รายงานวิจัยฉบับสมบูรณ์

โครงการ วิธีการประมาณค่าแบบซ้ำของตัวดำเนินการไม่เชิงเส้นสำหรับ การแก้ปัญหาอสมการการแปรผัน และปัญหาเชิงดุลยภาพนัยทั่วไป

โดย

ดร.รัตนาพร วังคีรี

มิถุนายน 2555

สัญญาเลขที่ MRG5380244

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และ มหาวิทยาลัยนเรศวร

บทคัดย่อ

รหัสโครงการ: MRG5380244

ชื่อโครงการ: วิธีการประมาณค่าแบบซ้ำของตัวดำเนินการไม่เชิงเส้นสำหรับการแก้ปัญหาอสมการการแปรผันและปัญหาเชิงดุลยภาพนัยทั่วไป

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บทคัดย่อ: ในงานวิจัยนี้ ผู้วิจัยได้นำเสนอวิธีการประมาณค่าแบบซ้ำหลายๆแบบ เพื่อใช้สำหรับการประมาณค่าจุดตรึงร่วมของกลุ่มนับได้ของการส่งไม่เชิงเส้นแบบต่าง ๆ และเพื่อแก้ไขปัญหาในทางคณิตศาสตร์หลายแบบตัวอย่างเช่น ปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป และระบบของปัญหาเชิงดุลยภาพในทั้งปริภูมิฮิลเบิร์ต และ ปริภูมิบานาค

คำหลัก : วิธีการประมาณค่าแบบซ้ำ; จุดตรึงร่วม; การส่งแบบไม่ขยาย; อสมการการแปรผัน; ปัญหาเชิงดุลยภาพ

Abstract

Project Code: MRG5380244

Project Title: Iterative approximation methods of nonlinear operators for sloving variational inequality problems and generalized equilibriums

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The purposes of this research are to introduce several new iterative approximation methods for approximating the common fixed points of the countable families of nonlinear mappings and solving many mathematical problems such as variational inequality problems, equilibrium problems, mixed equilibrium problems, generalized mixed equilibrium problems and system of equilibrium problems in both Hilbert spaces and Banach spaces

Keywords: Iterative approximation methods, Common fixed point, Nonexpansive mappings, Variational inequality, Equilibrium Problems.

บทนำ

ทฤษฎีจุดตรึง (fixed point theory) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน (functional analysis) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการ และ การพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ๆ ที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆนั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน (basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ทฤษฎีจุดตรึงนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับ **การมีคำตอบ** (existence of solution) และ **การมีเพียงคำตอบเดียว** ของสมการ (uniqueness of solution) ตลอดจนการคิดค้นหาวิธีในการประมาณหาคำตอบของสมการต่างๆ ดังนั้นการศึกษาทฤษฎีต่างๆ ที่เกี่ยวข้องกับการมีจุดตรึงของการส่งต่างๆ และ การหาระเบียบวิธีต่างๆที่ใช้ในการประมาณค่าคำตอบนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์กลุ่มหนึ่งจำนวนมากให้ความสนใจศึกษา เมื่อศึกษาการมีคำตอบของสมการต่างๆแล้ว ปัญหาที่น่าสนใจต่อไปก็คือ เราจะหาคำตอบของสมการต่างๆนั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้นระเบียบวิธีการกระทำซ้ำของจุดตรึง(fixed-point iterations) ต่างๆที่ใช้ในการหาคำตอบ และ ประมาณคำตอบ เพื่อนำไปประยุกต์ใช้เกี่ยวกับการแก้ปัญหาในเรื่องของสมการตัวดำเนินการไม่เชิงเส้น (nonlinear operator equations) ในเรื่องของแก้ปัญหาอสมการการแปรผัน (variational inequality problem (VIP)) และแก้สมการหาคำตอบของปัญหาเชิงดุลยภาพ(equilibrium problems (EP)) ปัญหาที่ดีที่สุด(optimizations problems) ทั้งในปริภูมิฮิลเบิร์ตและปริภูมิบานาค ซึ่งปัญหาดังกล่าวเป็นปัญหาที่สำคัญที่มีประโยชน์มากมายในสาขาวิชาต่างๆ เช่นสาขาวิชาฟิสิกส์ คณิตศาสตร์ประยุกต์ วิศวกรรม และสาขาทางเศรษฐศาสตร์

จากความสำคัญข้างต้นเป็นผลให้นักคณิตศาสตร์จึงได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ซึ่งการวิจัยเกี่ยวกับวิธีการทำซ้ำของจุดตรึง และการประมาณค่าจุดตรึงที่สำคัญนั้นสามารถนำมาเป็นเครื่องมือในการแก้สมการหาคำตอบของปัญหาเชิงดุลยภาพ ดังเช่นใน ปี 1997 Combettes และ Hirstoaga ได้เริ่มต้นศึกษาและใช้วิธีประมาณค่าแบบซ้ำในการค้นหาผลเฉลยของปัญหาเชิงดุลยภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้ม (strong convergence theorems) ในปริภูมิฮิลเบิร์ต นอกจากนั้นแล้วยังมีนักคณิตศาสตร์อีกมากมาย นำวิธีการประมาณค่าแบบซ้ำดังกล่าวมาประยุกต์ใช้ในการแก้ปัญหาอสมการการแปรผัน ปัญหาค่าน้อยสุด และปัญหาอื่นๆ ทางคณิตศาสตร์

ดังนั้นการคิดค้นเพื่อให้เกิดวิธีการประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ๆ และทฤษฎีการลู่เข้าสู่จุดตรึงจึงเป็นองค์ความรู้ใหม่ที่จะได้รับ นอกจากนั้นแล้วยังสามารถใช้วิธีการประมาณค่าดังกล่าวเพื่อประยุกต์ใช้หาคำตอบของปัญหาเชิงดุลยภาพ และ ปัญหาอสมการการแปรผัน และปัญหาอื่นๆ ทางคณิตศาสตร์ ซึ่งองค์ความรู้ใหม่ที่ได้นั้นจะเป็นพื้นฐานที่สำคัญในการพัฒนาสาขาวิชา

การวิเคราะห์เชิงฟังก์ชันและสาขาวิชาอื่นๆ ที่เกี่ยวข้อง ดังที่ได้กล่าวมาแล้วข้างต้นอันจะเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ในงานวิจัยนี้ ผู้วิจัยได้นำเสนอวิธีการประมาณค่าแบบซ้ำหลายๆแบบ เพื่อใช้สำหรับการประมาณค่าจุดตรึงร่วมของการส่งไม่เชิงเส้นแบบต่าง ๆ และเพื่อแก้ไขปัญหาในทางคณิตศาสตร์หลายแบบตัวอย่างเช่น ปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป และระบบของปัญหาเชิงดุลยภาพในทั้งปริภูมิฮิลเบิร์ต และ ปริภูมิบานาค

เนื้อหาโครงการโดยสรุป

วิธีการประมาณค่าแบบซ้ำของตัวดำเนินการไม่เชิงเส้นสำหรับการแก้ปัญหาอสมการการแปรผัน และปัญหาเชิงดุลยภาพนัยทั่วไป

(Executive Summary : ITERATIVE APPROXIMATION METHODS OF NONLINEAR OPERATORS FOR SLOVING VARIATIONAL INEQUALITY PROBLEMS AND GENERALIZED EQUILIBRIUM PROBLEMS)

1. ความสำคัญและที่มาของปัญหา

วิธีการประมาณค่าแบบซ้ำของจุดตรึง (Iterative approximation method of fixed point) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน (functional analysis) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง การคิดค้นเพื่อให้ได้มาซึ่งวิธีการประมาณค่าแบบต่างๆที่ใช้ในการประมาณค่าจุดตรึงของการส่งแบบต่างๆ หรือประมาณค่าคำตอบของอสมการต่างๆนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์จำนวนมากให้ความสนใจศึกษา เนื่องจากมีนักนักคณิตศาสตร์กลุ่มหนึ่งที่ได้ศึกษาการมีคำตอบของ จุดตรึงของการส่งแบบต่างๆ และคำตอบของอสมการต่างๆ แล้ว ปัญหาที่น่าสนใจต่อไปก็คือ เราจะค้นหาคำตอบนั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจที่ศึกษา โดยได้คิดค้นระเบียบวิธีการกระทำซ้ำของจุดตรึง (fixed-point iterations) สำหรับการส่งแบบต่างๆ หรือ วิธีการประมาณค่าแบบซ้ำ (Iterative approximation methods) เพื่อใช้ในการหาคำตอบ และ การประมาณค่าคำตอบให้กับการส่งต่างๆเหล่านั้น พร้อมทั้งนำผลที่ได้ไปประยุกต์ใช้เกี่ยวกับการแก้ปัญหาในเรื่องของ การหาผลเฉลยของอสมการการแปรผัน (variational inequality problem (VIP)) ผลเฉลยของปัญหาเชิงดุลยภาพ (equilibrium problems (EP)) ปัญหาเชิงดุลยภาพแบบผสม (mixed equilibrium problems (MEP)) ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป (generalized mixed equilibrium problems (GMEP)) รวมทั้ง ระบบปัญหาเชิงดุลยภาพ (system of equilibrium problems(SEP)) ปัญหาที่ดีที่สุด (optimizations problems) และ ปัญหาค่าน้อยที่สุด (minimizations problems) ทั้งในปริภูมิฮิลเบิร์ตและปริภูมิบานาค ซึ่งองค์ความรู้ใหม่ที่ได้จากการศึกษาทฤษฎีเกี่ยวกับปัญหาดังกล่าวนั้นเป็นแบบจำลองพื้นฐาน และ เครื่องมือที่สำคัญในการศึกษาเกี่ยวกับทั้งปัญหาเชิงเส้นและไม่เชิงเส้น (linear and nonlinear problems) ซึ่งปัญหาทั้งสองดังกล่าวถือเป็นปัญหาหลักในการศึกษาทั้งในแง่ของวิทยาศาสตร์บริสุทธิ์ (pure sciences) และวิทยาศาสตร์ประยุกต์ (applied sciences) อย่างเช่นทางด้านช่างเครื่อง(mechanics) ฟิสิกส์(physics) การหาค่าที่ดีที่สุด และการควบคุม (optimization and control) การเงิน(finance) นิเวศวิทยา(ecology) เครือข่าย(network) กำหนดการไม่เชิงเส้น(nonlinear programming), ทฤษฎีเกี่ยวกับเกมส์(game theory) เศรษฐศาสตร์และการเคลื่อนย้ายเชิงดุลยภาพ(economics and transportation equilibrium) วิทยาศาสตร์ของวิศวกรรมศาสตร์(engineering science) เป็นต้น

การศึกษาทฤษฎีบทเกี่ยวกับอสมการการแปรผันนั้น ต้องอาศัยเทคนิคความรู้ที่ผสมผสานกันของ การวิเคราะห์เชิงนูน (convex analysis), การวิเคราะห์เชิงฟังก์ชัน (functional analysis) และการวิเคราะห์เชิงตัวเลข (numerical analysis) มาประกอบกันเพื่อให้ได้ซึ่งคำตอบของอสมการแปรผัน ซึ่งวิธีที่นิยมใช้กันอย่างแพร่หลาย คือ การคิดค้นวิธีการประมาณค่าแบบซ้ำโดยใช้เทคนิคการฉาย (projection technique) (ดู [4], [6], [11], [12], [20-32], [52-54]) ซึ่งจากการแก้ปัญหาโดยใช้วิธีการดังกล่าว ทำให้นักคณิตศาสตร์ได้มองเห็นถึงความสัมพันธ์ระหว่างปัญหาอสมการการแปรผัน และ ปัญหาของจุดตรึง(fixed point problem) นอกจากนั้นแล้ว ในปี 1997 Combettes และ Hirstoaga [9] คิดค้นวิธีการประมาณค่าแบบซ้ำเพื่อการหาผลเฉลยของปัญหาเชิงดุลยภาพ (EP) ซึ่งเป็นนัยทั่วไปของปัญหาอสมการการแปรผัน โดยใช้ปัญหาของจุดตรึงมาช่วยในการแก้ไขปัญหาดังกล่าว อีกทั้งปัญหาเชิงดุลยภาพนี้ถือเป็นเครื่องมือสำคัญยิ่งในการแก้ปัญหาในทางฟิสิกส์ หรือแม้กระทั่งในทางเศรษฐศาสตร์ ซึ่งสามารถดูได้จากเอกสารอ้างอิง [6] , [10] และ [18] เป็นผลให้ต่อมาในปี 2008

Peng และ Yao [30] ได้ศึกษาและคิดค้นวิธีประมาณค่าแบบซ้ำเพื่อใช้ในการค้นหาผลเฉลยร่วมของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป (GMEP) และ ปัญหาจุดตรึงของการส่งไม่ขยายในปริภูมิฮิลเบิร์ต นอกจากนั้นแล้วในปี 2009 Saeidi [39] ได้ศึกษาถึงปัญหาที่เป็นนัยทั่วไปกว่าปัญหา EP ซึ่งถูกเรียกว่าระบบของปัญหาเชิงดุลยภาพ (SEP)

จากที่กล่าวมาข้างต้นจะเห็นได้ว่าการศึกษาทฤษฎีบทเกี่ยวกับวิธีการประมาณค่าแบบซ้ำสำหรับแก้ไขปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป รวมทั้ง ระบบปัญหาเชิงดุลยภาพ สำหรับการส่งแบบไม่ขยายนั้นนับเป็นหัวข้อการศึกษาที่น่าสนใจและมีประโยชน์ความสำคัญเป็นอย่างยิ่ง

ดังนั้นในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการ และ การพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ๆ ที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆนั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน(basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

2. วัตถุประสงค์ของโครงการ

2.1 นำเสนอวิธีการประมาณค่าแบบซ้ำชนิดใหม่ๆที่เป็นนัยทั่วไป เพื่อหาผลเฉลยของปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป รวมทั้ง ระบบปัญหาเชิงดุลยภาพ และ ปัญหาจุดตรึงของการส่งไม่ขยายบนปริภูมิฮิลเบิร์ต เพื่อก่อให้เกิดทฤษฎีบท และองค์ความรู้ใหม่ๆ ในด้านการวิเคราะห์เชิงฟังก์ชัน

2.2 นำเสนอวิธีการประมาณค่าแบบซ้ำเพื่อหาคำตอบร่วมของปัญหาในข้อที่ 5.1 และปัญหาจุดตรึงของการส่งไม่เชิงเส้นแบบอื่นๆ ทั้งบนปริภูมิฮิลเบิร์ต และปริภูมิบานาค เพื่อก่อให้เกิดทฤษฎีบท และองค์ความรู้ใหม่ๆ ที่กว้างขวางมากขึ้น

2.3 ศึกษาถึงบทประยุกต์ของผลลัพธ์ที่ได้จากข้อ 2.1 และ 2.2 เพื่อนำไปสู่การแก้ปัญหาด้านศาสตร์และด้านอื่นๆ เช่น ช่างเครื่อง ฟิสิกส์ การเงิน นิเวศวิทยา เครือข่าย การหาค่าที่ดีที่สุดและการควบคุม กำหนดการไม่เชิงเส้น ทฤษฎีเกี่ยวกับเกมส์ เศรษฐศาสตร์และการเคลื่อนย้ายเชิงดุลยภาพ วิทยาศาสตร์ของวิศวกรรมศาสตร์ เป็นต้น

3. ระเบียบวิธีวิจัย

3.1 ค้นคว้าหาเอกสาร ตำรา วารสาร และ เอกสารสิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยด้าน ทฤษฎีบทเกี่ยวกับการประมาณค่าและทฤษฎีบทจุดตรึง จากแหล่งข้อมูลต่างๆ

3.2 ศึกษาเงื่อนไขที่จำเป็นและเพียงพอต่างๆ ที่เกี่ยวกับการประมาณค่าจุดตรึงของการส่งแบบไม่ขยายหรือการส่งแบบอื่นๆ จากเอกสารที่ได้รับหรืองานวิจัยต่าง ๆ

3.3 ศึกษารูปแบบต่าง ๆ ของวิธีการประมาณค่า เพื่อแก้ไขปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป หรือปัญหาค่าน้อยสุดในปริภูมิฮิลเบิร์ตและปริภูมิบานาคจากเอกสารที่เกี่ยวข้อง

3.4 โดยการอาศัยองค์ความรู้ที่สำคัญต่าง ๆ ที่ได้จากการศึกษาตามระเบียบวิธีตามข้อ 3.1 – 3.3 และประสบการณ์ที่ได้จากการแลกเปลี่ยนความคิดเห็นและปรึกษาผู้เชี่ยวชาญทั้งในประเทศและผู้เชี่ยวชาญต่างประเทศที่มีการเชื่อมโยงการทำวิจัยกันอยู่เพื่อก่อให้เกิดแนวทางในการคิดค้นทฤษฎีบทใหม่ๆ และองค์

ความรู้ใหม่ ๆ ที่เกี่ยวกับวิธีการประมาณค่าแบบต่าง ๆ เพื่อแก้ไขปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป หรือปัญหาค่าน้อยสุด ตามวัตถุประสงค์ที่กำหนดไว้ในข้อ 2.1 - 2.2

4. แผนการดำเนินงานวิจัยตลอดโครงการในแต่ละช่วง 6 เดือน

ปีที่1 (2553)

กิจกรรมและขั้นตอนดำเนินงาน	**2553 (6 เดือนแรก)**					
	1	2	3	4	5	6
1. ค้นคว้าหาเอกสารที่เกี่ยวข้องกับวิธีการประมาณแบบซ้ำของจุดตรึงของการส่งแบบไม่ขยายหรือ การส่งไม่เชิงเส้นแบบอื่น ๆ	■	■				
2. ศึกษาความรู้พื้นฐานเกี่ยวกับกับวิธีการประมาณแบบซ้ำของจุดตรึงของการส่งแบบไม่ขยายหรือ การส่งไม่เชิงเส้นแบบอื่น ๆ จากเอกสารที่ได้รับ			■	■		
3. คิดค้นและวิจัยเพื่อนำเสนอวิธีการประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ ๆ เพื่อใช้ในการหาผลเฉลยของปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ตามวัตถุประสงค์ 2.1					■	■
4. รายงานความก้าวหน้าของโครงการใน 6 เดือนแรก						■
กิจกรรมและขั้นตอนดำเนินงาน	**2553 (6 เดือนหลัง)**					
	7	8	9	10	11	12
1. เดินทางไปหาเอกสารที่เกี่ยวข้องเพิ่มเติม	■					
2. คิดค้นและวิจัยเพื่อนำเสนอวิธีการประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ ๆ เพื่อใช้ในการหาผลเฉลยของปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ในปริภูมิฮิลเบิร์ต เพิ่มเติม	■	■	■	■	■	

3.เขียน และ พิมพ์ผลงานวิจัยเกี่ยวกับวิธีการประมาณค่าแบบซ้ำของจุดตรึง รวมทั้งส่งผลงานเพื่อลงตีพิมพ์ในวารสารนานาชาติที่มีค่า Impact factor					■	■
4. รายงานความก้าวหน้าของโครงการใน 6 เดือนหลัง						■

ปีที่ 2 (2554)

กิจกรรมและขั้นตอนดำเนินงาน	**2554 (6 เดือนแรก)**					
	1	2	3	4	5	6
1. เดินทางเพื่อค้นหาเอกสาร วารสาร เพิ่มเติม	■					
2. ศึกษาบทความวิจัยที่เกี่ยวกับวิธีการประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ๆ เพื่อใช้ในการ**หาผลเฉลยร่วม** ของปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไปในปริภูมิบานาค เพิ่มเติม	■	■	■	■	■	
3. เดินทางไปหาเอกสาร วารสาร เพิ่มเติม						■
4. รายงานความก้าวหน้าของโครงการใน 6 เดือนแรก ปีที่ 2						■
กิจกรรมและขั้นตอนดำเนินงาน	**2554 (6 เดือนหลัง)**					
	7	8	9	10	11	12
1. คิดค้นและวิจัยเพื่อนำเสนอวิธีการประมาณ ประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ๆ เพื่อใช้ในการ**หาผลเฉลยร่วม** ของปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ในปริภูมิบานาคตามวัตถุประสงค์ 2.2 เพิ่มเติม	■	■				
2. เขียนและส่งผลงานเพื่อลงตีพิมพ์ในวารสาร			■	■		

นานาชาติ						
3. เขียน และ พิมพ์ รายงานฉบับสมบูรณ์ของโครงการ						
4. ส่งรายงานฉบับสมบูรณ์ต่อ สกว.						

