รายงานวิจัยฉบับสมบูรณ์

โครงการวิธีการทำซ้ำหาจุดตรึงร่วมสำหรับการส่งแบบไม่เชิงเส้นในปริภูมิบานาค
Common Fixed Points Iterations for Nonlinear Mappings in Banach Spaces

โดย ดร.กมลรัตน์ แนมมณี และคณะ

พฤษภาคม 2553

สัญญาเลขที่ MRG5180011

รายงานวิจัยฉบับสมบูรณ์

โครงการวิธีการทำซ้ำหาจุดตรึงร่วมสำหรับการส่งแบบไม่เชิงเส้นในปริภูมิบานาค
Common Fixed Points Iterations for Nonlinear Mappings in Banach Spaces

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยเรื่อง วิธีการทำซ้ำหาจุดตรึงร่วมสำหรับการส่งแบบไม่เชิงเส้นในปริภูมิบานาค (MRG5180011) นี้ สำเร็จลุล่วงด้วยดีจากการได้รับทุนอุดหนุนการวิจัยจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และสำนักงานคณะกรรมการอุดมศึกษา (สกอ.) ประจำปี 2551-2553 และขอขอบคุณ ศาสตราจารย์ ดร.สุเทพ สวนใต้ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ นักวิจัยที่ปรึกษา ที่ได้ให้คำแนะนำและข้อเสนอแนะในการทำวิจัยด้วยดีตลอดมา

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Project Title: Common Fixed Points Iterations for Nonlinear Mappings in Banach Spaces
(ชื่อโครงการ) วิธีการทำซ้ำหาจุดตรึงร่วมสำหรับการส่งแบบไม่เชิงเส้นในปริภูมิบานาค

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Abstract

The purposes of this research are to create new knowledge of fixed point theorems and construct several new iterative methods for approximating a fixed point and a common fixed point of nonlinear mappings in Banach spaces. We establish some convergence theorems of such iterative methods for finding a fixed point and a common fixed point of nonlinear mappings.

Keywords: Uniformly convex / Iterative method / Nonexpansive mapping / Asymptotically nonexpansive mapping / Asymptotically k-strict pseudo-contractive mapping

บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้ คือ การสร้างองค์ความรู้ใหม่ของทฤษฎีบทจุดตรึง และการสร้างระเบียบวิธีการทำซ้ำชนิดใหม่ต่างๆ เพื่อใช้ในการประมาณค่าจุดตรึงและจุดตรึงร่วมของการส่งแบบไม่เชิงเส้นในปริภูมิบานาค นอกจากนี้ เรายังได้สร้างทฤษฎีบทการลู่เข้าของระเบียบวิธีการทำซ้ำดังกล่าว สำหรับการหาจุดตรึงและจุดตรึงร่วมของการส่งแบบไม่เชิงเส้นอีกด้วย

คำสำคัญ : ปริภูมินูนแบบเอกรูป / ระเบียบวิธีการทำซ้ำ / การส่งแบบไม่ขยาย / การส่งแบบไม่ขยายเชิงกำกับ / การส่งแบบหดเทียมโดยแท้เชิงกำกับ

หน้าสรุปโครงการ (Executive Summary)

<u>ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่</u>

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1. ชื่อโครงการ (ภาษาไทย) วิธีการทำซ้ำหาจุดตรึงร่วมสำหรับการส่งแบบไม่เชิงเส้นในปริภูมิบานาค

(ภาษาอังกฤษ) Common Fixed Points Iterations for Nonlinear Mappings in Banach Spaces

2. ชื่อหัวหน้าโครงการ หน่วยงานที่สังกัด ที่อยู่ หมายเลขโทรศัพท์ โทรสาร และ e-mail

ชื่อ นาย กมลรัตน์ แนมมณี **คุณวุฒิ** วิทยาศาสตรดุษฎีบัณฑิต (วท.ด. คณิตศาสตร์)

ตำแหน่ง อาจารย์

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3. สาขาวิชาที่ทำการวิจัย การวิเคราะห์เชิงฟังก์ชัน (Functional Analysis) ทาง Fixed point theory

5. ระยะเวลาดำเนินงาน 2 **ปี**

6. ได้เสนอโครงการนี้ หรือโครงการที่มีส่วนเหมือนกับเรื่องนี้บางส่วนเพื่อขอทุนต่อแหล่งทุนอื่นที่ใดบ้าง

✔ ไม่ได้เสนอต่อแหล่งทุนอื่น

☐ เสนอต่อ ....

ชื่อโครงการที่เสนอ ....

....

กำหนดทราบผล (หรือสถานภาพที่ทราบ) ....

....

7. ปัญหาที่ทำการวิจัย และความสำคัญของปัญหา

ทฤษฎีจุดตรึงนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับ **การมีคำตอบ** (existence of solution) และ **การมีเพียงคำตอบเดียว** ของสมการ (uniqueness of solution) ตลอดจนการคิดค้นหาวิธีในการประมาณหาคำตอบของสมการต่างๆ

ดังนั้นการศึกษาทฤษฎีต่างๆ ที่เกี่ยวข้องกับการมีจุดตรึงของการส่งต่างๆ และ การหาระเบียบวิธีต่างๆ ที่ใช้ในการประมาณค่าคำตอบนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์กลุ่มหนึ่งจำนวนมากให้ความสนใจศึกษา เมื่อศึกษาการมีคำตอบของสมการต่างๆแล้ว ปัญหาที่น่าสนใจต่อไปก็คือ เราจะหาคำตอบของสมการต่างๆ นั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้นระเบียบวิธีการกระทำซ้ำของจุดตรึง(fixed-point iterations) ต่างๆ ที่ใช้ในการหาคำตอบ และ ประมาณคำตอบ เพื่อนำไปประยุกต์ใช้เกี่ยวข้องกับการแก้ปัญหาในเรื่องของสมการตัวดำเนินการไม่เชิงเส้น

(nonlinear operator equations) ในเรื่องของแก้ปัญหาอสมการแปรผัน (variational inequality problem (VIP)) และแก้สมการหาคำตอบของปัญหาดุลภาพ(Equilibrium Problems (EP)) ปัญหาที่ดีที่สุด (optimizations problems) ปัญหาน้อยที่สุด (minimizations problems) ทั้งในปริภูมิฮิลเบิตร์และปริภูมิบานาค ซึ่งปัญหาดังกล่าวเป็นปัญหาที่สำคัญที่มีประโยชน์มากมายในสาขาวิชาต่างๆ เช่นสาขาวิชาฟิสิกส์ คณิตศาสตร์ประยุกต์ วิศวกรรม และสาขาทางเศรษฐศาสตร์

ทฤษฎีบทสำคัญที่สนับสนุนการมีอยู่จริงของการส่งมีดังต่อไปนี้ Banach Contraction Mapping Principle, Brouwer's Fixed Point Theorem, Schauder's Fixed Point Theorem และ Kirk's Theorem โดยผลของทฤษฎีบทเหล่านี้ ให้ความจริงที่สำคัญคือ การส่งมีจุดตรึง ในปี 1965 นักคณิตศาสตร์ที่ชื่อ Felix Brower ได้พิสูจน์ว่า ถ้า C เป็นเซตย่อยคอนเวกซ์ปิดที่มีขอบเขต ของปริภูมิฮิลแบร์ต E และ T เป็นการส่งไม่ขยายจาก C ไปยัง C แล้ว T มีจุดตรึงใน C ยิ่งไปกว่านั้น ยังให้ผลที่เป็นจริงนี้ด้วยสำหรับปริภูมิบานาคคอนเวกซ์เอกรูป (uniformly convex Banach space) และในปีเดียวกัน W.A. Kirk ได้พิสูจน์ว่า สำหรับ X ที่เป็นปริภูมิบานาคสะท้อน (reflexive Banach space) และ C เป็นเซตย่อยคอนเวกซ์ปิดของ X ซึ่งมีสมบัติ normal structure และให้ T เป็นการส่งไม่ขยายจาก C ไปยัง C แล้ว T มีจุดตรึงใน C

ดังนั้น เมื่อมีทฤษฎีบทมาสนับสนุนการมีจริงของจุดตรึงของการส่งแล้ว ปัญหาในการประมาณค่าจุดตรึงของการส่ง จึงเป็นปัญหาที่นักคณิตศาสตร์จำนวนมากให้ความสนใจศึกษาค้นคว้าหาวิธีในการประมาณค่าจุดตรึงนั้น เริ่มในปี 1953 W.R. Mann ได้สร้างระเบียบวิธีการทำซ้ำเพื่อหาจุดตรึงของการส่งในปริภูมิฮิลแบร์ต โดยเรียกระเบียบวิธีทำซ้ำนี้ว่า "Mann Iteration Process" ต่อมาในปี 1974 และในปี 1976 S. Ishikawa ได้สร้างระเบียบวิธีการทำซ้ำแบบใหม่เพื่อหาจุดตรึงของการส่งในปริภูมิบานาค โดยเรียกระเบียบวิธีทำซ้ำนี้ว่า "Ishikawa Iteration Process" ซึ่งระเบียบวิธีการทำซ้ำเพื่อหาจุดตรึงทั้งสองนี้มีชื่อเสียงอย่างมากและได้ถูกนำไปศึกษาและประยุกต์ใช้งานกันอย่างแพร่หลาย

ในงานวิจัยนี้ ได้นิยามและศึกษาระเบียบวิธีทำซ้ำแบบใหม่เพื่อหาจุดตรึงร่วม และหาเงื่อนไขที่พอเพียงสำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีทำซ้ำแบบใหม่ที่นิยามขึ้น สำหรับการส่งแบบไม่ขยายในปริภูมิบานาค อีกทั้งยังได้นิยามและศึกษาระเบียบวิธีทำซ้ำแบบใหม่พื่อหาจุดตรึงร่วม และหาเงื่อนไขที่พอเพียงสำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีทำซ้ำแบบใหม่ที่นิยามขึ้น สำหรับการส่งแบบไม่ขยายเชิงกำกับในปริภูมิบานาค ซึ่งผลงานที่ได้จากงานวิจัยนี้เป็นการขยายผลงานต่างๆ มากมายของการวิจัยในสาขานี้ รวมไปถึงการนำไปประยุกต์ใช้สำหรับการหาคำตอบของสมการที่สำคัญๆ ในสาขาต่างๆ ซึ่งมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการ และ มีประโยชน์อย่างมากในการพัฒนาทางด้านวิทยาศาสตร์พื้นฐาน (Basic Science) และวิทยาศาสตร์เชิงประยุกต์ (Apply Science) อันถือเป็นพื้นฐานที่สำคัญในการพัฒนาประเทศชาติต่อไป

8. วัตถุประสงค์

1. นิยามและศึกษาระเบียบวิธีการทำซ้ำใหม่เพื่อหาจุดตรึง สำหรับการส่งแบบไม่ขยายในปริภูมิบานาค

2. หาเงื่อนไขที่พอเพียง สำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำใหม่ไปยังจุดตรึงสำหรับการส่งแบบไม่ขยาย

3. นิยามและศึกษาระเบียบวิธีทำซ้ำใหม่เพื่อหาจุดตรึงร่วม สำหรับการส่งแบบไม่ขยายเชิงกำกับในปริภูมิบานาค

4. หาเงื่อนไขที่พอเพียง สำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำใหม่ไปยังจุดตรึงร่วมสำหรับการส่งแบบไม่ขยายเชิงกำกับ

5. ประยุกต์ระเบียบวิธีทำซ้ำที่สร้างขึ้นข้างต้นสำหรับการแก้ปัญหาสมการที่สำคัญๆ ต่างๆ

9. ระเบียบวิธีวิจัย

ขั้นที่ 1. ค้นคว้ารวบรวมตำรา เอกสาร และบทความทางวิชาการของงานวิจัยที่เกี่ยวข้องจากเอกสารอ้างอิง โดยการสืบค้นข้อมูลทาง Internet และห้องสมุด

ขั้นที่ 2. หาข้อมูลเพิ่มเติมด้วยการเข้าร่วมฟังสัมมนาและประชุมทางวิชาการ ในเรื่องที่เกี่ยวข้องกับงานวิจัย

ขั้นที่ 3. ดำเนินการพิสูจน์ทฤษฎีบทที่เกี่ยวข้องตามวัตถุประสงค์ที่ตั้งไว้ โดยการนิยามระเบียบวิธีการทำซ้ำแบบใหม่เพื่อหาจุดตรึง และจุดตรึงร่วม

ขั้นที่ 4. ทำการวิเคราะห์ และศึกษาระเบียบวิธีการทำซ้ำแบบใหม่ที่นิยามขึ้นเพื่อหาเงื่อนไขที่พอเพียงสำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำแบบใหม่ สำหรับการส่งที่ไม่ขยายและการส่งที่ไม่ขยายแบบเชิงเส้นกำกับในปริภูมิบานาค และประยุกต์ระเบียบวิธีทำซ้ำที่สร้างขึ้นสำหรับการแก้ปัญหาสมการที่สำคัญๆ ต่าง ๆ

ขั้นที่ 5. รวบรวมข้อมูล เพื่อสรุปผล เตรียมเอกสารสำหรับการตีพิมพ์ และ เขียนรายงานการวิจัย

ขั้นที่ 6. สรุปผล จัดพิมพ์เอกสาร เขียนรายงานวิจัย และเผยแพร่งานวิจัย

10. จำนวนโครงการที่ผู้สมัครกำลังดำเนินการอยู่ โดยขอให้ระบุระยะเวลาเริ่มต้นและสิ้นสุดของแต่ละ โครงการ แหล่งทุน และงบประมาณสนับสนุนที่ได้รับ เวลาที่ใช้ทำโครงการวิจัยในแต่ละโครงการเป็นกี่ชั่วโมงต่อสัปดาห์ ทั้งในฐานะหัวหน้าโครงการ ผู้ร่วมโครงการของแต่ละโครงการที่กำลังดำเนินการอยู่

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บทที่ 1
บทนำ (Introduction)

ทฤษฎีจุดตรึง (fixed point theory) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน (functional analysis) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการและ การพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ ๆที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่าง ๆนั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน (basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ทฤษฎีจุดตรึงนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับ การมีคำตอบ (existence of solution) และ การมีเพียงคำตอบเดียว ของสมการ (uniqueness of solution) ตลอดจนการคิดค้นหาวิธีในการประมาณหาคำตอบของสมการต่าง ๆ ดังนั้น การศึกษาทฤษฎีต่าง ๆ ที่เกี่ยวข้องกับการมีจุดตรึงของการส่งต่าง ๆ และ การหาระเบียบวิธีต่าง ๆที่ใช้ในการประมาณค่าคำตอบนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์กลุ่มหนึ่งจำนวนมากให้ความสนใจศึกษา เมื่อศึกษาการมีคำตอบของสมการต่าง ๆแล้ว ปัญหาที่น่าสนใจต่อไปก็คือ เราจะหาคำตอบของสมการต่าง ๆนั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้นระเบียบวิธีการกระทำซ้ำของจุดตรึง(fixed-point iterations) ต่าง ๆที่ใช้ในการหาคำตอบ และ ประมาณคำตอบ เพื่อนำไปประยุกต์ใช้เกี่ยวข้องกับการแก้ปัญหาในเรื่องของสมการตัวดำเนินการไม่เชิงเส้น (nonlinear operator equations) ในเรื่องของแก้ปัญหาอสมการแปรผัน (variational inequality problem (VIP)) และแก้สมการหาคำตอบของปัญหาดุลภาพ(equilibrium problems (EP)) ปัญหาที่ดีที่สุด(optimizations problems) ปัญหาน้อยที่สุด (minimizations problems) ทั้งในปริภูมิฮิลเบิตร์และปริภูมิบานาค ซึ่งปัญหาดังกล่าวเป็นปัญหาที่สำคัญที่มีประโยชน์มากมายในสาขาวิชาต่าง ๆ เช่นสาขาวิชาฟิสิกส์ คณิตศาสตร์ประยุกต์ วิศวกรรม และสาขาทางเศรษฐศาสตร์

จากความสำคัญข้างต้นเป็นผลให้นักคณิตศาสตร์จึงได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ซึ่งการวิจัยเกี่ยวกับการกระทำซ้ำของจุดตรึงและการประมาณค่าจุดตรึงที่สำคัญนั้นสามารถนำมาแก้สมการหาคำตอบของปัญหาดุลภาพ เช่น ใน ปี 1997 Combettes และ Hirstoaga ได้เริ่มต้นศึกษาและใช้วิธีการทำซ้ำในการหาการประมาณค่าที่ดีที่สุดเพื่อแก้ปัญหาดุลภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้ม (strong convergence theorems) และมีนักคณิตศาสตร์อีกมากมาย นำทฤษฎีบทการทำซ้ำดังกล่าวมาประยุกต์ใช้ในการแก้อสมการแปรผัน

ระเบียบวิธีการทำซ้ำเพื่อหาจุดตรึงสำหรับการส่งแบบไม่ขยายในปริภูมิบานาค ที่มีชื่อเสียงและได้ถูกนำมาศึกษาโดยนักคณิตศาสตร์หลายท่าน[20, 29, 39, 41, 43] และประยุกต์ใช้งานกันอย่างแพร่หลาย

