The Plasticity Zone At Mode I Crack Tip

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Abstract. *In this paper, the plasticity zone at mode I crack tip based on the analytical way is presented for a semi-infinite crack. An isotropic, elastic / perfectly plastic solid is considered under both plane stress and plane strain conditions. The Tresca yield criteria is also applied for this purpose. The analytical solutions are finally given using Maple & Matlab softwares.*

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1. Introduction

The study of plasticity zone at the crack tip for an isotropic, elastic/perfectly plastic solid is one of the top concerns in mechanics [1-7]. Many research works focused on the relationship between shape, plasticity size at crack tip and the growth of fatigue crack in the structure. The first theoretical works on the size and shape of the plasticity zone at crack tip were provided by Irwin and Dugdale in accordance with [8-10]. In this paper, we re-determine the plasticity zone at the crack tip only for mode I as shown in Fig.*1* based on the Tresca yield criteria [11-18] when it is expressed as invariants, respectively. The analytical calculation process will be supported by Maple & Matlab softwares.

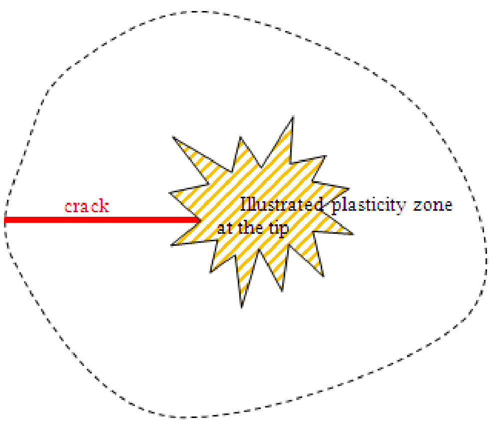


Fig. 1: The semi-infinite crack of mode I (*opening*).

2. Formulation

The stress field in mode I is presented in polar coordinates [1] as depicted in Fig. 2.

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\beta}{2}\right)\left[1-\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{3\beta}{2}\right)\right] \tag{1}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\beta}{2}\right)\left[1+\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{3\beta}{2}\right)\right] \tag{2}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{3\beta}{2}\right) \tag{3}$$

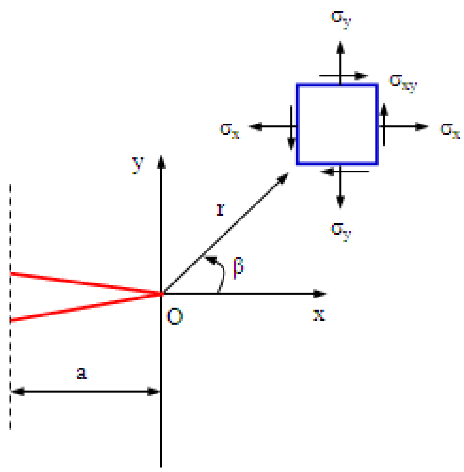


Fig. 2: The crack tip expressed in polar coordinates

Besides,

$$\sigma_z = 0, \quad \text{for plane stress} \tag{4}$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right), \quad \text{for plane strain} \tag{5}$$

$$\sigma_{xz} = \sigma_{yz} = 0 \tag{6}$$

The Tresca yield criterion is given under invariant form as follow [1, 5]

$$4I_2^3 - 27I_3^2 - 36\sigma_{ys}^2 I_2^2 + 96\sigma_{ys}^4 I_2 - 64\sigma_{ys}^6 = 0 \tag{7}$$

; where σ_{ys} is called yield stress in shear and I_2, I_3 are invariants

$$I_2 = \frac{1}{6}\left(\sigma_x - \sigma_y\right)^2 + \frac{1}{6}\left(\sigma_y - \sigma_z\right)^2 + \frac{1}{6}\left(\sigma_z - \sigma_x\right)^2 + \left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2\right) \quad (8)$$

and

$$I_3 = \frac{1}{3^3}\left(2\sigma_x - \sigma_y - \sigma_z\right)\left(2\sigma_y - \sigma_x - \sigma_z\right)\left(2\sigma_z - \sigma_x - \sigma_y\right) + \sigma_{xy}\sigma_{yz}\sigma_{zx} + \sigma_{xz}\sigma_{zy}\sigma_{yx} - \frac{1}{3}\left(2\sigma_y - \sigma_x - \sigma_z\right)\sigma_{xz}\sigma_{zx} - \frac{1}{3}\left(2\sigma_z - \sigma_x - \sigma_y\right)\sigma_{xy}\sigma_{yx} - \frac{1}{3}\left(2\sigma_x - \sigma_y - \sigma_z\right)\sigma_{yz}\sigma_{zy} \quad (9)$$

