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Original Article

Developing a finite difference hybrid method for solving second order initial-value problems for the Volterra type integro-differential equations

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Abstract

It is well known that the study of many processes of the natural sciences can be reduced to solving Volterra integro-differential equations. Recent studies on certain problems of the environment such as the HIV virus, bird flu virus, and diseases associated with mutations of viruses have become relevant. A solution to such problems is associated with finding solutions of VIDEs. There are several classes of methods for solving IDEs. In contrast to the known methods, this paper developed the finite difference hybrid method by a combination of power series and the shifted Legendre polynomial through a block method which is self-starting and helped in eliminating the problem inherent with finding special predictors to estimate y' in the integrators. The method was analyzed and the result revealed that the method is consistent, zero stable and convergent. Some test examples were considered and the results compared favorably with some existing methods.

Keywords: Volterra integro differential equations, finite difference method, hybrid method, shifted legendre polynomials, trapezoidal rule

1. Introduction

There has been growing interest in recent times in the field of integro-differential equations; these equations which involve both differential and integral operators of an unknown function contained in the same equation are classified into Fredholm and Volterra integral equation. Volterra integro-differential equations (VIDEs) contain the unknown function $y(x)$ and one of its derivatives $y^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign respectively with at least one of the limits of integration being a variable, while it is a fixed point number for that of Fredholm type.

Integro-differential equations play an important role in many branches of linear and non-linear functional analysis and their applications are found in the theory of engineering, mechanics, physics, chemistry, biology, economics, and electrostatics. The mentioned integro-differential equations are usually difficult to solve analytically, so approximation methods are required to obtain the solution for both the linear and nonlinear integro-differential equations (Gherjalar & Mohammadikia, 2012; Huesin *et al*., 2008; Mehdiyeva *et al.*, 2013).

Many researchers have studied and discussed different methods for obtaining the numerical solution of VIDEs of various order. Day (1967) used the trapezoidal and Euler's rules for the solution of first order VIDEs, Linz (1969) developed a linear multistep method for the solution of first order VIDEs and the application of orthonormal Bernstein polynomials to construct an efficient scheme for solving fractional stochastic integro-differential equation was discussed by Farshid and Nasrin (2017). Farshid, Saeed, and Emran (2015) developed a method for solving nonlinear fractional integro-differential equations of the Volterra type using novel mathematical

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matrices. The spectral method for Volterra functional integro-differential equations of neutral type was treated in Sedaghat, Ordokhani, and Dehghan (2014). The mixed interpolation and collocation methods for first and second order VIDEs with periodic solution were introduced in Brunner, Makroglou, and Miller (1997). The numerical solution of a non-linear Volterra integro-differential equation via Runge-Kutta-Verner method was developed by Filiz **(**2013**).** The Legendre spectral collocation method for neutral and high-order VIDE was investigated by Wei and Chen (2014). A numerical framework for solving high-order pantograph-delay VIDEs and the numerical solution of optimal control problem of the non-linear Volterra integral equations via generalized hat functions were respectively discussed by Farshid, Saeed, and Emran *(*2016*)* and Farshid and Elham (2016). Similarly, an improved method based on Haar wavelets for the numerical solution of nonlinear integral and integro-differential equations of first and higher orders was developed by Siraj-ul-Islam, Aziz, and Al-Fhaid (2014). The improved method resulted in computational efficiency and simple applicability of the earlier methods. In addition to this, the new approach was extended from IDEs of first order to IDEs of higher orders with initial and boundary conditions. Unlike the earlier methods where the kernel function was approximated by two-dimensional Haar wavelets, the kernel function in the present case is approximated by one-dimensional Haar wavelets. The modified approach is easily extendable to higher order IDEs. Farshid *et al.* *(*2015*)* introduced the numerical solution of integro-differential equations by using rationalized Haar functions methods. Though these methods have the advantage of being simple in implementation, they have the disadvantage of not being continuous at all interior points of the integration interval.