ประกอบด้วย "*Mann iteration process*" ซึ่งสร้างขึ้นในปี 1953 โดยนักคณิตศาสตร์ชื่อ *W. R. Mann* [24] และ "*Ishikawa iteration process*" ซึ่งสร้างขึ้นในปี 1974 โดย *S. Ishikawa* [13]

ในปี 1965 *F.E. Browder* [2] ได้พิสูจน์ว่า ถ้า C เป็นเซตย่อยปิด คอนเวกซ์ ที่มีขอบเขต ของปริภูมิ ฮิลแบร์ต X และ $T:C\to C$ เป็นการส่งไม่ขยาย แล้ว T จะมีจุดตรึงใน C ยิ่งไปกว่านั้นยังแสดงผลที่ได้นี้เป็นจริงด้วยเมื่อ X เป็นปริภูมิบานาคคอนเวกซ์เอกรูป *(uniformly convex Banach space)* ในปีเดียวกัน W. A. Kirk ได้แสดงว่า ถ้า X เป็นปริภูมิบานาคสะท้อน และ C เป็นเซตย่อยปิด คอนเวกซ์ ที่มีขอบเขต ของ X ซึ่งมีสมบัติ normal structure และให้ $T:C\to C$ เป็นการส่งไม่ขยาย แล้ว T มีจุดตรึงใน C

ในปี 1953 *W.R. Mann* [24] ได้สร้างระเบียบวิธีทำซ้ำเพื่อหาจุดตรึงของการส่ง T โดยนิยามดังนี้ สำหรับ $n\ge 1$ ให้ $x_1\in C,\quad x_{n+1}=\alpha_n Tx_n+(1-\alpha_n)x_n$ เมื่อลำดับ $\{\alpha_n\}\subset[0,1]$ ซึ่งเรียกระเบียบวิธีทำซ้ำนี้ว่า *Mann iteration process* ต่อมา *S.Ishikawa* พิสูจน์ว่า ถ้า C เป็นเซตย่อยคอนเวกซ์กระชับของปริภูมิบานาค X และ $T:C\to C$ เป็นการส่งไม่ขยาย และ $\{x_n\}$ นิยามโดย $x_1\in C,\quad x_{n+1}=\alpha_n Tx_n+(1-\alpha_n)x_n$ เมื่อลำดับ $\{\alpha_n\}\subset[0,1]$ ซึ่งสอดคล้องเงื่อนไข $\sum_{n=1}^{\infty}\alpha_n=\infty,\quad \lim_{x\to\infty} sub\ \alpha_n<1$ แล้ว ลำดับ $\{x_n\}$ ลู่เข้าแบบเข้มไปยังจุดตรึงของการส่ง T ยิ่งไปกว่านั้น *S.Ishikawa* ได้สร้างระเบียบวิธีทำซ้ำเพื่อหาจุดตรึงของการส่ง T โดยนิยามดังนี้ สำหรับ $n\ge 1$ ให้ $x_1\in C$, $\begin{array}{l} \mathrm{y_n}=\beta_n Tx_n+(1-\beta_n)x_n \\ x_{n+1}=\alpha_n Ty_n+(1-\alpha_n)x_n \end{array}$, $n\ge 1$ เมื่อลำดับ $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ ซึ่งเรียกระเบียบวิธีทำซ้ำนี้ว่า *Ishikawa iteration process* ในปี 1979 *S. Reich* [33] ได้พิสูจน์ว่า ถ้า C เป็นเซตย่อยคอนเวกซ์ปิดที่ไม่ใช่เซตว่างของปริภูมบานาคคอนเวกซ์เอกรูป X ภายใต้นอร์มที่เป็น *Fre'chet differentiable* และ $T:C\to C$ เป็นการส่งแบบไม่ขยาย แล้ว ลำดับ $\{x_n\}$ นิยามโดย *Mann iteration process* จะลู่เข้าแบบอ่อนไปยังจุดตรึงของการส่ง T เมื่อ $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$ ในปี 1993 *K.K. Tan* และ *H. K. Xu* [39] พิสูจน์ว่า ถ้า C เป็นเซตย่อยคอนเวกซ์ปิดที่ไม่ใช่เซตว่างของปริภูมบานาคคอนเวกซ์เอกรูป X ซึ่งสอดคล้องเงื่อนไขของ *Opial (Opial's condition)*[29] หรือภายใต้นอร์มที่เป็น *Fre'chet differentiable* และ $T:C\to C$ เป็นการส่งไม่ขยาย แล้ว ลำดับ $\{x_n\}$ นิยามโดย *Ishikawa iteration process* จะลู่เข้าแบบอ่อนไปยังจุดตรึงของการส่ง T เมื่อ ลำดับ $\{\alpha_n\}$ และ $\{\beta_n\}$ สอดคล้องเงื่อนไข $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty,\quad \sum_{n=1}^{\infty}\beta_n(1-\beta_n)<\infty$ และ $\limsup_{n\to\infty}\beta_n<1$ ต่อมาในปี 1998 *W. Takahashi* และ *G.E. Kim* [41] พิสูจน์ว่า ถ้า X ปริภูมบานาคคอนเวกซ์เอกรูปซึ่งสอดคล้องเงื่อนไขของ *Opial* หรือภายใต้นอร์มที่เป็น *Fre'chet differentiable* และ C เป็น เซตย่อยคอนเวกซ์ปิดที่ไม่ใช่เซตว่างของ X และ $T:C\to C$ เป็นการส่งแบบไม่ขยาย ที่มีจุดตรึง แล้ว ลำดับ $\{x_n\}$ นิยามโดย *Ishikawa iteration process* จะลู่เข้าแบบอ่อนไปยังจุดตรึงของการส่ง T เมื่อ ลำดับ $\{\alpha_n\}$ และ $\{\beta_n\}$ สอดคล้องเงื่อนไข $\alpha_n\in[a,b],\beta_n\in[0,b]$ หรือ $\alpha_n\in[a,1],\beta_n\in[a,b]$ สำหรับบางค่า a,b ซึ่ง $0<a\le b<1$

การส่งแบบไม่ขยายเชิงกำกับเป็นการส่งที่เป็นการวางนัยทั่วไปของการส่งแบบไม่ขยาย ซึ่งนิยามได้ดังต่อไปนี้ ให้ X เป็นปริภูมินอร์ม และ C เป็นเซตย่อยที่ไม่เป็นเซตว่างของ X การส่ง $T:C\to C$ จะถูกเรียกว่า การส่งแบบไม่ขยายเชิงกำกับ ถ้า มี ลำดับ $\{k_n\}\subset[1,\infty), k_n\to 1$ เมื่อ $n\to\infty$ ซึ่งทำให้ $\|T^n x-T^n y\|\le k_n\|x-y\|$ สำหรับทุกๆ x,y ที่เป็นสมาชิกของ C และทุก ๆ $n\ge 1$ เราพบว่าถ้า $k_n=1$ แล้ว T คือการส่งที่ไม่ขยาย และการส่ง T จะถูกเรียกว่าเป็น *uniformly* L*-Lipschitzian* ถ้า มีค่าคงตัวที่เป็นบวก L ซึ่งทำให้ $\|T^n x-T^n y\|\le L\|x-y\|$ สำหรับทุกๆ x,y ที่เป็นสมาชิกของ C และทุก ๆ $n\ge 1$ เราพบความจริงว่า ถ้าการส่ง T เป็นการส่งแบบไม่ขยายเชิงกำกับ แล้วการส่ง T จะเป็น *uniformly* L*-Lipschitzian* เมื่อ $L=\sup\{k_n:n\ge 1\}$ ซึ่งมีนักคณิตศาสตร์หลายท่านที่ศึกษาการส่งนี้กับระเบียบวิธีการทำซ้ำต่างๆ [1. 5, 6, 34, 38, 40, 45] ในปี 1972 K. Goebel และ W.A. Kirk [10] ได้พิสูจน์ทฤษฎีบทที่สำคัญคือ ถ้า C เป็น เซตย่อยคอนเวกซ์ปิดที่มีขอบเขตและไม่ใช่เซตว่างของ ปริภูมิบานาคคอนเวกซ์เอกรูป X และ T เป็นการส่งแบบไม่ขยายเชิงกำกับจาก C ไปยัง C แล้ว T จะมีจุดตรึงใน C ต่อมาในปี 1991 *J. Schu* [35] ได้สร้างระเบียบวิธีทำซ้ำที่เรียกว่า *“The Modified Mann Iteration Method”* และ *“The Modified Ishikawa Iteration Method”* เมื่อ C เป็นเซตย่อยคอนเวกซ์ที่ไม่ใช่เซตว่างของปริภูมิบานาค X $T:C\to C$ และ ลำดับ $\{\alpha_n\},\{b_n\}$ เป็นสมาชิกของ $[0,1]$ ซึ่งระเบียบวิธีการทำซ้ำดังกล่าวนิยามได้ดังต่อนี้ สำหรับ $x_1\in C,$

The Modified Mann Iteration Method: $x_{n+1}=\alpha_n T^n x_n+(1-\alpha_n)x_n,$ เมื่อ $n\ge 1$

The Modified Ishikawa Iteration Method:

$$\begin{aligned} y_n &= b_n T^n x_n+(1-b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n+(1-\alpha_n)x_n, \end{aligned} \quad \text{เมื่อ } n\ge 1$$

นอกจากนี้ *J.Schu* ได้พิสูจน์ว่า สำหรับปริภูมิฮิลแบร์ต H ถ้า C เป็นเซตย่อยคอนเวกซ์ปิดที่มีขอบเขตและไม่ใช่เซตว่างของ H และให้ $T:C\to C$ เป็นการส่งไม่ขยายแบบเชิงเส้นกำกับและต่อเนื่องแบบบริบูรณ์ ซึ่งลำดับ $\{k_n\}$ สอดคล้องเงื่อนไข $k_n\ge 1$ และ $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$ กำหนดให้ ลำดับ $\{\alpha_n\}$ อยู่ใน $[0,1]$ ซึ่งสอดคล้องเงื่อนไข $\varepsilon\le\alpha_n\le 1-\varepsilon$ สำหรับบางค่าของ $\varepsilon>0$ และทุกค่าของ $n\ge 1$ แล้ว ลำดับ $\{x_n\}$ นิยามโดย *The Modified Mann Iteration Method* ลู่เข้าแบบเข้มไปยังจุดตรึงของการส่ง T ในปี 1994 *B.E.Rhoades* [32] ได้แสดงผลที่ได้นี้เป็นจริง สำหรับ *The Modified Ishikawa Iteration Method* และสำหรับ X ที่เป็นปริภูมิบานาคคอนเวกซ์เอกรูป ต่อมาในปี 2000 *M.O. Osilike* และ *S.C. Aniagbosor* [30] ได้พิสูจน์ว่า สำหรับ ปริภูมิบานาคคอนเวกซ์เอกรูป X ที่สอดคล้องเงื่อนไขของ *Opial* ลำดับ $\{x_n\}$ นิยามโดย *The Modified Ishikawa Iteration Method* ลู่เข้าแบบอ่อนไปยังจุดตรึงของการส่ง T และในปีเดียวกัน *B.L. Xu* และ *M.Aslam Noor* [45] ได้พิสูจน์ว่า ลำดับ $\{x_n\},\{y_n\},\{z_n\}$ นิยามโดย *Noor Iteration Method:* สำหรับ $x_1\in C,$

$$\begin{aligned} z_n &= a_n T^n x_n+(1-a_n)x_n \\ y_n &= b_n T^n z_n+(1-b_n)x_n, \quad \text{เมื่อ } n\ge 1 \\ x_{n+1} &= \alpha_n T^n y_n+(1-\alpha_n)x_n, \end{aligned}$$

เมื่อ $\{a_n\},\{b_n\},\{\alpha_n\}$ เป็นลำดับที่เหมาะสมใน$[0,1]$

ภายใต้เงื่อนไขที่พอเพียงบางอย่าง ระเบียบวิธีการทำซ้ำนี้ลู่เข้าแบบเข้มไปยังจุดตรึงของการส่ง T

ในปี 2005 ศ.ดร.สุเทพ สวนใต้ [38] ได้สร้างระเบียบวิธีการทำซ้ำที่เรียกว่า The *Modified Noor Iteration Method* นิยามดังต่อไปนี้ สำหรับ $x_1 \in C,$

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n)x_n, \qquad \text{เมื่อ } n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n)x_n, \end{aligned}$$

เมื่อ $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\}$ เป็นลำดับที่เหมาะสมใน $[0,1]$ และได้พิสูจน์ว่าภายใต้เงื่อนไขที่พอเพียงบางอย่าง ลำดับ $\{x_n\},\{y_n\},\{z_n\}$ จะลู่เข้าแบบเข้มไปยังจุดตรึงของการส่ง T นอกจากนั้นยังได้พิสูจน์ว่าสำหรับปริภูมิบานาคคอนเวกซ์เอกรูป X ที่สอดคล้องเงื่อนไขของ *Opial* ลำดับ $\{x_n\}$ นิยามโดย *The Modified Noor Iteration Method* ลู่เข้าแบบอ่อนไปยังจุดตรึงของการส่ง T และในปีต่อมา K. Nammanee, M. A. Noor and S. Suantai [26] และได้ขยายงานของ ศ.ดร. สุเทพ สวนใต้ โดยนิยามระเบียบวิธีการทำซ้ำที่เรียกว่า *The Modified Noor Iteration Method with Errors* ดังนี้ สำหรับ $x_1 \in C,$

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n, \qquad \text{เมื่อ } n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n, \end{aligned}$$

เมื่อ $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\mu_n\},\{\lambda_n\}$ เป็นลำดับที่เหมาะสมใน $[0,1]$ และ $\{u_n\},\{v_n\},\{w_n\}$ เป็นลำดับที่มีขอบเขตใน C และได้ทฤษฎีบทการลู่เข้าทั้งแบบเข้มและแบบอ่อนของระเบียบวิธีการทำซ้ำนี้สำหรับการส่งแบบไม่ขยายเชิงกำกับไปยังจุดตรึงของ T

นอกจากนี้แล้วยังมีนักคณิตศาสตร์หลายท่านที่ศึกษาทฤษฎีบทการลู่เข้าสู่จุดตรึงร่วม(Common fixed point theorem) ของระเบียบวิธีการทำซ้ำต่างๆ สำหรับการส่งแบบไม่ขยายและแบบไม่ขยายเชิงกำกับ อาทิเช่น ในปี 1986 Das และ Debata [7] ได้นิยามระเบียบวิธีการทำซ้ำขึ้นมาใหม่ซึ่งทั่วไปกว่าระเบียบวิธีการทำซ้ำของ Mann และ Ishikawa สำหรับสองการส่งแบบไม่ขยาย S และ T ดังนี้ สำหรับ $x_1 \in C,$

$$\begin{aligned} y_n &= (1-b_n)x_n + b_n S x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \end{aligned} \quad \text{เมื่อ } n \geq 1$$

ในปี 2005 Khan และ Fukhar-ud-din [15] ได้ขยายระเบียบวิธีการทำซ้ำนี้โดยนิยามดังนี้

$$\begin{aligned} y_n &= (1-a_n-b_n)x_n + a_n S x_n + b_n u_n, \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_n T y_n + \beta_n v_n, \end{aligned} \quad \text{เมื่อ } n \geq 1$$

เมื่อ $\{a_n\},\{b_n\},\{\alpha_n\},\{\beta_n\}$ เป็นลำดับที่เหมาะสมใน $[0,1]$ และ $\{u_n\},\{v_n\}$ เป็นลำดับที่มีขอบเขตใน C และได้พิสูจน์ทฤษฎีการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำนี้สู่จุดตรึงร่วมของ S และ T ต่อมาในปี 2007 Fukhar-ud-din และ Khan [8] ได้ขยายระเบียบวิธีการทำซ้ำของตัวเอง โดยนิยามดังนี้ สำหรับ $x_1 \in C,$

$$\begin{aligned} x_n &= (1-a_n-b_n)x_n + a_n T_1 x_n + b_n u_n, \\ y_n &= (1-c_n-d_n)x_n + c_n T_2 x_n + d_n v_n, \qquad \text{เมื่อ } n \geq 1 \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_n T_3 y_n + \beta_n w_n, \end{aligned}$$

เมื่อ $\{a_n\},\{b_n\},\{c_n\},\{d_n\},\{\alpha_n\},\{\beta_n\}$ เป็นลำดับที่เหมาะสมใน $[0,1]$ และ $\{u_n\},\{v_n\},\{w_n\}$ เป็นลำดับที่มีขอบเขตใน C และได้พิสูจน์ทฤษฎีการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำนี้สู่จุดตรึงร่วมของสามการส่งแบบไม่ขยาย T_1,T_2 และ T_3 และเมื่อปี 2006 Jeong และ Kim [14] ได้นิยามระเบียบวิธีการทำซ้ำขึ้นใหม่และได้พิสูจน์ ทฤษฎีการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำสู่จุดตรึงร่วมของสองการส่งแบบไม่ขยายเชิงกำกับ T_1 และ T_2 ซึ่งระเบียบวิธีการทำซ้ำดังกล่าวนิยามดังนี้ สำหรับ $x_1 \in C,$

$$\begin{aligned} y_n &= (1-a_n-b_n)x_n + a_nSx_n + b_nu_n, \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_nTy_n + \beta_nv_n, \end{aligned} \quad \text{เมื่อ} \quad n \ge 1$$

เมื่อ $\{a_n\},\{b_n\},\{\alpha_n\},\{\beta_n\}$ เป็นลำดับที่เหมาะสมใน $[0,1]$ และ $0<\delta \le a_n+b_n, \alpha_n+\beta_n \le 1-\delta <1$ และ $\{u_n\},\{v_n\}$ เป็นลำดับที่มีขอบเขตใน C

ดังนั้นงานวิจัยนี้ จะนิยามและศึกษาระเบียบวิธีทำซ้ำแบบใหม่เพื่อหาจุดตรึงร่วม และหาเงื่อนไขที่พอเพียง สำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีทำซ้ำแบบใหม่ที่นิยามขึ้นสำหรับการส่งแบบไม่ขยายในปริภูมิบานาค อีกทั้งยังจะนิยามและศึกษาระเบียบวิธีทำซ้ำแบบใหม่เพื่อหาจุดตรึงร่วม และหาเงื่อนไขที่พอเพียง สำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีทำซ้ำแบบใหม่ที่นิยามขึ้นสำหรับการส่งแบบไม่ขยายเชิงกำกับในปริภูมิบานาค ซึ่งผลงานที่ได้จากงานวิจัยนี้เป็นการขยายผลงานต่างๆ มากมายของการวิจัยในสาขานี้ รวมไปถึงการนำไปประยุกต์ใช้สำหรับการหาคำตอบของสมการที่สำคัญๆ ในสาขาต่างๆ ซึ่งมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการ และ มีประโยชน์อย่างมากในการพัฒนาทางด้านวิทยาศาสตร์พื้นฐาน *(Basic Science)* และวิทยาศาสตร์เชิงประยุกต์ *(Apply Science)* อันถือเป็นพื้นฐานที่สำคัญในการพัฒนาประเทศชาติต่อไป

CHAPTER 2

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapter.

