รายงานวิจัยฉบับสมบูรณ์

โครงการ ระเบียบวิธีเชิงสถิติสำหรับการวิเคราะห์ข้อมูลทางชีววิทยา สิ่งแวดล้อมและการแพทย์: ทฤษฎีและการประยุกต์ II

Statistical Methods of data Analysis in Biology Environment and Medicine: Theory and Applications II

โดย รองศาสตราจารย์ ดร. มนต์ทิพย์ เทียนสุวรรณ

พฤษภาคม 2556

สัญญาเลขที่ BRG5380004

รายงานวิจัยฉบับสมบูรณ์

โครงการ ระเบียบวิธีเชิงสถิติสำหรับการวิเคราะห์ข้อมูลทางชีววิทยา สิ่งแวดล้อมและการแพทย์: ทฤษฎีและการประยุกต์ II

Statistical Methods of data Analysis in Biology Environment and Medicine: Theory and Applications II

ผู้วิจัย	สังกัด
1. รศ.ดร. มนต์ทิพย์ เทียนสุวรรณ	มหาวิทยาลัยมหิดล
2. ดร. สาธินี เลิศประไพ	มหาวิทยาลัยบูรพา
3. ดร. สุขุมาล สาริกะวณิช	ม.เทคโนโลยีพระจอมเกล้าธนบุรี
4. น.ส. หนึ่งฤทัย สนธิมูล	มหาวิทยาลัยมหิดล
5. นายวุฒิชัย ศรีโสดาพล	มหาวิทยาลัยมหิดล
6. น.ส. ภัทรพร ตัสโต	มหาวิทยาลัยมหิดล
7. น.ส. รัชนีกร ทบประดิษฐ์	มหาวิทยาลัยมหิดล
8. Mr. Ananta R Dhungana	มหาวิทยาลัยมหิดล

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

Contract Number BRG5380004

Project Title: Statistical Methods of data Analysis in Biology Environment and Medicine: Theory and Applications II

Principal Investigator: Assoc. Prof. Dr. Montip Tiensuwan

Address: Department of Mathematics Faculty of Science, Mahidol University, Rama 6 Road, Bangkok, Thailand 10400

Telphone:02-2015538, 02-2015340, **Fax:** 02-2015343, **e-mail**: montip.tie@mahidol.ac.th

Abstract

Data analysis is an integral part of statistics, and statistical methods to analyze data arising from various sources abound in the literature. Nevertheless, the need for new methods of data analysis or application of conventional methods to new area often arises because of emerging new and difficult data collection methods and innovative statistical problems. In the context of biology environment and medicine, we focused on nutritional assessment based on prices, and eleven nutrient indicators, namely energy, carbohydrates, fats, proteins, calcium, iron, vitamin A, thiamin, riboflavin, vitamin C, and niacin from groups of vegetables, foods, fruits, and desserts. Nutrient indicators and prices are assessed by the Multiple Criteria Decision Making (MCDM) and Electre methods. The overall index average is ranked from the highest to the lowest based on these indicators. Hence, the MCDM and Electre methods are shown to be feasible measures to compare and rank the types of vegetables, foods, fruits, and desserts based on nutrient indicators and prices within each category. A time series analysis is adopted as a technique for forecasting of monthly rainfall and water level along the Chao Phraya River in Thailand. The Box-Jenkins technique is used for identifying the parameters of an Autoregressive Integrated Moving Average (ARIMA) model. The Akaike information criterion, the Schwarz's Bayesian criterion and the mean square error are used throughout to test for simplification of any particular model. The periodogram analysis is used to confirm the existence of a seasonal period in the ARIMA model. The ARIMA with seasonal model possibly predicts the monthly rainfall and water level along the Chao Phraya River one year ahead with acceptable accuracy. The nonlinear mixed effects model is also proposed for repeated measures data to describe longitudinal changes in Carcinoembryonic Antigen (CEA) levels of colorectal cancer patients over time. The CEA level of colorectal cancer patients is regarded as the marker choice for monitoring. Parameters of the proposed model are estimated by

using Lindstrom and Bates (LB) and the Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithms. The results show that the estimates of the recurrent time (ϕ_2) by using LB and SAEM are equal to 17.4 and 22.8 months, respectively. Further, the residual sum of squares of the proposed model by using the LB and SAEM algorithms are also compared. Dose response models are the mathematical functions that relate the dose to the measure of observed effect. Three models are considered, namely, the multistage Weibull, logistic and log-logistic models. The study focuses on estimating parameters and comparing estimates in these three models based on the method of maximum likelihood and Berkson's minimum chi-square method. The two methods are also applied to four real experimental data sets for each of the three models. Lastly, we focused on apply the concept of sufficiency and Fisher information for the comparison of experiments of different bivariate normal populations with arbitrarily specified means and variances and with different forms of the correlation parameter rho$\left(\rho\right)$ and to apply the concept of sufficiency for the simple linear regression model $y_x = \alpha + \beta x + e_x$, $e_x \sim N\left(0, \sigma^2\right), x \in [-1,1]$ non – stochastic regression and for the one-way random normal model.

Keywords: Akaike's information criteria, asymptotic comparison, Berkson minimum chi-square, bivariate/trivariate normal populations, Box-Jenkins models, carcinoembryonic antigen, electre method, Fisher information, fixed effects, information, linear models, logistic model, log-logistic model, maximum likelihood, most uniform allocation, multistage Weibull model, multiple criteria decision making method, nonlinear mixed effects model, nutrient indicators, one-way random effects model, optimal designs, prices, random effects, repeated measures, Schwartz's Bayesian criterion, sufficient experiments, sufficiency, variance of white noise

Contract Number BRG5380004

ชื่อโครงการ: ระเบียบวิธีเชิงสถิติสำหรับการวิเคราะห์ข้อมูลทางชีววิทยา สิ่งแวดล้อมและการแพทย์: ทฤษฎีและการประยุกต์ II

หัวหน้าโครงการ: รองศาสตราจารย์ ดร. มนต์ทิพย์ เทียนสุวรรณ

ที่อยู่: ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหิดล ถนนพระรามที่ 6 กรุงเทพมหานคร 10400

โทรศัพท์: 02-2015538, 02-2015340, **Fax:** 02-2015343, **e-mail**: montip.tie@mahidol.ac.th

บทคัดย่อ

การวิเคราะห์ข้อมูลคือภาคจำนวนเต็มของสถิติและวิธีเชิงสถิติในการวิเคราะห์ข้อมูลที่เกิดขึ้นจากแหล่งแตกต่างกันมีมากมายในเอกสารข้อมูลหรือวรรณกรรม อย่างไรก็ตามความต้องการวิธีการใหม่ๆสำหรับการวิเคราะห์ข้อมูลหรือการประยุกต์วิธีธรรมดาที่นำไปสู่เนื้อหาใหม่จะเกิดขึ้นเสมอๆเพราะข้อมูลใหม่ที่ปรากฏและวิธีการเก็บรวบรวมข้อมูลที่ได้ยากและปัญหาเชิงสถิติใหม่ๆ ในส่วนข้อมูลทางชีววิทยา สิ่งแวดล้อมและการแพทย์ ผู้วิจัยได้ให้ความสนใจการประเมินตัวชี้วัดราคาและสารอาหารทั้งหมด 11 ชนิด ได้แก่ พลังงาน คาร์โบไฮเดรต ไขมัน โปรตีน แคลเซียม เหล็ก วิตามินเอ วิตามินบี 1 วิตามินบี 2 วิตามินซี และวิตามินบี 3 จากกลุ่มของผัก อาหาร ผลไม้ และขนมหวาน ตัวชี้วัดสารอาหารได้ถูกประเมินและเปรียบเทียบโดยใช้วิธีเกณฑ์การตัดสินใจพหุคูณและวิธี Electre ค่าเฉลี่ยดัชนีทั้งหมด ได้ถูกจัดอันดับจากค่าสูงสุดไปยังค่าต่ำสุด โดยอาศัยตัวชี้วัดสารอาหารและราคาทั้งหมด ดังนั้น จะเห็นได้ว่าวิธีเกณฑ์การตัดสินใจพหุคูณและวิธี Electre เหมาะสมในการเปรียบเทียบและการวัดจัดอันดับสำหรับผัก อาหาร ผลไม้ และขนมหวานแต่ละชนิดในแต่ละประเภท อนุกรมเวลาได้นำมาใช้สำหรับการพยากรณ์ปริมาณน้ำฝนและระดับน้ำรายเดือนของแม่น้ำเจ้าพระยาในประเทศไทย เทคนิคบอกซ์-เจนกินส์ได้ถูกนำมาใช้ในการระบุพารามิเตอร์ของตัวแบบการถดถอยในตัวเองรวมค่าเฉลี่ยเคลื่อนที่ (ARIMA) เกณฑ์ที่ใช้ในการพิจารณาความเหมาะสมของตัวแบบ คือ เกณฑ์สารสนเทศของอากาอิเคะ (AIC) เกณฑ์ชวาร์ซเบเชียน (SBC) และค่าประมาณความแปรปรวนของส่วนรบกวน การวิเคราะห์แผนภาพเป็นคาบได้ถูกนำมาใช้เพื่อยืนยันการมีอยู่ของคาบฤดูกาลในตัวแบบ ARIMA ตัวแบบARIMAแบบฤดูกาลอาจจะใช้ทำนายปริมาณน้ำฝนรายเดือนและระดับน้ำรายเดือนของแม่น้ำเจ้าพระยา หนึ่งปีล่วงหน้าด้วยความแม่นที่ยอมรับได้ ผู้วิจัยได้นำเสนอตัวแบบอิทธิพลผสมไม่เชิงเส้นสำหรับข้อมูลที่มีการวัดซ้ำในช่วงเวลา เพื่ออธิบายการเปลี่ยนแปลงระดับ Carcinoembryonic Antigen (CEA) ของผู้ป่วยโรคมะเร็งลำไส้ใหญ่ ระดับCEAของผู้ป่วยโรคมะเร็งลำไส้ใหญ่ได้ใช้เป็นตัวบ่งชี้ตัวหนึ่งสำหรับการดูแลผู้ป่วยโรคมะเร็งลำไส้ใหญ่ กล่าวคือถ้าระดับCEAในผู้ป่วยโรคมะเร็งลำไส้ใหญ่ ภายหลังได้รับการผ่าตัด มีค่าสูงขึ้นเรื่อยๆ บ่งถึงว่าอาจมีการกลับมาเป็นซ้ำของโรคอีก ผู้วิจัยได้เสนอตัวแบบอิทธิพลผสมไม่เชิงเส้นของระดับCEA ของผู้ป่วยมะเร็งลำไส้ใหญ่ โดยอิทธิพลผสมเกิดจากอิทธิพลคงที่และ

อิทธิพลสุ่ม พารามิเตอร์ ϕ_2 ในตัวแบบที่นำเสนอใช้แทนเวลาเมื่อเกิดการกลับมาเป็นซ้ำของโรค(จำนวนเดือนภายหลังการผ่าตัด) พารามิเตอร์ในตัวแบบประมาณค่าใช้ขั้นตอนวิธี Lindstrom and Bates (LB) และ Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) ผลการศึกษาพบว่า ตัวประมาณค่าโดยใช้ขั้นตอนวิธีLB และ SAEM ให้ค่ากลับมาเป็นซ้ำเมื่อผู้ป่วยได้รับการผ่าตัดไปแล้ว 17.4 และ 22.8 เดือน ตามลำดับ ตัวแบบการตอบสนองขนาดยาคือฟังก์ชันทางคณิตศาสตร์ที่เกี่ยวข้องกับขนาดยาและการวัดผลกระทบค่าสังเกต ตัวแบบที่พิจารณา 3 ตัวแบบคือ ตัวแบบไวบุลล์หลายขั้นตอน ตัวแบบลอจิสติกและตัวแบบลอก-ลอจิสติก การศึกษาครั้งนี้ให้ความสนใจกับการประมาณค่าพารามิเตอร์และการเปรียบเทียบตัวประมาณค่าจากตัวแบบทั้ง 3 ตัวแบบดังกล่าวด้วยวิธีภาวะน่าจะเป็นสูงสุดและวิธีไคกำลังสองต่ำสุดของเบอร์คสัน วิธีประมาณค่าพารามิเตอร์ทั้งสองวิธีได้ประยุกต์ใช้กับข้อมูล 4 ชุดการทดลองสำหรับแต่ละตัวแบบ สุดท้ายผู้วิจัยให้ความสนใจเชิงแนวคิดของความเพียงพอและสารสนเทศของฟิชเชอร์สำหรับการเปรียบเทียบการทดลองที่แตกต่างของประชากรแบบทวิปรกติ เมื่อกำหนดค่าเฉลี่ยและค่าแปรปรวนในรูปแบบต่างๆของพารามิเตอร์สหสัมพันธ์ (ρ) และประยุกต์แนวคิดความเพียงพอสำหรับตัวแบบถดถอยเชิงเส้นอย่างง่าย $y_x = \alpha + \beta x + e_x, e_x \sim N\left(0, \sigma^2\right), x \in [-1,1]$ การถดถอยไม่ใช่เชิงสุ่ม และสำหรับตัวแบบเชิงสุ่มปรกติทางเดียว

คำสำคัญ: เกณฑ์สารสนเทศของอากาอิเคะ, การเปรียบเทียบขนาดตัวอย่างใหญ่, วิธีไคกำลังสองต่ำสุดของเบอร์คสัน, ประชากรปรกติสอง/สามตัวแปร, ตัวแบบบอกซ์-เจนกินส์**,** สารก่อภูมิต้านทานคาร์ซิโนเอมไบรโอนิค, วิธีอิเลคเตอร์, สารสนเทศของฟิชเชอร์, อิทธิพลตรึง, สารสนเทศ, ตัวแบบเชิงเส้น, ตัวแบบลอจิสติก, ตัวแบบลอก-ลอจิสติก, ความควรจะเป็นสูงสุด, การจัดสรรเอกรูปยิ่ง, ตัวแบบไวบุลล์หลายขั้นตอน, การตัดสินใจเกณฑ์พหุคูณ, ตัวแบบอิทธิพลผสมไม่เชิงเส้น, ตัวชี้วัดสารอาหาร, ตัวแบบอิทธิพลสุ่มทางเดียว, การออกแบบที่เหมาะสม, ราคา, อิทธิพลสุ่ม, การวัดซ้ำ, เกณฑ์ชวาร์ซเบเชียน, การทดลองพอเพียง, พอเพียง, ความแปรปรวนรบกวน

Executive Summary

Project Title: Statistical Methods of Data Analysis in Biology, Environment and Medicine: Theory and Applications II
(วิธีเชิงสถิติสำหรับการวิเคราะห์ข้อมูลทางชีววิทยา สิ่งแวดล้อม และการแพทย์ : ทฤษฎีและการประยุกต์ II)

Principal Investigator: Assoc.Prof. Dr.Montip Tiensuwan
Department of Mathematics, Faculty of Science, Mahidol University, Rama 6 Road, Bangkok, 10400 Tel.: 02-2015340, 02-2015538 Fax: 02 2015343
e-mail: montip.tie@mahidol.ac.th

Objectives

We proposed and/or applied statistical methods of data analysis on multiple criterion decision making (MCDM) to nutrient and vitamin in food, fruit, vegetable, soft drink in term of quantity, quality and price; forecasted rainfall and water level in each province along the Chao Praya river by using Box-Jenkins models and Fourier analysis; proposed nonlinear mixed effects models of colorectal cancer patients to find out the recurrent time of cancer patients; estimated parameters on dose response relationship and applications; and proposed comparison of sufficient experiments.
We will use various data sets to test our methods, i.e. for numerical examples and for applications in biology, environment and medicine.

Following the objectives the research has been done as follows.

1. Multiple Criterion Decision Making (MCDM)

We have collected the data and examined the nutrient and vitamin in food, fruit, vegetables, soft drink in term of quantity, quality and price.

2. Rainfall and water level forecasting

2.1 Forecasting rainfall and water level in each province along the Chao Praya River separately by using Box Jenkins models. **2.2** Using the Fourier analysis in order to confirm the seasonal period of Box Jenkins models.

3. Nonlinear mixed effects models

3.1 Estimating the parameters, σ^2, θ and $\sum_\theta$, of the nonlinear mixed effects models by NLME and SAEM algorithms. **3.2** Computing the variance of estimators of the proposed model. **3.3** Comparing the variance of estimators based on the NLME and SAEM.

4. Dose Response Relationship with Applications

4.1 Estimated parameters on the dose response models, i.e. multistage Weibull, logistic and log-logistic models involving two parameters θ_0 and θ_1 and evaluating estimators **4.2** Deriving the properties of estimators by using the maximum likelihood and Berkson's minimum chi-square methods in large sample. **4.3** Comparing the estimators based on the maximum likelihood and Berkson's minimum chi-square methods by using asymptotic comparison of MSE for Case I: θ_1 is known and θ_0 is unknown, Case II: θ_0 is known and θ_1 is unknown Case III: θ_0 and θ_1 are unknown

5. Comparison of experiments

5.1 Sufficiency in bivariate and trivariate normal populations. **5.2** Sufficiency in linear and quadratic regression models with/without intercept term. **5.3** Sufficiency on one-way random effect models.

Contents

Contract Number BRG5380004

Final Report

==

Report Period 31 May 2010 – 30 May 2013

1. Project Title: **Statistical Methods of data Analysis in Biology Environment and Medicine: Theory and Applications II**

(ระเบียบวิธีเชิงสถิติสำหรับการวิเคราะห์ข้อมูลทางชีววิทยาสิ่งแวดล้อมและการแพทย์ : ทฤษฎีและการประยุกต์ II)

2. Research Team:

2.1 Principal Investigator:

Assoc. Prof. Dr. Montip Tiensuwan

Address: Department of Mathematics Faculty of Science, Mahidol University, Rama 6 Road, Bangkok, Thailand 10400

Telphone: 02-2015538, 02-2015340, **Fax:** 02- 2015343,

e-mail: montip.tie@mahidol.ac.th

Devoted Time per week: 40 hours

2.2 Team Members:

2.2.1 Miss Satinee Lertprapai, Ph.D.

Address: Department of Mathematics, Faculty of Science, Burapha University

Telephone: 038-745900 Ext.3099 **Fax:** 038-745900 Ext. 3078**,**

e-mail: satineel@yahoo.com

Devoted Time per week: 20 hours

2.2.2 Miss Sukuman Sarikavanij, Ph.D.

Address: Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi

Telephone: 02-4708822, 02-4708997 **Fax:** 02-4284025, **e-mail:** sukuman@hotmail.com

Devoted Time per week: 20 hours

2.3 Ph.D. Students:

(1) Miss Nuengruithai Sontimool

(2) Mr. Wuttichai Srisodaphol

(3) Mr. Ananta R. Dhungana

2.4 M.Sc. Students:

(1) Miss Pattaraporn Tusto

(2) Miss Ratchaneekorn Thoppradid

2.5 Expert Contacts:

(1) Prof. Bimal K. Sinha (The University of Maryland Baltimore County, USA)

(2) Prof. Bikas K. Sinha (Stat-Math Unit, Indian Statistical Institute, Kolkata, India)

(3) Assoc.Prof. Timothy E. O'Brien (Loyola University Chicago, USA)

3. Research Field: Mathematical Science (Mathematical Statistics, Statistical Modelling, Data Analysis)

4. Keywords: Multiple criteria decision making(MCDM), Box-Jenkins ARIMA models, Fourier analysis, nonlinear mixed effects models, dose response models, sufficient experiments, bivariate normal models

5. Background and Rationale

Data analysis is an integral part of statistics, and statistical methods to analyze data arising from various sources abound in the literature. Nevertheless, the need for new methods of data analysis or application of conventional methods to new area often arises because of emerging new and difficult data collection methods and innovative statistical problems. For example, in the context of environmental science, it is necessary to study the overall pollution level or state of the environment in a given region on the basis of several key factors such as the pollution level of air, land, water, and the like. It is thus necessary to suitably and meaningfully define an aggregate index which captures the essential features of the component indices and yet represents an overall picture of the environment. Data Envelopment Analysis (DEA) and its variations such as Multiple Criterion Decision Making (MCDM) are recent statistical techniques which are used to compute this aggregate index. In another context, Time series analysis is one of the statistical analyses and one of the most powerful techniques used to develop a model and produce a reliable forecast, which is used in many applications such as: economic forecasting, sales forecasting, meteorology, quality control, inventory studies, census analysis and etc. There are many methods of model fitting including the following: Box-Jenkins ARIMA models, Box-Jenkins multivariate models and Fourier analysis. As a third application, we mention that in the context of mathematical modeling based on non-linear mixed effects (NLME) models of cancer and non-linear dose response models i.e. estimate parameters of the proposed model by using the algorithm of Linstrom and Bates and the stochastic approximation version of EM (Expectation and Maximization), i.e. SAEM algorithm. Further the maximum likelihood, Berkson's minimum chi-square, and Bayesian methods are going to use for estimating parameters of the proposed non-linear dose response models. Lastly comparison of some experiments is an interested problem which can be further developed and applied in some theory and practical cases.

We believe that there are ample opportunities to carry out several statistical data analyses projects along the above lines which would be very helpful in the context of Thailand.

6. Project Objectives

The main objectives of this research project are given as the followings.

6.1 Data Envelopment Analysis (DEA) and Multiple Criterion Decision Making (MCDM)

Comparing several competing estimates in terms of their biases or variances or mean squared errors or tests in terms of their local and global powers is a common practice in statistics. This is an extremely important problem because of the nonexistence of a *best* estimate or test which can be used on all occasions. Although the fascinating area of *decision theory* deals with this problem, and notions of admissibility and minimaxity along with the property of Bayesness play a crucial role, nevertheless other different approaches are also welcome and often contribute significantly in this domain. DEA/MCDM is one such procedure which has proved it self to be very valuable during the last few years.

DEA/MCDM is based on the premise that to compare and possibly rank several *facilities* (estimates, tests, machines, products, etc) in regard to their *chemical emissions* (bias, variance, MSE, price, defects, etc), there does not exist a facility which is uniformly the best or the worst for all possible scenarios (for all values of unknown parameters). One can then define what are called *ideal* and *anti-ideal* facilities, and compute the distances of any facility from these special facilities. It should be noted that ideal and anti-ideal facilities may not exist in practice. One can then compute an index of performance of any given facility based on its distance from ideal and anti-ideal facilities, and rank all the facilities under the condition that a facility is considered good if its distance from ideal is small as well as its distance from anti-idfeal is large. This is essentially the basis of DEA/MCDM.

A specific research project in connection with DEA/MCDM is outlined below. We will examine the nutrient and vitamins in foods, fruits, vegetables, soft drinks in term of quantity, quality and price and investigate which is the best or the worst among each kind of food, fruit, and vegetable, and make some recommendations for consumer to select. The technical tool used for this purpose will be Multiple Criterion Decision Making (MCDM), which is a variation of DEA. We will apply all the methods mentioned above to carry out an extensive MCDM strategy.

6.2 Rainfall and water level forecasting

As mentioned above, rainfall and water level forecasting are important topics in meteorology in Thailand. The prime minister mentioned that rainfall forecasting in Thailand is not accuracy. Not only there was a flood in Chiangmai province in 2005 but also there lack of water in eastern part of Thailand. Therefore the main objectives of this study are outlined below.

1. Forecasting rainfall and water level in each province along the Chao Praya river separately by using Box Jenkins models.
2. Forecasting rainfall and water level in each province along the Chao Praya river by using Fourier analysis.

3. Comparing the results between Box Jenkins and Fourier analysis.

6.3 Nonlinear Mixed Effects Models

In the field of medical science, data usually consists of repeated measurements on individuals observed under varying conditions. For example, data from clinical trials are often longitudinal clinical studies; measurements are taken on each of number of subjects over time. Similarly, clinically useful to know when prostate-specific antigen (PSA) levels first begin to rise rapidly and to determine if the natural history of PSA progression is different in men with locally confined prostate cancers compared to men with metastatic tumors.

Prostate cancer is the most common cancer in male populations in many parts of the world. It is a slowing growing deadly cancer with very few signs and symptoms in the early stage. In the United States an estimated 244,000 new cases and 40,400 deaths from prostate cancer will occur in the United States in 1995 (Wingo *et al*., 1995).

Prostate Specific Antigen (PSA) is a glycoprotein that is produced by the prostatic epithelium and can be measured in serum samples by immunoassay. It is made by prostate cells and is released into the bloodstream. Large prostates have more PSA, so a rise in PSA means that the gland is enlarging rapidly, which can be a sign of cancer or is irritated by infection. Any manipulation of the prostate raises blood PSA levels (Webber *et al*., 1995 [14]).

Another example, a rising Carcinoembryonic Antigen (CEA) level indicates progression or recurrence of the cancer. CEA is a type of protein molecule that can be found in many different cells of the body, but is typically associated with certain tumors and the developing fetus (Benchimol *et al*., 1989). CEA is as a tumor marker, especially for cancers of the gastrointestinal tract. When the CEA level is abnormally high before surgery or other treatment, it is expected to fall to normal following successful to remove all of cancer.

By definition, studies of growth and decay involve repeated measurements taken on sample units, which could be human or animal subjects, plants, or cultures.

Modeling data of this kind usually involves characterization of the relationship between the measured response, y , and the repeated measurement factor, or covariate, x. In many applications, the proposed systematic relationship between y and x is nonlinear in unknown parameters of interest. The model has two types of parameters: global parameters that correspond to the fixed effects and parameters which vary among the population that correspond to the random effects.

We use nonlinear mixed effects model estimator (NLME) by Lindstrom and Bates algorithm, stochastic approximation version of the standard EM (Expectation and Maximization step) (SAEM) algorithm for estimating the parameters in the proposed model. The SAEM is very efficient for computing the maximum likelihood estimate of parameter and useful for fitting model that belong to the exponential family

The main objectives of this topic are:

1. Estimating the parameters, σ^2, θ and $\sum_{\theta}$, of the nonlinear mixed effects models by NLME and SAEM algorithms.
2. Computing the variance of estimators of the proposed model.
3. Comparing the variance of estimators based on the NLME and SAEM.
4. Deriving the test statistic to test the parameters of the proposed model.

6.4 Dose Response Relationship with Applications

Dose response models are mathematical relationships (functions) that relate (predict) between the dose and the measure of observed effect. The observed effect can be extremely complex, depending on a variety of factors including the absorption, metabolism and elimination of the drug; and the presence of other drugs of disease.

Dose response models have been used to model quantal response data. Quantal response data may be classified by one of two possibilities, e.g. dead or alive, with or without a particular type of tumor, normal or abnormal level of a hormone. Researchers have considered the experiment about carcinogen with animal (mice). They would like to know levels of the dose of the drug that the mice will get cancer. Dose response models have been used to utilize about these problems.

The multistage Weibull model, logistic model and log-logistic model are three popular dose response models. There are many examples of research that utilize dose response models 9Guess et.al., 1977; Feldstein, 1978. For example, first, these models have been centered on the problems to experimental animal-carcinogenesis data. To consider low-dose-rate extrapolations for DDT and chloroform are examples for using the models. Responders are animals with liver hepatomas and kidney epithelial tumors for

DDT and chloroform, respectively. Second, there are 37 breast cancer patients who underwent chemotherapy. The laboratory procedures are used to measure their tumor enzyme profile. The problem of the clinician is that of evaluating relationship between a patient's chance to respond to cytotoxic chemotherapy and the knowledge of the patient's enzyme activity profile. Third, the Harrogate benzopyrene skin painting experiment in mice where the responses are for infiltrating carcinomas; the units of dose rate are μ g of benzopyrene per week for the 69-week duration of experiment.

In Thailand, the most extreme cases of cancers that could occur in people are human papilloma, breast cancer, liver cancer and lung cancer. There are many factors leading to further progression of cancers, i.e., tobacco smoking, drinking, radiation, chemicals or virus. If we can obtain the patient's chance who will get cancer after they got carcinogen or the decreasing chance of cancer after it can be treated of the patients, these information are useful to decrease the mortality rate of people with cancer.

The main purposes of this study are as follows.

1. To study the dose response models, measure of increased risk and method for estimating and evaluating estimators.
2. To derive the properties of estimators by using the maximum likelihood, Berkson's minimum chi-square and Bayesian methods.
3. To derive the properties of the measure of increased risk.
4. To compare the estimators based on the maximum likelihood, Berkson's minimum chi-square and Bayesian methods by using MSE.

6.5 Comparison of Experiemnts

The purpose of this study is to apply the concept of sufficiency and Fisher information for the comparison of experiments of different bivariate normal populations with arbitrarily specified means and variances and with different forms of the correlation parameter rho(ρ) and to apply the concept of sufficiency for the simple linear regression model $y_x = \alpha + \beta x + e_x, e_x \sim N(0,\sigma^2), x \in [-1,1]$ non – stochastic regression and for the one-way random normal model. The main objectives of this study are:

1. Applying the appropriate stochastic transformation for the given bivariate normal population and examine the applicability of the concept of ‘sufficiency’ of one experiment for another.

2. Comparing the experiments in terms of Information of the parameter ‘rho’ provided by the given bivariate normal experiments.

3. Studying the concept to generate the random variable $\hat{y}_x$ having the same distribution as y_x for simple regression model $y_x = \alpha + \beta x + e_x$, $e_x \sim N\left(0,\sigma^2\right), x \in [-1,1]$ using the data at the two ends of $x \in [-1,1]$.

4. Studying the condition to generate independent random variables $\hat{y}_x$ having the same distribution as y_x for simple regression model $y_x = \alpha + \beta x + e_x$, $e_x \sim N\left(0,\sigma^2\right), x \in [-1,1]$ taking the different number of data at the two ends of $x \in [-1,1]$.

5. Applying the concept of sufficiency for one-way random normal model by using sufficient criteria.

7. Research Activities

The activities in this project have been progressing very well, keeping up with the proposed plan. These activities can be shown as follows.

7.1 Data Envelopment Analysis (DEA) and Multiple Criterion Decision Making (MCDM)

People, including professional decision makers, have dealt with multiple criteria for as long as there were decisions to be made (Zeleny, 1982). It has become more and more difficult to see the world around us in a unidimensional way and to use only a single criterion when judging what we see. We always compare, rank and order the objects of our choice with respect to various *criteria of choice*. But only in a very simple, straightforward, or routine situation can we assume that a single criterion of choice will be fully satisfactory. One of the methods to compare, rank and order several alternatives, such as countries or stations within a country in regard to their environmental qualities, or several estimators with respect to their mean squared errors, is based on the notion of “Multiple Criteria Decision Making (MCDM)”.

Multiple criteria decision making was introduced as a promising and important field of study in the early 1970’s (Fuller and Carlsson, 1996). Since then the number of contributions to theories and models, which could be used as a basis for more systematic and rational decision making with multiple criteria, has continued to grow at a steady rate. There has been a growing interest and activity in the area of multiple criteria decision making (Carlsson and Fuller, 1995). The field of multiple criteria decision making provides a set of mathematical techniques which helps to reach a decision in optimization problems in which more than one objective plays a role.

A typical MCDM problem involves a number of alternatives to be assessed and a number of criteria or indicators to assess the alternatives. Each alternative has a value for each indicator and based on these values the alternatives can be assessed and ranked. Multiple criteria decision making approaches seek to take explicit account of more than one criterion in supporting the decision process. It is becoming widely recognized that there are substantial benefits to be gained in practice from their use. However, many different approaches satisfying this description has been proposed leading to potential confusion. Some of these approaches are appropriate in different situations. Some are alternative ways of tackling the same type of problem. Moreover, multiple criteria decision making research has developed rapidly and has become a main area of research for dealing with complex decision problems which require the consideration of multiple objectives or criteria. Over the past twenty years, numerous multiple criterion decision methods have been developed which are able to solve such problems (Hanne, 2000). The following section is some literature concerned with MCDM.

As advocated by Hwang and Yoon (1981), Zeleny (1982), and Yoon and Hwang (1995), Multiple Criteria Decision Making (MCDM) is a body of techniques used for meaningful integration of component indices to an overall index in order to decide on the ranking of a number of '*locations*' from the best to worst. This is based on the premises that in the absence of a natural ideal '*location*', a best alternative would be the one which has the shortest distance from the hypothetical ideal '*location*' and at the same time farthest distance from the hypothetical anti-ideal (negative ideal) '*location*'

In particular, Yoon (1995) and Hwang and Yoon (1981) develop an MCDM approach called *technique for order preference by similarity to ideal solution* (TOPSIS), using the intuitive principle that the best alternatives should have the shortest distance from the ideal alternative and the farthest distance from the negative-ideal alternative.

In 1999, Filar *et al*. focused on the application of multiple criteria decision making (MCDM) techniques to assess the effectiveness of US environmental policies for the reduction of toxic release. They described in detail the above TOPSIS method. They used entropy as a basis to determine the importance weights and applied the MCDM technique to assess the state and movement of US toxic release of priority chemicals constituting the “33/50 program”.

In 2002, Maitra *et al*. discussed some extensions and generalizations of MCDM, and applied MCDM to data on air, water and land quality of the fifty US states, and similar indices for 106 countries in the United Nations human environmental indicators (HEI) study.

Many other applications of MCDM in a variety of different contexts can be found in 1991 Jung *et al*., in 1995 Carlsson *et al*., in 2004 Ageev *et al*., in 2001 Corner *et al*., in 2003 Steuer *et al*., Dong *et al*. in 2001, in 2001 Buchanan *et al*., in 1997 Kaliszewski *et al*., in 2002 Ortega *et al*., in 1995 Tarp *et al*., in 1997 Bakker *et al*., in 1992 Alvisi *et al*., in 2002 Karamouz *et al*., in 1999 Buchanan *et al*., in 2004 Lertprapai *et al*., in 2005 Tiensuwan *et al*., in 2006 Tiensuwan *et al*., in 2009 Lertprapai *et a.l*, and in 2010 Lertprapai *et al*..

1.1 Description of The Multiple Criteria Decision Making (MCDM) Method

The Multiple Criteria Decision Making (MCDM) method has recently been recognized as an efficient statistical method to combine component ‘indices’ arising from many ‘sources’ into a single overall meaningful index. Such an index can be effectively used to compare relevant ‘facilities’.

The Multiple Criteria Decision Making (MCDM) method is a procedure to integrate multiple indicators into a single meaningful and overall index by combining $(x_{i1},...,x_{iN})$ for any row i, $i=1,2,...,K$ across all indicators $j=1,2,...,N$. This is based on the premise that in the absence of a natural ideal location, a best alternative would be the one which has the shortest distance from the hypothetical ideal location.

We begin with the description of the problem. We are given a data matrix X with K rows and N columns $\left(X=(x_{ij}):K\times N\right)$

$$X = (x_{ij}) = \begin{pmatrix} x_{11} & \cdots & x_{1N} \\ x_{21} & \cdots & x_{2N} \\ \vdots & \ddots & \vdots \\ x_{K1} & \cdots & x_{KN} \end{pmatrix}$$

where the rows represent facilities which need to be compared or ranked with respect to the elements x_{ij}, the columns represent various sources of the elements x_{ij} and the x_{ij} themselves represent some quantitative information about the facilities. The MCDM method provides a statistical method to combine the elements in any row into a single value which can then be used to compare the rows on a linear scale.

We can define an Ideal Row (IDR) as one with the smallest observed value for each column as

$$IDR = (\min_i x_{i1}, ..., \min_i x_{iN}) = (u_1, ..., u_N). \tag{1.1}$$

and a Negative Ideal Row (NIDR) as one with the largest observed value for each column as

$$NIDR = (\max_i x_{i1}, ..., \max_i x_{iN}) = (v_1, ..., v_N). \tag{1.2}$$

For any given row i, we now compute the distance of each row from the Ideal row and from the Negative Ideal row based on the L_2-norm by using the formulae:

$$L_2(i, IDR) = \left[\sum_{j=1}^{N} \frac{(x_{ij} - u_j)^2 w_j}{\sum_{i=1}^{K} x_{ij}^2} \right]^{1/2}, \tag{1.3}$$

$$L_2(i, NIDR) = \left[\sum_{j=1}^{N} \frac{(x_{ij} - v_j)^2 w_j}{\sum_{i=1}^{K} x_{ij}^2} \right]^{1/2}, \tag{1.4}$$

where $w_1, w_2, ..., w_N$ are suitably chosen nonnegative weights between 0 and 1. An objective way to select the weights is to use Shannon's entropy (Shannon and Weaver, 1947) measure ϕ based on the proportions $p_{1j}, ..., p_{Kj}$ for the j th column where

$$p_{ij} = \frac{x_{ij}}{\sum_{i=1}^{K} x_{ij}}. \qquad (1.5)$$

For the j th column, ϕ_j is computed as

$$\phi_j = \frac{-\sum_{i=1}^{K} p_{ij} \log(p_{ij})}{\log(K)}. \qquad (1.6)$$

Obviously, it is assumed here that the x_{ij} are positive.

The quantity ϕ essentially provides a measure of closeness of the different proportions. The smaller the value of ϕ, the larger is the variation among the proportions for classifying the rows. So we can select the weights as

$$w_j = \frac{(1-\phi_j)}{\sum_{j=1}^{N}(1-\phi_j)} \quad , \qquad j = 1,\ldots,N. \qquad (1.7)$$

In addition to Shannon's entropy measure, we can also use the sample variance of these proportions, given by

$$s^2_{j/prop} = \frac{\sum_{i=1}^{K}(p_{ij} - \overline{p}_j)^2}{(K-1)}. \qquad (1.8)$$

If $\overline{x}_j$ and s_j^2 denote the mean and variance of x_{ij} in the j th column, $s^2_{j/prop}$ is directly proportional to $s_j^2 \big/ \overline{x}_j^2$, which is the square of the sample coefficient of the variation cv_j for the j th column. Therefore we propose to use $w_j = cv_j$ for all j.

The various rows are now ranked based on an overall index I computed as

$$I_i = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, \quad i = 1,\ldots,K.$$

$$I_i = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1,\ldots,K.$$

In addition to the L_2-norm we can also use the L_1-norm as a distance measure and rank the rows once again. The L_1-norm distance is defined as

$$L_1(i, IDR) = \sum_{j=1}^{N} \frac{|x_{ij} - u_j| w_j}{\sum_{i=1}^{K} x_{ij}}, \qquad (1.9)$$

$$L_1(i, NIDR) = \sum_{j=1}^{N} \frac{|x_{ij} - v_j| w_j}{\sum_{i=1}^{K} x_{ij}}, \tag{1.10}$$

where w_j are the appropriate weights.

We have weights of two kinds that is weights for use with Shannon's entropy measure ϕ and weights for use with coefficients of variation (cv). We are now using normalized values of combined indices for each category of L_1-norm and L_2-norm. Finally, we rank index I from large to small values.

1.2 Description of Electre Method

The Electre method requires more extensive computations than the MCDM method. It is used for comparing the status of two locations rather than ranking all of them together. We start with the $K \times N$ data matrix **X** of observations and proceed as follows:

Step 1: Transform matrix $\mathbf{X} = [X_1, X_2, ..., X_N]$ to matrix $\mathbf{R} = [R_1, R_2, ..., R_N]$,

where $R_i = \frac{X_i}{\|X_i\|_2}$.

Step 2: Transform matrix **R** to matrix **V** where **V = RW** where matrix

$\mathbf{W} = diag[w_1, w_2, ..., w_N]$ is based on the weight of the coefficient of variation (w_2).

Step 3: Construct two matrices **C** and **D**,

where $c_{ij} = \sum_{k: v_{ik} \geq v_{jk}} w_k$, and $d_{ij} = \frac{\max_{k: v_{ik} < v_{jk}} |V_{ik} - V_{jk}|}{\max_k |V_{ik} - V_{jk}|}$.

Compute $\bar{c} = \frac{\sum\sum_{i \neq j} c_{ij}}{K(K-1)}$, and $\bar{d} = \frac{\sum\sum_{i \neq j} d_{ij}}{K(K-1)}$.

Step 4: Construct matrices **F** and **G** such that

$f_{ij} = \begin{cases} 1 & ; c_{ij} \geq \bar{c} \\ 0 & ; otherwise \end{cases}$ and $g_{ij} = \begin{cases} 1 & ; d_{ij} \leq \bar{d} \\ 0 & ; otherwise \end{cases}$.

Step 5: Define matrix **E** where $e_{ij} = f_{ij} \cdot g_{ij}$.

It should be noted that the weights w_i are obtained as discussed and that $e_{ij} = 0$ means that row *i* is better than row *j* or $e_{ij} = 1$ means that column *j* is better than row *i*.

In this study, the Excel program is used for the computation of the MCDM and Electre methods.

1.3 Data Collection

An important aspect of nutrition is the daily intake of nutrients. Nutrients consist of various chemical substances in the food that makes up each person's diet. Many nutrients are essential for life, and an adequate amount of nutrients in the diet is necessary for providing energy, building and maintaining body organs, and for various metabolic processes. People depend on nutrients in their diet because the human body is not able to produce many of these nutrients or it cannot produce them in adequate amounts.