5. ผลงาน/หัวข้อเรื่องที่คาดว่าจะตีพิมพ์ในวารสารวิชาการระดับนานาชาติในแต่ละปี

ปีที่ 1 : ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : “Strong convergence of the iterative approximation methods for solving mixed equilibrium problems and fixed point problems of an nonexpansive mappings in Hilbert spaces”

วารสาร “Fixed point Theory and Applications” เป็นวารสารระดับนานาชาติ มี Impact factor = 0.728

ปีที่ 2 : ชื่อเรื่องที่คาดว่าจะตีพิมพ์ : เรื่อง “Strong convergence of the iterative approximation methods for solving generalized mixed equilibrium problems, variational inequality problems and fixed point problems in Banach spaces”

วารสาร “Applied Mathematics and Computation” เป็นวารสารระดับนานาชาติที่มี impact factor = 0.961

6. **งบประมาณโครงการ**

รายการ	ปีที่ 1	ปีที่ 2	รวม
1. หมวดค่าตอบแทน			
- ค่าตอบแทนหัวหน้าโครงการ	120,000	120,000	240,000
2. หมวดค่าวัสดุ			
- ค่าวัสดุสำนักงาน	10,000	10,000	20,000
- ค่าวัสดุคอมพิวเตอร์(เช่นซื้อ แผ่น CD หมึกปริ้นท์)	10,500	10,500	21,000
3. หมวดค่าใช้สอย			
- ค่าใช้จ่ายสำหรับเดินทางเพื่อทำวิจัยและนำเสนอผลงานวิจัยในประเทศได้แก่			
- ค่าพาหนะ	15,000	15,000	30,000
- ค่าที่พัก	13,500	13,500	27,000
- ค่าเบี้ยเลี้ยง	4,800	4,800	9,600
- ค่าส่งไปรษณีย์และค่าถ่ายเอกสาร	10,000	10,000	20,000
- ค่าตีพิมพ์วารสาร(Page charge)	25,000	25,000	50,000
4. หมวดค่าจ้าง			
- ค่าจ้างนิสิตช่วยปฏิบัติงานวิจัยจำนวน 1 คน(ให้ค้นคว้าหาเอกสารงานวิจัยที่เกี่ยวข้องและรวมถึงงานด้านอื่น ๆ ด้วย)	31,200	31,200	62,400
รวมงบประมาณโครงการ	240,000	240,000	480,000

เหตุผลในการจ้างนิสิตช่วยงาน

1. เพื่อช่วยในการสืบค้นข้อมูลใหม่ๆ ตามที่ได้รับหมอบหมายอยู่ตลอดเวลาเพราะงานทางด้านการคิดค้นทฤษฎีบท หรือองค์ความรู้ใหม่ ๆ จำเป็นที่ต้องทราบข้อมูลที่เป็นปัจจุบันให้มากที่สุด
2. เพื่อช่วยในการพิมพ์งาน (paper) หรือ พิมพ์รายงานต่างๆ
3. เพื่อเป็นการพัฒนาความรู้ในการวิจัยด้านนี้ให้กับนักศึกษา สำหรับเตรียมความพร้อมในการศึกษาระดับที่สูงขึ้นไป

รายละเอียดในการจ้างนิสิต

ค่าจ้างนิสิตช่วยปฏิบัติงานวิจัยจำนวน 1 คน โดยจ่ายให้นิสิตเป็นรายเดือน ๆ ละ 2,600 บาท เป็นเวลา 2 ปี รวมรายจ่ายที่ต้องจ้างนิสิตช่วยงานเป็นจำนวนเงิน 2,600 x 12 x 2= 62,400 บาท

วัตถุประสงค์ของโครงการ

1 นำเสนอวิธีการประมาณค่าแบบซ้ำชนิดใหม่ๆที่เป็นนัยทั่วไป เพื่อหาผลเฉลยของปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป รวมทั้ง ระบบปัญหาเชิงดุลยภาพ และ ปัญหาจุดตรึงของการส่งไม่ขยายบนปริภูมิฮิลเบิร์ต เพื่อก่อให้เกิดทฤษฎีบท และองค์ความรู้ใหม่ๆ ในด้านการวิเคราะห์เชิงฟังก์ชัน

2 นำเสนอวิธีการประมาณค่าแบบซ้ำเพื่อหาคำตอบร่วมของปัญหาในข้อที่ 5.1 และปัญหาจุดตรึงของการส่งไม่เชิงเส้นแบบอื่นๆ ทั้งบนปริภูมิฮิลเบิร์ต และปริภูมิบานาค เพื่อก่อให้เกิดทฤษฎีบทและองค์ความรู้ใหม่ๆ ที่กว้างขวางมากขึ้น

3 ศึกษาถึงบทประยุกต์ของผลลัพธ์ที่ได้จากข้อ 1 และ 2 เพื่อนำไปสู่การแก้ปัญหาด้านคณิตศาสตร์และด้านอื่นๆ เช่น ช่างเครื่อง ฟิสิกส์ การเงิน นิเวศวิทยา เครือข่าย การหาค่าที่ดีที่สุดและการควบคุม กำหนดการไม่เชิงเส้น ทฤษฎีเกี่ยวกับเกมส์ เศรษฐศาสตร์และการเคลื่อนย้ายเชิงดุลยภาพ วิทยาศาสตร์ของวิศวกรรมศาสตร์ เป็นต้น

ระเบียบวิธีวิจัย

1 ค้นคว้าหาเอกสาร ตำรา วารสาร และ เอกสารสิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยด้าน ทฤษฎีบทเกี่ยวกับการประมาณค่าและทฤษฎีบทจุดตรึงจากแหล่งข้อมูลต่างๆ

2 ศึกษาเงื่อนไขที่จำเป็นและเพียงพอต่างๆ ที่เกี่ยวกับการประมาณค่าจุดตรึงของการส่งแบบไม่ขยายหรือการส่งแบบอื่นๆ จากเอกสารที่ได้รับหรืองานวิจัยต่างๆ

3 ศึกษารูปแบบต่าง ๆ ของวิธีการประมาณค่า เพื่อแก้ไขปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ปัญหาเชิงดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป หรือปัญหาค่าน้อยสุดในปริภูมิฮิลเบิร์ตและปริภูมิบานาคจากเอกสารที่เกี่ยวข้อง

4 โดยการอาศัยองค์ความรู้ที่สำคัญต่าง ๆ ที่ได้จากการศึกษาตามระเบียบวิธีตามข้อ 1 – 3 และประสบการณ์ที่ได้จากการแลกเปลี่ยนความคิดเห็นและปรึกษาผู้เชี่ยวชาญทั้งในประเทศและผู้เชี่ยวชาญต่างประเทศที่มีการเชื่อมโยงการทำวิจัยกันอยู่เพื่อก่อให้เกิดแนวทางในการคิดค้นทฤษฎีบทใหม่ๆ และองค์ความรู้ใหม่ ๆ ที่เกี่ยวกับวิธีการประมาณค่าแบบต่างๆ เพื่อแก้ไขปัญหาอสมการการแปรผัน ปัญหาเชิงดุลยภาพ ดุลยภาพแบบนัยทั่วไป ปัญหาเชิงดุลยภาพแบบผสม ปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป ระบบของปัญหาเชิงดุลยภาพแบบผสมนัยทั่วไป หรือปัญหาค่าน้อยสุด ตามวัตถุประสงค์ที่กำหนดไว้ในข้อ 1 - 2

ผลการวิจัย

Rattanaporn Wangkeeree, A new hybrid projection algorithm basing on the shrinking projection method for a pair of asymptotically quasi-ϕ-nonexpansive mappings, *JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 14, NO.2(2012), 298-313.*

In this paper, motivated and inspired by the above research works, we introduce a new hybrid projection algorithm basing on the shrinking projection method for a pair of asymptotically quasi-ϕ-nonexpansive mappings to have strong convergence theorems for approximating the common element of the set of common fixed points of such two mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. We prove strong convergence theorem which is our main result.

The main results of this paper

Theorem 1. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \longrightarrow 1$ as $n \longrightarrow \infty$ and S be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \longrightarrow 1$ as $n \longrightarrow \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S) \cap VI(A,C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T^n x_n + \gamma_n J S^n w_n), \\ y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)\xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{0.1}$$

where $\xi_n = \alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + (k_n - 1)\theta_n$, $k_n = \max\{k_n^T, k_n^S\}$ for all $n \geq 1$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

(C1) $\lim_{n \longrightarrow \infty} \delta_n = 0$;

(C2) $\{r_n\} \subset [a,b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;*

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

 (a) $\liminf_{n \longrightarrow \infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n \longrightarrow \infty} \alpha_n \gamma_n > 0$ *and*

 (b) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ *and* $\liminf_{n \longrightarrow \infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.

2. R. Wangkeeree, U. Kamraksa, R. Wangkeeree , A General composite algorithms for solving general equilibrium problems and fixed point problems in Hilbert spaces, *Abstract and Applied Analysis, Volume 2011, Article ID 976412, 25 pages, doi:10.1155/2011/976412*

In this paper, motivated by the above results, we introduce a general iterative scheme below in a real Hilbert space H, with the initial guess $x_0 \in C$ chosen arbitrary,

$$\begin{cases} \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n F) S_{i(n+1)}^{p(n+1)} x_n, \\ x_{n+1} = \mu_n P_C[y_n] + (1-\mu_n) u_n, n \geq 0. \end{cases} \tag{0.2}$$

where $p(n) = j+1$ if $jN < n \leq (j+1)N, j = 1, 2, \dots$ and $n = jN + i(n),\ i(n) \in \{1, 2, \dots, N\}$, C is a nonempty closed and convex subset of H, $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in [0,1], $\phi : C \times C \longrightarrow \mathbb{R}$ is a bifunction satisfying certain conditions, $S_1, S_2, \dots, S_N : C \longrightarrow C$ is a finite family of asymptotically nonexpansive mappings with sequences $\{1 + k_{p(n)}^{i(n)}\}$ respectively, $f : C \longrightarrow H$ is a contraction with coefficient $0 < \rho < 1$, F is δ-strongly monotone and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, γ is a positive real number such that $\gamma < \frac{1}{\rho}\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$ and A is an α-inverse strongly monotone mapping. We prove that the proposed algorithm converges strongly to $x^* \in \Omega$ which is the unique solution of the following variational inequality

$$\langle (F - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega.$$

In particular,

(I) if F is a strongly positive bounded linear operator on H, then x^* is the unique solution of the variational inequality ;

(II) if $F = I$, the identity mapping on H and $\gamma = 1$, then x^* is the unique solution of the variational inequality ;

(III) if $F = I$, the identity mapping on H and $f = 0$, then x^* is the unique solution of minimization problem.

The main results of this paper

Theorem 1. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1, S_2, \dots, S_N : C \longrightarrow C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{1 + k_{p(n)}^{i(n)}\}$ respectively, such that $k_{p(n)}^{i(n)} \longrightarrow 0$ as $n \longrightarrow \infty$, $h_n := \max_{1 \leq i(n) \leq N}\{k_{p(n)}^{i(n)}\}$ and $\Gamma := \cap_{i=1}^N Fix(S_i)$,

$$\Gamma = Fix(S_N S_{N-1} S_{N-2} \dots S_1) = Fix(S_1 S_N \dots S_2) = \cdots = Fix(S_{N-1} S_{N-2} \cdots S_1 S_N).$$

Let $A : C \longrightarrow H$ be an α-inverse strongly monotone mapping. Let $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4) such that $\Omega := EP \cap \Gamma$ is nonempty. Let $F : C \longrightarrow H$ be δ-strongly monotone and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, $f : C \longrightarrow H$ a ρ-contraction, γ a positive real number such that $\gamma < (1 - \sqrt{(1-\delta)/\lambda})/\rho$ and r a constant such that $r \in (0, 2\alpha)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by (0.2). Suppose that $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\lim_{n\longrightarrow\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\longrightarrow\infty} \frac{h_n}{\alpha_n} = 0$;

(C2) $0 < \liminf_{n\longrightarrow\infty} \mu_n \leq \limsup_{n\longrightarrow\infty} \mu_n < 1$ and $\lim_{n\longrightarrow\infty} \frac{\mu_{n+1} - \mu_n}{\alpha_{n+1}} = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{z\in B} \|S_{i(n+1)}^{p(n+1)} z - S_{i(n)}^{p(n)} z\| < \infty$, for each bounded subset B of C. Then, the sequence $\{x_n\}$ converges strongly to x^* of the following variational inequality

$$\langle (F - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega \tag{0.3}$$

or equivalently $\tilde{x} = P_\Omega(I - F + \gamma f)\tilde{x}$, where P_Ω is the metric projection of H onto Ω.

3. Rattanaporn Wangkeeree, Rabian Wangkeeree , Strong convergence theorems of the general iterative methods for nonexpansive semigroups in Banach spaces*, *J Glob Optim DOI 10.1007/s10898-011-9835-6.*

Let E be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping from E to E^*. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on E such that $Fix(\mathcal{S}) := \cap_{t\geq 0} Fix(T(t)) \neq \emptyset$, and f is a contraction on E with coefficient $0 < \alpha < 1$. Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and γ a positive real number such that $\gamma < \frac{1}{\alpha}\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$. When the sequences of real numbers $\{\alpha_n\}$ and $\{t_n\}$ satisfy some appropriate conditions, the three iterative processes given as follows :

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, n \geq 0,$$

$$y_{n+1} = \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n, n \geq 0,$$

and

$$z_{n+1} = T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n), n \geq 0$$

converge strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $Fix(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0, \ x \in Fix(\mathcal{S}).$$

Our results extend and improve corresponding ones of Li, Li and Su [S. Li, L. Li, and Y. Su, General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space, Nonlinear Analysis 70 (2009) 3065-3071] and Chen and He [R. Chen and H. He, Viscosity approximation of common fixed points of nonexpansive semigroups in Banach space, Applied Mathematics Letters 20 (2007) 751-757] and many others.

For solving the equilibrium problem, let us assume that the bifunction ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;
(A2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for any $x, y \in C$;
(A3) ϕ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\text{limsup}_{t\longrightarrow 0^+} \phi(tz + (1-t)x, y) \leq \phi(x, y);$$

(A4) $\phi(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The main results of this paper

Let E be a real Banach space. Let T be a nonexpansive mapping on E. For $f \in \Pi_E$ and F a δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{\alpha}\right\}$. For each $t \in (0, 1)$, the mapping $S_t : E \longrightarrow E$ defined by

$$S_t(x) = t\gamma f(x) + (I - tF)Tx, \ \ \forall x \in E$$

is a contraction mapping. Indeed, for any $x, y \in E$,

$$\begin{aligned} \|S_t(x) - S_t(y)\| &\leq \|t\gamma(f(x) - f(y)) + (I - tF)Tx - (I - tF)Ty\| \\ &\leq t\gamma\|f(x) - f(y)\| + (1 - t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\|x - y\| \\ &\leq \left(1 - t[(1 - \sqrt{\frac{1-\delta}{\lambda}}) - \gamma\alpha]\right)\|x - y\| \end{aligned} \quad (0.4)$$

Thus, by Banach contraction mapping principle, there exists a unique fixed point x_t in E that is

$$x_t = t\gamma f(x_t) + (I - tF)Tx_t. \tag{0.5}$$

Lemma 1. Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let T be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ a contraction mapping with coefficient $\alpha(0 < \alpha < 1)$, and let F be a δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{\alpha}\right\}$. Then the net $\{x_t\}$ defined by (0.5) converges strongly as $t \longrightarrow 0$ to a common fixed point $\tilde{x}$ in $F(T)$ which solves the variational inequality :

$$\langle (F - \gamma f)\tilde{x}, j(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{0.6}$$

Lemma 2. Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm and admits the duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$, i.e. $\varphi([0, 1]) \subset [0, 1]$. Let T be a nonexpansive mapping on E with $F(T) \neq \emptyset$ and $f \in \Pi_E$ a contraction mapping with coefficient $\alpha(0 < \alpha < 1)$, let A be a strongly positive bounded linear operator on E with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Assume that the net $\{x_t\}$ defined by (0.5) converges strongly to a common fixed point $\tilde{x}$ in $F(T)$ as $t \longrightarrow 0$. Suppose that $\{x_n\} \subset E$ is bounded and $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$. Then

$$\limsup_{n\longrightarrow\infty} \langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \leq 0. \tag{0.7}$$

Theorem 3. Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $Fix(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\longrightarrow\infty} t_n = \infty.$$

Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, $f : E \longrightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$ and γ a positive real number such that $\gamma < \frac{1}{\alpha}\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$. Then, the sequence $\{x_n\}$ defined by

$$x_0 = x \in E, x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, n \geq 0 \tag{0.8}$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $Fix(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0, \ x \in Fix(\mathcal{S}) \tag{0.9}$$

or equivalently $\tilde{x} = Q_{Fix(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{Fix(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $Fix(\mathcal{S})$.

Out Put จากโครงการวิจัยได้รับทุนจาก สกอ. สกว.และมหาวิทยาลัยนเรศวร

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติจำนวน 3 เรื่อง

(1) R. Wangkeeree, A new hybrid approximation algorithm based on the shrinking projection method for two asymptotically quasi-phi-nonexpansive mappings, (2012), VOL. 14, NO.2, 298-313.
วารสาร "Journal of Computational Analysis and Applications" (วารสารนานาชาติ)
Corresponding Author : R. Wangkeeree
Impact factor ปี 2010 = 0.453

(2) R. Wangkeeree, U. Kamraksa and **R. Wangkeeree**, A General composite algorithms for solving general equilibrium problems and fixed point problems in Hilbert spaces, Volume 2011, Article ID 976412, 25 pages
doi:10.1155/2011/976412.
ในวารสาร "Abstract and Applied Analysis" (วารสารนานาชาติ)
Corresponding Author : R. Wangkeeree
Impact factor ปี 2010 = 1.442

(3) **R. Wangkeeree**, and R. Wangkeeree, Strong convergence theorems of the general iterative methods for nonexpansive semigroups in Banach spaces, DOI 10.1007/s10898-011-9835-6.
ในวารสาร "**Journal of Global Optimization**" (วารสารนานาชาติ)
Corresponding Author : R. Wangkeeree
Impact factor ปี 2010 = 1.16

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ในด้านเชิงวิชาการเพื่อให้รู้ถึงวิธีการประมาณค่าคำตอบของปัญหาทางคณิตศาสตร์ในรูปแบบต่างๆ ทั้งในระดับพื้นฐาน และระดับขั้นสูง นอกจากนั้นแล้ว มีการนำไปใช้ประโยชน์ในเชิงสาธารณะ โดยทำให้มีการพัฒนาการเรียนการสอน และ มีเครือข่ายความร่วมมือ สร้างกระแสความสนใจในด้านการพัฒนาวิธีการประมาณค่าประมาณค่า ให้กว้างขวางมากยิ่งขึ้น

3. การเสนอผลงานในที่ประชุมวิชาการระดับนานาชาติจำนวน 1 ครั้ง

3.1 วันที่ 10 มีนาคม 2554-11 มีนาคม 2554

หัวข้อ: A new hybrid approximation algorithm based on the shrinking projection method for two asymptotically quasi-phi-nonexpansive mappings,

ชื่อการประชุม: The 16th Annual Meeting in Mathematics (AMM2011), Kosa Hotel, Khonkaen, Thailand.

ภาคผนวก 1

A new hybrid approximation algorithm based on the shrinking projection method for two asymptotically quasi-phi-nonexpansive mappings

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A NEW HYBRID APPROXIMATION ALGORITHM BASED ON THE SHRINKING PROJECTION METHOD FOR TWO ASYMPTOTICALLY QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce a new hybrid projection algorithm based on the shrinking projection method for approximating a common element of in the fixed point sets of two asymptotically quasi-ϕ-nonexpansive mappings and the solutions set of a variational inequality corresponding to an inverse-strongly monotone operator. The strong convergence theorem is established in real Banach spaces. Our results improve and extend the corresponding results announced by recent results.

Keywords: Asymptotically quasi-ϕ-nonexpansive mappings, Banach space, Generalzied projection, Shrinking projection method, Variational inequality.

AMS Subject Classification: 47H09, 47H10

1. Introduction

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be an operator. The classical variational inequality problem [19] for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. The solution set of (1.1) is denoted by $VI(A, C)$. Such a problem is connected with convex minimization problem, complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$. First, we recall that a mapping $A : C \to E^*$ is said to be:

(i) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$, for all $x, y \in C$.

(ii) *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \qquad \text{for all } x, y \in C.$$

Let J be the *normalized duality mapping* from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}.$$

It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping are given in [10, 32, 36].

Recall that a mappings $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

A mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \text{ for all } x \in C, y \in F(T).$$

A mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \text{ for all } x, y \in C.$$

A mapping T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - y\| \leq k_n\|x - y\|, \text{ for all } x \in C, y \in F(T).$$

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A mapping T is called *uniformly L-Lipschitzian continuous* if there exists a constant $L>0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|, \text{ for all } x,y\in C.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] in 1972. Since 1972, many authors have studied the weak and strong convergence of iterative processes for such a class of mappings.

Let C be a nonempty closed convex subset of a Hilbert space H and a mapping $P_C : H \to C$ is called the *metric projection* of H onto C if for each $x \in H$, there exists a unique element $Px \in C$ such that $\|x - Px\| = d(x, C)$.

If C is a nonempty closed convex subset of a Hilbert space H and let $P_C : H \to C$ be the *metric projection* of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator $\prod_C$ in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{1.2}$$

for all $x, y \in E$. Observe that, in a Hilbert space H, (1.2) reduces to $\phi(y,x) = \|x-y\|^2$ for all $x,y \in H$. The *generalized projection* $\Pi_C : E \to C$ is the mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y,x)$, that is, $\Pi_C x = x^*$, where x^* is the solution of the following minimization problem:

$$\phi(x^*, x) = \inf_{y\in C} \phi(y,x). \tag{1.3}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J (see, for example, [1, 2, 9, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y,x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.
(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$ for all $x,y,z \in E$.
(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$.
(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x,y) = 0 \text{ if and only if } x = y.$$

For more details see, for example, [10, 32]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the fixed point set of T. A point p in C is said to be an *asymptotic fixed point* of T [30] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T will be denoted by $\hat{F}(T)$. Recall the following definitions:

(i) A mapping $T : C \to C$ is called *relatively nonexpansive* [8, 9, 11] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.
(ii) A mapping $T : C \to C$ is said to be *relatively asymptotically nonexpansive* [1, 26] if $\hat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leqslant k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(iii) A mapping $T : C \to C$ is said to be *ϕ-nonexpansive* [25, 28, 38] if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$.
(iv) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* [25, 28, 38] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.
(v) A mapping $T : C \to C$ is said to be *asymptotically ϕ-nonexpansive* [38] if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x, y)$ for all $x, y \in C$.
(vi) A mapping $T : C \to C$ is said to be *asymptotically quasi-ϕ-nonexpansive* [38] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(vii) A mapping $T : C \to C$ is said to be *asymptotically regular* on C if, for any bounded subset D of C, the following equality holds:

$$\lim_{n\to\infty} \sup_{x\in D} \|T^{n+1}x - T^n x\| = 0.$$

(viii) A mapping $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Remark 1.1. The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the strong restriction that $\hat{F}(T) = F(T)$.

Remark 1.2. In a real Hilbert spaces, the class of (asymptotically) quasi-ϕ-nonexpansive mappings is coincides with the class of (asymptotically) quasi-nonexpansive mappings.

We give some examples which are closed and asymptotically quasi-ϕ-nonexpansive.

Example 1.3. (1). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}J$ is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then Π_C is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (see [4]). More precisely, let $t \in (0,1)$ and define a contraction $G_t : C \to C$ by $G_t x = tx_0 + (1-t)Tx$ for all $x \in C$, where $x_0 \in C$ is a fixed point in C. Applying Banach's Contraction Principle, there exists a unique fixed point x_t of G_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$ even if T has a fixed point. However, in the case of T having a fixed point, Browder [4] proved that the net $\{x_t\}$ defined by $x_t = tx_0 + (1-t)Tx_t$ for all $t \in (0,1)$ converges strongly to an element of $F(T)$ which is nearest to x_0 in a real Hilbert space. Motivated by Browder [4], Halpern [15] proposed the following iteration process:

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \geqslant 0 \tag{1.4}$$

and proved the following theorem.

Theorem H. *Let C be a bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}, 0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to an element of $F(T)$ which is nearest to u.*

Recently, Martinez-Yanes and Xu [21] has adapted Nakajo and Takahashi's [23] idea to modify the process (1.4) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary,} \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leqslant \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.5}$$

They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}x$.