Throughout this thesis, we let $\mathbb{R}$ stand for the set of all real numbers and $\mathbb{N}$ the set of all natural numbers.

2.1 Basic results

Definition 2.1.1. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : E \longrightarrow \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in E$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in E$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.1.2. Let $(E, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset E$ is said to *converge strongly* in X if there exists $x \in E$ such that $\lim\limits_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim\limits_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset E$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in B if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Definition 2.1.3. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.1.4. A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$

Definition 2.1.5. Let C be a nonempty subset of normed space X. A mapping $T : C \longrightarrow C$ is said to be *lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in C$

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.1.1}$$

The smallest number k for which 2.1.1 holds is called the *Lipschitz constant* of T.

Definition 2.1.6. A lipschitzian mapping $T : C \longrightarrow C$ with Lipschitz constant $k < 1$ is said to be a *contraction mapping.*

Definition 2.1.7. An element $x \in C$ is said to be a *fixed point* of a mapping $T : C \longrightarrow C$ iff $Tx = x$.

Definition 2.1.8. [**Banach's contraction mapping principle**] Let (M, d) be a complete metric spaces and let $T : M \longrightarrow M$ be a contraction. Then T has a unique fixed point x_0.

Definition 2.1.9. Let F and E be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow E$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on* F if T a is linear operator.

Definition 2.1.10. A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n\longrightarrow\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converge weakly to x.

Definition 2.1.11. Let F and E be normed spaces over the field $\mathbb{K}$ and $T : F \longrightarrow E$ a linear operator. T is said to be *bounded* on F, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in F$.

Definition 2.1.12. Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$; such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.1.13. Let F and E be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow E$ an operator and $c \in F$. We say that T is *continuous at* c if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If T is continuous at each $x \in F$, then T is said to be *continuous on* F.

Definition 2.1.14. Let X and Y be normed spaces. The mapping $T : X \longrightarrow Y$ is said to be *completely continuous* if and only if $T(C)$ is a compact subset of Y for every bounded subset C of X.

Definition 2.1.15. A mapping $T : C \longrightarrow C$ is said to be *semicompact* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x \in C$.

Definition 2.1.16. A subset C of a normed linear space X is said to be *convex set in* X if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

Definition 2.1.17. Let X be a real normed space and C a nonempty subset of X. A mapping $T : C \longrightarrow C$ is said to be

(a) *nonexpansive* whenever $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$;

(b) *asymptotically nonexpansive* on C if there exists a sequence $\{k_n\}$ in $[1, \infty)$, with $\lim_{n\longrightarrow\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \tag{2.1.2}$$

for all $x, y \in C$ and each $n \geq 1$;

(c) *strict pseudo-contractive mapping* [3] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{2.1.3}$$

for all $x, y \in C$. (If (2.1.3) holds, we also say that T is a k-strict pseudo-contraction.)

It is know that if T is 0-strict pseudo-contractive mapping, T is nonexpansive mapping.

(d) *asymptotically k-strict pseudo-contractive* if there exists a constant $0 \leq k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{2.1.4}$$

for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\gamma_n \geq 0$ for all n such that $\lim_{n \to \infty} \gamma_n = 0$.

Definition 2.1.18. [10] A Banach space X is said to be *uniformly convex* if for each $0 < \epsilon \leq 2$, there is $\delta > 0$ such that $\forall x, y \in X$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \epsilon$ imply $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 2.1.19. [10] Let X be a Banach space. Then *the modulus of convexity of X* $\delta : [0, 2] \longrightarrow [0, 1]$ defined as follows:

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}.$$

Theorem 2.1.20. [10] *Let X be a Banach space. Then X is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon > 0$.*

Theorem 2.1.21. [2] *Let C be a nonempty, closed, convex and bounded subset of uniformly convex Banach space E and let $T : C \longrightarrow C$ be nonexpansive mapping. Then T has a fixed point.*

Theorem 2.1.22. [10] *Let C be a nonempty, closed, convex and bounded subset of uniformly convex Banach space E and let $T : C \longrightarrow C$ be asymptotically nonexpansive mapping. Then T has a fixed point.*

Definition 2.1.23. [29] A Banach space X is said to satisfy *Opial's condition* if any sequence $\{x_n\}$ in C, $x_n \longrightarrow x$ weakly as $n \longrightarrow \infty$ implies that $\limsup_{n \longrightarrow \infty} \|x_n - x\| < \limsup_{n \longrightarrow \infty} \|x_n - y\|$ for all $y \in C$ with $y \neq x$.

Definition 2.1.24. [37] The mapping $T : C \longrightarrow C$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Definition 2.1.25. [15] Two mappings $T_1, T_2 : C \longrightarrow C$ where C a nonempty subset of X, is said to satisfy *condition* (A') if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - T_1x\| + \|x - T_2x\|) \geq f(d(x, F))$$

for all $x \in C$ where $F := F(T_1) \cap F(T_2)$ and $d(x, F) = \inf\{\|x - x^*\| : x^* \in F\}$.

Remark 2.1.26. Note that condition (A') reduces to condition (A) when $T_1 = T_2$.

We modify this condition for three mappings $T_1, T_2, T_3 : C \longrightarrow C$ as follows:

Definition 2.1.27. Three mappings $T_1, T_2, T_3 : C \longrightarrow C$ where C is a subset of X, is said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{3}(\|x - T_1x\| + \|x - T_2x\| + \|x - T_3x\|) \geq f(d(x, F))$$

for all $x \in C$ where $F := F(T_1) \cap F(T_2) \cap F(T_3)$.

Remark 2.1.28. Note that condition (A'') reduces to condition (A) when $T_1 = T_2 = T_3$.

Remark 2.1.29. [37] It is well known that every continuous and demicompact mapping must satisfy condition (A). Since every completely continuous $T : C \longrightarrow C$ is continuous and demicompact so that it satisfies condition (A).

Lemma 2.1.30. [39] *Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n = 1, 2, ...$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(i) $\lim_{n \longrightarrow \infty} a_n$ *exists;*

(ii) $\lim_{n \longrightarrow \infty} a_n = 0$, *whenever* $\liminf_{n \longrightarrow \infty} a_n = 0$.

Lemma 2.1.31 (Browder, [2])**.** *Let* X *be a uniformly convex Banach space,* C *a nonempty closed convex subset of* X, *and* $T : C \longrightarrow X$ *be nonexpansive mapping. Then* $I - T$ *is demiclosed at* 0, *i.e., if* $x_n \longrightarrow x$ *weakly and* $x_n - Tx_n \longrightarrow 0$ *strongly, then* $x \in F(T)$.

Lemma 2.1.32. [6] *Let* X *be a uniformly convex Banach space,* C *a nonempty closed convex subset of* X, *and* $T : C \longrightarrow C$ *be an asymptotically nonexpansive mapping. Then* $I - T$ *is demiclosed at* 0, *i.e., if* $x_n \longrightarrow x$ *weakly and* $x_n - Tx_n \longrightarrow 0$ *strongly, then* $x \in F(T)$, *where* $F(T)$ *is the set of fixeds point of* T.

Lemma 2.1.33. [38] *Let* X *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* X. *Let* $u, v \in X$ *be such that* $\lim_{n \longrightarrow \infty} \|x_n - u\|$ *and* $\lim_{n \longrightarrow \infty} \|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequence of* $\{x_n\}$ *which converge weakly to* u *and* v, *respectively, then* $u = v$.

Lemma 2.1.34. [44] *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \longrightarrow [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|),$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, where

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda).$$

Lemma 2.1.35. [6] *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \longrightarrow [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Definition 2.1.36. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

Lemma 2.1.37. *[25] There holds the identity in a Hilbert space H:*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H$.

(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x||^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$ for all $x, y \in H$ and $\lambda \in [0, 1]$.

Lemma 2.1.38. *[19] Let T be an asymptotically k-strict pseudo-contractive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties*

(i) $x_n \rightharpoonup z$ and

(ii) $Tx_n - x_n \to 0$.

Then $(I - T)z = 0$.

Lemma 2.1.39. *[31] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.1.40. *[19] Assume that C is a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically k-strict pseudo-contraction. Then for each $n \geq 1$, T^n satisfies the Lipschitz condition:*

$$\|T^n x - T^y\| \leq L_n\|x - y\|$$

for all $x, y \in C$, where $L_n = \frac{k+\sqrt{1+\gamma_n(1-k)}}{1-k}$.

CHAPTER 3

MAIN RESULTS

3.1 Approximating common Fixed points of nonexpansive mappings in a Banach Space

In this section, we introduce a three-step iterative scheme with errors for three nonexpansive mapping in a uniformly convex Banach space. Moreover, It is proved that if the mappings satisfied condition (A), then the iterative scheme converges strongly to a common fixed point of these mappings. And if one of these mappings is demicompact or completely continuous, then the iterative scheme converges strongly to a common fixed point of these mappings. It also shown that if the our space satisfies Opial' s condition, then we would have weak convergence theorem of our iterative scheme.

Let C be a nonempty closed convex subset of a Banach space X. Let T_1, T_2 and T_3 be three nonexpansive mappings from C to itself. Let F be the set of common fixed points of T_1, T_2 and T_3 that is $F = \bigcap_{i=1}^{3} F(T_i) = \{x \in C : x = T_1x = T_2x = T_3x\}$.

ALGORITHM 1.1. For a given $x_1 \in C$, compute sequences $\{z_n\}, \{y_n\}$ and $\{x_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= (1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n, \\ y_n &= (1 - c_n - d_n)z_n + c_nT_2z_n + d_nv_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n, \quad n \geq 1, \end{aligned} \tag{3.1.1}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = 0$, then Algorithm 1.1 reduces to

ALGORITHM 1.2. For a given $x_1 \in C$, compute sequences $\{y_n\}$ and $\{x_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1 - c_n - d_n)x_n + c_nT_2x_n + d_nv_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n, \quad n \geq 1, \end{aligned} \tag{3.1.2}$$

where $\{v_n\}, \{w_n\}$ are bounded sequences in C and $\{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = d_n = 0$, then Algorithm 1.1 reduces to

ALGORITHM 1.3. For a given $x_1 \in C$, compute sequences $\{x_n\}$ by the iterative schemes

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_nT_3x_n + \beta_nw_n, \quad n \geq 1, \tag{3.1.3}$$

where $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0,1]$. Algorithm 1.3 is known as a Mann-iterative scheme with error which is introduced by Xu [44].

In the sequel, the following lemmas are needed to prove our main results.

Lemma 3.1.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and $\{x_n\}$ be the sequences defined as in (3.1.1) and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. If $F \neq \emptyset$, then $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Assume that $F \neq \emptyset$. Let $p \in F$ and

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\},\ M_2 = \sup\{\|v_n - p\| :\geq 1\}$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_1, M_2, M_3\}.$$

We have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)y_n + \alpha_n T_3 y_n + \beta_n w_n - p\| \\
&= \|(1-\alpha_n-\beta_n)(y_n - p) + \alpha_n(T_3 y_n - p) + \beta_n(w_n - p)\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n - p\| + \alpha_n\|T_3 y_n - p\| + \beta_n\|w_n - p\| \\
&\leq \|y_n - p\| + \beta_n\|w_n - p\| \\
\|y_n - p\| &= \|(1-c_n-d_n)z_n + c_n T_2 z_n + d_n v_n - p\| \\
&= \|(1-c_n-d_n)(z_n - p) + c_n(T_2 z_n - p) + d_n(v_n - p)\| \\
&\leq (1-c_n-d_n)\|z_n - p\| + c_n\|T_2 z_n - p\| + d_n\|v_n - p\| \\
&\leq \|z_n - p\| + d_n\|v_n - p\| \\
\|z_n - p\| &= \|(1-a_n-b_n)x_n + b_n T_1 x_n + b_n u_n - p\| \\
&= \|(1-a_n-b_n)(x_n - p) + a_n(T_1 x_n - p) + b_n(u_n - p)\| \\
&\leq (1-a_n-b_n)\|x_n - p\| + b_n\|T_1 x_n - p\| + b_n\|u_n - p\| \\
&\leq \|x_n - p\| + b_n\|u_n - p\|
\end{aligned}$$

It follows that $\|x_{n+1} - p\| \leq \|x_n - p\| + b_n\|u_n - p\| + d_n\|v_n - p\| + \beta_n\|w_n - p\|$.

And so $\|x_{n+1} - p\| \leq \|x_n - p\| + M(b_n + d_n + \beta_n)$.

Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.1.30 implies that $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists. □

Lemma 3.1.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3.1.1) with $F \neq \emptyset$.*

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_3y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n+d_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_2z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n+b_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_1x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ *and* $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_2x_n - x_n\| = 0 = \lim_{n\longrightarrow\infty} \|T_3x_n - x_n\|$.

Proof. (i) From Lemma 3.1.1, we have $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for any $p \in F(T)$. It follows that $\{x_n - p\}, \{T_1x_n - p\}, \{z_n - q\}, \{T_2z_n - p\}$ and $\{y_n - p\}, \{T_3y_n - p\}$ are all bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. Now we set

$$\begin{aligned}
r_1 &= \sup\{\|x_n - p\| : n \geq 1\},\\
r_2 &= \sup\{\|T_1x_n - p\| : n \geq 1\},\\
r_3 &= \sup\{\|z_n - p\| : n \geq 1\},\\
r_4 &= \sup\{\|T_2z_n - p\| : n \geq 1\},\\
r_5 &= \sup\{\|y_n - p\| : n \geq 1\},\\
r_6 &= \sup\{\|T_3y_n - p\| : n \geq 1\},\\
r_7 &= \sup\{\|u_n - p\| : n \geq 1\},\\
r_8 &= \sup\{\|v_n - p\| : n \geq 1\},\\
r_9 &= \sup\{\|w_n - p\| : n \geq 1\},\\
r &= \max\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}.
\end{aligned}$$

By Lemma 2.1.35 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-a_n-b_n)x_n + a_nT_1x_n + b_nu_n - p\|^2 \\
&= \|(1-a_n-b_n)(x_n-p) + a_n(T_1x_n-p) + b_n(u_n-p)\|^2 \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_n\|T_1x_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\quad - a_n(1-a_n-b_n)g(\|T_1x_n-x_n\|) \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_n\|x_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\leq \|x_n-p\|^2 + r^2b_n - a_n(1-a_n-b_n)g(\|T_1x_n-x_n\|) \\
&\leq \|x_n-p\|^2 + r^2b_n, \\
\|y_n - p\|^2 &= \|(1-c_n-d_n)z_n + c_nT_2z_n + d_nv_n - p\|^2 \\
&= \|(1-c_n-d_n)(z_n-p) + c_n(T_2z_n-p) + d_n(v_n-p)\|^2 \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_n\|T_2z_n-p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T_2z_n-z_n\|) \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_n\|z_n-p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T_2z_n-z_n\|) \\
&\leq \|z_n-p\|^2 + r^2d_n - c_n(1-c_n-d_n)g(\|T_2z_n-z_n\|) \\
&\leq \|z_n-p\|^2 + r^2d_n
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n-\beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(T_3y_n-p) + \beta_n(w_n-p)\|^2 \\
&\leq (1-\alpha_n-d_n)\|y_n-p\|^2 + \alpha_n\|T_3y_n-p\|^2 + \beta_n\|w_n-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3y_n-y_n\|) \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2 + \alpha_n\|y_n-p\|^2 + \beta_n\|w_n-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3y_n-y_n\|) \\
&\leq \|y_n-p\|^2 + r^2\beta_n - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3y_n-y_n\|),
\end{aligned}$$

which lead to the following:

$$\alpha_n(1-\alpha_n-\beta_n)g(\|T_3y_n-y_n\|) \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n), \tag{3.1.4}$$

and

$$c_n(1-c_n-d_n)g(\|T_2z_n-z_n\|) \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n), \tag{3.1.5}$$

and

$$a_n(1-a_n-b_n)g(\|T_1x_n-x_n\|) \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n), \tag{3.1.6}$$

