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Short Communication

A simple closed-form formula for the conditional moments of the Ornstein-Uhlenbeck process

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Abstract

This paper derives a simple closed-form formula for the n^{th} conditional moment of the Ornstein-Uhlenbeck (O-U) process, for any positive integer n. The system of recursive ordinary differential equations (ODEs) associated with the n^{th} conditional moment of the O-U process is solved analytically. We also provide practitioners a pseudocode for an algorithm to compute the conditional moments and discuss the efficiency of our formula compared to solving the system of recursive ODEs using the direct method.

Keywords: O-U process, conditional expectation, closed-form formula

1. Introduction

The Ornstein-Uhlenbeck (O-U) process is often known as a mean-reverting process satisfying the stochastic differential equation (SDE)

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t, \tag{1.1}$$

where κ is the speed of adjustment of the rate towards its long term mean a, σ is the volatility control, and W_t is the standard Brownian motion under a probability (Ω, F, P) with a filtration $(F_t)_{t\geq 0}$. The most important feature which this process exhibits is the mean reversion, which means that if the rate X_t is bigger than the long term mean a, then the coefficient κ makes the drift become negative so that the rate will be pulled down in the equilibrium direction of a, and similarly if the rate is smaller than a. Therefore, κ is the speed of adjustment of the rate towards its long run mean. The solution of the SDE (1.1) can be obtained by Ito's lemma as

$$X_t = X_0 e^{-\kappa t} + \alpha\left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s} dW_s$$

for a given initial X_0 (Karatzas and Shreve, 1991; Levy, 2016) for more details). Moreover, the O-U process is a continuous time version of the first-order autoregressive process in discrete time and a special case of the Schwartz model.

The O-U process is interesting in finance because there are also compelling economic arguments in favor of mean reversion. When the interest rates are high, the economy tends to slow down and borrowers require less funds. Furthermore, the interest rates pull back to its equilibrium value and the rates decline. On the contrary when the rates are slow, there tends to be high demand for funds on the part of the borrowers and rates tend to increase. This process has been applied apparently under the title of the Vasicek (1997) to describe quantities such as interest rates where there is some underlying reason to ban indefinite growth and require mean reversion.

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In some cases of pricing financial derivatives based on commodities described by Schwartz model (1997), we may need to calculate a conditional moment in the form of

$$E^P\left[X_T^n \mid F_t\right] = E^P\left[X_T^n \mid X_t = x\right] \tag{1.2}$$

for any positive integer $n, 0 \le t \le T$ and $x > 0$ where we denote by $E^P\left[X \mid F_t\right]$, the conditional expectation of a random variable X with respect to the probability measure P and σ-field F_t. For example, Weraprasertsakun and Rujivan (2017) applied the Feynman-Kac theorem to obtain the n^{th} conditional moment of the O-U process in the form

$$E^P\left[X_T^n \mid X_t = x\right] = \left(\sum_{j=0}^{n} A_j^{(n)}(\tau)x^j\right)e^{-n\kappa\tau} \tag{1.3}$$

with $A_n^{(n)}(\tau) = 1$ by solving the system of recursive ordinary differential equations (ODEs)

$$\frac{dA_j^{(n)}}{d\tau} = \kappa(n-j)A_j^{(n)}(\tau) + (j+1)\kappa\alpha A_{j+1}^{(n)}(\tau) + \frac{1}{2}(j+1)(j+2)\sigma^2 A_{j+2}^{(n)}(\tau) \tag{1.4}$$

subject to the initial conditions

$$A_j^{(n)}(0) = 0 \text{ for all } j = n-1, n-2, \ldots, 0, \tag{1.5}$$

providing that $A_n^{(n)}(\tau) = 1$ and $A_{n+1}^{(n)}(\tau) = 0$ for all $\tau = T - t \ge 0$ with $n = 1, 2.$ Unfortunately, they solved the recursive ODEs (1.4) only for $n = 1, 2,$ and used the solutions to derive a closed-form formula for pricing variance swap on a commodity. Moreover, they did not derive explicit formulas for $A_j^{(n)}(\tau), j = n-1, n-2, ..., 0$ for *n* > = 3. Therefore, we shall complete their work by deriving a closed-form formula for $A_j^{(n)}(\tau), j = n-1, n-2, ..., 0$ for *n* > = 3. The result obtained in this paper will be useful for the researchers who try to find a closed-form formula for pricing skewness swaps, kurtosis swaps, or higher moment swaps.