In [27] (see also [22]), Qin and Su improved the result of Martinez-Yanes and Xu [21] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C \textit{ chosen arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, y_n) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.6}$$

If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

In [24], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for two relatively nonexpansive mappings:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary },\\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n),\\ y_n = J^{-1}(\delta_n Jx_0 + (1-\delta_n)Jz_n),\\ H_n = \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\},\\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\},\\ x_{n+1} = P_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and T, S are relatively nonexpansive mappings. They proved, under appropriate conditions on the parameters, that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a common fixed point of T and S.

Very recently, motivated by the above research work, Qin, Cho and Kang [25] introduced and considered the shrinking projection method which was first introduced by Takahashi et al. [35] in a real Hilbert space for two asymptotically quasi-ϕ-nonexpansive mappings in uniformly smooth and uniformly convex Banach spaces. To be more precise, they proved the following results.

Theorem QCK. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \to 1$ as $n \to \infty$ and let S be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrary},\\ C_1 = C, x_1 = \Pi_{C_1} x_0,\\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n),\\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n),\\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \le \phi(u, x_n) + (k_n - 1)\theta_n\},\\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases} \tag{1.8}$$

where $k_n = \max\{k_n^T, k_n^S\}$ for all $n \ge 1$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are sequences in $[0,1]$ which satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;
(C2) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$;
(C3) $0 \le \delta_n < 1$ *and* $\limsup_{n\to\infty} \delta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

The problem of finding a solution to variational inequalities for monotone mappings in Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [5, 16, 20] and the references therein. Iiduka and Takahashi [17] introduced the following algorithm for finding a solution of the variational inequality for an α-inverse-strongly monotone mapping A with $\|Ay\| \le \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$ in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_0 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \forall n \ge 0. \tag{1.9}$$

Assume that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$ where $1/c$ is the 2-uniformly convexity constant of E. They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} \Pi_{VI(A,C)}(x_n)$.

In this paper, motivated and inspired by the above research works, we introduce a new hybrid projection algorithm based on the shrinking projection method for two of asymptotically quasi-ϕ-nonexpansive mappings to have strong convergence theorems for approximating a common element in the fixed point sets of these two mappings and the solution set of the variational inequality for an inverse-strongly monotone operator in Banach spaces. Our results improve and extend the corresponding results announced by recent results.

2. Preliminaries

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [13, 29, 32, 33]. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [13, 32, 33] for more details.

We define the function $\delta : [0, 2] \to [0, 1]$ which is called the modulus of convexity of E as following

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in C, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}. \tag{2.1}$$

Then E is said to be 2-uniformly convex if there exists a constant $c > 0$ such that constant $\delta(\varepsilon) > c\varepsilon^2$ for all $\varepsilon \in (0, 2]$. Constant $\frac{1}{c}$ is called the 2-uniformly convexity constant of E. A 2-uniformly convex Banach space is uniformly convex, see [7, 34] for more details. We know the following lemma of 2-uniformly convex Banach spaces:

Lemma 2.1. [3, 6] *Let E be a 2-uniformly convex Banach, then for all x, y from any bounded set of E and $jx \in Jx, jy \in Jy$,*

$$\langle x - y, jx - jy\rangle \geq \frac{c^2}{2}\|x - y\|^2 \tag{2.2}$$

where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.

Now we present some definitions and lemmas which will be applied in the proof of the main result in the next section.

Lemma 2.2 (Kamimura and Takahashi [18])**.** *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.3 (Alber [2])**.** *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for any $y \in C$.*

Lemma 2.4 (Alber [2])**.** *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$$

for all $y \in C$.

Let E be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping J from E to E^*. Then J^{-1} is also single-valued, one to one, surjective, and it is the duality mapping from E^* to E. We need the following mapping V which studied in Alber [2],

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2 \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma:

Lemma 2.5 (Kamimura and Takahashi [18])**.** *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in (2.3). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 ([12, Lemma 1.4])**.** *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|), \tag{2.4}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping F of $[0,1)$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \forall y \in C\}.$$

Lemma 2.7. [31] *Let C be a nonempty closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $\Psi \subset E \times E^*$ be an operator defined as follows:*

$$\Psi v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C\ . \end{cases}$$

Then Ψ is maximal monotone and $\Psi^{-1}0 = VI(A, C)$.

3. Main Results

In this section, we prove strong convergence theorem which is our main result.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \to 1$ as $n \to \infty$ and S be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S) \cap VI(A, C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \ \text{chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T^n x_n + \gamma_n J S^n w_n), \\ y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)\xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\xi_n = \alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + (k_n - 1)\theta_n$, $k_n = \max\{k_n^T, k_n^S\}$ for all $n \geq 1$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

- (C1) $\lim_{n\to\infty} \delta_n = 0$;
- (C2) $\{r_n\} \subset [a, b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;*
- (C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*
  - (a) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$ *and*
  - (b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.

Proof. We divide the proof of Theorem 3.1 into four steps.
(I). We show first that the sequence $\{x_n\}$ is well defined. It is easily to seen that $VI(A, C)$ is closed and convex. By the same argument as in the proof of [38, Lemma 2.4], one can show that $F(T) \cap F(S)$ is closed and convex. Hence $\Omega := F(S) \cap F(T) \cap VI(A, C)$ is a nonempty, closed and convex subset of C. Consequently, Π_Ω is well defined.
Next, we prove by induction that C_n is closed and convex for all $n \geq 1$. It is obvious that $C_1 = C$ is

closed and convex. Suppose that C_j is closed and convex for some $j \in \mathbb{N}$. For any $u \in C_j$, we observe that

$$\phi(u, y_j) \leq \delta_j \phi(u, x_1) + (1-\delta_j)[\alpha_j \phi(u, x_{j-1}) + (1-\alpha_j)\phi(u, x_j) + (k_j - 1)\theta_j]$$

is equivalent to

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u, Jx_j\rangle + 2\delta_j\langle u, Jx_1\rangle - 2\langle u, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

It can be seen that C_{j+1} is closed. Next, we prove that C_{j+1} is convex. In fact, $u_1, u_2 \in C_{j+1}$, $\varrho \in (0,1)$. Put $u^* = \varrho u_1 + (1-\varrho)u_2$. From the definition of C_{j+1}, we have

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u_1, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u_1, Jx_j\rangle + 2\delta_j\langle u_1, Jx_1\rangle - 2\langle u_1, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j \end{aligned}$$

and

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u_2, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u_2, Jx_j\rangle + 2\delta_j\langle u_2, Jx_1\rangle - 2\langle u_2, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

Therefor we have

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle \varrho u_1, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle \varrho u_1, Jx_j\rangle + 2\delta_j\langle \varrho u_1, Jx_1\rangle - 2\langle \varrho u_1, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j \end{aligned}$$

and

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle (1-\varrho)u_2, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle (1-\varrho)u_2, Jx_j\rangle \\ & +2\delta_j\langle (1-\varrho)u_2, Jx_1\rangle - 2\langle (1-\varrho)u_2, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

Combining the last two inequalities, we obtain

$$\begin{aligned} & 2(1-\delta_j)\alpha_j \langle u^*, Jx_{j-1}\rangle + 2(1-\delta_j)(1-\alpha_j)\langle u^*, Jx_j\rangle + 2\delta_j\langle u^*, Jx_1\rangle - 2\langle u^*, Jy_j\rangle \\ \leq\ & \delta_j\|x_1\|^2 + (1-\delta_j)\alpha_j\|x_{j-1}\|^2 + (1-\delta_j)(1-\alpha_j)\|x_j\|^2 - \|y_j\|^2 + (1-\delta_j)(k_j-1)\theta_j. \end{aligned}$$

The convexity of C_j implies that $u^* \in C_j$. Therefor, we have $u^* \in C_{j+1}$. Then we conclude that C_{j+1} is closed and convex. Hence for each $n \geq 1$, C_n is closed and convex.

(II). Next, we show that $\Omega \subset C_n$ for all $n \geq 1$. In fact, $\Omega \subset C_1 = C$ is obvious. Suppose $\Omega \subset C_n$ for some $n \in \mathbb{N}$. Then, for all $q \in \Omega \subset C_n$, we know from Lemma 2.5 that

$$\begin{aligned} \phi(q, w_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - r_n Ax_n)) \\ &\leq \phi(q, J^{-1}(Jx_n - r_n Ax_n)) \\ &= V(q, Jx_n - r_n Ax_n) \\ &\leq V(q, (Jx_n - r_n Ax_n) + r_n Ax_n) - 2\langle J^{-1}(Jx_n - r_n Ax_n) - q, r_n Ax_n\rangle \\ &= V(q, Jx_n) - 2r_n\langle J^{-1}(Jx_n - r_n Ax_n) - q, Ax_n\rangle \\ &= \phi(q, x_n) - 2r_n\langle x_n - q, Ax_n\rangle + 2\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n\rangle. \end{aligned} \tag{3.2}$$

Since $q \in VI(A, C)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned} -2r_n\langle x_n - q, Ax_n\rangle &= -2r_n\langle x_n - q, Ax_n - Aq\rangle - 2r_n\langle x_n - q, Aq\rangle \\ &\leq -2\alpha r_n\|Ax_n - Aq\|^2. \end{aligned} \tag{3.3}$$

Therefore, from Lemma 2.1 and the assumption that $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$, we obtain that

$$\begin{aligned} 2\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - r_n Ax_n) - J^{-1}(Jx_n), -r_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - r_n Ax_n) - J^{-1}(Jx_n)\|\|r_n Ax_n\| \\ &\leq \frac{4}{c^2}\|JJ^{-1}(Jx_n - r_n Ax_n) - JJ^{-1}(Jx_n)\|\|r_n Ax_n\| \\ &= \frac{4}{c^2}\|(Jx_n - r_n Ax_n) - Jx_n\|\|r_n Ax_n\| \\ &= \frac{4}{c^2} r_n^2\|Ax_n\|^2 \end{aligned}$$

$$\leq \frac{4}{c^2} r_n^2 \|Ax_n - Aq\|^2. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) and using the condition that $r_n < c^2\alpha/2$, we get

$$\phi(q, w_n) \leq \phi(q, x_n) + 2r_n \left(\frac{2}{c^2} r_n - \alpha\right) \|Ax_n - Aq\|^2 \leq \phi(q, x_n). \tag{3.5}$$

Using (3.5) and Lemma 2.6, for each $q \in \Omega \subset C_n$, we obtain

$$\begin{aligned}
\phi(q, z_n) &= \phi(q, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n)) \\
&= \|q\|^2 - 2\alpha_n \langle q, Jx_{n-1}\rangle - 2\beta_n \langle q, JT^n x_n\rangle - 2\gamma_n \langle q, JS^n w_n\rangle \\
&\quad + \|\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_n \langle q, Jx_{n-1}\rangle - 2\beta_n \langle q, JT^n x_n\rangle - 2\gamma_n \langle q, JS^n w_n\rangle \\
&\quad + \alpha_n \|Jx_{n-1}\|^2 + \beta_n \|JT^n x_n\|^2 + \gamma_n \|JS^n w_n\|^2 \\
&= \alpha_n \phi(q, x_{n-1}) + \beta_n \phi(q, T^n x_n) + \gamma_n \phi(q, S^n w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n k_n^T \phi(q, x_n) + \gamma_n k_n^S \phi(q, w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n k_n \phi(q, x_n) + \gamma_n k_n \phi(q, w_n) \\
&\leq \alpha_n \phi(q, x_{n-1}) + \beta_n k_n \phi(q, x_n) + \gamma_n k_n \phi(q, x_n) \\
&= \alpha_n \phi(q, x_{n-1}) + (1-\alpha_n) k_n \phi(q, x_n) \\
&= \alpha_n \phi(q, x_{n-1}) + (1-\alpha_n) k_n \phi(q, x_n) + (1-\alpha_n)\phi(q, x_n) - (1-\alpha_n)\phi(q, x_n) \\
&= [\alpha_n \phi(q, x_{n-1}) + (1-\alpha_n)\phi(q, x_n)] + (1-\alpha_n)(k_n - 1)\phi(q, x_n) \\
&= [\alpha_n \phi(q, x_{n-1}) + (1-\alpha_n)\phi(q, x_n)] + (k_n - 1)\phi(q, x_n).
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}
\phi(q, y_n) &= \phi(q, J^{-1}(\delta_n Jx_1 + (1-\delta_n) Jz_n)) \\
&= \|q\|^2 - 2\delta_n \langle q, Jx_1\rangle - 2(1-\delta_n)\langle q, Jz_n\rangle + \|\delta_n Jx_1 + (1-\delta_n) Jz_n)\|^2 \\
&\leq \|q\|^2 - 2\delta_n \langle q, Jx_1\rangle - 2(1-\delta_n)\langle q, Jz_n\rangle + \delta_n \|x_1\|^2 + (1-\delta_n)\|z_n\|^2 \\
&= \delta_n \phi(q, x_1) + (1-\delta_n)\phi(q, z_n) \\
&\leq \delta_n \phi(q, x_1) + (1-\delta_n)([\alpha_n \phi(q, x_{n-1}) + (1-\alpha_n)\phi(q, x_n)] + (k_n - 1)\theta_n) \\
&= \delta_n \phi(q, x_1) + (1-\delta_n)\xi_n.
\end{aligned} \tag{3.7}$$

So, $q \in C_{n+1}$. Then by induction, $\Omega \subset C_n$ for all $n \geq 1$ and hence the sequence $\{x_n\}$ generated by (3.60) is well defined.

(III) Next, we show that $\{x_n\}$ is a convergent sequence in C. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - u, Jx_0 - Jx_n\rangle \geq 0, \quad \forall u \in C_n. \tag{3.8}$$

It follows from $\Omega \subset C_n$ for all $n \geq 1$ that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in \Omega. \tag{3.9}$$

From Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0),$$

for each $u \in \Omega \subset C_n$ and for all $n \geq 1$. Therefore the sequence $\{\phi(x_n, x_0)\}$ is bounded. Furthermore, since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \text{for all } n \geq 1.$$

This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing and hence $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Similarly, by Lemma 2.4, we have, for any positive integer m, that

$$\begin{aligned}
\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+m}, x_0) - \phi(x_n, x_0), \text{ for all } n \geq 1.
\end{aligned} \tag{3.10}$$

The existence of $\lim_{n\to\infty} \phi(x_n, x_0)$ implies that $\phi(x_{n+m}, x_n) \to 0$ as $n \to \infty$. From Lemma 2.2, we have

$$\|x_{n+m} - x_n\| \to 0, \text{ as } n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

(IV). Now, we will show that $p \in \Omega$. Indeed, taking $m = 1$ in (3.10), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

This implies that

$$\|x_{n+1} - x_{n-1}\| \le \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty.$$

Then

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_{n-1}\| = 0. \tag{3.13}$$

It follows from the last two inequalities that

$$\phi(x_{n+1}, x_{n-1}) \le \|x_{n+1}\|\|Jx_{n+1} - Jx_{n-1}\| + \|x_{n-1} - x_{n+1}\|\|x_{n-1}\| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

On the other hand, since $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \le \delta_n \phi(x_{n+1}, x_n) + (1-\delta_n)[\alpha_n \phi(x_{n+1}, x_{n-1}) + (1-\alpha_n)\phi(x_{n+1}, x_n) + (k_n - 1)\theta_n].$$

It follows from the condition (3.11), (3.14) and $\lim_{n\to\infty} k_n = 1$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.15}$$

Combining (3.12) and (3.15), we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.17}$$

On the other hand, noticing

$$\|Jy_n - Jz_n\| = \delta_n \|Jx_1 - Jz_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.19}$$

Using (3.12), (3.15) and (3.19) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.20}$$

Taking the constant $r = \sup_{n\ge1}\{\|x_{n+1}\|, \|T^n x_n\|, \|S^n w_n\|\}$, we have, from Lemma 2.6, that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying the inequality (2.4) and $g(0) = 0$.

Case I. Assume that (a) holds. Applying (2.4) and (3.5), we can calculate

$$\begin{aligned}
\phi(u, z_n) &= \phi(u, J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n)) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT^n x_n\rangle - 2\gamma_n\langle u, JS^n w_n\rangle \\
&\quad + \|\alpha_n Jx_{n-1} + \beta_n JT^n x_n + \gamma_n JS^n w_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT^n x_n\rangle - 2\gamma_n\langle u, JS^n w_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT^n x_n\|^2 + \gamma_n\|JS^n w_n\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT^n x_n\|) \\
&\le \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, T^n x_n) + \gamma_n\phi(u, S^n w_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT^n x_n\|) \\
&\le \alpha_n\phi(u, x_{n-1}) + \beta_n k_n\phi(u, x_n) + \gamma_n k_n\phi(u, w_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT^n x_n\|)
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\phi(u,x_{n-1})+\beta_nk_n\phi(u,x_n)+\gamma_nk_n\phi(u,x_n)\\
&\quad +2r_n\gamma_n\left(\frac{2}{c^2}r_n-\alpha\right)\|Ax_n-Au\|^2-\alpha_n\beta_ng(\|Jx_{n-1}-JT^nx_n\|)\\
&\le \alpha_n\phi(u,x_{n-1})+(1-\alpha_n)k_n\phi(u,x_n)+2r_n\gamma_n\left(\frac{2}{c^2}r_n-\alpha\right)\|Ax_n-Au\|^2\\
&\quad -\alpha_n\beta_ng(\|Jx_{n-1}-JT^nx_n\|). \qquad (3.21)
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_n\beta_ng(\|Jx_{n-1}-JT^nx_n\|) &\le \alpha_n[\phi(u,x_{n-1})-k_n\phi(u,x_n)]+k_n\phi(u,x_n)-\phi(u,z_n)\\
&\le \alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\phi(u,x_n)+\phi(u,x_n)-\phi(u,z_n)\\
&\le \alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n). \qquad (3.22)
\end{aligned}$$

We observe that

$$\begin{aligned}
&\alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n)\\
\le\ & \alpha_n[\|x_{n-1}\|^2-\|x_n\|^2-2\langle u,Jx_{n-1}-Jx_n\rangle]\\
&+\|x_n\|^2-\|z_n\|^2-2\langle u,Jx_n-Jz_n\rangle+(k_n-1)\theta_n\\
\le\ & \alpha_n[\|x_{n-1}-x_n\|(\|x_{n-1}\|+\|x_n\|)+2\|u\|\|Jx_{n-1}-Jx_n\|]\\
&+\|x_n-z_n\|(\|x_n\|+\|z_n\|)+2\|u\|\|Jx_n-Jz_n\|+(k_n-1)\theta_n.
\end{aligned}$$

It follows from (3.12), (3.17), (3.18) and (3.20) that

$$\lim_{n\to\infty}\alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n)=0. \qquad (3.23)$$

Applying $\liminf_{n\to\infty}\alpha_n\beta_n>0$ and (3.23) to (3.22), we get

$$\lim_{n\to\infty}g(\|Jx_{n-1}-JT^nx_n\|)=0.$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|Jx_{n-1}-JT^nx_n\|=0. \qquad (3.24)$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|x_{n-1}-T^nx_n\|=\lim_{n\to\infty}\|J^{-1}(Jx_{n-1})-J^{-1}(JT^nx_n)\|=0. \qquad (3.25)$$

From (3.12) and (3.25) , we have

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \qquad (3.26)$$

Note that

$$\|T^nx_n-p\|\le\|T^nx_n-x_n\|+\|x_n-p\|.$$

From (3.26), we obtain

$$\lim_{n\to\infty}\|T^nx_n-p\|=0. \qquad (3.27)$$

On the other hand, we have

$$\|T^{n+1}x_n-p\|\le\|T^{n+1}x_n-T^nx_n\|+\|T^nx_n-p\|\le\sup_{x\in\{x_n\}}\|T^{n+1}x-T^nx\|+\|T^nx_n-p\|.$$

The uniformly asymptotically regularity of T and the last inequality imply that

$$\lim_{n\to\infty}\|T^{n+1}x_n-p\|=0. \qquad (3.28)$$

That is $TT^nx_n\to p$ as $n\to\infty$. From the closedness of T, we see that $p\in F(T)$. In the same manner, we can apply the condition $\liminf_{n\to\infty}\alpha_n\gamma_n>0$ to conclude that

$$\lim_{n\to\infty}\|x_n-S^nw_n\|=0. \qquad (3.29)$$

Again, by (C2) and (3.21), we have

$$2\gamma_n\left(\alpha-\frac{2}{c^2}b\right)\|Ax_n-Au\|^2 \le \frac{1}{a}[\alpha_n[\phi(u,x_{n-1})-\phi(u,x_n)]$$

$$+(k_n-1)\theta_n+\phi(u,x_n)-\phi(u,z_n)]. \tag{3.30}$$

It follows from (3.23) and $\liminf_{n\to\infty}\gamma_n \geq \liminf_{n\to\infty}\beta_n\gamma_n > 0$ that

$$\liminf_{n\to\infty}\|Ax_n - Au\| \leq 0.$$

Since $\liminf_{n\to\infty}\|Ax_n - Au\| \geq 0$, we have

$$\lim_{n\to\infty}\|Ax_n - Au\| = 0. \tag{3.31}$$

From Lemma 2.4, Lemma 2.5, and (3.4), we have

$$\begin{aligned}
\phi(x_n,w_n) &= \phi(x_n,\Pi_C J^{-1}(Jx_n - r_nAx_n))\\
&\leq \phi(x_n,J^{-1}(Jx_n - r_nAx_n))\\
&= V(x_n,Jx_n - r_nAx_n)\\
&\leq V(x_n,(Jx_n - r_nAx_n)+r_nAx_n)\\
&\quad -2\langle J^{-1}(Jx_n - r_nAx_n)-x_n, r_nAx_n\rangle\\
&= \phi(x_n,x_n)+2\langle J^{-1}(Jx_n - r_nAx_n)-x_n, -r_nAx_n\rangle\\
&= 2\langle J^{-1}(Jx_n - r_nAx_n)-x_n, -r_nAx_n\rangle\\
&\leq \frac{4}{c^2}b^2\|Ax_n - Au\|^2.
\end{aligned} \tag{3.32}$$

It follows from (3.31) that

$$\lim_{n\to\infty}\phi(x_n,w_n) = 0. \tag{3.33}$$

Lemma 2.2 implies that

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.34}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|Jx_n - Jw_n\| = 0. \tag{3.35}$$

Combining (3.29) and (3.34), we also obtain

$$\lim_{n\to\infty}\|w_n - S^nw_n\| = 0. \tag{3.36}$$

Moreover

$$\|w_n - w_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|.$$

By (3.34) and (3.12), we have

$$\lim_{n\to\infty}\|w_n - w_{n+1}\| = 0. \tag{3.37}$$

Note that

$$\|S^nw_n - p\| \leq \|S^nw_n - w_n\| + \|w_n - x_n\| + \|x_n - p\|.$$

From (3.26), we obtain

$$\lim_{n\to\infty}\|S^nw_n - p\| = 0. \tag{3.38}$$

On the other hand, we have

$$\begin{aligned}
\|S^{n+1}w_n - p\| &\leq \|S^{n+1}w_n - S^nw_n\| + \|S^nw_n - p\|\\
&\leq \sup_{w\in\{w_n\}}\|S^{n+1}w - S^nw\| + \|S^nw_n - p\|.
\end{aligned}$$

The uniformly asymptotically regularity of S and the last inequality imply that

$$\lim_{n\to\infty}\|S^{n+1}w_n - p\| = 0. \tag{3.39}$$

That is $SS^nw_n \to p$ as $n\to\infty$. From the closedness of S, we see that $p \in F(S)$.