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0, 1)$ such that

$$0 < \eta < \alpha_n \text{ and } \alpha_n + \beta_n < \eta' < 1 \text{ for all } n \geq n_0.$$

This implies by (3.1.4) that

$$\eta(1-\eta')g(\|T_3y_n - y_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + r^2(b_n + d_n + \beta_n), \tag{3.1.7}$$

for all $n \geq n_0$. It follows from (3.1.7) that for $m \geq n_0$

$$\begin{aligned} \sum_{n=n_0}^{m} g(\|T_3y_n - y_n\|) &\leq \frac{1}{\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\quad + r^2\sum_{n=n_0}^{m}(b_n + d_n + \beta_n)\Bigg) \\ &\leq \frac{1}{\eta(1-\eta')}\Bigg(\|x_{n_0} - p\|^2 \\ &\quad + r^2\sum_{n=n_0}^{m}(b_n + d_n + \beta_n)\Bigg). \end{aligned} \tag{3.1.8}$$

By letting $m \longrightarrow \infty$ in inequality (3.1.8) we get that $\sum_{n=n_0}^{\infty} g(\|T_3y_n - y_n\|) < \infty$, which implies $\lim_{n\longrightarrow\infty} g(\|T_3y_n - y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\longrightarrow\infty}\|T_3y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, then by using the same argument as above with the inequality (3.1.5), it can be show that $\lim_{n\longrightarrow\infty}\|T_2z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then by using (3.1.6) and the same argument as in (i), it can be show that $\lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, by (i), (ii) and (iii) we have

$$\lim_{n\longrightarrow\infty}\|T_3y_n - y_n\| = \lim_{n\longrightarrow\infty}\|T_2z_n - z_n\| = \lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0. \tag{3.1.9}$$

From $z_n = (1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n$ and $y_n = (1 - c_n - d_n)z_n + c_nT_2z_n + d_nv_n$, we have

$$\begin{aligned} \|z_n - x_n\| &= \|(1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n - x_n\| \\ &= \|a_n(T_1x_n - x_n) + b_n(u_n - x_n)\| \\ &\leq a_n\|T_1x_n - x_n\| + b_n\|u_n - x_n\| \\ &\leq \|T_1x_n - x_n\| + 2rb_n, \end{aligned}$$

and $\|y_n - x_n\| \leq \|T_2z_n - z_n\| + \|T_1x_n - x_n\| + 2rb_n + 2rd_n$. Hence

$$\begin{aligned}\|T_2x_n - x_n\| &\leq \|x_n - T_2z_n\| + \|T_2z_n - T_2x_n\| \\ &\leq \|x_n - z_n\| + \|T_2z_n - z_n\| + \|z_n - x_n\| \\ &\leq \|T_2z_n - z_n\| + 2\|z_n - x_n\| \\ &\leq \|T_2z_n - z_n\| + 2\|T_1x_n - x_n\| + 4rb_n,\end{aligned}$$

and

$$\begin{aligned}\|T_3x_n - x_n\| &\leq \|x_n - T_3y_n\| + \|T_3y_n - T_3x_n\| \\ &\leq \|x_n - y_n\| + \|T_3y_n - y_n\| + \|y_n - x_n\| \\ &\leq \|T_3y_n - y_n\| + 2\|y_n - x_n\| \\ &\leq \|T_3y_n - y_n\| + 2\|T_2z_n - z_n\| \\ &\quad + 2\|T_1x_n - x_n\| + 4rb_n + 4rd_n.\end{aligned}$$

It follows from (3.1.9) that $\lim_{n\longrightarrow\infty}\|T_2x_n - x_n\| = \lim_{n\longrightarrow\infty}\|T_3x_n - x_n\| = 0$. □

Theorem 3.1.3. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined as in (3.1.1). If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. Let $p \in F$. By Lemma 3.1.1, $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let p_1 and p_2 be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 3.1.2(iii), we have $\lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0$. By Lemma 2.1.32, we have $I - T_1, I - T_2$ and $I - T_3$ are demiclosed with respect to zero, therefore $T_1p_i = p_i, T_2p_i = p_i$ and $T_3p_i = p_i$, (i=1, 2) hence $p_1, p_2 \in F$. By Lemma 3.1.1 $\lim_{n\longrightarrow\infty}\|x_n - p_1\|$ and $\lim_{n\longrightarrow\infty}\|x_n - p_2\|$ exist. Using Lemma 2.1.33 we obtain that $p_1 = p_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

Our next goal is to prove a strong convergence theorem of iterative scheme (3.1.1) to a common fixed point of T_1, T_2 and T_3.

Theorem 3.1.4. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C satisfying condition (A) and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and*

$\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n+d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n+b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\},\{y_n\},\{z_n\}$ be the sequences defined as in (3.1.1). If $F \neq \emptyset$, then $\{x_n\},\{y_n\},\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 3.1.2, we have

$$\lim_{n\longrightarrow\infty} \|T_3x_n - x_n\| = \lim_{n\longrightarrow\infty} \|T_2x_n - x_n\| = \lim_{n\longrightarrow\infty} \|T_1x_n - x_n\| = 0.$$

By condition (A), we have $f(d(x_n,F)) \leq \frac{1}{3}(\|x_n - T_1x_n\| + \|x_n - T_2x_n\| + \|x_n - T_3x_n\|)$.

It follows that $\lim_{n\longrightarrow\infty} f(d(x_n,F)) \leq 0$, hence $\lim_{n\longrightarrow\infty} f(d(x_n,F)) = 0$.

Since f is a nondecreasing function and $f(0)=0$, we obtain that $\lim_{n\longrightarrow\infty} d(x_n,F) = 0$. Thus we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $(y_k) \subset F$ such that $\|x_{n_k} - y_k\| < 2^{-k}$ for all $k \geq 1$. Then following the method of proof of Tan and Xu [39], we get that $\{y_k\}$ is a Cauchy sequence in F and so it converges. Let $y_k \longrightarrow y$. Since F is closed, $y \in F$ and then $x_{n_k} \longrightarrow y$. By Lemma (3.1.1) $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exits, $\forall p \in F$. It implies that $\lim_{n\longrightarrow\infty} \|x_n - y\| = 0$. Hence $\{x_n\}$ converses strongly to a common fixed point of T_1, T_2 and T_3.

Since $\|y_n - x_n\| \leq \|T_2z_n - z_n\| + \|T_1x_n - x_n\| + 2rb_n + 2rd_n$ and $\|z_n - x_n\| \leq \|T_1x_n - x_n\| + 2rb_n$, it follows from Lemma 3.1.2 (ii) and (iii) that $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|z_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, which imply that $\lim_{n\longrightarrow\infty} \|y_n - y\| = 0$ and $\lim_{n\longrightarrow\infty} \|z_n - y\| = 0$. □

Theorem 3.1.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\},\{b_n\},\{c_n\},\{d_n\},\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n+d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n+b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\},\{y_n\},\{z_n\}$ be the sequences defined as in (3.1.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\},\{y_n\}$, $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 3.1.1, $\{x_n\}$ is bounded. In addition, by Lemma 3.1.2, $lim_{n\longrightarrow\infty}\|x_n - T_1x_n\| = 0$, $lim_{n\longrightarrow\infty}\|x_n - T_2x_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|x_n - T_3x_n\| = 0$, then $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \longrightarrow p$ as $j \longrightarrow \infty$. Thus $\lim_{j\longrightarrow\infty}\|x_{n_j} - T_1x_{n_j}\| = \lim_{j\longrightarrow\infty}\|x_{n_j} - T_2x_{n_j}\| = \lim_{j\longrightarrow\infty}\|x_{n_j} - T_3x_{n_j}\| = 0$. It follows that $\lim_{j\longrightarrow\infty}\|x_{n_j} - p\| = 0$. This implies by Lemma 2.1.32 that $p \in F$. Furthermore, by Lemma 3.1.1, we get that $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists. Thus $lim_{n\longrightarrow\infty}\|x_n - p\| = 0$. The proof is completed. □

Theorem 3.1.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (3.1.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is demicompact, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose that T_1 is demicompact. By Lemma 3.1.2, $\lim_{n\longrightarrow\infty}\|x_n - T_1x_n\| = \lim_{n\longrightarrow\infty}\|x_n - T_2x_n\| = \lim_{n\longrightarrow\infty}\|x_n - T_3x_n\| = 0$. Then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in C$. It follows from Lemma 2.1.32 that $p \in F$. Since $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists by Lemma 3.1.1, it follows that $\lim_{n\longrightarrow\infty}\|x_n - p\| = 0$. □

3.2 Convergence theorems for asymptotically nonexpansive mappings in a Banach Space

This section is to construct an iterative scheme for approximating common fixed points of three asymptotically nonexpansive mappings to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space.

Let C be a nonempty closed convex subset of normed linear space X and $T_1, T_2, T_3 : C \longrightarrow C$ be three asymptotically nonexpansive self-mappings. For a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= (1 - a_n - b_n)x_n + a_nT_1^nx_n + b_nu_n, \\ y_n &= (1 - c_n - d_n)z_n + c_nT_2^nz_n + d_nv_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_nT_3^ny_n + \beta_nw_n, \quad n \geq 1, \end{aligned} \tag{3.2.1}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = 0$, then the scheme (3.2.1) reduces to the iterative schemes

$$\begin{aligned} y_n &= (1 - c_n - d_n)x_n + c_n T_2^n x_n + d_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T_3^n y_n + \beta_n w_n, \quad n \geq 1, \end{aligned} \tag{3.2.2}$$

where $\{v_n\}, \{w_n\}$ are bounded sequences in C and $\{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = d_n = 0$, then scheme (3.2.1) reduces to the modified Mann-iterative scheme iterative schemes

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T_3^n x_n + \beta_n w_n, \quad n \geq 1, \tag{3.2.3}$$

where $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

In the sequel, the following lemmas are needed to prove our main results.

The next lemma is crucial for proving the main theorems.

Lemma 3.2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and $\{x_n\}$ be the sequence defined as in (3.2.1) and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. If $F \neq \emptyset$, then $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Assume that $F \neq \emptyset$. Let $p \in F$ and

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \ M_2 = \sup\{\|v_n - p\| :\geq 1\}$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_1, M_2, M_3\}.$$

We have

$$
\begin{aligned}
\|z_n - p\| &= \|(1-a_n-b_n)x_n + b_nT_1^n x_n + b_n u_n - p\| \\
&= \|(1-a_n-b_n)(x_n-p) + a_n(T_1^n x_n - p) + b_n(u_n - p)\| \\
&\leq (1-a_n-b_n)\|x_n-p\| + b_n\|T_1^n x_n - p\| + b_n\|u_n-p\| \\
&\leq (1-a_n-b_n)\|x_n-p\| + b_nk_n\|T_1^n x_n - p\| + b_n\|u_n-p\| \\
&\leq (1+a_n(k_n-1))\|x_n-p\|b_n\|u_n-p\| \\
&\leq k_n\|x_n-p\| + b_n\|u_n-p\| \\
\|y_n - p\| &= \|(1-c_n-d_n)z_n + c_nT_2^n z_n + d_n v_n - p\| \\
&= \|(1-c_n-d_n)(z_n-p) + c_n(T_2^n z_n - p) + d_n(v_n - p)\| \\
&\leq (1-c_n-d_n)\|z_n-p\| + c_n\|T_2^n z_n - p\| + d_n\|v_n-p\| \\
&\leq (1-c_n-d_n)\|z_n-p\| + c_nl_n\|T_2^n z_n - p\| + d_n\|v_n-p\| \\
&\leq (1+c_n(l_n-1))\|z_n-p\| + d_n\|v_n-p\| \\
&\leq l_n\|z_n-p\| + d_n\|v_n-p\| \\
\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)y_n + \alpha_nT_3^n y_n + \beta_n w_n - p\| \\
&= \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(T_3^n y_n - p) + \beta_n(w_n - p)\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_n\|T_3^n y_n - p\| + \beta_n\|w_n-p\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_nm_n\|T_3^n y_n - p\| + \beta_n\|w_n-p\| \\
&\leq (1+\alpha_n(m_n-1))\|y_n-p\| + \beta_n\|w_n-p\| \\
&\leq m_n\|y_n-p\| + \beta_n\|w_n-p\| \\
&\leq m_n[l_n[k_n\|x_n-p\| + b_n\|u_n-p\|] + d_n\|v_n-p\|] + \beta_n\|w_n-p\| \\
&= m_nl_nk_n\|x_n-p\| + m_nl_nb_n\|u_n-p\|] + m_nd_n\|v_n-p\|] + \beta_n\|w_n-p\| \\
&\leq h_n^3\|x_n-p\| + M(b_n+d_n+\beta_n) \\
&= (1+(h_n^3-1))\|x_n-p\| + M(b_n+d_n+\beta_n),
\end{aligned}
$$

where M is some positive constant and $h_n = \max\{k_n, l_n, M_n\}$.

Notice that $\sum_{n=1}^{\infty}(h_n-1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(h_n^3-1) < \infty$. Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.1.30 implies that $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists. □

Lemma 3.2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0,\infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n-1) < \infty$, $\sum_{n=1}^{\infty}(l_n-1) < \infty$, $\sum_{n=1}^{\infty}(m_n-1) < \infty$, respectively and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3.2.1) with $F \neq \emptyset$.*

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, then $\lim_{n\longrightarrow\infty}\|T_3^n y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n \longrightarrow \infty} c_n \leq \limsup_{n \longrightarrow \infty}(c_n + d_n) < 1$*, then* $\lim_{n \longrightarrow \infty} \|T_2^n z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n \longrightarrow \infty} a_n \leq \limsup_{n \longrightarrow \infty}(a_n + b_n) < 1$*, then* $\lim_{n \longrightarrow \infty} \|T_1^n x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n \longrightarrow \infty} \alpha_n \leq \limsup_{n \longrightarrow \infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n \longrightarrow \infty} c_n \leq \limsup_{n \longrightarrow \infty}(c_n + d_n) < 1$ *and* $0 < \liminf_{n \longrightarrow \infty} a_n \leq \limsup_{n \longrightarrow \infty}(a_n + b_n) < 1$*, then* $\lim_{n \longrightarrow \infty} \|T_2^n x_n - x_n\| = 0 = \lim_{n \longrightarrow \infty} \|T_3^n x_n - x_n\|$.

