

CHAPTER 3 EPIDEMIC MODEL WITH TRANSPORT-RELATED INFECTION

This chapter is organized as follows. The model with transport-related infection is formulated based on *SEIRS* (susceptible - exposed - infected - recovered) in section 3.1. In section 3.2, the effect of transport-related infection is studied via mathematical analysis of model including stability of equilibria of the model.

3.1 Model Formulation

The epidemic model for transmission of a communicable disease with population travel between two cities is based on monitoring the dynamics of the sub-population (susceptible; $S_i(t)$, exposed(latent); $E_i(t)$, infected; $I_i(t)$, and recovered; $R_i(t)$, in the city i , $i = 1, 2$ at time t). Thus, the total population in city i at time t is given by $N_i = S_i(t) + E_i(t) + I_i(t) + R_i(t)$ for $i = 1, 2$. It is assumed that both cities are identical, i.e. the demographic parameters are the same for each city.

The population of susceptible individuals is increased by the recruitment of individuals which are all newborn into the population at the rate a and the loss of infection-acquired immunity among recovered individuals (at the rate α_2) and by the susceptible individuals of city j leave to city i ($j \neq i, i = 1, 2$) (at the rate α_1). In the other hand, it is decreased when the susceptible individuals in city i leave to city j (at the rate α_1) and by natural death (at the rate b). It is assumed that susceptible individuals can acquire exposed individuals via effective contacts with infected individuals. The disease is transmitted horizontally within and between cities according to standard the incidence rate (that is, the number of new cases of infection per unit time)

$$\frac{\beta S_i I_i}{N_i}, \quad \text{for } i = 1, 2,$$

where β is the transmission rate within a city.

This population is further decreased when the individuals in city j travel to city i , and the disease is transmitted with the incidence rate

$$\frac{\gamma(\alpha_1 S_j)(\alpha_1 I_j)}{(\alpha_1 S_j + \alpha_1 E_j + \alpha_1 I_j + \alpha_1 R_i)} = \frac{\gamma \alpha_1 S_j I_j}{N_j}, \quad \text{for } j = 1, 2,$$

where γ is the transport-related transmission rate.

Thus, the rate of change of population of susceptible class is given by

$$\frac{dS_i}{dt} = a - bS_i - \frac{\beta S_i I_i}{N_i} + \alpha_2 R_i - \alpha_1 S_i + \alpha_1 S_j - \frac{\gamma \alpha_1 S_j I_j}{N_j}. \quad (3.1)$$

The population of exposed individuals is generated by the infected of susceptible individuals at the rate $\frac{\beta S_i I_i}{N_i}$ and at the rate $\frac{\gamma \alpha_1 S_j I_j}{N_j}$ when the individuals in city j travel to city i . It is reduced by progression to symptoms development (at the rate c), travel to city j (at the rate α_1) and natural death (at the rate b). Thus

$$\frac{dE_i}{dt} = \frac{\beta S_i I_i}{N_i} - (b + c + \alpha_1) E_i + \alpha_1 E_j + \frac{\gamma \alpha_1 S_j I_j}{N_j}. \quad (3.2)$$

The population of infected individuals in city i is generated when exposed individuals develop symptoms (at the rate c), and when infected individuals of city j leave to city i (at the rate α_1). It is decreased by progression to the recovered class (at the rate d), natural death and disease induced mortality (at the rate e), and when infected individuals of city i move to city j (at the rate α_1). Thus,

$$\frac{dI_i}{dt} = cE_i - (e + d + \alpha_1) I_i + \alpha_1 I_j. \quad (3.3)$$

The population of recovered individuals is generated when infected individuals recover and move to the recovered class at the rate d , and when recovered individuals of city j leave to city i . It is decreased by the loss of infection-acquired immunity (at the rate α_2), by natural death (at the rate b), and when recovered individuals of city i move to city j (at the rate α_1). Thus,

$$\frac{dR_i}{dt} = dI_i - (b + \alpha_1 + \alpha_2) R_i + \alpha_1 R_j. \quad (3.4)$$

It is assumed that the individuals have no infectious force in the latent period and the exposed individuals cannot recover to susceptible individuals. The individuals who are travelling do not give birth and do not take die. Infected individuals do not recover during travel. Thus, the epidemic model for transmission of a communicable disease with population travel between two cities i_1 consists of the following system of non-linear differential equations:

$$\begin{aligned}
\frac{dS_1}{dt} &= a - bS_1 - \frac{\beta S_1 I_1}{N_1} + \alpha_2 R_1 - \alpha_1 S_1 + \alpha_1 S_2 - \frac{\gamma \alpha_1 S_2 I_2}{N_2}, \\
\frac{dE_1}{dt} &= \frac{\beta S_1 I_1}{N_1} - (b + c + \alpha_1) E_1 + \alpha_1 E_2 + \frac{\gamma \alpha_1 S_2 I_2}{N_2}, \\
\frac{dI_1}{dt} &= cE_1 - (e + d + \alpha_1) I_1 + \alpha_1 I_2, \\
\frac{dR_1}{dt} &= dI_1 - (b + \alpha_1 + \alpha_2) R_1 + \alpha_1 R_2, \\
\frac{dS_2}{dt} &= a - bS_2 - \frac{\beta S_2 I_2}{N_2} + \alpha_2 R_2 - \alpha_1 S_2 + \alpha_1 S_1 - \frac{\gamma \alpha_1 S_1 I_1}{N_1}, \\
\frac{dE_2}{dt} &= \frac{\beta S_2 I_2}{N_2} - (b + c + \alpha_1) E_2 + \alpha_1 E_1 + \frac{\gamma \alpha_1 S_1 I_1}{N_1}, \\
\frac{dI_2}{dt} &= cE_2 - (e + d + \alpha_1) I_2 + \alpha_1 I_1, \\
\frac{dR_2}{dt} &= dI_2 - (b + \alpha_1 + \alpha_2) R_2 + \alpha_1 R_1.
\end{aligned} \tag{3.5}$$

A flow diagram of the model is depicted in Figure 3.1. The standard incidence is used in the model. If initial conditions are set as $S_i(0) \geq 0$, $E_i(0) \geq 0$, $I_i(0) \geq 0$ and $R_i(0) \geq 0$, it is easy to check that all solutions of (3.5) are nonnegative (that is $S_i(0) \geq 0$, $E_i(0) \geq 0$, $I_i(0) \geq 0$ and $R_i(0) \geq 0$ for $t > 0$, $i = 1, 2$) under the assumption $0 \leq \gamma \leq 1$. Note that the last two terms in the first and fifth equations of (3.5) satisfy that

$$\alpha_1 S_i - \frac{\gamma \alpha_1 S_i I_i}{N_i} \geq 0, (i = 1, 2),$$

for any $S_i \geq 0$, $E_i \geq 0$, $I_i \geq 0$ and $R_i \geq 0$ when $0 \leq \gamma \leq 1$. This is reasonable from a biological point of view, since the first term $\alpha_1 S_i$ represents the susceptible individuals leaving city i and the second term $\frac{\gamma \alpha_1 S_i I_i}{N_i}$ denotes individuals in $\alpha_1 S_i$ becoming infected during travel from city i to j . Hence, the difference between these two numbers should be nonnegative. It is supposed that $0 \leq \gamma \leq 1$.