Case II. Assume that (b) holds. Using the inequalities (2.4) and (3.5), we obtain

$$\begin{aligned}
\phi(u,z_n) &= \phi(u,J^{-1}(\alpha_nJx_{n-1}+\beta_nJT^nx_n+\gamma_nJS^nw_n))\\
&= \|u\|^2 - 2\alpha_n\langle u,Jx_{n-1}\rangle - 2\beta_n\langle u,JT^nx_n\rangle - 2\gamma_n\langle u,JS^nw_n\rangle
\end{aligned}$$

$$
\begin{aligned}
& +\|\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n\|^2 \\
\leq\ & \|u\|^2-2\alpha_n\langle u, Jx_{n-1}\rangle-2\beta_n\langle u, JT^n x_n\rangle-2\gamma_n\langle u, JS^n w_n\rangle \\
& +\alpha_n\|Jx_{n-1}\|^2+\beta_n\|JT^n x_n\|^2+\gamma_n\|JS^n w_n\|^2 \\
& -\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|) \\
\leq\ & \alpha_n\phi(u, x_{n-1})+\beta_n\phi(u, T^n x_n)+\gamma_n\phi(u, S^n w_n) \\
& -\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|) \\
\leq\ & \alpha_n\phi(u, x_{n-1})+\beta_n k_n\phi(u, x_n)+\gamma_n k_n\phi(u, w_n) \\
& -\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|) \\
\leq\ & \alpha_n\phi(u, x_{n-1})+\beta_n k_n\phi(u, x_n)+\gamma_n k_n\phi(u, x_n) \\
& +2r_n\gamma_n(\frac{2}{c^2}r_n-\alpha)\|Ax_n-Au\|^2-\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|) \\
\leq\ & \alpha_n\phi(u, x_{n-1})+(1-\alpha_n)k_n\phi(u, x_n)+2r_n\gamma_n(\frac{2}{c^2}r_n-\alpha)\|Ax_n-Au\|^2 \\
& -\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|).
\end{aligned} \tag{3.40}
$$

This implies that

$$
\begin{aligned}
\beta_n\gamma_n g(\|JT^n x_n-JS^n w_n\|) &\leq \alpha_n[\phi(u, x_{n-1})-k_n\phi(u, x_n)]+k_n\phi(u, x_n)-\phi(u, z_n) \\
&\leq \alpha_n[\phi(u, x_{n-1})-\phi(u, x_n)]+(k_n-1)\phi(u, x_n)+\phi(u, x_n)-\phi(u, z_n) \\
&\leq \alpha_n[\phi(u, x_{n-1})-\phi(u, x_n)]+(k_n-1)\theta_n+\phi(u, x_n)-\phi(u, z_n).
\end{aligned} \tag{3.41}
$$

It follows from the condition $\liminf_{n\to\infty}\beta_n\gamma_n>0$ and (3.23) that

$$\lim_{n\to\infty} g(\|JT^n x_n-JS^n w_n\|)=0.$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|JT^n x_n-JS^n w_n\|=0. \tag{3.42}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|T^n x_n-S^n w_n\|=\lim_{n\to\infty}\|J^{-1}(JT^n x_n)-J^{-1}(JS^n w_n)\|=0. \tag{3.43}$$

On the other hand, we can calculate

$$
\begin{aligned}
\phi(T^n x_n, z_n) &= \phi(T^n x_n, J^{-1}(\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n)) \\
&= \|T^n x_n\|^2-2\langle T^n x_n, \alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n)\rangle \\
&\quad +\|\alpha_n Jx_{n-1}+\beta_n JT^n x_n+\gamma_n JS^n w_n)\|^2 \\
&\leq \|T^n x_n\|^2-2\alpha_n\langle T^n x_n, Jx_n\rangle-2\beta_n\langle T^n x_n, JT^n x_n\rangle-2\gamma_n\langle T^n x_n, JS^n w_n\rangle \\
&\quad +\alpha_n\|x_n\|^2+\beta_n\|T^n x_n\|^2+\gamma_n\|S^n w_n\|^2 \\
&\leq \alpha_n\phi(T^n x_n, x_n)+\gamma_n\phi(T^n x_n, S^n w_n).
\end{aligned} \tag{3.44}
$$

Observe that

$$
\begin{aligned}
\phi(T^n x_n, S^n w_n) &= \|T^n x_n\|^2-2\langle T^n x_n, JS^n w_n\rangle+\|S^n w_n\|^2 \\
&= \|T^n x_n\|^2-2\langle T^n x_n, JT^n x_n\rangle+2\langle T^n x_n, JT^n x_n-JS^n w_n\rangle+\|S^n w_n\|^2 \\
&\leq \|S^n w_n\|^2-\|T^n x_n\|^2+2\|T^n x_n\|\|JT^n x_n-JS^n w_n\| \\
&\leq \|S^n w_n-T^n x_n\|(\|S^n w_n\|+\|T^n x_n\|)+2\|T^n x_n\|\|JT^n x_n-JS^n w_n\|.
\end{aligned}
$$

It follows from (3.42) and (3.43) that

$$\lim_{n\to\infty}\phi(T^n x_n, S^n w_n)=0. \tag{3.45}$$

Applying $\lim_{n\to\infty}\alpha_n=0$ and (3.45) and the fact that $\{\phi(T^n x_n, x_n)\}$ is bounded to (3.44), we obtain

$$\lim_{n\to\infty}\phi(T^n x_n, z_n)=0. \tag{3.46}$$

From Lemma 2.2, one obtains

$$\lim_{n\to\infty}\|T^n x_n-z_n\|=0. \tag{3.47}$$

We observe that

$$\|T^n x_n - x_n\| \le \|T^n x_n - z_n\| + \|z_n - x_n\|.$$

This together with (3.20) and (3.47), we obtain

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.48}$$

From the uniformly asymptotically regularity of T, we obtain that

$$\lim_{n\to\infty} \|T^{n+1} x_n - p\| = 0. \tag{3.49}$$

That is $TT^n x_n \to p$ as $n \to \infty$. From the closedness of T, we see that $p \in F(T)$. By the same proof as in Case I, we obtain that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.50}$$

Hence $w_n \to p$ as $n \to \infty$ for each $i \in I$ and

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \tag{3.51}$$

Combining (3.43), (3.48) and (3.50), we also have

$$\lim_{n\to\infty} \|S^n w_n - w_n\| = 0. \tag{3.52}$$

From the uniformly asymptotically regularity of S, we obtain that

$$\lim_{n\to\infty} \|S^{n+1} w_n - p\| = 0. \tag{3.53}$$

That is $SS^n w_n \to p$ as $n \to \infty$. From the closedness of S, we see that $p \in F(S)$.

We next show that $p \in VI(C, A)$.

Let $\Psi \subset E \times E^*$ be an operator defined by :

$$\Psi v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.54}$$

By Lemma 2.7, Ψ is maximal monotone and $\Psi^{-1}0 = VI(A, C)$. Let $(v, w) \in G(\Psi)$, since $w \in \Psi v = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $x_n = \Pi_{C_n} x \in C_n \subset C$, we get

$$\langle v - x_n, w - Av\rangle \quad \ge \quad 0. \tag{3.55}$$

Since A is α-inverse-strong monotone, we have

$$\begin{aligned} \langle v - x_n, w\rangle \quad &\ge \quad \langle v - x_n, Av\rangle \\ &= \quad \langle v - x_n, Av - Ax_n\rangle + \langle v - x_n, Ax_n\rangle \\ &\ge \quad \langle v - x_n, Ax_n\rangle. \end{aligned} \tag{3.56}$$

On other hand, from $w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n)$ and Lemma 2.3, we have $\langle v - w_n, Jw_n - (Jx_n - r_n Ax_n\rangle \ge 0$, and hence

$$\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\rangle \le 0. \tag{3.57}$$

Because A is $\frac{1}{\alpha}$ constricted, it holds from (3.56) and (3.57) that

$$\begin{aligned} \langle v - x_n, w\rangle \quad &\ge \quad \langle v - x_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\rangle \\ &= \quad \langle v - w_n, Ax_n\rangle + \langle w_n - x_n, Ax_n\rangle - \langle v - w_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\rangle \\ &= \quad \langle w_n - x_n, Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\rangle \\ &\ge \quad -\|w_n - x_n\|.\|Ax_n\| - \|v - w_n\|.\frac{\|Jx_n - Jw_n\|}{a}, \end{aligned} \tag{3.58}$$

$\forall n \in \mathbb{N} \cup \{0\}$. By taking the limit as $n \to \infty$ in (3.58) and from (3.34) and (3.35), we have $\langle v - p, w\rangle \ge 0$ as $n \to \infty$. By the maximality of Ψ we obtain $p \in \Psi^{-1}0$ and hence $p \in VI(A, C)$. Hence we conclude that

$$p \in \Omega := F(T) \cap F(S) \cap VI(A, C).$$

Finally, we show that $p = \Pi_\Omega x_0$. Indeed, taking the limit as $n \to \infty$ in (3.9), we obtain

$$\langle p - z, Jx_0 - Jp\rangle \geq 0, \qquad \forall z \in \Omega \tag{3.59}$$

and hence $p = \Pi_\Omega x_0$ by Lemma 2.3. This complete the proof.

■

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T be a uniformly L-Lipschitzian continuous and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^T\} \subset [0,1)$ such that $k_n^T \to 1$ as $n \to \infty$ and S be a uniformly L-Lipschitzian continuous and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^S\} \subset [0,1)$ such that $k_n^S \to 1$ as $n \to \infty$. Assume that $\Omega := F(T) \cap F(S) \cap VI(A,C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by (3.60). Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions (C1), (C2) and (C3) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.*

Proof. Following the proof lines of Theorem 3.1, we can obtain the following facts :
(i) Ω is closed and convex;
(ii) $\Omega \subset C_n$ for all $n \geq 1$;
(iii) $\{x_n\}$ is a Cauchy sequence and then $\{x_n\}$ and $\{w_n\}$ are convergent sequences in C;
(iv) $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0 = \lim_{n\to\infty} \|w_n - w_{n+1}\|$;
(v) $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \|w_n - S^n w_n\|$.
It is sufficient to show that $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \|S^{n+1}w_n - S^n w_n\|$. We observe that

$$\begin{aligned}\|T^{n+1}x_n - T^n x_n\| &\leq \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\leq (1+L)\|x_{n+1} - x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - T^n x_n\|.\end{aligned}$$

This implies that $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0$. Similarly, we can show that $\lim_{n\to\infty} \|S^{n+1}w_n - S^n w_n\| = 0$. By Theorem 3.1, we have the desired conclusion. This completes the proof.

■

From the definition of quasi-ϕ-nonexpansive mappings, we see that every quasi-ϕ-nonexpansive mapping is asymptotically quasi-ϕ-nonexpansive with the constant sequence $\{1\}$. From the proof of Theorem 3.1, we have the following results immediately.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let T and S be a closed and quasi-ϕ-nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap VI(A,C)$ is nonempty and bounded. Let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in \Omega$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$\begin{cases} x_0 = x \in E, \text{ chosen arbitrary}, \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n JTx_n + \gamma_n JSw_n), \\ y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)\xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \qquad \forall n \geq 1, \end{cases} \tag{3.60}$$

where $\xi_n = \alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)$, and $\theta_n = \sup\{\phi(z, x_n) : z \in \Omega\}$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

- (C1) $\lim_{n\to\infty} \delta_n = 0$;
- (C2) $\{r_n\} \subset [a,b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;*
- (C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*
  - (a) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*
  - (b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$*, where* Π_Ω *is the generalized projection from* E *onto* Ω*.*

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ภาคผนวก 2

A General composite algorithms for solving general equilibrium problems and fixed point problems in Hilbert spaces

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Research Article

A General Composite Algorithms for Solving General Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

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We introduce a general composite algorithm for finding a common element of the set of solutions of a general equilibrium problem and the common fixed point set of a finite family of asymptotically nonexpansive mappings in the framework of Hilbert spaces. Strong convergence of such iterative scheme is obtained which solving some variational inequalities for a strongly monotone and strictly pseudocontractive mapping. Our results extend the corresponding recent results of Yao and Liou (2010).

1. Introduction

Let C be a nonempty, closed, convex subset of a real Hilbert space H. Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, for all $x, y \in C$. It is clear that any α-inverse-strongly monotone mapping is monotone and $1/\alpha$-Lipschitz continuous. Let $f : C \to H$ be a ρ-contraction, that is, there exists a constant $\rho \in [0,1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$ and asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\left\|S^n x - S^n y\right\| \leq (1 + k_n)\left\|x - y\right\|, \quad \forall x, y \in C. \tag{1.1}$$

Denote the set of fixed points of S by $\mathrm{Fix}(S)$. For asymptotically nonexpansive self-map S, it is well known that $\mathrm{Fix}(S)$ is closed and convex (see, e.g., [1]).

The class of asymptotically nonexpansive mappings which is an important generalization of that of nonexpansive mappings was introduced by Goebel and Kirk [1]. They established that if C is a nonempty, closed, convex, bounded subset of a uniformly convex Banach space E and S is an asymptotically nonexpansive self-mapping of C, then S has a fixed point in C.

Let $A : C \to H$ be a nonlinear mapping and $\phi : C \times C \to \mathbb{R}$ a bifunction. Consider a general equilibrium problem:

$$\text{Find } z \in C \text{ such that } \phi(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of all solutions of the general equilibrium problem (1.2) is denoted by EP, that is,

$$\mathrm{EP} = \{z \in C : \phi(z, y) + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\}. \tag{1.3}$$

If $A = 0$, then (1.2) reduces to the following equilibrium problem of finding $z \in C$ such that

$$\phi(z, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $\phi = 0$, then (1.2) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

We note that the problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others. See, for example, [2–5].

In 2005, Combettes and Hirstoaga [6] introduced an iterative algorithm of finding the best approximation to the initial data and proved a strong convergence theorem. In 2007, by using the viscosity approximation method, S. Takahashi and W. Takahashi [7] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping. Subsequently, algorithms constructed for solving the equilibrium problems and fixed point problems have further developed by some authors. In particular, Ceng and Yao [8] introduced an iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem (1.2) and the set of common fixed points of finitely many nonexpansive mappings. Maingé and Moudafi [9] introduced an iterative algorithm for equilibrium problems and fixed point problems. Yao et al. [10] considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of an infinite nonexpansive mappings. Noor et al. [11] introduced an iterative method for solving fixed point problems and variational inequality problems. Wangkeeree [12] introduced a new iterative scheme for finding the common element of the set of common fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of the variational inequality. Wangkeeree and Kamraksa [13] introduced an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Their results extend and improve many results in the literature. For some works related to the equilibrium problem, fixed point problems, and the variational inequality problem, please see [1–57] and the references therein.

However, we note that all constructed algorithms in [7, 9–13, 16, 57] do not work to find the minimum-norm solution of the corresponding fixed point problems and the equilibrium problems. Very recently, Yao and Liou [46] purposed some algorithms for finding the minimum-norm solution of the fixed point problems and the equilibrium problems. They first suggested two new composite algorithms (one implicit and one explicit) for solving the above minimization problem. To be more precisely, let C be a nonempty, closed, convex subset of H, $\phi : C \times C \to \mathbb{R}$ a bifunction satisfying certain conditions, and $S : C \to C$ a nonexpansive mapping such that $\Omega := \mathrm{Fix}(S) \cap \mathrm{EP} \neq \emptyset$. Let f be a contraction on a Hilbert space H. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{gathered}\phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ x_{n+1} = \mu_n P_C\left[\alpha_n f(x_n) + (1-\alpha_n)Sx_n\right] + (1-\mu_n)u_n, \quad n \geq 0,\end{gathered} \tag{1.6}$$

where A is an α-inverse strongly monotone mapping. They proved that if $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in [0,1] satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(ii) $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$ and $\lim_{n\to\infty}((\mu_{n+1}-\mu_n)/\alpha_{n+1}) = 0$, then, the sequence $\{x_n\}$ generated by (1.6) converges strongly to $x^* \in \Omega$ which is the unique solution of variational inequality

$$\langle (I-f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega. \tag{1.7}$$

In particular, if we take $f = 0$ in (1.6), then the sequence $\{x_n\}$ generated by

$$\begin{gathered}\phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ x_{n+1} = \mu_n P_C[(1-\alpha_n)Sx_n] + (1-\mu_n)u_n, \quad n \geq 0,\end{gathered} \tag{1.8}$$

converges strongly to a solution of the minimization problem which is the problem of finding x^* such that

$$x^* = \arg\min_{x\in\Omega}\|x\|^2, \tag{1.9}$$

where Ω stands for the intersection set of the solution set of the general equilibrium problem and the fixed points set of a nonexpansive mapping.

On the other hand, iterative approximation methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [25, 43, 44] and the references therein. Let B be a strongly positive bounded linear operator on H, that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2 \quad \forall\, x \in H. \tag{1.10}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H

$$\min_{x\in \mathrm{Fix}(S)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{1.11}$$

where b is a given point in H. In 2003, Xu [43] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n u, \quad n \geq 0, \tag{1.12}$$

converges strongly to the unique solution of the minimization problem (1.11) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [29] introduced the following iterative process for nonexpansive mappings (see [43] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved [29, 43] that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.13) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in H. \tag{1.14}$$

Recently, Marino and Xu [28] mixed the iterative method (1.12) and the viscosity approximation method (1.13) introduced by Moudafi [29] and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.15}$$

where B is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the certain conditions, then the sequence $\{x_n\}$ generated by (1.15) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in H \tag{1.16}$$

which is the optimality condition for the minimization problem: $\min_{x\in \mathrm{Fix}(S)}(1/2)\langle Bx, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Recall that a mapping $F : H \to H$ is called δ-*strongly monotone* if there exists a positive constant δ such that

$$\langle Fx - Fy, x - y\rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in H. \tag{1.17}$$

Recall also that a mapping F is called λ-*strictly pseudocontractive* if there exists a positive constant λ such that

$$\langle Fx - Fy, x - y\rangle \leq \|x - y\|^2 - \lambda\|(x - y) - (Fx - Fy)\|^2, \quad \forall x, y \in H. \tag{1.18}$$

It is easy to see that (1.18) can be rewritten as

$$\langle (I - F)x - (I - F)y, x - y\rangle \geq \lambda\|(I - F)x - (I - F)y\|^2. \tag{1.19}$$

Remark 1.1. If F is a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$, then F is $\overline{\gamma}$-strongly monotone and 12-strictly pseudocontractive. In fact, since F is a strongly positive, bounded, linear operator with coefficient $\overline{\gamma}$, we have

$$\langle Fx - Fy, x - y\rangle = \langle F(x - y), x - y\rangle \geq \overline{\gamma}\ \|x - y\|^2. \tag{1.20}$$

Therefore, F is $\overline{\gamma}$-strongly monotone. On the other hand,

$$\begin{aligned}\|(I - F)x - (I - F)y\|^2 &= \langle (x - y) - (Fx - Fy), (x - y) - (Fx - Fy)\rangle \\ &= \langle x - y, x - y\rangle - 2\langle Fx - Fy, x - y\rangle + \langle Fx - Fy, Fx - Fy\rangle \\ &= \|x - y\|^2 - 2\langle Fx - Fy, x - y\rangle + \|Fx - Fy\|^2 \\ &\leq \|x - y\|^2 - 2\langle Fx - Fy, x - y\rangle + \|F\|^2\|x - y\|^2.\end{aligned} \tag{1.21}$$

Since F is strongly positive if and only if $(1/\|F\|)F$ is strongly positive, we may assume, without loss of generality, that $\|F\| = 1$. From (1.21), we have

$$\begin{aligned}\langle Fx - Fy, x - y\rangle &\leq \|x - y\|^2 - \frac{1}{2}\|(I - F)x - (I - F)y\|^2 \\ &= \|x - y\|^2 - \frac{1}{2}\|(x - y) - (Fx - Fy)\|^2.\end{aligned} \tag{1.22}$$

Hence, F is 12-strictly pseudocontractive.

In this paper, motivated by the above results, we introduce a general iterative scheme below in a real Hilbert space H, with the initial guess $x_0 \in C$ chosen arbitrary:

$$\begin{aligned}&\phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &y_n = \alpha_n\gamma f(x_n) + (I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n, \\ &x_{n+1} = \mu_n P_C[y_n] + (1 - \mu_n)u_n, \quad n \geq 0,\end{aligned} \tag{1.23}$$

where $p(n) = j + 1$ if $jN < n \leq (j + 1)N$, $j = 1, 2, \ldots$ and $n = jN + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, C is a nonempty, closed, convex subset of H, $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in [0,1],

$\phi : C \times C \to \mathbb{R}$ is a bifunction satisfying certain conditions, $S_1, S_2, \ldots, S_N : C \to C$ is a finite family of asymptotically nonexpansive mappings with sequences $\{1 + k_{p(n)}^{i(n)}\}$, respectively, $f : C \to H$ is a contraction with coefficient $0 < \rho < 1$, F is δ-strongly monotone and λ-strictly pseudocontractive with $\delta + \lambda > 1$, γ is a positive real number such that $\gamma < (1/\rho)(1 - \sqrt{(1-\delta)/\lambda})$, and A is an α-inverse strongly monotone mapping. We prove that the proposed algorithm converges strongly to $x^* \in \Omega$ which is the unique solution of the following variational inequality:

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \Omega. \tag{1.24}$$

In particular,

(I) if F is a strongly positive bounded linear operator on H, then x^* is the unique solution of the variational inequality (1.16),

(II) if $F = I$, the identity mapping on H and $\gamma = 1$, then x^* is the unique solution of the variational inequality (1.14),

(III) if $F = I$, the identity mapping on H and $f = 0$, then x^* is the unique solution of minimization problem (1.9).