Although solving the system of recursive ODEs (1.4) subject to the initial conditions (1.5) can be done by using the direct method, it is a tedious task and consumes much computational time and effort due to the cumbersome nature of the recursive ODEs. Therefore, a main contribution of the paper is to provide a simple closed-form formula for the coefficient functions $A_j^{(n)}(\tau)$ for all $j = n-1, n-2, \ldots, 0,$ for any positive integer *n*. Furthermore, we provide practitioners a pseudocode for an algorithm to compute the coefficient functions and so are the conditional expectation (1.3). Finally, we discuss the efficiency of our formula compare to solving the system of recursive ODEs using the direct method.

2. Main Results

A simple formula for the coefficient functions $A_j^{(n)}(\tau)$ for all $j = n-1, n-2, \ldots, 0,$ for any positive integer *n*, can be obtained as shown in the following theorem.

Theorem 2.1 The solution (1.4) can be written in the form

$$A_j^{(n)}(\tau)=\left(\prod_{r=0}^{n-j-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{n-j}{2}\right\rfloor}\frac{1}{\kappa^l}\alpha^{n-j-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{n-j-l}\left(e^{\kappa\tau}+1\right)^l c_{n,j}^{(l)} \tag{2.1}$$

for $\tau=T-t\geq 0$, where $c_{n,j}^{(l)}$ is defined using $j=n-k$ as an index in

$$\bar{C}_{n,k}=\begin{bmatrix}c_{n,n-k}^{(0)}\\ c_{n,n-k}^{(1)}\\ \vdots\\ c_{n,n-k}^{\left(\left\lfloor\frac{k}{2}\right\rfloor\right)}\end{bmatrix}\in\mathrm{R}^{\left\lfloor\frac{k}{2}\right\rfloor+1}, \tag{2.2}$$

which is defined recursively on k as follow;

$$\bar{C}_{n,1}=\left[c_{n,n-1}^{(0)}\right]=[1],\qquad \bar{C}_{n,2}=\begin{bmatrix}c_{n,n-2}^{(0)}\\ c_{n,n-2}^{(1)}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{4}\end{bmatrix}, \tag{2.3}$$

for odd $k\geq 3$,

$$\bar{C}_{n,k}=\frac{1}{k}\left(\bar{C}_{n,k-1}+\frac{1}{2}\begin{bmatrix}0\\ \bar{C}_{n,k-2}\end{bmatrix}\right), \tag{2.4}$$

and for even $k\geq 4$,

$$\bar{C}_{n,k}=\frac{1}{k}\left(\begin{bmatrix}\bar{C}_{n,k-1}\\ 0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\ \bar{C}_{n,k-2}\end{bmatrix}\right). \tag{2.5}$$

Proof. It suffices to show that the solution of

$$\frac{dA_{n-k}^{(n)}}{d\tau}-k\kappa A_{n-k}^{(n)}(\tau)=\left((n-k)+1\right)\kappa\alpha A_{(n-k)+1}^{(n)}(\tau)+\frac{1}{2}\left((n-k)+1\right)\left((n-k)+2\right)\sigma^2 A_{(n-k)+2}^{(n)}(\tau) \tag{2.6}$$

with conditions (1.5) when $j=n-k$ is

$$A_{n-k}^{(n)}(\tau)=\left(\prod_{r=0}^{k-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\frac{1}{\kappa^l}\alpha^{k-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{k-l}\left(e^{\kappa\tau}+1\right)^l c_{n,n-k}^{(l)} \tag{2.7}$$

where $c_{n,n-k}^{(l)}$ for $k=1,\ldots,n$ is defined through (2.3)-(2.5).

