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Original Article

On a weighted fuzzy knowledge measure with its application to multi-attribute decision-making

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Abstract

In a real-life problem with some ambiguity, the 'fuzzy entropy' measures the total amount of ambiguity associated with the fuzzy set. Analogous to this, a fuzzy knowledge measure may be considered for the total amount of precision present in a fuzzy set. Cognitively, fuzzy entropy and fuzzy knowledge measure seem to be dual concepts. We establish the duality of weighted fuzzy knowledge measure and weighted fuzzy entropy through characterization theorems. In many situations, every element of the universe of discourse may not be equally important for the expert. Therefore, a certain weight may be assigned to a particular member of the universe of discourse. In this work, we introduce a weighted fuzzy knowledge measure and demonstrate the effectiveness of the weighted fuzzy knowledge measure through a comparative study. We also discuss the application of the proposed weighted fuzzy knowledge measure in multi-attribute decision-making (MADM).

Keywords: fuzzy set, fuzzy entropy, weighted fuzzy knowledge measure, MADM

1. Introduction

In the contemporary world, most of the real life problems are pragmatically same in context of decision, management and prediction. The large amount of data as information is always accompanied by a large uncertainty. These two elements constitute the building blocks to deal with many complex problems. Shannon (1948) provided a measure of uncertainty or information in a random system. Shannon's measure of information requires probability distribution or probability density function, which can be obtained for data under consideration. Zadeh (1965) provided the concepts of fuzzy sets and fuzzy logic to cope with a variety of problems that require soft human reasoning. De Luca and Termini (1972) obtained a measure of fuzzy entropy/fuzzy information in the spirit of Shannon's (1948) entropy. Fuzzy entropy gives the amount of ambiguity/vagueness present in a fuzzy set. Some authors also consider it as a measure of ignorance entailed in a fuzzy set. Shannon's entropy (1948) and De Luca and Termini (1972) entropy are structurally similar but operationally different. Bhandari and Pal (1993) reviewed some measures of fuzziness and introduced some new measures of fuzzy entropy. It has been observed that in some events, the subjective considerations play an important role in many practical conditions of probabilistic nature. To incorporate such considerations in uncertainty quantification, Belis and Guiasu (1968) took into account a utility distribution $U=(u_1,u_2,...,u_n)$ with $u_i>0$ for the probability distribution $P=(p_1,p_2,...,p_n)$ and suggested a qualitative-quantitative measure of information. Guiasu (1971) termed this measure the weighted entropy. Various studies in the literature (Bhaker & Hooda, 1993; Guiasu, 1971; Srivastav, 2011) investigated the weighted entropy and useful entropy synonymously. Zadeh (1968) in the context of a discrete probabilistic framework made the first attempt to evaluate the ambiguity associated with a fuzzy event. He defined the weighted entropy $H(A,P)$ of a fuzzy event A with respect to (X,P) as

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$$H(A,P) = -\sum_{i=1}^{n} \mu_A(x_i)\, p_i \log p_i$$

where $\mu_A(x_i)$ is the membership function of A, and p_i is the probability of occurrence of x_i. In this work, we intend to investigate systems where subjective considerations are utilized to model expert and knowledge-based systems. Hooda and Bajaj (2010) introduced the notion of useful fuzzy information measure, which is based on utility, by fusing probabilities of randomness and uncertainties of fuzziness having utilities. They also defined total useful fuzzy information by recognizing usefulness of an event along with random uncertainties and fuzzy uncertainties, and a measure of useful fuzzy directed divergence was introduced.

Let $X = \{x_1, x_2, ..., x_n\}$ be the finite universe of discourse and $A = \{(x_i, \mu_A(x_i)) \mid x_i \in X\}$ be a fuzzy subset of *X*. Then entropy of the fuzzy set *A* gives the total amount of imprecision or ambiguity present in the fuzzy set *A*. In this situation, equal importances of each member of the universe of discourse have been assumed. However, sometimes one or more members of the universe of discourse may not be equally important to an expert. In view of this fact some amount of importance or weight may be attached with the members of the universe of discourse. Let $\{u_1, u_2, ..., u_n\}$ be the weights associated with $\{x_1, x_2, ..., x_n\}$, respectively. Then in this situation, in order to compute the amount of ambiguity or imprecision in the fuzzy set A, the idea of weighted fuzzy entropy has been introduced by Prakash, Sharma, and Mahajan (2008). Since then, in the recent literature related to fuzzy information measures when weights are assigned to certain parameters, the sum of the weights is assumed to be one. Therefore, to follow the convention, in this work we use the terminology 'weighted fuzzy knowledge measure' for our proposed measure. Further, we shall consider the sum of weights equals to one. Due to the adaptability feature and Zadeh's (1972) linguistic hedges some existing entropy measures may not be suitable to handle all such problems where computation of the ambiguity or imprecision of fuzzy set is desired. Guo (2016) and Szmidt, Kacprzyk, and Buinowski (2014) recognised the duality of intuitionistic fuzzy entropy and proposed a soft complementary dual of intuitionistic fuzzy entropy. This soft complementary dual was termed the intuitionistic fuzzy knowledge measure. Recently, Lalotra and Singh (2018) provided a novel knowledge measure of intuitionistic fuzzy sets and provided some characterization results for deriving a class of intuitionistic fuzzy knowledge measures. Singh, Lalotra, and Sharma (2019) investigated the concept of knowledge measure as a dual of the entropy in a fuzzy environment.