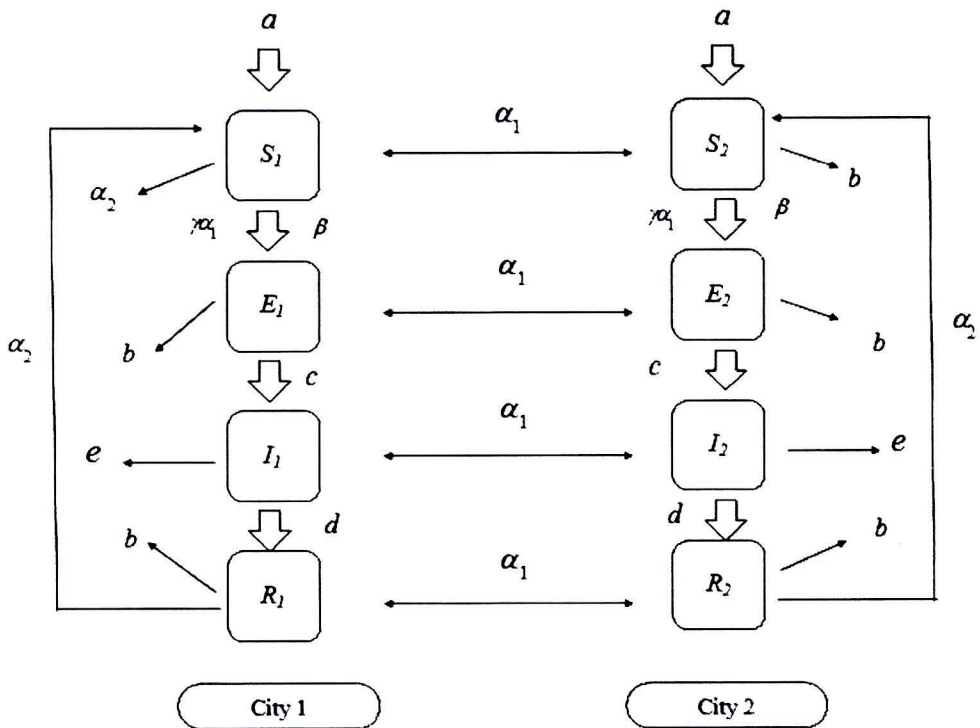


Figure 3.1 Schematic diagram of the *SEIRS* model for transmission of a communicable disease with population travel between two cities.

3.2 Analysis of the Model

In this section, the model (3.5) will be qualitatively analyzed into three cases which are no individual travels, only susceptible and exposed individuals travel, and all individuals travel. In each cases, the existence and locally stability of its associated equilibrium is investigated to understand the effect of transport-related infection for disease transmission.

3.2.1 No Individual Travels

The movement of individuals is neglected; consider the case where $\alpha_1 = 0$, model (3.5) can be reduced to the *SEIRS* model:

$$\begin{aligned}
 \frac{dS}{dt} &= a - \frac{\beta SI}{N} - bS + \alpha_2 R, \\
 \frac{dE}{dt} &= \frac{\beta SI}{N} - (b + c) E, \\
 \frac{dI}{dt} &= cE - (e + d) I, \\
 \frac{dR}{dt} &= dI - (b + \alpha_2) R.
 \end{aligned}
 \tag{3.6}$$

3.2.1.1 Disease-Free Equilibrium

In the absence of infection (that is $E = I = 0$), the model (3.6) has a unique disease-free equilibrium (DFE), P_0 , obtained by setting the derivatives in (3.6) of zero, given by $P_0 = P_0(S^0, 0, 0, 0)$ where

$$S^0 = \frac{a}{b}. \quad (3.7)$$

The basic reproduction number of (3.6) can be established using the next generation matrix operator method. Using the notation in Lemma 2.2 and Theorem 2.3, the matrices \mathcal{F} (for the new infection term) and \mathcal{V} (for the transition term) are given by

$$\mathcal{F} = \begin{pmatrix} -\frac{\beta SI}{N} \\ \frac{\beta SI}{N} \\ 0 \\ 0 \end{pmatrix} \quad (3.8)$$

and

$$\mathcal{V} = \begin{pmatrix} -a + bS - \alpha_2 R \\ (b + c)E \\ -cE + (e + d)I \\ -dI + (b + \alpha_2)R \end{pmatrix}, \quad (3.9)$$

respectively. The partial derivatives of (3.8) and (3.9) with respect to S, E, I, R , at disease-free equilibrium (P_0), give

$$D\mathcal{F}(P_0) = F = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad (3.10)$$

and

$$D\mathcal{V}(P_0) = V = \begin{pmatrix} b + c & 0 \\ -c & e + d \end{pmatrix}. \quad (3.11)$$

The matrices of (3.10) and (3.11) are of size 2×2 since there are two infectious classes.

Now, taking the inverse of matrix (3.11) leads to

$$V^{-1} = \begin{pmatrix} \frac{1}{b + c} & 0 \\ \frac{c}{(b + c)(e + d)} & \frac{1}{e + d} \end{pmatrix}. \quad (3.12)$$

Hence, the basic reproduction number for model (3.6) is

$$\mathcal{R}_0 = \rho(FV^{-1}) = \frac{\beta c}{(e + d)(b + c)}. \quad (3.13)$$

Here $\rho(FV^{-1})$ represents the spectral radius of the matrix FV^{-1} .

Theorem 3.1 The disease-free equilibrium (*DFE*), P_0 , of the system (3.6) is locally asymptotically stable (*LAS*) if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

Proof. The Jacobian of system (3.6) at *DFE*, $P_0(\frac{a}{b}, 0, 0, 0)$, is

$$J(P_0) = \begin{pmatrix} -b & 0 & -\beta & \alpha_2 \\ 0 & -b-c & \beta & 0 \\ 0 & c & -e-d & 0 \\ 0 & 0 & d & -b-\alpha_2 \end{pmatrix}, \quad (3.14)$$

the eigenvalues of (3.14) are $\lambda_1 = -b < 0$, $\lambda_2 = -(b + \alpha_2) < 0$, λ_3 and λ_4 are the roots of the quadratic equation:

$$f(\lambda) = \lambda^2 + (b + c + d + e)\lambda + (b + c)(e + d)(1 - \mathcal{R}_0) = 0.$$

It is easy to see that $\lambda_3 < 0$ and $\lambda_4 < 0$ if $\mathcal{R}_0 < 1$. Thus, by Theorem 2.1, P_0 is *LAS* if $\mathcal{R}_0 < 1$. □

The basic reproduction number [26], denoted as \mathcal{R}_0 , represents the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. If $\mathcal{R}_0 < 1$, then infected individuals in its entire period of infectivity will produce less than one infected individuals on average. Thus, disease will be wiped out of population. On the contrary, if $\mathcal{R}_0 > 1$, then infected individuals in its entire infective period having contact with susceptible individuals will produce more than one infected individuals on average, which will then lead to the disease invading the susceptible population.