For $k=1,$ the equation (2.6) is reduced to

$$\frac{dA_{n-1}^{(n)}}{d\tau}-\kappa A_{n-1}^{(n)}(\tau)=n\kappa\alpha,$$

with the solution subject to the initial condition (1.5) when $j=n-1,$

$$A_{n-1}^{(n)}(\tau)=e^{\kappa\tau}\left[-n\alpha e^{-\kappa\tau}+n\alpha\right].$$

This can be written in the form of (2.7) when $k=1$ as

$$A_{n-1}^{(n)}(\tau)=\left(\prod_{r=0}^{1-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{1}{2}\right\rfloor}\frac{1}{\kappa^{l}}\alpha^{1-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{1-l}\left(e^{\kappa\tau}+1\right)^{l}c_{n,n-1}^{(l)},$$

where $c_{n,n-1}^{(0)}=1$. For $k=2,$ the equation (2.6) becomes

$$\frac{dA_{n-2}^{(n)}}{d\tau}-2\kappa A_{n-2}^{(n)}(\tau)=(n-1)\kappa\alpha A_{n-1}^{(n)}(\tau)+\frac{1}{2}(n-1)n\sigma^{2},$$

with the solution subject to the initial condition (1.5) when $j=n-2$,

$$A_{n-2}^{(n)}(\tau)=n(n-1)e^{2\kappa\tau}\left[-\alpha^{2}e^{-\kappa\tau}+\frac{1}{2}\alpha^{2}e^{-2\kappa\tau}-\frac{1}{4\kappa}\sigma^{2}e^{-2\kappa\tau}+\frac{1}{2}\alpha^{2}+\frac{1}{4\kappa}\sigma^{2}\right].$$

By writing in the form of (2.7) when $k=2,$ we get

$$\begin{aligned}A_{n-2}^{(n)}(\tau)&=n(n-1)\left[\frac{1}{2}\alpha^{2}\left(e^{\kappa\tau}-1\right)^{2}+\frac{1}{4\kappa}\sigma^{2}\left(e^{\kappa\tau}-1\right)\left(e^{\kappa\tau}+1\right)\right]\\&=\left(\prod_{r=0}^{2-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{2}{2}\right\rfloor}\frac{1}{\kappa^{l}}\alpha^{2-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{2-l}\left(e^{\kappa\tau}+1\right)^{l}c_{n,n-2}^{(l)},\end{aligned}$$

where $c_{n,n-2}^{(0)}=\frac{1}{2}$ and $c_{n,n-2}^{(1)}=\frac{1}{4}.$ Similarly, for $k=3,$ the equation (2.6) becomes

$$\frac{dA_{n-3}^{(n)}}{d\tau}-3\kappa A_{n-3}^{(n)}(\tau)=(n-2)\kappa\alpha A_{n-2}^{(n)}(\tau)+\frac{1}{2}(n-2)(n-1)\sigma^{2}A_{n-1}^{(n)}(\tau)$$

with the solution in integral form

$$\begin{aligned}A_{n-3}^{(n)}(\tau)&=e^{3\kappa\tau}\int e^{-3\kappa\tau}(n-2)\kappa\alpha A_{n-2}^{(n)}(\tau)\,d\tau+e^{3\kappa\tau}\int e^{-3\kappa\tau}\frac{1}{2}(n-2)(n-1)\sigma^{2}A_{n-1}^{(n)}(\tau)\,d\tau\\&=e^{3\kappa\tau}\int e^{-3\kappa\tau}(n-2)\kappa\alpha\left(\left(\prod_{r=0}^{1}(n-r)\right)\sum_{l=0}^{1}\frac{1}{\kappa^{l}}\alpha^{2-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{2-l}\left(e^{\kappa\tau}+1\right)^{l}c_{n,n-2}^{(l)}\right)d\tau\\&\quad+e^{3\kappa\tau}\int e^{-3\kappa\tau}\frac{1}{2}(n-2)(n-1)\sigma^{2}\left(\left(\prod_{r=0}^{0}(n-r)\right)\sum_{l=0}^{0}\frac{1}{\kappa^{l}}\alpha^{1-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{1-l}\left(e^{\kappa\tau}+1\right)^{l}c_{n,n-1}^{(l)}\right)d\tau\\&=:R_{0}^{(1)}(\tau,3)+R_{1}^{(1)}(\tau,3)+R_{0}^{(2)}(\tau,3),\end{aligned}$$