In expert and knowledge based systems, the data are created from knowledge of the experts. If these data are fuzzy, then a certain amount of ambiguity is present in the data. If some weights are assigned to the parameters in a fuzzy system, then the amount of fuzzy uncertainty/vagueness associated with the fuzzy data set changes. As already discussed, the amount of ambiguity present in the fuzzy set is quantified with fuzzy entropy/knowledge measure. In this way, weighted fuzzy entropy or weighted fuzzy knowledge measure provides the revised level of ambiguity due to weight assignment. The following facts motivated us to consider the present study.

1) To the best of our knowledge, only few studies (Hooda & Bajaj, 2010; Prakash, Sharma, & Mahajan, 2008) are available in the literature related to weighted information in fuzzy settings.

2) Incorporation of weights in fuzzy data is expected to provide robust results.

3) Investigation of duality between weighted fuzzy knowledge measure and weighted fuzzy entropy.

4) In the existing studies, no illustrative study has been done for weighted fuzzy entropy/weighted fuzzy knowledge measure.

5) In order to validate the operational aspect of dualism in weighted fuzzy entropy and weighted fuzzy knowledge measure, the formulation of a weighted fuzzy knowledge measure is essential. Further, the investigation of superiority of the proposed weighted fuzzy knowledge measure is also required.

The main contributions of this paper are as follows.

1) We propose a weighted knowledge measure in fuzzy environment and investigate some of its properties.

2) In terms of characterization theorems, we establish the duality of weighted fuzzy knowledge measure and weighted fuzzy entropy measure.

3) We investigate the application of the proposed weighted knowledge measure in MADM with the help of a numerical example.

4) We demonstrate the effectiveness of the proposed weighted fuzzy knowledge measure from the point of view of linguistic hedges, duality and weight computation in MADM problems.

The present paper is organized as follows.

Section 2 presents some preliminary definitions. In Section 3, we propose a weighted fuzzy knowledge measure and prove some of its properties. We also show its application in MADM in section 4. In Section 5, we perform a comparative study of our proposed weighted fuzzy knowledge measure with some existing weighted fuzzy entropy measures. Finally, Section 6 covers the conclusions.

2. Preliminaries

In this section, we provide some basic definitions related to our work.

Definition 2.1. (Zadeh, 1965) Let $X = \{x_1, x_2, ..., x_n\}$ be a universal set, then a fuzzy subset A of the universal set X is defined as

$$A = \{(x_i, \mu_A(x_i)) \mid x_i \in X; \mu_A(x_i) : X \to [0,1]\},$$

where $\mu_A(x_i) : X \to [0,1]$ represents a membership function.

The number $\mu_A(x_i)$ describes the extent of presence of $x_i \in X$ in A.

Definition 2.2. (Zadeh, 1965) Let $F(X)$ denote the family of all fuzzy sets in universe X then the following operations are defined on $F(X)$:

1) Union:

$A \cup B = \mu_A(x_i) \vee \mu_B(x_i) = \max(\mu_A(x_i), \mu_B(x_i)),$

where $A, B \in F(X),$

2) Intersection:

$A \cap B = \mu_A(x_i) \wedge \mu_B(x_i) = \min(\mu_A(x_i), \mu_B(x_i)),$

where $A, B \in F(X),$

3) Complement:

$\mu_{\overline{A}}(x_i) = 1 - \mu_A(x_i),$ where $A \in F(X).$

Definition 2.3. (De Luca & Termini, 1972) Let $A \in F(X)$ then a measure of fuzziness $H(A)$ of a fuzzy set A should have the following four properties:

(E1) $H(A)$ is minimum iff A is crisp set,

(E2) $H(A)$ is maximum iff A is the most fuzzy set,

(E3) $H(A) \geq H(A^*)$, where A^* is a sharpened version

(E4) $H(A) = H(\overline{A})$, where $\overline{A}$ is the complement set of A.

Definition 2.4. (Singh, Lalotra, & Sharma, 2019) Let $A \in F(X)$ then a measure of knowledge in a fuzzy set A should satisfy the following properties:

(K1) $K(A)$ is maximum iff A is crisp set i.e., $\mu_A(x_i) = 0$ or 1 for all $x_i \in X$,

(K2) $K(A)$ is minimum iff A is the most fuzzy set i.e., $\mu_A(x_i) = 0.5$ for all $x_i \in X$,

(K3) $K(A^*) \geq K(A)$, where A^* is a sharpened version,

(K4) $K(A) = K(\overline{A})$, where $\overline{A}$ is the complement set of A.

Definition 2.5. (Parkash, Sharma, & Mahajan, 2008) Let $X = \{x_1, x_2, ..., x_n\}$ be the universal set and $U = \{u_1, u_2, ..., u_n\}$ be the weight vector associated with Xi.e., u_i denotes the weight of x_i with condition $u_i > 0$. Let $A \in F(X)$ then Parkash, Sharma, and Mahajan (2008) provided axioms to define weighted fuzzy entropy measure as follows:

$(E_U 1)$ $H(A;U)$ is minimum iff A is crisp set,

$(E_U 2)$ $H(A;U)$ is maximum iff A is the most fuzzy set,

$(E_U 3)$ $H(A;U) \geq H(A^*;U)$, where A^* is a sharpened version,

$(E_U 4)$ $H(A;U) = H(\overline{A};U)$, where $\overline{A}$ is the complement set of A.

In view of axioms of weighted fuzzy entropy (Parkash, Sharma, & Mahajan, 2008) and definition 2.4, we can propose the axiomatic definition of the weighted fuzzy knowledge measure.