3.2.1.2 Endemic Equilibrium

In the presence of infection ($E \neq 0$ and $I \neq 0$), the system (3.6) has an endemic equilibrium $P_*(S^*, E^*, I^*, R^*)$ where

$$S^* = \frac{a \{(b + \alpha_2)(c + d + e) + cd\}}{\Omega}, \quad (3.15)$$

$$E^* = \frac{a(\mathcal{R}_0 - 1)(b + \alpha_2)(e + d)}{\Omega}, \quad (3.16)$$

$$I^* = \frac{ac(\mathcal{R}_0 - 1)(b + \alpha_2)}{\Omega}, \quad (3.17)$$

$$R^* = \frac{acd(\mathcal{R}_0 - 1)(b + \alpha_2)}{\Omega}, \quad (3.18)$$

and

$$\Omega = (\mathcal{R}_0 - 1)(b(be + cd + d\alpha_2 + e\alpha_2 + bd + ce) + ce\alpha_2) + b(bc + \alpha_2 + be + cd + d\alpha_2 + e\alpha_2 + bd),$$

with

$$\begin{aligned} N^* &= S^* + E^* + I^* + R^* \\ &= \frac{a\mathcal{R}_0 \{(b + \alpha_2)(c + d + e) + cd\}}{\Omega} = \mathcal{R}_0 S^*. \end{aligned} \quad (3.19)$$



Clearly, it is evident from equations (3.15)–(3.18) that if $\mathcal{R}_0 < 1$, the model has no positive endemic equilibrium (Since E^* , I^* and R^* will be negative values which are biologically unrealistic). Therefore, the system (3.6) has an endemic equilibrium if $\mathcal{R}_0 > 1$.

Theorem 3.2 If $\mathcal{R}_0 > 1$, the endemic equilibrium, P_* , is *LAS*.

Proof. The Jacobian of system (3.6) at P_* is

$$J(P_*) = \begin{pmatrix} -b - \psi_1 & \psi_2 & -\psi_3 & \psi_2 + \alpha_2 \\ \psi_1 & -b - c - \psi_2 & \psi_3 & -\psi_2 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix}, \quad (3.20)$$

where

$$\psi_1 = \frac{\beta I^*(N^* - S^*)}{N^{*2}} = \frac{\beta c (\mathcal{R}_0 - 1)^2 (b + \alpha_2)}{\mathcal{R}_0^2 ((b + \alpha_2)(c + d + e) + cd)}, \quad (3.21)$$

$$\psi_2 = \frac{\beta S^* I^*}{N^{*2}} = \frac{\beta c (\mathcal{R}_0 - 1) (b + \alpha_2)}{\mathcal{R}_0^2 ((b + \alpha_2)(c + d + e) + cd)}, \quad (3.22)$$

$$\psi_3 = \frac{\beta S^*(N^* - I^*)}{N^{*2}} = \frac{(\beta c + \beta \mathcal{R}_0 (d + e))(b + \alpha_2) + \beta cd \mathcal{R}_0}{\mathcal{R}_0^2 ((b + \alpha_2)(c + d + e) + cd)}, \quad (3.23)$$

and $N^* = S^* + E^* + I^* + R^*$.

According to Lemma 2.1, the endemic equilibrium, P_* , is *LAS* if the following conditions hold:

- (i) $A_1 > 0$. Obviously, $\psi_i > 0$ for $i = 1, 2, 3$ when $\mathcal{R}_0 > 1$, and $a_{ii} < 0$ for $i = 1, 2, 3, 4$. Thus, $A_1 = -(a_{11} + a_{22} + a_{33} + a_{44}) > 0$.
- (ii) $A_2 = J_1 + J_2 + J_3$. Obviously, $a_{ii} < 0$ for $i = 1, 2, 3, 4$ then

$$J_1 = a_{44}a_{33} + a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11} > 0.$$

For J_2 , it is seen that

$$\begin{aligned} J_2 &= a_{33}a_{22} - a_{32}a_{23} \\ &= (b + c + \psi_2)(e + d) - c\psi_3 \\ &= \psi_2(e + d) + (b + c)(e + d) - c\psi_3 \\ &= \psi_2(e + d) + \beta c \left[\frac{1}{\mathcal{R}_0} - \frac{S^*(N^* - I^*)}{N^{*2}} \right]. \end{aligned}$$

From (3.19) gives $S^* = \frac{N^*}{\mathcal{R}_0}$ then

$$J_2 = \psi_2(e + d) + \beta c \left(\frac{S^* I^*}{N^{*2}} \right) = \psi_2(e + d + c) > 0,$$

and

$$J_3 = a_{22}a_{11} - a_{21}a_{12} = \psi_2 b + (b + c)(\psi_1 + b) > 0.$$

Thus, $A_2 = J_1 + J_2 + J_3 > 0$.

(iii) $A_3 = Q_1 + Q_2 + Q_3$. Since $-a_{33} > 0$, $-a_{44} > 0$, $J_2 > 0$ and $J_3 > 0$, this yields

$$Q_1 = -a_{44}(J_2 + J_3) > 0, \quad Q_2 = -a_{33}J_3 > 0$$

and

$$\begin{aligned} Q_3 &= -(a_{32}(a_{21}a_{13} + a_{43}a_{24}) + a_{11}(a_{44}a_{33} - a_{32}a_{23})) \\ &= \psi_2dc + (\psi_1 + b)(b + \alpha_2)(e + d) - bc\psi_3 \\ &= \psi_2c(b + d) + (e + d)[\psi_1(b + \alpha_2) + b\alpha_2] - cb(e + d). \end{aligned}$$

Furthermore,

$$Q_2 + Q_3 = \psi_2c(b + d) + (e + d)[\psi_1(b + \alpha_2) + b\alpha_2 + b(b + \psi_1) + \psi_1c + \psi_2b] > 0.$$

Thus, $A_3 = Q_1 + Q_2 + Q_3 > 0$.

(iv) $A_4 = \det(J(P_*)) = L_1 + L_2 + L_3$, Since $a_{33} < 0$, $a_{44} < 0$ and $J_3 > 0$, it is found that

$$\begin{aligned} L_1 &= a_{44}a_{33}J_3 = (b + \alpha_2)(e + d)J_3, \\ L_2 &= dc(b\psi_2 - \psi_1\alpha_2), \\ L_3 &= -(b + \alpha_2)c\psi_3b. \end{aligned}$$

It is verified that

$$\begin{aligned} A_4 &= b(e + d)(b + \alpha_2)(\psi_1 + \psi_2) + cb\psi_2(b + \alpha_2 + cd) \\ &\quad + b\psi_1(e + d)(b + \alpha_2 + c) + c\alpha_2e\psi_1 > 0. \end{aligned}$$

(v) $A_1A_2 - A_3 > 0$. Since $a_{ii} < 0$ for $i = 1, 2, 3, 4$, and $J_i > 0$ for $i = 1, 2, 3$, it follows that

$$\begin{aligned} A_1A_2 - A_3 &= -(a_{11} + a_{22})(J_1 + J_2 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 - Q_3 \\ &= -a_{11}(J_2 + J_3) - a_{22}(J_1 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 \\ &\quad - a_{11}(a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11}) + (b + \psi_2)J_2 \\ &\quad + c\psi_2(e + c) + bc\psi_3 > 0. \end{aligned}$$

(vi) Finally, it can be shown that $A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$, see Appendix A.

Hence, by Lemma 2.1, all eigenvalues of $J(P_*)$ have negative real part when $\mathcal{R}_0 > 1$. Thus, P_* is *LAS* whenever it exists by Theorem 2.1.

