pythagorean fuzzy set, (α, β) - Pythagorean fuzzy subgroup, (α, β) - Pythagorean fuzzy normal subgroup, (α, β) - Pythagorean fuzzy level subgroup

1. Introduction

Group theory is a branch of pure Mathematics which deals with the algebraic structure known as group. It is used in various fields of Mathematics such as Cryptography, Algebraic geometry, Harmonic analysis and many more. Not only in Mathematics, it is also used in Physics, Chemistry, Materials science etc. In 1965, in his pioneer paper Zadeh (Zadeh, 1965) first invented fuzzy sets to handle uncertainty in real life problems. This invention has made a huge worldwide impact on research. Utilizing the idea of a fuzzy set, in 1971 Rosenfeld (Rosenfeld, 1971) first defined fuzzy subgroup. In 1979, the notion of fuzzy subgroup was redefined by Anthony and Sherwood (Anthony & Sherwood, 1979; Anthony & Sherwood, 1982). Fuzzy level subgroup was introduced by Das (Das, 1981). Ajmal and Prajapati (Ajmal & Prajapati, 1992) gave the idea of fuzzy coset and fuzzy normal subgroup.

In 1986, adding non membership degree with membership degree Atanassov (Atanassov, 1986) introduced intuitionistic fuzzy set. Using this concept, in 2013 Yager (Yager, 2013) defined Pythagorean fuzzy set. This set gives a modern way to model vagueness and uncertainty with high precision and accuracy compared to intuitionistic fuzzy sets, which is more fruitful in many decision making problems. Some results related to it were given by Peng (Peng & Yang, 2015) and Yang (Peng & Yang, 2016). In 2018, Naz *et al.* (Naz, Ashraf & Akram, 2018) gave a novel approach to decision-making with Pythogorean fuzzy information. In 2019, Akram and Naz (Akram & Naz, 2019) used complex Pythogorean fuzzy environment for decision-making problems. Ejegwa (Ejegwa, 2019) gave an application of Pythogorean fuzzy set in career placements using max-min-max composition.

Intuitionistic fuzzy subgroup was first studied by Zhan and Tan (Zhan & Tan, 2004) in 2004. There are some situations when the sum of membership degree and non-membership degree of an element does not lie in [0,1]. In this situation, we opted for Pythagorean fuzzy set. But if Pythagorean fuzzy set fails then what to do. The solution is (α, β) -Pythagorean fuzzy set. Using (α, β) -PFS we have introduced a new fuzzy subgroup that is (α, β) -PFSG which is a generalization of intuitionistic fuzzy subgroup. The properties of (α, β) -PFSG are much more efficient than those of intuitionistic fuzzy subgroup.

In this paper, (α, β) -PFS is defined and (α, β)

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-PFSG is described in Section 3. (α,β) -PFC and (α,β) -PFNSG are discussed in Section 4. In Section 5, (α,β) -PFLSG and its properties are explained. Finally, dihedral group is modelled in terms of (α,β) - Pythagorean fuzzy subgroup in Section 6 as an application.

2. Preliminaries

Here, we recap some important definitions that will be used in later sections.

Definition 2.1 (Zadeh, 1965) Let S be a crisp set. A mapping μ from S to the closed interval [0,1] is called a fuzzy subset in S. For every $u \in S$, $\mu(u)$ is called membership degree of u in S.

Definition 2.2 (Rosenfeld, 1971) Let $(G,\circ)$ be a group and μ be a fuzzy subset of G. Then μ is said to be fuzzy subgroup of $(G,\circ)$ if the following conditions hold:

(i) $\mu(u \circ v) \geq \mu(u) \wedge \mu(v)$ for all $u, v \in G$

(ii) $\mu(u^{-1}) \geq \mu(u)$ for all $u \in G$.

Definition 2.3 (Atanassov, 1986) Let S be a crisp set. An intuitionistic fuzzy set (IFS) I in S is an object having the form $I = \{(u, \mu(u), \nu(u)|u \in S\}$ where $\mu(u) \in [0,1]$ and $\nu(u) \in [0,1]$ are membership degree and non-membership degree of $u \in S$ respectively, which satisfy $0 \leq \mu(u) + \nu(u) \leq 1$.

Definition 2.4 (Zhan & Tan, 2004) Let $(G,\circ)$ be a group and $I = \{(u, \mu(u), \nu(u)|u \in S\}$ be a IFS of G. Then I is said to be intuitionistic fuzzy subgroup (IFSG) of G if the following conditions hold:

(i) $\mu(u \circ v) \geq \mu(u) \wedge \mu(v)$ and $\nu(u \circ v) \leq \nu(u) \vee \nu(v)$ for all $u,v \in G$

(ii) $\mu(u^{-1}) \geq \mu(u)$ and $\nu(u^{-1}) \leq \nu(u)$ for all $u \in G$.

In 2013, Yager (Yager, 2013) defined Pythagorean fuzzy set (PFS) as a generalization of IFS.

Definition 2.5 (Yager, 2013) Let S be a crisp set. An Pythagorean fuzzy set (PFS) ψ in S is an object having the form $\psi = \{(u, \mu(u), \nu(u)|u \in S\}$ where $\mu(u) \in [0,1]$ and $\nu(u) \in [0,1]$ are membership degree and non-membership degree of $u \in S$ respectively, which satisfy the condition $0 \leq \mu^2(u) + \nu^2(u) \leq 1$.

Definition 2.6 Let $(G,\circ)$ be a group and $\psi = \{(u, \mu(u), \nu(u)|u \in S\}$ be a PFS of G. Then ψ is said to be Pythagorean fuzzy subgroup (PFSG) of G if the following conditions hold:

(i) $\mu^2(u \circ v) \geq \mu^2(u) \wedge \mu^2(v)$ and $\nu^2(u \circ v) \leq \nu^2(u) \vee \nu^2(v)$ for all $u, v \in G$

(ii) $\mu^2(u^{-1}) \geq \mu^2(u)$ and $\nu^2(u^{-1}) \leq \nu^2(u)$ for all $u \in G$.

Here, $\mu^2(u) = \{\mu(u)\}^2$ and $\nu^2(u) = \{\nu(u)\}^2$ for all $u \in G$.

3. $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ -Pythagorean Fuzzy Subgroup (PFSG)

Let us define (α,β) -Pythagorean fuzzy set as an extension of Pythagorean fuzzy set.

Definition 3.1 Let S be a crisp set and $\alpha, \beta \in [0,1]$ be such that $0 \leq \alpha^2 + \beta^2 \leq 1$. A (α,β) -Pythagorean fuzzy set ψ^* in S is an object having the form $\psi^* = \{(u, \mu^\alpha(u), \nu^\beta(u)|u \in S\}$where $\mu^\alpha(u) = \mu(u) \wedge \alpha$ and $\nu^\beta(u) = \nu(u) \vee \beta$ are membership degree and non-membership degree of $u \in S$ respectively, which satisfy the condition $0 \leq (\mu^\alpha(u))^2 + \left(\nu^\beta(u)\right)^2 \leq 1$.