Proof. (i) From Lemma 3.2.1, we know that $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists for any $p \in F$. It follow that $\{x_n - p\}, \{T_1^n x_n - p\}, \{z_n - q\}, \{T_2^n z_n - p\}$ and $\{y_n - p\}, \{T_3^n y_n - p\}$ are all bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. Now we set

$$\begin{aligned}
r_1 &= \sup\{\|x_n - p\| : n \geq 1\}, \\
r_2 &= \sup\{\|T_1^n x_n - p\| : n \geq 1\}, \\
r_3 &= \sup\{\|z_n - p\| : n \geq 1\}, \\
r_4 &= \sup\{\|T_2^n z_n - p\| : n \geq 1\}, \\
r_5 &= \sup\{\|y_n - p\| : n \geq 1\}, \\
r_6 &= \sup\{\|T_3^n y_n - p\| : n \geq 1\}, \\
r_7 &= \sup\{\|u_n - p\| : n \geq 1\}, \\
r_8 &= \sup\{\|v_n - p\| : n \geq 1\}, \\
r_9 &= \sup\{\|w_n - p\| : n \geq 1\}, \\
r &= \max\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}. \qquad (3.2.4)
\end{aligned}$$

By using Lemma 2.1.35 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-a_n-b_n)x_n + a_nT_1^nx_n + b_nu_n - p\|^2 \\
&= \|(1-a_n-b_n)(x_n-p) + a_n(T_1^nx_n - p) + b_n(u_n-p)\|^2 \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_n\|T_1^nx_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\quad - a_n(1-a_n-b_n)g(\|T_1^nx_n - x_n\|) \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_nk_n^2\|x_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\leq (1+a_n(k_n^2-1))\|x_n-p\|^2 + r^2b_n \\
&\leq k_n^2\|x_n-p\|^2 + r^2b_n, \\
\|y_n - p\|^2 &= \|(1-c_n-d_n)z_n + c_nT_2^nz_n + d_nv_n - p\|^2 \\
&= \|(1-c_n-d_n)(z_n-p) + c_n(T_2^nz_n - p) + d_n(v_n-p)\|^2 \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_n\|T_2^nz_n-p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T_2^nz_n - z_n\|) \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_nl_n^2\|z_n-p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T_2^nz_n - z_n\|) \\
&\leq (1+c_n(l_n^2-1))\|z_n-p\|^2 + r^2d_n \\
&\leq l_n^2\|z_n-p\|^2 + r^2d_n
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n-\beta_n)y_n + \alpha_nT_3^ny_n + \beta_nw_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(T_3^ny_n - p) + \beta_n(w_n-p)\|^2 \\
&\leq (1-\alpha_n-d_n)\|y_n-p\|^2 + \alpha_n\|T_3^ny_n-p\|^2 + \beta_n\|w_n-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3^ny_n - y_n\|) \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2 + \alpha_nm_n^2\|y_n-p\|^2 + \beta_n\|w_n-p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3^ny_n - y_n\|)
\end{aligned}$$

$$
\begin{aligned}
&\le (1+\alpha_n(m_n^2-1))\|y_n-p\|^2+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\le m_n^2\|y_n-p\|^2+r^2\beta_n-\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\le m_n^2[l_n^2\|z_n-p\|^2+r^2 d_n]+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\le m_n^2[l_n^2[k_n^2\|x_n-p\|^2+r^2 b_n]+r^2 d_n]+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&= m_n^2 l_n^2 k_n^2\|x_n-p\|^2+m_n^2 l_n^2 r^2 b_n+m_n^2 r^2 d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\le h_n^3\|x_n-p\|^2+M(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&= (1+(h_n^3-1))\|x_n-p\|^2+M(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|),
\end{aligned}
$$

which leads to the following:

$$
\begin{aligned}
\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (3.2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
m_n^2 c_n(1-c_n-d_n)g(\|T_2^n z_n-z_n\|) \le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (3.2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
m_n^2 l_n^2 a_n(1-a_n-b_n)g(\|T_1^n x_n-x_n\|) \le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (3.2.7)
\end{aligned}
$$

(i) If $0<\liminf_{n\longrightarrow\infty}\alpha_n\le\limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n)<1$, then there exist a positive integer n_0 and $\eta,\eta'\in(0,1)$ such that

$$0<\eta<\alpha_n \ \text{ and } \ \alpha_n+\beta_n<\eta'<1 \ \text{ for all } \ n\ge n_0.$$

This implies by (3.2.5) that

$$
\begin{aligned}
\eta(1-\eta')g(\|T_3^n y_n-y_n\|) \le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (3.2.8)
\end{aligned}
$$

for all $n \geq n_0$. It follows from (3.2.8) that for $m \geq n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|T_3^n y_n - y_n\|) &\leq \frac{1}{\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\quad + M\sum_{n=n_0}^{m}((h_n^3 - 1) + b_n + d_n + \beta_n)\Bigg) \\ &\leq \frac{1}{\eta(1-\eta')}\Bigg(\|x_{n_0} - p\|^2 \\ &\quad + M\sum_{n=n_0}^{m}((h_n^3 - 1) + b_n + d_n + \beta_n)\Bigg). \qquad (3.2.9)\end{aligned}$$

Let $m \longrightarrow \infty$ in inequality (3.2.9) we get that $\sum_{n=n_0}^{\infty} g(\|T_3^n y_n - y_n\|) < \infty$, and therefore $\lim_{n\longrightarrow\infty} g(\|T_3^n y_n - y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\longrightarrow\infty} \|T_3^n y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, then by the using a similar method, together with inequality (3.2.6), it can be show that $\lim_{n\longrightarrow\infty} \|T_2^n z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then by the using a similar method, together with inequality (3.2.7), it can be show that $\lim_{n\longrightarrow\infty} \|T_1^n x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, and $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$ by (i), (ii) and (iii) we have

$$\lim_{n\longrightarrow\infty} \|T_3^n y_n - y_n\| = \lim_{n\longrightarrow\infty} \|T_2^n z_n - z_n\| = \lim_{n\longrightarrow\infty} \|T_1^n x_n - x_n\| = 0. \quad (3.2.10)$$

From $z_n = (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n$ and $y_n = (1 - c_n - d_n)z_n + c_n T_2^n z_n + d_n v_n$, we have

$$\begin{aligned}\|z_n - x_n\| &= \|(1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n - x_n\| \\ &= \|a_n(T_1^n x_n - x_n) + b_n(u_n - x_n)\| \\ &\leq a_n\|T_1^n x_n - x_n\| + b_n\|u_n - x_n\| \\ &\leq \|T_1^n x_n - x_n\| + 2rb_n,\end{aligned}$$

and $\|y_n - x_n\| \leq \|T_2^n z_n - z_n\| + \|T_1^n x_n - x_n\| + 2rb_n + 2rd_n$. Hence

$$\begin{aligned}\|T_2^n x_n - x_n\| &\leq \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\| \\ &\leq \|x_n - z_n\| + \|T_2^n z_n - z_n\| + l_n\|z_n - x_n\| \\ &\leq \|T_2^n z_n - z_n\| + (1 + l_n)\|z_n - x_n\| \\ &\leq \|T_2^n z_n - z_n\| + (1 + l_n)[\|T_1^n x_n - x_n\| + 2rb_n],\end{aligned}$$

and

$$\begin{aligned}
\|T_3^n x_n - x_n\| &\leq \|x_n - T_3^n y_n\| + \|T_3^n y_n - T_3^n x_n\| \\
&\leq \|x_n - y_n\| + \|T_3^n y_n - y_n\| + m_n\|y_n - x_n\| \\
&\leq \|T_3^n y_n - y_n\| + (1+m_n)\|y_n - x_n\| \\
&\leq \|T_3^n y_n - y_n\| + (1+m_n)[\|T_2^n z_n - z_n\| \\
&\quad + \|T_1^n x_n - x_n\| + 2rb_n + 2rd_n]
\end{aligned}$$

It follows from (3.2.10) that $\lim_{n\longrightarrow\infty}\|T_2^n x_n - x_n\| = \lim_{n\longrightarrow\infty}\|T_3^n x_n - x_n\| = 0$. □

Lemma 3.2.3. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0,\infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty}\beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3.2.1)with $F \neq \emptyset$.*

If $0 < \liminf_{n\longrightarrow\infty}\alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then $\lim_{n\longrightarrow\infty}\|T_1 x_n - x_n\| = \lim_{n\longrightarrow\infty}\|T_2 x_n - x_n\| = \lim_{n\longrightarrow\infty}\|T_3 x_n - x_n\| = 0$.

Proof. By Lemma 3.2.2, we have

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\|T_3^n y_n - y_n\| = \lim_{n\longrightarrow\infty}\|T_2^n z_n - z_n\| &= \lim_{n\longrightarrow\infty}\|T_1^n x_n - x_n\| = 0, \\
\text{and} \quad \lim_{n\longrightarrow\infty}\|T_3^n x_n - x_n\| &= \lim_{n\longrightarrow\infty}\|T_2^n x_n - x_n\| = 0. \quad (3.2.11)
\end{aligned}$$

Also we obtain that $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ by the proof of lemma 3.2.2.

Since $x_{n+1} - y_n = \alpha_n(T_n^3 y_n - y_n) + \beta_n(w_n - y_n)$, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\
&\leq \|T_n^3 y_n - y_n\| + \beta_n\|w_n - y_n\| + \|y_n - x_n\|
\end{aligned}$$

And so

$$\begin{aligned}
\|x_{n+1} - T_1^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T_1^n x_{n+1} - T_1^n x_n\| + \|T_1^n x_n - x_n\| \\
&\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T_1^n x_n - x_n\| \\
&= (1+k_n)\|x_{n+1} - x_n\| + \|T_1^n x_n - x_n\| \\
&\leq (1+k_n)[\|T_n^3 y_n - y_n\| + \beta_n\|w_n - y_n\| + \|y_n - x_n\|] \\
&\quad + \|T_1^n x_n - x_n\|.
\end{aligned}$$

This together with (3.2.11) implies that $\|x_{n+1} - T_1^n x_{n+1}\| \longrightarrow 0$ (as $n \longrightarrow \infty$).

Thus

$$\begin{aligned}\|x_{n+1}-T_1x_{n+1}\| &\leq \|x_{n+1}-T_1^{n+1}x_{n+1}\|+\|T_1x_{n+1}-T_1^{n+1}x_{n+1}\|\\ &\leq \|x_{n+1}-T_1^{n+1}x_{n+1}\|+k_1\|x_{n+1}-T_1^{n}x_{n+1}\|\longrightarrow 0,\end{aligned}$$

which implies that $\lim_{n\longrightarrow\infty}\|T_1x_n-x_n\|=0$.

Similarly, we obtain that $\lim_{n\longrightarrow\infty}\|T_2x_n-x_n\|=\lim_{n\longrightarrow\infty}\|T_3x_n-x_n\|=0$ □

Now we give a weak convergence theorem for iteration (3.2.1).

Theorem 3.2.4. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $\{k_n\},\{l_n\},\{m_n\}\subset[0,\infty)$, $\lim_{n\longrightarrow\infty}k_n=1, \lim_{n\longrightarrow\infty}l_n=1, \lim_{n\longrightarrow\infty}m_n=1$, $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, $\sum_{n=1}^{\infty}(m_n-1)<\infty$, respectively and let $\{a_n\},\{b_n\},\{c_n\},\{d_n\},\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n\geq 1$ and $\sum_{n=1}^{\infty}b_n<\infty$, $\sum_{n=1}^{\infty}d_n<\infty$, $\sum_{n=1}^{\infty}\beta_n<\infty$ and*

(i) If $0<\liminf_{n\longrightarrow\infty}\alpha_n\leq\limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n)<1$,

(ii) If $0<\liminf_{n\longrightarrow\infty}c_n\leq\limsup_{n\longrightarrow\infty}(c_n+d_n)<1$,

(iii) If $0<\liminf_{n\longrightarrow\infty}a_n\leq\limsup_{n\longrightarrow\infty}(a_n+b_n)<1$.

For a given $x_1\in C$, let $\{x_n\}$ be the sequence defined as in (3.2.1). If $F\neq\emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. Let $p\in F$. By Lemma 3.2.1, $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let p_1 and p_2 be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 3.2.3, we have $\lim_{n\longrightarrow\infty}\|T_1x_n-x_n\|=0$. By Lemma 2.1.32, we have $I-T_1, I-T_2$ and $I-T_3$ are demiclosed with respect to zero, therefore $T_1p_i=p_i, T_2p_i=p_i$ and $T_3p_i=p_i$, (i=1, 2) hence $p_1,p_2\in F$. By Lemma 3.2.1 $\lim_{n\longrightarrow\infty}\|x_n-p_1\|$ and $\lim_{n\longrightarrow\infty}\|x_n-p_2\|$ exist. Using Lemma 2.1.33 we obtain that $p_1=p_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

Our next goal is to prove a strong convergence theorem:

Theorem 3.2.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $\{k_n\},\{l_n\},\{m_n\}\subset[0,\infty)$, $\lim_{n\longrightarrow\infty}k_n=1, \lim_{n\longrightarrow\infty}l_n=1, \lim_{n\longrightarrow\infty}m_n=1$, $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, $\sum_{n=1}^{\infty}(m_n-1)<\infty$, respectively and let $\{a_n\},\{b_n\},\{c_n\},\{d_n\},\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n\geq 1$ and $\sum_{n=1}^{\infty}b_n<\infty$, $\sum_{n=1}^{\infty}d_n<\infty$, $\sum_{n=1}^{\infty}\beta_n<\infty$ and*

(i) $0<\liminf_{n\longrightarrow\infty}\alpha_n\leq\limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n)<1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \le \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \le \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (3.2.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}$, $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 3.2.1, $\{x_n\}$ is bounded. In addition, by Lemma 3.2.3, $lim_{n\longrightarrow\infty} \|x_n - T_1x_n\| = 0$, $lim_{n\longrightarrow\infty}\|x_n - T_2x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - T_3x_n\| = 0$, then $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \longrightarrow p$ as $j \longrightarrow \infty$. Thus $\lim_{j\longrightarrow\infty} \|x_{n_j} - T_1x_{n_j}\| = \lim_{j\longrightarrow\infty} \|x_{n_j} - T_2x_{n_j}\| = \lim_{j\longrightarrow\infty} \|x_{n_j} - T_3x_{n_j}\| = 0$. It follows that $\lim_{j\longrightarrow\infty} \|x_{n_j} - p\| = 0$. This implies by Lemma 2.1.32 that $p \in F$. Furthermore, by Lemma 3.2.1, we get that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. Thus $lim_{n\longrightarrow\infty}\|x_n - p\| = 0$. □

Theorem 3.2.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty} (m_n - 1) < \infty$, respectively and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \le \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \le \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \le \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (3.2.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is demicompact, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose that T_1 is semicompact. By Lemma 3.2.3, $\lim_{n\longrightarrow\infty} \|x_n - T_1x_n\| = \lim_{n\longrightarrow\infty} \|x_n - T_2x_n\| = \lim_{n\longrightarrow\infty} \|x_n - T_3x_n\| = 0$. Then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in C$. It follows from Lemma 2.1.32 that $p \in F$. Since $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists by Lemma 3.2.1, it follows that $\lim_{n\longrightarrow\infty} \|x_n - p\| = 0$. □

3.3 Strong convergence theorems by hybrid method for asymptotically k-strict pseudo-contractive mapping in Hilbert space

This section, we introduce the hybrid method of modified Mann's iteration for an asymptotically k-strictly pseudo-contractive mapping to prove some strong convergence theorems for such mappings in a Hilbert space.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [3, 21, 24, 42]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [10, 42].

Our iteration method for finding a fixed point of an asymptotically k-strict pseudo-contractive mapping T is the modified Mann's iteration method studied in [21, 34, 35, 39] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \geq 0, \tag{3.3.1}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ line in the interval $(0,1)$.

In 2007, Takahashi, Takeuchi and Kubota [42] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.2}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then we prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2008, I. Inchan [12], introduce the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then him prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these fact, it is the purpose of this paper to introduce the modified Mann iteration processes for an asymptotically k-strictly pseudo-contractive mapping by idear in (3.3.3). Let C be a closed convex subset of a Hilbert space H, T be an asymptotically k-strictly pseudo-contractive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad\qquad\qquad\qquad +[k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\| + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.4}$$

where $\theta_n = (diam C)^2(1-\alpha_n)\gamma_n \to 0, \ (n \to \infty)$.

We shall prove that the iteration generated by (3.3.4) converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 3.3.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically k-strictly pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1-k$. Then $\{x_n\}$ generated by (3.3.4) converges strongly to $z_0 = P_{F(T)}x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. Then we have, for $u \in F(T) \subset C_m$

$$\begin{aligned} \|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\ &= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\ &= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\ &\le \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 + k\|x_m - T^m x_m\|^2] \\ &\quad -\alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\ &= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\ &\le \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2 \\ &\le \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m. \end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \le \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$. Then

$$\|y_m - z\|^2 = \|y_m - z_j + z_j - z\|^2$$

$$= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle$$
$$\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + \|z_j - z\|^2$$
$$+2\|y_m - z_j\|\|z_j - z\|.$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, we have

$$\|y_m - z\|^2 = \|y_m - (\alpha x + (1-\alpha)y)\|^2$$
$$= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2$$
$$= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2$$
$$\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m)$$
$$+(1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m)$$
$$-\alpha(1-\alpha)\|y - x\|^2$$
$$= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2$$
$$+[k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$$
$$= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$$
$$= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \text{ for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.3.5}$$

So, for $u \in F(T)$, we have

$$0 \leq \langle x_0 - x_n, x_n - u\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle$$
$$= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle$$
$$\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \text{ for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.3.6}$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0 \text{ for all } n \in \mathbb{N}. \tag{3.3.7}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - x_{n+1} \rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\| \|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \text{ for all } n \in \mathbb{N}. \tag{3.3.8}$$

From (3.3.6) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.3.7) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n \rangle - 2\langle x_0 - x_n, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3.9}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1 - \alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n, \tag{3.3.10}$$

By the definition of y_n, we have

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n) T^n x_n - x_n\| \\ &= (1 - \alpha_n)\|T^n x_n - x_n\|. \end{aligned}$$

From (3.3.10), we have

$$(1 - \alpha_n)^2 \|T^n x_n - x_n\|^2 = \|y_n - x_n\|^2$$

$$= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2$$

$$\leq \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\|$$

$$\leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + \|x_{n+1} - x_n\|^2$$

$$+2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\|$$

$$= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\|$$

$$+\|y_n - x_{n+1}\|) + \theta_n.$$

It follows that

$$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\|$$

$$+\|y_n - x_{n+1}\|) + \theta_n.$$

Hence

$$(1-k-\alpha_n)\|T^n x_n - x_n\| \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n. \quad (3.3.11)$$

From $\limsup_{n\to\infty} \alpha_n < 1-k$, we can chosen $\epsilon > 0$ such that $\alpha_n \leq 1-k-\epsilon$ for large enough n. From (3.3.9) and (3.3.11), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \quad (3.3.12)$$

Next, we show that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. From Lemma 2.1.40, we have

$$\|Tx_n - x_n\| \leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\|$$

$$+\|x_{n+1} - x_n\|$$

$$\leq L_1\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + L_{n+1})\|x_n - x_{n+1}\|. \quad (3.3.13)$$

From (3.3.9) and (3.3.12), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \quad (3.3.14)$$

By (3.3.13), Lemma 2.1.38 and boundedness of $\{x_n\}$ we obtain $\emptyset \neq \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\|x_0 - z_0\| \leq \|x_0 - w\| \leq \liminf_{n\to\infty} \|x_0 - x_n\|$$

$$\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\|x_n - z_0\|^2 = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2$$

$$\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.$$

This complete the proof. □

Using this Theorem 3.3.1, we have the following corollaries.