□

3.2.2 Only Susceptible and Exposed Individuals Travel

In the case where the infected and recovered individuals are inhibited from travelling to another city, and no have transport-related transmission rate, that is $\alpha_1 = \gamma = 0$,

model (3.5) becomes

$$\begin{aligned}
\frac{dS_1}{dt} &= a - bS_1 - \frac{\beta S_1 I_1}{N_1} + \alpha_2 R_1 - \alpha_1 S_1 + \alpha_1 S_2, \\
\frac{dE_1}{dt} &= \frac{\beta S_1 I_1}{N_1} - (b + c + \alpha_1) E_1 + \alpha_1 E_2, \\
\frac{dI_1}{dt} &= cE_1 - (e + d) I_1, \\
\frac{dR_1}{dt} &= dI_1 - (b + \alpha_2) R_1, \\
\frac{dS_2}{dt} &= a - bS_2 - \frac{\beta S_2 I_2}{N_2} + \alpha_2 R_2 - \alpha_1 S_2 + \alpha_1 S_1, \\
\frac{dE_2}{dt} &= \frac{\beta S_2 I_2}{N_2} - (b + c + \alpha_1) E_2 + \alpha_1 E_1, \\
\frac{dI_2}{dt} &= cE_2 - (e + d) I_2, \\
\frac{dR_2}{dt} &= dI_2 - (b + \alpha_2) R_2.
\end{aligned} \tag{3.24}$$

3.2.2.1 The Local Stability of *DFE*

The model (3.24) has a disease-free equilibrium given by

$$P_1\left(\frac{a}{b}, 0, 0, 0, \frac{a}{b}, 0, 0, 0\right). \tag{3.25}$$

Here, the next generation matrices F and V are given by

$$F = \begin{pmatrix} 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} b + c + \alpha_1 & 0 & -\alpha_1 & 0 \\ -c & e + d & 0 & 0 \\ -\alpha_1 & 0 & b + c + \alpha_1 & 0 \\ 0 & 0 & -c & e + d \end{pmatrix}, \text{ respectively.}$$

It follows that, using the next generation approach, the basic reproduction number for the model (3.24) denoted by $\mathcal{R}'_0 = \rho(FV^{-1})$

$$\mathcal{R}'_0 = \frac{\beta c}{(e + d)(b + c)}. \tag{3.26}$$

Note that $\mathcal{R}'_0 = \mathcal{R}_0$, see (3.13) and (3.26). The Jacobian of the model (3.24) is given by

$$J(P) = \begin{pmatrix} A_1 & B \\ B & A_2 \end{pmatrix}, \quad (3.27)$$

where P is an equilibrium of (3.24),

$$A_i = \begin{pmatrix} -b - \alpha_1 - \frac{\beta I_i(N_i - S_i)}{N_i^2} & \frac{\beta S_i I_i}{N_i^2} & -\frac{\beta S_i(N_i - I_i)}{N_i^2} & \frac{\beta S_i I_i}{N_i^2} + \alpha_2 \\ \frac{\beta I_i(N_i - S_i)}{N_i^2} & -b - c - \alpha_1 - \frac{\beta S_i I_i}{N_i^2} & \frac{\beta S_i(N_i - I_i)}{N_i^2} & -\frac{\beta S_i I_i}{N_i^2} \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix}$$

for $i = 1, 2$, and

$$B = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$



The local stability proof for P_1 is given below.

Theorem 3.3 The disease-free equilibrium, P_1 , of the model (3.24) is *LAS* whenever $\mathcal{R}'_0 < 1$ and unstable if $\mathcal{R}'_0 > 1$.

Proof. Evaluating the Jacobian of model (3.24) at P_1 gives

$$J(P_1) = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} -b - \alpha_1 & 0 & -\beta & \alpha_2 \\ 0 & -b - c - \alpha_1 & \beta & 0 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Cui *et al.*[17], the characteristic polynomial of $J(P_1)$ is given by

$$\begin{aligned} \det(J(P_1) - \lambda I) &= \det \begin{pmatrix} A - \lambda I & B \\ B & A - \lambda I \end{pmatrix} \\ &= \det \begin{pmatrix} A + B - \lambda I & B \\ A + B - \lambda I & A - \lambda I \end{pmatrix} \\ &= \det \begin{pmatrix} A + B - \lambda I & B \\ \mathbf{0} & A - B - \lambda I \end{pmatrix} \\ &= \det(A + B - \lambda I) \det(A - B - \lambda I), \end{aligned}$$

where $\mathbf{0}$ is a zero matrix and I is an identity matrix. Thus, the eigenvalues of $J(P_1)$ are identical to those of $A + B$ and $A - B$, where

$$A + B = \begin{pmatrix} -b & 0 & -\beta & \alpha_2 \\ 0 & -b - c & \beta & 0 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix},$$

and

$$A - B = \begin{pmatrix} -b - 2\alpha_1 & 0 & -\beta & \alpha_2 \\ 0 & -b - c - 2\alpha_1 & \beta & 0 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix}.$$

It is found that the eigenvalues of $A + B$ and $A - B$ are the roots of equations

$$\begin{aligned} f_{A+B}(\lambda) &= (\lambda + b)(\lambda + b + \alpha_2)(\lambda^2 + a_1\lambda + a_2) = 0, \\ f_{A-B}(\lambda) &= (\lambda + b + 2\alpha_1)(\lambda + b + \alpha_2)(\lambda^2 + a_3\lambda + a_4) = 0, \end{aligned}$$

respectively, where

$$\begin{aligned} a_1 &= b + c + d + e, \\ a_2 &= (b + c)(d + e)(1 - \mathcal{R}'_0), \\ a_3 &= (b + c + d + e + 2\alpha_1), \\ a_4 &= (e + d)(b + c + 2\alpha_1) \left[1 - \frac{\beta c}{(d + e)(b + c + 2\alpha_1)} \right]. \end{aligned}$$

It is easy to see that $a_1 > 1$, $a_2 > 1$, $a_3 > 1$, $a_4 > 1$ when $\mathcal{R}'_0 < 1$ and $\frac{\beta c}{(d + e)(b + c + 2\alpha_1)} < \mathcal{R}'_0 < 1$. These imply that, using the Routh–Hurwitz criterion (2.13), all eigenvalues of $A + B$ and $A - B$ have negative real part. Hence P_1 is *LAS* if $\mathcal{R}'_0 < 1$. □

3.2.2.2 The Existence and Local Stability of Endemic Equilibrium

In the presence of infection ($E_1, E_2, I_1, I_2 \neq 0$), model (3.24) has a unique endemic equilibrium given by $P_2(S^*, E^*, I^*, R^*, S^*, E^*, I^*, R^*)$ appearing in two cities when $\mathcal{R}'_0 > 1$. It is clear that the endemic equilibrium (P_2) is identical P_* (see (3.15)–(3.18)), as show in the following results.

Lemma 3.1 The model (3.24) has a unique endemic equilibrium, P_2 , if $\mathcal{R}'_0 > 1$.

The local stability of P_2 is now explored in the next theorem.

Theorem 3.4 The endemic equilibrium, P_2 , is *LAS* if $\mathcal{R}'_0 > 1$.

Proof. The Jacobian of (3.24) at P_2 is given by

$$J(P_2) = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} -b - \alpha_1 - \psi_1 & \psi_2 & -\psi_3 & \psi_2 + \alpha_2 \\ \psi_1 & -b - c - \alpha_1 - \psi_2 & \psi_3 & -\psi_2 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similar to the proof of Theorem 3.3, to calculate the eigenvalues of $J(P_2)$ is equivalent to calculate the eigenvalues of matrices $A + B$ and $A - B$. Since $A + B$ is the same as $A + B$ in the proof of Theorem 3.2, the eigenvalues of $A + B$ have negative real part if $\mathcal{R}'_0 > 1$. Future, the matrix $A - B$ is given by

$$A - B = \begin{pmatrix} -b - 2\alpha_1 - \psi_1 & \psi_2 & -\psi_3 & \psi_2 + \alpha_2 \\ \psi_1 & -b - c - 2\alpha_1 - \psi_2 & \psi_3 & -\psi_2 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix}.$$

For simplification, the entries of $A - B$ is denoted by a_{ij} for $i, j = 1, 2, 3, 4$. It is seen that the matrix $A - B$ is in the form (2.14). It suffices to check that matrix $A - B$ satisfies the conditions in Lemma 2.1. This is checked as following six steps.