where

$$\begin{aligned}R_{0}^{(1)}(\tau,3)&=\left(\prod_{r=0}^{2}(n-r)\right)\frac{1}{2}\kappa\alpha^{3}e^{3\kappa\tau}\int e^{-3\kappa\tau}\left(e^{\kappa\tau}-1\right)^{2}d\tau,\\R_{1}^{(1)}(\tau,3)&=\left(\prod_{r=0}^{2}(n-r)\right)\frac{1}{4}\alpha\sigma^{2}e^{3\kappa\tau}\int e^{-3\kappa\tau}\left(e^{\kappa\tau}-1\right)\left(e^{\kappa\tau}+1\right)d\tau,\\R_{0}^{(2)}(\tau,3)&=\left(\prod_{r=0}^{2}(n-r)\right)\frac{1}{2}\sigma^{2}\alpha e^{3\kappa\tau}\int e^{-3\kappa\tau}\left(e^{\kappa\tau}-1\right)d\tau.\end{aligned}$$

By integration, we obtain the solution

$$R_0^{(1)}(\tau,3)=\left(\prod_{r=0}^{2}(n-r)\right)\frac{1}{6}\alpha^3\left(e^{\kappa\tau}-1\right)^3+R_0^{(1)}(0,3),$$

$$R_1^{(1)}(\tau,3)+R_0^{(2)}(\tau,3)=\left(\prod_{r=0}^{2}(n-r)\right)\frac{1}{4\kappa}\alpha\sigma^2\left(e^{\kappa\tau}-1\right)^2\left(e^{\kappa\tau}+1\right)+R_1^{(1)}(0,3)+R_0^{(2)}(0,3).$$

By the initial condition (1.5) when $j=n-3$, $R_0^{(1)}(0,3)+R_1^{(1)}(0,3)+R_0^{(2)}(0,3)=0,$ and

$$\begin{aligned}A_{n-3}^{(n)}(\tau)&=R_0^{(1)}(\tau,3)+\left[R_1^{(1)}(\tau,3)+R_0^{(2)}(\tau,3)\right]\\&=\left(\prod_{r=0}^{2}(n-r)\right)\sum_{l=0}^{\left\lfloor\frac{3}{2}\right\rfloor}\frac{1}{\kappa^l}\alpha^{3-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{3-l}\left(e^{\kappa\tau}+1\right)^l c_{n,n-3}^{(l)},\end{aligned}$$

where $c_{n,n-3}^{(0)}=\frac{1}{6}$ and $c_{n,n-3}^{(1)}=\frac{1}{4}$, which is in the form of (2.7) when $k=3$. This shows that (2.4) holds for $k=3$,

$$\begin{bmatrix}c_{n,n-3}^{(0)}\\c_{n,n-3}^{(1)}\end{bmatrix}=\begin{bmatrix}\frac{1}{6}\\\frac{1}{4}\end{bmatrix}=\frac{1}{3}\left(\begin{bmatrix}\frac{1}{2}\\\frac{1}{4}\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}\right).$$

For $k=4$, the equation (2.6) becomes

$$\frac{dA_{n-4}^{(n)}}{d\tau}-4\kappa A_{n-3}^{(n)}(\tau)=(n-3)\kappa\alpha A_{n-3}^{(n)}(\tau)+\frac{1}{2}(n-3)(n-2)\sigma^2A_{n-2}^{(n)}(\tau)$$

with the solution in integral form

$$\begin{aligned}A_{n-4}^{(n)}(\tau)&=e^{4\kappa\tau}\int e^{-4\kappa\tau}(n-3)\kappa\alpha A_{n-3}^{(n)}(\tau)\,d\tau+e^{4\kappa\tau}\int e^{-4\kappa\tau}\frac{1}{2}(n-3)(n-2)\sigma^2A_{n-2}^{(n)}(\tau)\,d\tau\\&=:R_0^{(1)}(\tau,4)+R_1^{(1)}(\tau,4)+R_0^{(2)}(\tau,4)+R_1^{(2)}(\tau,4),\end{aligned}$$

where

$$R_0^{(1)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{6}\kappa\alpha^4e^{4\kappa\tau}\int e^{-4\kappa\tau}\left(e^{\kappa\tau}-1\right)^3d\tau,$$