This paper proposes a continuous method which allows evaluation at all interior points of the integration interval which recently appeared in Kamoh, Aboiyar, and Kimbir (2017). Techniques for the derivation of continuous linear multistep methods (LMMs) for direct solution of initial value problems of ordinary differential equations as discussed in the literature include collocation and interpolation using different basis functions among which are radial basis function, power series, Chebyshev polynomials, Legendre polynomials, Hermite polynomials and many others.

In this study, the Volterra type integro-differential equations considered with all the algorithms developed from the idea of interpolation and collocation suggested in ordinary differential equations by many scholars with some modifications, which include the introduction of the integral part $z(x)$ into ($y''(x) = f(x, y(x))$ for the construction of an approximate solution to the initial value problems of the Volterra type integro-differential equations of the form

$$y''(x) = f(x, y(x), z(x)), y(x_0) = y_0, y'(x_0) = g_0 \quad (1)$$

where $$z(x) = \int_{x_0}^{x} K(x, t, y(t))dt$$

There are various techniques for solving (1), e.g. Adomian decomposition method, Galerkin method, rationalized Haar functions method, He's homotopy perturbation method and Variational iteration method. The Adomian decomposition method is an analytical technique that evaluates the solution in the form of Adomian polynomials. This technique does not simplify or discretized the given problem and can be applied to both linear and non-linear problems. The Galerkin and rationalized Haar functions methods are numerical techniques which are not continuous and there are numerous different approaches for the solution of integro-differential equations based on these methods. The Variational iteration method is an analytical method that can be applied to various types of linear and nonlinear problems. In this method, a correction functional is constructed by a general Lagrange multiplier that can be identified optimally via the Variational theory.

Equation (1) has been studied by many scholars such as Shaw (1996, 2000), Tari, Saeedi, and Momeni-Masuleh (2013), Volterra (1959), Shaw and Garey (1997), Wazwaz (2011), Yalcinbas and Sezer (2000).

Suppose equation (1) has a unique solution on the segment [a, b] and satisfies the initial conditions

$$y(x_0) = y_0, y'(x_0) = g_0 \quad (2)$$

the numerical solution of (1) and (2) is then investigated by means of the constant step size h defined on the segment [a, b] divided into N equal parts by $x_i = x_0 + ih$ $(i = 0,1,2,3, \dots, N)$.

2. Methodology

The linear multistep methods of solution for second order initial value problems for ordinary differential equations is of the following form

$$\sum_{j=0}^{k-1} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \quad (3)$$

as discussed by scholars such as Fatunla (1991), Awoyemi and Kayode (2005), Adesanya, Anake, Bishop, and Osilagun (2009), Jator (2007), Jator and Li (2009), Yahaya and Badmus (2009), Awoyemi, Adebile, Adesanya, and Anake (2011), Gear (1964), Adeyeye and Omar (2016, 2017, 2018), and Alkasassbeh and Omar (2016, 2017) among others. It is adopted to solve systems of equations arising from the discretization of the second order initial value problems of the Volterra type (1). The idea adopted in approximating the exact solution $y(x)$ of (1) in the partition $I_n = a < x_0 < x_1 < \cdots < x_n = b$ of the integration interval $[a, b]$ with a constant step size h is the combination of the power series and the shifted Legendre polynomials, which widely used for their smooth properties in the approximation of functions (Higham, 2004). The shifted Legendre polynomial is used because of its flexibility in the choice of interval while the Legendre polynomials are restricted within the interval of $[-1, 1]$ as basis functions.

Consider the approximate solution of (1) given by the combination of power series $q_i(t)$ and the shifted Legendre polynomial $p_i(t)$ of the following form

$$y(x) = \sum_{i=0}^{m+s-1} c_i (q_i(t) + p_i(t)) \quad (4)$$

where $c_i \in \mathbb{R}$, $t = (x - x_n)$, $y \in C^2(a, b)$, m and s are collocation and interpolation points respectively.