Here we give some operations on (α, β) -PFS which are stated below:

Let $\psi_1^* = \{(u, \mu_1^\alpha(u), \nu_1^\beta(u)|u \in S\}$ and $\psi_2^* = \{(u, \mu_2^\alpha(u), \nu_2^\beta(u)|u \in S\}$ be two (α, β) -PFSs in S. Then the following holds:

- $\psi_1^* \cup \psi_2^* = \left\{\left(u, \mu_1^\alpha(u) \vee \mu_2^\alpha(u), \nu_1^\beta(u) \wedge \nu_2^\beta(u)\right) |u \in S\right\}$
- $\psi_1^* \cap \psi_2^* = \left\{\left(u, \mu_1^\alpha(u) \wedge \mu_2^\alpha(u), \nu_1^\beta(u) \vee \nu_2^\beta(u)\right) |u \in S\right\}$
- $\psi_1^* \subseteq \psi_2^*$ if $\mu_1^\alpha(u) \leq \mu_2^\alpha(u)$ and $\nu_1^\beta(u) \geq \nu_2^\beta(u)$ for all $u \in S$
- $\psi_1^* = \psi_2^*$ if $\mu_1^\alpha(u) = \mu_2^\alpha(u)$ and $\nu_1^\beta(u) = \nu_2^\beta(u)$ for all $u \in S$

Throughout this paper, we will write (α, β) -Pythagorean fuzzy set as (α, β) -PFS and it will be denoted by $\psi^* = (\mu^\alpha, \nu^\beta)$ instead of $\psi^* = \{(a, \mu^\alpha(a), \nu^\beta(a)|a \in S\}$.

Now by an example we will show that (α, β) -PFS is a more general form of a fuzzy set than IFS and PFS.

Example 3.2 Let us consider the crisp set $S = \{u, v, w\}$.

Define the membership degrees and non-membership degrees of the elements of S by

$$\mu(u) = 0.5,\ \mu(v) = 0.95,\ \mu(w) = 0.8$$

and $\nu(u) = 0.7,\ \nu(v) = 0.35,\ \nu(w) = 0.4$

Since $\mu(u) + \nu(u) > 1$ for all $u \in S$, then $I = \{(u, \mu(u), \nu(u)|u \in S\}$ is not an IFS of S.

Again $\mu^2(v) + \nu^2(v) = 1.025 > 1$. So $\psi = \{(u, \mu(u), \nu(u)|u \in S\}$ is not a PFS of S.

Now take $\alpha = 0.9$ and $\beta = 0.3$, then $\alpha^2 + \beta^2 = 0.9 < 1$.

Therefore

$$\mu^\alpha(u) = 0.5,\ \mu^\alpha(v) = 0.9,\ \mu^\alpha(w) = 0.8$$

and $\nu^\beta(u) = 0.7,\ \nu^\beta(v) = 0.35,\ \nu^\beta(w) = 0.4$

We can easily varify that $0 \leq (\mu^\alpha(u))^2 + \left(\nu^\beta(u)\right)^2 \leq 1$ for all $u \in S$.

Hence $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α, β) -PFS of S.

Now, we will define (α, β) -Pythagorean fuzzy subgroup (PFSG) as an extension of fuzzy subgroup and intuitionistic fuzzy subgroup (IFSG).

Definition 3.3 Let $(G,\circ)$ be a group and $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFS of G. Then ψ^* is said to be a (α, β) -PFSG of the group $(G,\circ)$ if the following conditions hold:

- $\mu^\alpha(u \circ v) \geq \mu^\alpha(u) \wedge \mu^\alpha(v)$ and $\nu^\beta(u \circ v) \leq \nu^\beta(u) \vee \nu^\beta(v)$ for all $u, v \in G$
- $\mu^\alpha(u^{-1}) \geq \mu^\alpha(u)$ and $\nu^\beta(u^{-1}) \leq \nu^\beta(u)$ for all $u \in G$.

Example 3.4 Let us take the set $G = \{1, -1, i, -i\}$. $(G, .)$ is a group, where '.' is the usual multiplication.

Define the membership degree and non-membership degree of the elements of G by

$$\mu(1) = 0.8,\ \mu(-1) = 0.6,\ \mu(i) = 0.5,\ \mu(-i) = 0.5$$

and $\nu(1) = 0.1,\ \nu(-1) = 0.2,\ \nu(i) = 0.6,\ \nu(-i) = 0.6$

We take $\alpha = 0.7$ and $\beta = 0.5$. Then (α, β) -PFS $\psi^* = (\mu^\alpha, \nu^\beta)$ of G is given by

$$\mu^\alpha(1) = 0.7,\ \mu^\alpha(-1) = 0.6,\ \mu^\alpha(i) = 0.5,\ \mu^\alpha(-i) = 0.5$$

and $\nu^{\beta}(1) = 0.5,\ \nu^{\beta}(-1) = 0.5,\ \nu^{\beta}(i) = 0.6,\ \nu^{\beta}(-i) = 0.6$

Here, $\mu^{\alpha}(i.-i) = \mu^{\alpha}(1) = 0.7$ and $\nu^{\beta}(i.-i) = \nu^{\beta}(1) = 0.5$

Now, $\mu^{\alpha}(i) \wedge \mu^{\alpha}(-i) = min\{0.5,0.5\} = 0.5$ and $\nu^{\beta}(i) \vee \nu^{\beta}(-i) = max\{0.6,0.6\} = 0.6$

So, $\mu^{\alpha}(i.-i) > \mu^{\alpha}(i) \wedge \mu^{\alpha}(-i)$ and $\nu^{\beta}(i.-i) < \nu^{\beta}(i) \vee \nu^{\beta}(-i)$

Also, $\mu^{\alpha}(i) = \mu^{\alpha}(-i)$ and $\nu^{\beta}(i) = \nu^{\beta}(-i)$

In the same manner, it can be shown that $\mu^{\alpha}(u.v) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$, $\nu^{\beta}(u.v) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$ for all $u, v \in G$ and $\mu^{\alpha}(u^{-1}) \geq \mu^{\alpha}(u)$, $\nu^{\beta}(u^{-1}) \leq \nu^{\beta}(u)$ for all $u \in G$. Hence ψ^{*} is a (α,β) -PFSG of the group $(G,.)$.

We can easily check that for a (α,β) -PFSG $\psi^{*} = (\mu^{\alpha}, \nu^{\beta})$ of a group $(G,\circ)$, the following results hold:

(i) $\mu^{\alpha}(e) \geq \mu^{\alpha}(u)$ and $\nu^{\beta}(e) \leq \nu^{\beta}(u) \forall u \in G$

(ii) $\mu^{\alpha}(u^{-1}) = \mu^{\alpha}(u)$ and $\nu^{\beta}(u^{-1}) = \nu^{\beta}(u) \forall u \in G$

where e is the identity element in G.

Now, we will show that every intuitionistic fuzzy subgroup *(*IFSG*)* of a group $(G,\circ)$ is also a (α,β) -Pythagorean fuzzy subgroup (PFSG) of the group $(G,\circ)$. But the converse is not true.