Nutrients are essential to the human diet if they meet two characteristics. First, omitting the nutrient from the diet leads to a nutritional deficiency and a decline in some aspect of health. Second, if the omitted nutrient is put back into the diet, the symptoms of nutritional deficiency will decline and the individual will return to normal, barring any permanent damage caused by its absence. There are six major of nutrients found in food: carbohydrates, proteins, lipids (fats and oils), vitamins (both fat-soluble and water-soluble), minerals and water.

The implication of MCDM and Electre methods are used to determine when the alternative is to evaluate a number. Which was based indicators and criteria for each. We decide to choose the best option. Based on those values in various ways.

Therefore, in this topic, we extend our study the MCDM and Electre methods and apply these techniques to the nutrients data in order to rank, which can then be ranked the foods from the best to the worst in terms of the various nutrients and prices.

We have collected data, that is UHT milk, UHT fruit juice, foods, fruits vegetables and deserts. In this study, nutrients data on vegetables, foods, fruits and desserts were collected by using the INMUCAL-Nutrients program from the Institute of Nutrition, Mahidol University. The nutrient indicators are energy, carbohydrate, fat, protein, calcium, iron, vitamin A, thiamin, riboflavin, vitamin C and niacin. Prices were collected on April 2, 2011 at Mahanak market, Bangkok.

The nutrients data were considered as amounts of nutrients derived from energy, carbohydrate, fat, protein, calcium, iron, vitamin A, thiamin, riboflavin, vitamin C and niacin from vegetables, foods, fruits and desserts. In addition, nutrient indicators were classified into four types as follows.

Type I Vegetables. Vegetables are separated into seven groups, namely blossom, chili, leaf and apex, pod, fruit, plant and tuber.

Type Π Foods. Foods are separated into five groups, namely stir frying, broth, blend, roast and rice noodles.

Type Ш Fruits. Fruits are separated into seven groups, namely foreign fruit, numerative noun for fruits, bunch fruit, apple, rose apple, orange and mongo.

Type IV Desserts. Desserts are separated into five groups, namely coconut milk, steam, fried, roast and conserved.

We have managed these data by classifying into various groups and investigated /find indicators in each group. Next step we will apply the MCDM and Electre methods to these data for each group.

1.4 Results

1.4.1 The MCDM Application

The results of the MCDM application to each type of vegetables, foods, fruits or desserts based on the eleven nutrient indicators and prices are given in the following sections.

1.4.1.1 Vegetables

Vegetables are classified into seven groups, namely blossom, chili, leaf and apex, pod, fruit, plant and tuber. The data matrix X with eleven rows of blossom vegetable types including abalone mushroom, chinese chives, chinese mustard green, cowslip creeper, dried mushroom, ear mushroom, onion flowers, sajor-caju mushroom, sesban, shiitake mushroom, straw mushroom and thirteen columns of nutrient indicators namely energy, carbohydrate, fat, protein, protein-vegetable, calcium, iron, iron-vegetable, vitamin A, thiamin, riboflavin, vitamin C, niacin and price. In matrix X, we use the actual nutrient indicator values and the reciprocal of price for each type of blossom vegetables for reasonable comparing and ranking.

The results of the overall index of blossom vegetables based on the L_1-norm and L_2-norm are shown in Tables 1.4.1- 4.

Table 1.4.1 Results on blossom vegetables based on the L_1-norm and L_2-norms by using the weight of Shannon's entropy (w_1).

Type	L_1(IDR)	L_1(NIDR)	L_1(index)	Rank	L_2(IDR)	L_2(NIDR)	L_2(index)	Rank
1	0.0636	0.2694	0.1909	5	0.1956	0.6223	0.2391	5
2	0.0711	0.2619	0.2136	4	0.2066	0.6171	0.2508	4
3	0.0856	0.2474	0.2572	3	0.2702	0.6074	0.3079	3
4	0.1165	0.2165	0.3497	2	0.3639	0.5481	0.3990	2
5	0.2482	0.0848	0.7455	1	0.6510	0.3378	0.6584	1
6	0.0165	0.3165	0.0496	11	0.0686	0.7145	0.0876	11
7	0.0357	0.2973	0.1073	10	0.1403	0.6847	0.1700	9
8	0.0382	0.2948	0.1147	9	0.1272	0.6709	0.1594	10
9	0.0488	0.2842	0.1466	7	0.1658	0.6508	0.2030	7
10	0.0427	0.2903	0.1281	8	0.1722	0.6704	0.2043	6
11	0.0543	0.2787	0.1630	6	0.1595	0.6319	0.2015	8

From Table 1.4.1, the overall index I with Shannon's entropy weight (w_1) based on the L_1-norm and L_2-norm are similar and we observe that most of the first rank of blossom vegetables has type 5 (dried mushroom), followed by type 4 (cowslip creeper) while the last rank has type 6 (ear mushroom).

Table 1.4.2 Results on blossom vegetables based on the L_1-norm and L_2-norm by using the weight of coefficient of variation (w_2).

Type	L_1(IDR)	L_1(NIDR)	L_1(index)	Rank	L_2(IDR)	L_2(NIDR)	L_2(index)	Rank
1	0.0754	0.4131	0.1544	5	0.2062	0.7735	0.2105	5
2	0.1181	0.3704	0.2418	4	0.2865	0.7375	0.2798	4
3	0.1478	0.3407	0.3026	2	0.3766	0.7209	0.3431	3
4	0.1434	0.3452	0.2935	3	0.3932	0.6905	0.3628	2
5	0.3442	0.1443	0.7046	1	0.7704	0.4358	0.6387	1
6	0.0215	0.4670	0.0440	11	0.0748	0.8610	0.0799	11
7	0.0565	0.4320	0.1157	8	0.1694	0.8195	0.1713	9
8	0.0472	0.4413	0.0966	10	0.1364	0.8175	0.1430	10
9	0.0581	0.4304	0.1189	7	0.1775	0.8018	0.1812	7
10	0.0487	0.4398	0.0998	9	0.1748	0.8222	0.1753	8
11	0.0695	0.4190	0.1422	6	0.1846	0.7757	0.1923	6

From Table 1.4.2, the overall index I with coefficient of variation (w_2) based on the L_1-norm and L_2-norm are similar and we observe that most of the first rank of blossom vegetables has type 5 (dried mushroom). The last rank has type 6 (ear- mushroom).

Table 1.4.3 Results of overall index I based on the L_1-norm and L_2-norm with w_1 and w_2 weights on blossom vegetables.

Type	I_i			
	w_1		w_2	
	L_1	L_2	L_1	L_2
1. Abalone mushroom	0.1909	0.2391	0.1544	0.2105
2. Chinese chives	0.2136	0.2508	0.2418	0.2798
3. Chinese mustard green	0.2572	0.3079	0.3026	0.3431
4. Cowslip creeper	0.3497	0.3990	0.2935	0.3628
5. Dried mushroom	0.7455	0.6584	0.7046	0.6387
6. Ear mushroom	0.0496	0.0876	0.0440	0.0799
7. Onion flowers	0.1073	0.1700	0.1157	0.1713
8. Sajor-caju mushroom	0.1147	0.1594	0.0966	0.1430
9. Sesban	0.1466	0.2030	0.1189	0.1812
10. Shiitake mushroom	0.1281	0.2043	0.0998	0.1753
11. Straw mushroom	0.1630	0.2015	0.1422	0.1923

As can be seen in Table 1.4.3, we cannot compare the mean and rank the overall indices I because of difference weights. Hence the normalized overall index based on the L_1-norm and L_2-norm with w_1 and w_2 weights are necessary before computing the mean and standard deviation for each type of blossom vegetables.

The results of the normalized overall index based on the L_1-norm and L_2-norm with w_1 and w_2 weights are shown in Table 1.4.4.

From Table 1.4.4, we observe that most of the first rank of blossom vegetables has dried mushroom with the mean of 0.7186, followed by cowslip creeper with the mean of 0.3658. The last rank has ear mushroom with the mean of 0.0677.

Similarly, the results of the MCDM method with other vegetable groups are shown in Tables 1.4.5- 10**.**

Table 1.4.4 Results of the MCDM method on blossom vegetables with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Abalone mushroom	0.1993	0.2400	0.1700	0.2175	0.2067	0.0296	5
2. Chinese chives	0.2229	0.2517	0.2662	0.2891	0.2575	0.0277	4
3. Chinese mustard green	0.2684	0.3090	0.3330	0.3546	0.3162	0.0369	3
4. Cowslip creeper	0.3651	0.4004	0.3230	0.3749	0.3658	0.0322	2
5. Dried mushroom	0.7781	0.6607	0.7755	0.6600	0.7186	0.0673	1
6. Ear mushroom	0.0517	0.0879	0.0484	0.0826	0.0677	0.0204	11
7. Onion flowers	0.1120	0.1706	0.1273	0.1770	0.1467	0.0320	9
8. Sajor-caju mushroom	0.1197	0.1600	0.1064	0.1477	0.1335	0.0247	10
9. Sesban	0.1530	0.2037	0.1308	0.1873	0.1687	0.0329	7
10. Shiitake mushroom	0.1337	0.2050	0.1098	0.1811	0.1574	0.0434	8
11. Straw mushroom	0.1701	0.2022	0.1566	0.1987	0.1819	0.0222	6

Table 1.4.5 Results of the MCDM method on chili vegetables with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Bell pepper	0.0597	0.0983	0.0628	0.1032	0.0810	0.0229	9
2. Bird pepper	0.1277	0.1490	0.1266	0.1484	0.1379	0.0125	6
3. Black pepper	0.1562	0.2507	0.1549	0.2458	0.2019	0.0535	3
4. Chili spur pepper	0.0153	0.0366	0.0174	0.0400	0.0273	0.0128	10
5. Dried goat pepper	0.6788	0.6397	0.6849	0.6407	0.6610	0.0242	1
6. Dried guinea pepper	0.6614	0.6196	0.6538	0.6146	0.6374	0.0237	2
7. Piment rouge	0.1537	0.2059	0.1534	0.2077	0.1802	0.0308	5
8. Piment vert	0.0658	0.1323	0.0711	0.1434	0.1031	0.0404	7
9. Sweet pepper	0.0553	0.1040	0.0585	0.1101	0.0820	0.0291	8
10. Yellow pepper	0.1619	0.2000	0.1648	0.2030	0.1824	0.0221	4

From Table 1.4.5, we observe that dried goat pepper has of the first rank of chili vegetables with the mean of 0.6610 while chili spur pepper has the last rank of the group with the mean of 0.0273.

Table 1.4.6 Results of the MCDM method on leaf and apex vegetables with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Acacia	0.3724	0.3803	0.3552	0.3737	0.3704	0.0107	2
2. Basil	0.3281	0.3377	0.3142	0.3301	0.3275	0.0098	4
3. Bitter gourd	0.3526	0.3454	0.3189	0.3258	0.3357	0.0159	3
4. Chinese chive	0.1083	0.1165	0.1040	0.1149	0.1109	0.0058	11
5. Coconut	0.0976	0.1306	0.1029	0.1362	0.1168	0.0194	10
6. Feuille de menthe	0.2854	0.2884	0.2752	0.2821	0.2828	0.0057	6
7. Gord gourd	0.1605	0.1854	0.1478	0.1752	0.1672	0.0165	9
8. Kaffir lime leaves	0.5502	0.4823	0.6124	0.5215	0.5416	0.0548	1
9. Lettuce	0.0744	0.0852	0.0693	0.0821	0.0778	0.0073	12
10.Sauropus androgynus	0.2842	0.3198	0.2637	0.3106	0.2946	0.0255	5
11.Stem of sweet basil	0.2658	0.3038	0.2551	0.3001	0.2812	0.0244	7
12.Stem of sweet basil	0.2028	0.1985	0.2001	0.1952	0.1992	0.0032	8

As can be seen in Table 1.4.6, the first rank of leaf and apex vegetables has kaffir lime leaves with the mean of 0.5416, followed by acacia with the mean of 0.3704. On the other hand, the last rank has for lettuce representing the 12^{th} with the mean of 0.0778.

Table 1.4.7 Results of the MCDM method on pod vegetables with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Green pea	0.4112	0.4078	0.4186	0.4110	0.4121	0.0046	3
2. Holland bean	0.3181	0.3081	0.3300	0.3265	0.3207	0.0098	4
3. Horse radish tree	0.4545	0.4383	0.4036	0.3983	0.4237	0.0271	2
4. Lotus stem	0.0575	0.1034	0.0363	0.0801	0.0693	0.0289	10
5. Ripe tamarind	0.5701	0.5059	0.5813	0.5055	0.5407	0.0407	1
6. Smooth loofah	0.1358	0.2321	0.1587	0.2467	0.1933	0.0543	7
7. Snake gourd	0.0862	0.1136	0.0960	0.1311	0.1067	0.0198	9
8. Sponge gourd	0.0552	0.0717	0.0660	0.0832	0.0690	0.0116	11
9. String bean	0.2401	0.2563	0.2711	0.2833	0.2627	0.0187	6
10. Winged bean	0.3048	0.3348	0.2733	0.3045	0.3044	0.0251	5
11. Young corn	0.1236	0.1745	0.1638	0.2141	0.1690	0.0372	8

From Table 1.4.7, ripe tamarind represents the first rank of pod vegetables with the mean of 0.5407 while sponge gourd has the last rank of the group with the mean of 0.0690.

From Table 1.4.8, the first rank of fruit vegetables has pea eggplant with the mean of 0.5665, followed by pumpkin with the mean of 0.4086. The last rank has long cucumber with the mean of 0.0681.

Table 1.4.8 Results of the MCDM method on fruit vegetables with normalized overall indices.

Type	I_i				**Mean**	**SD**	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Cherry tomatoes	0.1196	0.1582	0.1430	0.1745	0.1488	0.0233	10
2. Chinese bitter gourd	0.1985	0.1970	0.1844	0.1897	0.1924	0.0066	7
3. Cucumber	0.1032	0.1097	0.1094	0.1183	0.1101	0.0062	13
4. Eggplant	0.2040	0.2171	0.2349	0.2387	0.2237	0.0162	6
5. Green papaya	0.0973	0.1312	0.1146	0.1447	0.1220	0.0205	12
6. Lemon	0.2808	0.3469	0.2745	0.3372	0.3099	0.0375	3
7. Long cucumber	0.0429	0.0827	0.0532	0.0935	0.0681	0.0239	15
8. Long eggplant	0.1040	0.1362	0.1217	0.1552	0.1293	0.0217	11
9. Pea eggplant	0.6010	0.5065	0.6381	0.5204	0.5665	0.0634	1
10. Pumpkin	0.4580	0.4154	0.3877	0.3734	0.4086	0.0373	2
11. Purple eggplant	0.1583	0.1826	0.1758	0.2025	0.1798	0.0183	9
12. Small eggplant	0.1652	0.1959	0.1876	0.2176	0.1916	0.0217	8
13. Tomato	0.3236	0.3153	0.2729	0.2786	0.2976	0.0256	4
14. Wax gourd	0.0728	0.0958	0.0791	0.1012	0.0872	0.0134	14
15. Wild bitter gourd	0.2443	0.3234	0.2605	0.3295	0.2894	0.0433	5

As can be seen in Table 1.4.9, the first rank of plant vegetables has parsley with the mean of 0.4381, followed by water mimosa with the mean of 0.4367. The last rank has for chinese cabbage with the mean of 0.0652.

Table 1.4.9 Results of the MCDM method on plant vegetables with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Asparagus	0.1154	0.1447	0.0984	0.1339	0.1231	0.0204	12
2. Cabbage chick	0.2004	0.2341	0.1639	0.2169	0.2038	0.0300	9
3. Celery	0.2230	0.2352	0.2198	0.2295	0.2269	0.0068	7
4. Chinese cabbage	0.2036	0.2178	0.2072	0.2180	0.2117	0.0074	8
5. Chinese cabbage	0.0545	0.0769	0.0535	0.0759	0.0652	0.0130	15
6. Chinese kale	0.2449	0.2589	0.2482	0.2597	0.2529	0.0075	6
7. Coriander	0.2649	0.2695	0.2683	0.2695	0.2681	0.0022	5
8. Dill	0.3503	0.3385	0.3411	0.3299	0.3399	0.0084	3
9. Lettuce Taiwan	0.0956	0.1383	0.0997	0.1403	0.1185	0.0241	13
10. Parsley	0.4156	0.4023	0.4825	0.4519	0.4381	0.0363	1
11. Spinate	0.3585	0.3381	0.3324	0.3194	0.3371	0.0163	4
12. Spring onion	0.1116	0.1375	0.0957	0.1221	0.1167	0.0176	14
13. Swamp cabbage	0.1508	0.1706	0.1363	0.1609	0.1546	0.0147	11
14. Swamp morning glory	0.1901	0.2163	0.1774	0.2104	0.1985	0.0180	10
15. Water minosa	0.4667	0.4138	0.4549	0.4113	0.4367	0.0283	2

Table 1.4.10 Results of the MCDM method on tuber vegetables with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Banana blossom	0.1252	0.1320	0.1375	0.1444	0.1348	0.0082	13
2. Broccoli	0.3626	0.3574	0.3646	0.3486	0.3583	0.0071	2
3. Cabbage	0.1195	0.1213	0.1276	0.1250	0.1233	0.0036	14
4. Carrot	0.6093	0.4632	0.4762	0.3862	0.4837	0.0927	1
5. Cauliflower	0.1785	0.2212	0.2076	0.2290	0.2091	0.0222	8
6. Chinese cabbage	0.1271	0.1891	0.1564	0.2006	0.1683	0.0332	11
7. Curcuma	0.0773	0.1603	0.1236	0.1907	0.1380	0.0489	12
8. Fingerroot	0.2421	0.3143	0.3251	0.3428	0.3061	0.0442	4
9. Fresh ginger	0.0425	0.0674	0.0505	0.0745	0.0587	0.0148	18
10. Garlic	0.2057	0.2739	0.2890	0.3153	0.2710	0.0468	5
11. Old ginger	0.0552	0.0875	0.0758	0.0989	0.0794	0.0187	16
12. Onion	0.0535	0.0772	0.0666	0.0826	0.0700	0.0128	17

Table 1.4.10 Results of the MCDM method on tuber vegetables with normalized overall indices (cont.).

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
13. Potato	0.2286	0.2262	0.2584	0.2429	0.2390	0.0149	6
14. Radish	0.0696	0.0991	0.0951	0.1148	0.0947	0.0188	15
15. Red cabbage	0.4126	0.3514	0.3509	0.2992	0.3535	0.0464	3
16. Shallot	0.1252	0.2018	0.1611	0.2148	0.1757	0.0407	10
17. Sweet potato	0.1570	0.2025	0.1940	0.2205	0.1935	0.0268	9
18. Taro	0.1923	0.2390	0.2404	0.2612	0.2332	0.0291	7

From Table 1.4.10, we observe that most of the first rank of tuber vegetables has carrot with the mean of 0.4837. The last rank has fresh ginger with the mean of 0.0587.

1.4.1.2 Foods

Foods are classified into five groups, namely stir frying, broth, blend, roast and rice noodles, respectively. The results of the normalized overall index based on the L_1-norm and L_2-norm with w_1 and w_2 weights in each group is shown in Tables 1.4.11- 26.

Table 1.4.11 Results of the MCDM method on stir frying (vegetable) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.2640	0.3508	0.3047	0.3676	0.3218	0.0468	3
2	0.0472	0.1466	0.0803	0.1827	0.1142	0.0616	7
3	0.2379	0.2934	0.2673	0.3059	0.2761	0.0301	4
4	0.1101	0.1690	0.1297	0.1908	0.1499	0.0366	6
5	0.1684	0.2282	0.1923	0.2548	0.2109	0.0382	5
6	0.7502	0.6535	0.6943	0.6035	0.6753	0.0622	1
7	0.5180	0.5116	0.5417	0.5220	0.5233	0.0130	2

Note: 1: quick-fried water spinach seasoned with chili and soy sauce 2: mushroom fried with ginger
3: water minosa fried in oyster sauce 4: sugar pea fried in oyster sauce
5: chinese kale fried in oyster sauce 6: tofu fried with cabbage
7: cowslip creeper fried in oyster sauce

As can be seen in Table 1.4.11, the first rank of stir frying (vegetable) foods has tofu fried with cabbage with the mean of 0.6753, followed by cowslip creep fried in oyster sauce with a mean of 0.5233. On the other hand, the last rank has mushroom fried with ginger with the mean of 0.1142.

Table 1.4.12 Results of the MCDM method on stir frying (meat) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.4015	0.3972	0.3661	0.3640	0.3822	0.0199	2
2	0.1574	0.2174	0.1984	0.2419	0.2038	0.0357	8
3	0.1213	0.1860	0.1526	0.1983	0.1645	0.0347	9
4	0.7505	0.6225	0.6879	0.5756	0.6591	0.0764	1
5	0.2641	0.3037	0.2983	0.3169	0.2957	0.0225	3
6	0.1953	0.2644	0.2457	0.2858	0.2478	0.0386	5
7	0.1901	0.2443	0.2299	0.2711	0.2338	0.0338	6
8	0.1943	0.2345	0.2178	0.2487	0.2238	0.0234	7
9	0.0113	0.0262	0.0137	0.0307	0.0205	0.0094	10
10	0.2329	0.3088	0.2834	0.3464	0.2929	0.0476	4

Note: 1: stir fried pork with green peppers; 2: spicy meat and tomato dip; 3: pork fried roasted chili paste; 4: fried seafood in yellow curry; 5: spicy fried pork; 6: stir fried pork tenderloin with black peppercorn; 7: tofu and pork with chili paste; 8: pork fried in oyster sauce; 9: dried fried chicken; 10: fried fish with tamarind sauce

As can be seen in Table 1.4.12, the first rank of stir frying (meat) foods has fried seafood in yellow curry with the mean of 0.6591, followed by stir fried pork with green peppers with the mean of 0.3822. On the other hand, the last rank has dried fried chicken with the mean of 0.0205.

From Table 1.4.13, we observe that stir fried pork liver with black peppercorn has the first rank of stir frying (mixed) foods with the mean of 0.8476 while fried chinese kale with crispy pork has the last rank of the group with the mean of 0.0099.

Table 1.4.13 Results of the MCDM method on stir frying (mixed) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.0328	0.1110	0.0504	0.1326	0.0817	0.0477	12
2	0.0809	0.2327	0.1094	0.2401	0.1658	0.0825	3
3	0.0890	0.2456	0.1211	0.2549	0.1777	0.0849	2
4	0.0297	0.0846	0.0438	0.0984	0.0641	0.0326	23
5	0.0373	0.1021	0.0522	0.1193	0.0777	0.0392	15
6	0.0411	0.1049	0.0574	0.1165	0.0800	0.0364	13
7	0.0364	0.1202	0.0562	0.1381	0.0877	0.0491	10
8	0.0364	0.1299	0.0596	0.1554	0.0953	0.0564	7
9	0.0495	0.1659	0.0726	0.1768	0.1162	0.0645	5
10	0.0258	0.0950	0.0410	0.1110	0.0682	0.0412	20
11	0.0315	0.1086	0.0465	0.1180	0.0762	0.0435	17
12	0.0383	0.1275	0.0612	0.1504	0.0944	0.0532	8
13	0.0215	0.0832	0.0359	0.1009	0.0604	0.0377	25
14	0.0416	0.1009	0.0602	0.1170	0.0799	0.0350	14
15	0.0262	0.0915	0.0401	0.1040	0.0655	0.0381	22
16	0.0501	0.1150	0.0686	0.1299	0.0909	0.0377	9
17	0.9768	0.7662	0.9503	0.6972	0.8476	0.1372	1
18	0.0348	0.1136	0.0540	0.1306	0.0832	0.0461	11
19	0.0205	0.0508	0.0295	0.0592	0.0400	0.0180	27
20	0.0259	0.0656	0.0371	0.0761	0.0512	0.0236	26
21	0.0292	0.1030	0.0429	0.1102	0.0713	0.0412	19
22	0.0247	0.1094	0.0423	0.1330	0.0773	0.0521	16
23	0.0227	0.0835	0.0376	0.1019	0.0614	0.0374	24
24	0.0672	0.1918	0.1010	0.2178	0.1445	0.0718	4
25	0.0059	0.0264	0.0094	0.0309	0.0181	0.0123	28
26	0.0030	0.0158	0.0041	0.0168	0.0099	0.0074	29
27	0.0331	0.0854	0.0470	0.0999	0.0663	0.0314	21
28	0.0290	0.1041	0.0417	0.1111	0.0715	0.0422	18
29	0.0451	0.1333	0.0681	0.1542	0.1002	0.0519	6

Note: 1: chicken wing fried with cabbage pickled 2: stir fried chinese kale and pork with oyster sauce
3: sauted mixed vegetables with pork in oyster sauce 4: stir fried mung bean noodle
5: stir fried pork sausage with egg and garnished with seafood 6: stuffed omelet
7: thai rich tofu 8: chicken fried with cashews

9: stir fried pork with string bean in soybean paste white 10: shrimps fried with fresh coconut
11: fried pork minced and string bean with chili 12: seabass spicy stir fried
13: chicken stir fried with ginger 14: saute sponge gourd egg and shrimps
15: pork fried with bamboo shoots and chili 16: fried egg garnished with seafood
17: stir fried pork liver with black peppercorn 18: pork fried with chili, ginger and string bean
19: saute chayote tops and young leaves egg and shrimps 20: radish dried salted fried with egg
21: sweet and sour sauce fried with pork 22: catfish spicy stir fried
23: stir fried seabass with chinese celery 24: deep fried fish with sweet chili sauce
25: fried chinese kale with salted fish 26: fried chinese kale with crispy
27: stir fried crispy basil with pork and black preserved eggs
28: chicken stir fried with chili pepper 29: seafood spicy stir fried

Table 1.4.14 Results of the MCDM method on broth foods with normalized overall indices.

Type	I_i				**Mean**	**SD**	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.0259	0.0732	0.0473	0.1004	0.0617	0.0323	8
2	0.8465	0.6848	0.7767	0.6288	0.7342	0.0965	1
3	0.0540	0.1273	0.0645	0.1330	0.0947	0.0412	6
4	0.1373	0.2469	0.1715	0.2607	0.2041	0.0593	4
5	0.1132	0.1855	0.1596	0.2442	0.1756	0.0546	5
6	0.0458	0.0881	0.0581	0.0982	0.0726	0.0247	7
7	0.3165	0.4172	0.3841	0.4387	0.3891	0.0534	2
8	0.2532	0.2935	0.2482	0.2889	0.2709	0.0235	3

Note: 1: clear soup with bean curd and minced pork 2: clear soup with seaweed and minced pork
3: pork blood soup 4: bitter melon soup in sparerib broth
5: stuffed squid soup 6: mined pork soup with coccinia grandis
7: chinese vegetable stew 8: clear soup with pork kidney

From Table 1.4.14, clear soup with seaweed and minced pork represents the first rank of broth foods with the mean of 0.7342 while clear soup with bean curd and minced pork has the last rank of the group with the mean of 0.0617.

Table 1.4.15 Results of the MCDM method on broth (spicy) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.1964	0.0719	0.1747	0.0854	0.1321	0.0626	7
2	0.1850	0.0718	0.1676	0.0867	0.1278	0.0568	8
3	0.0652	0.0472	0.0737	0.0652	0.0628	0.0111	15
4	0.1410	0.0615	0.1360	0.0786	0.1043	0.0401	11
5	0.1348	0.0547	0.1267	0.0684	0.0961	0.0405	13
6	0.1198	0.0604	0.1190	0.0784	0.0944	0.0298	14
7	0.7762	0.2354	0.6803	0.2768	0.4922	0.2759	1
8	0.3673	0.1339	0.3199	0.1562	0.2443	0.1166	3
9	0.3651	0.1426	0.3090	0.1650	0.2454	0.1086	2
10	0.3081	0.1196	0.2482	0.1317	0.2019	0.0915	4
11	0.1591	0.0698	0.1420	0.0830	0.1135	0.0437	9
12	0.1457	0.0521	0.1290	0.0618	0.0972	0.0471	12
13	0.0797	0.0323	0.0722	0.0391	0.0558	0.0236	16
14	0.2838	0.1102	0.2335	0.1231	0.1876	0.0847	5
15	0.0664	0.0344	0.0635	0.0436	0.0520	0.0155	18
16	0.0649	0.0372	0.0607	0.0475	0.0525	0.0127	17
17	0.2364	0.0889	0.2065	0.1047	0.1591	0.0733	6
18	0.1390	0.0853	0.1271	0.1025	0.1135	0.0242	10
19	0.0481	0.0234	0.0446	0.0283	0.0361	0.0121	19

Note: 1: sour prawn soup; 2: hot and sour chicken soup
3: spicy soup with striped snakehead fish; 4: fresh mackerel soup
5: carp fish soup; 6: sour and spicy fish grilled
7: spicy seafood soup; 8: hot and sour pork spare ribs
9: steamed egg with tom yum; 10: sour and spicy fish crispy with tamarind apex
11: tom yum with multiply mushroom; 12: hot and sour snapper filet
13: tom yum canned mackerel in tomato sauce
14: hot and spicy pork rib hot pot with tamarind and thai herbs
15: thai hot soup with beef; 16: pork organ soup
17: sour and spicy soup of thai chicken; 18: sour and spicy smoked fish soup
19: hot and sour nile tilapia

As can be seen in Table 1.4.15, the first rank of broth (spicy) foods has spicy seafood soup with the mean of 0.4922, followed by steamed egg with tom yum with the mean of 0.2454. On the other hand, the last rank has for hot and sour nile tilapia representing the 19th rank with the mean of 0.0361.

Table 1.4.16 Results of the MCDM method on broth (coconut milk) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.1052	0.1690	0.0947	0.1584	0.1318	0.0373	8
2	0.4749	0.4470	0.4644	0.4396	0.4565	0.0161	2
3	0.1549	0.1640	0.1576	0.1675	0.1610	0.0058	7
4	0.2227	0.2630	0.2551	0.2879	0.2572	0.0269	5
5	0.2109	0.2415	0.2427	0.2682	0.2408	0.0235	6
6	0.4816	0.4606	0.4284	0.4278	0.4496	0.0263	3
7	0.2088	0.2999	0.2281	0.3080	0.2612	0.0501	4
8	0.6081	0.5614	0.6253	0.5666	0.5904	0.0313	1

Note: 1: sour soup with shrimp and morning glory; 2: fish organs sour soup; 3: spicy vegetable and prawn soup; 4: chicken and eggplant in spicy soup; 5: un-coconut curry fish balls; 6: pork with vegetables curry; 7: hot yellow fish curry; 8: shrimp and fried egg sour soup

From Table 1.4.16, shrimp and fried egg sour soup represents the first rank of broth (coconut milk) foods with the mean of 0.5904 while sour soup with shrimp and morning glory has the last rank of group by with the mean of 0.1318.

Table 1.4.17 Results of the MCDM method on broth (without coconut milk) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.2232	0.4286	0.2557	0.4551	0.3407	0.1181	2
2	0.0141	0.0364	0.0164	0.0403	0.0268	0.0135	8
3	0.0078	0.0170	0.0087	0.0187	0.0131	0.0056	11
4	0.0178	0.0495	0.0201	0.0522	0.0349	0.0185	7
5	0.0178	0.0813	0.0201	0.0907	0.0525	0.0389	6
6	0.0330	0.0704	0.0377	0.0773	0.0546	0.0225	5
7	0.0383	0.1472	0.0441	0.1637	0.0983	0.0663	3

Table 1.4.17 Results of the MCDM method on broth (without coconut milk) foods with normalized overall indices (cont.).

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
8	0.0451	0.1301	0.0535	0.1442	0.0932	0.0512	4
9	0.0086	0.0359	0.0102	0.0402	0.0237	0.0167	10
10	0.0126	0.0344	0.0145	0.0379	0.0248	0.0132	9
11	0.9718	0.8716	0.9628	0.8505	0.9142	0.0621	1

Note: 1: soup with chicken, galangal root and coconut 2: soup shrimp and galangal in coconut milk
3: hot thai curry with chicken and bamboo shoot 4: curry with chicken
5: savory curry with pork 6: muslim-style curry with chicken and potatoes
7: green chicken curry 8: pork curry with water spinach
9: northern style pork curry with garlic 10: chicken curry with banana stalk
11: whisker sheatfish chu chee curry

From Table 1.4.17, the first rank of broth (without coconut milk) foods has whisker sheatfish chu chee curry with the mean of 0.9142, followed by soup with chicken, galangal root and coconut with the mean of 0.3407. The last rank has hot thai curry with chicken and bamboo shoot with the mean of 0.0131.

Table 1.4.18 Results of the MCDM method on blend (vegetable) foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.1621	0.2347	0.1982	0.2605	0.2139	0.0430	3
2	0.1324	0.2147	0.1739	0.2518	0.1932	0.0515	4
3	0.0232	0.0744	0.0326	0.0848	0.0537	0.0304	9
4	0.6676	0.6242	0.6911	0.6288	0.6529	0.0320	1
5	0.1030	0.1300	0.1081	0.1424	0.1209	0.0185	7
6	0.6855	0.6435	0.6309	0.5956	0.6389	0.0371	2
7	0.0848	0.1771	0.1074	0.2016	0.1427	0.0555	6
8	0.1335	0.1642	0.1563	0.1829	0.1592	0.0204	5
9	0.0664	0.1201	0.0787	0.1310	0.0990	0.0314	8

Note: 1: wing bean spicy and sour salad 2: chinese kale and seafood in spicy sauce
3: white jelly fungus in spicy sauce 4: thai spicy water minosa salad

5: crispy water convolvulus 6: fried gord gourd spicy salad
7: lemongrass and windbetal leaves spicy salad 8: yum hua plee
9: spicy long eggplant salad

From Table 1.4.18, thai spicy water minosa salad represents the first rank of blend (vegetable) foods with the mean of 0.6529 while white jelly fungus in spicy sauce has the last rank of the group with the mean of 0.0537.

Table 1.4.19 Results of the MCDM method on blend (meat) foods with normalized overall indices.

Type	I_i				**Mean**	**SD**	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.0284	0.0878	0.0324	0.0995	0.0620	0.0368	5
2	0.0154	0.0497	0.0189	0.0703	0.0386	0.0262	7
3	0.0460	0.1035	0.0536	0.1327	0.0840	0.0413	2
4	0.0124	0.0327	0.0145	0.0424	0.0255	0.0145	10
5	0.0235	0.1021	0.0258	0.1104	0.0655	0.0472	4
6	0.0203	0.1162	0.0235	0.1286	0.0721	0.0582	3
7	0.0094	0.0605	0.0139	0.0796	0.0408	0.0346	6
8	0.0057	0.0380	0.0080	0.0528	0.0261	0.0231	9
9	0.0054	0.0174	0.0062	0.0226	0.0129	0.0085	15
10	0.0147	0.0260	0.0159	0.0319	0.0221	0.0083	11
11	0.0074	0.0162	0.0080	0.0204	0.0130	0.0064	14
12	0.9974	0.9716	0.9965	0.9597	0.9813	0.0187	1
13	0.0146	0.0202	0.0151	0.0221	0.0180	0.0037	12
14	0.0084	0.0206	0.0095	0.0270	0.0164	0.0090	13
15	0.0140	0.0545	0.0163	0.0614	0.0365	0.0249	8

Note: 1: spicy cockle salad 2: spicy striped snakehead fish crispy salad
3: crispy catfish salad with green mango 4: sour steamed pork sausage salad
5: spicy fried egg and bacon 6: spicy pork small intestine
7: tuna spicy salad 8: spicy chicken feet salad
9: sliced grilled pork tenderloin salad old thai style 10: chinese sausage salad
11: sour canned fish salad 12: spicy friture salad
13: spicy snake skin gourami 14: shreded chicken spicy salad
15: spicy oyster salad

As can be seen in Table 1.4.19, the first rank of blend (meat) foods has spicy friture salad with the mean of 0.9813, followed by crispy catfish salad with green mango with the mean of 0.0840. On the other hand, the last rank has sliced grilled pork tenderloin salad old thai style with the mean of 0.0129.

Table 1.4.20 Results of MCDM method on blend (mixed) foods with normalized overall indices.

Type	I_i				**Mean**	**SD**	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.8508	0.7965	0.8780	0.8167	0.8355	0.0361	1
2	0.1026	0.1620	0.1096	0.1663	0.1351	0.0337	3
3	0.0552	0.1242	0.0639	0.1323	0.0939	0.0399	6
4	0.0759	0.1452	0.0773	0.1482	0.1117	0.0405	4
5	0.0793	0.1260	0.0869	0.1324	0.1062	0.0269	5
6	0.0622	0.0787	0.0610	0.0775	0.0699	0.0096	7
7	0.4953	0.5204	0.4403	0.4793	0.4838	0.0336	2
8	0.0198	0.0675	0.0239	0.0762	0.0469	0.0292	9
9	0.0316	0.0731	0.0360	0.0827	0.0559	0.0258	8

Note: 1: mungbean noodle salad; 2: spicy combination seafood salad; 3: mixed crispy spicy salad; 4: boiled egg spicy salad; 5: instant noodle yam; 6: egg preserved spicy salad; 7: yum yai; 8: stir fried chicken with pomelo salad; 9: hot spicy cavilla salad

From Table 1.4.20, mungbean noodle salad represents the first rank of blend (mixed) foods with the mean of 0.8355 while stir fried chicken with pomelo salad has the last rank of the group with the mean of 0.0469.

Table 1.4.21 Results of the MCDM method on roast foods with normalized overall indices.

Type	I_i				Mean	SD	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.1123	0.1462	0.1283	0.1581	0.1362	0.0201	9
2	0.1118	0.1632	0.1218	0.1771	0.1435	0.0316	8
3	0.2228	0.2693	0.2712	0.2972	0.2651	0.0310	5
4	0.2004	0.2255	0.2102	0.2251	0.2153	0.0122	6

Table 1.4.21 Results of the MCDM method on roast foods with normalized overall indices(cont.).

Type	I_i				Mean	SD	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
5	0.2527	0.3396	0.3069	0.3626	0.3154	0.0477	4
6	0.5222	0.4582	0.4777	0.4305	0.4721	0.0386	2
7	0.1160	0.1994	0.1614	0.2345	0.1778	0.0509	7
8	0.5586	0.5024	0.5845	0.5005	0.5365	0.0418	1
9	0.0941	0.1493	0.1215	0.1691	0.1335	0.0327	10
10	0.4551	0.4103	0.3651	0.3560	0.3966	0.0456	3
11	0.0846	0.1436	0.1136	0.1645	0.1266	0.0349	11

Note: 1: shrimps with glass noodles 2: roasted chicken with soy sauce

3: roasted chicken with ketchup 4: baked pork spare rib

5: roasted mungbean with termite mushroom 6: roasted crab with chili paste fried in oil

7: roasted chicken wing

8: one whole piece of roasted young mountain pork bathed in wild honey

9: grilled lemongrass chicken 10: egg hen water add steamed

11: roasted chicken with lemon

From Table 1.4.21, the first rank of roast foods has one whole piece of roasted young mountain pork bathed in wild honey with the mean of 0.5365, followed by roasted crab with chili paste fried in oil with the mean of 0.4721. The last rank has roasted chicken with lemon with the mean of 0.1266.

Table 1.4.22 Results of the MCDM method on steamed foods with normalized overall indices.

Type	I_i				**Mean**	**SD**	**Rank**
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.1195	0.1785	0.1441	0.2034	0.1614	0.0370	8
2	0.1386	0.1651	0.1566	0.1784	0.1597	0.0167	10
3	0.0810	0.1022	0.0703	0.0939	0.0869	0.014	12
4	0.3126	0.3471	0.3092	0.3443	0.3283	0.0202	5
5	0.1875	0.2220	0.1962	0.2251	0.2077	0.0187	7
6	0.1185	0.1873	0.1368	0.1988	0.1604	0.0387	9
7	0.1199	0.1174	0.1237	0.1206	0.1204	0.0026	11
8	0.4360	0.3948	0.4043	0.3734	0.4021	0.0260	3

Table 1.4.22 Results of the MCDM method on steamed foods with normalized overall indices(cont.).

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
9	0.4590	0.4079	0.4217	0.3838	0.4181	0.0315	2
10	0.4482	0.4077	0.4616	0.4125	0.4325	0.0265	1
11	0.2326	0.2763	0.2472	0.2796	0.2589	0.0228	6
12	0.3780	0.3834	0.4076	0.3997	0.3922	0.0138	4

Note: 1: ear mushroom with many materials 2: spotted featherback steamed
3: snapper steamed with lemon 4: fish steamed with tofu
5: fish steamed with vegetable and chili paste 6: silver pomfret steamed
7: nile tilapia steamed with vegetable 8: salmon steamed with lemon
9: salmon steamed with soy sauce
10: steamed fish with coconut milk in a banana leaf wrapping
11: chicken steamed with ear mushroom in soy sauce paste 12: giant seaperch steamed

From Table 1.4.22, we observe that the first rank of steamed foods has steamed fish with coconut milk in a banana leaf wrapping with the mean of 0.4325 while snapper steamed with lemon has the last rank of the group with the mean of 0.0869.