Definition 2.6. Let $A \in F(X)$, then a weighted knowledge measure for a fuzzy set A should satisfy the following properties:

$(K_U 1)$ $K(A;U)$ is maximum iff A is crisp set i.e., $\mu_A(x_i) = 0$ or 1 for all $x_i \in X$,

$(K_U 2)$ $K(A;U)$ is minimum iff A is the most fuzzy set i.e., $\mu_A(x_i) = 0.5$ for all $x_i \in X$,

$(K_U 3)$ $K(A^*;U) \geq K(A;U)$, where A^* is a sharpened version ,

$(K_U 4)$ $K(A;U) = K(\overline{A};U)$, where $\overline{A}$ is the complement set of A.

In the next section, we introduce a weighted fuzzy knowledge measure.

3. Weighted Fuzzy Knowledge Measure

Let $X=\{x_1,x_2,...,x_n\}$ be finite universe of discourse. $U=\{u_1,u_2,...,u_n\}$ be the weights associated with the elements of the universe of discourse such that $\sum_{i=1}^{n}u_i=1$, $u_i>0$. Then we define a weighted fuzzy knowledge measure as follows:

$$K(A;U)=\sum_{i=1}^{n}2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1 \qquad (1)$$

In the next theorem, we establish the validity of weighted fuzzy knowledge measure $K(A;U)$.

Theorem 3.1. $K(A;U)$ defined in Eq. (1) is a valid weighted fuzzy knowledge measure.

Proof. (K_U1) Firstly suppose $K(A;U)=1$

$$\Rightarrow\sum_{i=1}^{n}2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1=1$$

$$\Rightarrow\sum_{i=1}^{n}u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]=1$$

which is possible when $\mu_A(x_i)=0$ or 1.

Conversely, suppose $\mu_A(x_i)=0$ or 1 for all $x_i\in X$.

Then $K(A;U)=1$

Therefore, $K(A;U)$ is maximum if and only if A is a crisp set.

(K_U2) Firstly suppose that A is the most fuzzy set i.e. $\mu_A(x_i)=0.5$ for all $x_i\in X$.

Then,

$$K(A;U)=\sum_{i=1}^{n}2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1$$

$$=\sum_{i=1}^{n}u_i-1$$

$$=0.$$

Conversely, suppose that $K(A;U)$ is minimum. Then

$$K(A;U)=0$$

$$\Rightarrow\sum_{i=1}^{n}2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1=0$$

$$\Rightarrow\sum_{i=1}^{n}u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]=\frac{1}{2}$$

which is possible if $\mu_A(x_i)=\frac{1}{2}\ \forall\ 1\le i\le n$.

Therefore, $K(A;U)=0$ if and only if $\mu_A(x_i)=0.5$.

(K_U3) Let A^* be a sharpened version of A, i.e.,

1) *If* $\mu_A(x_i)\ge 0.5$, then $\mu_A(x_i)\le\mu_{A^*}(x_i)$;

2) $\mu_A(x_i)\ge\mu_{A^*}(x_i)$, otherwise.

Now,

$$K(A;U)=\sum_{i=1}^{n}2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1$$

Therefore, $\frac{\partial K(A;U)}{\partial\mu_A(x_i)}=u_i[8\mu_A(x_i)-4]$.

Case 1. When $0<\mu_A(x_i)<0.5$. Then

$$\frac{\partial K(A;U)}{\partial\mu_A(x_i)}<0.$$

Therefore, $K(A;U)$ is decreasing function of $\mu_A(x_i)$ satisfying $0<\mu_A(x_i)<0.5$.

Case 2. When $0.5<\mu_A(x_i)<1$. Then

$$\frac{\partial K(A;U)}{\partial\mu_A(x_i)}>0.$$

Therefore, $K(A;U)$ is increasing function of $\mu_A(x_i)$ satisfying $0.5<\mu_A(x_i)<1$.

Since $K(A;U)$ is decreasing function of $\mu_A(x_i)$ in [0, 0.5] and increasing function of $\mu_A(x_i)$ in [0.5,1]. Therefore,

$\mu_A(x_i)\le\mu_{A^*}(x_i)\Rightarrow K(A^*;U)\ge K(A;U)$ in [0,0.5] and

$\mu_A(x_i)\ge\mu_{A^*}(x_i)\Rightarrow K(A^*;U)\ge K(A;U)$ in [0.5,1].

Therefore, $K(A^*;U)\ge K(A;U)$.

(K_U4) We have

$$K(\bar{A};U)=\sum_{i=1}^{n}2u_i\left[\mu_{\bar{A}}^2(x_i)+(1-\mu_{\bar{A}}(x_i))^2\right]-1$$

$$= \sum_{i=1}^{n} 2u_i\left[(1-\mu_A(x_i))^2 + (1-1+\mu_A(x_i))^2\right] - 1$$

$$= \sum_{i=1}^{n} 2u_i\left[(1-\mu_A(x_i))^2 + (1-1+\mu_A(x_i))^2\right] - 1$$

$$= K(A;U).$$

$$\therefore\ K(A;U) = K(\bar{A};U).$$

Hence, $K(A;U)$ is a valid weighted knowledge measure.

Theorem 3.2. Let $K(A;U)$ and $K(B;U)$ be weighted know-ledge measures of fuzzy sets A and B respectively. Then $K(A\cup B;U)+K(A\cap B;U)=K(A;U)+K(B;U).$

Proof. We prove the result for two cases.