The second derivative of (4) is then substituted in (1) to obtain a differential system of the form

$$y''(x) = \sum_{i=0}^{m+s-1} c_i \left(q_i''(t) + p_i''(t) \right) \tag{5}$$

Evaluating (5) at the collocation points $x_{n+r}, r = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$ and evaluating (4) at the interpolation points x_n and $x_{n+\frac{4}{5}}$ respectively; gives a system of nonlinear algebraic equations of the following form

$$AX = B \tag{6}$$

where

$$A = \begin{bmatrix} (q_0+p_0)(0) & (q_1+p_1)(0) & (q_2+p_2)(0) & \cdots & (q_7+p_7)(0) \\ (q_0+p_0)(\frac{4}{5}h) & (q_1+p_1)(\frac{4}{5}h) & (q_2+p_2)(\frac{4}{5}h) & \cdots & (q_7+p_7)(\frac{4}{5}h) \\ (q_0+p_0)''(h) & (q_1+p_1)''(0) & (q_2+p_2)''(0) & & (q_7+p_7)''(0) \\ (q_0+p_0)''(\frac{1}{5}h) & (q_1+p_1)''(\frac{1}{5}h) & (q_2+p_2)''(\frac{1}{5}h) & \cdots & (q_7+p_7)''(\frac{1}{5}h) \\ (q_0+p_0)''(\frac{2}{5}h) & (q_1+p_1)''(\frac{2}{5}h) & (q_2+p_2)''(\frac{2}{5}h) & \cdots & (q_7+p_7)''(\frac{2}{5}h) \\ (q_0+p_0)''(\frac{3}{5}h) & (q_1+p_1)''(\frac{3}{5}h) & (q_2+p_2)''(\frac{3}{5}h) & \cdots & (q_7+p_7)''(\frac{3}{5}h) \\ (q_0+p_0)''(\frac{4}{5}h) & (q_1+p_1)''(\frac{4}{5}h) & (q_2+p_2)''(\frac{4}{5}h) & \cdots & (q_7+p_7)''(\frac{4}{5}h) \\ (q_0+p_0)''(h) & (q_1+p_1)''(h) & (q_2+p_2)''(h) & \cdots & (q_7+p_7)''(h) \end{bmatrix} X = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix}, B = \begin{bmatrix} y_n \\ y_{n+\frac{4}{5}} \\ f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} \tag{7}$$

Solving for $c_i's$, $i = 0(1)7$ in (6) using inverse of a matrix method which are then substituted into (4), gives a continuous implicit method;

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{8}$$

Evaluating (8) at $t = \frac{1}{5}h, \frac{2}{5}h, \frac{3}{5}h$ and h with its first derivative evaluated at $t = 0, \frac{1}{5}h, \frac{2}{5}h, \frac{3}{5}h$ and h with $t = (x_n - x)$ and the results substituted in (8) to obtain the following discrete schemes

$$y_{n+\frac{1}{5}} = y_n + \frac{1}{5}hy_n' + \frac{h^2}{252000}\left(107 f_{n+1} - 682 f_{n+\frac{4}{5}} + 1882 f_{n+\frac{3}{5}} - 3044 f_{n+\frac{2}{5}} + 4315 f_{n+\frac{1}{5}} + 2462 f_n\right) \tag{9}$$

$$y_{n+\frac{2}{5}} = y_n + \frac{2}{5}hy_n' + \frac{h^2}{15750}\left(10 f_{n+1} - 101 f_{n+\frac{4}{5}} + 272 f_{n+\frac{3}{5}} - 370 f_{n+\frac{2}{5}} + 1088 f_{n+\frac{1}{5}} + 355 f_n\right) \tag{10}$$

$$y_{n+\frac{3}{5}} = y_n + \frac{3}{5}hy_n' + \frac{h^2}{28000}\left(45 f_{n+1} - 288 f_{n+\frac{4}{5}} + 870 f_{n+\frac{3}{5}} - 72 f_{n+\frac{2}{5}} + 3501 f_{n+\frac{1}{5}} + 984 f_n\right) \tag{11}$$

$$y_{n+\frac{4}{5}} = y_n + \frac{4}{5}hy_n' + \frac{h^2}{7875}\left(16 f_{n+1} - 80 f_{n+\frac{4}{5}} + 608 f_{n+\frac{3}{5}} + 176 f_{n+\frac{2}{5}} + 1424 f_{n+\frac{1}{5}} + 376 f_n\right) \tag{12}$$