Corollary 3.3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be k-strictly pseudo-contractive mapping of C into itself for some $0 \leq k < 1$ such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{3.3.15}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$. Then $\{x_n\}$ generated by (3.3.15) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3.3. *[12] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.16}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (3.3.16) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3.4. *([42] Theorem 4.1) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.3.17}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

CHAPTER 4

CONCLUSIONS

4.1 Outputs 3 papers (Supported by TRF : MRG5180011)

1. K. Nammanee and S. Suantai, Approximating common fixed points of nonexpansive mappings in a Banach Space, **Thai journal of mathematics**, 6(2) (2008) 391–400.

2. KAMONRAT NAMMANEE and SUTHEP SUANTAI, CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN A BANACH SPACE, **JP journal of Fixed Point Theory and Applications**, 3(3), (2008) 219–236.

3. I. Inchan and K. Nammanee*, Strong convergence theorems by hybrid method for asymptotically k-strict pseudo-contractive mapping in Hilbert space, **Nonlinear Analysis : Hybrid System**, 3(4) (2009) 380–385.

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APPENDIX

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Approximating common fixed points of nonexpansive mappings in a Banach Space

K. Nammanee[1] and S. Suantai

Abstract : In this paper, we introduce a three-step iterative scheme with errors for three nonexpansive mappings in a uniformly convex Banach space. The new iterative scheme includes Mann iterative scheme with errors as spacial case. The results obtained in this paper present weak and strong convergence theorems of such schemes to a common fixed points of these mappings.

Keywords : three-step iterations, nonexpansive mapping, uniformly convex Banach space, Opial's condition, condition (A), completely continuous, demicompact
2000 Mathematics Subject Classification : 47H10, 47H09, 46B20

1 Introduction

In recent years, one-step and two-step iterative schemes (including Mann and Ishikawa iteration processes as the most important cases) have been studied extensively by many authors to solve the nonlinear operator equations in Hilbert spaces and Banach spaces; see [6, 9, 13, 14, 15, 18, 19] and the references therein. Noor [10, 11] introduced and analyzed three-step iterative methods to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. A similar idea goes back to the so-called -schemes introduced by Glowinski and Le Tallec [4] to find a zero of sum of two (or more) maximal monotone operators by using the Lagrangian multiplier. Glowinski and Le Tallec [4] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation, and they showed that three-step approximations perform better numerically. Haubruge et al. [5] studied the convergence analysis of threestep schemes of Glowinski and Le Tallec [4] and applied these schemes to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Inspired and motivated by these facts, we suggest a new class of three-step iterative schemes for solving the nonlinear equation Tx = x for nonexpansive mappings in Banach space in this paper. Our schemes can be viewed as an extension for Mann-iterative scheme with errors.

Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, i.e., $F(T) = \{x \in C : x = Tx\}$. Let T_1, T_2 and T_3 be three nonexpansive mappings from C to itself. Let F be the set

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of common fixed points of T_1, T_2 and T_3 that is $F = \bigcap_{i=1}^{3} F(T_i) = \{x \in C : x = T_1x = T_2x = T_3x\}$.

ALGORITHM 1.1. For a given $x_1 \in C$, compute sequences $\{z_n\}, \{y_n\}$ and $\{x_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= (1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n, \\ y_n &= (1 - c_n - d_n)z_n + c_nT_2z_n + d_nv_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = 0$, then Algorithm 1.1 reduces to

ALGORITHM 1.2. For a given $x_1 \in C$, compute sequences $\{y_n\}$ and $\{x_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1 - c_n - d_n)x_n + c_nT_2x_n + d_nv_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{v_n\}, \{w_n\}$ are bounded sequences in C and $\{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = d_n = 0$, then Algorithm 1.1 reduces to

ALGORITHM 1.3. For a given $x_1 \in C$, compute sequences $\{x_n\}$ by the iterative schemes

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_nT_3x_n + \beta_nw_n, \quad n \geq 1, \tag{1.3}$$

where $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$. Algorithm 1.3 is known as a Mann-iterative scheme with error which is introduced by Xu [19].

For a suitable choice of $a_n, b_n, c_n, d_n, \alpha_n$ and β_n one can obtain a number of new and known iterative schemes for solving nonlinear equations in Banach space and Hilbert space. The purpose of this paper is to use the iterative scheme (1.1) for approximating common fixed points of three nonexpansive mappings and to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space.

2 Preliminaries

Recall that a Banach space X is said to be uniformly convex if for each $\epsilon \in [0, 2]$, the modulus of convexity of X given by:

$\delta(\epsilon) = \inf\{1 - \frac{1}{2}\|x - y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}$

satisfies the inequality $\delta(\epsilon) > 0$ for all $\epsilon > 0$.

A Banach space X is said to satisfy *Opial's condition* [12] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$.

A mapping $T : C \to C$ is said to be *demicompact* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x \in C$.

A mapping $T : C \to X$ ia called *demiclosed* with respect to y if for each sequence $\{x_n\}$ in C and each $x \in X, x_n \to x$ weakly and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

A family $\{T_i : C \to C, i = 1, 2, 3, ..., n\}$ of maps is said to satisfy condition (A) (Definition 4.1. [3]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{n}(\sum_{i=1}^{n} \|x - T_i x\|) \geq f(d(x, F))$$

for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F = \bigcap_{i=1}^{n} F(T_i)\}$.

It is remarked that the condition (A) reduces to the condition (I) in ([16]) when $T_i = T$ for $i = 1, 2, 3, ..., n$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.1 (Tan and Xu, [18]). *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, ...,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim_{n\to\infty} a_n$ exists.
(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.2 (Cho, Zhou and Guo, [2]). *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \lambda\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 2.3 (Browder, [1]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to X$ be nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.*

Lemma 2.4 (Suantai, [17]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3 Main Results

In this section, we prove weak and strong convergence theorems of the three-step iterative scheme (1.1) for nonexpansive mappings in a uniformly convex Banach space. In order to prove this, the following lemmas are needed.

Lemma 3.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and $\{x_n\}$ be the sequences defined as in (1.1) and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. If $F \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Assume that $F \neq \emptyset$. Let $p \in F$ and

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \; M_2 = \sup\{\|v_n - p\| :\geq 1\}$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_1, M_2, M_3\}.$$

We have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n)y_n + \alpha_n T_3 y_n + \beta_n w_n - p\| \\
&= \|(1 - \alpha_n - \beta_n)(y_n - p) + \alpha_n(T_3 y_n - p) + \beta_n(w_n - p)\| \\
&\leq (1 - \alpha_n - \beta_n)\|y_n - p\| + \alpha_n\|T_3 y_n - p\| + \beta_n\|w_n - p\| \\
&\leq \|y_n - p\| + \beta_n\|w_n - p\| \\
\|y_n - p\| &= \|(1 - c_n - d_n)z_n + c_n T_2 z_n + d_n v_n - p\| \\
&= \|(1 - c_n - d_n)(z_n - p) + c_n(T_2 z_n - p) + d_n(v_n - p)\| \\
&\leq (1 - c_n - d_n)\|z_n - p\| + c_n\|T_2 z_n - p\| + d_n\|v_n - p\| \\
&\leq \|z_n - p\| + d_n\|v_n - p\| \\
\|z_n - p\| &= \|(1 - a_n - b_n)x_n + b_n T_1 x_n + b_n u_n - p\| \\
&= \|(1 - a_n - b_n)(x_n - p) + a_n(T_1 x_n - p) + b_n(u_n - p)\| \\
&\leq (1 - a_n - b_n)\|x_n - p\| + b_n\|T_1 x_n - p\| + b_n\|u_n - p\| \\
&\leq \|x_n - p\| + b_n\|u_n - p\|
\end{aligned}$$

It follows that $\|x_{n+1} - p\| \leq \|x_n - p\| + b_n\|u_n - p\| + d_n\|v_n - p\| + \beta_n\|w_n - p\|$.
And so $\|x_{n+1} - p\| \leq \|x_n - p\| + M(b_n + d_n + \beta_n)$.
Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.1 implies that $\lim_{n\to\infty}\|x_n - q\|$ exists. □

Lemma 3.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1) with $F \neq \emptyset$.*

(i) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\lim_{n\to\infty}\|T_3 y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, then $\lim_{n\to\infty}\|T_2 z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$, then $\lim_{n\to\infty}\|T_1 x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$, then $\lim_{n\to\infty}\|T_2 x_n - x_n\| = 0 = \lim_{n\to\infty}\|T_3 x_n - x_n\|$.

Proof. (i) From Lemma 3.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$. It follows that $\{x_n - p\}, \{T_1 x_n - p\}, \{z_n - q\}, \{T_2 z_n - p\}$ and $\{y_n - p\}, \{T_3 y_n - p\}$ are all bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption.

Now we set

$$\begin{aligned} r_1 &= \sup\{\|x_n - p\| : n \geq 1\}, \\ r_2 &= \sup\{\|T_1x_n - p\| : n \geq 1\}, \\ r_3 &= \sup\{\|z_n - p\| : n \geq 1\}, \\ r_4 &= \sup\{\|T_2z_n - p\| : n \geq 1\}, \\ r_5 &= \sup\{\|y_n - p\| : n \geq 1\}, \\ r_6 &= \sup\{\|T_3y_n - p\| : n \geq 1\}, \\ r_7 &= \sup\{\|u_n - p\| : n \geq 1\}, \\ r_8 &= \sup\{\|v_n - p\| : n \geq 1\}, \\ r_9 &= \sup\{\|w_n - p\| : n \geq 1\}, \\ r &= \max\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}. \end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned} \|z_n - p\|^2 &= \|(1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n - p\|^2 \\ &= \|(1 - a_n - b_n)(x_n - p) + a_n(T_1x_n - p) + b_n(u_n - p)\|^2 \\ &\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|T_1x_n - p\|^2 + b_n\|u_n - p\|^2 \\ &\quad - a_n(1 - a_n - b_n)g(\|T_1x_n - x_n\|) \\ &\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|x_n - p\|^2 + b_n\|u_n - p\|^2 \\ &\leq \|x_n - p\|^2 + r^2b_n - a_n(1 - a_n - b_n)g(\|T_1x_n - x_n\|) \\ &\leq \|x_n - p\|^2 + r^2b_n, \\ \|y_n - p\|^2 &= \|(1 - c_n - d_n)z_n + c_nT_2z_n + d_nv_n - p\|^2 \\ &= \|(1 - c_n - d_n)(z_n - p) + c_n(T_2z_n - p) + d_n(v_n - p)\|^2 \\ &\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|T_2z_n - p\|^2 + d_n\|v_n - p\|^2 \\ &\quad - c_n(1 - c_n - d_n)g(\|T_2z_n - z_n\|) \\ &\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|z_n - p\|^2 + d_n\|v_n - p\|^2 \\ &\quad - c_n(1 - c_n - d_n)g(\|T_2z_n - z_n\|) \\ &\leq \|z_n - p\|^2 + r^2d_n - c_n(1 - c_n - d_n)g(\|T_2z_n - z_n\|) \\ &\leq \|z_n - p\|^2 + r^2d_n \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n - p\|^2 \\ &= \|(1 - \alpha_n - \beta_n)(y_n - p) + \alpha_n(T_3y_n - p) + \beta_n(w_n - p)\|^2 \\ &\leq (1 - \alpha_n - d_n)\|y_n - p\|^2 + \alpha_n\|T_3y_n - p\|^2 + \beta_n\|w_n - p\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g(\|T_3y_n - y_n\|) \\ &\leq (1 - \alpha_n - \beta_n)\|y_n - p\|^2 + \alpha_n\|y_n - p\|^2 + \beta_n\|w_n - p\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g(\|T_3y_n - y_n\|) \\ &\leq \|y_n - p\|^2 + r^2\beta_n - \alpha_n(1 - \alpha_n - \beta_n)g(\|T_3y_n - y_n\|), \end{aligned}$$

which lead to the following:

$$\begin{aligned} \alpha_n(1 - \alpha_n - \beta_n)g(\|T_3y_n - y_n\|) \leq\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ & + r^2(b_n + d_n + \beta_n), \end{aligned} \tag{3.1}$$

and

$$c_n(1-c_n-d_n)g(\|T_2z_n-z_n\|) \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n), \tag{3.2}$$

and

$$a_n(1-a_n-b_n)g(\|T_1x_n-x_n\|) \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n), \tag{3.3}$$

(i) If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0,1)$ such that

$$0 < \eta < \alpha_n \text{ and } \alpha_n+\beta_n < \eta' < 1 \text{ for all } n \geq n_0.$$

This implies by (3.1) that

$$\eta(1-\eta')g(\|T_3y_n-y_n\|) \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n), \tag{3.4}$$

for all $n \geq n_0$. It follows from (3.4) that for $m \geq n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|T_3y_n-y_n\|) &\leq \frac{1}{\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) \\ &\quad + r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n)\Bigg) \\ &\leq \frac{1}{\eta(1-\eta')}\Bigg(\|x_{n_0}-p\|^2 \\ &\quad + r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n)\Bigg).\end{aligned} \tag{3.5}$$

By letting $m \to \infty$ in inequality (3.5) we get that $\sum_{n=n_0}^{\infty} g(\|T_3y_n-y_n\|) < \infty$, which implies $\lim_{n\to\infty} g(\|T_3y_n-y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|T_3y_n-y_n\| = 0$.

(ii) If $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n+d_n) < 1$, then by using the same argument as above with the inequality (3.2), it can be show that $\lim_{n\to\infty}\|T_2z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n+b_n) < 1$, then by using (3.3) and the same argument as in (i), it can be show that $\lim_{n\to\infty}\|T_1x_n-x_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n+d_n) < 1$ and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n+b_n) < 1$, by (i), (ii) and (iii) we have

$$\lim_{n\to\infty}\|T_3y_n-y_n\| = \lim_{n\to\infty}\|T_2z_n-z_n\| = \lim_{n\to\infty}\|T_1x_n-x_n\| = 0. \tag{3.6}$$

From $z_n = (1-a_n-b_n)x_n + a_nT_1x_n + b_nu_n$ and $y_n = (1-c_n-d_n)z_n + c_nT_2z_n + d_nv_n$, we have

$$\begin{aligned}\|z_n-x_n\| &= \|(1-a_n-b_n)x_n + a_nT_1x_n + b_nu_n - x_n\| \\ &= \|a_n(T_1x_n-x_n) + b_n(u_n-x_n)\| \\ &\leq a_n\|T_1x_n-x_n\| + b_n\|u_n-x_n\| \\ &\leq \|T_1x_n-x_n\| + 2rb_n,\end{aligned}$$

and $\|y_n - x_n\| \leq \|T_2 z_n - z_n\| + \|T_1 x_n - x_n\| + 2rb_n + 2rd_n$. Hence

$$\begin{aligned} \|T_2 x_n - x_n\| &\leq \|x_n - T_2 z_n\| + \|T_2 z_n - T_2 x_n\| \\ &\leq \|x_n - z_n\| + \|T_2 z_n - z_n\| + \|z_n - x_n\| \\ &\leq \|T_2 z_n - z_n\| + 2\|z_n - x_n\| \\ &\leq \|T_2 z_n - z_n\| + 2\|T_1 x_n - x_n\| + 4rb_n, \end{aligned}$$

and

$$\begin{aligned} \|T_3 x_n - x_n\| &\leq \|x_n - T_3 y_n\| + \|T_3 y_n - T_3 x_n\| \\ &\leq \|x_n - y_n\| + \|T_3 y_n - y_n\| + \|y_n - x_n\| \\ &\leq \|T_3 y_n - y_n\| + 2\|y_n - x_n\| \\ &\leq \|T_3 y_n - y_n\| + 2\|T_2 z_n - z_n\| \\ &\quad + 2\|T_1 x_n - x_n\| + 4rb_n + 4rd_n. \end{aligned}$$

It follows from (3.6) that $\lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0$. □

Theorem 3.3. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} (\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty} (c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} (a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined as in (1.1). If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. Let $p \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let p_1 and p_2 be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 3.2(iii), we have $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$. By Lemma 2.3, we have $I - T_1, I - T_2$ and $I - T_3$ are demiclosed with respect to zero, therefore $T_1 p_i = p_i, T_2 p_i = p_i$ and $T_3 p_i = p_i$, (i=1, 2) hence $p_1, p_2 \in F$. By Lemma 3.1 $\lim_{n\to\infty} \|x_n - p_1\|$ and $\lim_{n\to\infty} \|x_n - p_2\|$ exist. Using Lemma 2.4 we obtain that $p_1 = p_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

Our next goal is to prove a strong convergence theorem of iterative scheme (1.1) to a common fixed point of T_1, T_2 and T_3.