By substituting equations from (1) to (6) into (7) to solve for r, the simplified equation under cubic form is described as follow:

$$r^3 + f_1(\beta)r^2 + f_2(\beta)r + f_3(\beta) = 0 \quad (10)$$

The Cardano method is used to solve (10) in combination with Maple & Matlab softwares. Equation (10) becomes:

$$t^3 + p(\beta)t + q(\beta) = 0 \quad (11)$$

; where

$$t = r + \frac{f_1(\beta)}{3} \quad (12)$$

$$p(\beta) = f_2(\beta) - \frac{f_1^2(\beta)}{3} \quad (13)$$

$$q(\beta) = f_3(\beta) + \frac{2f_1^3(\beta) - 9f_1(\beta)f_2(\beta)}{27} \quad (14)$$

The condition $Dis = \left[p^3(\beta)/27 + q^2(\beta)/4\right] < 0$ is applied to get three real solutions as below:

$$r_1(\beta) = -\frac{f_1(\beta)}{3} + 2\sqrt{\frac{|p(\beta)|}{3}}\cos\left(\frac{\varphi}{3}\right) \quad (15)$$

$$r_2(\beta) = -\frac{f_1(\beta)}{3} - 2\sqrt{\frac{|p(\beta)|}{3}}\cos\left(\frac{\varphi - \pi}{3}\right) \quad (16)$$

$$r_3(\beta) = -\frac{f_1(\beta)}{3} - 2\sqrt{\frac{|p(\beta)|}{3}}\cos\left(\frac{\varphi + \pi}{3}\right) \quad (17)$$

with

$$\varphi = \arccos\left(-\frac{q(\beta)}{2\times\sqrt{\left(\frac{|p(\beta)|}{3}\right)^3}}\right) \quad (18)$$

For plane stress case, the value of Dis can be rewritten as follow:

$$Dis = -\frac{\left(\cos^{16}\left(\frac{\beta}{2}\right)\right)\left(23\sin\left(\frac{\beta}{2}\right) - 3^2\sin\left(\frac{3\beta}{2}\right)\right)^2 K_I^{12}}{3^3\sigma_{ys}^{12}2^{20}\pi^6} \quad (19)$$

Obviously, it's less than 0. Then, the three real solutions under non-dimensionalized forms are given

$$\bar{r}_1(\beta) = -\frac{1}{6}\left(\cos^2\left(\frac{\beta}{2}\right)\right)\left(-10 + 6\cos(\beta)\right)\cos^2\left(\frac{1}{6}\cos^{-1}(\psi)\right) \quad (20)$$

$$\bar{r}_2(\beta) = \frac{1}{3\times 2^3}\left(\begin{array}{l} 7 + 4\cos(\beta) - 3\cos(2\beta) + \\ +2\left(\cos^2\left(\frac{\beta}{2}\right)\right)\left(-10 + 6\cos(\beta)\right) \end{array}\right) \times\cos\left(\frac{1}{3}\left(\pi + \cos^{-1}(\psi)\right)\right) \quad (21)$$

$$\bar{r}_3(\beta) = \frac{1}{3\times 2^3}\left(\begin{array}{l} 7 + 4\cos(\beta) - 3\cos(2\beta) + \\ +2\left(\cos^2\left(\frac{\beta}{2}\right)\right)\left(-10 + 6\cos(\beta)\right) \end{array}\right) \times\cos\left(\frac{1}{3}\left(-\pi + \cos^{-1}(\psi)\right)\right) \quad (22)$$

; where

$$\psi = \frac{1324 - 2070\cos(\beta) + 756\cos(2\beta) + 54\cos(3\beta)}{\left(-10 + 6\cos(\beta)\right)^3} \quad (23)$$