$$R_1^{(1)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{4}\alpha^2\sigma^2e^{4\kappa\tau}\int e^{-4\kappa\tau}\left(e^{\kappa\tau}-1\right)^2\left(e^{\kappa\tau}+1\right)d\tau,$$

$$R_0^{(2)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{4}\alpha^2\sigma^2e^{4\kappa\tau}\int e^{-4\kappa\tau}\left(e^{\kappa\tau}-1\right)^2d\tau,$$

$$R_1^{(2)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{8\kappa}\sigma^4e^{4\kappa\tau}\int e^{-4\kappa\tau}\left(e^{\kappa\tau}-1\right)\left(e^{\kappa\tau}+1\right)d\tau.$$

By integration, we obtain the solution

$$R_0^{(1)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{24}\alpha^4\left(e^{\kappa\tau}-1\right)^4+R_0^{(1)}(0,4),$$

$$R_1^{(1)}(\tau,4)+R_0^{(2)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{8\kappa}\alpha^2\sigma^2\left(e^{\kappa\tau}-1\right)^3\left(e^{\kappa\tau}+1\right)+R_1^{(1)}(0,4)+R_0^{(2)}(0,4),$$

$$R_1^{(2)}(\tau,4)=\left(\prod_{r=0}^{3}(n-r)\right)\frac{1}{32\kappa^2}\sigma^4\left(e^{\kappa\tau}-1\right)^2\left(e^{\kappa\tau}+1\right)^2+R_1^{(2)}(0,4).$$

By the initial condition (1.5) when $j=n-4$, $R_0^{(1)}(0,4)+R_1^{(1)}(0,4)+R_0^{(2)}(0,4)+R_1^{(2)}(0,4)=0,$ and

$$A_{n-4}^{(n)}(\tau)=R_0^{(1)}(\tau,4)+\left[R_1^{(1)}(\tau,4)+R_0^{(2)}(\tau,4)\right]+R_1^{(2)}(\tau,4)$$
$$=\left(\prod_{r=0}^{3}(n-r)\right)\sum_{l=0}^{\left\lfloor\frac{4}{2}\right\rfloor}\frac{1}{\kappa^l}\alpha^{4-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{4-l}\left(e^{\kappa\tau}+1\right)^l c_{n,n-4}^{(l)},$$

where $c_{n,n-4}^{(0)}=\frac{1}{24}, c_{n,n-4}^{(1)}=\frac{1}{8}$ and $c_{n,n-4}^{(2)}=\frac{1}{32}$, which is on the form of (2.7) when $k=4$. This show that (2.5) holds for $k=4$,

$$\begin{bmatrix} c_{n,n-4}^{(0)} \\ c_{n,n-4}^{(1)} \\ c_{n,n-4}^{(2)} \end{bmatrix}=\begin{bmatrix} \frac{1}{24} \\ \frac{1}{8} \\ \frac{1}{32} \end{bmatrix}=\frac{1}{4}\left(\begin{bmatrix} \frac{1}{6} \\ \frac{1}{4} \\ 0 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}\right).$$

For other k, the solution of (2.6) in integral form is

$$A_{n-k}^{(n)}(\tau):=Q_1(\tau)+Q_2(\tau),$$

where

$$Q_1(\tau)=e^{k\kappa\tau}\int e^{-k\kappa\tau}\left((n-k)+1\right)\kappa\alpha A_{(n-k)+1}^{(n)}(\tau)\,d\tau,$$
$$Q_2(\tau)=e^{k\kappa\tau}\int e^{-k\kappa\tau}\frac{1}{2}\left((n-k)+1\right)\left((n-k)+2\right)\sigma^2 A_{(n-k)+2}^{(n)}(\tau)\,d\tau.$$