Theorem 3.4. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C satisfying condition (A) and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} (\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty} (c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} (a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (1.1). If $F \neq \emptyset$, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 3.2, we have

$$\lim_{n\to\infty}\|T_3x_n - x_n\| = \lim_{n\to\infty}\|T_2x_n - x_n\| = \lim_{n\to\infty}\|T_1x_n - x_n\| = 0.$$

By condition (A), we have $f(d(x_n, F)) \leq \frac{1}{3}(\|x_n - T_1x_n\| + \|x_n - T_2x_n\| + \|x_n - T_3x_n\|)$.
It follows that $\lim_{n\to\infty} f(d(x_n, F)) \leq 0$, hence $\lim_{n\to\infty} f(d(x_n, F)) = 0$.
Since f is a nondecreasing function and $f(0) = 0$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$. Thus we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $(y_k) \subset F$ such that $\|x_{n_k} - y_k\| < 2^{-k}$ for all $k \geq 1$. Then following the method of proof of Tan and Xu [18], we get that $\{y_k\}$ is a Cauchy sequence in F and so it converges. Let $y_k \to y$. Since F is closed, $y \in F$ and then $x_{n_k} \to y$. By Lemma (3.1) $\lim_{n\to\infty}\|x_n - p\|$ exits, $\forall p \in F$. It implies that $\lim_{n\to\infty}\|x_n - y\| = 0$. Hence $\{x_n\}$ converses strongly to a common fixed point of T_1, T_2 and T_3.

Since $\|y_n - x_n\| \leq \|T_2z_n - z_n\| + \|T_1x_n - x_n\| + 2rb_n + 2rd_n$ and $\|z_n - x_n\| \leq \|T_1x_n - x_n\| + 2rb_n$, it follows from Lemma 3.2 (ii) and (iii) that $\|y_n - x_n\| \to 0$ as $n \to \infty$ and $\|z_n - x_n\| \to 0$ as $n \to \infty$, which imply that $\lim_{n\to\infty}\|y_n - y\| = 0$ and $\lim_{n\to\infty}\|z_n - y\| = 0$. □

Theorem 3.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, and

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (1.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded. In addition, by Lemma 3.2, $lim_{n\to\infty}\|x_n - T_1x_n\| = 0$, $lim_{n\to\infty}\|x_n - T_2x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_3x_n\| = 0$, then $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \to p$ as $j \to \infty$. Thus $\lim_{j\to\infty}\|x_{n_j} - T_1x_{n_j}\| = \lim_{j\to\infty}\|x_{n_j} - T_2x_{n_j}\| = \lim_{j\to\infty}\|x_{n_j} - T_3x_{n_j}\| = 0$. It follows that $\lim_{j\to\infty}\|x_{n_j} - p\| = 0$. This implies by Lemma 2.3 that $p \in F$. Furthermore, by Lemma 3.1, we get that $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus $lim_{n\to\infty}\|x_n - p\| = 0$. The proof is completed. □

Theorem 3.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, and

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (1.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is demicompact, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose that T_1 is demicompact. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0$. Then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in C$. It follows from Lemma 2.3 that $p \in F$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 3.1, it follows that $\lim_{n\to\infty} \|x_n - p\| = 0$. □

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CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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Abstract

In this paper, we introduce a three-step iterative scheme with errors for three asymptotically nonexpansive mappings in a uniformly convex Banach space. The new iterative scheme includes modified Mann iterative scheme with errors as special case. The results obtained in this paper present weak and strong convergence theorems of such schemes to a common fixed point of these mappings.

1. Introduction

Fixed-point iteration processes for asymptotically nonexpansive

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mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors; see [2-11]. Many of them are used widely to study the approximate solutions of the certain problems; see [7, 8, 13]. In 2000, Noor [6] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. In 2002, Xu and Noor [14] introduced and studied a three-step scheme to approximate fixed point of asymptotically nonexpansive mappings in a Banach space. Cho et al. [2] extended their schemes to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Suantai [11] defined a new three-step iteration which is an extension of Noor iterations and gave some weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in uniformly Banach space.

Let C be a nonempty closed convex subset of normed linear space X. Then a self-mapping $T : C \to C$ is said to be *nonexpansive* if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. A self-mapping $T : C \to C$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [0, \infty)$, $\lim_{n \to \infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|$$

for all $x, y \in C$ and each $n \geq 1$.

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, but the converse does not hold. It was proved in [3] that if X is uniformly convex and C is bounded, closed and convex, then every asymptotically nonexpansive mapping has a fixed point. Inspired and motivated by these facts, we suggest a three-step iterative scheme defined as follows.

Let $T_1, T_2, T_3 : C \to C$ be three asymptotically nonexpansive self-mappings. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$z_n = (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n,$$

$$y_n = (1 - c_n - d_n)z_n + c_n T_2^n z_n + d_n v_n,$$

$$x_{n+1} = (1 - \alpha_n - \beta_n)y_n + \alpha_n T_3^n y_n + \beta_n w_n, \quad n \geq 1, \tag{1.1}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = 0$, then the scheme (1.1) reduces to the iterative schemes

$$y_n = (1 - c_n - d_n)x_n + c_n T_2^n x_n + d_n v_n,$$

$$x_{n+1} = (1 - \alpha_n - \beta_n)y_n + \alpha_n T_3^n y_n + \beta_n w_n, \quad n \geq 1, \tag{1.2}$$

where $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C and $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = d_n = 0$, then scheme (1.1) reduces to the modified Mann-iterative scheme

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T_3^n x_n + \beta_n w_n, \quad n \geq 1, \tag{1.3}$$

where $\{w_n\}$ are bounded sequences in C and $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to establish strong convergence theorems under only one map which is completely continuous or demicompact and weak convergence theorems under space satisfying Opial's condition in a uniformly convex Banach space.

Recall that a Banach space X is said to be *uniformly convex* if for each $\varepsilon \in [0, 2]$, the modulus of convexity of X given by:

$$\delta(\varepsilon) = \inf\left\{1 - \frac{1}{2}\| x - y \| : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| \geq \varepsilon\right\}$$

satisfies the inequality $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$.

A Banach space X is said to satisfy *Opial's condition* [8] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \| x_n - x \| < \limsup_{n\to\infty} \| x_n - y \|.$$

A mapping $T : C \to C$ is said to be *demicompact* if, for any sequence $\{x_n\}$ in C such that $\| x_n - Tx_n \| \to 0$ as $n \to \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x \in C$.

A mapping $T : C \to X$ is called *demiclosed* with respect to y if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \to x$ weakly and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 (Tan and Xu [12]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, \dots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n \to \infty} a_n$ *exists.*

(2) $\lim_{n \to \infty} a_n = 0$ *whenever* $\liminf_{n \to \infty} a_n = 0$.

Lemma 1.2 (Cho et al. [2]). *Let X be a uniformly convex Banach space and*

$$B_r = \{x \in X : \| x \| \le r\}, \quad r > 0.$$

Then there exists a continuous, strictly increasing and convex function

$$g : [0, \infty) \to [0, \infty), \quad g(0) = 0$$

such that

$$\| \lambda x + \beta y + \gamma z \|^2 \le \lambda \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 - \lambda \beta g(\| x - y \|),$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 1.3 (Chidume et al. [1]). *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to X$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 1.4 (Suantai [11]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty}\| x_n - u \|$ and $\lim_{n\to\infty}\| x_n - v \|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$. In the sequel, $\bigcap_{n=1}^{3} F(T_i)$ will be denoted by F.*

2. Main Results

In this section, we prove weak and strong convergence theorems of the three-step iterative scheme (1.1) for three asymptotically nonexpansive mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

The next lemma is crucial for proving the main theorems.

Lemma 2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be three asymptotically nonexpansive self-mappings of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $\lim_{n\to\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and $\{x_n\}$ be the sequence defined as in (1.1) and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. If $F \neq \varnothing$, then $\lim_{n\to\infty}\| x_n - p \|$ exists for all $p \in F$.*

Proof. Assume that $F \neq \varnothing$. Let $p \in F$. Then we have

$$
\begin{aligned}
\| z_n - p \| &= \| (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n - p \| \\
&\leq (1 - a_n - b_n)\| x_n - p \| + a_n\| T_1^n x_n - p \| + b_n\| u_n - p \| \\
&\leq (1 - a_n - b_n)\| x_n - p \| + a_n k_n\| x_n - p \| + b_n\| u_n - p \| \\
&\leq (1 + a_n(k_n - 1))\| x_n - p \| + b_n\| u_n - p \| \\
&\leq k_n\| x_n - p \| + b_n\| u_n - p \|,
\end{aligned}
$$

$$\begin{aligned}\| y_n - p \| &= \| (1 - c_n - d_n) z_n + c_n T_2^n z_n + d_n v_n - p \| \\ &\le (1 - c_n - d_n) \| z_n - p \| + c_n \| T_2^n z_n - p \| + d_n \| v_n - p \| \\ &\le (1 - c_n - d_n) \| z_n - p \| + c_n l_n \| z_n - p \| + d_n \| v_n - p \| \\ &\le (1 + c_n (l_n - 1)) \| z_n - p \| + d_n \| v_n - p \| \\ &\le l_n \| z_n - p \| + d_n \| v_n - p \|,\end{aligned}$$

$$\begin{aligned}\| x_{n+1} - p \| &= \| (1 - \alpha_n - \beta_n) y_n + \alpha_n T_3^n y_n + \beta_n w_n - p \| \\ &\le (1 - \alpha_n - \beta_n) \| y_n - p \| + \alpha_n \| T_3^n y_n - p \| + \beta_n \| w_n - p \| \\ &\le (1 - \alpha_n - \beta_n) \| y_n - p \| + \alpha_n m_n \| y_n - p \| + \beta_n \| w_n - p \| \\ &\le (1 + \alpha_n (m_n - 1)) \| y_n - p \| + \beta_n \| w_n - p \| \\ &\le m_n \| y_n - p \| + \beta_n \| w_n - p \| \\ &\le m_n [l_n [k_n \| x_n - p \| + b_n \| u_n - p \|] + d_n \| v_n - p \|] \\ &\quad + \beta_n \| w_n - p \| \\ &= m_n l_n k_n \| x_n - p \| + m_n l_n b_n \| u_n - p \| \\ &\quad + m_n d_n \| v_n - p \| + \beta_n \| w_n - p \| \\ &\le h_n^3 \| x_n - p \| + M(b_n + d_n + \beta_n) \\ &= (1 + (h_n^3 - 1)) \| x_n - p \| + M(b_n + d_n + \beta_n),\end{aligned}$$

where M is some positive constant and $h_n = \max\{k_n, l_n, m_n\}$.

Notice that $\sum_{n=1}^{\infty} (h_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty} (h_n^3 - 1) < \infty$. Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 1.1 it implies that $\lim_{n \to \infty} \| x_n - p \|$ exists.

Lemma 2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be three* *asymptotically nonexpansive self-mappings of* C *with* $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $\lim_{n\to\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, *respectively and* $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *such that* $a_n + b_n$, $c_n + d_n$ *and* $\alpha_n + \beta_n$ *are in* $[0, 1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. *For a given* $x_1 \in C$, *let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in* (1.1) *with* $F \neq \varnothing$.

(i) *If* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *then*

$$\lim_{n\to\infty}\| T_3^n y_n - y_n \| = 0.$$

(ii) *If* $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, *then*

$$\lim_{n\to\infty}\| T_2^n z_n - z_n \| = 0.$$

(iii) *If* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$, *then*

$$\lim_{n\to\infty}\| T_1^n x_n - x_n \| = 0.$$

(iv) *If* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

$$0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$$

and

$$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1,$$

then

$$\lim_{n\to\infty}\| T_2^n x_n - x_n \| = 0 = \lim_{n\to\infty}\| T_3^n x_n - x_n \|.$$

Proof. (i) From Lemma 2.1, we know that $\lim_{n\to\infty}\| x_n - p \|$ exists for any $p \in F$. It follows that $\{x_n - p\}, \{T_1^n x_n - p\}, \{z_n - q\}, \{T_2^n z_n - p\}$ and $\{y_n - p\}, \{T_3^n y_n - p\}$ are all bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. We shall use L to denote the positive different constants appearing in the following reasoning.

By using Lemma 1.2, we have

$$\begin{aligned}
\| z_n - p \|^2 &= \| (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n - p \|^2 \\
&= \| (1 - a_n - b_n)(x_n - p) + a_n(T_1^n x_n - p) + b_n(u_n - p) \|^2 \\
&\le (1 - a_n - b_n)\| x_n - p \|^2 + a_n\| T_1^n x_n - p \|^2 + b_n\| u_n - p \|^2 \\
&\quad - a_n(1 - a_n - b_n)g(\| T_1^n x_n - x_n \|) \\
&\le (1 - a_n - b_n)\| x_n - p \|^2 + a_n k_n^2\| x_n - p \|^2 + b_n\| u_n - p \|^2 \\
&\le (1 + a_n(k_n^2 - 1))\| x_n - p \|^2 + Lb_n \\
&\le k_n^2\| x_n - p \|^2 + Lb_n,
\end{aligned}$$

$$\begin{aligned}
\| y_n - p \|^2 &= \| (1 - c_n - d_n)z_n + c_n T_2^n z_n + d_n v_n - p \|^2 \\
&= \| (1 - c_n - d_n)(z_n - p) + c_n(T_2^n z_n - p) + d_n(v_n - p) \|^2 \\
&\le (1 - c_n - d_n)\| z_n - p \|^2 + c_n\| T_2^n z_n - p \|^2 + d_n\| v_n - p \|^2 \\
&\quad - c_n(1 - c_n - d_n)g(\| T_2^n z_n - z_n \|) \\
&\le (1 - c_n - d_n)\| z_n - p \|^2 + c_n l_n^2\| z_n - p \|^2 + d_n\| v_n - p \|^2 \\
&\quad - c_n(1 - c_n - d_n)g(\| T_2^n z_n - z_n \|) \\
&\le (1 + c_n(l_n^2 - 1))\| z_n - p \|^2 + Ld_n \\
&\le l_n^2\| z_n - p \|^2 + Ld_n,
\end{aligned}$$

and so

$$\begin{aligned}
\| x_{n+1} - p \|^2 &= \| (1 - \alpha_n - \beta_n)y_n + \alpha_n T_3^n y_n + \beta_n w_n - p \|^2 \\
&= \| (1 - \alpha_n - \beta_n)(y_n - p) + \alpha_n(T_3^n y_n - p) + \beta_n(w_n - p) \|^2 \\
&\le (1 - \alpha_n - d_n)\| y_n - p \|^2 + \alpha_n\| T_3^n y_n - p \|^2 + \beta_n\| w_n - p \|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n)g(\| T_3^n y_n - y_n \|)
\end{aligned}$$

$$\leq (1-\alpha_n-\beta_n)\| y_n - p\|^2 + \alpha_n m_n^2 \| y_n - p\|^2 + \beta_n \| w_n - p\|^2$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$\leq (1+\alpha_n(m_n^2-1))\| y_n - p\|^2 + L\beta_n$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$\leq m_n^2 \| y_n - p\|^2 + L\beta_n - \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$\leq m_n^2[l_n^2\| z_n - p\|^2 + Ld_n] + L\beta_n$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$\leq m_n^2[l_n^2[k_n^2\| x_n - p\|^2 + Lb_n] + Ld_n] + L\beta_n$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$\leq m_n^2 l_n^2 k_n^2 \| x_n - p\|^2 + Lb_n + Ld_n + L\beta_n$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$\leq h_n^6 \| x_n - p\|^2 + L(b_n + d_n + \beta_n)$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|)$$

$$= (1+(h_n^6-1))\| x_n - p\|^2 + L(b_n + d_n + \beta_n)$$

$$- \alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|),$$

which leads to the following:

$$\alpha_n(1-\alpha_n-\beta_n)g(\| T_3^n y_n - y_n \|) \leq \| x_n - p\|^2 - \| x_{n+1} - p\|^2$$
$$+ L((h_n^6-1) + b_n + d_n + \beta_n), \qquad (2.1)$$

$$c_n(1-c_n-d_n)g(\| T_2^n z_n - z_n \|) \leq \| x_n - p\|^2 - \| x_{n+1} - p\|^2$$
$$+ L((h_n^6-1) + b_n + d_n + \beta_n) \qquad (2.2)$$

and

$$a_n(1 - a_n - b_n)g(\| T_1^n x_n - x_n \|) \le \| x_n - p \|^2 - \| x_{n+1} - p \|^2$$

$$+ L((h_n^6 - 1) + b_n + d_n + \beta_n). \qquad (2.3)$$

(i) If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0, 1)$ such that

$$0 < \eta < \alpha_n \quad \text{and} \quad \alpha_n + \beta_n < \eta' < 1 \quad \text{for all } n \ge n_0.$$

This implies by (2.1) that

$$\eta(1 - \eta')g(\| T_3^n y_n - y_n \|) \le \| x_n - p \|^2 - \| x_{n+1} - p \|^2$$

$$+ L((h_n^6 - 1) + b_n + d_n + \beta_n), \qquad (2.4)$$

for all $n \ge n_0$. It follows from (2.4) that for $m \ge n_0$

$$\sum_{n=n_0}^{m} g(\| T_3^n y_n - y_n \|) \le \frac{1}{\eta(1 - \eta')}\left(\sum_{n=n_0}^{m}(\| x_n - p \|^2 - \| x_{n+1} - p \|^2)\right.$$

$$\left. + L\sum_{n=n_0}^{m}((h_n^6 - 1) + b_n + d_n + \beta_n)\right)$$

$$\le \frac{1}{\eta(1-\eta')}\left(\| x_{n_0} - p \|^2\right.$$

$$\left. + L\sum_{n=n_0}^{m}((h_n^6 - 1) + b_n + d_n + \beta_n)\right). \qquad (2.5)$$

Let $m \to \infty$ in inequality (2.5). Then we get that

$$\sum_{n=n_0}^{\infty} g(\| T_3^n y_n - y_n \|) < \infty$$

and therefore $\lim_{n\to\infty} g(\| T_3^n y_n - y_n \|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty} \| T_3^n y_n - y_n \| = 0.$$

(ii) If $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, then by using a similar method, together with inequality (2.2), it can be shown that $\lim_{n\to\infty}\| T_2^n z_n - z_n \| = 0$.