The results presented in this paper extend and improve the main results in Yao and Liou [46], Marino and Xu [28], and many others.

2. Preliminaries

Let C be a nonempty, closed, convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \tag{2.2}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered} \langle x - P_C x, y - P_C x \rangle \leq 0, \\ \|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \end{gathered} \tag{2.3}$$

for all $x \in H, y \in C$. For more details, see [39]. We will make use of the following well-known result.

Lemma 2.1. *Let H be a Hilbert space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \tag{2.4}$$

Throughout this paper, we assume that a bifunction $\phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$,

(A2) ϕ is monotone, that is, $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0}\phi(tz + (1 - t)x, y) \leq \phi(x, y)$,

(A4) for each $x \in C$, the mapping $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

Lemma 2.2 (see [6]). *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)–(A4). Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

Further, if $T_r(x) = \{z \in C : \phi(z, y) + (1/r)\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following hold:

(i) *T_r is single-valued and T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \tag{2.6}$$

(ii) EP *is closed and convex and* EP = Fix(T_r).

Lemma 2.3 (see [30]). *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ a constant. Then, one has*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.7}$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.4 (see [45]). *Let S be an asymptotically nonexpansive mapping defined on a bounded, closed, convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\|Sx_n - x_n\| \to 0$ as $n \to \infty$, then $x \in \mathrm{Fix}(S)$.*

Lemma 2.5 (see [44]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0, \tag{2.8}$$

where $\{\alpha_n\}, \{\sigma_n\}$, and $\{\gamma_n\}$ are nonnegative real sequences satisfying the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $\limsup_{n\to\infty}\sigma_n \leq 0$,

(iii) $\sum_{n=1}^{\infty}\gamma_n < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

Lemma 2.6 (see [41]). *Let E be a strictly convex Banach space and C a closed, convex subset of E. Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of nonexpansive mappings of C into itself such that the set of common fixed points of $S_1, S_2, \ldots, S_N$ is nonempty. Let $T_1, T_2, \ldots, T_N : C \to C$ be mappings given by*

$$T_i = (1 - \alpha_i) I + \alpha_i S_i, \quad \forall i = 1, 2, \ldots, N, \tag{2.9}$$

where I denotes the identity mapping on C. Then, the finite family $\{T_1, T_2, \ldots, T_N\}$ satisfies the following:

$$\begin{gathered}\bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \bigcap_{i=1}^{N} \operatorname{Fix}(S_i), \\ \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_N T_{N-1} T_{N-2} \cdots T_1) = \operatorname{Fix}(T_1 T_N \cdots T_2) = \operatorname{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{gathered} \tag{2.10}$$

The following lemma can be found in [35, Lemma 2.7]. For the sake of the completeness, we include its proof in a Hilbert space's version.

Lemma 2.7. *Let H be a real Hilbert space and $F : H \to H$ a mapping.*

(i) *If F is δ-strongly monotone and λ-strictly pseudocontractive with $\delta + \lambda > 1$, then $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.*

(ii) *If F is δ-strongly monotone and λ-strictly pseudocontractive with $\delta + \lambda > 1$, then for any fixed number $\tau \in (0,1)$, $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1-\delta)/\lambda})$.*

Proof. (i) For any $x, y \in H$, we have

$$\lambda \|(I - F)x - (I - F)y\|^2 \le \|x - y\|^2 - \langle Fx - Fy, x - y \rangle \le (1 - \delta)\|x - y\|^2, \quad \forall x, y \in H. \tag{2.11}$$

Thus,

$$\|(I - F)x - (I - F)y\| \le \sqrt{\frac{1-\delta}{\lambda}} \|x - y\|, \quad \forall x, y \in H. \tag{2.12}$$

Since $\delta + \lambda > 1$, we have $(1-\delta)/\lambda \in (0,1)$. Hence, $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.

(ii) Since $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$, we have for any $\tau \in (0,1)$,

$$\begin{aligned}\|x - y - \tau(Fx - Fy)\| &= \|(1-\tau)(x - y) + \tau[(I - F)x - (I - F)y]\| \\ &\le (1 - \tau)\|x - y\| + \tau\|(I - F)x - (I - F)y\|\end{aligned}$$

$$\leq (1-\tau)\|x-y\| + \tau\sqrt{\frac{1-\delta}{\lambda}}\|x-y\|$$
$$= \left(1-\tau\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|, \quad \forall x,y \in H. \tag{2.13}$$

Hence, $I-\tau F$ is contractive with constant $1-\tau(1-\sqrt{(1-\delta)/\lambda})$. □

Lemma 2.8. *Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{1+k_{p(n)}^{i(n)}\}$, respectively, such that $k_{p(n)}^{i(n)} \to 0$ as $n \to \infty$. Then, there exists a sequence $\{h_n\} \subset [0,\infty)$ with $h_n \to 0$ as $n \to \infty$ such that*

$$\left\|S_{i(n)}^{p(n)}x - S_{i(n)}^{p(n)}y\right\| \leq (1+h_n)\|x-y\|, \quad \forall x,y \in C, \tag{2.14}$$

where $p(n) = j+1$ if $jN < n \leq (j+1)N$, $j = 1,2,\ldots$ and $n = jN + i(n)$; $i(n) \in \{1,2,\ldots,N\}$.

Proof. Define the sequence $\{h_n\}$ by $h_n := \max\{k_{p(n)}^{i(n)} : 1 \leq i(n) \leq N\}$ and the result follows immediately. □

In the rest of our discussion in this paper, we will assume that $p(n) = j+1$ if $jN < n \leq (j+1)N$, $j = 1,2,\ldots$ and $n = jN + i(n)$; $i(n) \in \{1,2,\ldots,N\}$ and $h_n := \max\{k_{p(n)}^{i(n)} : 1 \leq i(n) \leq N\}$ for all $n \geq 1$, and for each $n \geq 1$, $n = (p(n)-1)N + i(n)$.

3. Main Results

Now, we are a position to state and prove our main results.

Theorem 3.1. *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Let $S_1, S_2, \ldots, S_N : C \to C$ be a finite family of asymptotically nonexpansive mappings with sequences $\{1+k_{p(n)}^{i(n)}\}$, respectively, such that $k_{p(n)}^{i(n)} \to 0$ as $n \to \infty$, $h_n := \max_{1\leq i(n)\leq N}\{k_{p(n)}^{i(n)}\}$ and $\Gamma := \bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$,*

$$\Gamma = \mathrm{Fix}(S_N S_{N-1} S_{N-2} \cdots S_1) = \mathrm{Fix}(S_1 S_N \cdots S_2) = \cdots = \mathrm{Fix}(S_{N-1} S_{N-2} \cdots S_1 S_N). \tag{3.1}$$

Let $A : C \to H$ be an α-inverse strongly monotone mapping. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)–(A4) such that $\Omega := \mathrm{EP} \cap \Gamma$ is nonempty. Let $F : C \to H$ be δ-strongly monotone and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : C \to H$ a ρ-contraction, γ a positive real number such that $\gamma < (1-\sqrt{(1-\delta)/\lambda})/\rho$, and r a constant such that $r \in (0, 2\alpha)$. For $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by (1.23)*. Suppose that $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(h_n/\alpha_n) = 0$,

(C2) $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$ *and* $\lim_{n\to\infty}((\mu_{n+1}-\mu_n)/\alpha_{n+1}) = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{z\in B}\|S_{i(n+1)}^{p(n+1)}z - S_{i(n)}^{p(n)}z\| < \infty$, *for each bounded subset* B *of* C. *Then, the sequence* $\{x_n\}$ *converges strongly to* x^* *of the following variational inequality:*

$$\langle (F-\gamma f)x^*, x-x^*\rangle \geq 0, \quad x\in\Omega \tag{3.2}$$

or equivalently $\tilde{x} = P_\Omega(I-F+\gamma f)\tilde{x}$, *where* P_Ω *is the metric projection of* H *onto* Ω.

Proof. First, we rewrite the sequence $\{x_n\}$ by the following:

$$x_{n+1} = \mu_n P_C[y_n] + (1-\mu_n)T_r(x_n - rAx_n), \quad n\geq 0, \tag{3.3}$$

where the mapping T_r is defined in Lemma 2.2. Pick $z\in\Omega$ and $u_n = T_r(x_n - rAx_n)$. The nonexpansivity of T_r and $I-rA$ implies that

$$\begin{aligned}\|u_n - z\| &= \|T_r(x_n - rAx_n) - T_r(z-rAz)\| \\ &\leq \|x_n - z\|, \quad \forall z\in\Omega.\end{aligned} \tag{3.4}$$

Setting $\overline{\gamma} := (1-\sqrt{(1-\delta)/\lambda})$ and using Lemma 2.7(ii), we have

$$\begin{aligned}\|y_n - z\| &= \left\|\alpha_n\gamma(f(x_n) - Fz) + (I-\alpha_n F)\left(S_{i(n+1)}^{p(n+1)}x_n - z\right)\right\| \\ &\leq \alpha\alpha_n\gamma\|x_n - z\| + \alpha_n\left\|\gamma f(x_n) - Fz\right\| + (1-\alpha_n\overline{\gamma})(1+h_{n+1})\|x_n - z\| \\ &= \left[1-\alpha_n(\overline{\gamma}-\alpha\gamma) + (1-\alpha_n\overline{\gamma})h_{n+1}\right]\|x_n - z\| + \alpha_n\left\|\gamma f(x_n) - Fz\right\|.\end{aligned} \tag{3.5}$$

By our assumptions, we have $(1-\alpha_n\overline{\gamma})(h_{n+1}/\alpha_n) \to 0$ as $n\to\infty$. We can assume, without loss of generality, that $(1-\alpha_n\overline{\gamma})(h_{n+1}/\alpha_n) < (1/2)(\overline{\gamma}-\alpha\gamma)$. Applying Lemma 2.7, we can calculate the following:

$$\begin{aligned}\|x_{n+1} - z\| &= \|\mu_n(P_C[y_n] - z) + (1-\mu_n)(u_n - z)\| \\ &\leq \mu_n\left\|P_C[y_n] - z\right\| + (1-\mu_n)\|u_n - z\| \\ &\leq \mu_n\left\|y_n - z\right\| + (1-\mu_n)\|x_n - z\| \\ &\leq \mu_n\left[1-\alpha_n(\overline{\gamma}-\alpha\gamma) + (1-\alpha_n\overline{\gamma})h_{n+1}\right]\|x_n - z\| \\ &\quad + \mu_n\alpha_n\left\|\gamma f(x_n) - Fz\right\| + (1-\mu_n)\|x_n - z\| \\ &= \left[1-\mu_n\alpha_n\left[(\overline{\gamma}-\alpha\gamma) - (1-\alpha_n\overline{\gamma})\frac{h_{n+1}}{\alpha_n}\right]\right]\|x_n - z\| + \mu_n\alpha_n\left\|\gamma f(x_n) - Fz\right\| \\ &\leq \left[1-\frac{1}{2}\mu_n\alpha_n(\overline{\gamma}-\alpha\gamma)\right]\|x_n - z\| + \frac{\mu_n\alpha_n(1/2)(\overline{\gamma}-\alpha\gamma)}{(1/2)(\overline{\gamma}-\alpha\gamma)}\left\|\gamma f(x_n) - Fz\right\|.\end{aligned} \tag{3.6}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{2\|\gamma f(x_0) - F(z)\|}{\overline{\gamma} - \alpha\gamma}\right\}. \tag{3.7}$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{f(x_n)\}$, and $\{y_n\}$ are all bounded.

Next, we show that

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0. \tag{3.8}$$

From (1.23), we have

$$\begin{aligned}
\|y_{n+N} - y_{n+N-1}\| &= \Big\|\alpha_{n+N}\gamma f(x_{n+N}) + (I - \alpha_{n+N}F)S_{i(n+N+1)}^{p(n+N+1)} x_{n+N} \\
&\quad -\alpha_{n+N-1}\gamma f(x_{n+N-1}) - (I - \alpha_{n+N-1}F)S_{i(n+N)}^{p(n+N)} x_{n+N-1}\Big\| \\
&= \Big\|\alpha_{n+N}\gamma\big(f(x_{n+N}) - f(x_{n+N-1})\big) + (\alpha_{n+N} - \alpha_{n+N-1})\gamma f(x_{n+N-1}) \\
&\quad + (I - \alpha_{n+N}F)\Big(S_{i(n+N+1)}^{p(n+N+1)} x_{n+N} - S_{i(n+N+1)}^{p(n+N+1)} x_{n+N-1}\Big) \\
&\quad + [(I - \alpha_{n+N}F) - (I - \alpha_{n+N-1}F)]S_{i(n+N+1)}^{p(n+N+1)} x_{n+N-1} \\
&\quad +(I - \alpha_{n+N-1}F)\Big(S_{i(n+N+1)}^{p(n+N+1)} x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\Big)\Big\| \\
&\leq \alpha_{n+N}\gamma\alpha\|x_{n+N} - x_{n+N-1}\| + |\alpha_{n+N} - \alpha_{n+N-1}|\gamma\|f(x_{n+N-1})\| \\
&\quad + \big(1 - \alpha_{n+N}\overline{\gamma}\big)(1 + h_{n+N+1})\|x_{n+N} - x_{n+N-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n+N}|\|F\|\Big\|S_{i(n+N+1)}^{p(n+N+1)} x_{n+N-1}\Big\| \\
&\quad + \big(1 - \alpha_{n+N-1}\overline{\gamma}\big)\Big\|S_{i(n+N+1)}^{p(n+N+1)} x_{n+N-1} - S_{i(n+N)}^{p(n+N)} x_{n+N-1}\Big\| \\
&\leq \alpha_{n+N}\gamma\alpha\|x_{n+N} - x_{n+N-1}\| + |\alpha_{n+N} - \alpha_{n+N-1}|\gamma\|f(x_{n+N-1})\| \\
&\quad + \big(1 - \alpha_{n+N}\overline{\gamma}\big)(1 + h_{n+N+1})\|x_{n+N} - x_{n+N-1}\| \\
&\quad + |\alpha_{n+N-1} - \alpha_{n+N}|\|F\|\Big\|S_{i(n+N+1)}^{p(n+N+1)} x_{n+N-1}\Big\| \\
&\quad + \sup_{x\in\{x_n:n\in\mathbb{N}\}}\Big\|S_{i(n+N+1)}^{p(n+N+1)} x - S_{i(n+N)}^{p(n+N)} x\Big\|,
\end{aligned} \tag{3.9}$$

and from (3.3), we have

$$\begin{aligned}\|x_{n+N+1}-x_{n+N}\| &= \big\|\mu_{n+N}P_C[y_{n+N}]+(1-\mu_{n+N})u_{n+N}-\mu_{n+N-1}P_C[y_{n+N-1}]\\&\quad-(1-\mu_{n+N-1})u_{n+N-1}\big\|\\&= \big\|\mu_{n+N}(P_C[y_{n+N}]-P_C[y_{n+N-1}])+(\mu_{n+N}-\mu_{n+N-1})P_C[y_{n+N-1}]\\&\quad+(1-\mu_{n+N})(u_{n+N}-u_{n+N-1})+(\mu_{n+N-1}-\mu_{n+N})u_{n+N-1}\big\|\\&\leq \mu_{n+N}\big\|y_{n+N}-y_{n+N-1}\big\|+(1-\mu_{n+N})\|u_{n+N}-u_{n+N-1}\|\\&\quad+\big|\mu_{n+N}-\mu_{n+N-1}\big|\left(\big\|P_C[y_{n+N-1}]\big\|+\|u_{n+N-1}\|\right),\\\|u_{n+N}-u_{n+N-1}\| &= \|T_r(x_{n+N}-rAx_{n+N})-T_r(x_{n+N-1}-rAx_{n+N-1})\|\\&\leq \|(x_{n+N}-rAx_{n+N})-(x_{n+N-1}-rAx_{n+N-1})\|\\&\leq \|x_{n+N}-x_{n+N-1}\|.\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}\|x_{n+N+1}-x_{n+N}\| &\leq \mu_{n+N}\alpha_{n+N}\gamma\alpha\|x_{n+N}-x_{n+N-1}\|+\mu_{n+N}|\alpha_{n+N}-\alpha_{n+N-1}|\gamma\big\|f(x_{n+N-1})\big\|\\&\quad+\mu_{n+N}(1-\alpha_{n+N}\overline{\gamma})(1+h_{n+N+1})\|x_{n+N}-x_{n+N-1}\|\\&\quad+\mu_{n+N}|\alpha_{n+N-1}-\alpha_{n+N}|\|F\|\left\|S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1}\right\|\\&\quad+\mu_{n+N}\sup_{x\in\{x_n:n\in\mathbb{N}\}}\left\|S_{i(n+N+1)}^{p(n+N+1)}x-S_{i(n+N)}^{p(n+N)}x\right\|\\&\quad+(1-\mu_{n+N})\|x_{n+N}-x_{n+N-1}\|\\&\quad+\big|\mu_{n+N}-\mu_{n+N-1}\big|\left(\big\|P_C[y_{n+N-1}]\big\|+\|u_{n+N-1}\|\right)\\&\leq (1-\mu_{n+N}\alpha_{n+N}(\overline{\gamma}-\gamma\alpha))\|x_{n+N}-x_{n+N-1}\|+\mu_{n+N}\alpha_{n+N}\\&\quad\times\Bigg[\left(\frac{h_{n+N+1}}{\alpha_{n+N}}+h_{n+N+1}\overline{\gamma}\right)M\\&\qquad+\left|1-\frac{\alpha_{n+N-1}}{\alpha_{n+N}}\right|\gamma\big\|f(x_{n+N-1})\big\|+\left|\frac{\alpha_{n+N-1}}{\alpha_{n+N}}-1\right|\|F\|\left\|S_{i(n+N+1)}^{p(n+N+1)}x_{n+N-1}\right\|\\&\qquad+\frac{1}{\mu_{n+N}}\left|\frac{\mu_{n+N}-\mu_{n+N-1}}{\alpha_{n+N}}\right|\left(\big\|P_C[y_{n+N-1}]\big\|+\|u_{n+N-1}\|\right)\Bigg]\\&\quad+\sup_{x\in\{x_n:n\in\mathbb{N}\}}\left\|S_{i(n+N+1)}^{p(n+N+1)}x-S_{i(n+N)}^{p(n+N)}x\right\|.\end{aligned} \tag{3.11}$$

By Lemma 2.5, we obtain that

$$\lim_{n\to\infty}\|x_{n+N+1}-x_{n+N}\|=0. \tag{3.12}$$

Furthermore,

$$\|x_{n+N} - x_n\| \le \|x_{n+N} - x_{n+N-1}\| + \|x_{n+N-1} - x_{n+N-2}\| + \cdots + \|x_{n+1} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Hence,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0. \tag{3.14}$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.15}$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\mu_n (P_C[y_n] - z) + (1-\mu_n)(u_n - z)\right\|^2 \\
&\le \mu_n \left\|P_C[y_n] - z\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&\le \mu_n \left\|y_n - z\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n \left\|\alpha_n \gamma f(x_n) + (I - \alpha_n F) S_{i(n+1)}^{p(n+1)} x_n - z\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n \left\|\alpha_n \gamma f(x_n) - \alpha_n F(z) + (I - \alpha_n F) S_{i(n+1)}^{p(n+1)} x_n - (I - \alpha_n F) z\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&\le \mu_n \left\|(I - \alpha_n F) S_{i(n+1)}^{p(n+1)} x_n - (I - \alpha_n F) z\right\|^2 + \mu_n \alpha_n^2 \left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\mu_n \alpha_n \left\langle (I - \alpha_n F) S_{i(n+1)}^{p(n+1)} x_n - (I - \alpha_n F) z, \gamma f(x_n) - F(z) \right\rangle + (1-\mu_n)\|u_n - z\|^2 \\
&\le \mu_n (1 - \alpha_n \overline{\gamma})^2 (1 + h_{n+1})^2 \|x_n - z\|^2 + \mu_n \alpha_n^2 \left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\alpha_n \mu_n (1 - \alpha_n \overline{\gamma}) \left\|\gamma f(x_n) - F(z)\right\| \|x_n - z\| + (1-\mu_n)\|u_n - z\|^2 \\
&\le \mu_n (1 - \alpha_n \overline{\gamma}) \left(1 + 2h_{n+1} + h_{n+1}^2\right) \|x_n - z\|^2 + \mu_n \alpha_n^2 \left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\alpha_n \mu_n (1 - \alpha_n \overline{\gamma}) \left\|\gamma f(x_n) - F(z)\right\| \|x_n - z\| + (1-\mu_n)\|u_n - z\|^2 \\
&= \mu_n (1 - \alpha_n \overline{\gamma}) (1 + h_{n+1}^*) \|x_n - z\|^2 + \mu_n \alpha_n^2 \left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\alpha_n \mu_n (1 - \alpha_n \overline{\gamma}) \left\|\gamma f(x_n) - F(z)\right\| \|x_n - z\| + (1-\mu_n)\|u_n - z\|^2,
\end{aligned} \tag{3.16}$$

where $h_{n+1}^* = 2h_{n+1} + h_{n+1}^2$. From Lemma 2.3, we get

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\
&\le \|(x_n - rAx_n) - (z - rAz)\|^2 \\
&\le \|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2.
\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.16), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \mu_n(1 - \alpha_n\overline{\gamma})(1 + h_{n+1}^*)\|x_n - z\|^2 + \mu_n\alpha_n^2\left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\left\|\gamma f(x_n) - F(z)\right\|\|x_n - z\| \\
&\quad + (1 - \mu_n)\left[\|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2\right] \\
&= \left(1 - \alpha_n\mu_n\left(\overline{\gamma} - \frac{h_{n+1}^*}{\alpha_n}\right)\right)\|x_n - z\|^2 - \alpha_n\mu_n h_{n+1}\overline{\gamma}\|x_n - z\|^2 \\
&\quad + \mu_n\alpha_n^2\left\|\gamma f(x_n) - F(z)\right\|^2 + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\left\|\gamma f(x_n) - F(z)\right\|\|x_n - z\| \\
&\quad + (1 - \mu_n)r(r - 2\alpha)\|Ax_n - Az\|^2.
\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}
(1 - \mu_n)r(2\alpha - r)\|Ax_n - Az\|^2 &\le \left(1 - \alpha_n\mu_n\left(\overline{\gamma} - \frac{h_{n+1}^*}{\alpha_n}\right)\right)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad + \mu_n\alpha_n^2\left\|\gamma f(x_n) - F(z)\right\|^2 + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma}) \\
&\quad \times \left\|\gamma f(x_n) - F(z)\right\|\|x_n - z\| \\
&\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \mu_n\alpha_n^2\left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\left\|\gamma f(x_n) - F(z)\right\|\|x_n - z\| \\
&\le (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\| \\
&\quad + \mu_n\alpha_n^2\left\|\gamma f(x_n) - F(z)\right\|^2 \\
&\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\left\|\gamma f(x_n) - F(z)\right\|\|x_n - z\|.
\end{aligned} \tag{3.19}$$