Table 1.4.23 Results of the MCDM method on grilled foods with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.7694	0.6921	0.8285	0.7284	0.7546	0.0585	1
2	0.3288	0.3990	0.3227	0.3970	0.3619	0.0418	3
3	0.5476	0.6015	0.4577	0.5584	0.5413	0.0604	2

Note: 1: chicken grilled with herb 2: fish grilled with lemon sauce
3: fish grilled with soy sauce

As can be seen in Table 1.4.23, the first rank of grilled foods has chicken grilled with herb with the mean of 0.7546, followed by fish grilled with soy sauce with the mean of 0.5413. The last rank has fish grilled with lemon sauce representing the third with the mean of 0.3619.

Table 1.4.24 Results of the MCDM method on rice noodle food (water) with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.0769	0.1560	0.1217	0.1856	0.1350	0.0467	8
2	0.0896	0.1783	0.1397	0.2141	0.1554	0.0534	7
3	0.1757	0.2492	0.2352	0.2912	0.2378	0.0477	4
4	0.1255	0.2180	0.1998	0.2666	0.2025	0.0586	6
5	0.0498	0.1042	0.0757	0.1246	0.0886	0.0327	9
6	0.0328	0.0782	0.0513	0.0983	0.0652	0.0289	11
7	0.6572	0.5553	0.5762	0.4839	0.5681	0.0713	1
8	0.1764	0.2554	0.2260	0.2753	0.2333	0.0430	5
9	0.5803	0.4805	0.5256	0.4372	0.5059	0.0613	2
10	0.0259	0.0659	0.0427	0.0833	0.0544	0.0252	12
11	0.0321	0.0998	0.0565	0.1289	0.0793	0.0433	10
12	0.3666	0.4455	0.4448	0.4628	0.4299	0.0430	3

Note: 1: small size rice noodle fresh with beef 2: wide rice noodle with fish ball swamp cabbage
3: boiled vermicelli 4: small size rice noodle with stewed beef ball beef and soup
5: rice noodle fine thread stewed with beef and beef soup
6: big size rice noodle with stewed beef ball beef and soup
7: big size rice noodle with pork and soup 8: wheat noodle with roasted pork
9: wheat noodle with meat ball and ball fried
10: fried rice noodle with dark soy sauce topped with shirmp
11: rice noodle fine thread with stewed beef 12: guay-jub northeasterm style

From Table 1.4.24, big sized rice noodle with pork and soup represents the first rank of rice noodle (water) foods with the mean of 0.5681 while fried rice noodle with dark soy sauce topped with shirmp has the last rank of the group with the mean of 0.0544.

From Table 1.4.25, we observe that the first rank of rice noodle (dried) food has wide rice noodle with pork egg and soy sauce with the mean of 0.5445. The last rank has small size rice noodle fried with stewed beef ball with the mean of 0.1186.

Table 1.4.25 Results of the MCDM method on rice noodle (dried) food with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.5859	0.5310	0.5533	0.5079	0.5445	0.0332	1
2	0.4722	0.4838	0.5017	0.4970	0.4887	0.0133	3
3	0.5474	0.5195	0.5449	0.5149	0.5317	0.0169	2
4	0.0860	0.1326	0.1052	0.1507	0.1186	0.0287	8
5	0.0892	0.1676	0.0997	0.1757	0.1330	0.0449	7
6	0.2908	0.3244	0.2906	0.3270	0.3082	0.0202	4
7	0.1426	0.1727	0.1418	0.1742	0.1578	0.0181	6
8	0.1176	0.1825	0.1405	0.1978	0.1596	0.0371	5

Note: 1: wide rice noodle with pork egg and soy sauce 2: pad thai
3: macaroni fried with pork 4: small size rice noodle fried with stewed beef ball
5: rice noodle fried fine thread with stewed beef ball 6: saute mungbean noodle and hen egg
7: small size rice noodle with pork 8: rice noodle fried with pork and spicy

Table 1.4.26 Results of the MCDM method on fermented rice noodle food with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.2187	0.2632	0.2949	0.3176	0.2736	0.0429	4
2	0.2308	0.2492	0.2122	0.2376	0.2325	0.0155	8
3	0.2685	0.2749	0.2658	0.2735	0.2707	0.0043	5
4	0.2679	0.2892	0.2789	0.3008	0.2842	0.0141	3
5	0.1795	0.2098	0.1913	0.2212	0.2005	0.0186	9
6	0.6587	0.5929	0.6397	0.5719	0.6158	0.0402	1
7	0.2421	0.2863	0.2480	0.2868	0.2658	0.0241	6
8	0.2321	0.2858	0.1995	0.2648	0.2456	0.0378	7
9	0.4201	0.3875	0.4153	0.3860	0.4023	0.0180	2

Note: 1: rice vermicelli with tofu fried in coconut cream 2: fermented rice noodle with minced pork
3: fermented rice noodle with fish curry 4: fermented rice noodle with fish soup
5: fermented rice noodle with chicken feet 6: fermented rice noodle with southern curry
7: fermented rice noodle with chicken curry 8: fermented rice noodle with fish organs soup
9: fermented rice noodle with green curry chicken

From Table 1.4.26, the first rank of fermented rice noodle food has fermented rice noodle with southern curry with the mean of 0.6158, followed by fermented rice noodle with green curry chicken with the mean of 0.4023. On the other hand, the last rank has fermented rice noodle with chicken feet with the mean of 0.2005.

1.4.1.3 Fruits

Fruits are classified into seven groups, namely foreign fruit, numerative noun fruit, bunch fruit, apple, rose apple, orange and mango, respectively. The results of the normalized overall index based on the L_1-norm and L_2-norm with w_1 and w_2 weights in each group is shown in Tables 1.4.27- 33.

Table 1.4.27 Results of the MCDM method on foreign fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Pears	0.0974	0.2143	0.1240	0.2288	0.1661	0.0652	5
2. Cantaloupe	0.5856	0.5506	0.5227	0.5043	0.5408	0.0354	2
3. Hard - type ripe persimmon	0.6132	0.5551	0.5209	0.4993	0.5471	0.0497	1
4. Kiwi	0.4552	0.4857	0.5498	0.5371	0.5070	0.0443	3
5. Cherries eating raw	0.2539	0.3270	0.3712	0.3944	0.3366	0.0618	4

As can be seen in Table 1.4.27, the first rank of foreign fruits is hard-type ripe persimmon with the mean of 0.5471, followed by cantaloupe with the mean of 0.5408. The last rank has pears with the mean of 0.1661.

Table 1.4.28 Results of the MCDM method on numerative noun for fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Dragon fruit	0.0945	0.1318	0.0996	0.1335	0.1149	0.0206	14
2. Papaya ripe	0.2858	0.2696	0.2296	0.2343	0.2548	0.0273	5
3. Pomelo thong dee	0.0973	0.1009	0.0967	0.0998	0.0987	0.0020	17
4. Jackfruit mature	0.1419	0.1449	0.1551	0.1589	0.1502	0.0081	11
5. Watermelon red	0.3276	0.3104	0.2590	0.2720	0.2923	0.0321	3
6. Guava common	0.1477	0.2193	0.1490	0.2149	0.1827	0.0397	8

Table 1.4.28 Results of the MCDM method on numerative noun for fruits with normalized overall indices(cont.).

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
7. Native mature melon	0.0576	0.0759	0.0575	0.0745	0.0664	0.0102	24
8. Pineapple	0.0731	0.0809	0.0800	0.0849	0.0797	0.0049	22
9. Sapodilla common	0.1555	0.1826	0.1542	0.1778	0.1675	0.0148	9
10. Sweet tamarind	0.5569	0.4548	0.6019	0.4791	0.5232	0.0682	1
11. Gandaria marian plum	0.2262	0.2159	0.2001	0.2000	0.2105	0.0128	7
12. Yumbean	0.0757	0.0980	0.0873	0.1095	0.0926	0.0145	20
13. Toddy palm	0.1746	0.2579	0.1998	0.2798	0.2281	0.0491	6
14. Rambutan	0.0868	0.1052	0.0983	0.1128	0.1008	0.0111	16
15. Mangosteen	0.0617	0.0759	0.0682	0.0810	0.0717	0.0085	23
16. Litchi	0.0776	0.1019	0.0863	0.1078	0.0934	0.0139	19
17. Banana khai variety	0.1302	0.1449	0.1408	0.1586	0.1436	0.0117	13
18. Pomegranate solft seed	0.1273	0.1605	0.1370	0.1640	0.1472	0.0179	12
19. Durian mon-thong	0.4097	0.3784	0.3949	0.3677	0.3877	0.0185	2
20. Jujube thai variety	0.2503	0.2878	0.2617	0.2918	0.2729	0.0201	4
21. Sala	0.0747	0.1004	0.0772	0.1026	0.0887	0.0148	21
22. Sugar apple	0.1418	0.1548	0.1567	0.1646	0.1545	0.0094	10
23. Salak palm fruit	0.0799	0.1152	0.0922	0.1224	0.1024	0.0198	15
24. Carambola	0.0853	0.1086	0.0869	0.1089	0.0974	0.0131	18

From Table 1.4.28, we observe that most of the first rank of numerative noun fruits has sweet tamarind with the mean of 0.5232. The last rank has native mature melon with the mean of 0.0664.

Table 1.4.29 Results of the MCDM method on bunch fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Grapes green	0.1662	0.2463	0.1452	0.2224	0.1951	0.0472	5
2. Rambai	0.7422	0.6362	0.7496	0.6391	0.6918	0.0626	1
3. Star gooseberry	0.3129	0.3772	0.3200	0.3851	0.3488	0.0376	3
4. Longan	0.0957	0.1884	0.1490	0.2332	0.1666	0.0585	6
5. Longkong	0.1620	0.2926	0.2156	0.3203	0.2477	0.0723	4
6. Langsad	0.5368	0.5208	0.4959	0.4865	0.5100	0.0230	2

From Table 1.4.29, rambai represents the first rank of bunch fruits with the mean of 0.6918 while longan has the last rank of the group with the mean of 0.1666.

Table 1.4.30 Results of MCDM method on apple fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Apple	0.9833	0.9786	0.9851	0.9797	0.9817	0.0030	1
2. Green apple	0.1810	0.2028	0.1703	0.1961	0.1875	0.0146	2
3. Fuji apple	0.0165	0.0349	0.0245	0.0423	0.0295	0.0114	3

From Table 1.4.30, the first rank of apple fruits has apple with the mean of 0.9817, followed by green apple with the mean of 0.1875. On the other hand, the last rank has fuji apple with the mean of 0.0295.

Table 1.4.31 Results of the MCDM method on rose apple fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Rose apple red color variety	0.2329	0.3746	0.2467	0.3737	0.3070	0.0778	2
2. Rose apple green	0.9428	0.8252	0.9180	0.7961	0.8706	0.0709	1
3. Rose apple	0.1956	0.2999	0.2285	0.3224	0.2616	0.0595	3
4. Rose apple green color variety	0.1362	0.2980	0.2101	0.3501	0.2486	0.0946	4

As can be seen in Table 1.4.31, the first rank of rose apple fruits has rose apple green with the mean of 0.8706, followed by rose apple red color variety with the mean of 0.3070. On the other hand, the last rank has rose apple green color variety with the mean of 0.2486.

From Table 1.4.32, we observe that the first rank of orange fruits has sweet orange with the mean of 0.6820. On the other hand, the rank has kalanchoe with the mean of 0.3514.

Table 1.4.32 Results of the MCDM method on orange fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Kalanchoe	0.2952	0.3617	0.3548	0.3938	0.3514	0.0412	3
2. Mandarin	0.7034	0.6900	0.5431	0.6035	0.6350	0.0756	2
3. Sweet orange	0.6466	0.6270	0.7610	0.6933	0.6820	0.0596	1

Table 1.4.33 Results of the MCDM method on mango fruits with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Unripe mango kaew variety	0.3917	0.4417	0.4028	0.4385	0.4187	0.0252	3
2. Unripe mango pimsaen variety	0.1601	0.2227	0.2052	0.2639	0.2130	0.0430	5
3. Ripe mango rad variety	0.1599	0.1982	0.1764	0.2112	0.1864	0.0228	7
4. Ripe mango kaew variety	0.4515	0.4724	0.4348	0.4548	0.4534	0.0154	2
5. Ripe mango nam-dok-mai variety	0.7321	0.6177	0.7051	0.5948	0.6624	0.0664	1
6. Ripe mango pimsaen-mun variety	0.1410	0.2236	0.1829	0.2555	0.2007	0.0497	6
7. Unripe mango kiew-sa-weya variety	0.1890	0.2476	0.2120	0.2600	0.2272	0.0326	4

From Table 1.4.33, we observe that the first rank of mango fruits has ripe mango nam-dok-mai variety with the mean of 0.6624 while unripe mango rad variety has the last rank of the group with the mean of 0.1864.

1.4.1.4 Desserts

Desserts are classified into five groups, namely coconut milk, steamed, fried, roast and conserved, respectively. The results of the normalized overall index based on the L_1-norm and L_2-norm with w_1 and w_2 weights in each group is shown in Tables 1.4.34- 38.

Table 1.4.34 Results of the MCDM method on coconut milk desserts with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.1796	0.2189	0.1639	0.2022	0.1912	0.0243	7
2	0.0596	0.0819	0.0690	0.0960	0.0766	0.0158	12
3	0.4487	0.4347	0.4608	0.4386	0.4457	0.0117	2
4	0.0306	0.0884	0.0444	0.1050	0.0671	0.0353	13
5	0.0591	0.1156	0.0736	0.1301	0.0946	0.0337	10
6	0.3735	0.3769	0.3911	0.3845	0.3815	0.0079	3
7	0.0421	0.1051	0.0565	0.1208	0.0811	0.0378	11
8	0.0433	0.0762	0.0530	0.0882	0.0652	0.0206	14
9	0.0068	0.0328	0.0123	0.0512	0.0258	0.0203	19
10	0.0252	0.0754	0.0370	0.0925	0.0575	0.0317	16
11	0.3681	0.3701	0.3749	0.3686	0.3704	0.0031	4
12	0.0376	0.0545	0.0400	0.0573	0.0473	0.0100	17
13	0.5525	0.4562	0.5328	0.4433	0.4962	0.0545	1
14	0.2803	0.2912	0.2478	0.2681	0.2719	0.0186	5
15	0.0142	0.0523	0.0202	0.0610	0.0369	0.0232	18
16	0.0416	0.0706	0.0505	0.0799	0.0607	0.0177	15
17	0.2401	0.2714	0.2458	0.2706	0.2570	0.0164	6
18	0.0771	0.1100	0.0870	0.1205	0.0987	0.0200	9
19	0.1686	0.2079	0.1727	0.2069	0.1890	0.0213	8

Note: 1: banana in coconut milk; 2: tab tim krob with coconut milk and syrup
3: black sticky rice with coconut milk and syrup; 4: taro balls in coconut milk
5: taro in coconut milk; 6: tao suan
7: rice drops in sweet coconut milk; 8: sago with coconut milk
9: nata de coco in syrup; 10: mungbean thread in coconut milk

Note (cont.): 11: boiled cowpea seeds with sugar and coconut milk
12: boiled rice flour chunk with coconut milk
13: rice green flake with coconut milk and syrup; 14: native melon in coconut milk
15: jelly with coconut cream pandanus leaves flavor; 16: sticky rice in coconut milk
17: kanom tue pap; 18: kanom tom khaw
19: kanom tom daeng

From Table 1.4.34, the first rank of coconut milk desserts has rice green flake with coconut milk and syrup with the mean of 0.4962, followed by black sticky rice with coconut milk and syrup with the mean of 0.4457. On the other hand, the last rank has nata de coco in syrup with the mean of 0.0258.

Table 1.4.35 Results of the MCDM method on steamed desserts with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.0045	0.0229	0.0076	0.0288	0.0160	0.0118	21
2	0.1811	0.2406	0.1878	0.2365	0.2115	0.0314	7
3	0.2169	0.3247	0.2432	0.3329	0.2794	0.0581	3
4	0.0841	0.1935	0.1046	0.2083	0.1476	0.0624	10
5	0.0922	0.1794	0.1075	0.1920	0.1428	0.0502	11
6	0.2376	0.2912	0.2625	0.3051	0.2741	0.0301	4
7	0.0104	0.0354	0.0152	0.0433	0.0261	0.0158	18
8	0.1903	0.2438	0.2061	0.2500	0.2226	0.029	6
9	0.7464	0.5548	0.7253	0.5374	0.6410	0.1101	1
10	0.1362	0.1948	0.1443	0.1962	0.1679	0.0321	9
11	0.0520	0.0819	0.0535	0.0799	0.0668	0.0163	13
12	0.2588	0.2568	0.2433	0.2400	0.2497	0.0095	5
13	0.0347	0.0715	0.0454	0.0903	0.0605	0.0252	15
14	0.0405	0.0604	0.0428	0.0605	0.0511	0.0109	17
15	0.0056	0.0272	0.0094	0.0340	0.0190	0.0137	20
16	0.0161	0.0747	0.0283	0.0973	0.0541	0.0383	16
17	0.0255	0.0842	0.0386	0.1032	0.0629	0.0368	14
18	0.1902	0.1979	0.1875	0.1961	0.1929	0.0049	8
19	0.3411	0.3234	0.3354	0.3158	0.3289	0.0115	2
20	0.1005	0.1893	0.0926	0.1750	0.1394	0.0499	12
21	0.0059	0.0315	0.0099	0.0395	0.0217	0.0163	19

Note: 1: banana paste
2: taro paste
3: sticky rice flour coconut milk and sugar boiled
4: mock fruits
5: egg custard baked
6: mock jackfruit seed
7: coconut pudding with water chestnut
8: egg yolk dropped in heavy syrup
9: gold threads egg yolk strained in heavy syrup
10: kanom piak poon
11: kanom touy
12: kanom Sali
13: kanom chun
14: kanom num dok mai
15: tong muan
16: sam pan nee
17: glutinous rice steeped in coconut milk

18: sticky rice with shrimp and coconut meat 19: sticky rice with custard
20: steamed mixture of pumpkin rice flour and coconut 21: steamed toddy palm cake

From Table 1.4.35, gold threads egg yolk strained in heavy syrup represents the first rank of steamed desserts with the mean of 0.6410 while banana paste has the last rank of the group with the mean of 0.0160.

Table 1.4.36 Results of the MCDM method on fried desserts with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1	0.0124	0.0956	0.0314	0.1463	0.0714	0.0613	7
2	0.2753	0.3701	0.2746	0.3616	0.3204	0.0526	3
3	0.1179	0.2113	0.1314	0.2216	0.1706	0.0534	6
4	0.4139	0.4377	0.3926	0.4173	0.4154	0.0185	2
5	0.8340	0.7143	0.8382	0.7113	0.7744	0.0712	1
6	0.0324	0.0726	0.0425	0.0895	0.0592	0.0265	8
7	0.0065	0.0467	0.0149	0.0676	0.0339	0.0283	10
8	0.0054	0.0568	0.0122	0.0797	0.0385	0.0357	9
9	0.1521	0.1895	0.1637	0.1972	0.1756	0.0212	5
10	0.0040	0.0289	0.0091	0.0430	0.0212	0.0180	11
11	0.1378	0.2451	0.1433	0.2430	0.1923	0.0598	4

Note: 1: deep fried banana 2: kanom khai nok krata
3: doughnut with sugar 4: glutinous rice cooked deep fried
5: krong krang krob kem 6: kanom nang led
7: taro slided deep fried sugar coated 8: banana slided deep fried sugar coated
9: kanom khai hong 10: deep fried rice cake
11: large soft rice pancake topped with shredded

From Table 1.4.36, we observe that the first rank of fried desserts has krong krang krob kem with the mean of 0.7744. On the other hand, the last rank has deep fried rice cake with the mean of 0.0212.

Table 1.4.37 Results of the MCDM method on roast desserts with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Kanom ping	0.1210	0.1384	0.1265	0.1472	0.1333	0.0118	6
2. Kanom krok	0.4442	0.5087	0.5284	0.5420	0.5058	0.0433	2
3. Tokyo snack similar to pancake	0.3716	0.3942	0.3717	0.3883	0.3815	0.0116	3
4. Kanom babin	0.0927	0.1581	0.0894	0.1464	0.1216	0.0357	7
5. Cake sponge commercially prepared	0.1004	0.1996	0.1263	0.2222	0.1621	0.0580	5
6. Kanom kleeb lamduan	0.1192	0.2418	0.1632	0.2708	0.1987	0.0699	4
7. Coconut meat and rice flour in coconut milk	0.0464	0.1424	0.0719	0.1711	0.1079	0.0585	8
8. Kanom naa nual	0.7842	0.6505	0.7148	0.6003	0.6874	0.0797	1

As can be seen in Table 1.4.37, the first rank of roast desserts has kanom naa nual with the mean of 0.6874, followed by kanom krok with the mean of 0.5058. On the other hand, the last rank has coconut meat and rice flour in coconut milk representing the 8th with the mean of 0.1079.

Table 1.4.38 Results of the MCDM method on conserved desserts with normalized overall indices.

Type	I_i				Mean	SD	Rank
	w_1		w_2				
	L_1	L_2	L_1	L_2			
1. Pumpkin in syrup	0.4602	0.4817	0.4284	0.4569	0.4568	0.0219	2
2. Cassava boiled in syrup	0.8060	0.6945	0.8177	0.6968	0.7537	0.0672	1
3.Toddy palm in heavy syrup	0.0097	0.0852	0.0363	0.1536	0.0712	0.0632	5
4. Artocarpus altilis in concentrated syrup	0.3513	0.4320	0.3549	0.4258	0.3910	0.0439	3
5. Mango pickled sweet	0.1231	0.3028	0.1436	0.3176	0.2218	0.1026	4

From Table 1.4.38, cassava boiled in syrup represents the first rank of conserved desserts with the mean of 0.7537 while toddy palm in heavy syrup has the last rank of the group with the mean of 0.0712.

1.4.2 Electre Method

We apply the Electre method on vegetables, foods, fruits and desserts based on eleven nutrient indicators and prices by using the following procedure.

Given a data matrix X with K rows and N columns $\left(X = (x_{ij}) : K \times N\right)$, where the row represents each type of vegetable, food, fruit or dessert to be compared with respect to the element x_{ij} and the columns represent the eleven nutrient indicators and prices for each type of vegetable, food, fruit or dessert.

Step 1: Transform matrix $\mathbf{X} = \left[X_1, X_2, ..., X_N\right]$ to matrix $\mathbf{R} = \left[R_1, R_2, ..., R_N\right]$,

where $R_i = \dfrac{X_i}{\|X_i\|_2}$.

Step 2: Transform matrix **R** to matrix **V** where **V = RW** where matrix

$\mathbf{W} = diag\left[w_1, w_2, ..., w_N\right]$ is based on the weight of the coefficient of variation(w_2).

Step 3: Construct two matrices **C** and **D**,

$$\text{where } c_{ij} = \sum_{k:v_{ik} \geq v_{jk}} w_k \quad , \qquad \text{and} \quad d_{ij} = \frac{\max_{k:v_{ik} < v_{jk}} \left|V_{ik} - V_{jk}\right|}{\max_k \left|V_{ik} - V_{jk}\right|}. \tag{1.11}$$

$$\text{Compute } \bar{c} = \frac{\sum\sum_{i \neq j} c_{ij}}{K(K-1)} \quad , \quad \text{and} \quad \bar{d} = \frac{\sum\sum_{i \neq j} d_{ij}}{K(K-1)}. \tag{1.12}$$

Step 4: Construct matrices **F** and **G** such that

$$f_{ij} = \begin{cases} 1 & ; c_{ij} \geq \bar{c} \\ 0 & ; otherwise \end{cases} \qquad \text{and} \qquad g_{ij} = \begin{cases} 1 & ; d_{ij} \leq \bar{d} \\ 0 & ; otherwise \end{cases}. \tag{1.13}$$

Step 5: Define matrix **E** where $e_{ij} = f_{ij} \cdot g_{ij}$. (1.14)

It should be noted that the weights w_i are obtained as discussed and $e_{ij} = 0$ means that row *i* is better than row *j* or $e_{ij} = 1$ means that column *j* is better than row *i*.

The results of the Electre method applied to each type of vegetable, food, fruit or dessert are based on the eleven nutrient indicators and prices as given in the following sections.

1.4.2.1 Vegetables

Vegetables are classified into seven groups, namely blossom, chili, leaf and apex, pod, fruit, plant and tuber. The data matrix X has eleven rows for blossom vegetable types including abalone mushroom, chinese chives, chinese mustard green, cowslip creeper, dried mushroom, ear mushroom, onion flowers, sajor-caju mushroom, sesban, shiitake mushroom, straw mushroom and thirteen columns of nutrient indicators namely energy, carbohydrate, fat, protein, protein-vegetable, calcium, iron, iron-vegetable, vitamin A, thiamin, riboflavin, vitamin C, niacin and price. In matrix X, we use the actual nutrient indicator values and the reciprocal of price for each type of blossom vegetables for reasonable comparison.

Finally, the results of the **E**-matrices for seven groups of vegetables are shown in Table 1.4.39.

Table 1.4.39 E-matrices of the seven groups of vegetable

blossom

Type	1	2	3	4	5	6	7	8	9	10	11	Rank
1	1	1	1	0	0	0	0	1	0	0	0	3
2	1	1	1	1	1	0	0	1	0	0	0	2
3	1	1	1	1	1	1	0	1	1	0	0	1
4	1	1	1	1	0	0	0	1	1	0	0	2
5	1	1	1	1	1	0	0	1	1	0	1	1
6	0	0	0	0	0	0	0	0	0	0	0	6
7	0	1	0	0	0	0	0	0	0	0	0	5
8	0	1	0	0	0	0	0	0	0	0	0	5
9	0	1	1	0	0	0	0	0	0	0	0	4
10	0	1	0	0	0	0	0	0	0	0	0	5
11	1	1	0	0	0	0	0	0	0	0	0	4

Note: 1: abalone mushroom 2: chinese chives 3: chinese mustard green 4: cowslip creeper 5: dried mushroom 6: ear mushroom 7: onion flowers 8: sajor-caju mushroom 9: Sesban 10: shiitake mushroom 11: straw mushroom

We observe that the blossom vegetables, chinese mustard green and dried mushroom have the best type while chinese chives and cowslip creeper have the second best, respectively. The worst has ear mushroom.

chili

Type	1	2	3	4	5	6	7	8	9	10	Rank
1	0	0	0	0	1	0	0	0	0	0	4
2	1	0	1	0	1	0	0	0	0	0	3
3	1	1	1	0	1	0	0	0	1	0	2
4	0	0	0	0	0	0	0	0	0	0	5
5	1	1	1	1	1	1	0	1	1	0	1
6	1	1	1	1	1	1	0	1	1	0	1
7	1	1	1	0	1	0	0	0	1	0	2
8	0	0	0	0	1	0	0	0	0	0	4
9	0	0	0	0	1	0	0	0	0	0	4
10	1	1	1	0	1	0	0	0	1	0	2

Note: 1: bell pepper 2: bird pepper 3: black pepper 4: chili spur pepper 5: dried goat pepper 6: dried guinea pepper 7: piment rouge 8: piment vert 9: sweet pepper 10: yellow pepper

For the chili vegetables based on nutrient indicators and prices, we found that dried goat pepper and dried guinea pepper have the best while yellow pepper, black pepper and piment rouge have the second best. The worst has chili spur pepper.

leaf and apex

Type	1	2	3	4	5	6	7	8	9	10	11	12	Rank
1	1	0	1	0	1	0	1	1	0	0	1	0	3
2	0	1	1	1	1	0	1	1	0	0	1	0	2
3	0	0	1	0	1	0	1	1	0	0	1	0	4
4	0	0	0	0	0	0	1	0	0	0	0	0	8
5	0	0	0	0	0	0	0	0	0	0	0	0	9
6	0	1	1	0	1	0	1	1	0	0	1	0	3
7	0	0	0	0	0	0	1	1	0	0	0	0	7
8	1	1	1	1	1	1	1	1	0	1	1	1	1
9	0	0	0	0	0	0	0	0	0	0	0	0	9
10	0	1	0	0	0	0	1	1	0	0	1	0	5
11	0	0	1	0	1	0	1	1	0	0	1	0	4
12	0	0	0	0	1	0	1	1	0	0	0	0	6

Note: 1: acacia 2: basil 3: bitter gourd 4: chinese chive 5: coconut 6: feuille de menthe 7: gord gourd 8: kaffir lime leaves 9: lettuce 10: sauropus androgynous 11: stem of sweet basil 12: stem of sweet basil

From leaf and apex vegetables based on nutrient indicators and prices, we observe that the best has kaffir lime leaves and the second best is basil. The worst have coconut and lettuce.

pod

Type	1	2	3	4	5	6	7	8	9	10	11	Rank
1	0	0	1	1	1	1	0	0	0	0	1	3
2	0	0	1	0	1	1	0	0	0	0	1	4
3	0	1	1	1	1	1	1	0	0	0	1	2
4	0	0	0	0	0	0	0	0	0	0	0	7
5	1	1	1	1	1	1	1	1	0	0	1	1
6	0	0	0	0	1	1	0	0	0	0	1	5
7	0	0	0	0	0	1	0	0	0	0	0	6
8	0	0	0	0	0	0	0	0	0	0	0	7
9	0	0	1	0	1	1	0	0	0	0	1	4
10	0	0	0	0	1	1	0	0	0	0	1	5
11	0	0	1	0	1	1	0	0	0	0	1	4

Note: 1: green pea 2: holland bean 3: horse radish tree 4: lotus stem 5: ripe tamarind 6: smooth loofah 7: snake gourd 8: sponge gourd 9: string bean 10: winged bean 11: young corn

For pod vegetables based on nutrient indicators and prices, the best type has ripe tamarind and second best has horse radish tree. In addition, the worst types have lotus stem and sponge gourd.

fruit

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Rank
1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	6
2	1	0	0	0	1	1	0	0	1	0	0	0	0	1	1	4
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	8
4	1	1	0	0	1	0	0	0	1	0	0	0	0	1	1	4
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	8
6	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	6
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9
8	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	7
9	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1
10	1	1	1	0	1	1	0	0	1	1	0	0	1	1	1	2
11	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	6
12	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	6
13	1	0	1	0	1	0	0	0	1	1	0	0	0	1	1	3
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	8
15	0	1	0	0	0	0	0	0	1	0	0	0	0	1	1	5

Note: 1: cherry tomatoes 2: chinese bitter gourd 3: cucumber 4: eggplant 5: green papaya 6: lemon 7: long cucumber 8: long cucumber 9: pea eggplant 10: pumpkin 11: purple eggplant 12: small eggplant 13: tomato 14: wax gourd 15: wild bitter gourd

Next, for the fruit vegetables based on nutrient indicators and prices, the best type has pea eggplant and the second best has pumpkin. The worst type has long cucumber.

plant

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Rank
1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	8
2	0	0	0	1	0	0	1	1	0	0	0	0	1	0	0	6
3	0	0	0	1	0	0	1	1	0	1	0	0	1	0	0	5
4	0	0	0	1	0	0	1	1	0	1	0	0	1	0	0	5
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9
6	0	0	0	1	0	0	1	1	0	1	0	0	1	0	0	5
7	0	0	0	1	0	1	1	1	0	1	1	0	1	0	1	4
8	0	1	0	1	0	1	1	1	0	1	1	1	1	0	0	3
9	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	8
10	1	1	0	1	0	1	1	1	0	1	1	1	1	0	1	1
11	0	0	0	1	0	1	1	1	0	1	1	1	1	0	0	4
12	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	8
13	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	7
14	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	7
15	1	0	0	1	1	1	1	1	0	1	1	1	1	0	0	2

Note : 1: asparagus 2: cabbage chick 3: celery 4: chinese cabbage 5: chinese cabbage
6: chinese kale 7: coriander 8: dill 9: lettuce Taiwan 10: parsley 11: spinate
12: spring onion 13: swamp cabbage 14: swamp morning glory 15: water minosa

Moreover, the plant vegetables based on nutrient indicators and prices, the best type has parsley and the second best has water mimosa. The worst type has chinese cabbage.

tuber

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Rank
1	1	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0	6
2	1	1	1	0	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1
3	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	7
4	1	0	0	0	1	1	1	0	1	0	1	1	0	1	0	0	1	1	2
5	1	0	0	0	1	1	1	0	0	0	1	1	0	1	0	0	1	0	4
6	1	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	6
7	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	8
8	1	1	0	0	1	1	1	0	0	0	1	1	0	1	0	0	1	0	3
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
10	0	0	0	0	1	1	1	0	0	0	1	1	1	1	0	1	0	0	4
11	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	9
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
13	1	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0	6
14	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	7
15	1	1	0	0	1	1	1	0	0	0	1	1	1	1	0	0	1	0	2
16	0	0	0	0	1	1	1	0	0	0	1	0	0	1	0	0	1	0	5
17	0	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	1	0	6
18	1	1	0	0	1	1	1	0	0	0	1	0	1	1	0	0	0	0	4

Note: 1: banana blossom 2: broccoli 3: cabbage 4: carrot 5: cauliflower 6: chinese cabbage 7: curcuma 8: fingerroot 9: fresh ginger 10: garlic 11: old ginger 12: onion 13: potato 14: radish 15: red cabbage 16: shallot 17: sweet potato 18: taro

Finally, the tuber vegetables based on nutrient indicators and prices, broccoli has the best type while second best have carrot and red cabbage. The worst types have fresh ginger and onion.

1.4.2.2 Foods

Foods are classified into five groups, namely stir frying, broth, blend, roast and rice noodles, respectively. The results of the **E**-matrices in each food groups are shown in Table 1.4.40.

Table 1.4.40 E-matrices of the five groups of foods

stir frying (vegetable)

Type	1	2	3	4	5	6	7	Rank
1	0	1	0	1	1	0	0	3
2	0	0	0	0	0	0	0	5
3	0	1	0	1	1	0	0	3
4	0	0	0	0	0	0	0	5
5	0	0	0	1	0	0	0	4
6	1	1	1	1	1	0	0	1
7	0	1	1	1	1	0	0	2

Note: 1: quick-fried water spinach seasoned with chili and soy sauce 2: mushroom fried with ginger
3: water minosa fried in oyster sauce 4: sugar pea fried in oyster sauce
5: chinese kale fried in oyster sauce 6: tofu fried with cabbage
7: cowslip creeper fried in oyster sauce

For the stir frying (vegetable) foods, tofu fried with cabbage has the best type while mushroom fried with ginger, sugar pea fried in oyster sauce have the worst types.

stir frying (meat)

Type	1	2	3	4	5	6	7	8	9	10	Rank
1	0	1	1	0	0	1	1	1	1	0	2
2	0	0	0	0	0	0	1	0	1	0	5
3	0	0	0	0	0	0	0	0	1	0	6
4	1	1	1	0	1	1	1	1	1	1	1
5	0	1	1	0	0	1	1	1	1	0	2
6	0	0	1	0	0	0	0	0	1	0	5
7	0	0	1	0	0	0	0	0	1	0	5
8	0	0	1	0	0	0	1	0	1	0	4
9	0	0	0	0	0	0	0	0	0	0	7
10	0	1	1	0	0	1	0	1	1	0	3

Note: 1: stir fried pork with green peppers 2: spicy meat and tomato dip
3: pork fried roasted chili paste 4: fried seafood in yellow curry 5: spicy fried pork
6: stir fried pork tenderloin with black peppercorn 7: tofu and pork with chili paste
8: pork fried in oyster sauce 9: dried fried chicken 10: fried fish with tamarind sauce

As can be seen, the best type of stir frying (meat) foods has fried seafood in yellow curry and then spicy fried pork, stir fried pork with green peppers have the second best. The worst has dried fried chicken.

stir frying (mixed)

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	Rank
1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	1	0	1	1	1	0	0	9
2	0	0	0	1	0	0	1	1	0	1	1	1	1	0	1	0	0	1	1	0	1	1	1	1	1	1	0	1	0	3
3	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2
4	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	12
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	1	1	1	0	0	10
6	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	1	1	0	0	1	0	1	1	0	0	0	8
7	0	0	0	1	0	0	0	0	0	0	1	0	1	0	1	0	0	0	1	0	1	1	1	0	1	1	0	0	0	7
8	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	1	1	0	0	0	10
9	0	0	0	1	1	1	0	1	0	1	1	0	1	0	1	0	0	0	1	1	1	0	1	0	1	1	0	1	0	5
10	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	12
11	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	12
12	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	1	0	0	1	0	1	1	0	0	0	9
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	14
14	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	1	0	0	1	0	1	0	0	0	0	9
15	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	12
16	0	0	0	1	1	1	0	0	0	0	1	0	1	0	1	0	0	1	1	1	1	0	1	0	1	1	1	0	0	6

stir frying (mixed) (cont.)