Based on the same idea, we introduce $R_l^{(i)}$ by splitting $A_j^{(n)}$ in $Q_1(\tau)$ and $Q_2(\tau)$ as follows. By substituting $A_{(n-k)+1}^{(n)}(\tau)$ in $Q_1(\tau)$, we have

$$Q_1(\tau)=e^{k\kappa\tau}\int e^{-k\kappa\tau}\left((n-k)+1\right)\kappa\alpha\left[\left(\prod_{r=0}^{(k-1)-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\frac{1}{\kappa^l}\alpha^{(k-1)-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{(k-1)-l}\left(e^{\kappa\tau}+1\right)^l c_{n,n-(k-1)}^{(l)}\right]d\tau,$$
$$=\left(\prod_{r=0}^{k-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}c_{n,n-(k-1)}^{(l)}\frac{1}{\kappa^{l-1}}\alpha^{k-2l}\sigma^{2l}e^{k\kappa\tau}\int e^{-k\kappa\tau}\left(e^{\kappa\tau}-1\right)^{(k-1)-l}\left(e^{\kappa\tau}+1\right)^l d\tau,$$
$$=:\left(\prod_{r=0}^{k-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}R_l^{(1)}(\tau,k).$$

By substituting $A_{(n-k)+2}^{(n)}$ in $Q_2(\tau)$, we have

$$Q_2(\tau)=e^{k\kappa\tau}\int e^{-k\kappa\tau}\frac{1}{2}\left((n-k)+1\right)\left((n-k)+2\right)\sigma^2$$
$$\left[\left(\prod_{r=0}^{(k-2)-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}\frac{1}{\kappa^l}\alpha^{(k-2)-2l}\sigma^{2l}\left(e^{\kappa\tau}-1\right)^{(k-2)-l}\left(e^{\kappa\tau}+1\right)^l c_{n,n-(k-2)}^{(l)}\right]d\tau,$$
$$=\left(\prod_{r=0}^{k-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}c_{n,n-(k-2)}^{(l)}\frac{1}{2\kappa^l}\alpha^{(k-2)-2l}\sigma^{2l+2}e^{k\kappa\tau}\int e^{-k\kappa\tau}\left(e^{\kappa\tau}-1\right)^{(k-2)-l}\left(e^{\kappa\tau}+1\right)^l d\tau,$$
$$=:\left(\prod_{r=0}^{k-1}(n-\mathrm{r})\right)\sum_{l=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}R_l^{(2)}(\tau,k).$$

For odd k, the splitting of $R_l^{(i)}(\tau,k)$, for $i=1,2,$ are combined to obtain $A_{n-k}^{(n)}(\tau)$ according to the case of $k=3$, namely, by the shifting index of $R_l^{(2)}$,

$$A_{n-k}^{(n)}(\tau)=\left(\prod_{r=0}^{k-1}(n-r)\right)\left[\sum_{l=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}R_l^{(1)}(\tau,k)+\sum_{l=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}R_l^{(2)}(\tau,k)\right]$$

$$=\left(\prod_{r=0}^{k-1}(n-r)\right)\left[R_0^{(1)}(\tau,k)+\sum_{l=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left[R_l^{(1)}(\tau,k)+R_l^{(2)}(\tau,k)\right]\right].$$

By integration subject to initial condition (1.5) for $j=n-k,$ the solution can be written in the form of (2.7) where the coefficients $c_{n,n-k}^{(l)}$ satisfy (2.4). Similarly, the process follows the case of $k=4,$ i.e.,

$$A_{n-k}^{(n)}(\tau)=\left(\prod_{r=0}^{k-1}(n-r)\right)\left[\sum_{l=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}R_l^{(1)}(\tau,k)+\sum_{l=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}R_l^{(2)}(\tau,k)\right]$$

$$=\left(\prod_{r=0}^{k-1}(n-r)\right)\left[R_0^{(1)}(\tau,k)+\sum_{l=1}^{\left\lfloor\frac{k}{2}\right\rfloor-1}\left[R_l^{(1)}(\tau,k)+R_l^{(2)}(\tau,k)\right]+R_{\left\lfloor\frac{k-2}{2}\right\rfloor}^{(2)}(\tau,k)\right].$$

By integration subject to initial condition (1.5) for $j=n-k,$ the solution can be written in the form of (2.7) where the coefficients $c_{n,n-k}^{(l)}$ satisfy (2.5).

Remark 2.2 From the result, we can implement closed-form formula for the conditional moments as the following.