Since $\liminf_{n\to\infty}(1 - \mu_n)r(2\alpha - r) > 0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we derive

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.20}$$

From Lemma 2.2, we obtain

$$\begin{aligned}\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\le \langle (x_n - rAx_n) - (z - rAz), u_n - z\rangle \\ &= \frac{1}{2}\Big(\|(x_n - rAx_n) - (z - rAz)\|^2 + \|u_n - z\|^2 \\ &\quad -\|(x_n - z) - r(Ax_n - Az) - (u_n - z)\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - r(Ax_n - Az)\|^2\Big) \\ &= \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 + 2r\langle x_n - u_n, Ax_n - Az\rangle - r^2\|Ax_n - Az\|^2\Big).\end{aligned} \tag{3.21}$$

Thus, we deduce

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|. \tag{3.22}$$

By (3.16) and (3.22), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \mu_n(1 - \alpha_n\overline{\gamma})(1 + h^*_{n+1})\|x_n - z\|^2 + \mu_n\alpha_n^2\big\|\gamma f(x_n) - F(z)\big\|^2 \\ &\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\big\|\gamma f(x_n) - F(z)\big\|\|x_n - z\| \\ &\quad + (1 - \mu_n)\Big[\|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|\Big] \\ &\le \left(1 - \alpha_n\mu_n\left(\overline{\gamma} - \frac{h^*_{n+1}}{\alpha_n}\right)\right)\|x_n - z\|^2 + \mu_n\alpha_n^2\big\|\gamma f(x_n) - F(z)\big\|^2 \\ &\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\big\|\gamma f(x_n) - F(z)\big\|\|x_n - z\| \\ &\quad + (1 - \mu_n)\Big[-\|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\|\Big].\end{aligned} \tag{3.23}$$

Therefore,

$$\begin{aligned}(1 - \mu_n)\|x_n - u_n\|^2 &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \mu_n\alpha_n^2\big\|\gamma f(x_n) - F(z)\big\|^2 \\ &\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\big\|\gamma f(x_n) - F(z)\big\|\|x_n - z\| \\ &\quad + (1 - \mu_n)[2r\|x_n - u_n\|\|Ax_n - Az\|] \\ &\le (\|x_n - z\| - \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \mu_n\alpha_n^2\big\|\gamma f(x_n) - F(z)\big\|^2 \\ &\quad + 2\alpha_n\mu_n(1 - \alpha_n\overline{\gamma})\big\|\gamma f(x_n) - F(z)\big\|\|x_n - z\| \\ &\quad + 2r(1 - \mu_n)\|x_n - u_n\|\|Ax_n - Az\|.\end{aligned} \tag{3.24}$$

Since $\liminf_{n\to\infty}(1-\mu_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1}-x_n\| \to 0$ and $\|Ax_n - Az\| \to 0$, we derive that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.25}$$

Next, we show that

$$\lim_{n\to\infty}\left\|x_n - S_{i(n+N)}S_{i(n+N-1)}S_{i(n+N-2)}\cdots S_{i(n+2)}S_{i(n+1)}x_n\right\| = 0. \tag{3.26}$$

By using (3.14), it suffices to show that

$$\lim_{n\to\infty}\left\|x_{n+N} - S_{i(n+N)}S_{i(n+N-1)}S_{i(n+N-2)}\cdots S_{i(n+2)}S_{i(n+1)}x_n\right\| = 0. \tag{3.27}$$

Observe that

$$\begin{aligned}
\left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| &\le \|x_{n+N}-x_{n+N-1}\| + \left\|x_{n+N} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \\
&\le \|x_{n+N}-x_{n+N-1}\| + \mu_{n+N-1}\left\|P_C[y_{n+N-1}] - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \\
&\quad + (1-\mu_{n+N-1})\left\|u_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \\
&\le \|x_{n+N}-x_{n+N-1}\| + \mu_{n+N-1}\left\|y_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \\
&\quad + (1-\mu_{n+N-1})\left\|u_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \\
&\le \|x_{n+N}-x_{n+N-1}\| + \mu_{n+N-1}\alpha_{n+N-1} \\
&\quad \times \left(\|\gamma f(x_{n+N-1})\| + \left\|FS_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\|\right) \\
&\quad + (1-\mu_{n+N-1})\|u_{n+N-1}-x_{n+N-1}\| \\
&\quad + (1-\mu_{n+N-1})\left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\|.
\end{aligned} \tag{3.28}$$

Hence,

$$\begin{aligned}
\left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| &\le \frac{1}{\mu_{n+N-1}}\|x_{n+N}-x_{n+N-1}\| \\
&\quad + \alpha_{n+N-1}\left(\|\gamma f(x_{n+N-1})\| + \left\|FS_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\|\right) \\
&\quad + \frac{(1-\mu_{n+N-1})}{\mu_{n+N-1}}\|u_{n+N-1}-x_{n+N-1}\|.
\end{aligned} \tag{3.29}$$

From (3.14), (3.25), $\lim_{n\to\infty}\alpha_n = 0$, and (C2), we have

$$\left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

Since $S_{i(n)}$ is Lipschitz with constant $L_{i(n)}$ for each $i(n) \in \{1,2,\ldots,N\}$ and for $L = \max_{1\le i\le N}\{L_{i(n)}\}$, and for any positive number $n \ge 1$, $n = (p(n)-1)N + i(n)$, we have

$$\begin{aligned}\left\|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\right\| &\le \left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| + \left\|S_{i(n+N)}^{p(n+N)}x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\right\| \\ &\le \left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| + L\left\|S_{i(n+N)}^{p(n+N)-1}x_{n+N-1} - x_{n+N-1}\right\| \\ &\le \left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| \\ &\quad + L\Big(\left\|S_{i(n+N)}^{p(n+N)-1}x_{n+N-1} - S_{i(n)}^{p(n+N)-1}x_{n-1}\right\| \\ &\qquad + \left\|S_{i(n)}^{p(n+N)-1}x_{n-1} - x_{n-1}\right\| + \|x_{n-1} - x_{n+N-1}\|\Big).\end{aligned} \tag{3.31}$$

Since for each $n > N$, $n + N = n(\operatorname{mod} N)$, and also $n = (p(n)-1)N + i(n)$, so

$$n + N = (p(n) - 1 + 1)N + i(n) = (p(n+N) - 1)N + i(n+N), \tag{3.32}$$

that is,

$$p(n+N) - 1 = p(n), \qquad i(n+N) = i(n). \tag{3.33}$$

Hence,

$$\left\|S_{i(n+N)}^{p(n+N)-1}x_{n+N-1} - S_{i(n)}^{p(n+N)-1}x_{n-1}\right\| = \left\|S_{i(n)}^{p(n)}x_{n+N-1} - S_{i(n)}^{p(n)}x_{n-1}\right\| \le L\|x_{n+N-1} - x_{n-1}\|. \tag{3.34}$$

Also,

$$\left\|S_{i(n)}^{p(n+N)-1}x_{n-1} - x_{n-1}\right\| = \left\|S_{i(n)}^{p(n)}x_{n-1} - x_{n-1}\right\|. \tag{3.35}$$

Therefore, substituting (3.34) and (3.35) into (3.31), we have

$$\begin{aligned}\left\|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\right\| &\le \left\|x_{n+N-1} - S_{i(n+N)}^{p(n+N)}x_{n+N-1}\right\| + L^2\|x_{n+N-1} - x_{n-1}\| \\ &\quad + L\left\|S_{i(n)}^{p(n)}x_{n-1} - x_{n-1}\right\| + L\|x_{n-1} - x_{n+N-1}\|.\end{aligned} \tag{3.36}$$

From (3.30) and (3.14), we have

$$\lim_{n\to\infty}\left\|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\right\| = 0. \tag{3.37}$$

Also,

$$\left\|x_{n+N} - S_{i(n+N)}x_{n+N-1}\right\| \le \|x_{n+N} - x_{n+N-1}\| + \left\|x_{n+N-1} - S_{i(n+N)}x_{n+N-1}\right\|, \tag{3.38}$$

so that

$$x_{n+N-1} - S_{i(n+N)} x_{n+N-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.39}$$

Indeed, noting that each $S_{i(n)}$ is Lipschitzian and using (3.39), we can calculate the following:

$$\begin{gathered} x_{n+N} - S_{i(n+N) x_{n+N-1}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ S_{i(n+N)} x_{n+N-1} - S_{i(n+N)} S_{i(n+N-1)} x_{n+N-2} \quad \text{as } n \longrightarrow \infty, \\ \vdots \\ S_{i(n+N)} S_{i(n+N-1)} \cdots S_{i(n+2)} x_{n+1} - S_{i(n+N)} S_{i(n+N-1)} \cdots S_{i(n+2)} S_{i(n+1)} x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{gathered} \tag{3.40}$$

It follows from (3.40) that

$$x_{n+N} - S_{i(n+N)} S_{i(n+N-1)} \cdots S_{i(n+1)} x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.41}$$

Using (3.14), we have

$$x_n - S_{i(n+N)} S_{i(n+N-1)} \cdots S_{i(n+1)} x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.42}$$

Hence (3.26) is proved. Let $\Phi = P_\Omega$. Then, $\Phi(I - F - \gamma f)$ is a contraction on C. In fact, from Lemma 2.7(i), we have

$$\begin{aligned} \|\Phi(I - F - \gamma f)x - \Phi(I - F - \gamma f)y\| &\leq \|(I - F - \gamma f)x - (I - F - \gamma f)y\| \\ &\leq \|(I - F)x - (I - F)y\| + \gamma\|f(x) - f(y)\| \\ &\leq \sqrt{\frac{1-\delta}{\lambda}}\|x - y\| + \alpha\gamma\|x - y\| \\ &= \left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma\right)\|x - y\|, \quad \forall x, y \in C. \end{aligned} \tag{3.43}$$

Therefore, $\Phi(I - F - \gamma f)$ is a contraction on C with coefficient $(\sqrt{(1-\delta)/\lambda} + \alpha\gamma) \in (0, 1)$. Thus, by Banach contraction principal, $P_\Omega(I - F - \gamma f)$ has a uninique fixed point x^*, that is $P_\Omega(I - F - \gamma f)x^* = x^*$ which mean that x^* is the unique solution in Ω of the variational inequality (3.2). Next, we show that

$$\limsup_{n \to \infty} \langle \gamma f(x^*) - F x^*, x_n - x^* \rangle \leq 0. \tag{3.44}$$

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Fx^*, x_n - x^*\rangle = \lim_{j\to\infty}\left\langle \gamma f(x^*) - Fx^*, x_{n_j} - x^*\right\rangle. \tag{3.45}$$

Since $\{x_n\}$ is bounded, we may also assume that there exists some $\tilde{x} \in H$ such that $x_{n_j} \rightharpoonup \tilde{x}$. Since the family $\{S_i\}_{i=1}^N$ is finite, passing to a further subsequence if necessary, we may further assume, for some $i(n) \in \{1,2,\ldots,N\}$, it follows that

$$x_{n_j} - S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+1)}x_{n_j} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.46}$$

By Lemma 2.4, we obtain

$$\tilde{x} \in F\left(S_{i(n+N)}S_{i(n+N-1)}\cdots S_{i(n+1)}\right), \tag{3.47}$$

so this implies that $\tilde{x} \in \Gamma$. Next, we show $\tilde{x} \in \mathrm{EP}$. Since $u_n = T_r(x_n - rAx_n)$, for any $y \in C$, we have

$$\phi(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq 0. \tag{3.48}$$

From the monotonicity of F, we have

$$\frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq \phi(y, u_n), \quad \forall y \in C. \tag{3.49}$$

Hence,

$$\left\langle y - u_n, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\right\rangle \geq \phi(y, u_{n_i}), \quad \forall y \in C. \tag{3.50}$$

Put $z_t = ty + (1-t)\tilde{x}$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.50), we have

$$\begin{aligned}
\langle z_t - u_{n_i}, Az_t\rangle &\geq \langle z_t - u_{n_i}, Az_t\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\right\rangle + \phi(z_t, u_{n_i}) \\
&= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
&\quad + \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\right\rangle + \phi(z_t, u_{n_i}).
\end{aligned} \tag{3.51}$$

Note that $\|Au_{n_i} - Ax_{n_i}\| \leq (1/\alpha)\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, \mathrm{A}z_t - Au_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.51), we have

$$\langle z_t - \tilde{x}, Az_t\rangle \geq \phi(z_t, \tilde{x}). \tag{3.52}$$

From (A1), (A4), and (3.52), we also have

$$\begin{aligned} 0 &= \phi(z_t, z_t) \le t\phi(z_t, y) + (1-t)\phi(z_t, \tilde{x}) \\ &\le t\phi(z_t, y) + (1-t)\langle z_t - \tilde{x}, Az_t\rangle \\ &= t\phi(z_t, y) + (1-t)t\langle y - \tilde{x}, Az_t\rangle \end{aligned} \tag{3.53}$$

and, hence,

$$0 \le \phi(z_t, y) + (1-t)\langle Az_t, y - \tilde{x}\rangle. \tag{3.54}$$

Letting $t \to 0$ in (3.54) and using (A3), we have, for each $y \in C$,

$$0 \le \phi(\tilde{x}, y) + \langle y - \tilde{x}, A\tilde{x}\rangle. \tag{3.55}$$

This implies that $\tilde{x} \in \mathrm{EP}$. Therefore, $\tilde{x} \in \Omega$. Therefore,

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Fx^*, x_n - x^*\rangle = \langle \gamma f(x^*) - Fx^*, \tilde{x} - x^*\rangle \le 0. \tag{3.56}$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. From Lemma 2.7 and (1.23), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \left\|\mu_n(P_C[y_n] - x^*) + (1-\mu_n)(u_n - x^*)\right\|^2 \\ &\le \mu_n\left\|P_C[y_n] - x^*\right\|^2 + (1-\mu_n)\|u_n - x^*\|^2 \\ &\le \mu_n\left\|y_n - x^*\right\|^2 + (1-\mu_n)\|u_n - x^*\|^2 \\ &= \mu\left\|\alpha_n\gamma f(x_n) - \alpha_n F(x^*) + (I - \alpha_n F)S_{i(n+1)}^{p(n+1)}x_n - (I - \alpha_n F)x^*\right\|^2 \\ &\quad + (1-\mu_n)\|u_n - x^*\|^2 \\ &= (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\left\|\alpha_n(\gamma f(x_n) - F(x^*)) + (I - \alpha_n F)\left(S_{i(n+1)}^{p(n+1)}x_n - x^*\right)\right\|^2 \\ &\le (1-\mu_n)\|x_n - x^*\|^2 + \mu_n(1-\alpha_n\overline{\gamma})^2(1+h_{n+1})^2\|x_n - x^*\|^2 \\ &\quad + 2\mu_n\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\ &\le (1-\mu_n)\|x_n - x^*\|^2 + \mu_n(1-\alpha_n\overline{\gamma})\left(1 + 2h_{n+1} + h_{n+1}^2\right)\|x_n - x^*\|^2 \\ &\quad + 2\mu_n\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\ &= (1-\mu_n)\|x_n - x^*\|^2 + \mu_n(1-\alpha_n\overline{\gamma})(1 + h_{n+1}^*)\|x_n - x^*\|^2 \\ &\quad + 2\mu_n\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \end{aligned}$$

$$\leq \left(1 - \mu_n \alpha_n \left(\overline{\gamma} - \frac{h_{n+1}^*}{\alpha_n}\right)\right) \|x_n - x^*\|^2$$
$$+ 2\mu_n \alpha_n \left(\overline{\gamma} - \frac{h_{n+1}^*}{\alpha_n}\right) \left[\frac{1}{(\overline{\gamma} - h_{n+1}^*/\alpha_n)} \langle \gamma f(x_n) - F(x^*), x_{n+1} - x^* \rangle\right], \tag{3.57}$$

where $h_{n+1}^* = 2h_{n+1} + h_{n+1}^2$. Hence, all conditions of Lemma 2.5 are satisfied. Therefore, $x_n \to x^*$. This completes the proof. □

The following example shows that there exist the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfying the conditions (C1) and (C2) of Theorem 3.1.

Example 3.2. For each $n \geq 0$, let $\alpha_n = 1/(n+1)$ and $\mu_n = 1/2 + 1/(n+1)$. Then, it is easy to obtain $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$, $0 < 1/2 = \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n = 1/2 < 1$ and $\lim_{n\to\infty}((\mu_{n+1} - \mu_n)/\alpha_{n+1}) = 0$. Hence, conditions (C1) and (C2) of Theorem 3.1 are satisfied.

Corollary 3.3. *Let $C, H, A, \phi, \Omega, f, F, r$ be as in Theorem 3.1. Let $S_1, S_2, \ldots, S_N : C \to C$ be a family of nonexpansive mappings. Let $T_1, T_2, \ldots, T_N : C \to C$ be mappings defined by* (2.9). *For $T_n := T_{n \bmod N}$, let the sequence $\{x_n\}$ be generated by*

$$\phi(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \mu_n P_C[\alpha_n \gamma f(x_n) + (1 - \alpha_n F) T_n x_n] + (1 - \mu_n) u_n, \quad n \geq 0. \tag{3.58}$$

Assume that $\sum_{n=1}^{\infty} \sup_{z \in B} \|T_{n+1} z - T_n z\| < \infty$ for each bounded subset B of C and the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$ *and* $\lim_{n\to\infty}((\mu_{n+1} - \mu_n)/\alpha_{n+1}) = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* of the following variational inequality:

$$\langle (F - \gamma f) x^*, x - x^* \rangle \geq 0, \quad x \in \Omega, \tag{3.59}$$

or equivalently $\tilde{x} = P_\Omega (I - F + \gamma f)\tilde{x}$, where P_Ω is the metric projection of H onto Ω.

Proof. By Lemma 2.6, we have

$$\bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_N T_{N-1} T_{N-2} \cdots T_1) = \operatorname{Fix}(T_1 T_N \cdots T_2) = \operatorname{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N). \tag{3.60}$$

Therefore, the result follows from Theorem 3.1. □

Remark 3.4. As in [58, Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying the condition $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C by using convex combination of a general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

Setting $\gamma = 1$, $F = I$, and $S_n \equiv S$, a nonexpansive mapping, in Corollary 3.3, we obtain the following result.

Corollary 3.5 ([46], Theorem 3.7). *Let C, H, A, ϕ, f, r be as in Theorem 3.1. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := \mathrm{EP} \cap \mathrm{Fix}(S) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{gathered}\phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \mu_n P_C\left[\alpha_n f(x_n) + (1-\alpha_n)Sx_n\right] + (1-\mu_n)u_n, \quad n \geq 0.\end{gathered} \tag{3.61}$$

Assume the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

(C2) $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$ *and* $\lim_{n\to\infty}((\mu_{n+1} - \mu_n)/\alpha_{n+1}) = 0$.

Then, the sequence $\{x_n\}$ converges strongly to x^* of the following variational inequality:

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in \Omega. \tag{3.62}$$

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ภาคผนวก 3

Strong convergence theorems of the general iterative methods for nonexpansive semigroups in Banach spaces

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The general iterative methods for nonexpansive semigroups in Banach spaces

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Abstract Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on E such that $Fix(\mathcal{S}) := \cap_{t\geq 0} Fix(T(t)) \neq \emptyset$, and f is a contraction on E with coefficient $0 < \alpha < 1$. Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. When the sequences of real numbers $\{\alpha_n\}$ and $\{t_n\}$ satisfy some appropriate conditions, the three iterative processes given as follows :

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad n \geq 0,$$
$$y_{n+1} = \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n, \quad n \geq 0,$$

and

$$z_{n+1} = T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n), \quad n \geq 0$$

converge strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $Fix(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0, \quad x \in Fix(\mathcal{S}).$$

Our results extend and improve corresponding ones of Li et al. (Nonlinear Anal 70:3065–3071, 2009) and Chen and He (Appl Math Lett 20:751–757, 2007) and many others.

Keywords General iterative method · Nonexpansive semigroup · Reflexive Banach space · Uniformly Gâteaux differentiable norm · Fixed point

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1 Introduction

Let E be a real Banach space. A mapping T of E into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in E$. We denote by $Fix(T)$ the set of fixed points of T. A mapping $f : E \longrightarrow E$ is called α-contraction, if there exists a constant $0 < \alpha < 1$ such that $\| f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in E$. A family $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ of mappings of E into itself is called a *nonexpansive semigroup* on E if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in E$;
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in E$ and $t \geq 0$;
(iv) for all $x \in E$, the mapping $t \mapsto T(t)x$ is continuous.

We denote by $Fix(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$Fix(\mathcal{S}) := \{x \in E : T(t)x = x, 0 \leq t < \infty\} = \cap_{t\geq 0} Fix(T(t)).$$

In [1], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space

$$x_n = \alpha_n x + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ is a sequence of positive real numbers which diverges to ∞. Under certain restrictions on the sequence $\{\alpha_n\}$, Shioji and Takahashi [1] proved strong convergence of the sequence $\{x_n\}$ to a member of $F(\mathcal{S})$. In [2], Shimizu and Takahashi studied the strong convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N} \tag{1.2}$$

in a real Hilbert space where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Banach space E and $\lim_{n\longrightarrow\infty} t_n = \infty$. Using viscosity method, Chen and Song [3] studied the strong convergence of the following iterative method for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ with $Fix(\mathcal{S}) \neq \emptyset$ in a Banach space :

$$x_{n+1} = \alpha_n f(x) + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where f is a contraction. Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval $(0, t)$. Suzuki [4] was the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.4}$$

for the nonexpansive semigroup case. In 2002, Benavides, Acedo and Xu [5] in a uniformly smooth Banach space, showed that if $\mathcal{S}$ satisfies an asymptotic regularity condition and $\{\alpha_n\}$ fulfills the control conditions $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n\longrightarrow\infty} \frac{\alpha_n}{\alpha_{n+1}} = 0$, then then both the implicit iteration process (1.4) and the explicit iteration process (1.5)

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

converge to a same point of $F(\mathcal{S})$. In 2005, Xu [6] studied the strong convergence of the implicit iteration process (1.1) and (1.4) in a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping. Recently Chen and He [7] introduced the viscosity approximation process :

$$x_{n+1} = \alpha_n f(x_n) + (1-\beta_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.6}$$

where f is a contraction, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and a nonexpansive semigroup $\{T(t) : t \geq 0\}$. The strong convergence theorem of $\{x_n\}$ is proved in a reflexive Banach space which admits a weakly sequentially continuous duality mapping. In [8], Chen and He introduced and studied modified Mann iteration for nonexpansive mapping in a uniformly convex Banach space.