Case 1. When $\mu_A(x_i)\ge\mu_B(x_i)$. We have

$$K(A\cup B;U)+K(A\cap B;U)$$

$$= \sum_{i=1}^{n} 2u_i\left[\mu_{A\cup B}^2(x_i)+(1-\mu_{A\cup B}(x_i))^2\right]-1+\sum_{i=1}^{n} 2u_i\left[\mu_{A\cap B}^2(x_i)+(1-\mu_{A\cap B}(x_i))^2\right]-1$$

$$= \sum_{i=1}^{n} 2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1+\sum_{i=1}^{n} 2u_i\left[\mu_B^2(x_i)+(1-\mu_B(x_i))^2\right]-1$$

$$= K(A;U)+K(B;U).$$

Case 2. When $\mu_B(x_i)\ge\mu_A(x_i)$. We have

$$= \sum_{i=1}^{n} 2u_i\left[\mu_{A\cup B}^2(x_i)+(1-\mu_{A\cup B}(x_i))^2\right]-1+\sum_{i=1}^{n} 2u_i\left[\mu_{A\cap B}^2(x_i)+(1-\mu_{A\cap B}(x_i))^2\right]-1$$

$$= \sum_{i=1}^{n} 2u_i\left[\mu_B^2(x_i)+(1-\mu_B(x_i))^2\right]-1+\sum_{i=1}^{n} 2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1$$

$$= K(B;U)+K(A;U).$$

Therefore, $K(A\cup B;U)+K(A\cap B;U)=K(A;U)+K(B;U).$

Hence, the desired result follows.

Now to obtain more knowledge measures, we prove a characterization theorem.

Theorem 3.3. Let $F:[0,1]\to[0,1]$ be a mapping and the function $K_F: F(X)\to[0,1]$ be defined by $K_F(A;U)=\sum_{i=1}^{n} u_i F(\mu_A(x_i))$, then

1) K_F satisfy $(K_U 1)$ if and only if $F(0)=1=F(1),\ F(x)\neq 0\ \forall x\in(0,1),$
2) K_F satisfy $(K_U 2)$ if and only if $F(x)>F\left(\frac{1}{2}\right),\ \forall x\in[0.1]-\left\{\frac{1}{2}\right\},$

3) K_F satisfy $(K_U 3)$ if and only if $F(x)$ is increasing on [0.5,1] and $F(x)$ is decreasing on [0,0.5],

4) K_F satisfy $(K_U 4)$ if and only if $F(x) = F(1-x)\ \forall x \in [0,1]$.

Proof. The proof is similar to that of Theorem 3.3 in Singh, Lalotra, and Sharma (2019).

Theorem 3.4. Let $X = \{x_1, x_2, ..., x_n\}$ and KF : F(X) → [0,1] be $K_F(A;U) = \sum_{i=1}^{n} u_i\ F(\mu_A(x_i))$.

Then KF(*A;U*) satisfies $K_U 1 - K_U 4$ if and only if K_F for some function F:[0,1]→[0,1] satisfies 1 – 4 (as given in Theorem 3.3).

Theorem 3.5. Let $\phi : [0,1] \to [0,1]$ be a mapping and the function $E_\phi : F(X) \to [0,1]$ be defined by $E_\phi(A;U) = \sum_{i=1}^{n} u_i \phi(\mu_A(x_i))$,

then

1) E_ϕ satisfy $(E_U 1)$ if and only if $\phi(0) = 1 = \phi(1),\ \phi(x) \neq 0,\ \forall x \in (0,1)$,

2) E_ϕ satisfy $(E_U 2)$ if and only if $E(x) < E\left(\frac{1}{2}\right),\ \forall x \in [0.1] - \left\{\frac{1}{2}\right\}$,

3) E_ϕ satisfy $(E_U 3)$ if and only if $\phi(x)$ is decreasing on [0.5,1] and $\phi(x)$ is increasing on [0,0.5],

4) E_ϕ satisfy $(K_U 4)$ if and only if $\phi(x) = \phi(1-x),\ \ \forall x \in [0,1]$.

Theorem 3.6. Let $X = \{x_1, x_2, ..., x_n\}$ and $E_\phi : F(X) \to [0,1]$ be $E_\phi(A;U) = \sum_{i=1}^{n} u_i\ \phi(\mu_A(x_i))$. Then $E_\phi(A;U)$ satisfies $E_U 1 - E_U 4$ if and only if E_ϕ for some function $\phi : [0,1] \to [0,1]$ satisfies 1 - 4 (as given in Theorem 3.5).

Now, with the help of Theorem 3.4 and 3.6 we observe that

$$\phi(x) = 1 - F(x)$$

which further implies

$$E_\phi(A;U) = 1 - K_F(A;U)$$

This is an exact dual equation which connects the weighted fuzzy entropy and the weighted fuzzy knowledge measure.

3.1 Principle of minimum weighted fuzzy knowledge measure

Here, we provide an application of the newly introduced weighted fuzzy knowledge measure to study of the minimum weighted fuzzy knowledge. For this, we study the following problem:

Problem 1. Minimize

$$K(A;U) = \sum_{i=1}^{n} 2u_i \left[\mu_A^2(x_i) + (1-\mu_A(x_i))^2\right] - 1 . \qquad (2)$$

subject to the constraint

$$\sum_{i=1}^{n} \mu_A(x_i) = a . \qquad (3)$$

Consider the following Lagrangian

$$L=\frac{1}{n}\sum_{i=1}^{n}2u_i\left[\mu_A^2(x_i)+(1-\mu_A(x_i))^2\right]-1+\lambda\left\{\sum_{i=1}^{n}\mu_A(x_i)-a\right\}. \tag{4}$$

Thus, $\frac{\partial L(A;U)}{\partial \mu_A(x_i)}=0$ gives $\mu_A(x_i)=\frac{1}{2}\left[1-\frac{\lambda}{4u_i}\right]$.