Theorem 3.5 If $I = (\mu, \nu)$ is a IFSG of a group $(G,\circ)$, then $\psi^{*} = (\mu^{\alpha}, \nu^{\beta})$ is a (α,β) -PFSG of the group $(G,\circ)$.

Proof. Since $I = (\mu, \nu)$ is a IFSG of a group $(G,\circ)$, then for all $u, v \in G$, we have

$\mu(u \circ v) \geq \mu(u) \wedge \mu(v)$, $\nu(u \circ v) \leq \nu(u) \vee \nu(v)$ and $\mu(u^{-1}) \geq \mu(u)$, $\nu(u^{-1}) \leq \nu(u)$.

Here $\mu(u) \in [0,1]$ and $\nu(u) \in [0,1]$ for all $u \in G$. Many cases arise. We will study some cases

Case 1: Let $\mu(u) > \mu(v)$ and $\nu(u) > \nu(v)$ for all $u, v \in G$. For $\alpha, \beta \in [0,1]$ we have, $\mu^{\alpha}(u) \geq \mu^{\alpha}(v)$ and $\nu^{\beta}(u) \geq \nu^{\beta}(v)$ for all $u, v \in G$.

Now, $\mu(u \circ v) \geq \mu(u) \wedge \mu(v) = \mu(v) \Rightarrow \mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(v) = \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$

i.e, $\mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$

Again, $\nu(u \circ v) \leq \nu(u) \vee \nu(v) = \nu(u) \Rightarrow \nu^{\beta}(u \circ v) \leq \nu^{\beta}(u) = \nu^{\beta}(v) \vee \nu^{\beta}(v)$

i.e, $\nu^{\beta}(u \circ v) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Case 2: Let $\mu(u) < \mu(v)$ and $\nu(u) < \nu(v)$ for all $u, v \in G$.So, $\mu^{\alpha}(u) \leq \mu^{\alpha}(v)$ and$\nu^{\beta}(u) \leq \nu^{\beta}(v)$ for all $u, v \in G$.

Now, $\mu(u \circ v) \geq \mu(u) \wedge \mu(v) = \mu(u) \Rightarrow \mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(u) = \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$

i.e, $\mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$

Again, $\nu(u \circ v) \leq \nu(u) \vee \nu(v) = \nu(v) \Rightarrow \nu^{\beta}(u \circ v) \leq \nu^{\beta}(v) = \nu^{\beta}(u) \vee \nu^{\beta}(v)$

i.e, $\nu^{\beta}(u \circ v) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Case 3: Let $\mu(u) = \mu(v)$ and $\nu(u) = \nu(v)$ for all $u, v \in G$. So, $\mu^{\alpha}(u) = \mu^{\alpha}(v)$ and $\nu^{\beta}(u) = \nu^{\beta}(v)$ for all $u, v \in G$.

Now, $\mu(u \circ v) = \mu(u) = \mu(v) \Rightarrow \mu^{\alpha}(u \circ v) = \mu^{\alpha}(u) = \mu^{\alpha}(v)$

i.e, $\mu^{\alpha}(u \circ v) = \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$

Again, $\nu(u \circ v) = \nu(u) = \nu(v) \Rightarrow \nu^{\beta}(u \circ v) = \nu^{\beta}(u) = \nu^{\beta}(v)$

i.e, $\nu^{\beta}(u \circ v) = \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

In this way, considering all the cases we can easily show that

$\mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$ and $\nu^{\beta}(u \circ v) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v), \forall u, v \in G$.

Again, $\mu(u^{-1}) \geq \mu(u)$ and $\nu(u^{-1}) \leq \nu(u)$ for all $u \in G$. Since $\mu(u) \in [0,1]$ and $\nu(u) \in [0,1]$,

$\mu^{\alpha}(u^{-1}) \geq \mu^{\alpha}(u)$ and $\nu^{\beta}(u^{-1}) \leq \nu^{\beta}(u)$ for all $u \in G$.

Hence $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is a (α, β) -PFSG of the group $(G, \circ)$.

Theorem 3.6 If $\psi = (\mu, \nu)$ is a PFSG of a group $(G, \circ)$, then $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is a (α, β) -PFSG of the group $(G, \circ)$.

Proof. This proof is similar to the previous theorem.

Example 3.7 Let us consider the Klein's 4-group $G = \{e, a, b, c\}$, where $a^2 = b^2 = c^2 = e$ and $ab = c$, $bc = a$, $ca = b$, with e the identity element of the group.

Define the membership degree and non-membership degree of the elements of G by

$$\mu(e) = 0.98,\ \mu(c) = 0.95,\ \mu(a) = 0.9,\ \mu(b) = 0.9$$

and $$\nu(e) = 0.32,\ \nu(c) = 0.35,\ \nu(a) = 0.5,\ \nu(b) = 0.5$$

Here we can see that for all $u \in G, \mu(u) + \nu(u) > 1$. So $I = \{(u, \mu(u), \nu(u)) | u \in G\}$ is not a IFS of G. Therefore I is not a IFSG of the Klein's 4-group G.

Again we can check that for all $u \in G, \mu^2(u) + \nu^2(u) > 1$. So $\psi = \{(u, \mu(u), \nu(u)) | u \in G\}$ is not a PFS of G. Therefore ψ is not a PFSG of the Klein's 4-group G

But for every $\alpha \leq \{\mu(x) | x \in G\}$ and $\beta \geq \{\nu(x) | x \in G\}$ be such that $\alpha^2 + \beta^2 \in [0,1]$ as $\alpha = 0.85$ and $\beta = 0.5$, we can easily verify that $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is a (α, β) -PFSG of the Klein's 4-group G.

Remark 3.8 IFSG $\subsetneq$ PFSG $\subsetneq$ $(\alpha, \beta) -$ PFSG.

Proposition 3.9 Let $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ be a (α, β) -PFS of a group $(G, \circ)$. Then for all $u, v \in G$, ψ^* is a (α, β) -PFSG of $(G, \circ)$ if and only if $\mu^{\alpha}(u \circ v^{-1}) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$ and $\nu^{\beta}(u \circ v^{-1}) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Proof. Let $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ be a (α, β) -PFSG of a group $(G, \circ)$.

So for all $u, v \in G$, $\mu^{\alpha}(u \circ v^{-1}) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v^{-1}) = \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$ and

$\nu^{\beta}(u \circ v^{-1}) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v^{-1}) = \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Conversely, let us assume that for all $u, v \in G$, $\mu^{\alpha}(u \circ v^{-1}) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$ and

$\nu^{\beta}(u \circ v^{-1}) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Now for all $u \in G$, $\mu^{\alpha}(e) = \mu^{\alpha}(u \circ u^{-1}) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(u^{-1}) = \mu^{\alpha}(u)$ and

$\nu^{\beta}(e) = \nu^{\beta}(u \circ u^{-1}) \leq \nu^{\beta}(u) \vee \nu^{\beta}(u^{-1}) = \nu^{\beta}(u)$, where e is the identity element of G.