(i) $A_1 > 0$. From (3.21)–(3.23), it obvious that $a_{ii} < 0$ for $i = 1, 2, 3, 4$, and $\psi_i > 0$ for $i = 1, 2, 3$ when $\mathcal{R}'_0 > 1$. Thus, $A_1 = -(a_{11} + a_{22} + a_{33} + a_{44}) > 0$.

(ii) $A_2 = J_1 + J_2 + J_3 > 0$. Obviously $a_{ii} < 0 (i = 1, 2, 3, 4)$, then

$$J_1 = a_{44}a_{33} + a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11} > 0.$$

For J_2 and J_3 , it is found that

$$\begin{aligned} J_2 &= a_{33}a_{22} - a_{32}a_{23} \\ &= (b + c + 2\alpha_1 + \psi_2)(e + d) - c\psi_3 \\ &= (\psi_2 + 2\alpha_1)(e + d) + c\psi_2 > 0, \end{aligned}$$

and

$$\begin{aligned} J_3 &= a_{22}a_{11} - a_{21}a_{12} \\ &= \psi_2(b + 2\alpha_1) + (b + c + 2\alpha_1)(\psi_1 + b + 2\alpha_1) > 0. \end{aligned}$$

Thus, $A_2 > 0$.

(iii) $A_3 = Q_1 + Q_2 + Q_3 > 0$. Since $J_2 > 0$, $J_3 > 0$ and $a_{33} < 0$, $a_{44} < 0$, these yield

$$Q_1 = -a_{44}(J_2 + J_3) > 0 \quad \text{and} \quad Q_2 = -a_{33}J_3 > 0.$$

For Q_3 , it can be verified that

$$\begin{aligned} Q_3 &= -(a_{32}(a_{21}a_{13} + a_{43}a_{24}) + a_{11}(a_{44}a_{33} - a_{32}a_{23})) \\ &= (\psi_1 + b + 2\alpha_1)(b + \alpha_2)(e + d) + dc\psi_2 - c(b + 2\alpha_1)\psi_3 \\ &= dc\psi_2 + (b + 2\alpha_1)c\psi_2 + (e + d)[\psi_1(b + \alpha_2) + \alpha_2(b + 2\alpha_1)] \\ &\quad - c(b + 2\alpha_1)(e + d), \end{aligned}$$

where

$$\begin{aligned} Q_2 + Q_3 &= c\psi_2(b + d + 2\alpha_1) + (e + d)[\psi_1(b + \alpha_2) + \alpha_2(b + 2\alpha_1)] \\ &\quad + (e + d)[\psi_2(b + 2\alpha_1) + \psi_1(b + c + 2\alpha_1) + (b + 2\alpha_1)^2] > 0. \end{aligned}$$

Thus, $A_3 = Q_1 + Q_2 + Q_3 > 0$.

(iv) $A_4 = \det(A - B) = L_1 + L_2 + L_3$, where

$$\begin{aligned} L_1 &= a_{44}a_{33}J_3 = (b + \alpha_2)(e + d)J_3, \\ L_2 &= dc(\psi_2(b + 2\alpha_1) - \psi_1\alpha_2), \\ L_3 &= -(b + \alpha_2)(b + 2\alpha_1)c\psi_3. \end{aligned}$$

It is verified that

$$\begin{aligned} L_1 + L_2 + L_3 &= (b + \alpha_2)(e + d)(b + 2\alpha_1)(\psi_1 + \psi_2) + c\psi_2(b + 2\alpha_1)(b + \alpha_2 + cd) \\ &\quad \psi_1(b + 2\alpha_1)(e + d)(b + \alpha_2 + c) + c\alpha_2e\psi_1 > 0, \end{aligned}$$

then $A_4 > 0$.

(v) $A_1A_2 - A_3 > 0$. Since $J_i > 0$ for $i = 1, 2, 3$, and $a_{ii} < 0$ for $i = 1, 2, 3, 4$,

$$\begin{aligned} A_1A_2 - A_3 &= -(a_{11} + a_{22})(J_1 + J_2 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 - Q_3 \\ &= -a_{11}(J_2 + J_3) - a_{22}(J_1 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 \\ &\quad - a_{11}(a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11}) + (b + \psi_2 + 2\alpha_1)J_2 \\ &\quad + ce(\psi_2 + 2\alpha_1) + c^2\psi_2 + (b + 2\alpha_1)c\psi_3 > 0. \end{aligned}$$

(vi) Finally, it can be shown that $A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$, see Appendix B.

From (i)–(v), all the eigenvalues of $A - B$ have negative real part. Since all the eigenvalues of $A - B$ and $A + B$ have negative real part whenever $\mathcal{R}'_0 > 1$, P_2 is *LAS* whenever it exists according to Theorem 2.1.

□

3.2.3 All Individuals Travel Between Two Cities

In this section, the full model (3.5) is studied when all individuals can travel between two cities. It is easy to see that a disease-free equilibrium, P_1 is the same as a disease-free equilibrium of the models (3.6) and (3.24), and is given by (3.25).

According the same processes as described in section 3.2.2, the next generation matrices F and V are given by

$$F = \begin{pmatrix} 0 & \beta & 0 & \gamma \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma \alpha_1 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} b+c+\alpha_1 & 0 & -\alpha_1 & 0 \\ -c & e+d+\alpha_1 & 0 & -\alpha_1 \\ -\alpha_1 & 0 & b+c+\alpha_1 & 0 \\ 0 & -\alpha_1 & -c & e+d+\alpha_1 \end{pmatrix}.$$

It follows that, using the next generation approach, the reproduction number of the model (3.5), denoted by $\mathcal{R}_{0\gamma}$, is

$$\mathcal{R}_{0\gamma} = \mathcal{R}_0 + \frac{\gamma \alpha_1 c}{(b+c)(d+e)}. \quad (3.28)$$

Consequently, using Theorem 2.3, the following results is established.

Lemma 3.2 The disease-free equilibrium, P_1 , of the model (3.5) is *LAS* if $\mathcal{R}_{0\gamma} < 1$, and unstable if $\mathcal{R}_{0\gamma} > 1$.

The model (3.5) has a unique endemic equilibrium, denote by $P_\gamma^*(S_\gamma^*, E_\gamma^*, I_\gamma^*, R_\gamma^*, S_\gamma^*, E_\gamma^*, I_\gamma^*, R_\gamma^*)$, appears in two cities where

$$S_\gamma^* = \frac{a \{(b + \alpha_2)(c + d + e) + cd\}}{\Omega_\gamma}, \quad (3.29)$$

$$E_\gamma^* = \frac{a(\mathcal{R}_{0\gamma} - 1)(b + \alpha_2)(e + d)}{\Omega_\gamma}, \quad (3.30)$$

$$I_\gamma^* = \frac{ac(\mathcal{R}_{0\gamma} - 1)(b + \alpha_2)}{\Omega_\gamma}, \quad (3.31)$$

$$R_\gamma^* = \frac{acd(\mathcal{R}_{0\gamma} - 1)(b + \alpha_2)}{\Omega_\gamma}, \quad (3.32)$$

and

$$\begin{aligned} \Omega_\gamma &= (\mathcal{R}_{0\gamma} - 1)(b(be + cd + d\alpha_2 + e\alpha_2 + bd + ce) + ce\alpha_2) \\ &\quad + b(bc + c\alpha_2 + be + cd + d\alpha_2 + e\alpha_2 + bd). \end{aligned}$$