From Eq. (3), we get

$$a=\sum_{i=1}^{n}\frac{1}{2}\left[1-\frac{\lambda}{4u_i}\right] \tag{5}$$

From Eq. (5), we get

$$\lambda=\frac{4\left(\frac{n}{2}-a\right)}{\sum_{i=1}^{n}\frac{1}{u_i}}. \tag{6}$$

Therefore, $\mu_A(x_i)=\frac{1}{2}\left[\frac{\left(\frac{n}{2}-a\right)}{u_i\sum_{i=1}^{n}\frac{1}{u_i}}+1\right]=\frac{1}{2}\left[\frac{\left(\frac{n}{2}-a\right)}{u_ik}+1\right]$, where $k=\sum_{i=1}^{n}\frac{1}{u_i}$.

Also, $\frac{\partial^2 K(A;U)}{\partial^2 \mu_A^2(x_i)}=8u_i>0$.

Therefore, $K(A;U)$ is minimum at $\mu_A(x_i)=\frac{2}{4u_i}\left[\left(\frac{n-a+ku_i}{k}\right)^2+\left(\frac{a-n+3ku_i}{k}\right)^2\right]-1$.

Special case: When each of the members has equal importance, then u_i may be considered as $\frac{1}{n}$.

Thus, $\mu_A(x_i)=\frac{1}{2}\left[\frac{n-a+2}{2n}\right]$.

Therefore, $K(A;U)$ is minimum at $\mu_A(x_i)=\frac{2an-n^2+2}{4}$.

4. Application of Weighted Knowledge Measure in MADM

Here, we discuss a MADM problem that involves some available alternatives and a set of attributes associated with each of the alternative. The basic MADM problem involves the selection of best alternative when weight of each attribute is known. Sometimes weights of attributes corresponding to the alternatives under consideration are not given. Then we derive the weights of attributes using some mathematical model. Fuzzy entropy model has been prevalently used by many authors. In the following, we consider weighted fuzzy knowledge measure based model for objective weight determination of attributes under consideration.

Let $X=\{x_1,x_2,...,x_n\}$ be the set of n-alternatives and $A=\{a_1,a_2,...,a_m\}$ be the set of m-attributes. Let $U=\{u_1,u_2,...,u_n\}$ be the weight vector associated with the members of the universe of discourse. A stepwise procedure to solve the MADM problem described above is as follows:

Step 1. Let $M=\lfloor\mu(x_i,a_j)\rfloor$ be the fuzzy decision matrix.

Step 2. Then for fuzzy entropy model the weight vector associated with attributes is given by

$$w_j = \frac{1 - E(a_j;U)}{m - \sum_{j=1}^{m} E(a_j;U)};\ j = 1,2,...,m. \tag{7}$$

Now, in the context of knowledge measure equation (7) can be modified as follows:

$$w_j = \frac{K(a_j;U)}{\sum_{j=1}^{m} K(a_j;U)};\ j = 1,2,...,m. \tag{8}$$

Step 3. The score function for each of the alternatives is given by

$$S(x_i) = \sum_{j=1}^{m} \mu_{ij} \times w_j, \tag{9}$$

Here, rating of alternatives x_i under attributes a_j is denoted by μ_{ij}. The score values obtained from equation (9) determine the best alternative. The alternative with highest score is considered as the best alternative.

Now, we demonstrate the above proposed algorithm in a numerical example.

Example 4.1. Suppose an exporter intends to locate markets in different countries. Let there be five markets (alternatives) $(x_1, x_2, x_3, x_4, x_5)$. The four attributes considered by an exporter regarding the market under consideration are: $a_1 =$ transportation charges, $a_2 =$ demand, $a_3 =$ customs duty, $a_4 =$ consumer diversity. Let $U = \{0.3, 0.15, 0.1, 0.25, 0.2\}$ be the weights given to the various alternatives as per their political stability.

The ratings of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ are given by the fuzzy decision matrix $M = [\mu(x_i, a_j)]_{5\times4} = \lfloor \mu_{ij} \rfloor_{5\times4}$ given by

$$\mathrm{M} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.621 & 0.2 & 0.3 \\ 0.7 & 0.7 & 0.2 & 0.84 \\ 0.12 & 0.41 & 0.6 & 0.5 \\ 0.3 & 0.7 & 0.9 & 0.1 \\ 0.3 & 0.2 & 0.8 & 0.2 \end{pmatrix} \end{array}$$

In this example we compare the results of the proposed weighted fuzzy knowledge measure and the following three weighted fuzzy entropies.

$$E_1(A;U) = -\sum_{i=1}^{n} u_i \left[\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))\right],$$

(Parkash, Sharma, & Mahajan, 2008)

$$E_2(A;U) = \sum_{i=1}^{n} u_i \left[\sin \frac{\pi \mu_A(x_i)}{2} + \sin \frac{\pi(1 - \mu_A(x_i))}{2} - 1\right],$$ (Parkash, Sharma, & Mahajan, 2008)

And

$$E_3(A;U) = \sum_{i=1}^{n} u_i \left[\cos \frac{\pi \mu_A(x_i)}{2} + \cos \frac{\pi(1 - \mu_A(x_i))}{2} - 1\right].$$

(Parkash, Sharma, & Mahajan, 2008).

On implementing the procedure of MADM on various weighted fuzzy entropies and the proposed weighted fuzzy knowledge measure for the decision matrix and weight vector *U*, we obtain different weights of criteria using different weighted fuzzy entropies and our proposed weighted fuzzy knowledge measure shown in Table 1. We have the following observations from this table.

1. For the weighted fuzzy entropy measures $E_1(A;U)$, $E_2(A;U)$ and $E_3(A;U)$, the range of values of the weights of alternatives is very small. For example, the highest and lowest weights in $E_1(A;U)$ are 0.2814 and 0.2187, respectively. Here, the difference is 0.0627. Therefore, these weighted fuzzy entropies are suitable for the decision-makers with no evident preferences towards various criteria.