Similarly, for plane strain case, the value of Dis can also be rewritten as below

$$Dis = \frac{(1-2\nu)^2\cos^{12}\left(\frac{\beta}{2}\right)\left(-1 + \cos(\beta)\right)}{3^3\sigma_{ys}^{12}\pi^6 2^{21}} \times\left(1 - 8\nu + 8\nu^2 + \cos(\beta)\right)^2 \times\left(-7 - 8\nu + 8\nu^2 + 9\cos(\beta)\right)^2 K_I^{12} \quad (24)$$

This value is less than 0 because of the fact that $\left(-1 + \cos(\beta)\right) < 0$. The three real solutions under non-dimensionalized forms are also given

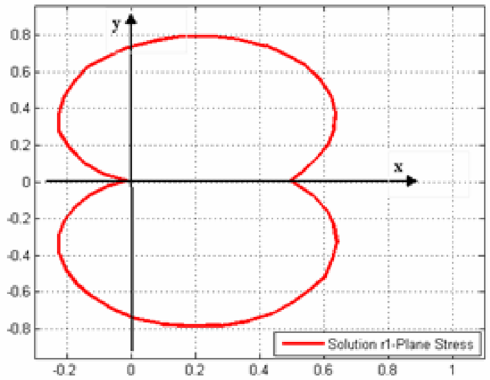
$$\bar{r}_1(\beta) = \frac{1}{2}\sin^2(\beta) \quad (25)$$

$$\bar{r}_2(\beta) = -\frac{1}{4}\left(\cos^2\left(\frac{\beta}{2}\right)\right)\left(-3 + 8\nu - 8\nu^2 + \cos(\beta)\right) + \left(\frac{1}{2} - \nu\right)\left(\cos\left(\frac{\beta}{2}\right)\right)\left(\sin(\beta)\right) \quad (26)$$

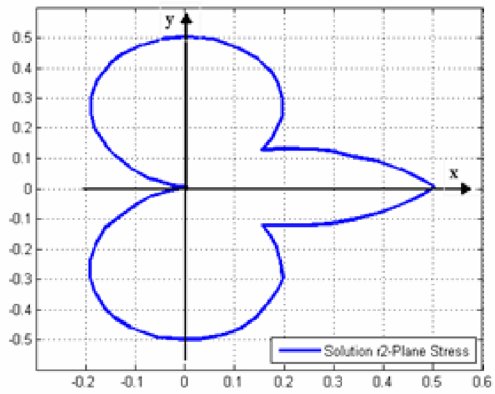
$$\bar{r}_3(\beta) = -\frac{1}{4}\left(\cos^2\left(\frac{\beta}{2}\right)\right)\left(-3 + 8\nu - 8\nu^2 + \cos(\beta)\right) - \left(\frac{1}{2} - \nu\right)\left(\cos\left(\frac{\beta}{2}\right)\right)\left(\sin(\beta)\right) \quad (27)$$

3. Results and Discussions

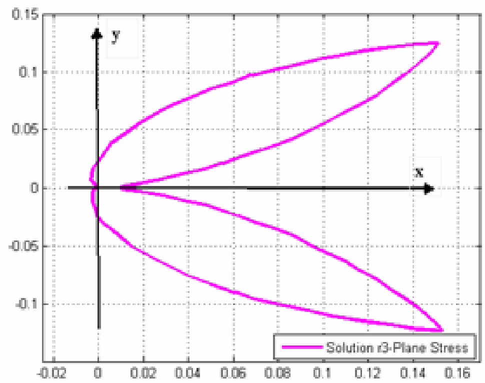
After having the three analytical solutions above, we proceed to draw plasticity zones based on Maple & Matlab software. For plane stress case, Fig.3 and Fig.4 show the plasticity zone for each non-dimensionalized solution $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$ and for the superposition of three zones, respectively. From physical perspective, the plasticity zone must be the outermost zone as depicted by red curve. We can also call it as the boundary of plasticity and this is the main objective of this paper.