The Jacobian matrix of system (3.5) is given by

$$J(P) = \begin{pmatrix} A_1 & B_2 \\ B_1 & A_2 \end{pmatrix}, \quad (3.33)$$

where

$$A_i = \begin{pmatrix} -b - \alpha_1 - \frac{\beta I_i(N_i - S_i)}{N_i^2} & \frac{\beta S_i I_i}{N_i^2} & -\frac{\beta S_i(N_i - I_i)}{N_i^2} & \frac{\beta S_i I_i}{N_i^2} + \alpha_2 \\ \frac{\beta I_i(N_i - S_i)}{N_i^2} & -b - c - \alpha_1 - \frac{\beta S_i I_i}{N_i^2} & \frac{\beta S_i(N_i - I_i)}{N_i^2} & -\frac{\beta S_i I_i}{N_i^2} \\ 0 & c & -e - d - \alpha_1 & 0 \\ 0 & 0 & d & -b - \alpha_1 - \alpha_2 \end{pmatrix}$$

and

$$B_i = \begin{pmatrix} \alpha_1 - \frac{\gamma \alpha_1 I_i(N_i - S_i)}{N_i^2} & \frac{\gamma \alpha_1 S_i I_i}{N_i^2} & -\frac{\gamma \alpha_1 S_i(N_i - I_i)}{N_i^2} & \frac{\gamma \alpha_1 S_i I_i}{N_i^2} \\ \frac{\gamma \alpha_1 I_i(N_i - S_i)}{N_i^2} & \alpha_1 - \frac{\gamma \alpha_1 S_i I_i}{N_i^2} & \frac{\gamma \alpha_1 S_i(N_i - I_i)}{N_i^2} & -\frac{\gamma \alpha_1 S_i I_i}{N_i^2} \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{pmatrix}$$

and $N_i = S_i + E_i + I_i + R_i$, for $i = 1, 2$. Next, the local stability of P_γ^* is analyzed in the next theorem.

Theorem 3.5 The endemic equilibrium, P_γ^* , of (3.5) is LAS if $\mathcal{R}_{0\gamma} > 0$.

Proof. The Jacobian matrix of (3.5) at P_γ^* is given by

$$J(P_\gamma^*) = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} -b - \alpha_1 - \psi_1 & \psi_2 & -\psi_3 & \psi_2 + \alpha_2 \\ \psi_1 & -b - c - \alpha_1 - \psi_2 & \psi_3 & -\psi_2 \\ 0 & c & -e - d - \alpha_1 & 0 \\ 0 & 0 & d & -b - \alpha_1 - \alpha_2 \end{pmatrix},$$

$$B = \begin{pmatrix} \alpha_1 - \mu_1 & \mu_2 & -\mu_3 & \mu_2 \\ \mu_1 & \alpha_1 - \mu_2 & \mu_3 & -\mu_2 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{pmatrix},$$

with

$$\begin{aligned}
\psi_1 &= \frac{\beta I_\gamma^*(N_\gamma^* - S_\gamma^*)}{N_\gamma^{*2}} = \frac{\beta c(\mathcal{R}_{0\gamma} - 1)^2 (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}, \\
\psi_2 &= \frac{\beta S_\gamma^* I_\gamma^*}{N_\gamma^{*2}} = \frac{\beta c(\mathcal{R}_{0\gamma} - 1) (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}, \\
\psi_3 &= \frac{\beta S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} = \frac{(\beta c + \beta \mathcal{R}_{0\gamma}(d + e))(b + \alpha_2) + \beta cd \mathcal{R}_{0\gamma}}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}, \\
\mu_1 &= \frac{\gamma \alpha_1 I_\gamma^*(N_\gamma^* - S_\gamma^*)}{N_\gamma^{*2}} = \frac{\gamma \alpha_1 c(\mathcal{R}_{0\gamma} - 1)^2 (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}, \\
\mu_2 &= \frac{\gamma \alpha_1 S_\gamma^* I_\gamma^*}{N_\gamma^{*2}} = \frac{\gamma \alpha_1 c(\mathcal{R}_{0\gamma} - 1) (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}, \\
\mu_3 &= \frac{\gamma \alpha_1 S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} = \frac{(\gamma \alpha_1 c + \gamma \alpha_1 \mathcal{R}_{0\gamma}(d + e))(b + \alpha_2) + \gamma \alpha_1 cd \mathcal{R}_{0\gamma}}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)},
\end{aligned}$$

and $N_\gamma^* = S_\gamma^* + E_\gamma^* + I_\gamma^* + R_\gamma^*$. The eigenvalues of $J(P_\gamma^*)$ is equivalent to calculate the eigenvalues of $A + B$ and $A - B$ as in the following. First, according to Lemma 2.1, the matrix $A + B$:

$$A + B = \begin{pmatrix} -b - \theta_1 & \theta_2 & -\theta_3 & \theta_2 + \alpha_2 \\ \theta_1 & -b - c - \theta_2 & \theta_3 & -\theta_2 \\ 0 & c & -e - d & 0 \\ 0 & 0 & d & -b - \alpha_2 \end{pmatrix},$$

where $\theta_1 = \psi_1 + \mu_1$, $\theta_2 = \psi_2 + \mu_2$, $\theta_3 = \psi_3 + \mu_3$ is checked into six step. For simplification, the entries of $A + B$ are denoted by a_{ij} for $i, j = 1, 2, 3, 4$.

(i) $A_1 > 0$. Obviously, $\theta_i > 0$ for $i = 1, 2, 3$ if $\mathcal{R}_{0\gamma} > 1$, and $a_{ii} < 0$ for $i = 1, 2, 3, 4$. Thus, $A_1 = -(a_{11} + a_{22} + a_{33} + a_{44}) > 0$.

(ii) $J_i > 0$ for $i = 1, 2, 3$. Since $a_{ii} < 0$ for $i = 1, 2, 3, 4$, then

$$\begin{aligned}
J_1 &= a_{44}a_{33} + a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11} > 0, \\
J_2 &= a_{33}a_{22} - a_{32}a_{23} = \theta_2(e + d + c) > 0, \\
J_3 &= a_{22}a_{11} - a_{21}a_{12} = (b + c)(b + \theta_1) + b\theta_2 > 0.
\end{aligned}$$

(iii) $A_3 = Q_1 + Q_2 + Q_3 > 0$. Since $J_2 > 0$, $J_3 > 0$, $-a_{44} > 0$, and $-a_{33} > 0$, then $Q_1 = -a_{44}(J_2 + J_3) > 0$ and $Q_2 = -a_{33}J_3 > 0$, respectively.

For Q_3 , it is verified that

$$\begin{aligned}
Q_3 &= -(a_{32}(a_{21}a_{13} + a_{43}a_{24}) + a_{11}(a_{44}a_{33} - a_{32}a_{23})) \\
&= \theta_2 dc + (\theta + b)(b + \alpha_2)(e + d) - bc\theta_3 \\
&= \theta_2 c(b + d) + (e + d)[\theta_1(b + \alpha_2) + b\alpha_2] - cb(e + d).
\end{aligned}$$

It follows that

$$Q_2 + Q_3 = \theta_2 c(b + d) + (e + d)[\theta_1(b + \alpha_2) + b\alpha_2 + b(b + \theta_1) + \theta_1 c + \theta_2 b] > 0.$$

Thus, $A_3 = Q_1 + Q_2 + Q_3 > 0$.