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	Rank
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
18	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	1	0	0	0	11
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	15
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	15
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	14
22	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	13
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	13
24	1	0	0	0	1	0	1	1	0	1	1	0	1	0	1	0	0	1	0	1	1	1	1	0	1	1	0	1	0	4
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	15
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	13
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	14
29	0	0	0	1	0	0	0	1	0	1	0	0	1	0	0	0	0	0	1	1	0	0	1	0	1	0	0	0	0	8

Note: 1: chicken wing fried with cabbage pickled 2: stir fried chinese kale and pork with oyster sauce
3: sauted mixed vegetables with pork in oyster sauce 4: stir fried mung bean noodle
5: stir fried pork sausage with egg and garnished with seafood 6: stuffed omelet 7: thai rich tofu
8: chicken fried with cashews 9: stir fried pork with string bean in soybean paste white
10: shrimps fried with fresh coconut 11: fried pork minced and string bean with chili
12: seabass spicy stir fried 13: chicken stir fried with ginger
14: saute sponge gourd egg and shrimps 15: pork fried with bamboo shoots and chili
16: fried egg garnished with seafood 17: stir fried pork liver with black peppercorn
18: pork fried with chili, ginger and string bean
19: saute chayote tops and young leaves egg and shrimps 20: radish dried salted fried with egg
21: sweet and sour sauce fried with pork 22: catfish spicy stir fried
23: stir fried seabass with chinese celery 24: deep fried fish with sweet chili sauce
25: fried chinese kale with salted fish 26: fried chinese kale with crispy pork
27: stir fried crispy basil with pork and black preserved eggs
28: chicken stir fried with chili pepper 29: seafood spicy stir fried

For stir frying (mixed) foods, the best type has stir fried pork liver with black peppercorn. The worst type has fried chinese kale with crispy pork.

broth

Type	1	2	3	4	5	6	7	8	Rank
1	0	0	0	0	0	0	0	0	7
2	1	0	1	1	1	1	1	1	1
3	1	0	0	0	0	0	0	0	6
4	1	0	1	0	0	1	0	0	4
5	1	0	0	0	0	1	0	0	5
6	1	0	0	0	0	0	0	0	6
7	1	0	1	1	1	1	0	0	2
8	1	0	1	0	1	1	0	0	3

Note: 1: clear soup with bean curd and minced pork 2: clear soup with seaweed and minced pork
3: pork blood soup 4: bitter melon soup in sparerib broth 5: stuffed squid soup
6: mined pork soup with coccinia grandis 7: chinese vegetable stew
8: clear soup with pork kidney

For broth foods, the best type has clear soup with seaweed and minced pork. The worst type has clear soup with bean curd and minced pork.

broth (spicy)

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Rank
1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	1	1	0	0	1	7
2	0	0	1	0	1	0	0	0	0	0	0	1	1	0	1	1	0	0	1	6
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12
4	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	8
5	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	8
6	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	8
7	1	1	1	1	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	0	0	0	0	1	1	1	0	1	1	1	1	1	2
9	1	0	1	0	1	1	0	0	0	0	1	1	1	0	1	1	1	1	1	3
10	0	1	1	1	1	1	0	0	0	0	1	1	1	0	1	1	0	1	1	3
11	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	8
12	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	9
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	10
14	0	1	1	1	1	1	0	0	0	0	1	1	1	0	1	1	0	0	1	4
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	11
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12
17	0	1	1	1	1	1	0	0	0	0	0	1	1	0	1	1	0	0	1	5
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12

Note: 1: sour prawn soup 2: hot and sour chicken soup 3: spicy soup with striped snakehead fish
4: fresh mackerel soup 5: carp fish soup 6: sour and spicy fish grilled 7: spicy seafood soup
8: hot and sour pork spare ribs 9: egg water add steamed with tom yum
10: sour and spicy fish crispy with tamarind apex 11: tom yum with multiply mushroom
12: hot and sour snapper filet 13: tom yam canned mackerel in tomato sauce
14: hot and spicy pork rib hot pot with tamarind and thai herbs 15: thai hot soup with beef
16: pork organ soup 17: sour and spicy soup of thai chicken
18: sour and spicy smoked fish soup 19: hot and sour nile tilapia

As can be seen, the best type from this group of broth (spicy) foods has spicy seafood soup. The worst have spicy soup with striped snakehead fish, pork organ soup, sour and spicy smoked fish soup and hot and sour nile tilapia.

broth (coconut milk)

Type	1	2	3	4	5	6	7	8	Rank
1	0	0	0	0	0	0	0	0	5
2	1	0	1	1	1	0	0	0	2
3	0	0	0	0	0	0	0	0	5
4	1	0	1	0	1	0	0	0	3
5	0	0	1	0	0	0	0	0	4
6	1	0	1	0	1	0	1	0	2
7	1	0	0	0	0	0	0	0	4
8	1	1	1	1	1	0	1	0	1

Note: 1: sour soup with shrimp and morning glory 2: fish organs sour soup
3: spicy vegetable and prawn soup 4: chicken and eggplant in spicy soup
5: un-coconut curry fish balls 6: pork with vegetables curry 7: hot yellow fish curry
8: shrimp and fried egg sour soup

Next, for another group of the broth (coconut milk) foods, the best type has shrimp and fried egg sour soup, the worst types have sour soup with shrimp and morning glory and spicy vegetable and prawn soup.

broth (without coconut milk)

Type	1	2	3	4	5	6	7	8	9	10	11	Rank
1	0	1	1	1	1	1	1	1	1	1	0	1
2	0	0	1	0	0	0	0	0	0	0	0	6
3	0	0	0	0	0	0	0	0	0	0	0	7
4	0	0	1	0	0	0	0	0	1	0	0	5
5	0	1	1	1	0	0	0	0	1	1	0	3
6	0	1	1	0	0	0	0	0	1	1	0	4
7	0	1	1	1	1	1	0	0	1	1	0	2
8	0	1	1	1	1	1	0	0	1	1	0	2
9	0	0	0	0	0	0	0	0	0	0	0	7
10	0	0	0	0	0	0	0	0	0	0	0	7
11	0	1	1	1	1	1	1	1	1	1	0	1

Note: 1: soup with chicken, galangal root and coconut 2: soup shrimp and galangal in coconut milk
3: hot thai curry with chicken and bamboo shoot 4: curry with chicken
5: savory curry with pork 6: muslim-style curry with chicken and potatoes
7: green chicken curry 8: pork curry with water spinach 9: northern style pork curry with garlic
10: chicken curry with banana stalk 11: whisker sheatfish chu chee curry

From the broth (without coconut milk) foods, whisker sheatfish chu chee curry and soup with chicken, galangal root and coconut have the best types and the worst have hot thai curry with chicken and bamboo shoot, northern style pork curry with garlic and chicken curry with banana stalk.

blend (vegetable)

Type	1	2	3	4	5	6	7	8	9	Rank
1	0	1	1	0	1	0	1	0	1	2
2	0	0	1	0	1	0	1	1	1	2
3	0	0	0	0	0	0	0	0	0	5
4	1	1	1	0	1	0	1	1	1	1
5	0	0	1	0	0	0	0	0	0	4
6	1	1	1	0	1	0	1	1	1	1
7	0	0	1	0	0	0	0	0	0	4
8	0	0	1	0	1	0	0	0	0	3
9	0	0	1	0	0	0	0	0	0	4

Note: 1: wing bean spicy and sour salad 2: chinese kale and seafood in spicy sauce
3: white jelly fungus in spicy sauce 4: thai spicy water minosa salad

Note (cont.): 5: crispy water convolvulus 6: fried gord gourd spicy salad
7: lemongrass and windbetal leaves spicy salad 8: yum hua plee
9: spicy long eggplant salad

As can be seen, the best types of blend (vegetable) foods h thaive spicy water minosa salad and fried gord gourd spicy salad. The worst has white jelly fungus in spicy sauce.

blend (meat)

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Rank
1	0	0	0	1	0	0	0	1	1	0	1	0	1	1	1	3
2	0	0	0	1	0	0	0	1	1	1	1	0	1	1	0	3
3	0	1	0	1	0	0	1	1	1	1	1	0	1	1	0	2
4	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	6
5	0	0	0	1	0	0	1	1	1	0	0	0	0	1	1	4
6	0	0	0	1	0	0	1	1	0	0	1	0	0	0	0	5
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	7
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
10	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	7
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
12	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
13	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	6
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
15	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	6

Note: 1: spicy cockle salad 2: spicy striped snakehead fish crispy salad
3: crispy catfish salad with green mango 4: sour steamed pork sausage salad
5: spicy fried egg and bacon 6: spicy pork small intestine 7: tuna spicy salad
8: spicy chicken feet salad 9: sliced grilled pork tenderloin salad old thai style
10: chinese sausage salad 11: sour canned fish salad 12: spicy friture salad
13: spicy snake skin gourami 14: shreded chicken spicy salad 15: spicy oyster salad

Next, for the blend (meat) foods, the best type has spicy friture salad and the worst types have spicy chicken feet salad, sliced grilled pork tenderloin salad old thai style, sour canned fish salad and shreded chicken spicy salad.

blend (mixed)

Type	1	2	3	4	5	6	7	8	9	Rank
1	0	1	1	1	1	1	0	1	1	1
2	0	0	0	0	0	1	0	1	1	3
3	0	0	0	0	0	1	0	1	1	3
4	0	0	1	0	0	1	0	1	1	2
5	0	0	0	0	0	1	0	0	0	4
6	0	0	0	0	0	0	0	0	0	5
7	0	1	1	1	1	1	0	1	1	1
8	0	0	0	0	0	0	0	0	0	5
9	0	0	0	0	0	0	0	1	0	4

Note: 1: mungbean noodle salad 2: spicy combination seafood salad 3: mixed crispy spicy salad
4: boiled egg spicy salad 5: instant noodle yam 6: egg preserved spicy salad
7: yum yai 8: stir fried chicken with pomelo salad 9: hot spicy cavilla salad

From the blend (mixed) foods, mungbean noodle salad and yum yai have the best types and the worst have stir fried chicken with pomelo salad and egg preserved spicy salad.

roast

Type	1	2	3	4	5	6	7	8	9	10	11	Rank
1	0	0	0	0	0	0	0	0	0	0	1	4
2	0	0	0	0	0	0	0	0	0	0	1	4
3	1	1	0	0	0	0	0	0	1	0	1	3
4	1	1	0	0	0	0	0	0	1	0	1	3
5	1	1	1	0	0	0	1	0	1	0	1	2
6	1	1	0	1	1	0	1	0	1	1	1	1
7	0	0	0	0	0	0	0	0	1	0	0	4
8	1	1	1	1	1	0	1	0	1	0	1	1
9	0	0	0	0	0	0	0	0	0	0	0	5
10	1	1	0	0	1	0	1	0	1	0	1	2
11	0	0	0	0	0	0	0	0	0	0	0	5

Note: 1: shrimps with glass noodles 2: roasted chicken with soy sauce 3: roasted chicken with ketchup
4: baked pork spare rib 5: roasted mungbean with termite mushroom
6: roasted crab with chili paste fried in oil 7: roasted chicken wing
8: one whole roasted young mountain pork bathed in wild honey 9: grilled lemongrass chicken
10: egg hen water add steamed 11: roasted chicken with lemon

As can be seen, the best types of roast foods have roasted crab with chili paste fried in oil and one whole roasted piece of young mountain pork bathed in wild honey. The worst have grilled lemongrass chicken and roasted chicken with lemon.

steamed

Type	1	2	3	4	5	6	7	8	9	10	11	12	Rank
1	0	1	1	0	0	0	0	0	0	0	0	0	6
2	0	0	1	0	0	0	0	0	0	0	0	0	7
3	0	0	0	0	0	0	0	0	0	0	0	0	8
4	1	1	1	0	0	1	1	0	0	0	1	0	3
5	1	1	1	0	0	1	1	0	0	0	0	0	4
6	0	0	1	0	0	0	0	0	0	0	0	0	7
7	0	0	1	0	0	0	0	0	0	0	0	0	7
8	1	1	1	0	1	1	1	0	0	0	1	0	2
9	1	1	1	0	1	1	1	0	0	0	1	1	1
10	1	1	1	0	1	1	1	0	0	0	1	0	2
11	0	1	1	0	0	1	1	0	0	0	0	0	5
12	1	1	1	0	1	1	1	0	0	0	1	0	2

Note: 1: mushroom with many materials 2: spotted featherback steamed
3: steamed snapper with lemon 4: steamed fish with tofu
5: steamed fish with vegetable and chili paste 6: silver pomfret steamed
7: steamed nile tilapia with vegetable 8: steamed salmon with lemon
9: steamed salmon with soy sauce 10: steamed fish with coconut milk in a banana leaf wrapping
11: steamed chicken with ear mushroom in soy sauce paste 12: steamed giant seaperch

From the data of steamed foods, the best type has steamed salmon with soy sauce, steamed fish with coconut milk in a banana leaf wrapping and steamed giant seaperch have the second best while the worst has steamed snapper with lemon.

grilled

Type	1	2	3	Rank
1	0	1	0	1
2	0	0	0	2
3	0	0	0	2

Note: 1: chicken grilled with herb 2: fish grilled with lemon sauce 3: fish grilled with soy sauce

As can be seen, the best type has chicken grilled with herb. The worst types have fish grilled with lemon sauce and fish grilled with soy sauce.

rice noodle (water)

Type	1	2	3	4	5	6	7	8	9	10	11	12	Rank
1	0	0	0	0	1	1	0	0	0	1	1	0	5
2	0	0	0	0	1	1	0	0	0	1	1	0	5
3	1	1	0	1	1	1	0	0	0	1	1	0	3
4	0	1	0	0	1	1	0	0	0	1	1	0	4
5	0	0	0	0	0	0	0	0	0	1	1	0	7
6	0	0	0	0	0	0	0	0	0	1	0	0	8
7	1	1	1	1	1	1	0	1	0	1	1	0	1
8	1	0	0	0	1	0	0	0	0	0	1	0	6
9	1	1	1	1	1	1	0	0	0	1	1	1	1
10	0	0	0	0	0	0	0	0	0	0	0	0	9
11	0	0	0	0	0	0	0	0	0	1	0	0	8
12	1	1	1	1	1	1	0	0	0	1	1	0	2

Note: 1: small size rice noodle fresh with beef 2: wide rice noodle with fish ball swamp cabbage
3: boiled vermicelli 4: small size rice noodle with beef ball beef and stewed soup
5: rice noodle fine thread with beef and beef stewed soup
6: big size rice noodle with beef ball beef and stewed soup
7: big size rice noodle with pork and soup 8: wheat noodle with roasted pork
9: wheat noodle with meat ball and ball fried
10: fried rice noodle with dark soy sauce topped with shirmp
11: rice noodle fine thread with stewed beef 12: guay-jub northeasterm style

From the rice noodle (water) foods, big sized rice noodle with pork and soup and wheat noodle with fried meat ball have the best types. The worst has fried rice noodle with dark soy sauce topped with shirmp.

rice noodle (dried)

Type	1	2	3	4	5	6	7	8	Rank
1	0	0	0	1	1	1	1	1	1
2	0	0	0	1	1	1	1	1	1
3	0	0	0	1	1	1	1	1	1
4	0	0	0	0	0	0	0	0	4
5	0	0	0	0	0	0	0	0	4
6	0	0	0	1	1	0	1	0	2
7	0	0	0	0	0	0	0	0	4
8	0	0	0	1	0	0	0	0	3

Note: 1: wide rice noodle with pork egg and soy sauce 2: pad thai
3: macaroni fried with pork 4: small size rice noodle fried with stewed beef ball
5: fried fine thread rice noodle with stewed beef ball 6: saute mungbean noodle and hen egg
7: small size rice noodle with pork 8: rice noodle fried with pork and spicy

For the rice noodle (dried) foods wide rice noodle with pork egg and soy sauce, pad thai and macaroni fried with pork have the best types and the worst types have small sized rice noodle fried with stewed beef ball, fried fine thread rice noodle with stewed beef ball and small sized rice noodle with pork.

fermented rice noodle

Type	1	2	3	4	5	6	7	8	9	Rank
1	0	0	0	0	0	0	0	0	0	5
2	0	0	0	0	0	0	0	0	0	5
3	0	0	0	1	1	0	0	0	0	4
4	0	1	0	0	1	0	0	0	0	4
5	0	0	0	0	0	0	0	0	0	5
6	1	1	1	1	1	0	1	1	0	1
7	0	0	1	1	1	0	0	0	0	3
8	0	0	0	0	0	0	0	0	0	5
9	1	1	1	0	1	0	1	1	0	2

Note: 1: rice vermicelli with tofu fried in coconut cream 2: fermented rice noodle with minced pork
3: fermented rice noodle with fish curry 4: fermented rice noodle with fish soup
5: fermented rice noodle with chicken feet 6: fermented rice noodle with southern curry
7: fermented rice noodle with chicken curry 8: fermented rice noodle with fish organs soup
9: fermented rice noodle with green curry chicken

As can be seen, the best type of fermented rice noodle food has fermented rice noodle with southern curry. The worst have rice vermicelli with tofu fried in coconut cream, fermented rice noodle with minced pork, fermented rice noodle with chicken feet and fermented rice noodle with fish organs soup.

1.4.2.3 Fruits

Fruits are classified into seven groups, namely foreign fruit, numerative noun fruit, bunch fruit, apple, rose apple, orange and mango, respectively. The results of the **E**-matrices in each group is shown in Table 1.4.41.

Table 1.4.41 E-matrices of the five group of fruits

foreign fruits

Type	1	2	3	4	5	Rank
1	0	0	0	0	0	3
2	1	0	0	0	0	2
3	1	0	0	0	0	2
4	1	0	0	0	1	1
5	0	0	0	0	0	3

Note: 1: pear 2: cantaloupe 3: hard-type ripe persimmon 4: kiwi 5: cherries eaten raw

From the foreign fruits, kiwi has the best type while pear and cherries eaten raw have the worst types.

numerative noun for fruits

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	Rank
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	7
2	1	0	0	0	0	0	1	1	0	0	0	1	1	1	1	1	0	0	0	0	1	0	1	1	3
3	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	9
4	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	7
5	0	0	1	0	0	0	1	0	0	0	0	1	1	0	1	1	0	0	0	0	1	0	0	1	5
6	1	0	1	0	0	0	1	1	0	0	0	1	0	1	1	1	0	0	0	0	1	0	1	1	3
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	10
9	1	0	1	0	0	0	1	1	0	0	0	1	0	1	1	1	0	1	0	0	1	0	1	0	3
10	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1
11	0	0	1	0	0	1	1	1	0	0	0	1	0	1	1	1	0	0	0	0	1	0	0	1	4
12	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	9
13	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	1	0	1	0	0	1	0	1	1	5
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	10
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
17	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	1	0	1	1	6
18	1	0	1	0	0	0	1	1	0	0	0	0	0	1	1	1	0	0	0	0	1	0	1	1	4
19	1	0	1	1	0	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	2
20	1	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	2

numerative noun for fruits (cont.)

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	Rank
21	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
22	0	0	1	0	0	0	1	1	0	0	0	1	0	1	1	1	0	0	0	0	1	0	1	1	4
23	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
24	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	8

Note: 1: dragon fruit 2: papaya ripe 3: pomelo thong dee 4: jackfruit mature 5: watermelon red 6: guava common 7: native mature melon 8: pineapple 9: Sapodilla common 10: tamarind sweet 11: gandaria marian plum 12: yumbean 13: toddy palm 14: rambutan 15: mangosteen 16: litchi 17: banana khai variety 18: pomegranate solft seed 19: durian mon-thong 20: jujube thai variety 21: sala 22: sugar apple 23: salak palm fruit 24: carambola

As can be seen, the best type of numerative noun fruits is sweet tamarind and then mon-thong durian and Jujube thai variety have the second best types. The worst have native mature melon, mangosteen and litchi.

bunch fruit

Type	1	2	3	4	5	6	Rank
1	0	0	0	0	0	0	3
2	1	0	1	1	1	0	1
3	1	0	0	1	1	0	2
4	0	0	0	0	0	0	3
5	0	0	0	0	0	0	3
6	1	0	0	1	1	0	2

Note: 1: grapes green 2: rambai 3: star gooseberry 4: longan 5: longkong 6: langsad

As can be seen, the best type of bunch fruits has rambai. The worst have longan, longkong and grapes green.

apple

Type	1	2	3	Rank
1	0	1	1	1
2	0	0	1	2
3	0	0	0	3

Note: 1: apple 2: green apple 3: fuji apple

For the apple fruits, apple has the best type. The worst type has fuji apple.

rose apple

Type	1	2	3	4	Rank
1	0	0	0	1	2
2	1	0	1	1	1
3	0	0	0	0	3
4	0	0	0	0	3

Note: 1: rose apple red color variety 2: green rose apple 3: rose apple
4: rose apple green color variety

As can be seen, green rose apple has the best type in the group. The worst have rose apple and rose apple green color variety.

orange

Type	1	2	3	Rank
1	0	0	0	2
2	0	0	0	2
3	1	0	0	1

Note: 1: kalanchoe 2: mandarin 3: sweet orange

For the data of the orange group in the fruit, sweet orange has the best type and then kalanchoe and mandarin tave the worst.

mango

Type	1	2	3	4	5	6	7	Rank
1	0	1	1	0	0	0	1	2
2	0	0	1	0	0	0	0	3
3	0	0	0	0	0	0	0	4
4	0	0	0	0	0	0	0	4
5	1	1	1	0	0	1	1	1
6	0	0	0	0	0	0	0	4
7	0	0	1	0	0	0	0	3

Note: 1: unripe mango kaew variety 2: unripe mango pimsaen variety 3: unripe mango rad variety
4: ripe mango kaew variety 5: ripe mango nam-dok-mai variety
6: ripe mango pimsaen-mun variety 7: unripe mango kiew-sa-weya variety

From the fruit of the mango group, ripe mango nam-dok-mai variety has the best type while unripe mango rad variety, ripe mango kaew variety and ripe mango pimsaen-mun variety have the worst types.

1.4.2.4 Desserts

Desserts are classified into five groups, namely coconut milk, steamed, fried, roast and conserved, respectively. The results of the **E**-matrices in each group are shown in Table 1.4.42.

Table 1.4.42 E-matrices of the five group of desserts

coconut milk

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Rank
1	0	0	0	1	0	0	1	0	1	1	0	1	0	0	1	1	0	0	0	5
2	0	0	0	1	1	0	0	1	1	1	0	0	0	0	1	0	0	0	0	6
3	1	1	0	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1
4	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	8
5	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	10
6	1	1	0	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1
7	0	0	0	1	1	0	0	0	1	1	0	1	0	0	1	0	0	0	0	6
8	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	9
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
10	0	0	0	1	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	8
11	1	1	0	1	1	0	1	1	1	1	0	1	0	1	1	1	0	1	1	2
12	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	9
13	1	1	0	1	1	0	1	1	1	1	0	1	0	1	1	1	0	1	1	2
14	0	0	0	1	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0	6
15	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	10
16	0	0	0	1	1	0	0	0	1	1	0	0	0	0	1	0	0	0	0	7

coconut milk (cont.)

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Rank
17	1	1	0	1	1	0	1	1	1	1	0	1	0	1	1	1	0	1	1	2
18	1	0	0	1	1	0	1	1	1	1	0	1	0	1	1	1	0	0	1	3
19	0	1	0	1	1	0	1	1	1	1	0	1	0	0	1	1	0	0	0	4

Note: 1: banana in coconut milk 2: tab tim krob with coconut milk and syrup 3: sticky rice black with coconut milk and syrup 4: taro balls in coconut milk 5: taro in coconut milk 6: tao suan 7: rice drops in sweet coconut milk 8: sago with coconut milk 9: nata de coco in syrup 10: mungbean thread in coconut milk 11: boiled cowpea seeds with sugar and coconut milk 12: boiled rice flour chunk with coconut milk 13: rice green flake with coconut milk and syrup 14: native melon in coconut milk 15: jelly with coconut cream pandanus leaves flavor 16: sticky rice in coconut milk 17: kanom tue pap 18: kanom tom khaw 19: kanom tom daeng

As can be seen, sticky rice black with coconut milk and syrup and tao suan have the best types in the group. The worst has nata de coco in syrup.

steamed

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	Rank
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15
2	1	0	0	1	1	0	1	0	0	1	1	0	1	1	1	1	1	0	0	1	1	3
3	1	0	0	1	1	0	1	0	0	1	1	0	1	1	1	1	1	0	0	0	1	4
4	1	0	0	0	1	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	6
5	1	0	0	0	0	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	7
6	1	0	0	1	1	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	5
7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	13
8	1	0	0	0	1	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	6
9	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
10	1	0	0	0	0	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	7
11	1	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	0	0	0	0	1	9
12	1	0	0	1	1	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	5
13	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	12
14	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	12
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14
16	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	12
17	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	1	11
18	1	0	0	0	0	0	1	0	0	0	1	0	1	1	1	1	1	0	0	0	1	7
19	1	1	0	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	0	1	1	2
20	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0	0	1	10
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15

Note: 1: banana paste 2: taro paste 3: sticky rice flour coconut milk and sugar boiled 4: mock fruits
5: egg custard baked 6: mock jackfruit seed 7: coconut pudding with water chestnut
8: egg yolk dropped in heavy syrup 9: gold threads egg yolk strained in heavy syrup
10: kanom piak poon 11: kanom touy 12: kanom sali 13: kanom chun 14: kanom num dok mai
15: tong muan 16: sam pan nee 17: glutinous rice steeped in coconut milk
18: sticky rice with shrimp and coconut meat 19: sticky rice with custard
20: steamed mixture of pumpkin rice flour and coconut 21: steamed toddy palm cake

As can be seen, gold thread egg yolk strained in heavy syrup has the best type in the group. The worst have banana paste and steamed toddy palm cake.

fried

Type	1	2	3	4	5	6	7	8	9	10	11	Rank
1	0	0	0	0	0	0	1	1	0	1	0	5
2	1	0	0	0	0	1	1	1	0	1	0	3
3	1	0	0	0	0	1	1	1	0	1	0	3
4	1	0	1	0	0	1	1	1	0	1	1	2
5	1	0	1	1	0	1	1	1	1	1	1	1
6	0	0	0	0	0	0	1	1	0	1	0	5
7	0	0	0	0	0	0	0	0	0	1	0	6
8	0	0	0	0	0	0	0	0	0	1	0	6
9	0	0	0	0	0	1	1	1	0	1	0	4
10	0	0	0	0	0	0	0	0	0	0	0	7
11	1	0	0	0	0	1	1	1	0	1	0	3

Note: 1: deep fried banana 2: kanom khai nok krata 3: doughnut with sugar
4: glutinous rice cooked deep fried 5: krong krang krob kem 6: kanom nang led
7: taro slided deep fried sugar coated 8: banana slided deep fried sugar coated
9: deep fried kanom khai hong 10: rice cake 11: large soft rice pancake topped with shredded

From the fried desserts, krong krang krob kem has the best type while rice cake has the worst type.

roast

Type	1	2	3	4	5	6	7	8	Rank
1	0	0	0	0	0	0	0	0	4
2	1	0	0	1	1	1	1	0	2
3	1	0	0	1	1	1	1	0	2
4	0	0	0	0	0	0	1	0	3
5	0	0	0	0	0	0	0	0	4
6	0	0	0	0	0	0	1	0	3
7	0	0	0	0	0	0	0	0	4
8	1	0	1	1	1	1	1	0	1

Note: 1: kanom ping 2: kanom krok 3: tokyo snack similar to pancake 4: kanom babin
5: cake prepared sponge commercially 6: kanom kleeb lamduan
7: coconut meat and rice flour in coconut milk 8: kanom naa nual

For the roast desserts, kanom naa nual has the best type and then cake prepared sponge commercially, coconut meat and rice flour in coconut milk and kanom ping have the worst.

conserved

Type	1	2	3	4	5	Rank
1	0	0	1	0	1	1
2	0	0	1	0	1	1
3	0	0	0	0	0	3
4	0	0	1	0	1	1
5	0	0	1	0	0	2

Note: 1: pumpkin in syrup 2: cassava boiled in syrup 3: toddy palm in heavy syrup
4: artocarpus altilis in concentrated syrup 5: mango pickled sweet

As can be seen, pumpkin in syrup, cassava boiled in syrup and artocarpus altilis in concentrated syrup have the best types in the group of conserved dessert. The worst has toddy palm in heavy syrup.

1.5 Conclusion

The MCDM method provides us with a natural technique to integrate the various columns of a data matrix so that each row is endowed with a single overall index, summarizing the different component indices, thus making a ranking of the rows and hence their comparison feasible. For nutrient data, we can rank types of vegetable, food, fruit and dessert with respect to all such indicators to see the best and the worst types in terms of nutrient.

The Electre method is used for comparing the status of two locations rather than ranking all of them together. Pairwise comparison is used where for each pair one is better than the other. However the row with the largest number of zero elements will be the best one and then with the smallest will be the worst one.

By comparing the results of both methods, we observe that the Multiple Criteria Decision Making (MCDM) method is better than the Electre method because we compute the mean and standard deviation of the overall index based on the L_1-norm and the L_2-norm with two weights and the averages of the overall index are ranked from the large value to the small value while the Electre method provides the E matrix with element having only value equal to 0 or 1.

The data for vegetables, foods, fruits and desserts using the Multiple Criteria Decision Making (MCDM) and Electre methods can provide a guidelines for consumers to choose the best value nutrients and lower prices of vegetables, foods, fruits and desserts. Data should be considered for each amount of nutrient in each of the four groups, based on eleven nutrient indicators, namely energy, carbohydrate, fat, protein, calcium, iron, vitamin A, thiamin, riboflavin, vitamin C and niacin.

References

1. Ageev I., Trenev V. and Leontiev S. Multiple criteria decision making in the adaptive mechanism, The 17th International Conference on Multiple Criteria Decision Making, Whistler, British Columbia, Canada, August 6-11, 2004.
2. Alvisi F., Malagoli C. and Regazzi D. Multifru: Multiple-criteria decision making in orchard management. Third International Symposium on Computer Modeling in Fruit Research and Orchard Management, Palmerston North, New Zealand; 1992.

3. Bakker E. and Sanden P.A. Application of multiple criteria decision making for the design and analysis of greenhouse cropping systems. Second International Smposium on Models for plant growth, Environmental control and farm management in protected cultivation, Wageningen, Netherlands, August 25-28, 1997.
4. Buchanan J., Sheppard P. and Vanderpooten D.(1999). Project ranking using ELECTRE III. Research Report 99-01, Department of Management System, University of Waikato, New Zealand.
5. Buchanan J. and Kock N.(2001). Information overload: A decision making perspective. Springer Verlag, Berlin, 49-59.
6. Carlsson C. and Fuller R. (1995). Multiple criteria decision making: The case for interdependence. Computers and Operations Research, 22: 251-260.
7. Corner J., Buchanan J. and Henig M.(2001). A dynamic model for structuring decision problems. Journal of Multi-Criteria Decision Analysis, 10: 129-141.
8. Dong J., Zhang D. and Nagurney A.(2002). Supply chain networks with multicriteria decision makers. Transportation and Traffic Theory in the 21st Century, Amsterdam, Netherlands, 170-196.
9. Filar J.A., Ross N.P. and Wu M.L. (1999). Environmental assessment based on multiple indicators. Technical Report, Department of Applied Mathematics, University of South Australia.
10. Fuller R. and Carlsson C. (1996). Fuzzy multiple criteria decision making: Recent developments. Fuzzy sets and systems, 78: 139-153.
11. Hanne T. (2000). Intelligent strategies for meta multiple criteria decision making. International Series in Operations Research and Management Science 33, Hardbound.
12. Hwang C.L. and Yoon K. (1981). Multiple attribute decision making: Methods and applications, A State-of-the-Art-Survey, Lecture Note in Economics and Mathematical System, Springer-Verlag,Berlin.
13. Jung E.S. and Freivalds A. (1991). Multiple criteria decision-making for the resolution of conflicting ergonomic knowledge in manual materials handling. Ergonomics, 34(11): 1351-6.
14. Karamouz M. (2002). Monitoring and evaluation scheme using the multiple-criteria decision-making technique: Application to irrigation projects. Journal of Irrigation and Drainage Engineering, 128: 341-350.

15. Kaliszewski I. and Michalowski W. (1997). Psychological stability of solutions in the multiple criteria decision problems. Interim Report to IIASA.
16. Lertprapai, S. ;Tiensuwan, M. ; and Sinha, B. (2004). A Statistical Approach to Combining Environmental Indices with an Application to Air Pollution Data from Bangkok, Thailand. PJS, Vol. 20(2), pp. 245-261.
17. Lertprapai, S. ;Tiensuwan, M. ; and Sinha, B. (2004). On a comparison of three estimators of binomial variance by multiple criteria decision making method. IJSS, Vol.3, pp.105-117.
18. Lertprapai, S.; Tiensuwan, M.; and Sinha, B.(2004). On a comparison of two standard estimates of a binomial proportion based on multiple criteria decision making method. JSTA, Vol.3, No.2, pp.141-149.
19. Lertprapai S, Tiensuwan M. (2009). An application of multiple criteria decision making in combining environmental indices of five air pollution indicators. Thailand Statistician. 7(2), pp.131-145.
20. Lertprapai, S., Tiensuwan, M. (2010). An application of multiple criteria decision making to water quality data in Thailand. Far East Journal of Applied Mathematics. 40(2), pp.103-124.
21. Maitra R., Ross N.P. and Sinha B.K.(2002). On some aspects of data integration techniques with applications. Technical Report, Department of Mathematics and Statistics, University of Maryland Baltimore County.
22. Ortega J.F. (2002). Multi-criteria decision making for low income & labor market: A literature review, UTC-UIC Working Paper.
23. Shannon C.E. and Weaver W. (1947). The Mathematical theory of communication. The University of Illinois Press, Urbaba, Illinois
24. Sinha Bikas K and Shah KR. (2003). On some aspects of data integration techniques with environmental applications. Environmetrics, 14: 409-416.
25. Sinha Bikas K and Sinha Bimal K. (2002). On some aspects of MCDM with environmental applications.
26. Steuer R. E. and Na P. (2003). Multiple criteria decision making combined with finance: A categorized bibliographic study. European Journal of Operational Research, 150(3): 496-515.
27. Tarp P. and Helles F. (1995). Multi-criteria decision-making in forest management planning: An overview. Journal of Forest Economics, 1(3): 273-306.
28. Tiensuwan, M.; Lertprapai, S. and Sinha, B. (2005). On a comparison of four estimates of a common mean by multiple criteria decision making method. JSR, Vol.39, No.2, pp.23-35.

29. Tiensuwan M.; Lertprapai S. and Sinha B.(2006). On a comparison of several competing estimates of a univariate normal mean by multiple criteria decision making method. Communications in Statistics: Simulation and Computation, Vol. 35, No.4, pp.877-891.
30. Yoon K. and Hwang C.L. (1995). Multiple attribute decision making: An introduction. Sage Publication, California.
31. Zeleny M. (1982). Multiple criteria decision making. McGraw-Hill, New York.

7.2 Rainfall and Water Level Forecasting

Rivers and canals are important and supportive for living. They are used for consumption, agriculture and communication. When many canals flow to join together then this is called "the River". There are many main rivers in Thailand such as Ping, Wang, Yom, Nan, Chao Phraya etc. However one of the most important rivers is the Chao-Phraya river because it is the largest river in Thailand. The Chao-Phraya begins at the confluence of the Ping and Nan rivers. The two rivers converge at Paknampo district in Nakhonsawan province, which is the north of Thailand. The Chao-Phraya flows through Chainat, Singburi, Angthong, Ayutthaya, Pathumthani, Nonthaburi, Bangkok and ends in Samut Prakarn. It is 372 kilometers long from north to south.

A water level of the Chao Phraya river depends on many factors such as river flow, rainfall, tide, sea level etc. One of the important factors is rainfall which relates to water-related problems such as flood and drought. Rainfall information is an important water resource for food production plan, agricultural purposes, helping ensure the safety of all people and all activity plans. So an accurate prediction of monthly rainfall and water level can help to solve these problems.

A time series is a collection of observations recorded at intervals of time. In real world data, there are many data which are in a time series such as daily stock prices, weekly sales figures, monthly bookings for an airline or the quarterly gross national product. Time series forecasting is the use of a model to forecast future data based on known past data. There are several methods used to forecast the model of time series data such as Box-Jenkins, Exponential smoothing (single, double, triple), Batch Least Squares, Artificial Neural Networks etc.

In this subtopic, we examine two time series namely, rainfall and water level by using Box-Jenkins model for forecasting, since the Box-Jenkins method is the most suitable method for forecasting rainfall data (Panichkitkosolkul, 2009). At each station, rainfall and water level is represented along the Chao Phraya river. These stations are forecasted by using the Box-Jenkins ARIMA model in order to obtain the best model for forecasting, for each station, the rainfall and water level.

This study focuses on the data of monthly rainfall amounts and water level along the Chao-Phraya river of Thailand in 1976-2006. The data of monthly rainfall is selected from nine representative stations while the data of water level is selected from four representative stations. In this study, we wish to obtain the best model of rainfall and water level series for each station located in the province along the Chao Phraya river by using Akaike's Information Criterion (AIC) and Schwarz's Bayesian Criterion (SBC). All the predicted stations are shown in Table 2.1 and Figure 2.1, respectively.

Table 2.1 Station location and period of rainfall and water level

Data	Station	Location	Period
Rainfall	1	Nakhonsawan	1976-2003
	2	Chainat	1976-2006
	3	Sing Buri	1976-2002
	4	Ang Thong	1976-2006
	5	Ayutthaya	1976-2006
	6	Pathum Thani	1976-2002
	7	Nonthaburi	1976-2006
	8	Bangkok	1976-2006
	9	Samutprakarn	1976-2006
Water level	1	Nakhonsawan	1976-2003
	2	Chainat	1976-2006
	3	Sing Buri	1976-2002
	4	Ang Thong	1976-2006

For water level series, we use only four locations because we can measure the water level at only four locations which are not related with the sea level.

In time series analysis, there are several techniques to forecast time series data such as artificial neural networks, batch least squares, exponential smoothing (single, double, triple), Box-Jenkins etc. Many authors have used these methods for rainfall and water level forecasting. A review of univariate time series models research is presented as follows.

In 1998, Lee et al. used artificial neural networks to predict rainfall in Switzerland. They divided the region into 4 sub-areas that is two larger areas and two smaller areas. The two larger areas are predicted by radial basis function (RBF) networks while the two smaller areas used a simple linear model. The predicted results indicated that the RBF network was better than linear models.

In 2000, Jung et al. studied the prediction of hourly rainfall surface by using multi-layer neural networks with hidden layers. The criterions which are used to consider accuracy are the normalized root mean square error (NRMSE) and the cross correlation coefficients. The results showed that the neural network was the best for one and two hour predictions.

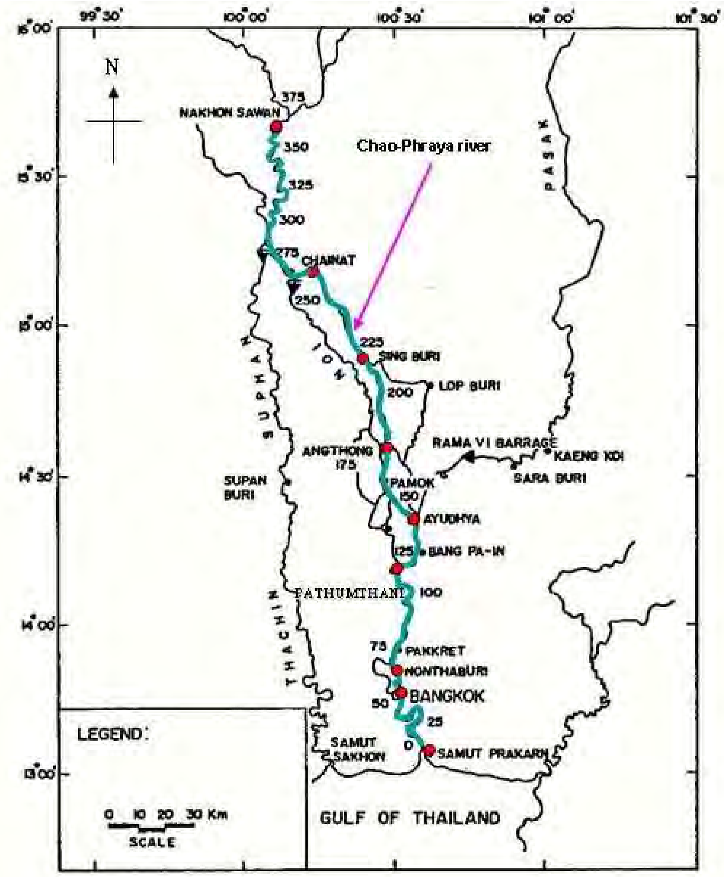


Figure 2.1 Location of rainfall and water level along the Chao Phraya river(Onodera, 1986)

In 2000, Sosa et al. used ARIMA models to predict rain attenuation in a Mexican tropical area namely, the Villahermosa region. By using the ARIMA model and MINITAB software, they had 25 different models which were 16 non-seasonal and 9 seasonal models. After taking a square root transformation in the original series, it was shown that the modified data should be from the seasonal model. The best model is selected by choosing the smallest residual average and residual variance. The ARIMA (0, 0, 0) × (0, 1, 1) was the best model to forecast rain attenuation in the Mexican tropical area.

In 2000, Tingsachali used a neural network model with a back-propagation algorithm (BP) to forecast Chao-Phraya river flood level in Bangkok. The networks are divided into two, the first had three layer networks with five input nodes, five hidden nodes and one output node. The second had ten input nodes, five hidden nodes and one output node. It was shown that the second network gave better accuracy than the first network. This study suggested that the neural networks model were appropriate for forecasting hourly water level and discharge for more time steps ahead in the Chao Phraya river.

In 2003, Sen presented new forecast models for Indian south-west monsoon season rainfall. The forecast during 1988-2002 used a 16- parameter power regression statistical model. It found that the forecast deviated in 2002, so the model was adjusted. The parameters were revised from 16 to 8 and 10 parameters. A new 8-parameter probabilistic model was used for giving a qualitative forecast while a new 10-parameter power regression model was used for the purpose of a long-range forecast update to be issued in mid-July. The comparative chart showed that 8-parameter and 10-parameter models were better than 16-parameter models.

In 2004, Mebrhatu et al. developed a simple statistical model to forecast the peak rainfall months of July-August in the highlands of Eritrea. They used rainfall amounts from the previous November-December and five values of South Indian ocean sea surface temperature from the previous year which were from January-February, March-April, May-June, September-October and November-December. The Jack-knife method of cross-validation and hit rate were used for consideration of the accuracy of the model. The results showed that this model can be describe and validate the pattern of the rainfall for the highlands of Eritrea.

In 2004, Boochabun et al. presented new techniques to study rainfall and river flow. The rainfall and discharge time series across Thailand was analyzed by using Dynamic Harmonic Regression (DHR) models. This model had a little error of forecast; it was a fitted model for forecasting the rainfall and discharge data.

In 2005, Weesakul et al. developed a mathematical model for forecasting annual rainfall in Thailand. The method in the study was Box-Jenkins techniques. The results showed that the ARIMA model was appropriate for most rainfall stations while the ARMA model was fitted for

8 stations from the total 31 rainfall stations. A mean relative accuracy for planning was proposed.

In 2008, Zaw et al. presented an empirical statistical modeling of rainfall prediction over Myanmar. They compared two methods between multi-variable polynomial regression (MPR) and multiple linear regression (MLR). Experiments showed that the model forecasting with MPR was closer to the actual values than MLR.

This review of the literature shows that there is no model developed to forecast one year ahead for rainfall and water level stations along the Chao Phraya river in Thailand. In this study, we focus on forecasting the amount of monthly rainfall and water level for locations along the Chao Phraya river by using Box-Jenkins techniques. The SAS® software version 9.1 is used to analyze the seasonal ARIMA models with the volume of monthly rainfall and water level.

2.1 Time Series Procedure

A time series is a set of observations obtained by measuring a single variable over a time period. Analyzing time series is to try to discover systematic patterns of the series as a mathematical model which can be explained by past data. The Box-Jenkins technique is one of the most popular strategies for building a model that we use to analyze our time series data. Figure below summarizes the iterative approach to model building for forecasting which is employed in this topic.

In model building of Box-Jenkins, there are three major stages to consider.

Stage 1 Identification

A tentative model is selected by considering the historical data, autocorrelation and partial autocorrelation function of the data to find out trends or seasonal patterns.