2. For the weighted fuzzy knowledge measure $K(A;U)$ the different values between minimum and maximum weights

Table 1. Weight vectors corresponding to various weighted fuzzy entropies and the proposed weighted fuzzy knowledge measure.

	w_1	w_2	w_3	w_4
$K(A;U)$	0.1552	0.1523	0.3465	0.3460
$E_1(A;U)$	0.2187	0.2187	0.2814	0.2813
$E_2(A;U)$	0.2355	0.2349	0.2648	0.2648
$E_3(A;U)$	0.2355	0.2349	0.2648	0.2648

are significantly larger than the different values calculated by the weighted fuzzy entropies under consideration. Thus, the weighted fuzzy knowledge measure $K(A;U)$ is suitable for decision-makers with evident preferences towards criteria.

The score values of various alternatives are given in Table 2.

Thus, we observe that the best alternative due to our proposed weighted knowledge measure $K(A;U)$ and some existing entropies E_1, E_2 and E_3 remains same. However, there is variation in preference of the worst two alternatives.

In the next section, we demonstrate the superiority/significance of our proposed weighted fuzzy knowledge measure on some existing weighted fuzzy entropy measures through a comparative study.

5. Comparative Study

5.1 Comparison based on linguistic hedges

A fuzzy set is always equipped with certain level of vagueness/ambiguity. In order to model systems having uncertain information due to ambiguity, some structured linguistic terminologies (linguistic variables) were introduced by Zadeh (1972). The linguistic terms like 'very', 'more or less', 'slightly', 'very very', etc., represent structured ambiguity to be dealt with in an expert-based system. Zadeh (1972) quantitatively represented these structured linguistic terms as follows:

'$A^{\frac{1}{2}}$' - 'More or Less Large,

'A^2' -'Very Large',

'A^3' -'Quite Very Large' and

'A^4'-'Very Very Large'.

To represent the above quantified structured linguistic terms in general, Zadeh (1972) suggested the following notation:

$$A^n = \begin{cases} CON(A), & n > 1; \\ DIL(A), & n < 1. \end{cases}$$

Some prominent illustrative studies in this regard are:
De, Biswas, and Roy (2000), Hung and Yang (2006), Hwang and Yang (2008), Verma and Sharma (2014), and Xia and Xu (2012).

Now we present a comparative analysis of our proposed weighted fuzzy knowledge measure in an illustrative example.

Example 5.1. For a fuzzy set A in X, the modifier is given as

$$A^n = \left\{ x, (\mu_A(x))^n) \middle| x \in X \right\}$$

Consider a fuzzy set A in $X = \{1,2,3,4,5\}$ given by

$$A = \{(1,0.3),(2,0.70),(3,0.35),(4,0.5),(5,0.2)\}.$$

Let the weight vector corresponding to X be $U = \{0.3, 0.15, 0.1, 0.25, 0.2\}$.

The following fuzzy sets can be generated with the help of above operations:

$$A^{\frac{1}{2}} = \{(1, 0.5477), (2, 0.8367), (3, 0.5916), (4, 0.7071), (5, 4472)\},$$

$$A^2 = \{(1, 0.0900), (2, 0.4900), (3, 0.1225), (4, 0.2500), (5, 0.0400)\},$$

$$A^3 = \{(1, 0.0270), (2, 0.3430), (3, 0.0429), (4, 0.1250), (5, 0.0080)\},$$

and

$$A^4 = \{(1, 0.0081), (2, 0.2401), (3, 0.0150), (4, 0.0625), (5, 0.0016)\}.$$

Table 2. Preference orders for various weighted fuzzy entropies and the proposed weighted fuzzy knowledge measure.

	S_1	S_2	S_3	S_4	S_5	Preference order
$K(A;U)$	0.3451	0.6445	0.4620	0.4997	0.4235	$S(x_2) > S(x_4) > S(x_3) > S(x_5) > S(x_1)$
$E_1(A;U)$	0.3858	0.6550	0.4254	0.5001	0.3907	$S(x_2) > S(x_4) > S(x_3) > S(x_5) > S(x_1)$
$E_2(A;U)$	0.3958	0.6575	0.4158	0.4997	0.4235	$S(x_2) > S(x_4) > S(x_3) > S(x_1) > S(x_5)$
$E_3(A;U)$	0.3958	0.6575	0.4158	0.4997	0.4235	$S(x_2) > S(x_4) > S(x_3) > S(x_1) > S(x_5)$

In view of the mathematical operations, the following order should be followed by the weighted fuzzy entropies:

$$E(A^{\frac{1}{2}};U) > E(A;U) > E(A^2;U) > E(A^3;U) > E(A^4;U). \quad (10)$$

Now in the context of knowledge, the weighted fuzzy knowledge measure should follow this order:

$$K(A^{\frac{1}{2}};U) < K(A;U) < K(A^2;U) < K(A^3;U) < K(A^4;U). \quad (11)$$

For the fuzzy entropies $E_1(A;U), E_2(A;U)$ and $E_3(A;U)$, the comparative results are shown in Table 3.

Now, from Table 3 we observe that

$E_1(A^{\frac{1}{2}};U) > E_1(A;U) > E_1(A^2;U) > E_1(A^3;U) > E_1(A^4;U),$

$E_2(A^{\frac{1}{2}};U) > E_2(A;U) > E_2(A^2;U) > E_2(A^3;U) < E_2(A^4;U),$

$E_3(A^{\frac{1}{2}};U) > E_3(A;U) > E_3(A^2;U) > E_3(A^3;U) < E_3(A^4;U),$

and $K(A^{\frac{1}{2}};U) < K(A;U) < K(A^2;U) < K(A^3;U) < K(A^4;U).$

Here, we observed that only the entropy $E_1(A;U)$ satisfies the requirement (10) and $K(A;U)$ satisfies (11), which are the desired requirements.