(iv) $A_4 = \det(J) = L_1 + L_2 + L_3 > 0$, Since $a_{33} < 0$, $a_{44} < 0$, and $J_3 > 0$, it is found that

$$\begin{aligned} L_1 &= a_{44}a_{33}J_3 = (b + \alpha_2)(e + d)J_3, \\ L_2 &= dc(b\theta_2 - \theta_1\alpha_2), \\ L_3 &= -(b + \alpha_2)c\theta_3b. \end{aligned}$$

It is verified that

$$\begin{aligned} L_1 + L_2 + L_3 &= b(e + d)(b + \alpha_2)(\theta_1 + \theta_2) + cb\theta_2(b + \alpha_2 + cd) \\ &\quad + b\theta_1(e + d)(b + \alpha_2 + c) + c\alpha_2e\theta_1 > 0. \end{aligned}$$

Thus, $A_4 = L_1 + L_2 + L_3 > 0$.

(v) $A_1A_2 - A_3 > 0$. From (i)–(iii), it can be seen that

$$\begin{aligned} A_1A_2 - A_3 &= -(a_{11} + a_{22})(J_1 + J_2 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 - Q_3 \\ &= -a_{11}(J_2 + J_3) - a_{22}(J_1 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 \\ &\quad - a_{11}(a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11}) + (b + \theta_2)J_2 \\ &\quad + c\theta_2(e + c) + bc\theta_3 > 0. \end{aligned}$$

(vi) Finally, it can be shown that $A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$, see Appendix C.

By Lemma 2.1, all eigenvalues of $A + B$ have negative real part when $\mathcal{R}_{0\gamma} > 1$.

Next, the matrix $A - B$ is given by

$$A - B = \begin{pmatrix} -b - 2\alpha_1 - \phi_1 & \phi_2 & -\phi_3 & \phi_2 + \alpha_2 \\ \phi_1 & -b - c - 2\alpha_1 - \phi_2 & \phi_3 & -\phi_2 \\ 0 & c & -e - d - 2\alpha_1 & 0 \\ 0 & 0 & d & -b - 2\alpha_1 - \alpha_2 \end{pmatrix}$$

where $\phi_1 = \psi_1 - \mu_1$, $\phi_2 = \psi_2 - \mu_2$, $\phi_3 = \psi_3 - \mu_3$.

For convenience, the entries of $A - B$ is denoted by a_{ij} for $i, j = 1, 2, 3, 4$, and the eigenvalues of $A - B$ is explored as follows.

(i) $A_1 > 0$. Obviously,

$$\begin{aligned} a_{33} &= -e - d - 2\alpha_1 < 0, \\ a_{44} &= -b - \alpha_2 - 2\alpha_1 < 0. \end{aligned}$$

Since $0 < \mathcal{R}_{0\gamma} - 1 < \mathcal{R}_{0\gamma} < \mathcal{R}_{0\gamma}^2$ when $\mathcal{R}_{0\gamma} > 1$, and also noting that $0 \leq \gamma \leq 1$, these give

$$\begin{aligned} a_{11} &= -b - \psi_1 - 2 \left(2 - \frac{\gamma c (\mathcal{R}_{0\gamma} - 1)^2 (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)} \right) \alpha_1 < 0 \\ \text{and } a_{22} &= -b - c - \psi_2 - \left(2 - \frac{\gamma c (\mathcal{R}_{0\gamma} - 1) (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)} \right) \alpha_1 < 0. \end{aligned}$$

Hence, $A_1 = -(a_{11} + a_{22} + a_{33} + a_{44}) > 0$.

(ii) $J_i > 0$, $i = 1, 2, 3$. Since $a_{ii} < 0$ for $i = 1, 2, 3, 4$, $J_1 = a_{44}a_{33} + a_{44}a_{22} + a_{44}a_{11} + a_{33}a_{11} > 0$.

For J_2 , it is seen that

$$\begin{aligned} J_2 &= (e + d + 2\alpha_1)(b + c + \phi_2 + 2\alpha_1) - c\phi_3 \\ &= (e + d + 2\alpha_1)(b + 2\alpha_1 + c) + (e + d + 2\alpha_1)\frac{(\beta - \gamma\alpha_1)S_\gamma^*I_\gamma^*}{N_\gamma^{*2}} \\ &\quad - \frac{(\beta - \gamma\alpha_1)cS_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}}. \end{aligned}$$

To test $J_2 > 0$, it can be shown as the following two case:

Case 1: $\beta \geq \gamma\alpha_1$,

$$\begin{aligned} J_2 &\geq (e + d + 2\alpha_1)(b + 2\alpha_1 + c) - \frac{(\beta + \gamma\alpha_1)cS_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}}, \\ &\geq (b + c)(e + d) - \frac{(\beta + \gamma\alpha_1)cS_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}}, \\ &\geq (\beta + \gamma\alpha_1)c \left(\frac{1}{\mathcal{R}_{0\gamma}} - \frac{S_\gamma^*}{N_\gamma^*} + \frac{S_\gamma^*I_\gamma^*}{N_\gamma^{*2}} \right), \\ &\geq \frac{(\beta + \gamma\alpha_1)cS_\gamma^*I_\gamma^*}{N_\gamma^{*2}} > 0. \end{aligned}$$



Case 2: $\beta < \gamma\alpha_1$,

$$\begin{aligned} J_2 &= (e + d + 2\alpha_1)(b + 2\alpha_1 + c) + (e + d + 2\alpha_1)\frac{(\beta - \gamma\alpha_1)S_\gamma^*I_\gamma^*}{N_\gamma^{*2}} \\ &\quad + \frac{(\gamma\alpha_1 - \beta)cS_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}}, \\ &\geq (e + d + 2\alpha_1)(b + 2\alpha_1 + c) + (e + d + 2\alpha_1)\frac{(\beta - \gamma\alpha_1)S_\gamma^*I_\gamma^*}{N_\gamma^{*2}}, \\ &\geq (e + d)(b + c) + 4\alpha_1^2 - (e + d + 2\alpha_1)\frac{(\gamma\alpha_1)S_\gamma^*I_\gamma^*}{N_\gamma^{*2}}, \\ &\geq 2\alpha_1^2 \left(2 - \frac{\gamma c (\mathcal{R}_{0\gamma} - 1) (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)} \right), \\ &\quad + (e + d)c \left(1 - \frac{\gamma\alpha_1 (\mathcal{R}_{0\gamma} - 1) (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)} \right) > 0. \end{aligned}$$

Therefore, $J_2 > 0$, from case 1 and case 2.

For J_3 , it can be verified that

$$\begin{aligned} J_3 &= (b + c + 2\alpha_1)(b + 2\alpha_1 + \phi_1) + \phi_2(b + 2\alpha_1) \\ &= (b + c + 2\alpha_1)(\phi_1 + \alpha_1) + (b + c + 2\alpha_1)(b + \alpha_1) + (b + 2\alpha_1)\phi_2 \\ &= (b + c + 2\alpha_1)(\alpha_1 + \phi_1) + (b + 2\alpha_1)(b + \alpha_1 + \phi_2) + c(b + \alpha_1). \end{aligned}$$

Noting that

$$\alpha_1 + \phi_1 = \left(1 - \frac{\gamma c (\mathcal{R}_{0\gamma} - 1)^2 (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}\right) \alpha_1 > 0, \quad (3.34)$$

$$\alpha_1 + \phi_2 = \left(1 - \frac{\gamma c (\mathcal{R}_{0\gamma} - 1) (b + \alpha_2)}{\mathcal{R}_{0\gamma}^2 ((b + \alpha_2)(c + d + e) + cd)}\right) \alpha_1 > 0, \quad (3.35)$$

when $0 \leq \gamma \leq 1$ and $\mathcal{R}_{0\gamma} > 0$. Hence, $J_3 > 0$.