Table 1. Algorithm of the coefficient functions.

```
Input: n, x, κ, α, σ, t, T
Output: the n^th conditional moment
   1.  Set C̄_{n,1} = {1}
   2.  Set C̄_{n,2} = {1/2, 1/4}
   3.  For k = 3 to n do
   4.      If k is odd then
   5.         Compute )2.4(
   6.      else
   7.         Compute )2.5(
   8.      EndIf
   9.  EndFor
   10. Compute )2.1(
   11. Compute )1.3(
```

We give the example by using (1.3) and (2.1) to derive the special cases of the conditional moments when $n=1,2,3.$

Example 2.3 Set $\tau=T-t.$ The first conditional moment is

$$E^P\left[X_T \mid X_t=x\right]=\left(x+A_0^{(1)}(\tau)\right)e^{-\kappa\tau},$$

where

$$A_0^{(1)}(\tau)=\alpha\left(e^{\kappa\tau}-1\right).$$

The second conditional moment is

$$E^P\left[X_T^2 \mid X_t=x\right]=\left(x^2+A_1^{(2)}(\tau)x+A_0^{(2)}(\tau)\right)e^{-2\kappa\tau},$$

where

$$A_1^{(2)}(\tau)=2\alpha\left(e^{\kappa\tau}-1\right),$$
$$A_0^{(2)}(\tau)=2\left(\frac{1}{2}\alpha^2\left(e^{\kappa\tau}-1\right)^2+\frac{1}{4\kappa}\sigma^2\left(e^{\kappa\tau}-1\right)\left(e^{\kappa\tau}+1\right)\right).$$

The third conditional moment is

$$E^P\left[X_T^3 \mid X_t=x\right]=\left(x^3+A_2^{(3)}(\tau)x^2+A_1^{(3)}(\tau)x+A_0^{(3)}(\tau)\right)e^{-3\kappa\tau},$$

where

$$A_2^{(3)}(\tau)=3\alpha\left(e^{\kappa\tau}-1\right),$$
$$A_1^{(3)}(\tau)=6\left(\frac{1}{2}\alpha^2\left(e^{\kappa\tau}-1\right)^2+\frac{1}{4\kappa}\sigma^2\left(e^{\kappa\tau}-1\right)\left(e^{\kappa\tau}+1\right)\right),$$
$$A_0^{(3)}(\tau)=6\left(\frac{1}{6}\alpha^3\left(e^{\kappa\tau}-1\right)^3+\frac{1}{4\kappa}\alpha\sigma^2\left(e^{\kappa\tau}-1\right)^2\left(e^{\kappa\tau}+1\right)\right).$$

3. Efficiency of Closed-form Formula

In this section, analytical formula (2.1) is compared with the formula (1.4) proposed by Weraprasertsakun and Rujivan (2017) in terms of computational time for obtaining the conditional moments for $n=5,6\ldots,20$ based on the program Mathematica V9.0 in the form of symbolic parameters. The computations are performed under Microsoft Windows 10 64-bit, quad-processor Intel Core i7 3.4 GHz machine with 32 GB main memory and the results are displayed in Figure 1.

The formula from WR consumed more time when *n* increased from 5 to 20 and increased exponentially from 0.328 to 13.906 sec with a total time of 74.563 sec (Figure 1). However, our formula only consumed 0.016 sec for the total, which was extremely fast at around 4,000 times faster.

This result simplifies the result from Weraprasertsakun and Rujivan (2017) by providing an analytical formula for computing conditional moments which is easier and faster to use without solving the system of recursive ordinary differential equations.

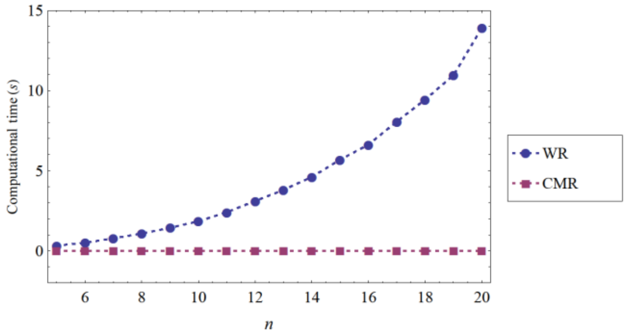

Figure 1. Comparison of computational times between Weraprasertsakun and Rujivan (WR) and our formula (CMR).

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