When $B = \{(1,0.2),(2,0.80),(3,0.35),(4,0.6),(5,0.3)\}$, from Table 4 we observe that

$E_1(B^{\frac{1}{2}};U) > E_1(B;U) > E_1(B^2;U) > E_1(B^3;U) > E_1(B^4;U),$

$E_2(B^{\frac{1}{2}};U) > E_2(B;U) > E_2(B^2;U) > E_2(B^3;U) < E_2(B^4;U),$

$E_3(B^{\frac{1}{2}};U) > E_3(B;U) > E_3(B^2;U) > E_3(B^3;U) < E_3(B^4;U),$

and $K(B^{\frac{1}{2}};U) < K(B;U) < K(B^2;U) < K(B^3;U) < K(B^4;U).$

Here, we observe that only the entropy $E_1(B;U)$ satisfies the requirement (10) and $K(B;U)$ satisfies (11).

When $C = \{(1,0.1),(2,0.60),(3,0.35),(4,0.5),(5,0.3)\}$, from Table 5 we observe that

$E_1(C^{\frac{1}{2}};U) > E_1(C;U) > E_1(C^2;U) > E_1(C^3;U) > E_1(C^4;U),$

Table 3. Results of measures of fuzziness with different information measures.

Fuzzy set	$E_1(A;U)$	$E_2(A;U)$	$E_3(A;U)$	$K(A;U)$
$A^{\frac{1}{2}}$	0.6297	0.3631	0.3631	0.1192
A	0.6130	0.3483	0.3483	0.1530
A^2	0.4061	0.2075	0.2075	0.4906
A^3	0.2549	0.1211	0.1211	0.7011
A^4	0.1654	0.3483	0.3483	0.8151

Table 4. Results of measures of fuzziness with different information measures.

Fuzzy set	$E_1(B;U)$	$E_2(B;U)$	$E_3(B;U)$	$K(A;U)$
$B^{\frac{1}{2}}$	0.5956	0.3387	0.3387	0.1773
B	0.5804	0.3227	0.3227	0.2130
B^2	0.4094	0.2138	0.2138	0.4768
B^3	0.2909	0.1496	0.1496	0.6338
B^4	0.2187	0.3227	0.3227	0.7278

Table 5. Results of measures of fuzziness with different information measures.

Fuzzy set	$E_1(C;U)$	$E_2(C;U)$	$E_3(C;U)$	$K(C;U)$
$C^{\frac{1}{2}}$	0.6238	0.3564	0.3564	0.1338
C	0.5587	0.3128	0.3128	0.2390
C^2	0.3531	0.1818	0.1818	0.5539
C^3	0.2174	0.1006	0.1006	0.7504
C^4	0.1338	0.3128	0.3128	0.8613

$E_2(C^{\frac{1}{2}};U) > E_2(C;U) > E_2(C^2;U) > E_2(C^3;U) < E_2(C^4;U),$

$E_3(C^{\frac{1}{2}};U) > E_3(C;U) > E_3(C^2;U) > E_3(C^3;U) < E_3(C^4;U),$

and $K(C^{\frac{1}{2}};U) < K(C;U) < K(C^2;U) < K(C^3;U) < K(C^4;U).$

Here, we observe that the entropy $E_1(C;U)$ satisfies the requirement (10) and $K(C;U)$ satisfies (11). So, the performance of the proposed weighted fuzzy knowledge measure is better than of some of the existing weighted fuzzy entropy measures.

5.2 Comparison based on duality

We have derived a dual equation between weighted fuzzy entropy and weighted fuzzy knowledge measure as follows:

$$K(A;U)=1-E(A;U).$$

If the fuzzy knowledge measure determines the knowledge in A and the entropy measure determines ignorance/ambiguity present in a fuzzy set precisely. Theoretically, a normal weighted fuzzy knowledge measure and normal weighted fuzzy entropy of fuzzy set sums up to 1. Due to adaptive nature of fuzzy entropies, the equation of duality does not hold in many practical situations. Thus, in a given situation if sum of the weighted fuzzy entropy and weighted fuzzy knowledge measure is close to 1 then such measures are good to use in practical problems.

In Table 3, we observe for entropy $E_1(A;U)$ and proposed weighted fuzzy knowledge measure $K(A;U)$, we have

$$E_1(A^{\frac{1}{2}};U)+K(A^{\frac{1}{2}};U)=0.7489,$$

$$E_1(A;U)+K(A;U)=0.766,$$

$$E_1(A^2;U)+K(A^2;U)=0.8967,$$

$$E_1(A^3;U)+K(A^3;U)=0.956,$$

$$E_1(A^4;U)+K(A^4;U)=0.9805.$$

In case of a fuzzy set A, we observe that the sum of weighted fuzzy entropy E_1 and weighted fuzzy knowledge measure K is close to 1 for A^3 and A^4. Therefore, significantly good amount of information is captured by the entropy-knowledge pair (E_1, K) if the fuzzy set A is concentrated more and more. For $A^{\frac{1}{2}}$ and A, the sum of weighted fuzzy entropy E_1 and weighted fuzzy knowledge measure K is significantly less than one. Therefore, there is a loss of information regarding the fuzzy sets $A^{\frac{1}{2}}$ and A. Similar results are observed in case of entropies E_2 and E_3 for the fuzzy sets *A*, B and *C*. Thus our proposed weighted fuzzy knowledge measure enables us to identify the suitability of the weighted fuzzy entropy in a given situation (for concentrated fuzzy sets in the present case). In this way, weighted fuzzy entropy-knowledge pair captures the information entailed in a fuzzy set in more comprehensive manner than the weighted fuzzy entropy/knowledge measure alone.