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2016}\left(11 f_{n+1} + 50 f_{n+\frac{4}{5}} + 250 f_{n+\frac{3}{5}} + 100 f_{n+\frac{2}{5}} + 475 f_{n+\frac{1}{5}} + 122 f_n\right) \tag{13}$$

$$y'_{n+\frac{1}{5}} = y'_n + \frac{h}{7200}\left(27f_{n+1} - 173f_{n+\frac{4}{5}} + 482f_{n+\frac{3}{5}} - 798f_{n+\frac{2}{5}} + 1427f_{n+\frac{1}{5}} + 475f_n\right) \quad (14)$$

$$y'_{n+\frac{2}{5}} = y'_n + \frac{h}{450}\left(f_{n+1} - 6f_{n+\frac{4}{5}} + 14f_{n+\frac{3}{5}} + 14f_{n+\frac{2}{5}} + 129f_{n+\frac{1}{5}} + 28f_n\right) \quad (15)$$

$$y'_{n+\frac{3}{5}} = y'_n + \frac{h}{800}\left(3f_{n+1} - 21f_{n+\frac{4}{5}} + 114f_{n+\frac{3}{5}} + 114f_{n+\frac{2}{5}} + 219f_{n+\frac{1}{5}} + 51f_n\right) \quad (16)$$

$$y'_{n+\frac{4}{5}} = y'_n + \frac{h}{225}\left(14f_{n+\frac{4}{5}} + 64f_{n+\frac{3}{5}} + 24f_{n+\frac{2}{5}} + 64f_{n+\frac{1}{5}} + 14f_n\right) \quad (17)$$

$$y'_{n+1} = y'_n + \frac{h}{288}\left(19f_{n+1} + 75f_{n+\frac{4}{5}} + 50f_{n+\frac{3}{5}} + 50f_{n+\frac{2}{5}} + 75f_{n+\frac{1}{5}} + 19f_n\right) \quad (18)$$

3. Analysis of the Method

3.1 Order and error constant

Expanding (9-18) in Taylor's series and collecting like terms in powers of h, the order and error constant are respectively obtained as follows;

$$\check{C}_0 = \check{C}_1 = \cdots = \check{C}_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \check{C}_8 = \begin{bmatrix} -2.1058201\times10^{-8} \\ 4.7762963\times10^{-7} \\ -8.0571429\times10^{-8} \\ -1.0835979\times10^{-7} \\ -1.4550265\times10^{-7} \\ -1.826455\times10^{-7} \\ -1.2529101\times10^{-7} \\ -1.6571429\times10^{-7} \\ -1.0835979\times10^{-7} \\ -2.9100529\times10^{-7} \end{bmatrix}$$

Hence the block method has order $\check{p} = 6$ and error constant $\check{C}_8$

3.2 Consistency

The linear multistep method (9-18) is said to be consistent if the following conditions hold:

(i) has order $\check{p} \geq 1$,

(ii) $\sum_{j=0}^{k} \check{\alpha}_j = 0$,

(iii) $\sum_{j=0}^{k} j\check{\alpha}_j = \sum_{j=0}^{k} \check{\beta}_j$,

(iv) $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$

Following Lambert (1973) and Fatunla (1991), a necessary and sufficient condition for a linear multistep method to be consistent is to satisfy condition (i) above. Based on this condition, the block method is consistent since $\check{p} = 6 > 1$.