Stage 2 Estimation

The tentative model is fitted by the data to estimate its parameters and testing them for significance. If the estimated parameters are not satisfactory on statistical grounds, we have to return to the identification stage.

Stage 3 Diagnosis

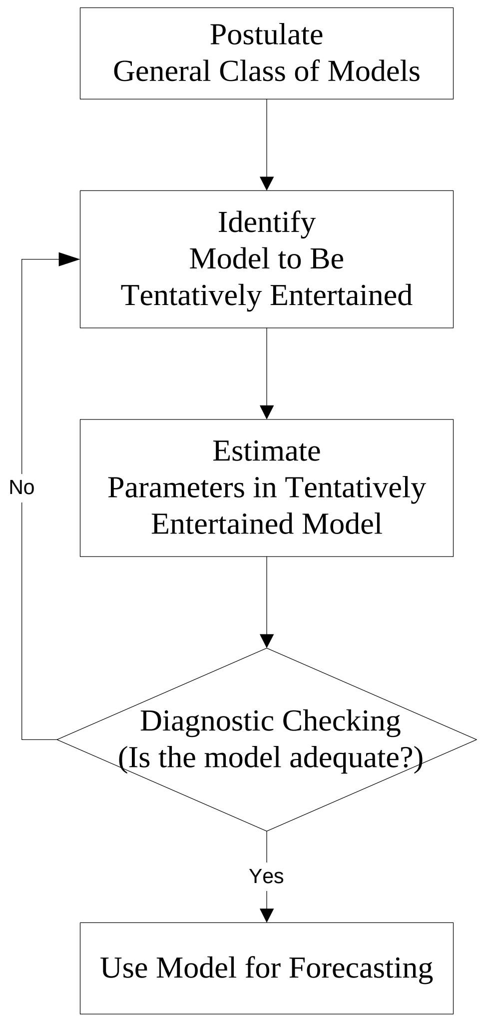


Figure 2.2 Flow Diagram of Box-Jenkins Method.

Source : Box GEP, Jenkins GM, Reinsel GC. Time series analysis forecasting and control. 3rd ed. New Jersey: Prentice-Hall; 1994. p17

The tentative model is examined as to how well it fits the data. Various diagnostics are used to check with which confidence the model can be used or whether we should return to the first stage. In this stage, we have to assess model adequacy by checking whether the model assumptions are satisfied. The basic assumption is that the residuals a_t are white noise. To check whether the residuals are white noise, we use the Ljung-Box test (1978). Figure 2.2 summarizes the iterative approach to model building for forecasting.

Box-Jenkins models or ARIMA models stand for Autoregressive Integrated Moving Average model which combines three types of processes: Autoregression (AR); differencing to remove the integration (I) of the series; and moving average (MA). The most general ARIMA model is traditionally written as ARIMA(p,d,q) while neglecting seasonality, where p is the order of autoregression, d is the degree of differencing, and q is the order of moving average involved. We can examine the pth and qth order by a sample autocorrelation function (ACF) and partial autocorrelation function (PACF) of any time series.

Many business, economics and agriculture time series contain a seasonal phenomenon that repeats itself after a regular period of time. The smallest time period for this repetitive phenomenon is called the seasonal period. Seasonal phenomenon is mostly caused by the weather. For the model with seasonality, we have to include the seasonal period relationship in the general ARIMA model. The seasonal ARIMA models are denoted as $ARIMA(p,d,q)\times(P,D,Q)_s$, where P is the order of seasonal autoregression, D is the degree of seasonal differencing, Q is the order of seasonal moving average and the sub-index s refers to the seasonal period. We can also examine the Pth and Qth order by the sample autocorrelation function (ACF) and partial autocorrelation function (PACF) patterns at the seasonal lags or the multiples of the seasonal period.

We review theoretical background for Box-Jenkins models as follows.

2.1.1 Sample Autocorrelation Function

In a time series data set with n observations, we define Z_t to be values of each datum for t = 1, 2, ..., n. For a stationary process $\{Z_t\}$, we have the mean $E(Z_t) = \mu$ and variance $Var(Z_t) = E(Z_t-\mu)^2 = \sigma^2$, which are constant, and the covariance $Cov(Z_t,Z_s)$, which are functions only of the time difference |t-s|. Hence, in this case, we write the covariance between Z_t and Z_{t+k} as

$$\gamma_k = Cov(Z_t, Z_{t+k}) = E(Z_t - \mu)(Z_{t+k} - \mu)$$

and the correlation between Z_t and Z_{t+k} as

$$\rho_k = \frac{Cov(Z_t, Z_{t+k})}{\sqrt{Var(Z_t)}\sqrt{Var(Z_{t+k})}} = \frac{\gamma_k}{\gamma_0}$$

where we note that Var(Z_t) = Var(Z_{t+k}) = γ_0. As functions of k, γ_k is called the autocovariance function and ρ_k is called the autocorrelation function (ACF) in time series analysis since they represented the covariance and correlation between Z_t and Z_{t+k} from the same process, separated only by k time lags.

In practice, ρ_i (i = 1, 2, …, k) are unknown and are replaced by their sample estimates $\hat{\rho}_i$. For a given observed time series $Z_1, Z_2, \ldots, Z_n$, the sample ACF is defined as

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0} = \frac{\sum_{t=1}^{n-k}(Z_t - \overline{Z})(Z_{t+k} - \overline{Z})}{\sum_{t=1}^{n}(Z_t - \overline{Z})^2}, \qquad k = 0, 1, 2, \ldots$$

where $\overline{Z} = \sum_{t=1}^{n} Z_t / n$ is the sample mean of the series. For processes in which $\rho_k = 0$ for k > m the standard error of $\hat{\rho}_k$ by Bartlett's approximation (Bartlett, 1946) is

$$S_{\hat{\rho}_k} = \sqrt{\frac{1}{n}(1 + 2\hat{\rho}_1 + \cdots + 2\hat{\rho}_m)}\ .$$

2.1.2 Sample Partial Autocorrelation Function

The conditional correlation Corr(Z_t, Z_{t+k} | $Z_{t+1}, \ldots, Z_{t+k-1}$) is usually referred to as the partial autocorrelation in time series analysis. Wei (1990) has shown the value of PACF by considering the regression model, where the dependent variable Z_{t+k} from a zero mean stationary is regressed on k lagged variables Z_{t+k-1}, Z_{t+k-2}, …, Z_t, i.e.,

$$Z_{t+k} = \phi_{k1} Z_{t+k-1} + \phi_{k2} Z_{t+k-2} + \cdots + \phi_{kk} Z_t + e_{t+k}$$

where ϕ_{ki} denotes the *i*th regression parameter and e_{t+k} is a normal error term uncorrelated with Z_{t+k-j} for j ≥ 1. Multiplying Z_{t+k-j} on both sides of the above regression equation and taking the expectation, we get

$$\gamma_j = \phi_{k1}\gamma_{j-1} + \phi_{k2}\gamma_{j-2} + \cdots + \phi_{kk}\gamma_{j-k}$$

and hence, $$\rho_j = \phi_{k1}\rho_{j-1} + \phi_{k2}\rho_{j-2} + \cdots + \phi_{kk}\rho_{j-k}$$

For j = 1, 2, …,k, we have the following system of equations :

$$\begin{aligned}
\rho_1 &= \phi_{k1}\rho_0 + \phi_{k2}\rho_1 + \cdots + \phi_{kk}\rho_{k-1} \\
\rho_2 &= \phi_{k1}\rho_1 + \phi_{k2}\rho_0 + \cdots + \phi_{kk}\rho_{k-2} \\
&\vdots \\
\rho_k &= \phi_{k1}\rho_{k-1} + \phi_{k2}\rho_{k-2} + \cdots + \phi_{kk}\rho_0.
\end{aligned}$$

Using Cramer's rule successively for k = 1, 2, …. , we have

$$\phi_{11} = \rho_1$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}}$$

$$\phi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}}$$

$$\vdots$$

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}.$$

ϕ_{kk} is usually referred to as the partial autocorrelation function (PACF). Durbin(1960) computes the sample PACF $\hat{\phi}_{kk}$ by a recursive method starting with $\hat{\phi}_{11} = \hat{\rho}_1$ as follows.

$$\hat{\phi}_{k+1,k+1} = \frac{\hat{\rho}_{k+1} - \sum_{j=1}^{k} \hat{\phi}_{kj}\hat{\rho}_{k+1-j}}{1 - \sum_{j=1}^{k} \hat{\phi}_{kj}\hat{\rho}_j}$$

and

$$\hat{\phi}_{k+1,j} = \hat{\phi}_{kj} - \hat{\phi}_{k+1,k+1}\hat{\phi}_{k,k+1-j}, \qquad j = 1, 2, \ldots, k.$$

The standard error of $\hat{\phi}_{kk}$ can be approximated by

$$S_{\hat{\phi}_{kk}} \approx \frac{1}{\sqrt{n}}.$$

2.1.3 Autoregressive Processes

In an autoregressive process, each value in a series is a linear function of the preceding value or values. In a first-order autoregressive process only the single preceding value is used; in a second-order process the two preceding values are used; and so on. These processes are commonly indicated by the notation AR(p), where the number in parentheses indicates the order. AR(p) expresses the current time series Z_t as a function of the past time series value Z_{t-1}, Z_{t-2}, …,Z_{t-p} as follows :

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} + a_t ,$$

where $\dot{Z}_t = Z_t - \mu$

ϕ_i is the autoregressive parameter; i = 1, 2, …, p, and

a_t is a zero mean Gaussian white noise process.

We can express the form of the backshift operators

$$\phi_p(B)Z_t = a_t,$$

where $\phi_p(B) = (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)$

and the backshift operator $B^i Z_t = Z_{t-i}$.

The coefficient ϕ is estimated from the observed series and indicates how strongly each value depends on the preceding value. Since $\sum_{j=1}^{p} |\phi_j| < \infty$, the process is always invertible. To be stationary the roots of $\phi_p(B) = 0$ must lie outside of the unit circle.

For a stationary process, the ACF tails off as exponential decay or damped sine wave. The PACF of AR(p) process cuts off after lag p.

2.1.4 Moving Average Processes

In a moving average, each value is determined by the average of the current disturbance or error and one or more previous disturbances. The order of the moving average process specifies how many previous disturbances are averaged into the new value. In the standard notation, a MA(q) or ARIMA(0,0,q) process uses q previous disturbances along with the current one. In another way, a MA(q) is represented by the current time series Z_t and is a function of the current and the past random shocks or errors a_t, a_{t-1}, a_{t-2}, …, a_{t-q} as follows.

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

where $\dot{Z}_t = Z_t - \mu$,

θ_j is the autoregressive parameter ; j = 1, 2, …, p,

a_t is a zero mean Gaussian white noise process.

We can also express the form of the backshift operators

$$\dot{Z}_t = \theta_q(B)a_t,$$

where $\theta_q(B) = (1-\theta_1 B - \cdots - \theta_q B^q)$ and $B^j a_t = a_{t-j}$.

A finite moving average process is always stationary. The moving average process is invertible if the roots of $\theta_q(B) = 0$ lie outside the unit circle.

For an invertible process, the ACF of MA(q) cuts off after lag q. The PACF tails off as exponential decay or damped sine wave.

2.1.5 Autoregressive Integrated Moving Average (ARIMA) Models

The General ARIMA Models

Most built models may be necessary to include both autoregressive and moving average terms in a model which are more parsimonious than pure autoregressive or pure moving average terms. This leads to the following useful mixed autoregressive moving average (ARIMA) process

$$\phi_p(B)\dot{Z}_t = \theta_q(B)a_t.$$

The stationary process which results from a properly differenced homogeneous nonstationary series has the differenced series $(1-B)^d Z_t$. In general, we can write the autoregressive integrated moving average (ARIMA) model as follows.

$$\phi_p(B)(1-B)^d Z_t = \theta_0 + \theta_q(B)a_t,$$

where $\phi_p(B) = 1-\phi_1 B - \cdots - \phi_p B^p$, $\theta_q(B) = 1-\theta_1 B - \cdots - \theta_q B^q$

and $\theta_0 = \mu(1-\phi_1 - \cdots - \phi_p)$.

For the invertible MA operator, we require that the roots of $\theta_q(B) = 0$ lie outside the unit circle. For the stationary AR operator, we require that the roots of $\phi_q(B) = 0$ also lie outside the unit circle. Also, we assume that $\theta_q(B) = 0$ and $\phi_q(B) = 0$ share no common factors. The parameter θ_0 plays very different roles for d = 0 and d > 0, the original process is stationary if θ_0

is related to the mean of the process, i.e. $\theta_0 = \mu(1-\phi_1 - \cdots - \phi_p)$. However, when $d \geq 1$ θ_0 is called the deterministic trend term which is often omitted from the model unless it is really needed.

The Seasonal ARIMA Models

The seasonal stationary ARIMA process is written as

$$\Phi_P(B^s)\phi_p(B)\dot{Z}_t = \theta_q(B)\Theta_Q(B^s)a_t .$$

The seasonal stationary process which results from a properly differenced and seasonal differenced homogeneous nonstationary series has the differenced series $(1-B)^d Z_t$ and $(1-B^s)^D Z_t$. The following equation is well known as the Box-Jenkins multiplicative seasonal ARIMA model:

$$\Phi_P(B^s)\phi_p(B)(1-B)^d(1-B^s)^D\dot{Z}_t = \theta_q(B)\Theta_Q(B^s)a_t$$

where $\dot{Z}_t = \begin{cases} Z_t - \mu & \text{if } d = D = 0 \\ Z_t & \text{otherwise,} \end{cases}$

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps},$$

and $\Theta_Q(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_Q B^{Qs}$.

We usually call $\phi_p(B)$ and $\theta_q(B)$ the regular autoregressive and moving average factors and $\Phi_P(B^s)$ and $\Theta_Q(B^s)$ the seasonal autoregressive and moving average factors, respectively. In general, the above equation is denoted as ARIMA(p,d,q)×(P,D,Q)$_s$, where the subindex s refers to the seasonal period.

2.1.6 Data Transformation

The most commonly used transformation are differencing and variance stabilizing transformation. Differencing is replacing each value in the series by a difference between that value and the preceding value. We use the differencing when a series is nonstationary. First, we determine the plotting time series data $Z_1, Z_2, ..., Z_n$. If a series is not stationary, its average level varies in the short term or the short-term variation is greater in some places than in others. We must transform the series until we obtain a series that is stationary.

Using differencing in our nonstationary data, we should first investigate whether the series is stationary in variance by applying variance-stabilizing transformations before taking differencing. The power transformation of Box-Cox (1964) is

$$T(Z_t) = Z_t^{(\lambda)} = \frac{Z_t^{\lambda} - 1}{\lambda}$$

where Z_t is an observation at time t and λ is called the transformation parameter. Some commonly used values of λ and their associated transformations are as follows in Table 2.2.

Table 2.2 The associated transformation of λ

Values of λ (lambda)	**Transformation**
–1.0	$\frac{1}{Z_t}$
–0.5	$1/\sqrt{Z_t}$
0.0	$\ln Z_t$
0.5	$\sqrt{Z_t}$
1.0	Z_t (no transforms)

Source: Wei WWS. Time series analysis. America: Addison-Wesley; 1990. p84

We apply the transformation that gives the minimum of residual sum of squares

$$S(\lambda) = \sum_{t=1}^{n} (Z_t(\lambda) - \hat{\mu})^2$$

where $\hat{\mu}$ is the corresponding sample mean of the transformed series $Z_t(\lambda)$.

For the data without the stationary in variance, there is no need to transform Z_t (λ = 1), and we have to consider whether it is the stationary in means or trends. Diminishing that factor is the way to investigate the other effects of ARIMA model. The differencing is denoted by

$$Z_t - Z_{t-1} = (1-B)^d Z_t$$

where Z_t is the value of data for t = 1, 2, ..., n observations and d is either 0,1 or 2 and B is back shift operator.

In the seasonal data, we must first difference the seasonal period which is denoted by

$$Z_t - Z_{t-sD} = (1-B^s)^D Z_t$$

where s is the seasonal period and D is either 0,1 or 2 and B is back shift operator.

2.2 Analyzing ARIMA Models

2.2.1 Identification

To identify the process underlying a series, we first determine the plotting time series data Z_1, Z_2, …, Z_n. In general, we need a minimum of n = 50 observations to identify the appropriate model (Wei, 1990). Plotting data gives a good idea about whether the series contains a trend, seasonality or outliers or is nonstationary. Variance stabilizing transformations or differencing are needed for nonstationary data. To confirm a necessary degree of differencing we have to compute and examine the sample ACF and the sample PACF of the original series or transformed series. If the sample ACF decay very slowly and the sample PACF cuts off after lag 1, this indicates that differencing is needed.

For a stationary process with nonseasonality, we examine the sample ACF and PACF of the properly transformed and differenced series to identify the order of p and q. Usually, the needed order of these p and q are less than or equal to 2 (Wei, 1990). We identify the order p and q by matching the patterns in the sample ACF and PACF with the theoretical pattern of known models (Wei, 1990). Table 2.3 summarizes the important characteristics for selecting p and q.

To identify the seasonal ARIMA model, first transform the data by a variance stabilizing transformation and differencing, if it is necessary. Then determining the order of P and Q by matching the patterns in Table 2.3 at lags of multiples of seasonal period. These orders are usually also less than or equal 2.

To test whether the deterministic trend parameter θ_0 is needed, we can test for its inclusion by comparing the sample mean, $\bar{W}$, of the transformed series with its approximate standard error $S_{\bar{W}}$. If the t-ratio test

$$T = \bar{W} / S_{\bar{W}},$$

where $$S_{\bar{W}} = [\frac{\hat{\gamma}_0}{n}(1 + 2\hat{\rho}_1 + 2\hat{\rho}_2 + \cdots + 2\hat{\rho}_k)]^{1/2}$$

is not significant at the α significant level, then the deterministic trend term θ_0 is not needed.

Table 2.3 Characteristics of theoretical ACF and PACF for stationary processes

Process	ACF	PACF
AR(p)	Tails off as exponential Decay or damped sine wave	Cuts off after lags p
MA(q)	Cuts off after lag q	Tails off as exponential Decay or damped sine wave
ARMA(p,q)	Tails off after lag (q-p)	Tails off after lag (p-q)

Source: Wei WWS. Time series analysis. America: Addison-Wesley; 1990. p106

2.2.2 Estimation and Forecasting

The next step of analyzing ARIMA models is to estimate the parameters in the model. Estimation methods which are more efficient and commonly used in time series analysis are the maximum likelihood method and the nonlinear estimation method. There are several statistical packages to obtain tentative models and estimate parameters. The maximum likelihood method is used in SPSS for windows to analyze ARIMA models in this study.

One of the most important objectives in the analysis of a time series is to forecast its future values. Most forecasting results are derived from a general theory of linear prediction developed by Kolmogorov (1939, 1941), Wiener (1949), Kalman (1960), Yaglom (1962), and Whittle (1983), among others. To forecast l-steps ahead of the future value Z_{n+l} as a linear combination of the observations Z_n, Z_{n-1}, Z_{n-2}, ….,we can let the minimum mean square error forecast $\hat{Z}_n(l)$ of Z_{n+l} be

$$\hat{Z}_n(l) = E(Z_{n+l} \mid Z_n, Z_{n-1}, \ldots)$$

where $\hat{Z}_n(l)$ is usually read as the l-step ahead of the forecast of Z_{n+l} at the forecast origin n.

The forecast error is

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = a_{n+l},$$

and

$$e_{n-j}(l) = Z_{n+l-j} - \hat{Z}_{n-j}(l) = a_{n+l-j}$$

which are made at the same lead time l but different origins n and n-j for j < l. This is also true for the forecast errors for different lead times made from the same time origin.

The (1-α)100% forecast limits are

$$\hat{Z}_n(l) \quad \pm \quad z_{\alpha/2}\left[1+\sum_{j=1}^{l-1}\varphi_j^2\right]^{1/2} \sigma_a \,,$$

where $z_{\alpha/2}$ is the standard normal deviate such that $P(Z > z_{\alpha/2}) = \alpha/2$ and φ_j are the weighted values of the individual ARIMA model. The l-step ahead forecasting data are also calculated by the statistical package.

2.2.2 Diagnosis Checking

Time series model building is an iterative procedure. Starting with model identification followed by estimated parameters, on this section we have to assess model adequacy by checking whether the model assumptions are satisfied. The basic assumption is that the $\{a_t\}$ are white noise. That is, the a_t's are uncorrelated random shocks with zero mean and constant variance. For any estimated model, the residuals $\hat{a}_t$'s are estimates of these unobserved white noise a_t's.

To check whether the residuals are white noise, we compute the sample ACF and PACF of the residual to see whether they do not form any pattern and are all statistically insignificant, i.e., within two standard deviations for $\alpha = 0.05$.

Box and Pierce (1970) proposed the most widely used test, the portmanteau test lack of fit which uses all the residual sample ACF's as a unit to contrast the null hypothesis:

$$H_0 : \rho_1 = \rho_2 = \equiv = \rho_K = 0$$

against the general alternative

$$H_1 : \text{not all } \rho_j = 0.$$

Based on the residual correlogram, Ljung and Box (1978) suggested the modified Q statistic

$$Q = n(n+2)\sum_{j=1}^{K}(n-k)^{-1}\hat{\rho}_k^2$$

where n denotes the length of the series after any differencing and K is the number of lags. Box and Pierce (1970) show that, under H_0, Q is asymptotically distributed as chi-squared with (K-m) degrees of freedom where m is the number of parameters estimated in the model.

2.3 Model Selection Criteria

There are several adequate models that can be used to represent a given data set which residuals from all adequate models are white noise. Model selection criteria based on residuals is another method to make a more accurate decision for adequate models. For the statistical package SPSS for windows, it introduces 2 criteria for consideration.

1. **AIC** : Akaike's Information Criterion

Akaike (1973, 1974) introduced an information criterion to assess the quality of the model fitting. For a statistical model of M parameters AIC is defined as

$$AIC(M) = n\ln\hat{\sigma}_a^2 + n(1+\ln 2\pi) + 2M.$$

The optimal order of the model is chosen by the value of M, which is a function of p and q. Models that yield a minimum value for the AIC criterion are to be used.

2. **SBC** : Schwartz's Bayesian Criterion

Schwartz (1978) suggested the following Bayesian criterion of model selection which is

$$SBC(M) = n\ln\hat{\sigma}_a^2 + M\ln n,$$

where M is the number of parameters in the model and n is the effective number of observations that is equivalent to the number of residuals that can be calculated from the series. The models with minimum SBC are used.

2.4 Model Selection Based on Forecasting Errors

Mostly, several adequate models can represent a series. The ultimate choice of a model may depend on the goodness of fit, such as the residual mean square or model selection criteria. If the main purpose of a model is to forecast future values, alternative criteria of model selection can be based on forecast errors.

The l-step ahead forecast error is $e_l = Z_{n+l} - \hat{Z}_n(l)$ where n is the forecast origin larger than or equal to the length of the series. Wei (1990) introduced the comparison that is usually based on the following summary statistics.

1. Mean percentage error, which is also referred to as bias since it measures forecast bias, $MPE = \left(\frac{1}{M}\sum_{l=1}^{M}\frac{e_l}{Z_{n+l}}\right)100\%$.

2. Mean square error $MSE = \frac{1}{M}\sum_{l=1}^{M}e_l^2$.

3. Mean absolute error $MAE = \frac{1}{M}\sum_{l=1}^{M}|e_l|$.

4. Mean absolute percentage error $\text{MAPE} = (\frac{1}{M}\sum_{l=1}^{M}\left|\frac{e_l}{Z_{n+l}}\right|)100\%$.

These summary statistics are used to indicate the adequate model which is given the minimum of summary statistics.

For Box-Jenkins models with SAS system, we can use SAS/ETS software with the procedures, PROC ARIMA to analyze the model and select an appropriate model for the data. We can use PROC ARIMA to fit the general ARIMA and seasonal ARIMA model, forecast future values, and provide forecast intervals.

2.5 Rainfall and Water Level Forecasting

Univariate Time Series Models

In this topic, two time series data sets, monthly rainfall and monthly water level, are analyzed by using the Box-Jenkins method. The model is fitted and forecasted by using Autoregressive Integrated Moving Average (ARIMA) model procedures with SAS® software version 9.1. To ensure the best model for each series, we use the model selection criteria, that is an Akaike's Information Criterion (AIC) and Schwartz's Bayesian Criterion (SBC). The model which has the minimum value of AIC and SBC is chosen to be the best ARIMA model and it is used to forecast one year ahead for each series.

2.5.1 Monthly Rainfall Data

The monthly rainfall series include data from nine locations, that is Nakhonsawan, Chainat, Singburi, Angthong, Ayutthaya, Pathumthani, Nonthaburi, Bangkok and Samutprakarn. The results for each location are shown as follows.

Nakhonsawan Location

The monthly rainfall series from 1976 to 2003 is plotted in Figure 2.3. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.4. The power transformation analysis indicates that it is not necessary to transform the data.

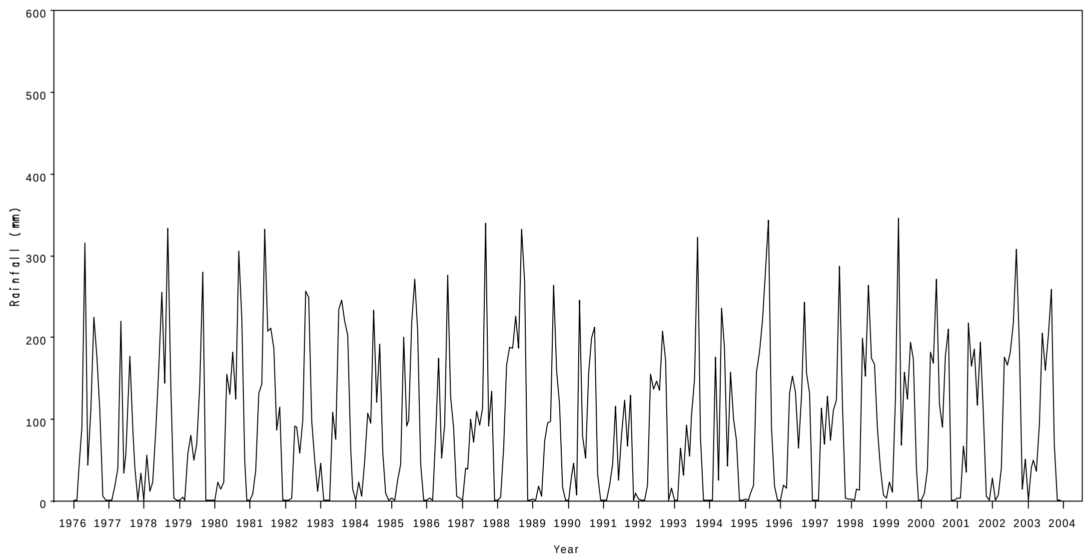


Figure 2.3 Monthly rainfall series at Nakhonsawan location

Table 2.4 Residual mean square error from the power transformation of monthly rainfall series at Nakhonsawan location

λ	Residual mean square error (RMSE)
-1.0	2651900000000
-0.5	7391100000000
0	35724.89
0.5	5290.58
1.0	5190.47

The sample ACF shows a damping sine-cosine wave and the sample PACF have large spike at multiples of the seasonal period of 12 months, implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

The sample ACF and sample PACF for seasonal differencing at periods of 12 months show a large spike at lag 12, at lag 24 and a possible one at lag 36 whereas the sample ACF has a large spike at lag 12. We use the theoretical ACF and PACF patterns to identify the model. These show the pattern of one spike in ACF and the PACF tails off which appears at the seasonal lags and indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. The residual ACF and PACF are close to zero. These indicate that the series is nonstationary and differencing is required.

The differenced residual sample ACF has a large spike at lag 1 whereas the differenced residual sample PACF cuts off at lag 1. Thus all tentative models of monthly rainfall series at Nakhonsawan location are ARIMA (1, 1, 0) × (0, 1, 1)$_{12}$, ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$ and ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$.

Table 2.5 Summary of tentative models fitted for monthly rainfall at Nakhonsawan location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 1, 0) × (0, 1, 1)$_{12}$		
	$(1+ \underset{(0.04867)}{0.48969}\, B)(1-B)(1-B^{12})Z = (1 - \underset{(0.03484)}{0.79023}\, B^{12})\, a_t$	4851.344
2. ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$		
	$(1-B)(1-B^{12})Z = (1 - \underset{(0.0080401)}{B})(1 - \underset{(0.03398)}{0.80963}\, B^{12})\, a_t$	3549.353
3. ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$		
	$(1-\underset{(0.05624)}{0.12088}\, B)(1-B)(1-B^{12})Z = (1 - \underset{(0.0092256)}{B})(1 - \underset{(0.03410)}{0.80592}\, B^{12})\, a_t$	3526.721

The parameters of all tentative models are estimated and shown in Table 2.5, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters for all models are considered at a 0.05 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.6. Model 1 is not adequate for the monthly rainfall at Nakhonsawan location because the residuals are not white noise.

For model selection, we use AIC and SBC to select the best model. From Table 2.7, we see that it is difficult to decide which model has the minimum AIC and SBC, therefore an alternative criteria for model selection is based on estimates of the variance of white noise($\hat{\sigma}_a^2$). The ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ model is selected to describe the monthly rainfall series at Nakhonsawan location with the minimum estimated variance of white noise.

Table 2.6 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Nakhonsawan location

Model	Q statistic	p-value
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$	53.39	0.0002
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	21.57	0.4858
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	19.34	0.5633

Table 2.7 Summary of AIC and SBC values for monthly rainfall at Nakhonsawan location

Model	AIC (Rank)	SBC (Rank)
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	3558.998 (2)	3566.554 (1)
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	3557.924 (1)	3569.257 (2)

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Nakhonsawan location are shown in Table 2.8 while actual and forecasted values are shown in Figure 2.4.

Table 2.8 Forecasted values for ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Nakhonsawan location

Date	Rainfall	Forecasted values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2004	1.0	5.5085	-110.8863	121.9034	-4.5085
Feb 2004	70.3	14.2531	-102.9891	131.4952	56.0469
Mar 2004	1.0	41.5961	-75.6583	158.8506	-40.596
Apr 2004	33.1	64.5860	-52.6687	181.8406	-31.486
May 2004	112.8	174.4849	57.2302	291.7395	-61.685
June 2004	128.9	162.4709	45.2163	279.7256	-33.571
July 2004	159.8	157.5917	40.3370	274.8463	2.2083
Aug 2004	159.7	165.8081	48.5535	283.0628	-6.1081
Sep 2004	239.0	235.0189	117.7643	352.2736	3.9811
Oct 2004	58.6	132.9679	15.7133	250.2226	-74.368
Nov 2004	1.0	19.2307	-98.0240	136.4853	-18.231
Dec 2004	1.0	10.8402	-106.4145	128.0948	-9.8402

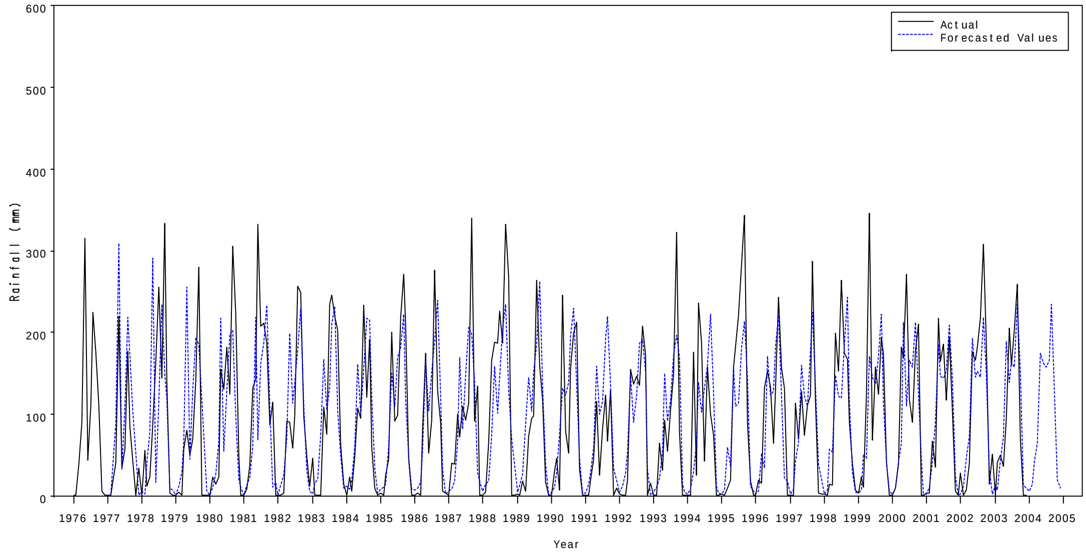


Figure 2.4 Actual and Forecasted values for ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Nakhonsawan location from January 1976 to December 2004

Chainat Location

The monthly rainfall series of Chainat location between January 1976 to December 2006 is plotted in Figure 2.5. It shows that this series is stationary in the mean. To investigate whether the series is stationary in variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.9. The power transformation analysis indicates that a square root transformation is needed for monthly rainfall series at Chainat location as shown in Figure 2.6.

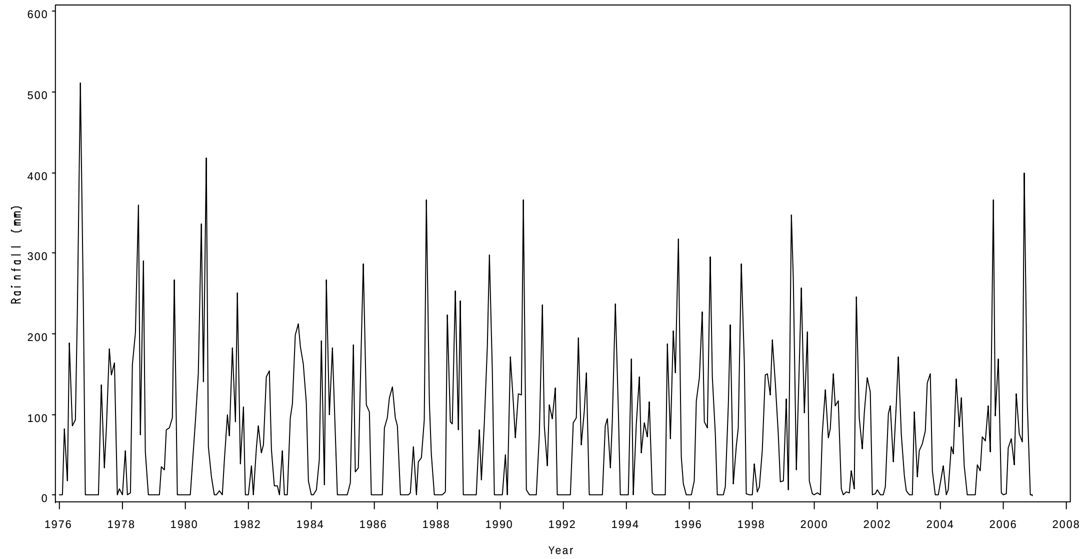


Figure 2.5 Monthly rainfall series at Chainat location

Table 2.9 Residual mean square errors from the power transformation of monthly rainfall at Chainat location

λ	Residual mean square error (RMSE)
-1.0	6774273605.33
-0.5	129875112000000
0	38491.5
0.5	6425.91
1.0	6447.77

The sample ACF for square root transformed monthly rainfall shows a damping sine-cosine wave and the sample PACF for square root transformed monthly rainfall has large spike at multiples of seasonal periods of 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

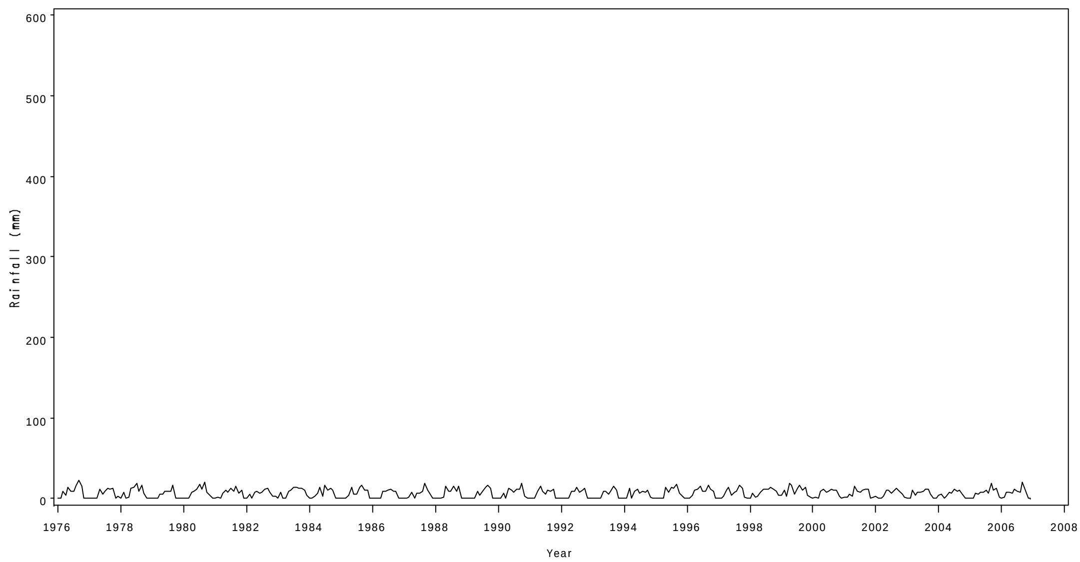


Figure 2.6 Square root transformed monthly rainfall series at Chainat location

The sample ACF for seasonal differencing at period 12 months of monthly rainfall series has a large spike at lag 12 and the sample PACF for seasonal differencing at period 12 months of monthly rainfall series shows a large spike at lag 12, at lag 24 and a possible one at lag 36. It indicates that the ARIMA $(0, 1, 1)_{12}$ model is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. The residual ACF and PACF cut off at lag 5. Hence, the tentative models are ARIMA $(5, 0, 0)\times(0, 1, 1)_{12}$, ARIMA $(0, 0, 5)\times(0, 1, 1)_{12}$ and ARIMA $(5, 0, 5) \times (0, 1, 1)_{12}$.

We consider that the residual sample ACF and PACF are close to zero. These indicate that the series is not stationary and differencing is required. The first difference residual sample ACF cuts off after lag 1 whereas the first difference residual sample PACF has a large spike at lag 1 or 4. The tentative models are ARIMA $(1, 1, 0) \times (0, 1, 1)_{12}$, ARIMA $(4, 1, 0) \times (0, 1, 1)_{12}$ and ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$.

Non-differenced models, ARIMA $(5, 0, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 0, 5) \times (0, 1, 1)_{12}$ and ARIMA $(5, 0, 5) \times (0,1,1)_{12}$, need the t-ratio test for checking whether a deterministic trend parameter θ_0 should be in the model. All non-differenced models do not need the deterministic trend term because the t-ratio test of each model is not significant at 0.05 significance level.

Table 2.10 Summary of tentative models fitted for monthly rainfall at Chainat location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$		
	$(1-0.18529B^5)(1-B^{12})\sqrt{Z_t} = (1-0.77240B^{12})a_t$ (0.05234) (0.03499)	10.9655
2. ARIMA (0, 0, 5) × (0, 1, 1) $_{12}$		
	$(1-B^{12})\sqrt{Z_t} = (1+0.18477B^5)(1-0.76975B^{12})a_t$ (0.05259) (0.03515)	10.9657
3. ARIMA (5, 0, 5) × (0, 1, 1) $_{12}$		
	$(1-0.62477B-0.3438B^4+0.37131B^5)(1-B^{12})\sqrt{Z_t} =$ (0.16549) (0.19370) (0.20060) $(1-0.57110B-0.42411B^4+0.56531B^5)(1-0.75791B^{12})a_t$ (0.15658) (0.17313) (0.17516) (0.37773)	10.84721
4. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$		
	$(1+0.52838B)(1-B)(1-B^{12})\sqrt{Z_t} = (1-0.80618B^{12})a_t$ (0.04473) (0.03348)	15.8029
5. ARIMA (4, 1, 0) × (0, 1, 1) $_{12}$		
	$(1+0.81384B+0.57103B^2+0.42698B^3+0.30988B^4)(1-B)(1-B^{12})\sqrt{Z_t} =$ (0.05046) (0.06157) (0.06196) (0.04907) $(1-0.78450B^{12})a_t$ (0.03477)	12.5011
6. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$		
	$(1-B)(1-B^{12})\sqrt{Z_t} = (1-0.96273B)(1-0.79144B^{12})a_t$ (0.01599) (0.03333)	11.3443

Hence, the parameters of all tentative models are estimated and shown in Table 2.10, where the values in the parentheses under each the estimates refer to the standard errors of those estimates. The estimation of all models gives all parameters being significant at 0.05 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.11. The models 4, 5 and 6 (ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$, ARIMA (4, 1, 0) × (0, 1, 1) $_{12}$, ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$) are not adequate for the monthly rainfall at Chainat location because the residuals are not white noise.