Again for all $u \in G$, $\mu^{\alpha}(u^{-1}) = \mu^{\alpha}(e \circ u^{-1}) \geq \mu^{\alpha}(e) \wedge \mu^{\alpha}(u^{-1}) = \mu^{\alpha}(u)$ and

$\nu^{\beta}(u^{-1}) = \nu^{\beta}(e \circ u^{-1}) \leq \nu^{\beta}(e) \vee \nu^{\beta}(u^{-1}) = \nu^{\beta}(u)$.

Therefore for all $u, v \in G$, $\mu^{\alpha}(u \circ v) = \mu^{\alpha}(u \circ (v^{-1})^{-1}) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v^{-1}) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$ and

$\nu^{\beta}(u \circ v) = \nu^{\beta}(u \circ (v^{-1})^{-1}) \leq \nu^{\beta}(u) \vee \mu^{\alpha}(v^{-1}) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Hence $\psi^* = (\mu^{\alpha}, \nu^{\beta})$ is a (α, β) -PFSG of the group $(G, \circ)$.

Now we will check whether the union and intersection of two (α, β) -PFSGs of a group $(G, \circ)$ is a (α, β) -PFSG of G.

Theorem 3.10 Intersection of two (α,β) -PFSGs of a group $(G,\circ)$ is a (α,β) -PFSG of the group $(G,\circ)$.

Proof. Let $\psi_1^* = (\mu_1^\alpha, \nu_1^\beta)$ and $\psi_2^* = (\mu_2^\alpha, \nu_2^\beta)$ be two (α,β) -PFSGs of a group $(G,\circ)$.

Then $\psi^* = \psi_1^* \cap \psi_2^* = (\mu^\alpha, \nu^\beta)$, where $\mu^\alpha(u) = \mu_1^\alpha(u) \wedge \mu_2^\alpha(u)$ and $\nu^\beta(u) = \nu_1^\beta(u) \vee \nu_2^\beta(u)$ for all $u \in G$.

Now for all $u, v \in G$

$$\begin{aligned}
\mu^\alpha(u \circ v^{-1}) &= \mu_1^\alpha(u \circ v^{-1}) \wedge \mu_2^\alpha(u \circ v^{-1}) \\
&\geq (\mu_1^\alpha(u) \wedge \mu_1^\alpha(v)) \wedge (\mu_2^\alpha(u) \wedge \mu_2^\alpha(v)) \\
&= (\mu_1^\alpha(u) \wedge \mu_2^\alpha(u)) \wedge (\mu_1^\alpha(v) \wedge \mu_2^\alpha(v)) \\
&= \mu^\alpha(u) \wedge \mu^\alpha(v)
\end{aligned}$$

and

$$\begin{aligned}
\nu^\beta(u \circ v^{-1}) &= \nu_1^\beta(u \circ v^{-1}) \vee \nu_2^\beta(u \circ v^{-1}) \\
&\leq \left(\nu_1^\beta(u) \vee \nu_1^\beta(v)\right) \vee \left(\nu_2^\beta(u) \vee \nu_2^\beta(v)\right) \\
&= \left(\nu_1^\beta(u) \vee \nu_2^\beta(u)\right) \vee \left(\nu_1^\beta(v) \vee \nu_2^\beta(v)\right) \\
&= \nu^\beta(u) \vee \nu^\beta(v)
\end{aligned}$$

Therefore $\psi^* = \psi_1^* \cap \psi_2^*$ is a (α,β) -PFSG of $(G,\circ)$.

Hence, intersection of two (α,β) -PFSGs of a group is also a (α,β) -PFSG of the group.

Corollary 1. Intersection of a family of (α,β) -PFSGs of a group $(G,\circ)$ is also a (α,β) -PFSG of the group $(G,\circ)$.

Remark 3.11 Union of two (α,β) -PFSGs of a group may not be a (α,β) -PFSG of that group.

Example 3.12 Let us take the group $G = (\mathcal{Z},+)$, the group of all integers under usual addition.

Let $\psi_1 = (\mu_1, \nu_1)$ and $\psi_2 = (\mu_2, \nu_2)$ be two PFSs of G defined by

$$\mu_1(u) = \begin{cases} 0.3, & when \;\; u \in 5\mathcal{Z} \\ 0, & elsewhere \end{cases}$$

$$\nu_1(u) = \begin{cases} 0, & when \;\; u \in 5\mathcal{Z} \\ 0.5, & elsewhere \end{cases}$$

$$\mu_2(u) = \begin{cases} 0.15, & when \;\; u \in 3\mathcal{Z} \\ 0, & elsewhere \end{cases}$$

$$\nu_2(u) = \begin{cases} 0.2, & when \;\; u \in 3\mathcal{Z} \\ 0.3 & elsewhere \end{cases}$$

Let $\alpha = 0.25$ and $\beta = 0.1$. Then $\psi^* = \psi_1^* \cup \psi_2^* = (\mu^\alpha, \nu^\beta)$, where

$$\mu^\alpha(u) = \begin{cases} 0.25, & when \;\; u \in 5\mathcal{Z} \\ 0.15, & when \;\; u \in 3\mathcal{Z} - 5\mathcal{Z} \\ 0, & elsewhere \end{cases}$$

$$\nu^\beta(u) = \begin{cases} 0.1, & when \;\; u \in 5\mathcal{Z} \\ 0.2, & when \;\; u \in 3\mathcal{Z} - 5\mathcal{Z} \\ 0.3, & elsewhere \end{cases}$$

Here, $\mu^\alpha(5 + (-3)) = \mu^\alpha(2) = 0$ but $\mu^\alpha(5) \wedge \mu^\alpha(-3) = min\{0.25, 0.15\} = 0.15$

So, $\mu^\alpha(5 + (-3)) \ngeq \mu^\alpha(5) \wedge \mu^\alpha(-3)$.

Again, $\nu^\beta(5 + (-3)) = \nu^\beta(2) = 0.3$ but $\nu^\beta(5) \vee \nu^\beta(-3) = max\{0.1, 0.2\} = 0.2$

So, $\nu^\beta(5 + (-3)) \nleq \nu^\beta(5) \vee \nu^\beta(-3)$.

Hence $\psi^* = \psi_1^* \cup \psi_2^* = (\mu^\alpha, \nu^\beta)$ is not a (α, β) -PFSG of $G = (\mathcal{Z}, +)$.

Definition 3.13 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFSG of a group $(G,\circ)$. Then for all $u, v \in G$, ψ^* is said to be normalized (α, β) -PFSG of G, if the following conditions hold:

(i) $\mu^\alpha(u \circ v) \geq \mu^\alpha(u) \wedge \mu^\alpha(v)$ and $\nu^\beta(u \circ v) \leq \nu^\beta(u) \vee \nu^\beta(v)$

(ii) $\mu^\alpha(u^{-1}) = \mu^\alpha(u)$ and $\nu^\beta(u^{-1}) = \nu^\beta(u)$

(iii) $\mu^\alpha(e) = 1$ and $\nu^\beta(e) = 0$

where e is the identity element in G.