(a)



(b)



(c)

Fig. 3: Plasticity zones for mode I plane stress of (a) $\overline{r}_1$ solution; (b) $\overline{r}_2$ solution and (c) $\overline{r}_3$ solution

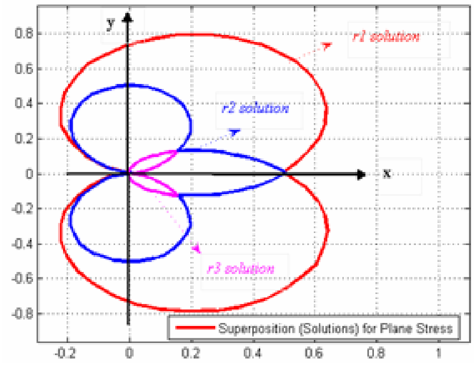


Fig. 4: the superposition of three zones of particular Fig.3(a)-3(c)

Completely similar for the plane strain case, we will also draw the plasticity zones for three non-dimensionalized analytical solutions $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$ in the form of separation or superposition as shown in Fig.5 and Fig.6 with only value $v = 0.3$.

It could be seen from Fig. 5-6 that the outermost zone or the plasticity zone for this case is the combination between the red zone and part of the blue zone that overcomes the red zone.

4. Conclusion

The plasticity zone at mode I crack tip of an isotropic, elastic/perfectly plastic solid for both plane stress and plane strain conditions are given in this study. The results in this paper also provide an accurate view of the plasticity zone at crack tip of mode I related to the concept of semi-infinite crack.

Acknowledgements

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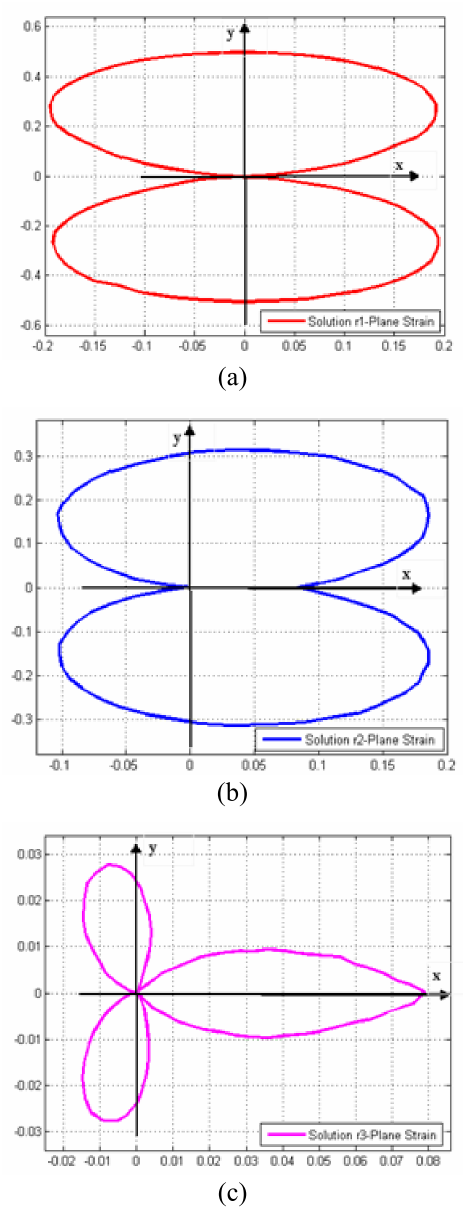

Fig. 5: Plasticity zones for mode I plane strain of (a) $\overline{r}_1$ solution; (b) $\overline{r}_2$ solution and (c) $\overline{r}_3$ solution (when ν = 0.3)

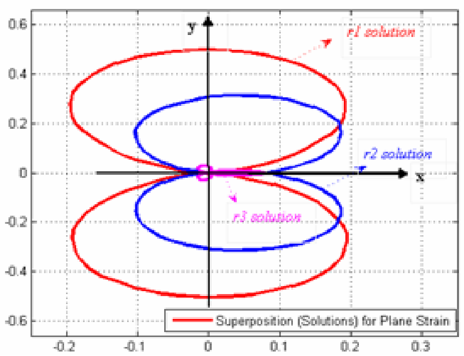

Fig. 6: the superposition of three zones of particular Fig.5(a)-5(c) (when ν = 0.3)

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Biography

Hoang Lan Ton-That was born in Vietnam in 1978. He received the M.Sc. degree in 2003. He is currently an assistant professor at Department of Civil Engineering, HCMC University of Architecture, Vietnam.