For model selection, we use AIC and SBC to select the best model. From Table 2.12, we see that the first model ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$ has the minimum AIC and SBC. Similarly, if we use the variance of white noise ($\hat{\sigma}_a^2$) to select the best model then ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$ has minimum variance of white noise. Thus the model ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$ is selected to describe the monthly rainfall series at Chainat location.

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$ model for monthly rainfall at Chainat location are shown in Table 2.13 while actual and forecast values are shown in Figure 2.7.

Table 2.11 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Chainat location

Model	Q statistic	p-value
1. ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$	24.91	0.3016
2. ARIMA (0, 0, 5) × (0, 1, 1) $_{12}$	24.84	0.3046
3. ARIMA (5, 0, 5) × (0, 1, 1) $_{12}$	20.67	0.2414
4. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$	83.22	0.0001
5. ARIMA (4, 1, 0) × (0, 1, 1) $_{12}$	36.46	0.0093
6. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	37.70	0.0198

Table 2.12 Summary of AIC and SBC values for monthly rainfall at Chainat location

Model	AIC (Rank)	SBC (Rank)
1. ARIMA (5, 0, 0) × (0, 1, 1) $_{12}$	1875.276 (1)	1883.037 (1)
2. ARIMA (0, 0, 5) × (0, 1, 1) $_{12}$	1875.282 (2)	1883.043 (2)
3. ARIMA (5, 0, 5) × (0, 1, 1) $_{12}$	1876.330 (3)	1903.493 (3)

Table 2.13 Forecasted values for ARIMA (5, 0, 0) × (0, 1, 1)12 of monthly rainfall at Chainat location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	0	0.908	26.1031	61.959	-0.908
Feb 2007	0	11.951	8.3607	101.789	-11.951
Mar 2007	1.8	27.301	1.3698	139.479	-25.501
Apr 2007	69.9	26.510	1.5507	137.716	43.39
May 2007	220.9	80.290	6.3798	240.447	140.61
June 2007	138	81.625	6.1957	246.195	56.375
July 2007	51.9	95.799	10.4839	270.253	-43.899
Aug 2007	47.1	93.571	9.7596	266.522	-46.471
Sep 2007	167.9	227.685	72.6175	471.891	-59.785
Oct 2007	212.2	89.282	8.4155	259.288	122.918
Nov 2007	1.8	15.776	6.2935	114.497	-13.976
Dec 2007	0	1.121	26.5019	64.980	-1.121

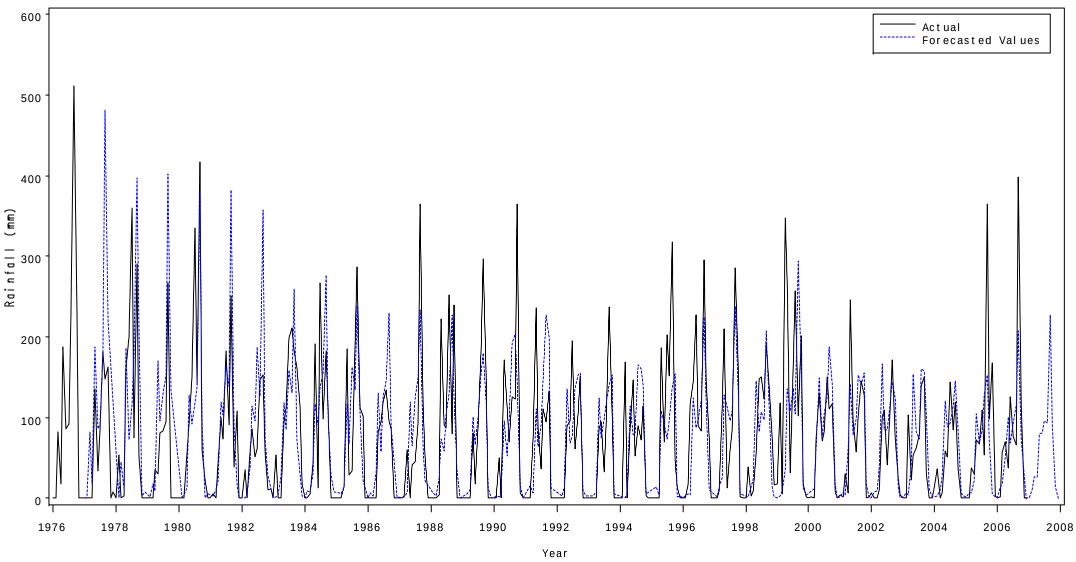


Figure 2.7 Actual and forecasted values for ARIMA (5, 0, 0) × (0, 1, 1)12 of monthly rainfall at Chainat location from January 1976 to December 2007

Singburi Location

The monthly rainfall series of Singburi location in January 1977 to December 2002 is plotted in Figure 2.8. It shows that the series is stationary in the mean. To investigate whether the series is stationary in variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.14. The power transformation analysis indicates that it is not necessary to transform the data.

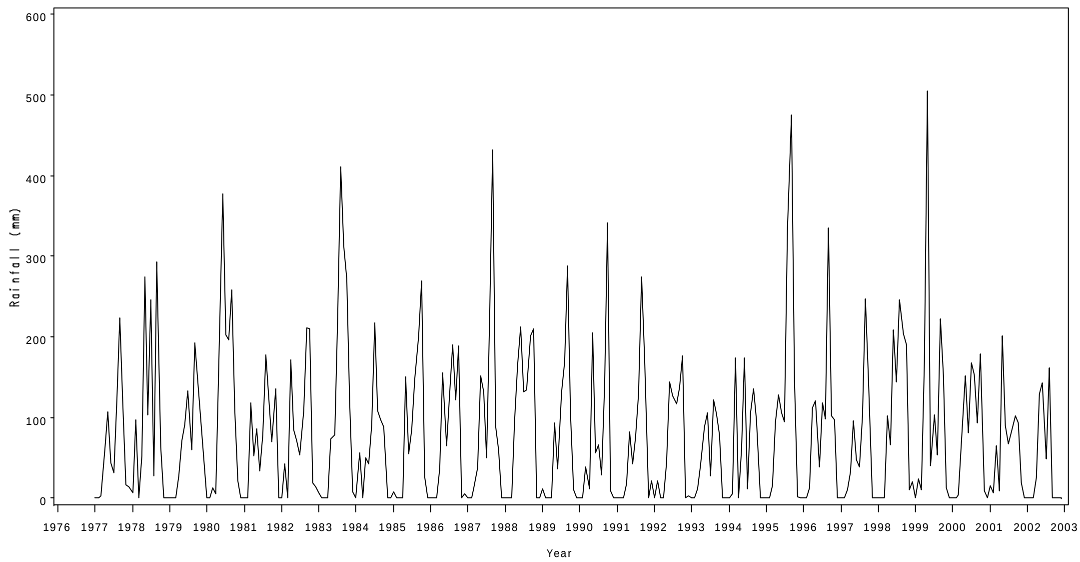


Figure 2.8 Monthly rainfall series of Singburi location

Table 2.14 Residual mean square error from the power transformation of monthly rainfall series at Singburi location

λ	Residual mean square error (RMSE)
-1.0	6635204.48
-0.5	267171000000000000
0	46505.48
0.5	6476.70
1.0	6430.52

We consider the sample ACF and sample PACF of monthly rainfall data and find out that the sample ACF shows a damping sine-cosine wave and the sample PACF has a large spike at multiples of seasonal period 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

We investigate the sample ACF and PACF for seasonal differencing at period 12 months. The sample ACF has a large spike at lag 12 and the sample PACF show a large spike at lag 12, at lag 24 and a possible one at lag 36. It indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

To identify the nonseasonal components of the ARIMA model, the sample residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. We see that the sample ACF cuts off after lag 1 and PACF tails off. Hence the tentative model is the ARIMA $(0, 0, 1) \times (0, 1, 1)_{12}$ model. Alternatively, we may consider the first difference residual of the ARIMA $(0, 1, 1)_{12}$ model. We see that the sample ACF and the sample PACF have large spikes at lag 1. Hence, the tentative models are ARIMA $(1, 1, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$ and ARIMA $(1, 1, 1) \times (0, 1, 1)_{12}$.

Non-differenced models, ARIMA $(0, 0, 1) \times (0, 1, 1)_{12}$, need the t-ratio test for checking whether a deterministic trend parameter θ_0 should be in the model. It shows that the ARIMA $(1, 0, 0) \times (0, 1, 1)_{12}$ model does not need the deterministic trend term because the t-ratio test of the model is not significant at 0.05 significance level.

Table 2.15 Summary of tentative models fitted for monthly rainfall at Singburi location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA $(0, 0, 1) \times (0, 1, 1)_{12}$		
	$(1-B^{12})Z_t = (1+\underset{(0.05852)}{0.14936}B)(1-\underset{(0.02446)}{0.92399}B^{12})a_t$	4583.831
2. ARIMA $(1, 1, 0) \times (0, 1, 1)_{12}$		
	$(1+\underset{(0.05383)}{0.44789}B)(1-B)(1-B^{12})Z_t = (1-\underset{(0.02589)}{0.91636}B^{12})a_t$	6401.820
3. ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$		
	$(1-B)(1-B^{12})Z_t = (1-\underset{(0.00168)}{B})(1-\underset{(0.03381)}{0.89760}B^{12})a_t$	4829.532
4. ARIMA $(1, 1, 1) \times (0, 1, 1)_{12}$		
	$(1-\underset{(0.05896)}{0.19547}B)(1-B)(1-B^{12})Z_t = (1-\underset{(0.0019926)}{B})(1-\underset{(0.03099)}{0.91077}B^{12})a_t$	4696.02

The parameters of all tentative models are estimated and presented in Table 2.15, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters for all models are considered at 0.05 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.16. Model 2 is not adequate for the monthly rainfall at Singburi location because the residuals are not white noise.

For model selection, we use AIC and SBC to select the best model. From Table 2.17, we see that model 4 has the minimum AIC and SBC. Thus the model ARIMA (1, 1, 1) × $(0, 1, 1)_{12}$ is selected to describe the monthly rainfall series at Singburi location.

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (1, 1, 1) × $(0, 1, 1)_{12}$ model for monthly rainfall at Singburi location are shown in Table 2.18 while actual and forecasted values are shown in Figure 2.9.

Table 2.16 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Singburi location

Model	Q statistic	p-value
1. ARIMA (0, 0, 1) × $(0, 1, 1)_{12}$	15.54	0.8381
2. ARIMA (1, 1, 0) × $(0, 1, 1)_{12}$	52.47	0.0003
3. ARIMA (0, 1, 1) × $(0, 1, 1)_{12}$	24.52	0.3205
4. ARIMA (1, 1, 1) × $(0, 1, 1)_{12}$	15.80	0.7806

Table 2.17 Summary of AIC and SBC values for monthly rainfall at Singburi location

Model	AIC (Rank)	SBC (Rank)
1. ARIMA (0, 0, 1) × $(0, 1, 1)_{12}$	3235.957 (3)	3243.276 (3)
3. ARIMA (0, 1, 1) × $(0, 1, 1)_{12}$	3205.661 (2)	3212.952 (2)
4. ARIMA (1, 1, 1) × $(0, 1, 1)_{12}$	3198.718 (1)	3209.655 (1)

Table 2.18 Forecasted values for ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Singburi location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2003	0	0.457	-134.826	137.741	-0.457
Feb 2003	0	3.374	-134.632	143.380	-3.374
Mar 2003	18.3	17.582	-120.533	157.697	0.718
Apr 2003	48.9	53.954	-84.165	194.073	-5.054
May 2003	135.7	146.931	8.811	287.050	-11.231
June 2003	101.7	95.167	-42.952	235.286	6.533
July 2003	106.8	88.374	-49.745	228.494	18.426
Aug 2003	145.6	138.885	0.766	279.004	6.715
Sep 2003	207.9	184.760	46.641	324.879	23.14
Oct 2003	140	129.712	-8.408	269.831	10.288
Nov 2003	21.2	15.242	-122.877	155.361	5.958
Dec 2003	3.1	1.284	-136.835	141.403	1.816

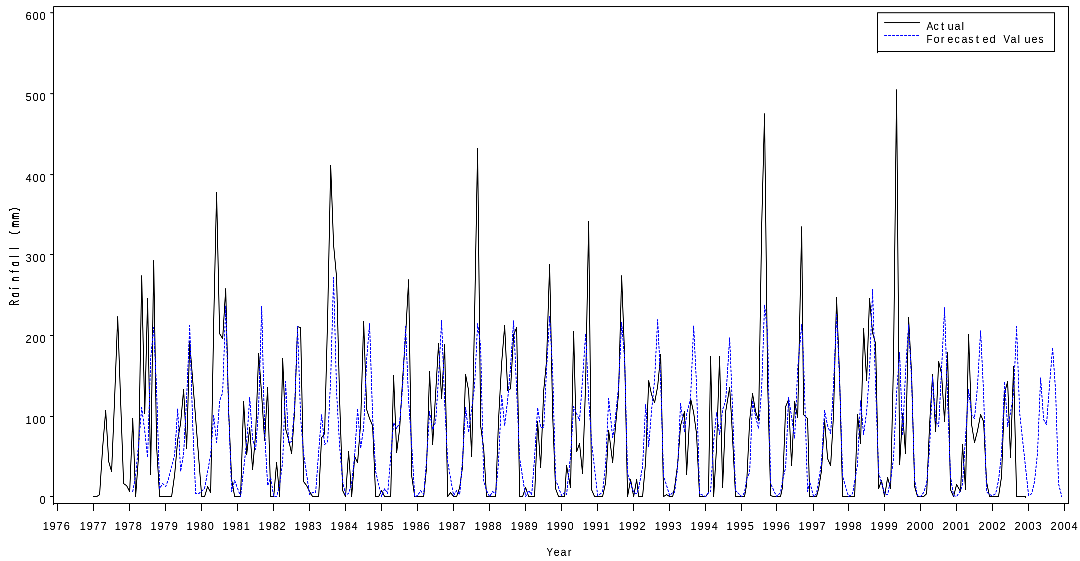


Figure 2.9 Actual and forecasted values for ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Singburi location from January 1976 to December 2003

Angthong Location

The monthly rainfall series of Angthong location between January 1976 to December 2006 is plotted in Figure 2.10. It shows that this series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the

SAS® system software version 9.1 which is shown in Table 2.19. The power transformation analysis indicates that it is not necessary to transform the data.

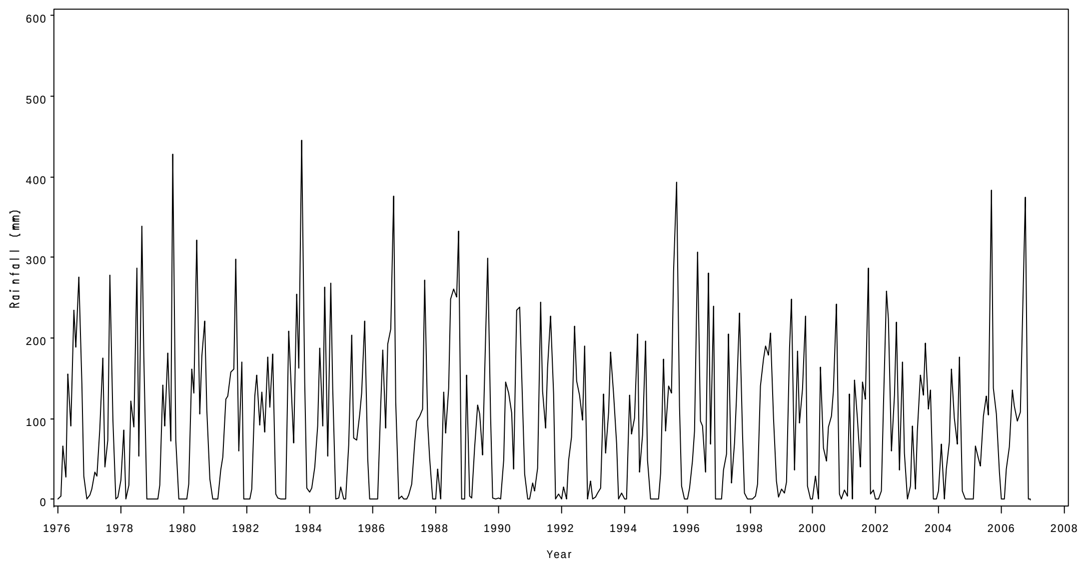


Figure 2.10 Monthly rainfall series at Angthong location

Table 2.19 Residual mean square error from the power transformation of monthly rainfall at Angthong location

λ	Residual mean square error (RMSE)
-1.0	18404643312.48
-0.5	370457125900000000
0	60605.41
0.5	8305.88
1.0	8173.41

We consider the sample ACF and sample PACF of monthly rainfall data. The sample ACF shows a damping sine-cosine wave and the sample PACF has large spike at multiples of seasonal period 12 months implying that a seasonal differencing $(1- B^{12})$ is needed in order to achieve stationarity.

We consider the sample ACF and PACF for seasonal differencing at period 12 months. The sample ACF has a large spike at lag 12 and the sample PACF shows large spikes at lag 12, at lag 24 and a possible one at lag 36. This indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered .The residual sample ACF and PACF are close to zero and we consider the first differenced residual of the ARIMA $(0, 1, 1)_{12}$ model.

The first differenced residual of the sample ACF cuts off after lag 1 and the sample PACF has large spikes at lag 1 and 5. Hence, the tentative models are ARIMA $(1, 1, 0) \times (0, 1, 1)_{12}$, ARIMA $(5, 1, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$, ARIMA $(1, 1, 1) \times (0, 1, 1)_{12}$ and ARIMA $(5, 1, 1) \times (0, 1, 1)_{12}$.

The parameters of all tentative models are estimated and shown in Table 2.20, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters ϕ_1 in model 4 and $\phi_1 - \phi_5$ in model 5 are not significant at 0.05 significance level. It implies that the model can be refitted with ϕ_1 and $\phi_1 - \phi_5$ removed if it is necessary. This would be considered as a tentative model 3.

Table 2.20 Summary of tentative models fitted for monthly rainfall at Angthong location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA $(1, 1, 0) \times (0, 1, 1)_{12}$		
	$\underset{(0.04402)}{(1+0.55558}\,B)(1-B)(1-B^{12})Z_t = (1-\underset{(0.02377)}{0.90751}\,B^{12})\,a_t$	7902.849
2. ARIMA $(5, 1, 0) \times (0, 1, 1)_{12}$		
	$(1+\underset{(0.05194)}{0.8768}B+\underset{(0.06717)}{0.6357}B^2+\underset{(0.07044)}{0.54293}B^3+\underset{(0.06969)}{0.44202}B^4+\underset{(0.05448)}{0.24488}B^5)(1-B)(1-B^{12})Z_t =$	
	$(1-\underset{(0.02455)}{0.90132}\,B^{12})\,a_t$	6279.693
3. ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$		
	$(1-B)(1-B^{12})Z_t = (1-\underset{(0.01188)}{0.97780}B)(1-\underset{(0.02589)}{0.89478}\,B^{12})\,a_t$	5637.677

Table 2.20 Summary of tentative models fitted for monthly rainfall at Angthong location (continued)

Model	Estimated parameters	$\hat{\sigma}_a^2$
4. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$		
	$(1+0.03764\,B)(1-B)(1-B^{12})\,Z_t = (1-0.97652B)(1-0.89547\,B^{12})\,a_t$ (0.05628) (0.01259) (0.02580)	5645.792
5. ARIMA (5, 1, 1) × (0, 1, 1) $_{12}$.		
	$(1+0.03415B-0.07757\,B^2+0.05625\,B^3+0.05611\,B^4+0.00251\,B^5)(1-B)(1-B^{12})\,Z_t =$ (0.05509) (0.05504) (0.05576) (0.05825) (0.05811)	
	$(1-0.97519B)(1-0.89945\,B^{12})\,a_t$ (0.01437) (0.02580)	5648.209

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.21. Models 1 and 2 are not adequate for the monthly rainfall at Angthong location because the residuals are not white noise. Hence model 3 is the best model for monthly rainfall series at Angthong location. Table 2.22 shows summary of AIC and SBC values for the best model of monthly rainfall at Angthong location.

Table 2.21 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Angthong location

Model	Q statistic	p-value
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$	62.65	<0.0001
2. ARIMA (5, 1, 0) × (0, 1, 1) $_{12}$	29.95	0.0379
3. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	20.56	0.5483

Table 2.22 Summary of AIC and SBC values for monthly rainfall at Angthong location

MODEL	AIC	SBC
3. ARIMA (5, 1, 1) × (0, 1, 1) $_{12}$	4127.163	4154.347

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Angthong location are shown in Table 2.23 and actual and forecasted values are shown in Figure 2.11.

Table 2.23 Forecasted values for ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Angthong location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	1	11.9398	-135.2230	159.1027	-10.94
Feb 2007	1	18.3671	-128.8320	165.5662	-17.367
Mar 2007	1	45.4696	-101.7657	192.7049	-44.47
Apr 2007	39.9	75.5637	-71.7079	222.8352	-35.664
May 2007	349.5	141.9006	-5.4072	289.2084	207.599
June 2007	122.4	128.0041	-19.3399	275.3481	-5.6041
July 2007	73.5	119.2910	-28.0892	266.6712	-45.791
Aug 2007	30.8	139.7486	-7.6678	287.1650	-108.95
Sep 2007	15.4	266.1273	118.6748	413.5799	-250.73
Oct 2007	80.1	173.1885	25.6997	320.6772	-93.089
Nov 2007	40.7	46.1192	-101.4057	193.6441	-5.4192
Dec 2007	5.2	15.7215	-131.8396	163.2825	-10.522

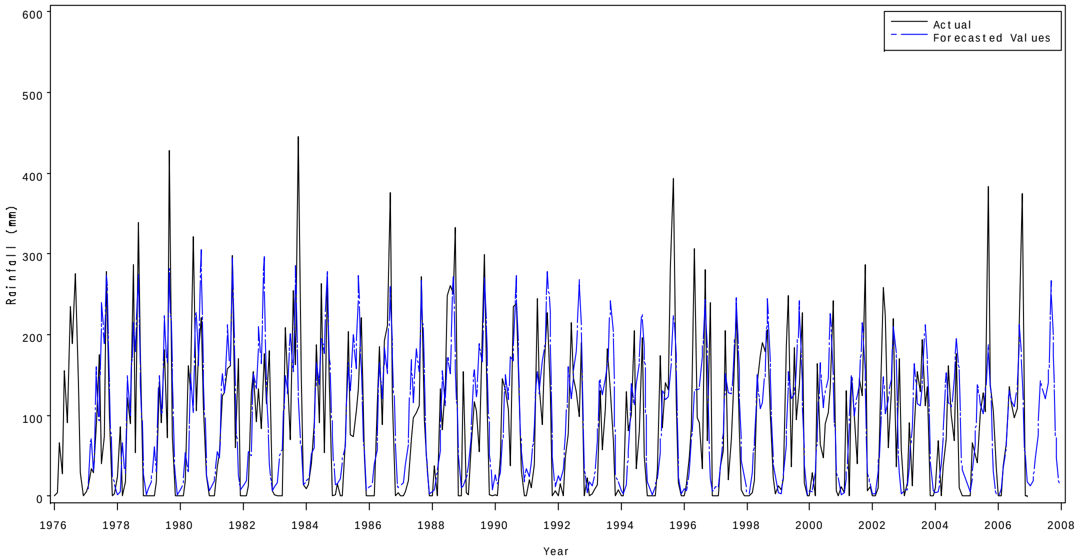


Figure 2.11 Actual and forecasted values for ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$ of rainfall at Angthong location from January 1976 to December 2007

Ayutthaya Location

The monthly rainfall series of Ayutthaya location between January 1976 to December 2006 is plotted in Figure 2.12. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 and are shown in Table 2.24. The power transformation analysis indicates that a square root transformation is needed. The square root transformed monthly rainfall series is shown in Figure 2.13.

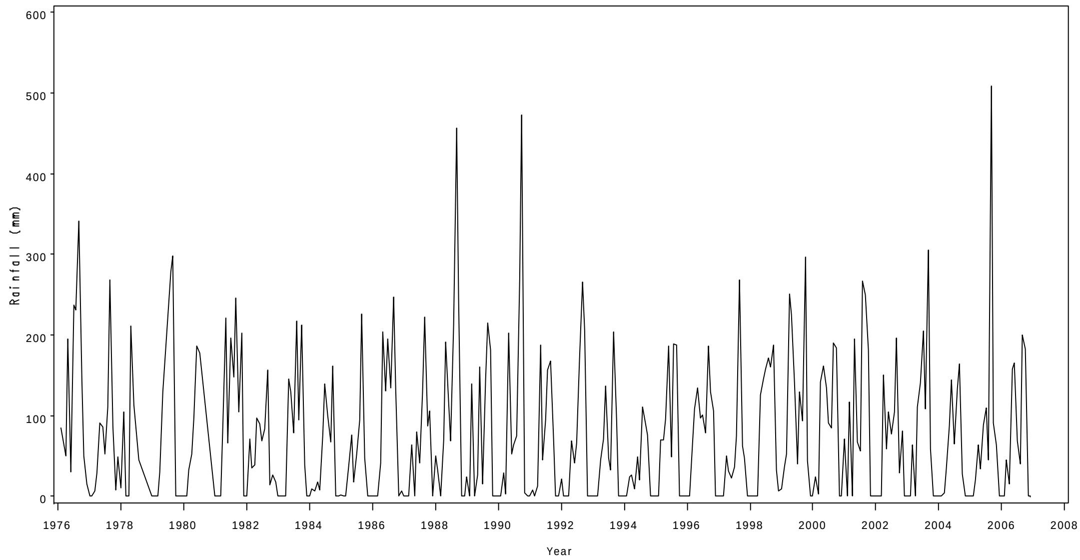


Figure 2.12 Monthly rainfall series at Ayutthaya location

Table 2.24 Residual mean square error from the power transformation of monthly rainfall at Ayutthaya location

λ	Residual mean square error (RMSE)
-1.0	493303362.11
-0.5	2304461300000000000000000
0	41668.67
0.5	6022.99
1.0	6078.39

The sample ACF shows a damping sine-cosine wave and the sample PACF has large spikes at multiples of seasonal period 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

The sample ACF has a large spike at lag 12 and the sample PACF shows large spike at lag 12, at lag 24 and a possible one at lag 36. This indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. The residual ACF and PACF are close to zero. Alternatively, we may consider the sample ACF and PACF of first seasonal differencing at period 12 months. We see that the sample ACF and PACF have large spikes at lag 1. Hence, all tentative models of monthly rainfall series at Ayutthaya location are ARIMA $(1, 1, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$ and ARIMA $(1, 1, 1) \times (0, 1, 1)_{12}$.

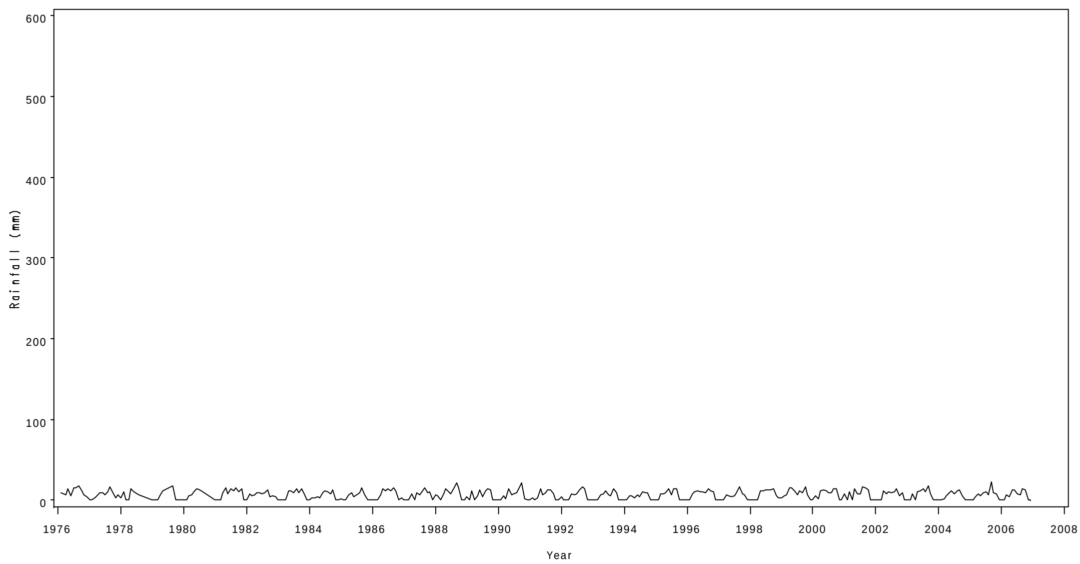


Figure 2.13 The square root transformed monthly rainfall series at Ayutthaya location

Parameters of all tentative models are estimated and shown in Table 2.25, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters for all models are considered at 0.05 significance level.

Table 2.25 Summary of tentative models fitted for monthly rainfall at Ayutthaya location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$		
	$(1+0.55163\,B)(1-B)(1-B^{12})\sqrt{Z_t} = (1-0.79399\,B^{12})\,a_t$	16.996
	(0.04620) (0.03548)	
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$		
	$(1-B)(1-B^{12})\sqrt{Z_t} = (1-0.95250\,B)(1-0.81134\,B^{12})\,a_t$	12.627
	(0.01779) (0.03434)	
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$		
	$(1+0.02675\,B)(1-B)(1-B^{12})\sqrt{Z_t} = (1-0.94636B)(1-0.81222\,B^{12})\,a_t$	12.659
	(0.05897) (0.01995) (0.03430)	

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.26. Model 1 is not adequate for the monthly rainfall at Ayutthaya location because the residuals are not white noise.

Table 2.26 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Ayutthaya location

Model	Q statistic	p-value
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$	53.41	0.0002
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	10.01	0.9862
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	9.88	0.9804

For model selection, we use AIC and SBC to select the best model. From Table 2.27, we see that the ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$ model has the minimum AIC and SBC. Hence it is selected to describe the monthly rainfall series at Ayutthaya with minimum estimated variance of white noise.

Table 2.27 Summary of AIC and SBC values for monthly rainfall at Ayutthaya location

Model	AIC (Rank)	SBC (Rank)
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	1759.210 (1)	1766.790 (1)
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	1761.030 (2)	1772.399 (2)

Table 2.28 Forecasted values for ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Ayutthaya location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	0	1.219	29.9788	71.474	-1.219
Feb 2007	0	0.715	32.0701	68.594	-0.715
Mar 2007	4.8	21.095	5.1980	136.445	-16.295
Apr 2007	84.8	35.389	0.9138	169.537	49.411
May 2007	354.1	97.274	8.5100	285.928	256.826
June 2007	109.0	112.779	13.4160	312.251	-3.779
July 2007	38.2	81.129	4.2057	258.381	-42.929
Aug 2007	35.4	89.282	6.1612	272.951	-53.882
Sep 2007	175.2	232.742	68.2491	498.002	-57.542
Oct 2007	83.5	99.200	8.8499	290.535	-15.7
Nov 2007	10.2	11.072	12.7334	110.615	-0.872
Dec 2007	0	-0.207	37.9458	63.064	0.207

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Ayutthaya location are shown in Table 2.28 while actual and forecasted values are shown in Figure 2.14.

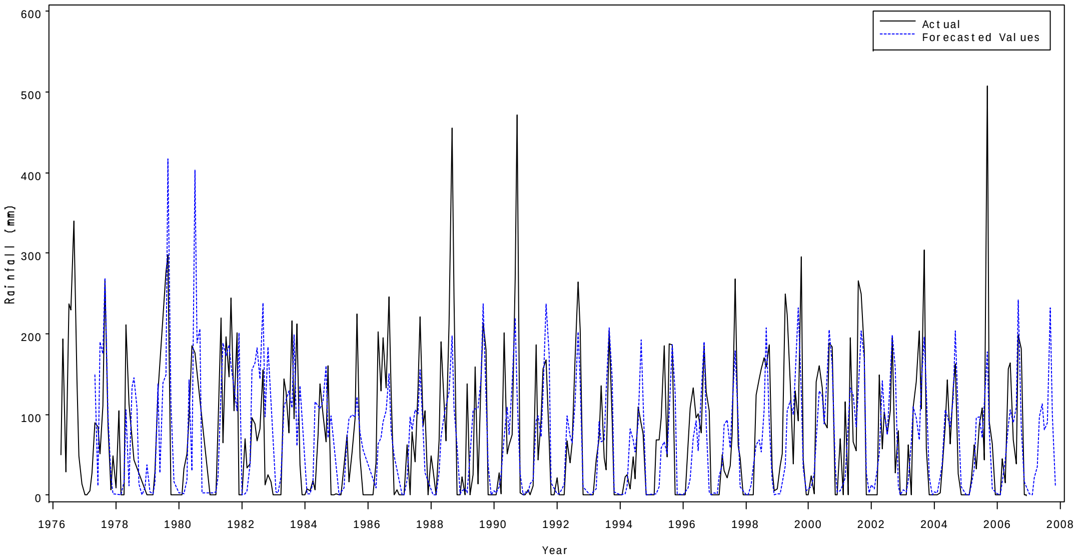


Figure 2.14 Actual and forecasted values for ARIMA (0, 1, 1) × $(0, 1, 1)_{12}$ of monthly rainfall at Ayutthaya location from January 1976 to December 2007

Pathumtani Location

The monthly rainfall series of Pathumtani location from January 1976 to December 2003 is plotted in Figure 2.15. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.29. The power transformation analysis indicates that it is not necessary to transform the data.

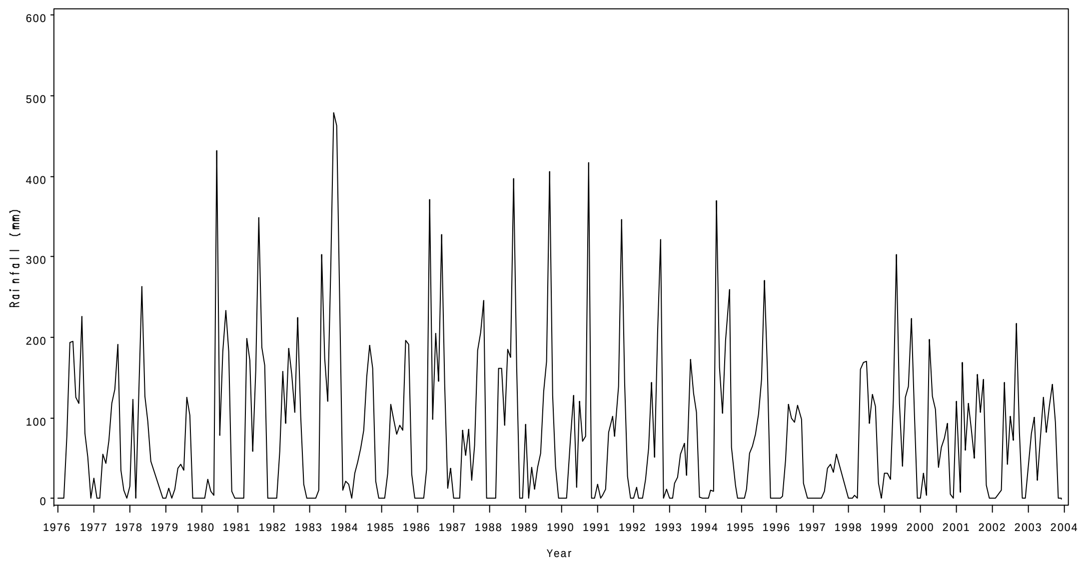


Figure 2.15 Monthly rainfall series at Pathumtani location

Table 2.29 Residual mean square error from the power transformation of monthly rainfall series at Pathumtani location

λ	Residual mean square error (RMSE)
-1.0	84599249060.22
-0.5	621843027500000
0	44534.83
0.5	6866.05
1.0	6785.24

The sample ACF shows a damping sine-cosine wave and the sample PACF has large spikes at multiples of seasonal period 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

The sample PACF still shows a large spike at lag 12, at lag 24 and a possible one at lag 36. The sample ACF has a large spike at lag 12. Comparing the theoretical ACF and PACF patterns, the pattern of one spike in ACF and the PACF tails off which appears at the seasonal lags indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. The residual ACF cuts off at lag 1 and the residual PACF cuts off at lag 1 or 2. Hence, all the tentative models are ARIMA $(1, 0, 0)\times(0, 1, 1)_{12}$, ARIMA $(2, 0, 0)\times(0, 1, 1)_{12}$, ARIMA $(0, 0, 1)\times(0, 1, 1)_{12}$, ARIMA $(1, 0, 1)\times(0, 1, 1)_{12}$ and ARIMA $(2, 0, 1)\times(0, 1, 1)_{12}$.

The parameters of all tentative models are estimated and shown in Table 2.30, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level. There is only one estimation of ϕ_2 in model 2 which is not significant.

Table 2.30 Summary of tentative models fitted for monthly rainfall at Pathumthani location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$		
	$(1-\underset{(0.05708)}{0.24296}B)(1-B^{12})Z_t = (1-\underset{(0.03318)}{0.82734}B^{12})a_t$	5721.007
2. ARIMA (2, 0, 0) × (0, 1, 1) $_{12}$		
	$(1-\underset{(0.05854)}{0.21874}B-\underset{(0.05919)}{0.10361}B^2)(1-B^{12})Z_t = (1-\underset{(0.03318)}{0.82737}B^{12})a_t$	5680.740
3. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-B^{12})Z_t = (1+\underset{(0.05758)}{0.20161}B)(1-\underset{(0.03380)}{0.82057}B^{12})a_t$	5781.594
4. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-\underset{(0.12588)}{0.72846}B)(1-B^{12})Z = (1-\underset{(0.15530)}{0.52847}B)(1-\underset{(0.03295)}{0.82977}B^{12})a_t$	5636.292

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.31. All models are adequate for the monthly rainfall at Pathumthani location.

For model selection, we use AIC and SBC to select the best model. From Table 2.32, we see that the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model has the minimum AIC whereas the ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$ model has the minimum SBC. We give the estimate variance of white noise ($\hat{\sigma}_a^2$) in Table 2.30. The minimum estimate variance of white noise ($\hat{\sigma}_a^2$) of the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model is less than that of the ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$ model. Hence the

ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model is selected to describe the monthly rainfall series at Pathumtani location.

Table 2.31 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Pathumtani location

Model	Q statistic	p-value
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	32.64	0.0671
3. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$	33.45	0.0558
4. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	31.63	0.0638

Table 2.32 Summary of AIC and SBC values for monthly rainfall at Pathumthani location

Model	AIC (Rank)	SBC (Rank)
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	3483.396 (2)	3490.823 (1)
3. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$	3486.588 (3)	3494.015 (3)
4. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	3479.867 (1)	3491.008 (2)

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Pathumtani location are shown in Table 2.33 while actual and forecasted values are shown in Figure 2.16.

Table 2.33 Forecasted values for ARIMA (1, 0, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Pathumtani location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2004	0	17.0626	-130.0822	164.2074	-17.063
Feb 2004	80.6	17.9229	-132.1355	167.9813	62.6771
Mar 2004	100.3	88.2630	-63.3189	239.8448	12.037
Apr 2003	22.3	51.5072	-100.8768	203.8913	-29.207
May 2004	73.9	128.3915	-24.4165	281.1995	-54.492
June 2004	124.9	96.0559	-56.9766	249.0884	28.8441
July 2004	82.5	82.4227	-70.7288	235.5742	0.0773
Aug 2004	114.2	109.1829	-44.0317	262.3975	5.0171
Sep 2004	141.6	155.9382	2.6901	309.1863	-14.338
Oct 2004	93.6	119.5989	-33.6670	272.8647	-25.999
Nov 2004	1	20.5350	-132.7403	173.8103	-19.535
Dec 2004	1	2.1763	-151.1039	155.4566	-1.1763

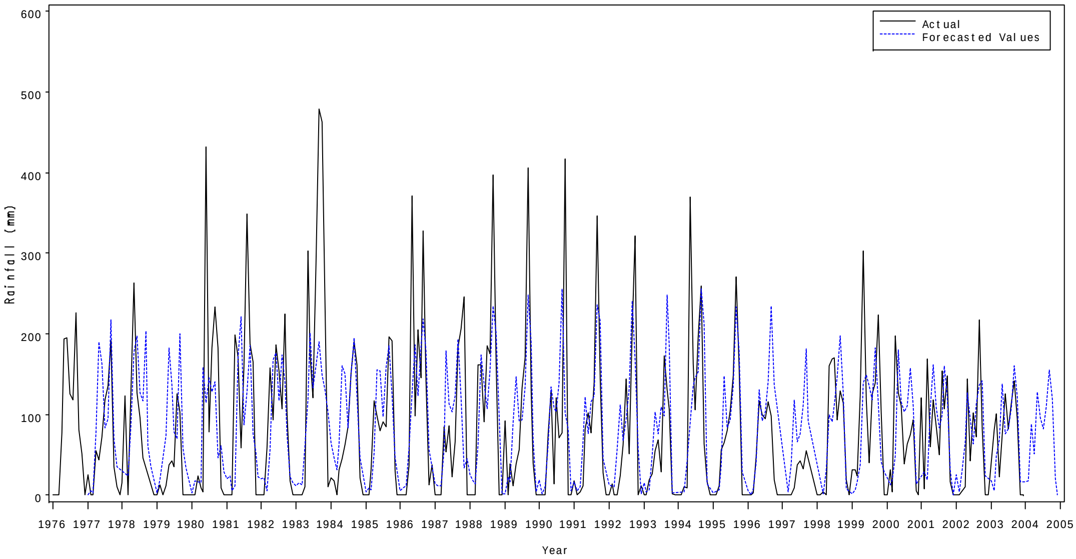


Figure 2.16 Actual and forecasted values for ARIMA (1, 0, 1) × $(0, 1, 1)_{12}$ of monthly rainfall at Pathumtani location from January 1976 to December 2004

Nonthaburi Location

The monthly rainfall series of Nonthaburi location from January 1976 to December 2006 is plotted in Figure 2.17. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.34. The power transformation analysis indicates that a square root transformation is needed. Figure 2.41 shows the plot of the transformed series.