We can easily verify that, for a (α, β) -PFSG $\psi^* = (\mu^\alpha, \nu^\beta)$ of a group $(G,\circ)$, the following properties hold: for all $u, v \in G$,

(i) $\mu^\alpha(u \circ v) = \mu^\alpha(u) \wedge \mu^\alpha(v)$, when $\mu(u) \neq \mu(v)$ and $\nu^\beta(u \circ v) = \nu^\beta(u) \vee \nu^\beta(v)$, when $\nu(u) \neq \nu(v)$

(ii) If $\mu^\alpha(u \circ v^{-1}) = \mu^\alpha(e)$ and $\nu^\beta(u \circ v^{-1}) = \nu^\beta(e)$ holds then $\mu^\alpha(u) = \mu^\alpha(v)$ and $\nu^\beta(u) = \nu^\beta(v)$, where e is the identity element in G.

Theorem 3.14 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFSG of a group $(G,\circ)$. Then the set $K = \{u \in G | \mu^\alpha(u) = \mu^\alpha(e)$ and $\nu^\beta(u) = \nu^\beta(e)\}$ forms a subgroup of the group $(G,\circ)$, where e is the identity element in G.

Proof. Here $K = \{u \in G | \mu^\alpha(a) = \mu^\alpha(e)$ and $\nu^\beta(a) = \nu^\beta(e)\}$. Clearly K is non empty, as $e \in K$.

To show, $(K,\circ)$ is a subgroup of $(G,\circ)$, we have to show that if $u, v \in K$, then $u \circ v^{-1} \in K$.

Let $u, v \in K$. Then $\mu^\alpha(u) = \mu^\alpha(e) = \mu^\alpha(v)$, $\nu^\beta(u) = \nu^\beta(e) = \nu^\beta(v)$.

Since $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α, β) -PFSG of $(G,\circ)$, then

$$\begin{aligned} \mu^\alpha(u \circ v^{-1}) &\geq \mu^\alpha(u) \wedge \mu^\alpha(v^{-1}) \\ &= \mu^\alpha(u) \wedge \mu^\alpha(v) \\ &= \mu^\alpha(e) \wedge \mu^\alpha(e) \\ &= \mu^\alpha(e) \end{aligned}$$

Also,

$$\begin{aligned} \nu^\beta(u \circ v^{-1}) &\leq \nu^\beta(u) \vee \nu^\beta(v^{-1}) \\ &= \nu^\beta(u) \vee \nu^\beta(v) \\ &= \nu^\beta(e) \vee \nu^\beta(e) \\ &= \nu^\beta(e) \end{aligned}$$

Again we have $\mu^\alpha(e) \geq \mu^\alpha(u \circ v^{-1})$ and $\nu^\beta(e) \leq \nu^\beta(u \circ v^{-1})$

Therefore, $\mu^\alpha(u \circ v^{-1}) = \mu^\alpha(e)$ and $\nu^\beta(u \circ v^{-1}) = \nu^\beta(e)$.

So, $u \circ v^{-1} \in K$.

Hence $(K,\circ)$ is a subgroup of $(G,\circ)$.

4. (α, β) -Pythagorean Fuzzy Coset (PFC) and (α, β) -Pythagorean Fuzzy Normal Subgroup (PFNSG)

In this section, we will define (α, β) -Pythagorean fuzzy coset, (α, β) -Pythagorean fuzzy normal subgroup and properties related to it.

Definition 4.1 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFSG of a group $(G,\circ)$. Then for $u \in G$, (α, β) -Pythagorean fuzzy left coset (PFLC) of ψ^* is the (α, β) -PFS $u\psi^* = (u\mu^\alpha, u\nu^\beta)$, defined by $(u\mu^\alpha)(m) = \mu^\alpha(u^{-1} \circ m)$, $(u\nu^\beta)(m) = \nu^\beta(u^{-1} \circ m)$ and (α, β) -Pythagorean fuzzy right coset (PFRC) of ψ^* is the (α, β) -PFS $\psi^* u = (\mu^\alpha u, \nu^\beta u)$, defined by $(\mu^\alpha u)(m) = \mu^\alpha(m \circ u^{-1})$, $(\nu^\beta u)(m) = \nu^\beta(m \circ u^{-1})$ for all $m \in G$.

Definition 4.2 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFSG of a group $(G,\circ)$. Then ψ^* is a (α, β) -Pythagorean fuzzy normal subgroup (PFNSG) of the group $(G,\circ)$ if every (α, β) -PFLC of ψ^* is also a (α, β) -PFRC of ψ^* in G.
Equivalently, $u\psi^* = \psi^* u$ for all $u \in G$.

Example 4.3 Let us take the group $G = (\mathcal{Z}_3, +_3)$, where '$+_3$' is addition of integers modulo 3.
Define the membership degree and non-membership degree of the elements of G by

$$\mu(0) = 0.9,\ \mu(1) = 0.7,\ \mu(2) = 0.7$$

and $$\nu(0) = 0.1,\ \nu(1) = 0.2,\ \nu(2) = 0.2$$

We take $\alpha = 0.8$ and $\beta = 0.1$. Now we can easily varify that $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α, β) -PFSG of the group $G = (\mathcal{Z}_3, +_3)$.

For $u = 1 \in G$, the (α, β) -PFLC of ψ^* is the (α, β) -PFS $1\psi^* = (1\mu^\alpha, 1\nu^\beta)$, defined by $(1\mu^\alpha)(m) = \mu^\alpha(1^{-1} +_3 m)$, $(1\nu^\beta)(m) = \nu^\beta(1^{-1} +_3 m)$ and the (α, β) -PFRC of ψ^* is the (α, β) -PFS $\psi^* 1 = (\mu^\alpha 1, \nu^\beta 1)$, defined by $(\mu^\alpha 1)(m) = \mu^\alpha(m +_3 1^{-1})$, $(\nu^\beta 1)(m) = \nu^\beta(m +_3 1^{-1})$ for all $m \in G$.

When $m = 0$, $(1\mu^\alpha)(0) = \mu^\alpha(1^{-1} +_3 0) = \mu^\alpha(2 +_3 0) = \mu^\alpha(2) = 0.7$and also, $(\mu^\alpha 1)(0) = \mu^\alpha(0 +_3 1^{-1}) = \mu^\alpha(0 +_3 2) = \mu^\alpha(2) = 0.7 (1\nu^\beta)(0) = \nu^\beta(1^{-1} +_3 0) = \nu^\beta(2 +_3 0) = \nu^\beta(2) = 0.2$ and $(\nu^\beta 1)(0) = \nu^\beta(0 +_3 1^{-1}) = \nu^\beta(0 +_3 2) = \nu^\beta(2) = 0.2$.
So $(1\mu^\alpha)(0) = (\mu^\alpha 1)(0)$ and $(1\nu^\beta)(0) = (\nu^\beta 1)(0)$.

Similarly, we can check that the result holds when $m = 1$ and 2. Therefore $(1\mu^\alpha)(m) = (\mu^\alpha 1)(m)$ and $(1\nu^\beta)(m) = (\nu^\beta 1)(m)$ for all $m \in G$. i.e, $1\psi^* = \psi^* 1$. In the same manner, it can be shown that $u\psi^* = \psi^* u$ for all $u \in G$.
Hence $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α, β) -PFNSG of the group $(\mathcal{Z}_3, +_3)$.