On the other hand, iterative approximation methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [9–13] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all} \quad x \in H. \tag{1.7}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.8}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2003, Xu ([10]) proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{1.9}$$

converges strongly to the unique solution of the minimization problem (1.8) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [14] introduced the following iterative iterative process for nonexpansive mappings (see [15] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.10}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved [14,15] that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.10) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in H. \tag{1.11}$$

Recently, Marino and Xu [16] mixed the iterative method (1.9) and the viscosity approximation method (1.10) and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.12}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the certain conditions, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in H \tag{1.13}$$

which is the optimality condition for the minimization problem: $\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e.,$h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Li et al. [17] introduced the following iterative procedures for the approximation of common fixed points of a one-parameter nonexpansive semigroup on a Hilbert space H :

$$x_0 = x \in H, x_{n+1} = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.14}$$

where A is a strongly positive bounded linear operator on H.

Let δ and λ be two positive real numbers such that $\delta, \lambda < 1$. Recall that a mapping F with domain $D(F)$ and range $R(F)$ in E is called δ-*strongly accretive* if, for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geq \delta\|x - y\|^2, \tag{1.15}$$

where J is the normalized duality mapping from E into the dual space E^*. Recall also that a mapping F is called λ-*strictly pseudo-contractive* if, for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|(x - y) - (Fx - Fy)\|^2. \tag{1.16}$$

It is easy to see that (1.16) can be rewritten as

$$\langle (I - F)x - (I - F)y, j(x - y)\rangle \geq \lambda\|(I - F)x - (I - F)y\|^2, \tag{1.17}$$

see [18].

Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \longrightarrow C$ be a mapping. T is said to be a λ-strictly pseudo-contraction in light of Browder and Petryshyn [19] if there exists a λ such that

$$\|Fx - Fy\|^2 \leq \|x - y\|^2 - \lambda\|(x - y) - (Fx - Fy)\|^2, \tag{1.18}$$

for every $x, y \in C$.

Remark 1.1 From (1.16) we can prove that if F is λ-strictly pseudo-contraction, then F is Lipschitz continuous with the Lipschitz constant $\frac{1+\lambda}{\lambda}$; see [20] for more details.

In this paper, motivated by the above results, we introduce and study the strong convergence theorems of the general iterative scheme $\{x_n\}$ defined by (1.19) in the framework of a reflexive Banach space E which admits a weakly sequentially continuous duality mapping :

$$x_0 = x \in E, x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad n \geq 0 \tag{1.19}$$

where F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, f is a contraction on E with coefficient $0 < \alpha < 1, \gamma$ is a positive real number such that $\gamma < \frac{1}{2\alpha}\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$ and $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ is a nonexpansive semigroup on E. The strong convergence theorems are proved under some appropriate control conditions on parameter $\{\alpha_n\}$ and $\{t_n\}$. Furthermore, by using these results, we obtain strong convergence theorems of the following new general iterative schemes $\{y_n\}$, and $\{z_n\}$ defined by

$$y_0 = y \in E, y_{n+1} = \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n, \quad n \geq 0 \tag{1.20}$$

and

$$z_0 = z \in E, \quad z_{n+1} = T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n), \quad n \geq 0. \tag{1.21}$$

The results presented in this paper extend and improve the main results in Li et al. [17] and Chen and He [7] and many others.

2 Preliminaries

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$ and let J denote the normalized duality mapping from E into E^* given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing and $\mathbb{N}$ denotes the set of all positive integer. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ (respectively $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. In a Banach space E, the following result (*the Subdifferential Inequality*) is well known ([28, Theorem 4.2.1]): $\forall x, y \in E, \forall j(x+y) \in J(x+y), \forall j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x+y\|^2 \leq \|x\|^2 + \langle y, j(x+y)\rangle. \tag{2.1}$$

A real Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if for all $\epsilon \in [0, 2]$, there exits $\delta_\epsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ with } \|x - y\| \geq \epsilon \text{ implies } \frac{\|x+y\|}{2} < 1 - \delta_\epsilon.$$

The following results are well known and can be founded in Ref. [28]:

(i) A uniformly convex Banach space E is reflexive and strictly convex ([28, Theorem 4.2.1 and 4.1.6])
(ii) If E is a strictly convex Banach space and $T : E \longrightarrow E$ is a non-expansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of E ([28, Theorem 4.5.3]).

Recall that the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*), if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each y in $S(E)$ the limit defined by (2.2) is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gáteaux differentiable*. It is well known that if E is a Banach space with a uniformly Gáteaux differentiable norm, then the mapping $J : E \longrightarrow E$ is single-valued and norm to weak star uniformly continuous on bounded sets of E [28, Theorem 4.3.6]. If C is a nonempty convex subset of a strictly convex Banach space E and $T : C \longrightarrow C$ is a nonexpansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of C [28, Theorem 4.5.3].

Now, we present the concept of uniformly asymptotically regular semigroup (also see [21,22]). Let C be a nonempty closed convex subset of a Banach space E, $\mathcal{S} = \{T(t) : 0$

$\leq t < \infty\}$ a continuous operator semigroup on C. Then $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset D of C,

$$\lim_{t \longrightarrow \infty} \sup_{x \in D} \|T(h)(T(t)x) - T(t)x\| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in [21, Examples 17, 18].

Lemma 2.1 *[3, Lemma 2.7] Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and D a bounded closed convex subset of C, and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. For each $h > 0$, set $\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x ds$, then*

$$\lim_{t \longrightarrow \infty} \sup_{x \in D} \|\sigma_t(x) - T(h)\sigma_t(x)\| = 0.$$

Example 2.2 The set $\{\sigma_t : t > 0\}$ defined by Lemma 2.1 is u.a.r. nonexpansive semigroup. In fact, it is obvious that $\{\sigma_t : t > 0\}$ is a nonexpansive semigroup. For each $h > 0$, we have

$$\begin{aligned}
\|\sigma_t(x) - \sigma_h\sigma_t(x)\| &= \left\|\sigma_t(x) - \frac{1}{h}\int_0^h T(s)\sigma_t(x)ds\right\| \\
&= \left\|\frac{1}{h}\int_0^h (\sigma_t(x) - T(s)\sigma_t(x))ds\right\| \\
&\leq \frac{1}{h}\int_0^h \|\sigma_t(x) - T(s)\sigma_t(x)\|ds.
\end{aligned}$$

Applying Lemma 2.1, we have

$$\lim_{t \longrightarrow \infty} \sup_{x \in D} \|\sigma_t(x) - \sigma_h\sigma_t(x)\| \leq \frac{1}{h}\int_0^h \lim_{t \longrightarrow \infty} \sup_{x \in D} \|\sigma_t(x) - T(s)\sigma_t(x)\|ds = 0.$$

Let C be a nonempty closed and convex subset of a Banach space E and D a nonempty subset of C. A mapping $Q : C \longrightarrow D$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t = 0$. A mapping $Q : C \longrightarrow D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.3 ([23,24]) *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C and $Q : C \longrightarrow D$ be a retraction. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Lemma 2.4 *[25, Lemma 2.3] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1 - t_n)a_n + t_n c_n + b_n, \forall n \geq 0,$$

where $\{t_n\}$, $\{b_n\}$, $\{c_n\}$ satisfying the restrictions:

(i) $\sum_{n=1}^{\infty} t_n = \infty$;
(ii) $\sum_{n=1}^{\infty} b_n < \infty$;
(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

The following lemma will be frequently used throughout the paper and can be found in [26].

Lemma 2.5 *[26, Lemma 2.7] Let E be a real smooth Banach space and $F : E \longrightarrow E$ a mapping.*

(i) *If F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, then $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\lambda}}$.*
(ii) *If F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, then for any fixed number $\tau \in (0, 1)$, $I - \tau F$ is contractive with constant $1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$.*

Let $\mathbb{N}$ be the set of positive integers and let l^{∞} be the Banach space of bounded valued functions on $\mathbb{N}$ with supremum norm. Let **LIM** be a linear continuous functional on l^{∞} and let $x = (a_1, a_2, ...) \in l^{\infty}$. Then sometimes, we denote by $\mathbf{LIM}_n(a_n)$ the value of $\mathbf{LIM}(x)$. We know that there exists a linear continuous functional **LIM** on l^{∞} such that $\|\mathbf{LIM}\| = \mathbf{LIM}(1) = 1$ and $\mathbf{LIM}(a_n) = \mathbf{LIM}(a_{n+1})$ for each $x = (a_1, a_2, ...) \in l^{\infty}$. Such a **LIM** is called a *Banach limit*. Let **LIM** be a Banach limit. Then,

$$\liminf_{n\longrightarrow\infty} a_n \leq \mathbf{LIM}(x) \leq \limsup_{n\longrightarrow\infty} a_n$$

for each $x = (a_1, a_2, ...) \in l^{\infty}$. Specially, if $a_n \longrightarrow a$, then $\mathbf{LIM}(x) = a$; see [28] for more details.

Proposition 2.6 *[28] Let E be a real Banach space which has a uniformly Gâteaux differentiable norm. Suppose that $\{x_n\}$ is a bounded sequence of E and let $\mathbf{LIM}_n$ be a Banach limit and $z \in E$. Then*

$$LIM_n\|x_n - z\|^2 = \inf_{y\in E} LIM_n\|x_n - y\|^2$$

if and only if

$$LIM_n\langle y - z, j(x_n - z)\rangle = 0, \quad \forall y \in E.$$

Let (X, d) be a metric space. A subset M of X is called a *Chebyshev* set, if for each $x \in X$, there exists a unique element $u \in M$ such that $d(x, u) = d(x, M)$, where $d(x, M) = \inf_{z\in M} d(x, z)$.

Theorem 2.7 ***(Day-James Theorem)*** *[27] E is a reflexive strictly convex Banach space if and only if every nonempty closed convex subset of E is Chebyshev set.*

Lemma 2.8 *[29, Lemma 3.2] Let C be a nonempty closed convex subset of E and let $\{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on C such that $Fix(\mathcal{S}) \neq \emptyset$ and at least there exists a $T(t)$ which is demicompact. Then, for each $x \in C$, there exists a sequence $\{T(t_j) : 0 \leq t_j < \infty, j \in \mathbb{N}\} \subset \{T(t) : 0 \leq t < \infty\}$ such that $\{T(t_j)x\}$ converges strongly to some point in $Fix(\mathcal{S})$, where $t_j \longrightarrow \infty$ as $j \longrightarrow \infty$.*

3 Main results

Let E be a real Banach space. Let T be a nonexpansive mapping on E. For $f \in \Pi_E$ and F a δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. For each $t \in (0, 1)$, the mapping $S_t : E \longrightarrow E$ defined by

$$S_t(x) = t\gamma f(x) + (I - tF)Tx, \quad \forall x \in E$$

is a contraction mapping. Indeed, for any $x, y \in E$,

$$\begin{aligned}\|S_t(x) - S_t(y)\| &\leq \|t\gamma(f(x) - f(y)) + (I - tF)Tx - (I - tF)Ty\| \\ &\leq t\gamma\|f(x) - f(y)\| + (1 - t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\|x - y\| \\ &\leq \left(1 - t[(1 - \sqrt{\frac{1-\delta}{\lambda}}) - \gamma\alpha]\right)\|x - y\|. \end{aligned} \tag{3.1}$$

Thus, by Banach contraction mapping principle, there exists a unique fixed point x_t in E that is

$$x_t = t\gamma f(x_t) + (I - tF)Tx_t. \tag{3.2}$$

Lemma 3.1 *Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let T be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ a contraction mapping with coefficient $\alpha (0 < \alpha < 1)$, and let F be a δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. Then the net $\{x_t\}$ defined by (3.2) converges strongly as $t \longrightarrow 0$ to a fixed point $\tilde{x}$ in $F(T)$ which solves the variational inequality :*

$$\langle (F - \gamma f)\tilde{x}, j(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{3.3}$$

Proof We first show that the uniqueness of a solution of the variational inequality (3.3). Suppose both $\tilde{x} \in F(T)$ and $x^* \in F(T)$ are solutions to (3.3), then

$$\langle (F - \gamma f)\tilde{x}, j(\tilde{x} - x^*)\rangle \leq 0 \tag{3.4}$$

and

$$\langle (F - \gamma f)x^*, j(x^* - \tilde{x})\rangle \leq 0. \tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$\langle (F - \gamma f)\tilde{x} - (F - \gamma f)x^*, j(\tilde{x} - x^*)\rangle \leq 0. \tag{3.6}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}\langle (F-\gamma f)x-(F-\gamma f)y, j(x - y)\rangle &= \langle Fx-Fy, j(x-y)\rangle-\gamma\langle f(x)- f(y), j(x - y)\rangle \\ &\geq \delta\|x - y\|^2 - \gamma\|f(x) - f(y)\|\|j(x - y)\| \\ &\geq \delta\|x - y\|^2 - \gamma\alpha\|x - y\|^2 \\ &= (\delta - \gamma\alpha)\|x - y\|^2 \geq 0. \end{aligned} \tag{3.7}$$

This together with (3.6) implies that $\tilde{x} = x^*$ and then the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (3.3). Next, we will prove that $\{x_t\}$ is bounded. Take a $p \in F(T)$, then we have

$$\begin{aligned}\|x_t - p\| &= \|t\gamma f(x_t) + (I - tF)Tx_t - p\| \\ &= \|(I - tF)Tx_t - (I - tF)p + t(\gamma f(x_t) - F(p))\| \\ &\leq (1 - t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\|x_t - p\| + t(\gamma\alpha\|x_t - p\| + \|\gamma f(p) - F(p)\|).\end{aligned}$$

It follows that

$$\|x_t - p\| \leq \frac{1}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \gamma\alpha}\|\gamma f(p) - F(p)\|.$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{FT(x_t)\}$. The definition of $\{x_t\}$ implies that

$$\|x_t - Tx_t\| = t\|\gamma f(x_t) - F(Tx_t)\| \longrightarrow 0 \text{ as } t \longrightarrow 0. \tag{3.8}$$

Assume that $t_n \longrightarrow 0$ as $n \longrightarrow \infty$. Set $x_n := x_{t_n}$ and define $\mu : E \longrightarrow [0, \infty)$ by

$$\mu(x) = \mathbf{LIM}_n\|x_n - x\|^2, x \in E,$$

where $\mathbf{LIM}_n$ is a Banach limit on l^∞. Since μ is continuous and convex and $\mu(z) \longrightarrow 0$ as $\|z\| \longrightarrow \infty$, and E is reflexive, μ attains its infimum over E. Let $u \in E$ be such that

$$\mathbf{LIM}_n\|x_n - u\|^2 = \inf_{x\in E}\mathbf{LIM}_n\|x_n - x\|^2$$

and let

$$K = \left\{x \in E : \mu(x) = \inf_{x\in E}\mathbf{LIM}_n\|x_n - x\|^2\right\}.$$

Then K is nonempty because $u \in K$. Furthermore, it easy to see that K is a closed convex subset of E. From (3.8), it follows that

$$\mu(Tx) = \mathbf{LIM}_n\|x_n - Tx\|^2 \leq \mathbf{LIM}_n\|Tx_n - Tx\|^2 \leq \mathbf{LIM}_n\|x_n - x\|^2 = \mu(x),$$

which implies that $T(K) \subset K$, that is, K is invariant under T. Let $p \in F(T)$. It follows from Day-James's theorem that there exists a unique element $z \in K$ such that

$$\|p - z\| = \inf_{x\in K}\|p - x\|. \tag{3.9}$$

Hence,

$$\|p - Tz\| = \|Tp - Tz\| \leq \|p - z\|, \tag{3.10}$$

and so $Tz = z$ since z is minimizer of μ over K. Hence $z \in F(T) \cap K$. For any $t \in (0, 1)$ and $y \in E$, then $z + t(y - Fz) \in E$. By Proposition 2.6, we have

$$\mathbf{LIM}_n\langle z + t(y - Fz) - z, j(x_n - z)\rangle = 0,$$

which implies that

$$\mathbf{LIM}_n\langle y - Fz, j(x_n - z)\rangle = 0, \quad \forall y \in E. \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}\|x_n - z\|^2 &= \|t_n \gamma f(x_n) + (I - t_n F)Tx_n - z\|^2 \\ &\le \|(I - t_n F)Tx_n - (I - t_n F)z\|^2 + 2t_n \langle \gamma f(x_n) - Fz, j(x_n - z)\rangle \\ &\le (1 - t_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2 \|x_n - z\|^2 + 2t_n \langle \gamma f(x_n) - Fz, j(x_n - z)\rangle \\ &\le (1 - t_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\|x_n - z\|^2 + 2t_n \langle \gamma f(x_n) - Fz, j(x_n - z)\rangle\end{aligned}$$

which implies that

$$\begin{aligned}\|x_n - z\|^2 &\le \frac{1}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} 2\langle \gamma f(x_n) - Fz, j(x_n - z)\rangle \\ &= \frac{1}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} 2\langle \gamma f(x_n) - y, j(x_n - z)\rangle + \frac{1}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} 2\langle y - Fz, j(x_n - z)\rangle.\end{aligned}$$

Applying (3.11) to the above inequality, we obtain that

$$\begin{aligned}\mathbf{LIM}_n \|x_n - z\|^2 &\le \frac{2}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} \mathbf{LIM}_n \langle \gamma f(x_n) - y, j(x_n - z)\rangle \\ &\quad + \frac{2}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} \mathbf{LIM}_n \langle y - Fz, j(x_n - z)\rangle \\ &= \frac{2}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} \mathbf{LIM}_n \langle \gamma f(x_n) - y, j(x_n - z)\rangle \\ &\le \frac{2}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)} \mathbf{LIM}_n \|\gamma f(x_n) - y\| \|x_n - z\|. \end{aligned} \tag{3.12}$$

Setting $y := \gamma f(z)$ in (3.12), we have

$$\begin{aligned}\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) \mathbf{LIM}_n \|x_n - z\|^2 &\le 2\mathbf{LIM}_n \|\gamma f(x_n) - \gamma f(z)\| \|x_n - z\| \\ &\le 2\gamma\alpha \mathbf{LIM}_n \|x_n - z\|^2\end{aligned}$$

and hence

$$\left(\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - 2\gamma\alpha\right) \mathbf{LIM}_n \|x_n - z\|^2 \le 0.$$

This implies that $\mathbf{LIM}_n \|x_n - z\|^2 = 0$, and then there exists a subsequence $\{x_{t_{n_j}}\}$ of $\{x_{t_n}\}$ such that $x_{t_{n_j}} \longrightarrow z$ as $j \longrightarrow \infty$, still denoted $\{x_n\}$. Next, we prove that z solves the variational inequality (3.3). From (3.2), we have

$$(F - \gamma f)x_t = \frac{1}{t}[(I - tF)Tx_t - (I - tF)x_t].$$

On the other hand, note for all $x, y \in E$,

$$\begin{aligned}\langle (I-T)x-(I-T)y, j(x-y)\rangle &= \langle x-y, j(x-y)\rangle - \langle Tx-Ty, j(x-y)\rangle\\ &= \|x-y\|^2 - \langle Tx-Ty, j(x-y)\rangle\\ &\geq \|x-y\|^2 - \|Tx-Ty\|\|x-y\|\\ &\geq \|x-y\|^2 - \|x-y\|^2 = 0.\end{aligned}$$

For $p \in F(T)$, we have

$$\begin{aligned}\langle (F-\gamma f)x_t, j(x_t-p)\rangle &= \frac{1}{t}\langle (I-tF)Tx_t-(I-tF)Tx_t, j(x_t-p)\rangle\\ &= -\frac{1}{t}\langle (I-T)x_t-(I-T)p, j(x_t-p)\rangle\\ &\quad+\langle Fx_t-FTx_t, j(x_t-p)\rangle\\ &\leq \langle Fx_t-FTx_t, j(x_t-p)\rangle.\end{aligned}$$

Replacing t with t_n and letting $n \longrightarrow \infty$, noticing that $Fx_{t_n} - FTx_{t_n} \longrightarrow Fz - Fz = 0$, we have that

$$\langle (F-\gamma f)z, j(z-p)\rangle \leq 0, \quad \forall p \in F(T).$$

That is, $z \in F(T)$ is a solution of (3.3). Then $z = \tilde{x}$. In summary, we have that each cluster point of $\{x_t\}$ converges strongly to $\tilde{x}$ as $t \longrightarrow 0$. This completes the proof. □

Lemma 3.2 *Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let T be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ a contraction mapping with coefficient $\alpha(0 < \alpha < 1)$, and let F be a δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$ and $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. Assume that the net $\{x_t\}$ defined by (3.2) converges strongly to a fixed point $\tilde{x}$ in $F(T)$ as $t \longrightarrow 0$. Suppose that $\{x_n\} \subset E$ is bounded and $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$. Then*

$$\limsup_{n\longrightarrow\infty} \langle \gamma f(\tilde{x}) - F(\tilde{x}), j(x_n - \tilde{x})\rangle \leq 0. \tag{3.13}$$

Proof We note that

$$\begin{aligned}x_t - x_n &= t\gamma f(x_t) + Tx_t - tFTx_t - x_n\\ &= t(\gamma f(x_t) - Fx_t) + (Tx_t - x_n) - t(FTx_t - Fx_t)\\ &= t(\gamma f(x_t) - Fx_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + t(Fx_t - FTx_t).\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_t - x_n\|^2\\ &\quad\leq \|Tx_t - Tx_n\|^2 + 2\langle t(\gamma f(x_t) - Fx_t) + (Tx_n - x_n) + t(Fx_t - FTx_t), j(x_t - x_n)\rangle\\ &\quad\leq \|x_t - x_n\|^2 + 2t\langle (\gamma f(x_t) - Fx_t), j(x_t - x_n)\rangle\\ &\qquad+ 2\langle Tx_n - x_n, j(x_t - x_n)\rangle + 2t\langle Fx_t - FTx_t, j(x_t - x_n)\rangle\\ &\quad\leq \|x_t - x_n\|^2 + 2t\langle (\gamma f(x_t) - Fx_t), j(x_t - x_n)\rangle\\ &\qquad+ 2\|Tx_n - x_n\|\|j(x_t - x_n)\| + 2t\|Fx_t - FTx_t\|\|j(x_t - x_n)\|\\ &\quad\leq \|x_t - x_n\|^2 + 2t\langle (\gamma f(x_t) - Fx_t), j(x_t - x_n)\rangle\\ &\qquad+ 2\|Tx_n - x_n\|\|x_t - x_n\| + 2t\|Fx_t - FTx_t\|\|x_t - x_n\|\end{aligned}$$

which implies

$$\begin{aligned}&\langle(\gamma f(x_t) - Fx_t), j(x_n - x_t)\rangle \\ &\leq \frac{\|Tx_n - x_n\|\|x_t - x_n\|}{t} + \|Fx_t - FTx_t\|\|x_t - x_n\|.\end{aligned} \tag{3.14}$$