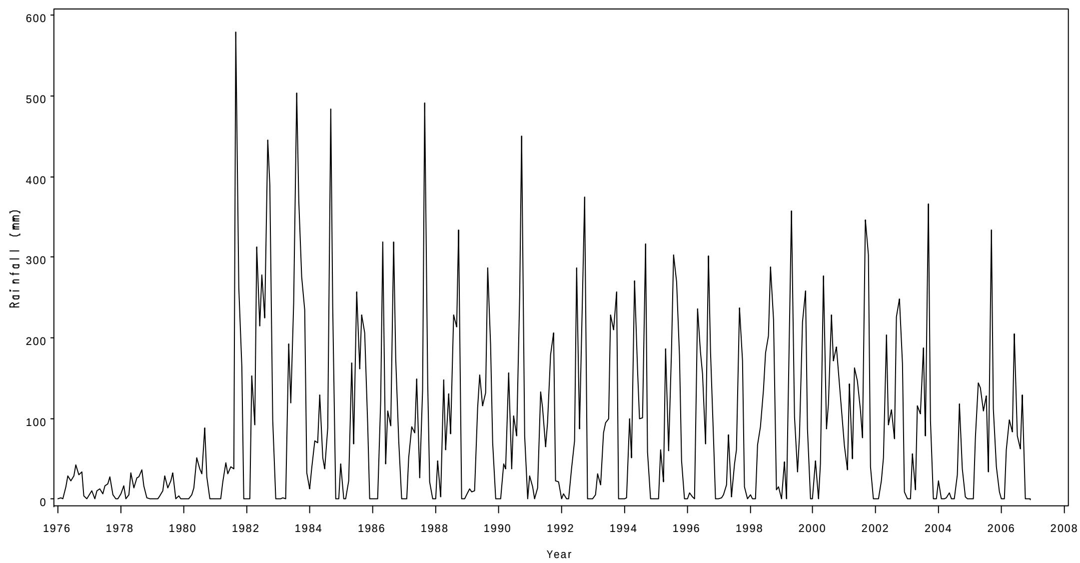


Figure 2.17 Monthly rainfall series at Nonthaburi location

Table 2.34 Residual mean square errors from the power transformation of monthly rainfall series at Nonthaburi location

λ	Residual mean square error (RMSE)
-1.0	6317021710.21
-0.5	5431799614.81
0	29744.09
0.5	8441.86
1.0	8518.80

The sample ACF shows a damped sine-cosine wave and the sample PACF has large spike at multiples of seasonal period 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

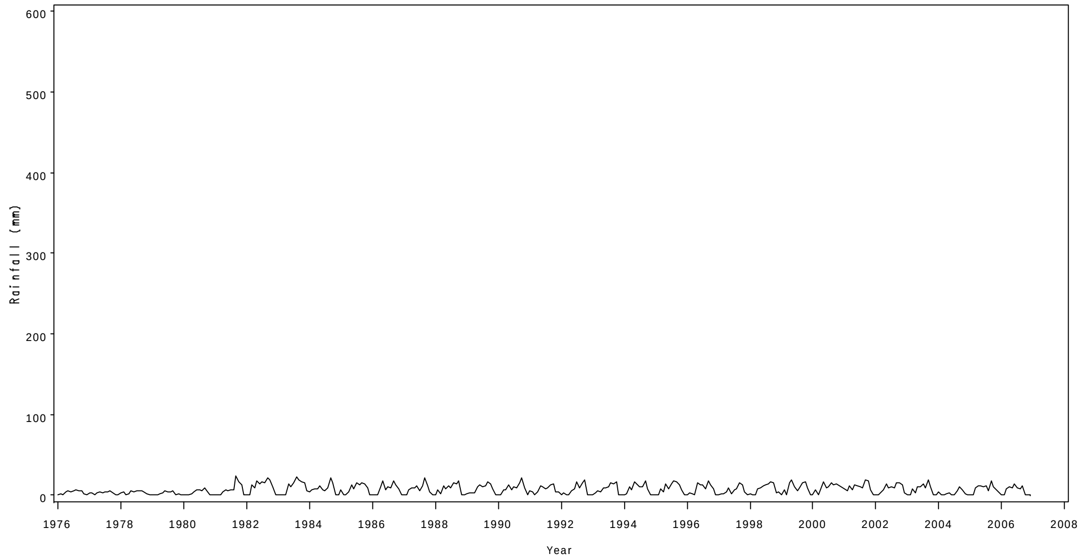


Figure 2.18 The square root transformed monthly rainfall series at Nonthaburi location

The sample ACF and PACF for seasonal differencing at period 12 months are considered. The sample ACF has a large spike at lag 12 and the sample PACF shows large spikes at lag 12, at lag 24 and a possible one at lag 36. It indicates that the ARIMA $(0, 1, 1)_{12}$ model is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. The residual ACF and PACF cut off at lag 1. Hence, all tentative models are ARIMA $(1, 0, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 0, 1) \times (0, 1, 1)_{12}$ and ARIMA $(1, 0, 1) \times (0, 1, 1)_{12}$.

The parameters of all tentative models are estimated and shown in Table 2.35, where the values in the parentheses under each of the estimates refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 or 0.01 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.36. Models 1, 2 and 3 are not adequate for the monthly rainfall series at Nonthaburi location because the residuals are not white noise. So we consider the first difference of the residual.

Table 2.35 Summary of tentative models fitted for monthly rainfall at Nonthaburi location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$		
12.842	$(1-0.41836B)(1-B^{12})\, Z_t = (1-0.77298B^{12})\, a_t$ (0.04851) (0.03602)	
2. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-B^{12})Z_t = (1 + 0.30166B)(1-0.72349B^{12})\, a_t$ (0.05058) (0.03834)	13.403
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-0.78799B)(1-B^{12})\, Z_t = (1-0.46749B)(1-0.79748\, B^{12})\, a_t$ (0.06567) (0.09340) (0.03446)	12.300

Table 2.36 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Nonthaburi location

Model	Q statistic	p-value
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	63.84	<0.0001
2. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$	101.99	<0.0001
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	42.17	0.0040

The differenced residual sample ACF and PACF have large spike at lag 1. Thus all tentative models are ARIMA(1, 1, 0) × (0, 1, 1)$_{12}$, ARIMA(0, 1, 1)×(0, 1, 1)$_{12}$ and ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$.

The parameters of all tentative models are estimated and shown in Table 2.37, where the values in the parentheses under each estimates refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.38. Models 1 and 2 are not adequate for the monthly rainfall at Nontaburi location because the residuals are not white noise.

Table 2.37 Summary of tentative models fitted for monthly rainfall at Nonthaburi location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$		
	$(1+0.40105B)(1-B)(1-B^{12})Z_t = (1-0.83799B^{12})a_t$ (0.04859) (0.03139)	14.893
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$		
	$(1-B)(1-B^{12})Z_t = (1-0.78376B)(1-0.80393B^{12})a_t$ (0.03309) (0.03354)	13.000
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$		
	$(1-0.28292B)(1-B)(1-B^{12})Z_t = (1-0.91804B)(1-0.82187B^{12})a_t$ (0.05918) (0.02451) (0.03264)	12.394

Table 2.38 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Nontaburi location

Model	Q statistic	p-value
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$	68.99	<0.0001
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	58.72	<0.0001
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	36.25	0.0221

Hence the best model is model 3. Table 2.39, shows summary AIC and SBC values for the best model of monthly rainfall series at Nonthaburi location.

The one year ahead forecasted values with 95 percent forecast limits of the ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Nonthaburi location are shown in Table 2.40 while actual and forecasted values are shown in Figure 2.49.

Table 2.39 Summary of AIC and SBC values for monthly rainfall at Nonthaburi location

Model	AIC	SBC
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	1925.477	1937.127

Table 2.40 Forecasted values for ARIMA (1, 1, 1) × $(0, 1, 1)_{12}$ of monthly rainfall at Nonthaburi location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	6.1	0.824	29.5387	59.854	5.276
Feb 2007	9.8	11.177	10.0398	100.059	-1.377
Mar 2007	28.3	26.481	1.9360	138.771	1.819
Apr 2007	46.4	26.019	2.0624	137.720	20.381
May 2007	132.6	80.145	5.7337	242.301	52.455
June 2007	89.1	81.551	5.5629	248.308	7.549
July 2007	105.4	95.844	9.7086	272.748	9.556
Aug 2007	116.3	93.691	9.0337	269.118	22.609
Sep 2007	252.5	227.134	70.3576	474.681	25.366
Oct 2007	181.2	89.259	7.6979	261.590	91.941
Nov 2007	42.8	14.782	7.8080	112.632	28.018
Dec 2007	2.6	1.005	29.9937	62.892	1.595

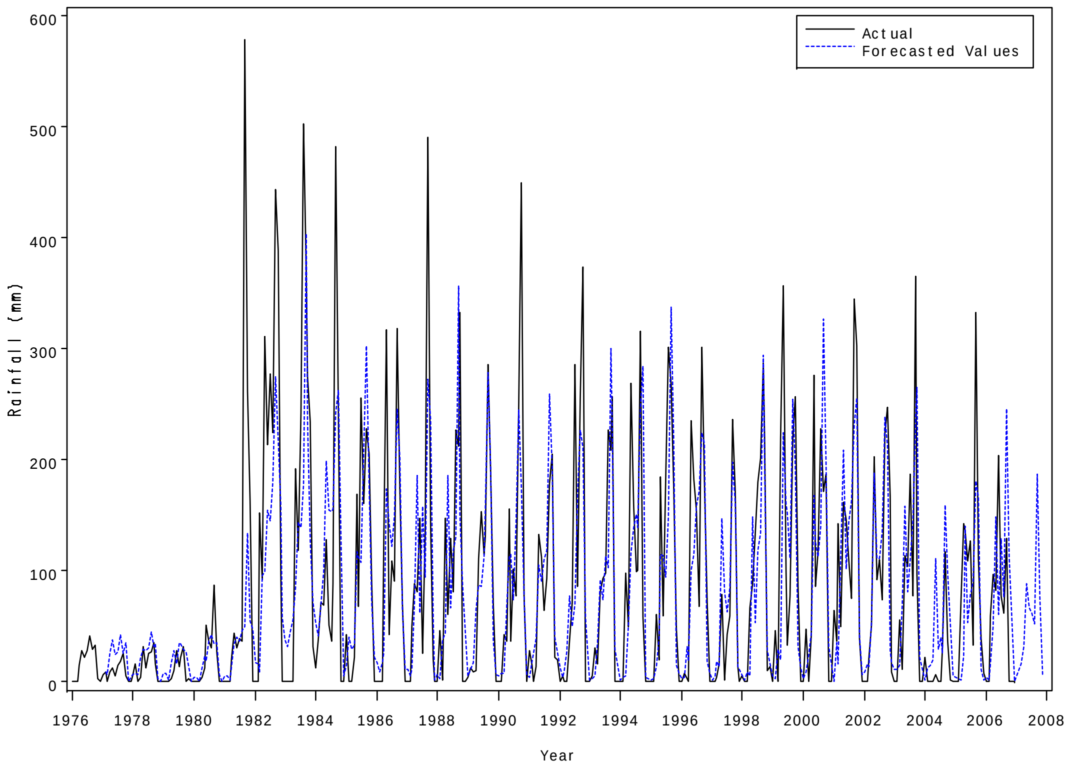


Figure 2.19 Actual and forecasted values for ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$of monthly rainfall at Nonthaburi location from January 1976 to December 2007

Bangkok Location

The monthly rainfall series of Bangkok location from January 1976 to December 2006 is plotted in Figure 2.20. It shows that this series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.41. The power transformation analysis indicates that it is not necessary to transform the data.

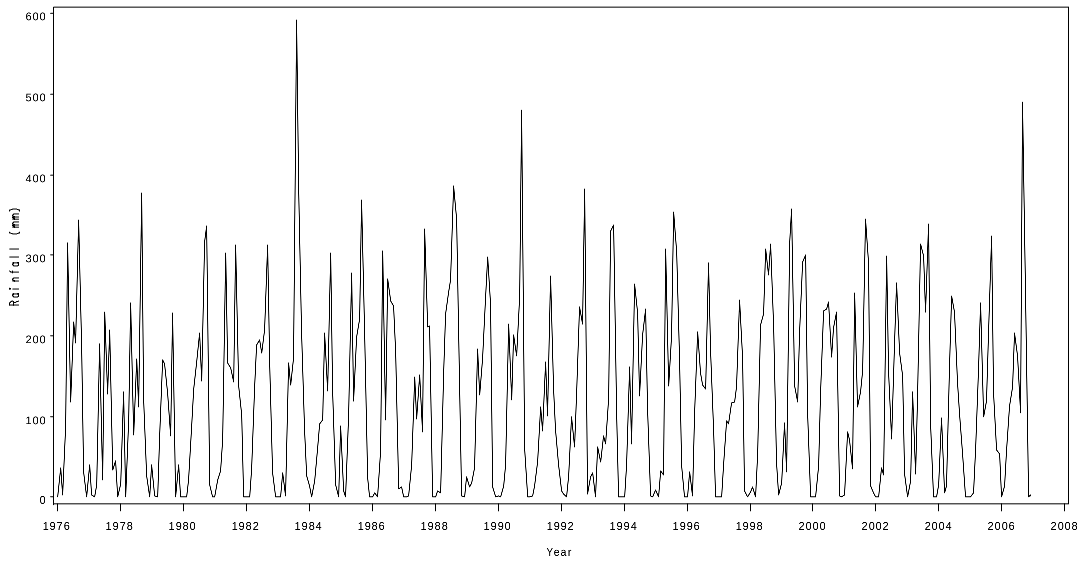


Figure 2.20 Monthly rainfall series at Bangkok location

Table 2.41 Residual mean square error from the power transformation of monthly rainfall series at Bangkok location

λ	Residual mean square error (RMSE)
-1.0	124292313.30
-0.5	313240216085.94
0	83118.00
0.5	8109.87
1.0	7841.06

The sample ACF shows a damping sine-cosine wave and the sample PACF has large spikes at multiples of seasonal period 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

We consider the sample ACF and PACF for seasonal differencing at period 12 months. The sample ACF has a large spike at lag 12 and the sample PACF shows large spikes at lag 12, at lag 24 and a possible one at lag 36. It indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are computed. The residual ACF and PACF are close to zero and we consider the first differenced residual of the ARIMA $(0, 1, 1)_{12}$ model.

The first differenced residual of the sample ACF cuts off after lag 1 and the sample PACF has large spikes at lag 1 or 4. Hence, all the tentative models are ARIMA (1, 1, 0) × (0, 1, 1)$_{12}$, ARIMA (4, 1, 0) × (0, 1, 1)$_{12}$, ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$, ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ and ARIMA (4, 1, 1) × (0, 1, 1)$_{12}$.

Table 2.42 Summary of tentative models for monthly rainfall at Bangkok location.

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 1, 0) × (0, 1, 1)$_{12}$		
	$\underset{(0.04793)}{(1+0.43872B)}(1-B)(1-B^{12})Z_t = (1-\underset{(0.02668)}{0.88263}B^{12})a_t$	6183.982
2. ARIMA (4, 1, 0) × (0, 1, 1)$_{12}$		
	$(1+\underset{(0.05205)}{0.63843}B+\underset{(0.06081)}{0.45793}B^2+\underset{(0.06091)}{0.29858}B^3+\underset{(0.05297)}{0.22579}B^4)(1-B^{12})Z_t =$	
	$(1-B)(1-\underset{(0.02740)}{0.87251}B^{12})a_t$	5322.739
3. ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$		
	$(1-B)(1-B^{12})Z_t = (1-\underset{(0.02605)}{0.87670}B)(1-\underset{(0.02709)}{0.87185}B^{12})a_t$	4909.798
4. ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$		
	$(1-\underset{(0.06016)}{0.15527}B)(1-B)(1-B^{12})Z_t = (1-\underset{(0.02585)}{0.91125}B)(1-\underset{(0.02901)}{0.8605}B^{12})a_t$	4830.067

The parameters of all tentative models are estimated and shown in Table 2.42, where the values in parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.43. Models 1 and 2 are not adequate for the monthly rainfall at Bangkok location because the residuals are not white noise.

For model selection, we use AIC and SBC to select the best model. From Table 2.44, model 4 has the minimum AIC and SBC. The model ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$ is selected to describe the monthly rainfall series at Bangkok location.

Table 2.43 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Bangkok location

Model	Q statistic	p-value
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$	52.94	0.0002
2. ARIMA (4, 1, 0) × (0, 1, 1) $_{12}$	32.62	0.0266
3. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	20.47	0.5539
4. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	13.60	0.8863

Table 2.44 Summary of AIC and SBC values for monthly rainfall at Bangkok location

Model	AIC (Rank)	SBC (Rank)
3. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$	4049.255 (2)	4057.011 (2)
4. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$	4044.403 (1)	4056.036 (1)

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Bangkok location are shown in Table 2.45 while actual and forecasted values are shown in Figure 2.21.

Table 2.45 Forecasted values for ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$ of monthly rainfall at Bangkok location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	3.7	5.4226	-176.9281	166.0828	-1.2774
Feb 2007	1	16.9730	-157.7238	191.6698	-15.973
Mar 2007	1	36.0593	-139.4871	211.6057	-35.059
Apr 2007	115.1	73.9337	-102.2328	250.1002	41.1663
May 2007	263.3	192.0030	15.2505	368.7556	71.297
June 2007	229.8	165.1739	-12.1580	342.5058	64.6261
July 2007	233.4	166.6020	-11.3066	344.5106	66.798
Aug 2007	152.5	174.6766	-3.8068	353.1599	-22.177
Sep 2007	254.3	292.4590	113.4028	471.5151	-38.159
Oct 2007	278.1	172.0350	-7.5922	351.6622	106.065
Nov 2007	23.1	34.2741	-145.9223	214.4705	-11.174
Dec 2007	1	43.9441	-204.0818	157.4459	-42.944

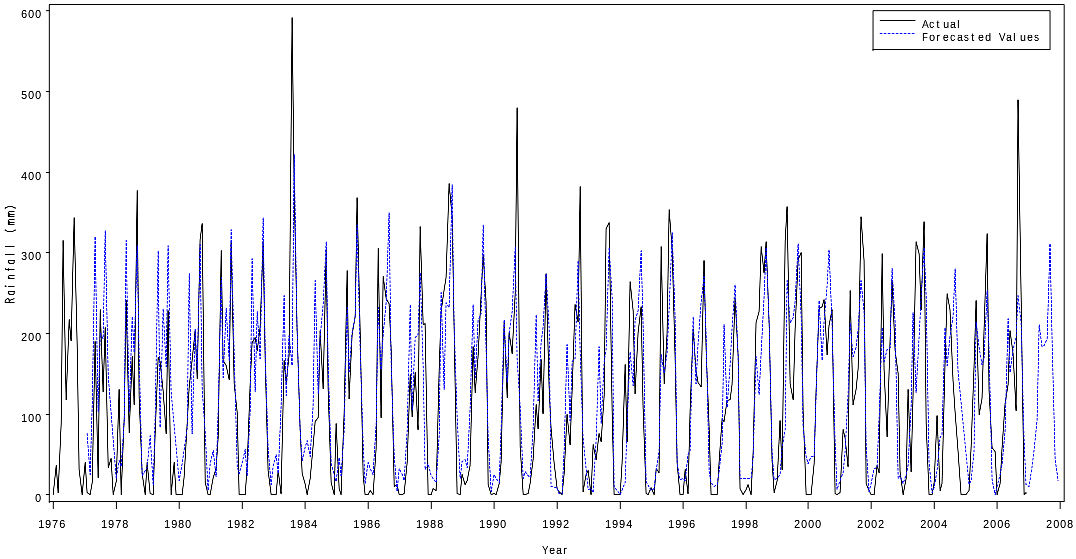


Figure 2.21 Actual and forecasted values for ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$ of monthly rainfall at Bangkok location from January 1976 to December 2007

Samutprakarn Location

The monthly rainfall series of Samutprakarn location from January 1976 to December 2006 is plotted in Figure 2.22. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.46. The power transformation analysis indicates that a square root transformation is needed. The transformation of the monthly rainfall series is shown in Figure 2.23.

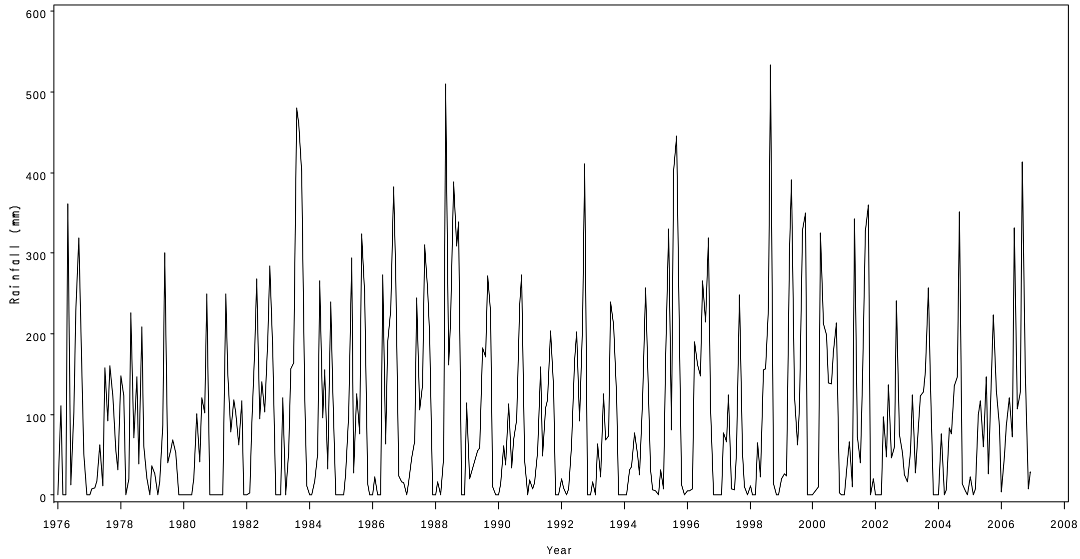


Figure 2.22 Monthly rainfall series at Samutprakarn location

Table 2.46 Residual mean square errors from the power transformation of monthly rainfall series at Samutprakarn location

λ	Residual mean square error (RMSE)
-1.0	96865055.01
-0.5	1013509000000000000
0	58835.01
0.5	10210.34
1.0	10238.47

We consider the sample ACF and sample PACF for square root transformed rainfall series. The sample ACF shows a damping sine-cosine wave and the sample PACF has large spike at multiples of seasonal period 12 months implying that a seasonal differencing $(1-B^{12})$ is needed in order to achieve stationarity.

The sample ACF and PACF for seasonal differencing at period 12 months are considered. The sample ACF has a large spike at lag 12 and the sample PACF shows large spikes at lag 12, at lag 24 and a possible one at lag 36. It indicates that the ARIMA $(0, 1, 1)_{12}$ model is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are computed. The residual ACF and PACF cut off at lag 1. Hence, the tentative models are ARIMA $(1, 0, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 0, 1) \times (0, 1, 1)_{12}$ and ARIMA $(1, 0, 1) \times (0, 1, 1)_{12}$.

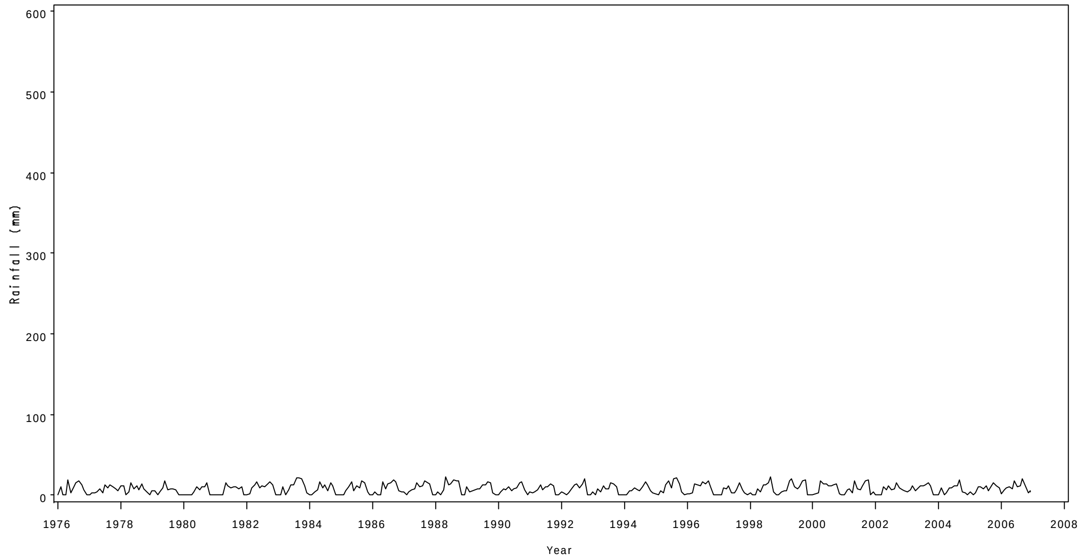


Figure 2.23 The square root transformed monthly rainfall series at Samutprakarn Location

Table 2.47 Summary of tentative models fitted for monthly rainfall at Samutprakarn location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 1, 0) × (0, 1, 1) $_{12}$		
	$(1+ 0.46546B)(1–B)\ (1-B^{12})\sqrt{Z_t} = (1-0.75055B^{12})\ a_t$ (0.04782) (0.03646)	21.964
2. ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$		
	$(1–0.88989B)(1- B^{12})\sqrt{Z_t} = (1 – B)(1– 0.78424\ B^{12})\ a_t$ (0.02412) (0.03368)	16.862
3. ARIMA (1, 1, 1) × (0, 1, 1) $_{12}$		
	$(1–0.12088\ B)(1–B)\ (1-B^{12})\sqrt{Z_t} =(1– B)(1– 0.80592\ B^{12})\ a_t$ (0.05624) (0.0092256) (0.03410)	16.801

The first differenced residuals of the sample ACF are considered. The sample ACF cuts off after lag 1 and the sample PACF has a large spike at lag 1. Hence, all tentative models are ARIMA (1, 1, 0)× (0, 1, 1)$_{12}$, ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$ and ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$. The parameters of all tentative models are estimated and shown in Table 2.47, where the values in the parentheses under the estimates refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level. The parameter θ_1 of model 3 is not significant, which implies that model 3 can be refitted with θ_1 removed if necessary. The model 3 becomes the tentative model 1.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.48. Model 1 is not adequate for the monthly rainfall at Samutprakarn location because the residuals are not white noise. Hence the best model is ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$. Table 2.49 shows summary of AIC and SBC values for the best model of monthly rainfall at Samutprakarn location.

Table 2.48 The Q statistic test for K = 24 of the tentative models for monthly rainfall at Samutprakarn location

Model	Q statistic	p-value
1. ARIMA (1, 1, 0) × (0, 1, 1)$_{12}$	78.20	0.0001
2. ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$	29.84	0.1224

Table 2.49 Summary of AIC and SBC values for monthly rainfall at Samutprakarn location

MODEL	AIC	SBC
2. ARIMA (0, 1, 1) × (0, 1, 1)$_{12}$	1978.367	1986.078

Table 2.50 Forecasted values for ARIMA (0, 1, 1) × (0, 1, 1)12 of monthly rainfall at Samutprakarn location

Date	Rainfall	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	43	8.408	24.812	123.556	34.592
Feb 2007	0	25.757	8.551	176.087	-25.757
Mar 2007	38.4	35.319	4.489	200.843	3.081
Apr 2007	158.8	83.550	1.003	302.362	75.25
May 2007	311.2	119.163	7.402	368.759	192.037
June 2007	282.9	147.172	15.084	418.666	135.728
July 2007	226.9	116.671	6.309	368.011	110.229
Aug 2007	69.7	120.957	7.078	377.384	-51.257
Sep 2007	122.1	334.339	97.666	715.130	-212.24
Oct 2007	175.6	103.752	3.093	350.100	71.848
Nov 2007	2.7	5.885	34.790	124.241	-3.185
Dec 2007	7.1	4.763	38.041	120.316	2.337

The one year ahead forecasted values with 95 percent forecast limits of the ARIMA (0, 1, 1) × (0, 1, 1) $_{12}$ model for monthly rainfall at Samutprakarn location are shown in Table 2.50 while actual and forecast values are shown in Figure 2.24.

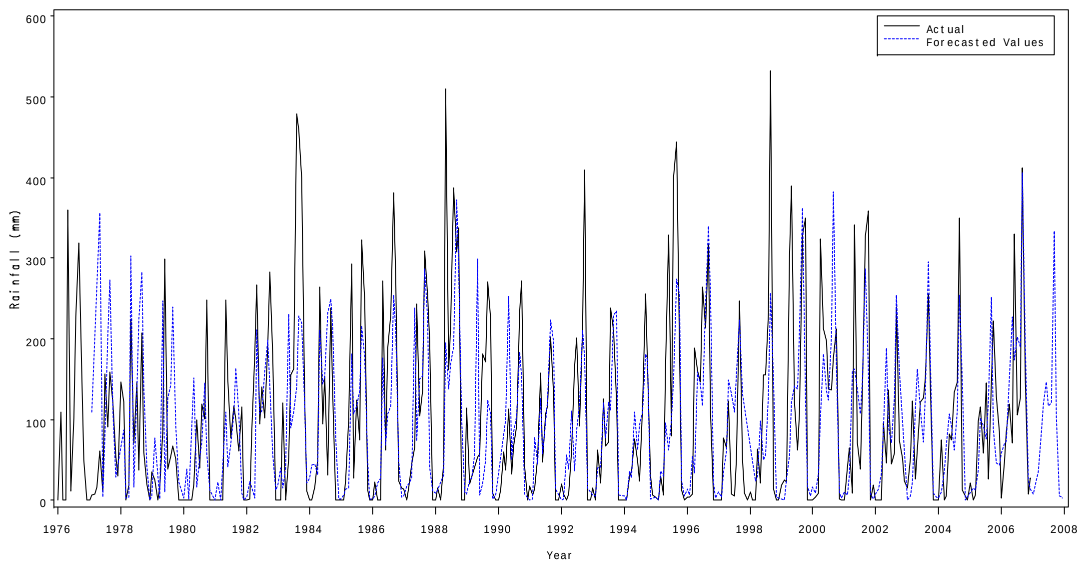


Figure 2.24 Actual and forecasted values for ARIMA (0, 1, 1) × (0, 1, 1)12 of monthly rainfall at Samutprakarn location from January 1976 to December 2007

2.5.2 Monthly Water level Data

The monthly water level series include data from four locations that is Nakhonsawan, Chainat, Singburi, and Angthong. The results of water level analysis for each location are shown as follows.

Nakhonsawan Location

The water level series of the Chao-Phraya River in Nakhonsawan location is plotted in Figure 2.25. This shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.51. The power transformation analysis indicates that it is not necessary to transform the data.

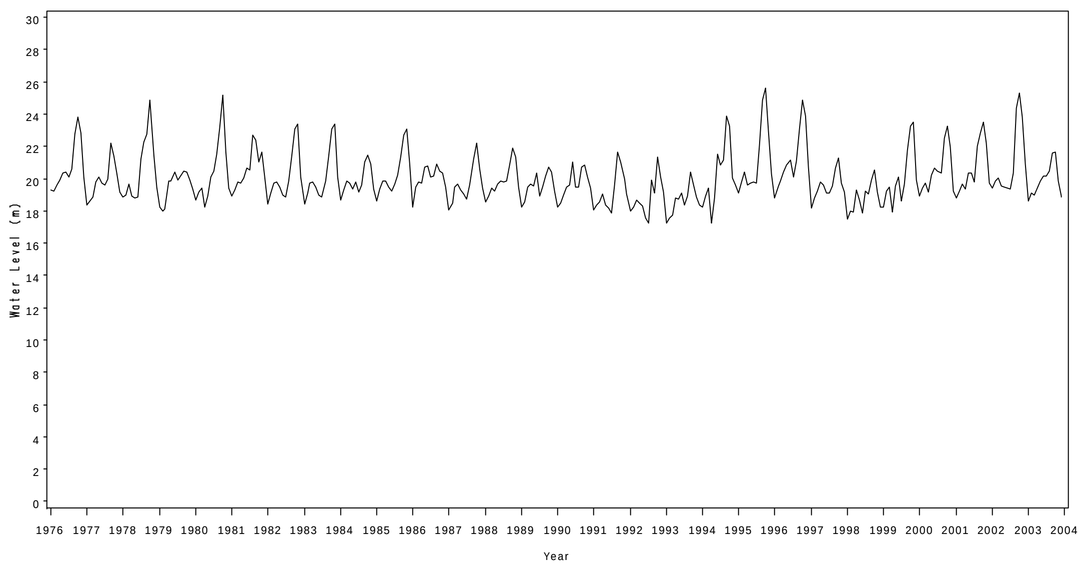


Figure 2.25 Water level of the Chao-Phraya River at Nakhonsawan location

Table 2.51 Residual mean square errors from the power transformation of water level at Nakhonsawan location

λ	Residual mean square error (RMSE)
-1.0	0.91753
-0.5	0.91254
0	0.90875
0.5	0.90624
1.0	0.90511

The sample ACF shows a damping sine-cosine wave and the sample PACF cuts off after lag 2. Hence the tentative model is AR(2). The estimates of parameters in the AR(2) model are shown in Table 2.52. To check model adequacy, we use a Q statistic. For K = 24, the Q statistic is 149.22, which is significant as $\chi^2_{0.05}(22) = 42.796$, the chi-square value at the significance level $\alpha = 0.05$ for the degrees of freedom = K-m = 24-2 = 22. Thus, we conclude that the AR(2) model is not adequate for the series. Next, we consider the sample ACF and PACF of the first differencing. The sample ACF shows large spikes at lag 12, 24, 36 and so on. Then we consider the differencing at period 12 months. The sample ACF and PACF of differencing at period 12 months. The sample ACF shows that it has a large spike at lag 12. To ensure that the first and twelfth differencing are needed, we consider the sample ACF and PACF of first and twelfth differencing. The sample ACF has a large spike at lag 12. Hence the model needs only the twelfth differencing. This indicates that the model ARIMA (0, 1, 1) $_{12}$ is appropriate.

Table 2.52 Summary of tentative models fitted for monthly water level at Nakhonsawan location

Model	Estimated parameters	$\hat{\sigma}^2_a$
1. AR (2)		
	$(1-1.03335B+0.47508B^2)Z_t = 20.01935 + a_t$ (0.04829) (0.04829) (0.11767)	0.916

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are computed. The residual ACF and residual PACF cut off after lag 1. The tentative models are ARIMA $(1, 0, 0) \times (0, 1, 1)_{12}$, ARIMA $(0, 0, 1) \times (0, 1, 1)_{12}$ and ARIMA $(1, 0, 1) \times (0, 1, 1)_{12}$.

Parameters of all tentative models are estimated and shown in Table 2.53, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level.

Table 2.53 Summary of tentative model fitted for monthly water level at Nakhonsawan location.

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$		
	$(1-0.69758B)(1-B^{12})\,Z_t = (1-0.88431B^{12})\,a_t$ (0.04000) (0.02653)	0.609
2. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-B^{12})Z_t = (1+0.61616B)(1-0.83628B^{12})\,a_t$ (0.04405) (0.03135)	0.708
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-0.61026\,B)(1-B^{12})Z_t = (1+0.16701\,B)\,(1+0.87965B^{12})\,a_t$ (0.06322) (0.07883) (0.02716)	0.604

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the *Q* statistic and p-values are given in Table 2.54. Models 1 and 2 are not adequate for the water level at Nakhonsawan location because the residuals are not white noise. Hence model 3 is the best model for the water level at Nakhonsawan location. Table 2.55 shows summary of AIC and SBC values for the best model of monthly water level at Nakhonsawan location.

Table 2.54 The Q statistic test for K = 24 of the tentative models for monthly water level at Nakhonsawan location.

Model	Q statistic	p-value
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	35.36	0.0355
2. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$	171.66	<0.0001
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	32.03	0.0582

Table 2.55 Summary of AIC and SBC values for monthly water level at Nakhonsawan location

Model	AIC	SBC
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	759.0837	770.426

The one year ahead forecast values with 95 percent forecast limits of the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model for monthly water level at Nakhonsawan location are shown in Table 2.56 while actual and forecasted values are shown in Figure 2.26.

Table 2.56 Forecasted values for ARIMA (1, 0, 1) × (1, 1, 0) $_{12}$ of water level at Nakhonsawan location

Date	Water Level	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2003	18.72	18.2085	16.6853	19.7317	0.5115
Feb 2003	18.82	18.8509	16.9217	20.7801	-0.0309
Mar 2003	18.81	19.2258	17.1658	21.2859	-0.4158
Apr 2003	18.85	19.1876	17.0809	21.2944	-0.3376
May 2003	19.02	19.5889	17.4651	21.7127	-0.5689
June 2003	20.39	19.8081	17.6780	21.9383	0.5819
July 2003	20.57	19.5107	17.3782	21.6432	1.0593
Aug 2003	21.42	20.3593	18.2259	22.4927	1.0607
Sep 2003	21.54	22.0773	19.9436	24.2110	-0.5373
Oct 2003	21.71	22.7555	20.6216	24.8893	-1.0455
Nov 2003	19.29	21.4914	19.3575	23.6253	-2.2014
Dec 2003	18.78	19.5850	17.4511	21.7189	-0.805

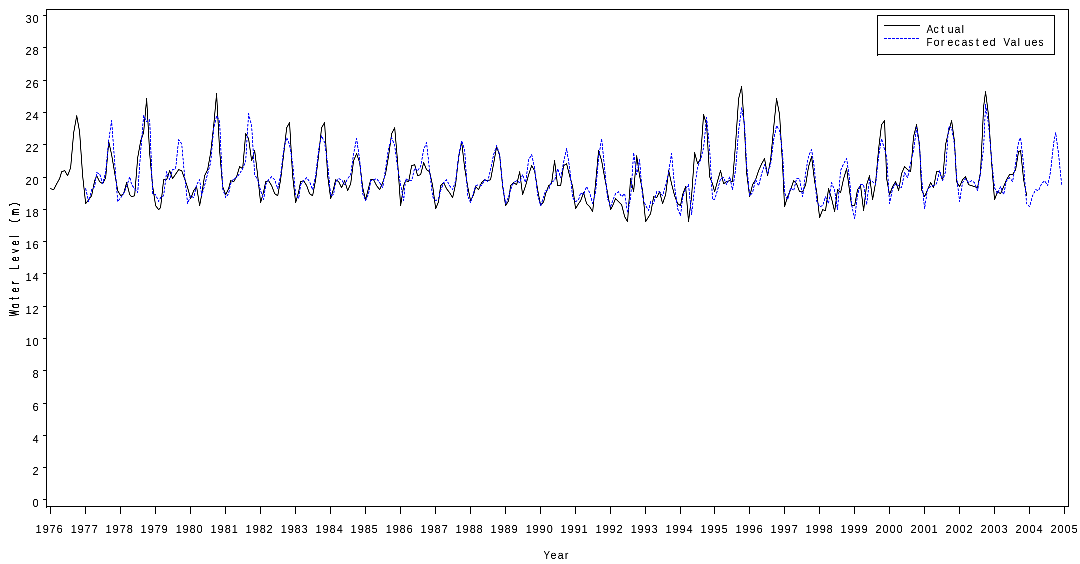


Figure 2.26 Actual and forecasted values for ARIMA (1, 0, 1) × (1, 1, 0) $_{12}$ of monthly water level at Nakhonsawan location from January 1976 to December 2003

Chainat Location

The water level series for the Chao Phraya River at Chainat location is plotted in Figure 2.27. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.57. The power transformation analysis indicates that a square root transformation is needed .