Proposition 4.4 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFSG of a group $(G,\circ)$. Then for all $u, v \in G$, ψ^* is a (α, β) -PFNSG of G if and only if $\mu^\alpha(u \circ v) = \mu^\alpha(v \circ u)$ and $\nu^\beta(u \circ v) = \nu^\beta(v \circ u)$.

Proof. Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α, β) -PFNSG of a group $(G,\circ)$. Then for all $u \in G$, $u\psi^* = \psi^* u$So for all $u, m \in G$, $(u\mu^\alpha)(m) = (\mu^\alpha u)(m)$ and $(u\nu^\beta)(m) = (\nu^\beta u)(m)$.
Therefore for all $u, m \in G$, $\mu^\alpha(u^{-1} \circ m) = \mu^\alpha(m \circ u^{-1})$ and $\nu^\beta(u^{-1} \circ m) = \nu^\beta(m \circ u^{-1})$.
So, $\mu^\alpha(u \circ v) = \mu^\alpha(u \circ (v^{-1})^{-1}) = \mu^\alpha((v^{-1})^{-1} \circ u) = \mu^\alpha(v \circ u)$ and$\nu^\beta(u \circ v) = \nu^\beta(u \circ (v^{-1})^{-1}) = \nu^\beta((v^{-1})^{-1} \circ u) = \nu^\beta(v \circ u)$.

Conversely, let for all $u, v \in G$, $\mu^\alpha(u \circ v) = \mu^\alpha(v \circ u)$ and $\nu^\beta(u \circ v) = \nu^\beta(v \circ u)$.
This gives, $\mu^\alpha(u \circ (v^{-1})^{-1}) = \mu^\alpha((v^{-1})^{-1} \circ u)$ and $\nu^\beta(u \circ (v^{-1})^{-1}) = \nu^\beta((v^{-1})^{-1} \circ u)$.
Put $v^{-1} = g$. Then for all $u, g \in G$, $\mu^\alpha(u \circ g^{-1}) = \mu^\alpha(g^{-1} \circ u)$ and $\nu^\beta(u \circ g^{-1}) = \nu^\beta(g^{-1} \circ u)$
$\Rightarrow (\mu^\alpha g)(u) = (g\mu^\alpha)(u)$ and $(\nu^\beta g)(u) = (g\nu^\beta)(u)$
$\Rightarrow \mu^\alpha g = g\mu^\alpha$ and $\nu^\beta g = g\nu^\beta$
$\Rightarrow \psi^* g = g\psi^*$ for all $g \in G$.

Hence ψ^* is a (α,β) -PFNSG of the group $(G,\circ)$.

Proposition 4.5 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α,β) -PFSG of a group $(G,\circ)$. Then for all $u, v \in G$, ψ^* is a (α,β) -PFNSG of G if and only if $\mu^\alpha(v \circ u \circ v^{-1}) = \mu^\alpha(u)$ and $\nu^\beta(v \circ u \circ v^{-1}) = \nu^\beta(u)$.

Proof. Let us assume that, $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α,β) -PFNSG of a group $(G,\circ)$.
Then for all $u, v \in G$, $\mu^\alpha(u \circ v) = \mu^\alpha(v \circ u)$ and $\nu^\beta(u \circ v) = \nu^\beta(v \circ u)$.
Therefore

$$\begin{aligned}
&\mu^\alpha(v \circ u \circ v^{-1}) = \mu^\alpha((v \circ u) \circ v^{-1}) \\
&= \mu^\alpha(v^{-1} \circ (v \circ u)) \qquad \text{(Using the above condition)} \\
&= \mu^\alpha(v^{-1} \circ v \circ u) \\
&= \mu^\alpha(e \circ u) \\
&= \mu^\alpha(u)
\end{aligned}$$

and

$$\begin{aligned}
&\nu^\beta(v \circ u \circ v^{-1}) = \nu^\beta((v \circ u) \circ v^{-1}) \\
&= \nu^\beta(v^{-1} \circ (v \circ u)) \qquad \text{(Using the above condition)} \\
&= \nu^\beta(v^{-1} \circ v \circ u) \\
&= \nu^\beta(e \circ u) \\
&= \nu^\beta(u)
\end{aligned}$$

Therefore if $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α,β) -PFNSG of a group $(G,\circ)$ then for all $u, k \in G$

$\mu^\alpha(v \circ u \circ v^{-1}) = \mu^\alpha(u)$ and $\nu^\beta(v \circ u \circ v^{-1}) = \nu^\beta(u)$.

Conversely, let for all $u, v \in G$, $\mu^\alpha(v \circ u \circ v^{-1}) = \mu^\alpha(u)$ and $\nu^\beta(v \circ u \circ v^{-1}) = \nu^\beta(u)$.
Therefore $u, v \in G$

$$\begin{aligned}
&\mu^\alpha(u \circ v) = \mu^\alpha(v^{-1} \circ v \circ u \circ v) \\
&= \mu^\alpha((v^{-1}) \circ (v \circ u) \circ (v^{-1})^{-1}) \\
&= \mu^\alpha(v \circ u) \qquad \text{(Using the above condition)}
\end{aligned}$$

and

$$\begin{aligned}
&\nu^\beta(u \circ v) = \nu^\beta(v^{-1} \circ u \circ u \circ v) \\
&= \nu^\beta((v^{-1}) \circ (v \circ u) \circ (v^{-1})^{-1}) \\
&= \nu^\beta(v \circ u) \qquad \text{(Using the above condition)}
\end{aligned}$$

Hence by using previous proposition 4.4, $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α,β) -PFNSG of the group $(G,\circ)$.

Theorem 4.6 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α,β) -PFNSG of a group $(G,\circ)$. Then$K = \{u \in G |$ $\mu^\alpha(u) = \mu^\alpha(e)$ and $\nu^\beta(u) = \nu^\beta(e)\}$ is a normal subgroup of the group $(G,\circ)$, where e is the identity element in G.

Proof. K is a non empty set, as $e \in K$.
By Proposition 3.15, we have K is a subgroup of the group $(G,\circ)$.
Let $u \in G$ and $k \in K$.
Since, $k \in K$, $\mu^\alpha(k) = \mu^\alpha(e)$ and $\nu^\beta(k) = \nu^\beta(e)$.
As $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α,β) -PFNSG of the group $(G,\circ)$, by Proposition 4.5, we have

$\mu^\alpha(u \circ k \circ u^{-1}) = \mu^\alpha(k)$ and $\nu^\beta(u \circ k \circ u^{-1}) = \nu^\beta(k)$ for all $u, k \in G$.

$\Rightarrow \mu^\alpha(u \circ k \circ u^{-1}) = \mu^\alpha(e)$ and $\nu^\beta(u \circ k \circ u^{-1}) = \nu^\beta(e)$ for all $u, k \in G$

$$\Rightarrow u \circ k \circ u^{-1} \in K$$

Therefore $(K,\circ)$ is a normal subgroup of the group $(G,\circ)$.

5. (α,β) -Pythagorean Fuzzy Level Subgroup (PFLSG)

In this section, we will describe (α,β) -Pythagorean fuzzy level subgroup and its properties.