Since $\{x_t\}$, $\{x_n\}$ and $\{Tx_n\}$ are bounded and $x_n - Tx_n \longrightarrow 0$, taking the upper limit as $n \longrightarrow \infty$ in (3.14), we get that

$$\limsup_{n\longrightarrow\infty}\langle(\gamma f(x_t) - Fx_t), j(x_n - x_t)\rangle \leq \|Fx_t - FTx_t\| \limsup_{n\longrightarrow\infty} \|x_t - x_n\|. \tag{3.15}$$

Taking the upper limit as $t \longrightarrow 0$ in (3.15), we have that

$$\limsup_{t\longrightarrow 0}\limsup_{n\longrightarrow\infty} \langle(\gamma f(x_t)-Fx_t), j(x_n-x_t)\rangle \leq \|F\tilde{x}-F\tilde{x}\| \limsup_{n\longrightarrow\infty} \|x_t-x_n\| = 0. \tag{3.16}$$

Since E has a uniformly Gâteaux differentiable norm, we obtain that j is single-valued and strong-weak* uniformly continuous on bounded subset of E. It follows that

$$\begin{aligned}&|\langle(\gamma f(\tilde{x}) - F\tilde{x}), j(x_n - \tilde{x})\rangle - \langle\gamma f(x_t) - Fx_t, j(x_n - x_t)\rangle| \\ &= |\langle(\gamma f(\tilde{x})-F\tilde{x}), j(x_n-\tilde{x})-j(x_n-x_t)\rangle + \langle\gamma f(\tilde{x})-\gamma f(x_t)+Fx_t-F\tilde{x}, j(x_n-x_t)\rangle| \\ &\leq |\langle(\gamma f(\tilde{x}) - F\tilde{x}), j(x_n - \tilde{x}) - j(x_n - x_t)\rangle| \\ &\quad + (\|\gamma f(\tilde{x}) - \gamma f(x_t)\| + \|Fx_t - F\tilde{x}\|)\,\|x_n - x_t\| \longrightarrow 0 \text{ as } t \longrightarrow 0.\end{aligned}$$

Hence, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall t \in (0, \delta)$, for all $n \in \mathbb{N}$, we have

$$\langle(\gamma f(\tilde{x}) - F\tilde{x}), j(x_n - \tilde{x})\rangle \leq \langle\gamma f(x_t) - Fx_t, j(x_n - x_t)\rangle + \varepsilon.$$

By (3.16), we get that

$$\begin{aligned}&\limsup_{n\longrightarrow\infty} \langle(\gamma f(\tilde{x}) - F\tilde{x}), j(x_n - \tilde{x})\rangle \\ &\quad = \limsup_{t\longrightarrow 0}\limsup_{n\longrightarrow\infty} \langle(\gamma f(\tilde{x}) - F\tilde{x}), j(x_n - \tilde{x})\rangle \\ &\quad \leq \limsup_{t\longrightarrow 0}\limsup_{n\longrightarrow\infty} \langle\gamma f(x_t) - Fx_t, j(x_n - x_t)\rangle + \varepsilon \\ &\quad \leq \varepsilon.\end{aligned}$$

Since ε is arbitrary, we get that

$$\limsup_{n\longrightarrow\infty} \langle(\gamma f(\tilde{x}) - F\tilde{x}), j(x_n - \tilde{x})\rangle \leq 0.$$

This complete the proof. □

Theorem 3.3 *Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let $\{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on C such that $Fix(\mathcal{S}) \neq \emptyset$ and at least there exists a $T(t)$ which is demicompact. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions*

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n\longrightarrow\infty} t_n = \infty.$$

Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta+\lambda > 1$, $f : E \longrightarrow E$ a contraction mapping with coefficient $\alpha \in (0, 1)$ and γ a positive real number such that

$0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. *Then, the sequence* $\{x_n\}$ *defined by* (1.19) *converges strongly to* $\tilde{x}$, *where* $\tilde{x}$ *is the unique solution in* $Fix(\mathcal{S})$ *of the variational inequality*

$$\langle (F-\gamma f)\tilde{x}, j(x-\tilde{x})\rangle \geq 0, \;\; x \in Fix(\mathcal{S}) \tag{3.17}$$

or equivalently $\tilde{x} = Q_{Fix(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, *where* $Q_{Fix(\mathcal{S})}$ *is the sunny nonexpansive retraction of* E *onto* $Fix(\mathcal{S})$.

Proof Note that $Fix(\mathcal{S})$ is a nonempty closed convex set. We first show that $\{x_n\}$ is bounded. Let $q \in Fix(\mathcal{S})$. Thus, by Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1}-q\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n F)T(t_n)x_n - (I-\alpha_n F)q - \alpha_n Fq\| \\
&\leq \alpha_n\|\gamma f(x_n) - Fq\| + \|I-\alpha_n F\|\,\|T(t_n)x_n - q\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(q)\| + \alpha_n\|\gamma f(q) - Fq\| + \|I-\alpha_n F\|\|x_n - q\| \\
&\leq \alpha_n\alpha\gamma\|x_n - q\| + \alpha_n\|\gamma f(q) - Fq\| + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-q\| \\
&= \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\alpha\gamma\right)\right)\|x_n-q\| + \alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\alpha\gamma\right) \\
&\quad\times\frac{\|\gamma f(q) - Fq\|}{1-\sqrt{\frac{1-\delta}{\lambda}}-\alpha\gamma} \\
&\leq \max\left\{\|x_n-q\|, \frac{1}{1-\sqrt{\frac{1-\delta}{\lambda}}-\alpha\gamma}\|\gamma f(q) - Fq\|\right\}, \quad \forall n \geq 0.
\end{aligned}$$

By induction, we get

$$\|x_n - q\| \leq \max\left\{\|x_0-q\|, \frac{1}{1-\sqrt{\frac{1-\delta}{\lambda}}-\alpha\gamma}\|\gamma f(q) - Fq\|\right\}, \quad n \geq 0.$$

This implies that $\{x_n\}$ is bounded, and hence so are $\{f(x_n)\}$ and $\{FT(t_n)x_n\}$. This implies that

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - T(t_n)x_n\| = \lim_{n\longrightarrow\infty}\alpha_n\|\gamma f(x_n) - FT(t_n)x_n\| = 0. \tag{3.18}$$

Since $\{T(t)\}$ is a u.a.r. nonexpansive semigroup and $\lim_{n\longrightarrow\infty} t_n = \infty$, we have, for all $h > 0$,

$$\lim_{n\longrightarrow\infty}\|T(h)(T(t_n)x_n) - T(t_n)x_n\| \leq \lim_{n\longrightarrow\infty}\sup_{x\in\{x_n\}}\|T(h)(T(t_n)x) - T(t_n)x\| = 0. \tag{3.19}$$

Hence, for all $h > 0$,

$$\begin{aligned}
\|x_{n+1} - T(h)x_{n+1}\| &\leq \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| \\
&\quad + \|T(h)T(t_n)x_n - T(h)x_{n+1}\| \\
&\leq 2\|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| \longrightarrow 0.
\end{aligned} \tag{3.20}$$

That is, for all $h > 0$,

$$\lim_{n\longrightarrow\infty}\|x_n - T(h)x_n\| = 0. \tag{3.21}$$

Since $\{T(t) : 0 \le t < \infty\}$ is a u.a.r. nonexpansive semigroup, by Lemma 2.8, for each $x \in E$, there exists a sequence $\{T(t_j) : 0 \le t_j < \infty, j \in \mathbb{N}\} \subset \{T(t) : 0 \le t < \infty\}$ such that $\{T(t_j)x\}$ converges strongly to some point in $Fix(\mathcal{S})$, where $t_j \longrightarrow \infty$ as $j \longrightarrow \infty$. Define a mapping T of E into itself by

$$Tx = \lim_{j \longrightarrow \infty} T(t_j)x.$$

By [29, Remark 3.4], we see that T is a nonexpansive mapping such that $F(T) = Fix(\mathcal{S})$. Applying (3.21), we can also get that

$$\begin{aligned}\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| &= \lim_{n \longrightarrow \infty} \lim_{j \longrightarrow \infty} \|x_n - T(t_j)x_n\| \\ &= \lim_{j \longrightarrow \infty} \lim_{n \longrightarrow \infty} \|x_n - T(t_j)x_n\| = 0. \end{aligned} \tag{3.22}$$

Let f be a contractive mapping of E and let the net $\{x_t\}$ be a unique point of E such that

$$x_t = t\gamma f(x_t) + (I - tF)Tx_t.$$

It follows from Lemma 3.1 that $\{x_t\}$ converges strongly as $t \longrightarrow 0$ to a fixed point $\tilde{x}$ in $F(T)$ which solves the variational inequality :

$$\langle (F - \gamma f)\tilde{x}, j(\tilde{x} - z)\rangle \le 0, z \in F(T).$$

By Lemma 3.2, we have

$$\limsup_{n \longrightarrow \infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\rangle \le 0. \tag{3.23}$$

Finally we shall show that $x_n \longrightarrow \tilde{x}$. For each $n \ge 0$, we have

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x} - \alpha_n F\tilde{x}\|^2 \\ &\le \|\alpha_n \gamma f(x_n) - \alpha_n F\tilde{x} + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x}\|^2 \\ &= \|(I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x}\|^2 + 2\alpha_n \langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle \\ &\le \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle \gamma f(x_n) - \gamma f(\tilde{x}), j(x_{n+1} - \tilde{x})\rangle \\ &\quad + 2\alpha_n \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle. \end{aligned} \tag{3.24}$$

On the other hand,

$$\begin{aligned}&\langle \gamma f(x_n) - \gamma f(\tilde{x}), j(x_{n+1} - \tilde{x})\rangle \\ &\quad \le \gamma\alpha \|x_n - \tilde{x}\| \|x_{n+1} - \tilde{x}\| \\ &\quad \le \gamma\alpha \|x_n - \tilde{x}\| \left[\sqrt{\left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n |\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle|}\right] \\ &\quad \le \gamma\alpha \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 \\ &\qquad + \gamma\alpha \|x_n - \tilde{x}\| \sqrt{2|\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle|} \sqrt{\alpha_n} \\ &\quad \le \gamma\alpha \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 + \sqrt{\alpha_n} M_0, \end{aligned} \tag{3.25}$$

where M_0 is a constant satisfying $M_0 \geq \gamma\alpha\|x_n - \tilde{x}\|\sqrt{2|\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle|}$. Substituting (3.25) in (3.24), we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq & \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n\gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \\
& \|x_n - \tilde{x}\|^2 \\
& + 2\alpha_n\sqrt{\alpha_n}M_0 + 2\alpha_n\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle \\
= & \left(1 - 2\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) + \alpha_n^2\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\right)\|x_n - \tilde{x}\|^2 \\
& + 2\alpha_n\gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - \tilde{x}\|^2 \\
& + 2\alpha_n\sqrt{\alpha_n}M_0 + 2\alpha_n\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle \\
= & \left(1 - 2\alpha_n\left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha\gamma + \alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\right)\|x_n - \tilde{x}\|^2 + \\
& + \alpha_n\left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\|x_n - \tilde{x}\|^2 + 2M_0\sqrt{\alpha_n}\right. \\
& \left. + 2\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle\right]. \\
= & (1 - \alpha_n\gamma_n)\|x_n - \tilde{x}\|^2 + \alpha_n\gamma_n\frac{\beta_n}{\gamma_n},
\end{aligned} \tag{3.26}
$$

where

$$
\gamma_n = 2\left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha\gamma + \alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]
$$

and

$$
\beta_n = \left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\|x_n - \tilde{x}\|^2 + 2M_0\sqrt{\alpha_n} + 2\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x})\rangle\right].
$$

It is easily to seen that $\sum_{n=1}^{\infty}\alpha_n\gamma_n = \infty$. Since $\{x_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\alpha_n = 0$, by (3.23), we obtain $\limsup_{n\longrightarrow\infty}\frac{\beta_n}{\gamma_n} \leq 0$. Applying Lemma 2.4 to (3.26) to conclude $x_n \longrightarrow \tilde{x}$ as $n \longrightarrow \infty$. This completes the proof. □

Using Theorem 3.3, we obtain the following two strong convergence theorems of new iterative approximation methods for a nonexpansive semigroup $\{T(t) : 0 \leq t < \infty\}$.

Corollary 3.4 *Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let $\{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on C such that $Fix(\mathcal{S}) \neq \emptyset$ and at least there exists a $T(t)$ which is demicompact. Suppose that the real sequences $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ satisfy the conditions*

$$
\lim_{n\longrightarrow\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty \quad \text{and} \quad \lim_{n\longrightarrow\infty}t_n = \infty.
$$

Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta+\lambda > 1$, $f : E \longrightarrow E$ a contraction mapping with coefficient $\alpha \in (0,1)$ and γ a positive real number such that $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. Then, the sequence $\{y_n\}$ defined by (1.20) converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $Fix(\mathcal{S})$ of the variational inequality

$$\langle (F-\gamma f)\tilde{x}, j(x-\tilde{x})\rangle \geq 0, \;\; x \in Fix(\mathcal{S})$$

or equivalently $\tilde{x} = Q_{Fix(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{Fix(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $Fix(\mathcal{S})$.

Proof Let $\{x_n\}$ be the sequence given by $x_0 = y_0$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad \forall n \geq 0.$$

Form Theorem 3.3, $x_n \longrightarrow \tilde{x}$. We claim that $y_n \longrightarrow \tilde{x}$. Indeed, we estimate

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| &\leq \alpha_n\gamma\|f(T(t_n)y_n) - f(x_n)\| + \|I - \alpha_n F\|\|T(t_n)x_n - T(t_n)y_n\| \\
&\leq \alpha_n\gamma\alpha\|T(t_n)y_n - x_n\| + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&\leq \alpha_n\gamma\alpha\|T(t_n)y_n - T(t_n)\tilde{x}\| + \alpha_n\gamma\alpha\|T(t_n)\tilde{x} - x_n\| \\
&\quad + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&\leq \alpha_n\gamma\alpha\|y_n - \tilde{x}\| + \alpha_n\gamma\alpha\|\tilde{x} - x_n\| + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&\leq \alpha_n\gamma\alpha\|y_n - x_n\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\| + \alpha_n\gamma\alpha\|\tilde{x} - x_n\| \\
&\quad + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&= \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\alpha\right)\right)\|x_n - y_n\| \\
&\quad + \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\alpha\right)\frac{2\alpha\gamma}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\alpha\right)}\|\tilde{x} - x_n\|.
\end{aligned}$$

It follows from $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\longrightarrow\infty}\|x_n - \tilde{x}\| = 0$ and Lemma 2.4 that $\|x_n - y_n\| \longrightarrow 0$. Consequently, $y_n \longrightarrow \tilde{x}$ as required. □

Corollary 3.5 *Let E be a real reflexive strictly convex Banach space which has uniformly Gâteaux differentiable norm. Let $\{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on C such that $Fix(\mathcal{S}) \neq \emptyset$ and at least there exists a $T(t)$ which is demicompact. Suppose that the real sequences $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ satisfy the conditions*

$$\lim_{n\longrightarrow\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty \quad \text{and} \quad \lim_{n\longrightarrow\infty} t_n = \infty.$$

Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta+\lambda > 1$, $f : E \longrightarrow E$ a contraction mapping with coefficient $\alpha \in (0,1)$ and γ a positive real number such that

$0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. *Then, the sequence* $\{z_n\}$ *defined by* (1.21) *converges strongly to* $\tilde{x}$, *where* $\tilde{x}$ *is the unique solution in* $Fix(\mathcal{S})$ *of the variational inequality*

$$\langle (F-\gamma f)\tilde{x}, j(x-\tilde{x})\rangle \geq 0, \;\; x \in Fix(\mathcal{S})$$

or equivalently $\tilde{x} = Q_{Fix(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, *where* $Q_{Fix(\mathcal{S})}$ *is the sunny nonexpansive retraction of* E *onto* $Fix(\mathcal{S})$.

Proof Define the sequences $\{y_n\}$ and $\{\beta_n\}$ by

$$y_n = \alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n \text{and} \beta_n = \alpha_{n+1} \text{ for all } n \in \mathbb{N}.$$

Taking $p \in Fix(\mathcal{S})$, we have

$$\begin{aligned}
\|z_{n+1} - p\| &= \|T(t_n)y_n - T(t_n)p\| \leq \|y_n - p\| \\
&= \|\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n - (I - \alpha_n F)p - \alpha_n F p\| \\
&\leq \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|z_n - p\| + \alpha_n\|\gamma f(z_n) - F(p)\| \\
&= \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|z_n - p\| \\
&\quad + \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\frac{\|\gamma f(z_n) - F(p)\|}{\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)}.
\end{aligned} \tag{3.27}$$

It follows from induction that

$$\|z_{n+1} - p\| \leq \max\left\{\|z_0 - p\|, \frac{\|\gamma f(z_0) - F(p)\|}{1 - \sqrt{\frac{1-\delta}{\lambda}}}\right\}, \quad n \geq 0. \tag{3.28}$$

Thus both $\{z_n\}$ and $\{y_n\}$ are bounded. We observe that

$$y_{n+1} = \alpha_{n+1}\gamma f(z_{n+1}) + (I - \alpha_{n+1}F)z_{n+1} = \beta_n \gamma f(T(t_n)y_n) + (I - \beta_n F)T(t_n)y_n.$$

Thus Corollary 3.4 implies that $\{y_n\}$ converges strongly to some point $\tilde{x}$. In this case, we also have

$$\|z_n - \tilde{x}\| \leq \|z_n - y_n\| + \|y_n - \tilde{x}\| = \alpha_n\|\gamma f(z_n) - F z_n\| + \|y_n - \tilde{x}\| \longrightarrow 0.$$

Hence the sequence $\{z_n\}$ converges strongly to some point $\tilde{x}$. This complete the proof. □

Using Theorem 3.3, Lemma 2.1 and Example 2.2, we have the following result.

Corollary 3.6 *Let* E *be a real uniformly convex Banach space which has uniformly Gâteaux differentiable norm. Let* $\{T(t) : 0 \leq t < \infty\}$ *be a nonexpansive semigroup on* C *such that* $Fix(\mathcal{S}) \neq \emptyset$ *and at least there exists a* $T(t)$ *which is demicompact. Suppose that the real sequences* $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ *satisfy the conditions*

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \;\; \text{and} \;\; \lim_{n\longrightarrow\infty} t_n = \infty.$$

Let F be δ-strongly accretive and λ-strictly pseudo-contractive with $\delta+\lambda > 1$, $f : E \longrightarrow E$ *a contraction mapping with coefficient* $\alpha \in (0, 1)$ *and* γ *a positive real number such that* $0 < \gamma < \min\left\{\frac{\delta}{\alpha}, \frac{1-\sqrt{\frac{1-\delta}{\lambda}}}{2\alpha}\right\}$. *Then the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)\frac{1}{t_n}\int_0^{t_n} T(t)x_n ds, \quad n \geq 0 \end{cases}$$

converges strongly to $\tilde{x}$, *where* $\tilde{x}$ *is the unique solution in* $Fix(\mathcal{S})$ *of the variational inequality*

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0, \ \ x \in Fix(\mathcal{S})$$

or equivalently $\tilde{x} = Q_{Fix(\mathcal{S})}((I - F + \gamma f)\tilde{x})$, *where* $Q_{Fix(\mathcal{S})}$ *is the sunny nonexpansive retraction of E onto* $Fix(\mathcal{S})$.

Corollary 3.7 *Let H be a real Hilbert space. Let* $\{T(t) : 0 \leq t < \infty\}$ *be a nonexpansive semigroup on C such that* $Fix(\mathcal{S}) \neq \emptyset$ *and at least there exists a* $T(t)$ *which is demicompact. Suppose that the real sequences* $\{\alpha_n\} \subset [0, 1]$, $\{t_n\} \subset (0, \infty)$ *satisfy the conditions*

$$\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \ \text{ and } \ \lim_{n\longrightarrow\infty} t_n = \infty.$$

Let $f : E \longrightarrow E$ *be a contraction mapping with coefficient* $\alpha \in (0, 1)$ *and A a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > \frac{1}{2}$ *and* $0 < \gamma < (1 - \sqrt{2 - 2\overline{\gamma}})/\alpha$. *Then the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(t)x_n ds, \quad n \geq 0 \end{cases}$$

converges strongly to $\tilde{x}$, *where* $\tilde{x}$ *is the unique solution in* $Fix(\mathcal{S})$ *of the variational inequality*

$$\langle (A - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0, \ \ x \in Fix(\mathcal{S})$$

or equivalently $\tilde{x} = Q_{Fix(\mathcal{S})}((I - A + \gamma f)\tilde{x})$, *where* $Q_{Fix(\mathcal{S})}$ *is the sunny nonexpansive retraction of E onto* $Fix(\mathcal{S})$.

Proof Since A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, we have

$$\langle Ax - Ay, x - y\rangle \geq \overline{\gamma}\|x - y\|^2.$$

Therefore, A is $\overline{\gamma}$-strongly accretive. On the other hand,

$$\begin{aligned} \|(I - A)x - (I - A)y\|^2 &= \langle (x - y) - (Ax - Ay), (x - y) - (Ax - Ay)\rangle \\ &= \langle x - y, x - y\rangle - 2\langle Ax - Ay, x - y\rangle + \langle Ax - Ay, Ax - Ay\rangle \\ &= \|x - y\|^2 - 2\langle Ax - Ay, x - y\rangle + \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\langle Ax - Ay, x - y\rangle + \|A\|^2\|x - y\|^2. \end{aligned}$$

Since A is strongly positive if and only if $(\frac{1}{\|A\|})A$ is strongly positive, we may assume, without loss of generality, that $\|A\| = 1$, so that

$$\langle Ax - Ay, x - y\rangle \leq \|x - y\|^2 - \frac{1}{2}\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2$$
$$-\frac{1}{2}\|(x - y) - (Ax - Ay)\|^2.$$

Hence A is $\frac{1}{2}$-strongly pseudo-contractive. Applying Corollary 3.6, we conclude the result. □

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