(iii) If $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$, then by using a similar method, together with inequality (2.3), it can be shown that $\lim_{n\to\infty}\| T_1^n x_n - x_n \| = 0$.

(iv) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$, from (i), (ii) and (iii) we have

$$\lim_{n\to\infty}\| T_3^n y_n - y_n \| = \lim_{n\to\infty}\| T_2^n z_n - z_n \|$$

$$= \lim_{n\to\infty}\| T_1^n x_n - x_n \| = 0. \tag{2.6}$$

From

$$z_n = (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n$$

and

$$y_n = (1 - c_n - d_n)z_n + c_n T_2^n z_n + d_n v_n,$$

we have

$$\begin{aligned}\| z_n - x_n \| &= \| (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n - x_n \| \\ &= \| a_n (T_1^n x_n - x_n) + b_n (u_n - x_n) \| \\ &\leq a_n \| T_1^n x_n - x_n \| + b_n \| u_n - x_n \| \\ &\leq \| T_1^n x_n - x_n \| + 2rb_n\end{aligned}$$

and

$$\| y_n - x_n \| \leq \| T_2^n z_n - z_n \| + \| T_1^n x_n - x_n \| + 2rb_n + 2rd_n.$$

Hence

$$\| T_2^n x_n - x_n \| \le \| x_n - T_2^n z_n \| + \| T_2^n z_n - T_2^n x_n \|$$

$$\le \| x_n - z_n \| + \| T_2^n z_n - z_n \| + l_n \| z_n - x_n \|$$

$$\le \| T_2^n z_n - z_n \| + (1 + l_n) \| z_n - x_n \|$$

$$\le \| T_2^n z_n - z_n \| + (1 + l_n)[\| T_1^n x_n - x_n \| + 2rb_n]$$

and

$$\| T_3^n x_n - x_n \| \le \| x_n - T_3^n y_n \| + \| T_3^n y_n - T_3^n x_n \|$$

$$\le \| x_n - y_n \| + \| T_3^n y_n - y_n \| + m_n \| y_n - x_n \|$$

$$\le \| T_3^n y_n - y_n \| + (1 + m_n) \| y_n - x_n \|$$

$$\le \| T_3^n y_n - y_n \| + (1 + m_n)[\| T_2^n z_n - z_n \|$$

$$+ \| T_1^n x_n - x_n \| + 2rb_n + 2rd_n].$$

It follows from (2.6) that

$$\lim_{n\to\infty} \| T_2^n x_n - x_n \| = \lim_{n\to\infty} \| T_3^n x_n - x_n \| = 0.$$

Lemma 2.3. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be three asymptotically nonexpansive self-mappings of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $\lim_{n\to\infty} m_n = 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, $\sum_{n=1}^{\infty} (m_n - 1) < \infty$, respectively and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1) with $F \ne \varnothing$.*

If

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1,$$

$$0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$$

and

$$0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + b_n) < 1,$$

then

$$\lim_{n\to\infty}\| T_1 x_n - x_n \| = \lim_{n\to\infty}\| T_2 x_n - x_n \|$$
$$= \lim_{n\to\infty}\| T_3 x_n - x_n \| = 0.$$

Proof. By Lemma 2.2, we have

$$\lim_{n\to\infty}\| T_3^n y_n - y_n \| = \lim_{n\to\infty}\| T_2^n z_n - z_n \|$$
$$= \lim_{n\to\infty}\| T_1^n x_n - x_n \| = 0,$$

and

$$\lim_{n\to\infty}\| T_3^n x_n - x_n \| = \lim_{n\to\infty}\| T_2^n x_n - x_n \| = 0. \tag{2.7}$$

Also we obtain that $\| y_n - x_n \| \to 0$ as $n \to \infty$ by the proof of Lemma 2.2.

Since

$$x_{n+1} - y_n = \alpha_n(T_n^3 y_n - y_n) + \beta_n(w_n - y_n),$$

we have

$$\| x_{n+1} - x_n \| \le \| x_{n+1} - y_n \| + \| y_n - x_n \|$$
$$\le \| T_n^3 y_n - y_n \| + \beta_n \| w_n - y_n \| + \| y_n - x_n \|.$$

And so

$$\| x_{n+1} - T_1^n x_{n+1} \| \le \| x_{n+1} - x_n \| + \| T_1^n x_{n+1} - T_1^n x_n \| + \| T_1^n x_n - x_n \|$$
$$\le \| x_{n+1} - x_n \| + k_n \| x_{n+1} - x_n \| + \| T_1^n x_n - x_n \|$$

$$= (1 + k_n)\| x_{n+1} - x_n \| + \| T_1^n x_n - x_n \|$$

$$\leq (1 + k_n)[\| T_n^3 y_n - y_n \| + \beta_n \| w_n - y_n \| + \| y_n - x_n \|]$$

$$+ \| T_1^n x_n - x_n \|.$$

This together with (2.7) implies that

$$\| x_{n+1} - T_1^n x_{n+1} \| \to 0 \quad (\text{as } n \to \infty).$$

Thus

$$\| x_{n+1} - T_1 x_{n+1} \| \leq \| x_{n+1} - T_1^{n+1} x_{n+1} \| + \| T_1 x_{n+1} - T_1^{n+1} x_{n+1} \|$$

$$\leq \| x_{n+1} - T_1^{n+1} x_{n+1} \| + k_1 \| x_{n+1} - T_1^n x_{n+1} \| \to 0,$$

which implies that $\lim_{n\to\infty} \| T_1 x_n - x_n \| = 0$.

Similarly, we obtain that

$$\lim_{n\to\infty} \| T_2 x_n - x_n \| = \lim_{n\to\infty} \| T_3 x_n - x_n \| = 0.$$

Now we give a weak convergence theorem for iteration (1.1).

Theorem 2.4. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be three asymptotically nonexpansive self-mappings of C with $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $\lim_{n\to\infty} m_n = 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, $\sum_{n=1}^{\infty} (m_n - 1) < \infty$, respectively and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} (\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty} (c_n + d_n) < 1$,

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} (a_n + b_n) < 1$.

For a given $x_1 \in C$, *let* $\{x_n\}$ *be the sequence defined as in* (1.1). *If* $F \neq \varnothing$, *then* $\{x_n\}$ *converges weakly to a common fixed point of* T_1, T_2 *and* T_3.

Proof. Let $p \in F$. By Lemma 2.1, $\lim_{n\to\infty}\| x_n - p \|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let p_1 and p_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 2.3, we have $\lim_{n\to\infty}\| T_1 x_n - x_n \| = 0$. By Lemma 1.3, we have $I - T_1$, $I - T_2$ and $I - T_3$ are demiclosed with respect to zero, therefore $T_1 p_i = p_i$, $T_2 p_i = p_i$ and $T_3 p_i = p_i$ $(i = 1, 2)$ hence $p_1, p_2 \in F$. By Lemma 2.1 $\lim_{n\to\infty}\| x_n - p_1 \|$ and $\lim_{n\to\infty}\| x_n - p_2 \|$ exist. Using Lemma 1.4 we obtain that $p_1 = p_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Our next goal is to prove a strong convergence theorem:

Theorem 2.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let* T_1, T_2 *and* T_3 *be three asymptotically nonexpansive self-mappings of C with* $\{k_n\}, \{l_n\}, \{m_n\} \subset [0, \infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $\lim_{n\to\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, *respectively and let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *such that* $a_n + b_n$, $c_n + d_n$ *and* $\alpha_n + \beta_n$ *are in* $[0, 1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, *let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in* (1.1). *If* $F \neq \varnothing$, *and one of* T_1, T_2 *and* T_3 *is completely continuous,* *then* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to a common fixed point of* T_1, T_2 *and* T_3.

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. In addition, by Lemma 2.3,

$$\lim_{n\to\infty}\| x_n - T_1x_n \| = 0,$$

$$\lim_{n\to\infty}\| x_n - T_2x_n \| = 0$$

and

$$\lim_{n\to\infty}\| x_n - T_3x_n \| = 0,$$

then $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, then there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \to p$ as $j \to \infty$. Thus

$$\lim_{j\to\infty}\| x_{n_j} - T_1x_{n_j} \| = \lim_{j\to\infty}\| x_{n_j} - T_2x_{n_j} \|$$
$$= \lim_{j\to\infty}\| x_{n_j} - T_3x_{n_j} \| = 0.$$

It follows that $\lim_{j\to\infty}\| x_{n_j} - p \| = 0$. This implies by Lemma 1.3 that $p \in F$. Furthermore, by Lemma 2.1, we get that $\lim_{n\to\infty}\| x_n - p \|$ exists. Thus $\lim_{n\to\infty}\| x_n - p \| = 0$.

Theorem 2.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let* T_1, T_2 *and* T_3 *be three asymptotically nonexpansive self-mappings of C with* $\{k_n\}, \{l_n\}, \{m_n\} \subset [0,\infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $\lim_{n\to\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, *respectively and let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *such that* $a_n + b_n$, $c_n + d_n$ *and* $\alpha_n + \beta_n$ *are in* $[0, 1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, *let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in* (1.1). *If* $F \neq \emptyset$, *and one of* T_1, T_2 *and* T_3 *is demicompact, then* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to a common fixed point of* T_1, T_2 *and* T_3.

Proof. Suppose that T_1 is semicompact. By Lemma 2.3,

$$\lim_{n\to\infty}\| x_n - T_1x_n \| = \lim_{n\to\infty}\| x_n - T_2x_n \|$$

$$= \lim_{n\to\infty}\| x_n - T_3x_n \| = 0.$$

Then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in C$. It follows from Lemma 1.3 that $p \in F$. Since $\lim_{n\to\infty}\| x_n - p \|$ exists by Lemma 2.1, it follows that

$$\lim_{n\to\infty}\| x_n - p \| = 0.$$

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Strong convergence theorems by hybrid method for asymptotically k-strict pseudo-contractive mapping in Hilbert space

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ABSTRACT

In this paper, we introduce the hybrid method of modified Mann's iteration for an asymptotically k-strict pseudo-contractive mapping. Then we prove that such a sequence converges strongly to $P_{F(T)}x_0$. This main theorem improves the result of Issara Inchan [I. Inchan, Strong convergence theorems of modified Mann iteration methods for asymptotically nonexpansive mappings in Hilbert spaces, Int. J. Math. Anal. 2 (23) (2008) 1135–1145] and concerns the result of Takahashi et al. [W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 341 (2008) 276–286], and many others.

1. Introduction

Let H be a real Hilbert space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. We know that a Hilbert space H satisfies Opial's condition [1], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Recall that a mapping $T : C \to C$ is said to be a strict pseudo-contractive mapping [2] if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

for all $x, y \in C$. (If (1.1) holds, we also say that T is a k-strict pseudo-contraction.)

It is known that if T is a 0-strict pseudo-contractive mapping, T is a nonexpansive mapping.

In this paper we will consider an iteration method of modified Mann for asymptotically k-strict pseudo-contractive mapping. We say that $T : C \to C$ is an asymptotically k-strict pseudo-contractive mapping if there exists a constant $0 \le k < 1$ satisfying

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.2}$$

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for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\gamma_n \geq 0$ for all n such that $\lim_{n\to\infty} \gamma_n = 0$. We see that if $k = 0$, then T is an asymptotically nonexpansive mapping. By Goebel and Kirk [3], T is an asymptotically nonexpansive mapping if there exists a sequence $\{\gamma_n\}$ of nonnegative numbers with $\lim_{n\to\infty} \gamma_n = 0$ and such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2, \tag{1.3}$$

for all $x, y \in C$ and all integers $n \geq 1$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [4–7]. However, Mann and Ishikawa iteration processes have only weak convergence even in Hilbert space: see [8,7].

Our iteration method for finding a fixed point of an asymptotically k-strict pseudo-contractive mapping T is the modified Mann's iteration method studied in [9–12] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \quad n \geq 0, \tag{1.4}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval (0, 1).

In 2007, Takahashi, Takeuchi and Kubota [7] introduced the modification of the Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then we prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2008, Inchan [13], introduced the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then he proves that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these facts, it is the purpose of this paper to introduce the modified Mann iteration processes for an asymptotically k-strict pseudo-contractive mapping by the idea in (1.6). Let C be a closed convex subset of a Hilbert space H, T be an asymptotically k-strict pseudo-contractive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1 - \alpha_n)]\|x_n - T^n x_n\| + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\theta_n = (\operatorname{diam} C)^2(1 - \alpha_n)\gamma_n \to 0$, $(n \to \infty)$.

We shall prove that the iteration generated by (1.7) converges strongly to $z_0 = P_{F(T)}x_0$.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

We collect some lemmas which will be used in the proof of the main result.

Lemma 2.1 ([14]). *The following identities hold in a Hilbert space H:*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$, $\forall x, y \in H$.

(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0, 1]$.

Lemma 2.2 ([15]). *Let T be an asymptotically k-strict pseudo-contractive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties*

(i) $x_n \rightharpoonup z$ *and*

(ii) $Tx_n - x_n \to 0$.

Then $(I - T)z = 0$.

Lemma 2.3 ([16]). *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if the following inequality holds*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.4 ([15]). *Assume that C is a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically k-strict pseudo-contraction. Then for each $n \geq 1$, T^n satisfies the Lipschitz condition:*

$$\|T^n x - T^y\| \leq L_n \|x - y\|$$

for all $x, y \in C$, where $L_n = \frac{k+\sqrt{1+\gamma_n(1-k)}}{1-k}$.

3. Main results

In this section, we prove strong convergence theorems by hybrid methods for asymptotically k-strict pseudo-contractive mappings in Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically k-strict pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen such that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (1.7) converges strongly to $z_0 = P_{F(T)}x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. Then we have, for $u \in F(T) \subset C_m$

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\
&= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 + k\|x_m - T^m x_m\|^2] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2 \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.
\end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It obviously follows that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$. Then

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\
&= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\
&\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.
\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0, 1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m$, we have

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\
&= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\
&\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m) \\
&\quad + (1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m) - \alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m \\
&= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m \\
&= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \quad \text{for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \quad \text{for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.1}$$

So, for $u \in F(T)$, we have

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - u\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \quad \text{for all } u \in F(T) \text{ and } n \in \mathbb{N}. \tag{3.2}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbb{N}$

$$\begin{aligned} 0 \leq \langle x_0 - x_n, x_n - x_{n+1}\rangle &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

From (3.2) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.3) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.5}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1 - \alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n, \tag{3.6}$$

By the definition of y_n, we have

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n)T^n x_n - x_n\| \\ &= (1 - \alpha_n)\|T^n x_n - x_n\|. \end{aligned}$$

From (3.6), we have

$$\begin{aligned} &(1 - \alpha_n)^2\|T^n x_n - x_n\|^2 = \|y_n - x_n\|^2 \\ &\quad = \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \end{aligned}$$

$$\begin{aligned}&\le \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\|\\ &\le \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\|\\ &= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n.\end{aligned}$$

It follows that

$$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \le 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n.$$

Hence

$$(1 - k - \alpha_n)\|T^n x_n - x_n\| \le 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n. \tag{3.7}$$

From $\limsup_{n\to\infty} \alpha_n < 1 - k$, we can choose $\epsilon > 0$ such that $\alpha_n \le 1 - k - \epsilon$ for large enough n. From (3.5) and (3.7), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.8}$$

Next, we show that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. From Lemma 2.4, we have

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|\\ &\le L_1\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + L_{n+1})\|x_n - x_{n+1}\|.\end{aligned} \tag{3.9}$$

From (3.5) and (3.8), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.10}$$

By (3.9), Lemma 2.2 and boundedness of $\{x_n\}$ we obtain $\emptyset \ne \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \le \|z_0 - x_0\|$ for all $n \ge 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| &\le \|x_0 - w\| \le \liminf_{n\to\infty} \|x_0 - x_n\|\\ &\le \limsup_{n\to\infty} \|x_0 - x_n\| \le \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\begin{aligned}\|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2\\ &\le 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This completes the proof. □

Using this Theorem 3.1, we have the following corollaries.

Corollary 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a k-strict pseudo-contractive mapping of C into itself for some $0 \le k < 1$ such that $F(T) \ne \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2\},\\ x_{n+1} = P_{C_{n+1}}x_0,\end{cases} \tag{3.11}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in [k, 1)$. Then $\{x_n\}$ generated by (3.11) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.3 ([13]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \ne \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n,\\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\},\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N},\end{cases} \tag{3.12}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0$ as $n \to \infty$ and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (3.12) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.4 (*[7, Theorem 4.1]*). *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.13}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

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