(iii) $A_3 = Q_1 + Q_2 + Q_3 > 0$. Obviously, $Q_1 = -a_{44}(J_2 + J_3) > 0$, $Q_2 = -a_{33}J_3 > 0$ and

$$\begin{aligned} Q_3 &= cd\phi_2 + (b + 2\alpha_1 + \phi_1)(b + 2\alpha_1 + \alpha_2)(e + d + 2\alpha_1) - (b + 2\alpha_1)c\phi_3 \\ &= cd(\alpha_1 + \phi_2) + (b + 2\alpha_1 + \phi_1)(b + 2\alpha_1 + \alpha_2)(e + d + 2\alpha_1) \\ &\quad - (b + 2\alpha_1)c\phi_3 - cd\alpha_1 \\ &> cd(\phi_2 + \alpha_1) + (b + 2\alpha_1)[(b + c)(e + d) - c\phi_3] \\ &\quad - c(b + 2\alpha_1 + \alpha_2)(e + d + 2\alpha_1) - \alpha_1 cd. \end{aligned}$$

It can be shown that

$$\begin{aligned} A_3 &= Q_1 + Q_2 + Q_3 \\ &> (b + 2\alpha_1) \left[(b + c)(e + d) - \frac{c(\beta - \gamma\alpha_1)S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right] > 0, \end{aligned}$$

as the following two case.

Case 1: $\beta \geq \gamma\alpha_1$,

$$\begin{aligned} A_3 &\geq (b + 2\alpha_1) \left[(b + c)(e + d) - \frac{c(\beta + \gamma\alpha_1)S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right] \\ &\geq \frac{(b + 2\alpha_1)(\beta + \gamma\alpha_1)cS_\gamma^*I_\gamma^*}{N_\gamma^{*2}} > 0. \end{aligned}$$

Case 2: $\beta < \gamma\alpha_1$,

$$A_3 \geq (b + 2\alpha_1) \left[(b + c)(e + d) + \frac{c(\gamma\alpha_1 - \beta)S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right] > 0.$$

From case 1 and case 2, it is clear that $A_3 > 0$.

(iv) $A_4 = \det(A - B) = L_1 + L_2 + L_3 > 0$, where

$$\begin{aligned} L_1 &= a_{44}a_{33}J_3 = (b + \alpha_2 + 2\alpha_1)(e + d + 2\alpha_1)J_3, \\ L_2 &= dc[\phi_2(b + 2\alpha_1) - \phi_1\alpha_2] \\ &= dc[(\phi_2 + \alpha_1)(b + 2\alpha_1)] - dc[\alpha_1(b + 2\alpha_1) + \phi_1\alpha_2], \\ L_3 &= -(b + \alpha_2 + 2\alpha_1)(b + 2\alpha_1)c\phi_3. \end{aligned}$$

Furthermore,

$$\begin{aligned}
L_1 + L_2 + L_3 &= (b + \alpha_2 + 2\alpha_1) [(e + 2\alpha_1)J_3 + (b + 2\alpha_1)c\phi_3] \\
&\quad + d(b + 2\alpha_1)(J_2 - c\alpha_1) + d\alpha_2(J_3 - c\phi_1) \\
&= (b + \alpha_2 + 2\alpha_1) \left[(e + 2\alpha_1)J_3 + c(b + 2\alpha_1) \frac{(\beta - \gamma\alpha_1)S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right] \\
&\quad + d(b + 2\alpha_1)(J_2 - c\alpha_1) + d\alpha_2(J_3 - c\phi_1) \\
&> (b + 2\alpha_2 + 2\alpha_1) \left[(e + 2\alpha_1) \left(J_3 - \frac{c\gamma\alpha_1 S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right) \right] \\
&\quad + d(b + 2\alpha_1)(J_2 - c\alpha_1) + d\alpha_2(J_3 - c\phi_1)
\end{aligned}$$

with

$$\begin{aligned}
J_3 + c\phi_3 &> J_3 - \frac{c\gamma\alpha_1 S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \\
&> c(b + \alpha_1) - \frac{c\gamma\alpha_1 S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \\
&> cb + c\alpha_1 \left(1 - \frac{\gamma\alpha_1 S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right) \\
&> cb + c\alpha_1 \left(1 - \frac{\gamma}{\mathcal{R}_{0\gamma}} \right) > 0, \tag{3.36} \\
J_2 - c\alpha_1 &= (e + d + \alpha_1)(b + c + \phi_2 + 2\alpha_1) + \alpha_1(b + \phi_2 + 2\alpha_1) - c\phi_3 \\
&> 0, \\
J_3 - c\phi_1 &= (b + 2\alpha_1)(\phi_1 + \alpha_1) + c\alpha_1 + (b + 2\alpha_1)(b + \alpha_1 + \phi_2) + c(b + \alpha_1) \\
&> 0.
\end{aligned}$$

Thus, $A_4 > 0$.

- (v) $A_1 A_2 - A_3 > 0$. From (i)–(iii), $J_i > 0$ for $i = 1, 2, 3$ and $a_{ii} < 0$ for $i = 1, 2, 3, 4$. It is found that

$$\begin{aligned}
A_1 A_2 - A_3 &= -(a_{11} + a_{22})(J_1 + J_2 + J_3) - a_{33}(J_1 + J_2) - a_{44}J_1 - Q_3 \\
&> -(a_{11} + a_{22})J_3 - a_{11}J_1 - a_{22} - Q_3 \\
&> -(a_{11} + a_{22})J_3 - a_{11}[J_1 - a_{33}a_{44}] - a_{11}a_{33}a_{44} \\
&\quad + (-a_{22} - c)J_2 + cJ_2 - Q_3 \\
&> -a_{11}a_{33}a_{44} - (a_{11} + a_{22})J_3 + cJ_2 - Q_3 \\
&> c(J_2 - \phi_2 d) + (2b + c + 2\alpha_1 + (\phi_1 + \alpha_1) + (\phi_2 + \alpha_1))J_3 \\
&\quad + (b + 2\alpha_1)c\phi_3 \\
&> c(J_2 - \phi_2 d) + (b + 2\alpha_1)(J_3 + c\phi_3) \\
&> c(J_2 - \phi_2 d) + (b + 2\alpha_1) \left(J_3 - \frac{c\gamma\alpha_1 S_\gamma^*(N_\gamma^* - I_\gamma^*)}{N_\gamma^{*2}} \right) \\
&> c(J_2 - \phi_2 d) + (b + 2\alpha_1) \left(1 - \frac{\gamma}{\mathcal{R}_{0\gamma}} \right) > 0,
\end{aligned}$$

where

$$J_2 - \phi_2 d > (e + d + 2\alpha_1)(\phi_2 + \alpha_1) - \phi_2 d > (e + 2\alpha_1)(\phi_2 + \alpha_1) + d\alpha_1 > 0.$$

Thus, $A_1 A_2 - A_3 > 0$.

(vi) Finally, $A_1 A_2 A_3 - A_3^2 - A_1^2 A_4 > 0$ is verified in Appendix D.

By Lemma 2.1, all the eigenvalues of $A - B$ have negative real part. Therefore, it can be concluded that all the eigenvalues of $A + B$ and $A - B$ have negative real part. These imply that P_γ^* is *LAS* when $\mathcal{R}_{0\gamma} > 1$ according to Theorem 2.1.

□