Definition 5.1 Let S be a crisp set. Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α,β) -PFS of S. For θ and $\tau \in [0,1]$, the set $\psi^*_{(\theta,\tau)} = \{u \in S | \mu^\alpha(u) \geq \theta$ and $\nu^\beta(u) \leq \tau\}$ is called a (α,β) -Pythagorean fuzzy level subset (PFLS) of the (α,β) -PFS ψ^* of S.

Proposition 5.2 Let $\psi_1^* = (\mu_1^\alpha, \nu_1^\beta)$ and $\psi_2^* = (\mu_2^\alpha, \nu_2^\beta)$ be two (α,β) -PFSs of a crisp set S. Then,

(i) $\psi^*_{(\theta,\tau)} \subseteq \psi^*_{(\epsilon,\delta)}$ if $\epsilon \leq \theta$ and $\tau \leq \delta$ for ϵ, τ, θ and $\delta \in [0,1]$.

(ii) $\psi_1^* \subseteq \psi_2^* \Rightarrow \psi^*_{1(\theta,\tau)} \subseteq \psi^*_{2(\theta,\tau)}$ for θ and $\tau \in [0,1]$.

Proof. (i) Let $u \in \psi^*_{(\theta,\tau)} \Rightarrow \ \mu^\alpha(u) \geq \theta,\ \nu^\beta(u) \leq \tau$.

Given, $\epsilon \leq \theta$ and $\tau \leq \delta$. So, $\epsilon \leq \theta \leq \mu^\alpha(u)$ and $\nu^\beta(u) \leq \tau \leq \delta$.

Therefore, $u \in \psi^*_{(\epsilon,\delta)}$.

Hence $\epsilon \leq \theta$ and $\tau \leq \delta \Rightarrow \psi^*_{(\theta,\tau)} \subseteq \psi^*_{(\epsilon,\delta)}$.

(ii) Since $\psi_1^* \subseteq \psi_2^*$, so $\mu_1^\alpha(u) \leq \mu_2^\alpha(u)$ and $\nu_1^\beta(u) \geq \nu_2^\beta(u)$ for all $u \in S$.

Let $u \in \psi^*_{1(\theta,\tau)}$. This implies that, $\mu_1^\alpha(u) \geq \theta$ and $\nu_1^\beta(u) \leq \tau$

$\Rightarrow \theta \leq \mu_1^\alpha(u) \leq \mu_2^\alpha(a)$ and $\nu_2^\beta(u) \leq \nu_1^\beta(u) \leq \tau$

$\Rightarrow \theta \leq \mu_2^\alpha(u)$ and $\nu_2^\beta(u) \leq \tau$

$\Rightarrow u \in \psi^*_{2(\theta,\tau)}$

Hence $\psi^*_{1(\theta,\tau)} \subseteq \psi^*_{2(\theta,\tau)}$ for θ and $\tau \in [0,1]$.

Proposition 5.3 Let $\psi^* = (\mu^\alpha, \nu^\beta)$ be a (α,β) -PFSG of a group $(G,\circ)$. Then the (α,β) -PFLS $\psi^*_{(\theta,\tau)}$ forms a subgroup of the group $(G,\circ)$, where $\theta \leq \mu^\alpha(e)$ and $\tau \geq \nu^\beta(e)$, e being the identity element in G.

Proof. We have, $\psi^*_{(\theta,\tau)} = \{u \in G | \mu^\alpha(u) \geq \theta$ and $\nu^\beta(u) \leq \tau\}$

So, $\psi^*_{(\theta,\tau)}$ is a non empty set, as $e \in \psi^*_{(\theta,\tau)}$.To show that $\psi^*_{(\theta,\tau)}$ is a subgroup of $(G,\circ)$, we have to show that for $u, v \in \psi^*_{(\theta,\tau)}$, $u \circ v^{-1} \in \psi^*_{(\theta,\tau)}$.

Let $u, v \in \psi^*_{(\theta,\tau)}$. Then $\mu^\alpha(u) \geq \theta$ &$\nu^\beta(u) \leq \tau$ and $\mu^\alpha(v) \geq \theta$ &$\nu^\beta(v) \leq \tau$.

Since $\psi^* = (\mu^\alpha, \nu^\beta)$ is a (α,β) -PFSG of the group $(G,\circ)$, then

$$\mu^\alpha(u \circ v^{-1}) \geq \mu^\alpha(u) \wedge \mu^\alpha(v^{-1})$$
$$= \mu^\alpha(u) \wedge \mu^\alpha(v))$$
$$\geq \theta \wedge \theta$$

$= \theta$

and,

$\nu^{\beta}(u \circ v^{-1}) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v^{-1})$

$= \nu^{\beta}(u) \vee \nu^{\beta}(v))$

$\leq \tau \vee \tau$

$= \tau$

Therfore, $u \circ v^{-1} \in \psi^*_{(\theta,\tau)}$.Hence $\psi^*_{(\theta,\tau)}$ is a subgroup of the group $(G,\circ)$.

Definition 5.4 The subgroup $\psi^*_{(\theta,\tau)}$ of the group $(G,\circ)$, is called the (α,β) -Pythagorean fuzzy level subgroup (PFLSG) of the (α,β) -PFSG $\psi^* = (\mu^{\alpha},\nu^{\beta})$.

Proposition 5.5 Let $\psi^* = (\mu^{\alpha},\nu^{\beta})$ be a (α,β) -PFS of a group $(G,\circ)$. If the (α,β) -PFLS $\psi^*_{(\theta,\tau)}$ is a subgroup of the group $(G,\circ)$, with $\theta \leq \mu^{\alpha}(e)$ and $\tau \geq \nu^{\beta}(e)$, where e is the identity element in G, then $\psi^* = (\mu^{\alpha},\nu^{\beta})$ is a (α,β) -PFSG of the group $(G,\circ)$.

Proof. Given that $\psi^* = (\mu^{\alpha},\nu^{\beta})$ is a (α,β) -PFS of G and $\psi^*_{(\theta,\tau)}$ is a subgroup of the group $(G,\circ)$. Let $u,v \in G$ and also assume that $\mu^{\alpha}(u) = \theta_1$, $\mu^{\alpha}(v) = \theta_2$ with $\theta_1 < \theta_2$ and $\nu^{\beta}(u) = \tau_1$, $\nu^{\beta}(v) = \tau_2$ with $\tau_1 > \tau_2$.

Therefore, $u \in \psi^*_{(\theta_1,\tau_1)}$ and $v \in \psi^*_{(\theta_2,\tau_2)}$.

Since, $\theta_1 < \theta_2$ and $\tau_1 > \tau_2$, then $\psi^*_{(\theta_2,\tau_2)} \subseteq \psi^*_{(\theta_1,\tau_1)}$. So, $v \in \psi^*_{(\theta_1,\tau_1)}$.