6. Conclusions

In this work, we have established the following:

1) Our suggested weighted fuzzy knowledge measure is more effective than some of the existing weighted fuzzy entropy measures while dealing with linguistic hedges.
2) The proposed weighted fuzzy knowledge measure was also found to have a dual character (operationally and structurally) with respect to the weighted fuzzy entropy.
3) In MADM problem, the importance of the proposed weighted fuzzy knowledge measure is explained in the situation when the decision maker has evident preferences towards criteria.
4) With the help of illustrative example, it has been observed that the information contained in a fuzzy set can be captured more comprehensively with the help of both weighted fuzzy entropy and weighted fuzzy knowledge measure.

In context of the present work, our future studies include:

1) The development of more flexible weighted fuzzy knowledge measures by inclusion of parameter(s) to handle the effect of external influences on the fuzzy system.
2) The development of a hybrid model dealing with qualitative information in ambiguous/vague environment using the notions of weighted fuzzy entropy and weighted fuzzy knowledge measure.

3) In the illustrative study in Section 5.2, it has been observed that the weighted fuzzy entropy-knowledge pair accounted for the information content of a fuzzy set in more concentrated form. Therefore, in this regard, there is a scope for the development of weighted fuzzy entropy and weighted fuzzy knowledge measure which provide complete information of a fuzzy set in concentrated as well as in dilated form.

4) The development of novel decision making models involving combined weights of criteria's as done in Peng and Garg (2018).

5) Extension of the present study in Pythagorean fuzzy environment as done in Peng and Selvachandran (2017) and Peng, Yuan, and Yang (2017).

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References

Belis, M., & Guiasu, S. (1968). A quantitative-qualitative measure of information in cybernetic systems. *IEEE Transactions in Information Theory IT, 14*, 593–594.

Bhaker, U. S., & Hooda, D. S. (1993). Mean value characterization of useful information measures. *Tamkang Journal of Mathematics*, *24*, 383–394.

Bhandari, D., & Pal, N. R. (1993). Some new information measures for fuzzy sets.*Information Sciences*, *67*, 204-228.

De Luca, A., & Termini, S. (1972). A definition of non-probabilistic entropy in the setting of fuzzy set theory. *Information and Control, 20*, 301–312.

De, S. K., Biswas R., & Roy A. R., (2000). Some operations on intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 114*, 477-484.

Guiasu, S. (1971). Weighted Entropy. *Reports on Mathematical Physics, 2*, 165-179.

Guo, K. (2016). Knowledge measure for Atanassov's intuitionistic fuzzy sets. *IEEE Transactions on Fuzzy Systems, 24*, 221072-1078.

Hooda, D. S., & Bajaj, R. K., (2010). Useful fuzzy measures of information, integrated ambiguity and directed divergence. *International Journal of General Systems, 39*, 647-658.

Hung, W. L., & Yang, M. S., (2006). Fuzzy entropy on intuitionistic fuzzy sets. *International Journal of Intelligent Systems, 21*, 443-451.

Hwang, C. H., & Yang, M. S. (2008). Entropy of fuzzy sets. *International Journal of Uncertainty Fuzziness Knowledge-Based Systems, 16*, 519-527.

Lalotra, S., & Singh, S. (2018). On a knowledge measure and an unorthodox accuracy measure of an intuitionistic fuzzy set(s) with their applications. *International Journal of Computational Intelligence Systems, 11*, 1338-1356.

Parkash, O., Sharma, P. K., & Mahajan, R. (2008). New measures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle. *Information Sciences, 178*, 2389-2395.

Peng, X., & Garg, H. (2018). Algorithms for interval-valued fuzzy soft sets in emergency decision making based on WDBA and CODAS with new information measure. *Computers & Industrial Engineering, 119*, 439-452.

Peng, X., & Selvachandran, G. (2017). Pythagorean fuzzy set: State of the art and future directions. *Artificial Intelligence Review*. doi:10.1007/s10462-017-9596-9

Peng, X., Yuan, H., & Yang, Y. (2017). Pythagorean fuzzy information measures and their applications. *International Journal of Intelligent Systems*, 1-39.

Renyi, A. (1961). On measures of entropy and information. Proceedings of 4th Berkeley Symposium Mathematics, Statistics and Probability, *1*, 547–561.

Shannon, C. E. (1948). The Mathematical theory of Communication. *Bell System Technical Journal, 27*, 423–467.

Singh, S., Lalotra, S., & Sharma, S. (2019). Dual concepts in fuzzy theory: Entropy and knowledge measure. *International Journal of Intelligent Systems*, 1-26, doi:10.1002/int.22085ss

Srivastav, A. (2011). Some new bonds of weighted entropy measures. *Cybernetics and Information Technologies*, *11*, 60–65.

Szmidt, E., Kacprzyk, J., & Buinowski, P. (2014). How to measure amount of knowledge conveyed by Atanassov's intuitionistic fuzzy sets. *Information Sciences*, *7*, 276-285.

Verma , R., & Sharma, B. D. (2014). On intuitionistic fuzzy entropy of order α. *Advances in Fuzzy Systems,* 1-8. Article ID 789890.

Xia, M., & Xu, Z. (2012). Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment. *Informaton Fusion*, *13*, 31-47.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*, 338–353.

Zadeh, L. A. (1968). Probability measures of fuzzy events. *Journal of Mathematical Analysis and Applications, 23*, 421–427.

Zadeh, L. A. (1972). A fuzzy-set-theoretic interpretation of linguistic hedges. *Journal of Cybernetics*, *2*, 4-34.