3.3 Zero stability

The block method (9-18) is said to be zero stable if the roots $z_r;\ r = 1, \ldots, n$ of the first characteristic polynomial $p(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfies $|z_r| \leq 1$and every root with $|z_r| = 1$ has multiplicity not exceeding the order of the differential equation in the limit as $h \to 0$. From the block method, we have $z^8(z^2 - 1) = 0$ and $z = (-1,1)$, showing that the method is zero stable.

3.4 Convergence

According to Lambert (1973) and Fatunla (1991), the block method is convergent since it is consistent and zero stable

4. Numerical Illustration

To achieve the validity of the proposed method, some standard test examples contained in the literature are considered.

Example 1. Consider a second order nonlinear Volterra integro differential equation

$$y''(x) = y(x)(4x^2 + 2) - x\left((1 - e^{\frac{x^2}{2}}\right) - \int_0^x xt\left(y'(x) + y(t)(1 - 2t)^{\frac{1}{2}}\right)dt,\ \ 0 \leq x \leq 1$$

$$y(0) = 1, y'(0) = 0$$

The exact solution is $y(x) = e^{x^2}$. The nonlinear example was solved by the proposed method. Table 1 summarizes the results.

Example 2. Consider a second order nonlinear Volterra integro differential equation

$$y''(x) + \int_0^x (y(s))^2 ds + \left(\frac{x}{2} - sinh(x) - \frac{1}{4} sinh(2x)\right) = 0, \ \ 0 \leq x \leq 1$$
$$y(0) = 0, y'(0) = 1$$

The exact solution is $y(x) = sinh(x)$. This example was solved using the proposed method. Table 2 summarizes the results.

Example 3. Consider a second order nonlinear Volterra integro differential equation

$$y''(x) - \int_0^x e^{-s} sinx y'(s) ds + y(s) = \left(\frac{1}{2} e^{-x} sin(2x) - sin(x)\right), \ \ 0 \leq x \leq 1$$
$$y(0) = -1, y'(0) = 1$$

The exact solution is $y(x) = sin(x) - cos(x)$. This example was solved using the proposed method. Table 3 summarizes the results.

5. Conclusions

In this paper, a finite difference hybrid method was developed for solving initial value problems for the Volterra-type intgro-differential equations of the second order by modifying the idea discussed for ordinary differential equations via interpolation and collocation techniques. The block method approach used in this study is self-starting it does not require finding special predictor to estimate y' in the integrators. The numerical results of some practical problems contained in the literature demonstrated the validity of the proposed method and the results compared favorably with some existing methods.

Table 3. Accuracy comparison of example 3 for $n = 100$.

Exact value $y(x_n)$	Proposed Method e_n	AL-Smadi *et al.* (2013) e_n
-0.895170748631198	7.19322×10^{-7}	4.76286×10^{-7}
-0.398157023286170	8.34389×10^{-5}	7.11599×10^{-7}
0.301168678939757	5.44122×10^{-6}	6.14195×10^{-8}
		$e_n = \lvert y_n - y(x_n)\rvert$

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Table 1. The exact solution is $y(x) = e^{x^2}$.

h	x_n	$y(x_n)$	y_n	e_n
0.05	0.05	1.00250312760580	1.00250292524674	2.0235906×10^{-7}
	0.20	1.04081077419239	1.04084664371519	3.5869523×10^{-5}
	0.50	1.28402541668774	1.28752749789383	3.5020812×10^{-3}
				$e_n = \lvert y_n - y(x_n)\rvert$

Table 2. The exact solution is $y(x) = sinh(x)$.

h	x_n	$y(x_n)$	y_n	e_n
0.01	0.01	0.100166750019844	0.1001664992415	$2.50395685 \times 10^{-7}$
	0.50	0.521095305493747	0.52117170760569	$7.64021119 \times 10^{-5}$
	1.00	1.17520119364380	1.17772464594854	$2.52345230 \times 10^{-3}$
				$e_n = \lvert y_n - y(x_n)\rvert$

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