Now, $u \in \psi^*_{(\theta_1,\tau_1)}$ and $v \in \psi^*_{(\theta_1,\tau_1)}$

$\Rightarrow u \circ v \in \psi^*_{(\theta_1,\tau_1)}$ $\big($Since $\psi^*_{(\theta_1,\tau_1)}$ is a subgroup of $(G,\circ)\big)$

Therefore, $\mu^{\alpha}(u \circ v) \geq \theta_1$ and $\nu^{\beta}(u \circ v) \leq \tau_1$

$\Rightarrow \mu^{\alpha}(u \circ v) \geq \theta_1 \wedge \theta_2$ and $\nu^{\beta}(u \circ v) \leq \tau_1 \vee \tau_2$ (Since $\theta_1 < \theta_2$ and $\tau_1 > \tau_2$)

$\Rightarrow \mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$ and $\nu^{\beta}(u \circ v) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$.

Again, $u \in \psi^*_{(\theta_1,\tau_1)} \Rightarrow u^{-1} \in \psi^*_{(\theta_1,\tau_1)}$ $\big($Since $\psi^*_{(\theta_1,\tau_1)}$ is a subgroup of $(G,\circ)\big)$

$\Rightarrow \mu^{\alpha}(u^{-1}) \geq \theta_1$ and $\nu^{\beta}(u^{-1}) \leq \tau_1$

$\Rightarrow \mu^{\alpha}(u^{-1}) \geq \mu^{\alpha}(u)$ and $\nu^{\beta}(u^{-1}) \leq \nu^{\beta}(u)$.

Since $u,v \in G$ are arbitrary then for all $u,v \in G$,

$\mu^{\alpha}(u \circ v) \geq \mu^{\alpha}(u) \wedge \mu^{\alpha}(v)$, $\nu^{\beta}(u \circ v) \leq \nu^{\beta}(u) \vee \nu^{\beta}(v)$ and $\mu^{\alpha}(u^{-1}) \geq \mu^{\alpha}(u)$, $\nu^{\beta}(u^{-1}) \leq \nu^{\beta}(u)$.

Hence $\psi^* = (\mu^{\alpha},\nu^{\beta})$ is a (α,β) -PFSG of the group $(G,\circ)$.

Proposition 5.6 Let $\psi^* = (\mu^{\alpha},\nu^{\beta})$ be a (α,β) -PFNSG of a group $(G,\circ)$. Then the PFLS $\psi^*_{(\theta,\tau)}$ forms a normal subgroup of the group $(G,\circ)$, where $\theta \leq \mu^{\alpha}(e)$ and $\tau \geq \nu^{\beta}(e)$, e is the identity element in G.

Proof. Since $\psi^* = (\mu^{\alpha},\nu^{\beta})$ is a (α,β) -PFNSG of the group $(G,\circ)$, then for all $u,v \in G$,

$\mu^{\alpha}(v \circ u \circ v^{-1}) = \mu^{\alpha}(u)$ and $\nu^{\beta}(v \circ u \circ v^{-1}) = \nu^{\beta}(u)$.

By Proposition 5.3, $\psi^*_{(\theta,\tau)} = \{u \in G | \mu^{\alpha}(u) \geq \theta$ and $\nu^{\beta}(u) \leq \tau\}$ is a subgroup of $(G,\circ)$.

Let $m \in G$ and $n \in \psi^*_{(\theta,\tau)}$. Then $\mu^{\alpha}(m \circ n \circ m^{-1}) = \mu^{\alpha}(n) \geq \theta$ and $\nu^{\beta}(m \circ n \circ m^{-1}) = \nu^{\beta}(n) \leq \tau$.

Therefore $m \circ n \circ m^{-1} \in \psi^*_{(\theta,\tau)}$.

Hence $\psi^*_{(\theta,\tau)}$ is a normal subgroup of the group $(G,\circ)$.

6. Application

Group theory has applications in many branches of Mathematics. It has lots of applications in Chemistry. No real life application has been proposed for fuzzy group theory from its invention until now. Here we have fuzzified the dihedral group, which can be useful in the study of molecule symmetry patterns.

6.1 Modelling of dihedral group in terms of (α,β) – Pythagorean fuzzy subgroup

Symmetry is a building block of group theory, and it is used in the study of molecules. All molecules have symmetry patterns. A highly asymmetric molecule is the tetrahedral carbon (CH_4) which is studied by use of the dihedral group. Now we will express the dihedral group in terms of (α,β) - Pythagorean fuzzy subgroup. We take the dihedral group D_4 (symmetries of square): $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$ where R_0 is rotation of 0^0, R_{90^o} is rotation of 90^0, R_{180^o} is rotation of 180^0, R_{270^o} is rotation of 270^0, H is flip about a horizontal axis, V is flip about a vertical axis, D is flip about the main diagonal and D' is flip about the other diagonal. Now we assign membership degrees and non-membership degrees to each element of D_4 as shown in Table 1.

Clearly $I = \{(u, \mu(u), \nu(u) | u \in D_4\}$ is not an intuitionistic fuzzy set as $\mu(R_0) + \nu(R_0) > 1$. Also, $\psi = \{(u, \mu(u), \nu(u) | u \in D_4\}$ is not a Pythagorean fuzzy set as $\mu^2(R_{180^o}) + \nu^2(R_{180^o}) > 1$.

Now we set $\alpha = 0.7, \beta = 0.5$. Then $\Psi^* = (\mu^\alpha, \nu^\beta)$ is a (α,β) - Pythagorean fuzzy set.

We take elements H and V. Now, $\mu^\alpha(H \circ V^{-1}) = \mu^\alpha(R_{180^\circ}) = 0.7 \geq \mu^\alpha(H) \wedge \mu^\alpha(V)$ and $\nu^\beta(H \circ V^{-1}) = \nu^\beta(R_{180^\circ}) = 0.5 \leq \nu^\beta(H) \vee \nu^\beta(V)$.

Similary we can check the conditions for other elements of D_4. Therefore $\Psi^* = (\mu^\alpha, \nu^\beta)$ is a (α,β) - Pythagorean fuzzy subgroup.

7. Conclusions

Pythagorean fuzzy set is the latest tool for depicting uncertainty. Here we have described (α,β) - Pythagorean fuzzy sets that are more fruitful than the intuitionistic fuzzy sets or the Pythagorean fuzzy sets. We gave a brief demonstration of (α,β) - Pythagorean fuzzy subgroup and proved that every intuitionistic fuzzy subgroup is an (α,β) - Pythagorean fuzzy subgroup but the converse need not be true. A necessary and sufficient condition for (α,β) - Pythagorean fuzzy set to be an (α,β) - Pythagorean fuzzy subgroup is given. Certain conditions are derived for (α,β) - Pythagorean fuzzy subgroup of a group to be a (α,β) - Pythagorean fuzzy normal subgroup of that group. (α,β) - Pythagorean fuzzy level subgroup is discussed. We have proved that (α,β) - PFLS is a normal subgroup of the given group. Finally, we represented the dihedral group D_4 in terms of an (α,β) - Pythagorean fuzzy subgroup.

Table 1. Membership and non-membership values of D_4

	R_0	R_{90^o}	R_{180^o}	R_{270^o}	H	V	D	D'
μ	0.9	0.7	0.95	0.7	0.75	0.8	0.85	0.8
ν	0.3	0.4	0.4	0.4	0.5	0.5	0.4	0.4

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