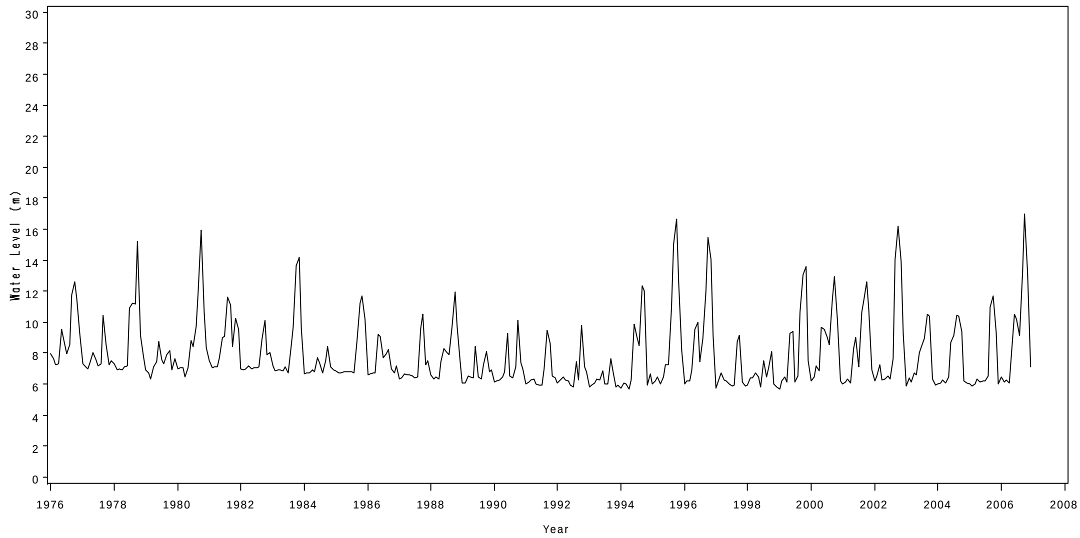


Figure 2.27 Monthly water level of the Chao-Phraya River at Chainat location

Table 2.57 Residual mean square errors from the power transformation of water level at Chainat location

λ	Residual mean square error (RMSE)
-1.0	2.22160
-0.5	2.17622
0	2.14040
0.5	2.12122
1.0	2.12632

The square root transformed water level of Chao Phraya River at Chainat location series is plotted. The sample ACF shows a damped sine-cosine wave and the sample PACF cuts off after lag 1. The appropriate model is AR(1). We compute the Q-statistic of the AR(1) model; it indicates that the model is not appropriate. Next, we consider the sample ACF and PACF of the first differencing. The sample ACF shows large spikes at lag 12, 24, 36 and so on. Then we consider the differencing at period 12 months. We considered the sample ACF and PACF of differencing at period 12 months. The sample ACF shows that it has a large spike at lag 12. To ensure that the first and twelfth differencing are needed, we consider the sample ACF and PACF of first and twelfth differencing. The sample ACF has a large spike at lag 12. Hence the model needs only the twelfth differencing. This indicates that the model ARIMA $(0, 1, 1)_{12}$ is appropriate.

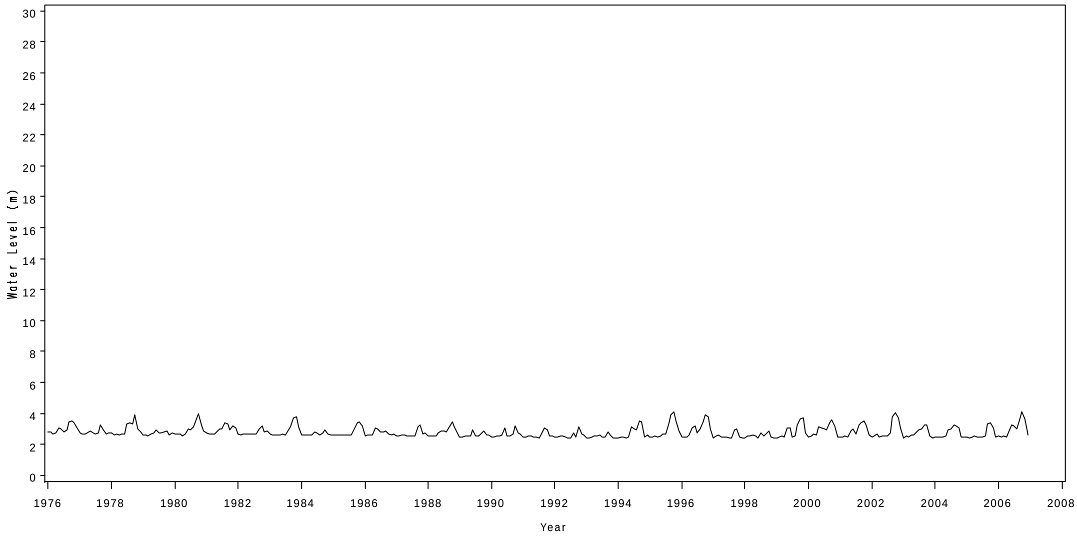


Figure 2.28 The square root transformed monthly water level of the Chao-Phraya River at Chainat location

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are computed. The residual ACF tails off after lag 2 and the residual PACF cuts off after lag 1. Hence the tentative models are ARIMA $(1, 0, 0)\times(0, 1, 1)_{12}$, ARIMA $(0, 0, 2)\times(0, 1, 1)_{12}$ and ARIMA $(1, 0, 2)\times(0, 1, 1)_{12}$.

Table 2.58 Summary of tentative models fitted for water level at Chainat location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$		
	$(1-0.67008\,B)(1-B^{12})\sqrt{Z_t} = (1 - 0.85349B^{12})\,a_t$ (0.03926) (0.02908)	0.044
2. ARIMA (0, 0, 2) × (0, 1, 1) $_{12}$		
	$(1-B^{12})\sqrt{Z_t} = (1+0.72921\,B+0.26594\,B^2)(1-0.83489B^{12})\,a_t$ (0.05112) (0.05131) (0.03052)	0.045
3. ARIMA (1, 0, 2) × (0, 1, 1) $_{12}$		
	$(1-0.62231B)(1-B^{12})\sqrt{Z_t} = (1+0.12762B-0.0561\,B^2)(1-0.84683B^{12})\,a_t$ (0.10871) (0.12365) (0.09414) (0.02971)	0.043

The parameters of all tentative models are estimated and shown in Table 2.58, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters for all models are considered at 0.05 significance level. The parameters ϕ_1 and ϕ_2 of model 3 are not significant. If we eliminate ϕ_1 and ϕ_2 from model 3, it becomes model 1.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.59. Model 2 is not adequate for the water level at Chainat location because the residuals are not white noise. Hence the model 3 is the best model for the water level series at Chainat location. Table 2.60 shows summary of AIC and SBC values for the best model of water level at Chainat location

Table 2.59 The Q statistic test for K = 24 of the tentative models for water level at Chainat location

Model	*Q* statistic	p-value
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	28.56	0.1578
2. ARIMA (0, 0, 2) × (0, 1, 1) $_{12}$	48.41	0.0006

Table 2.60 Summary of AIC and SBC values for water level at Chainat location

Model	AIC	SBC
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	-104.013	-96.2412

Table 2.61 Forecasted values for ARIMA (1, 0, 0) × (0, 1, 1)$_{12}$ of water level at Chainat location

Date	Water Level	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	6	6.3646	4.4668	8.5975	-0.3646
Feb 2007	6.04	6.4027	4.1519	9.1390	-0.3627
Mar 2007	5.98	6.5280	4.1173	9.4918	-0.548
Apr 2007	6.21	6.4064	3.9639	9.4322	-0.1964
May 2007	10.35	7.2096	4.5740	10.4422	3.1404
June 2007	8.89	8.1293	5.2993	11.5625	0.7607
July 2007	9.36	7.7141	4.9596	11.0746	1.6459
Aug 2007	8.42	8.2625	5.3985	11.7336	0.1575
Sep 2007	12.13	11.0002	7.6476	14.9605	1.1298
Oct 2007	13.61	12.2196	8.6690	16.3783	1.3904
Nov 2007	9.38	9.4668	6.3778	13.1639	-0.0868
Dec 2007	6.02	6.8502	4.2679	10.0408	-0.8302

The one year ahead forecasted values with 95 percent forecast limits of the ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$ model for water level at Chainat location are shown in Table 2.61 while actual and forecasted values are shown in Figure 2.29.

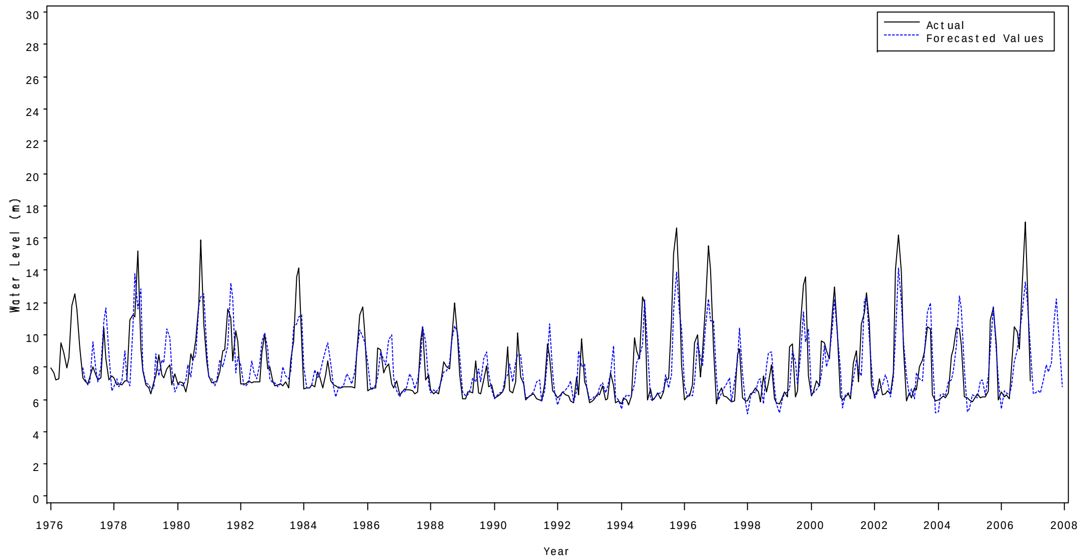


Figure 2.29 Actual and forecasted values for ARIMA (1, 0, 0) × (0, 1, 1)$_{12}$ of monthly water level at Chainat location from January 1976 to December 2007

Singburi Location

The water level series of the Chao-Phraya River at Singburi location is plotted in Figure 2.30. This shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.62. The power transformation analysis indicates that a square root transformation is needed.

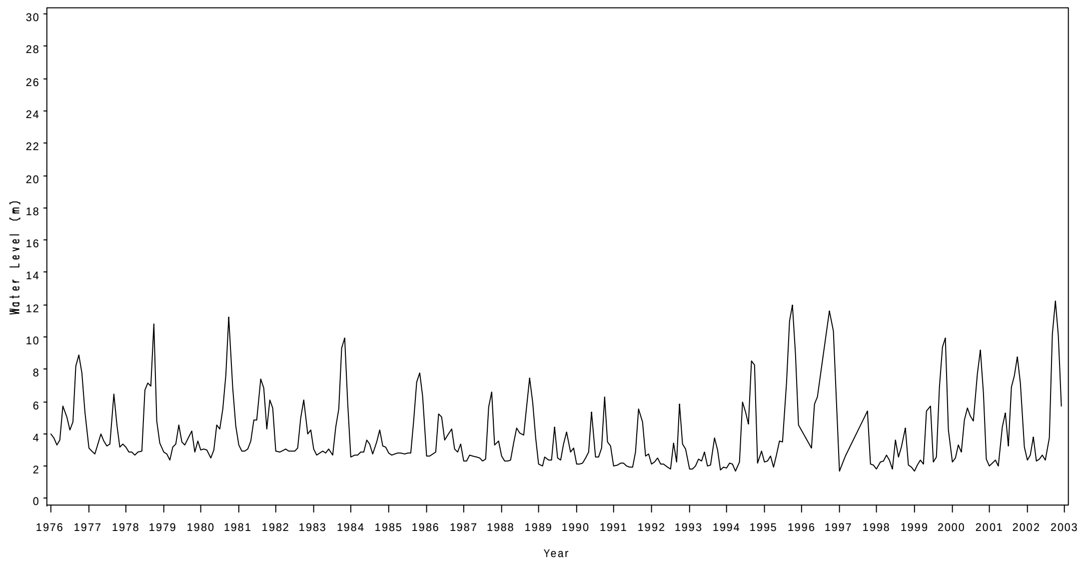


Figure 2.30 Water level of the Chao-Phraya River at Singburi location

Table 2.62 Residual mean square errors from the power transformation of water level at Singburi location

λ	Residual mean square error (RMSE)
-1.0	2.45988
-0.5	2.36734
0	2.30177
0.5	2.25589
1.0	2.25791

The square root transform of water level series of the Chao-Phraya River at Singburi location is plotted, the sample ACF shows a damped sine-cosine wave and the sample PACF cuts off after lag 1. Thus the appropriate model is AR(1). The tentative model AR(1) is not appropriate because for $K = 24$, the Q-statistic is $Q = 205.58$, which is significant as

$\chi^2_{0.05}(23) = 35.17$. Next, we consider the sample ACF and PACF of the first differencing. The sample ACF shows large spikes at lag 12, 24, 36 and so on. Then we consider the differencing at period 12 months. The sample ACF shows that it has a large spike at lag 12. To ensure that the first and twelfth differencing are needed, we consider the sample ACF and PACF of the first and twelfth differencing. The sample ACF has a large spike at lag 12. Hence the model needs only the twelfth differencing. This indicates that the model ARIMA (0, 1, 1) $_{12}$ is appropriate.

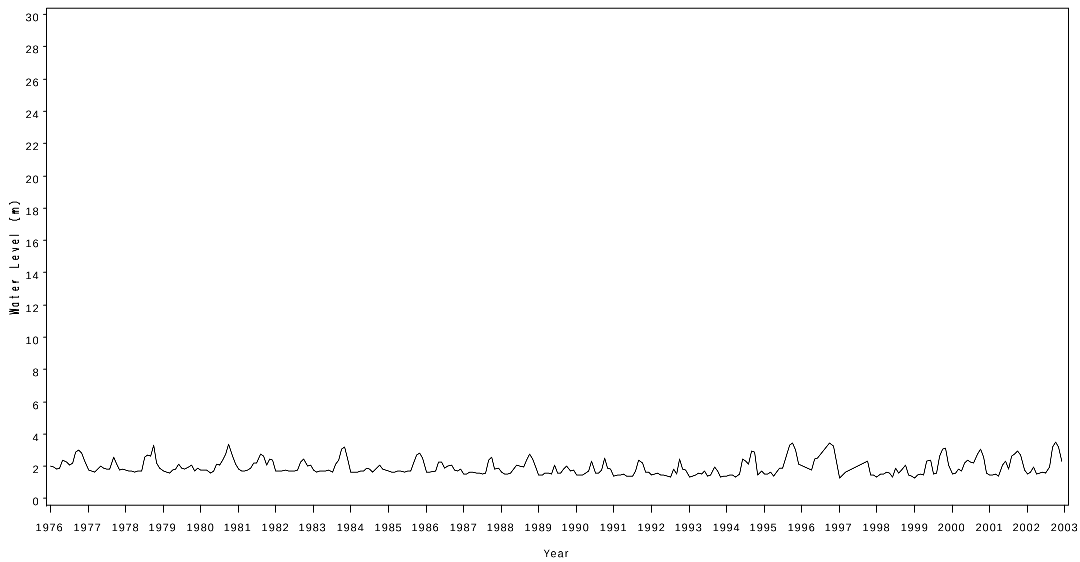


Figure 2.31 The square root transform of the water level of the Chao-Phraya River at Singburi location

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are computed. The residual ACF tails off after lag 1 and the residual PACF cuts off after lag 1. Hence the tentative models are ARIMA $(1, 0, 0)\times(0, 1, 1)_{12}$, ARIMA $(0, 0, 1)\times(0, 1, 1)_{12}$ and ARIMA $(1, 0, 1)\times(0, 1, 1)_{12}$.

The parameters for all tentative models are estimated and shown in Table 2.63, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level.

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.64. The model 2 is not adequate for the water level at Singburi location because the residuals are not white noise.

According to AIC and SBC, in Table 2.65, we see that the ARIMA(1, 0, 1)×(0, 1, 1)$_{12}$ has minimum AIC while the ARIMA(1, 0, 0)×(0, 1, 1)$_{12}$ has minimum SBC.

Alternatively, we consider the minimum variance of estimate. We found that variance of estimate of ARIMA(1, 0, 0)×(0, 1, 1)$_{12}$ equals the variance of estimate of ARIMA (1, 0, 1)×(0, 1, 1)$_{12}$. Hence we choose the model ARIMA(1, 0, 0)×(0, 1, 1)$_{12}$ to describe the water level series at Singburi location due to parsimony parameters.

Table 2.63 Summary of estimated parameters for tentative models of water level at Singburi location

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 0, 0) × (0, 1, 1)$_{12}$		
	$(1-0.60132\,B)(1-B^{12})\sqrt{Z_t} = (1 - 0.71133B^{12})\,a_t$ (0.04724) (0.04359)	0.098
2. ARIMA (0, 0, 1) × (0, 1, 1)$_{12}$		
	$(1-B^{12})\sqrt{Z_t} = (1+0.60889B)(1-0.68609B^{12})\,a_t$ (0.04726) (0.04866)	0.104
3. ARIMA (1, 0, 1) × (0, 1, 1)$_{12}$		
	$(1-0.46291B)(1-B^{12})\sqrt{Z_t} = (1+0.21255B)(1-0.69923B^{12})\,a_t$ (0.08283) (0.09903) (0.04789)	0.098

Table 2.64 The Q statistic test for K = 24 of the tentative models for water level at Singburi location

Model	Q statistic	p-value
1. ARIMA (1, 0, 0) × (0, 1, 1)$_{12}$	29.60	0.1285
2. ARIMA (0, 0, 1) × (0, 1, 1)$_{12}$	60.31	<0.0001
3. ARIMA (1, 0, 1) × (0, 1, 1)$_{12}$	26.99	0.1712

Table 2.65 Summary of AIC and SBC values for water level at Singburi location

Model	AIC (Rank)	SBC (Rank)
1. ARIMA (1, 0, 0) × (0, 1, 1)$_{12}$	150.5144 (2)	157.8263 (1)
3. ARIMA (1, 0, 1) × (0, 1, 1)$_{12}$	150.4274 (1)	161.3954 (2)

The one year ahead forecasted values with 95 percent forecast limits of the ARIMA(1, 0, 0)×(0, 1, 1) $_{12}$ model for water level at Singburi location are shown in Table 2.66 while actual and forecasted values are shown in Figure 2.32.

Table 2.66 Forecasted values for ARIMA (1, 0, 0) × (0, 1, 1)$_{12}$ of water level at Singburi location

Date	Water Level	Forecast	95% Confidence Limit		Error
			Lower	Upper	
Jan 2003	1.89	3.1055	1.31651	5.6507	-1.2155
Feb 2003	2.4	3.0154	1.03841	6.0219	-0.6154
Mar 2003	2.11	3.3643	1.17305	6.6839	-1.2543
Apr 2003	2.89	2.6699	0.75876	5.7452	0.2201
May 2003	2.81	3.5858	1.26893	7.0797	-0.7758
June 2003	4.19	3.6714	1.31658	7.2079	0.5186
July 2003	4.78	3.2681	1.07865	6.6410	1.5119
Aug 2003	5.31	4.0116	1.52147	7.6857	1.2984
Sep 2003	6.76	6.9196	3.46344	11.5601	-0.1596
Oct 2003	6.83	9.2170	5.13676	14.4816	-2.387
Nov 2003	2.5	7.0930	3.58625	11.7840	-4.593
Dec 2003	1.97	3.7626	1.36938	7.3401	-1.7926

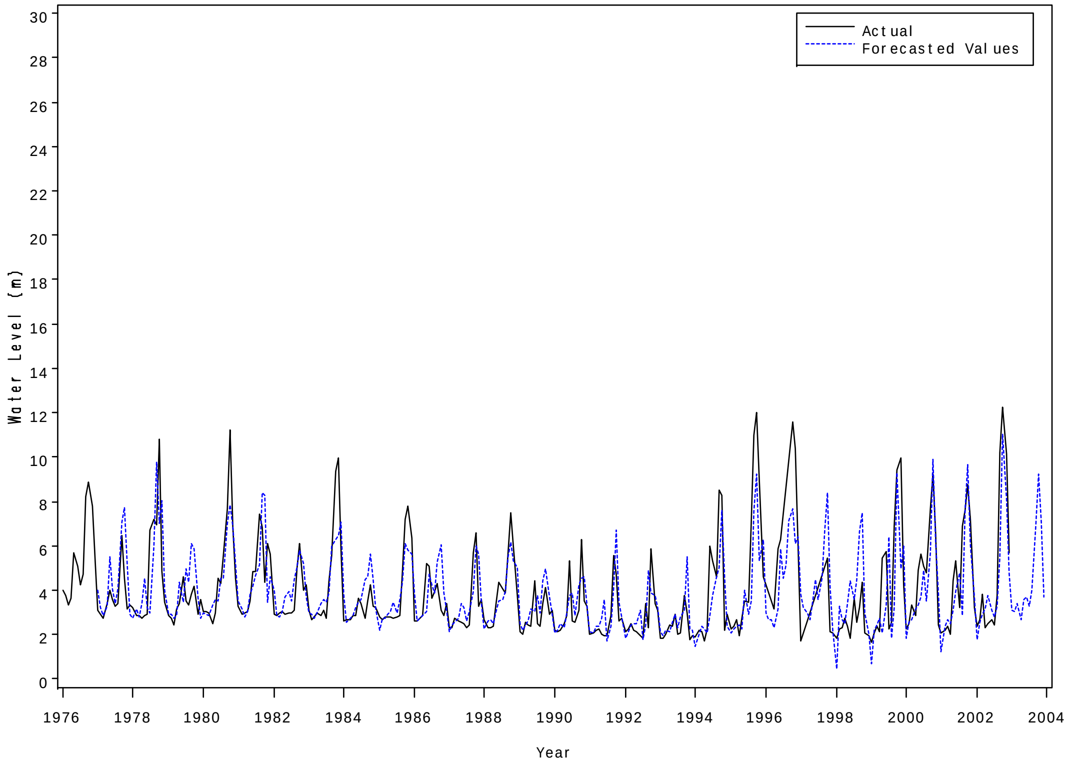


Figure 2.32 Actual and forecasted values of monthly water level at Singburi location from January 1976 to December 2003

Angthong Location

The water level series of the Chao-Phraya River at Angthong location is plotted in Figure 2.33. It shows that the series is stationary in the mean. To investigate whether the series is stationary in the variance, we use the Box-Cox power transformation. The preliminary residual mean square errors are calculated by using the power transformation via the SAS® system software version 9.1 as shown in Table 2.67. The power transformation analysis indicates that it is not necessary to transform the data.

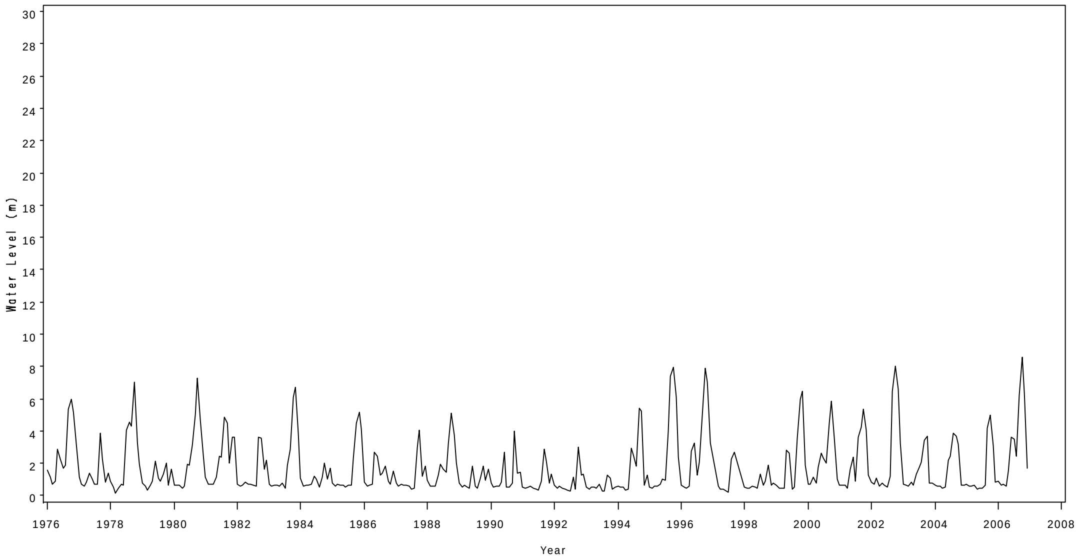


Figure 2.33 The water level of the Chao-Phraya River at Angthong location

Table 2.67 Residual mean square errors from the power transformation of water level at Angthong location

λ	Residual mean square error (RMSE)
-1.0	217664484.87
-0.5	2.06
0	1.49
0.5	1.37
1.0	1.32

The sample ACF shows a damped sine-cosine wave and the sample PACF cuts off after lag 1. Thus the appropriate model is AR(1). The tentative model AR(1) is not appropriate because for K = 24, the Q-statistic is Q = 174.48, which is significant as $\chi^2_{0.05}(23) = 35.17$. Next, we consider the sample ACF and PACF of the first differencing. The sample ACF shows large spikes at lag 12, 24, 36 and so on. Then we consider the differencing at period 12 months. The sample ACF shows that it has a large spike at lag 12. To ensure that the first and twelfth differencing are needed, we consider the sample ACF and PACF of first and twelfth differencing. The sample ACF has a large spike only at lag 12. Hence the model needs only the twelfth differencing. This indicates that the ARIMA $(0, 1, 1)_{12}$ model is appropriate.

To identify the nonseasonal components of the ARIMA model, the residual ACF and PACF of ARIMA $(0, 1, 1)_{12}$ are considered. The residual ACF tails off after lag 1 and the residual PACF also cuts off after lag 1. Hence the tentative models are ARIMA (1, 0, 0)×(0, 1, 1)$_{12}$, ARIMA (0, 0, 1)×(0, 1, 1)$_{12}$ and ARIMA (1, 0, 1)×(0, 1, 1)$_{12}$.

The parameters of all tentative models are estimated and shown in Table 2.68, where the values in the parentheses under each estimate refer to the standard errors of those estimates. The estimation of parameters of all models are considered at 0.05 significance level.

Table 2.68 Summary of tentative model fitted for water level at Angthong location.

Model	Estimated parameters	$\hat{\sigma}_a^2$
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$		
	$(1-0.66394B)(1-B^{12})Z_t = (1-0.81532B^{12})a_t$ (0.03872) (0.03341)	1.103
2. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-B^{12})Z_t = (1+0.60708B)(1-0.75416B^{12})a_t$ (0.04308) (0.03828)	1.223
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$		
	$(1-0.5295B)(1-B^{12})Z_t = (1+0.24755B)(1-0.80922B^{12})a_t$ (0.06604) (0.07776) (0.03378)	1.076

To check model adequacy, we consider whether the residuals of the model are white noise by using a Q statistic test with K = 24. The values of the Q statistic and p-values are given in Table 2.69. The model 2 is not adequate for the water level at Angthong location because the residuals are not white noise.

For model selection, we use two criteria, AIC and SBC, to select the best model. From Table 2.70, we see that the minimum AIC and SBC occur for the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model. Hence the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model is used to describe the monthly water level series at Angthong location.

Table 2.69 The Q statistic test for K = 24 of the tentative models for water level at Angthong location

Model	Q statistic	p-value
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	27.02	0.2106
2. ARIMA (0, 0, 1) × (0, 1, 1) $_{12}$	64.73	<0.0001
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	20.96	0.4613

Table 2.70 Summary of AIC and SBC values for water level at Angthong location

Model	AIC (Rank)	SBC (Rank)
1. ARIMA (1, 0, 0) × (0, 1, 1) $_{12}$	1029.560 (2)	1037.276 (2)
3. ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$	1021.894 (1)	1033.468 (1)

Table 2.71 Forecasted values for ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ of water level at Angthong location

Date	Water Level	Forecasted Values	95% Confidence Limit		Error
			Lower	Upper	
Jan 2007	0.86	0.6185	-1.4147	2.6516	0.2415
Feb 2007	0.93	0.5466	-2.0281	3.1214	0.3834
Mar 2007	0.74	0.6349	-2.0723	3.3422	0.1051
Apr 2007	0.68	0.5626	-2.1806	3.3058	0.1174
May 2007	3.92	1.0234	-1.7298	3.7766	2.8966
June 2007	2.38	1.8732	-0.8828	4.6292	0.5068
July 2007	2.93	1.6393	-1.1175	4.3962	1.2907
Aug 2007	2.04	2.0241	-0.7329	4.7812	0.0159
Sep 2007	5.65	4.4117	1.6546	7.1688	1.2383
Oct 2007	6.87	5.3782	2.6210	8.1353	1.4918
Nov 2007	3.39	3.3949	0.6378	6.1520	-0.0049
Dec 2007	1.45	1.3285	-1.4286	4.0856	0.1215

The one year ahead forecasted values with 95 percent forecast limits of the ARIMA (1, 0, 1) × (0, 1, 1) $_{12}$ model for water level at Angthong location are shown in Table 2.70 while actual and forecast values are shown in Figure 2.34.

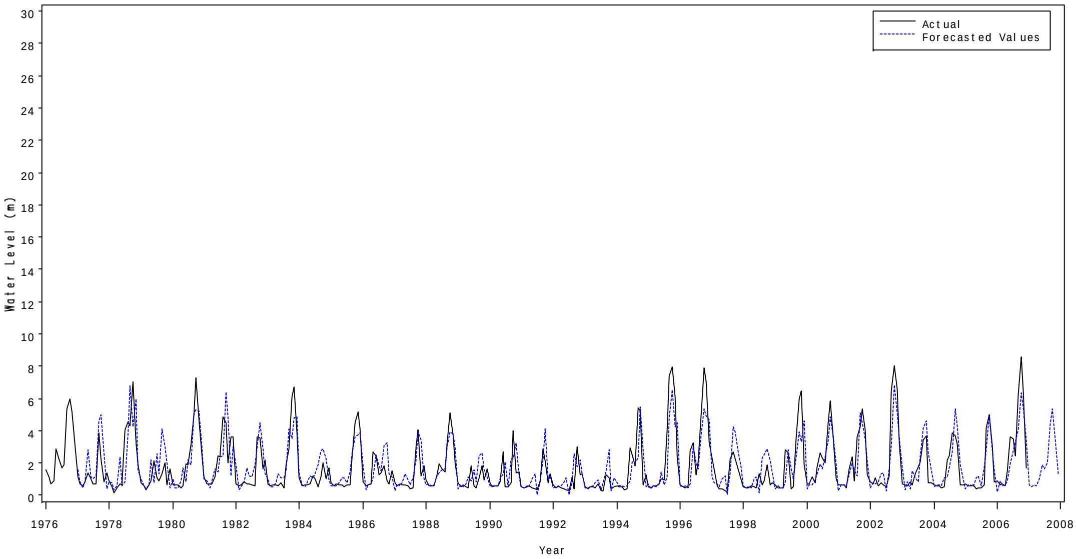
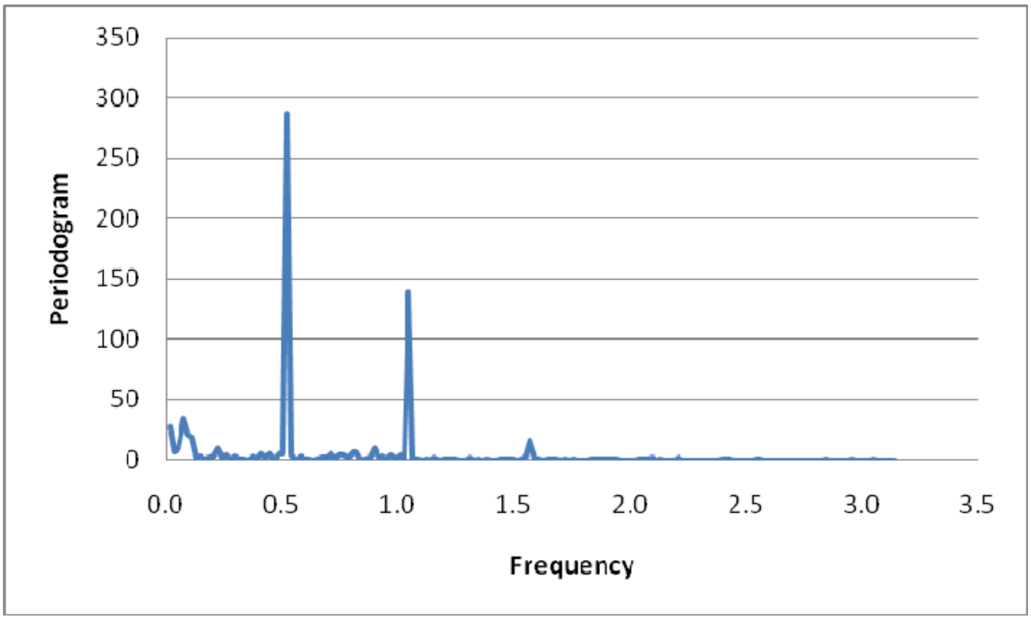


Figure 2.34 Actual and forecasted values for ARIMA (1, 0, 1) × (0, 1, 1)12 of monthly water level at Angthong location from January 1976 to December 2007

Periodogram Analysis

Periodogram analysis of rainfall in Nakhonsawan

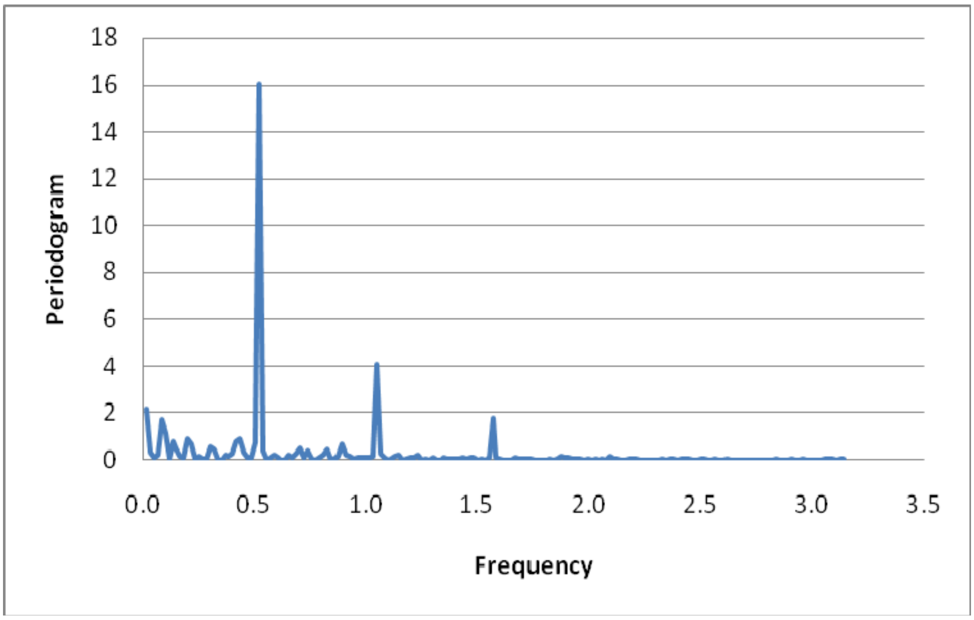


The periodogram is clearly dominated by a very large peak at frequency

$\omega_{28} = \frac{2\pi(28)}{336} = 0.52360$, $\omega_{56} = \frac{2\pi(56)}{336} = 1.04720$, and $\omega_{84} = \frac{2\pi(84)}{336} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_{28}} = 12$ months, $P = \frac{2\pi}{\omega_{56}} = 6$ months, and $P = \frac{2\pi}{\omega_{84}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,333) = 3.00$, and the periodogram is significant only at frequency $\omega_{28} = 0.52360$, $\omega_{56} = 1.04720$, and $\omega_{84} = 1.57080$.

Periodogram analysis of water level in Chainat

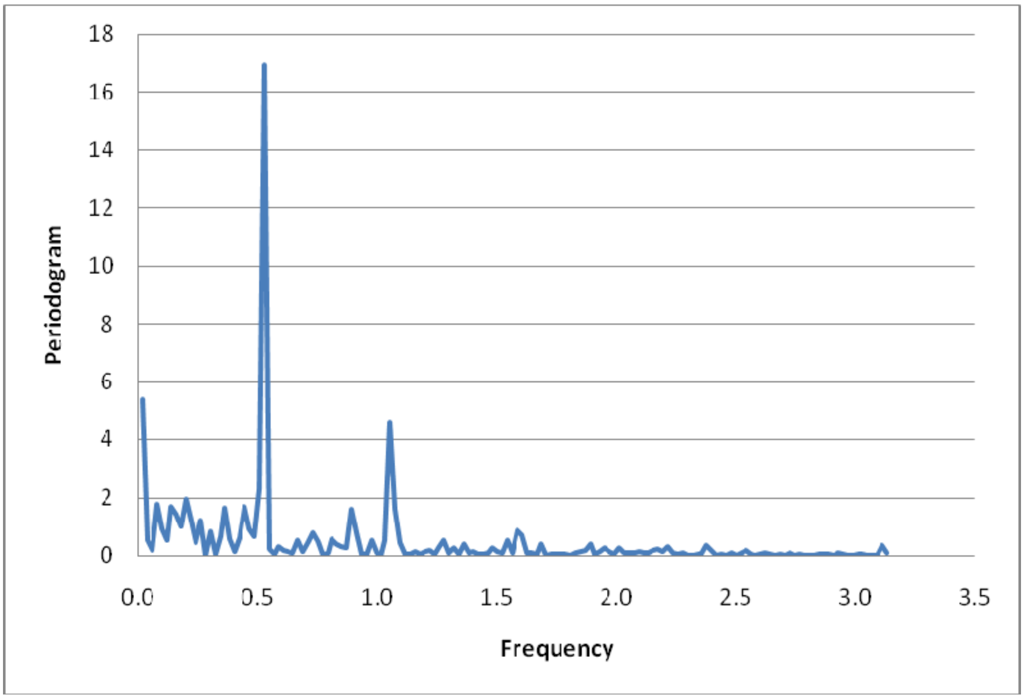


The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \frac{2\pi(1)}{372} = 0.01689$, $\omega_5 = \frac{2\pi(5)}{372} = 0.08445$, $\omega_6 = \frac{2\pi(6)}{372} = 0.10134$, $\omega_8 = \frac{2\pi(8)}{372} = 0.13512$, $\omega_{12} = \frac{2\pi(12)}{372} = 0.20268$, $\omega_{25} = \frac{2\pi(25)}{372} = 0.42226$, $\omega_{26} = \frac{2\pi(26)}{372} = 0.43915$, $\omega_{31} = \frac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \frac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \frac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_1} = 372$ months, $P = \frac{2\pi}{\omega_5} = 74.4$ months, $P = \frac{2\pi}{\omega_6} = 62$ months, $P = \frac{2\pi}{\omega_8} = 46.5$ months, $P = \frac{2\pi}{\omega_{12}} = 31$ months, $P = \frac{2\pi}{\omega_{25}} = 14.88$ months, $P = \frac{2\pi}{\omega_{26}} = 14.31$ months, $P = \frac{2\pi}{\omega_{31}} = 12$ months, $P = \frac{2\pi}{\omega_{62}} = 6$ months, and $P = \frac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,369) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.01689$, $\omega_5 = 0.08445$, $\omega_6 = 0.10134$, $\omega_8 = 0.13512$, $\omega_{12} = 0.20268$, $\omega_{25} = 0.42226$, $\omega_{26} = 0.43915$, $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of water level in Singburi



The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \frac{2\pi(1)}{309} = 0.02033$, $\omega_4 = \frac{2\pi(14)}{309} = 0.08134$, $\omega_7 = \frac{2\pi(7)}{309} = 0.14234$, $\omega_8 = \frac{2\pi(8)}{309} = 0.16267$,

$\omega_{10} = \frac{2\pi(10)}{309} = 0.20334$, $\omega_{18} = \frac{2\pi(18)}{309} = 0.36601$, $\omega_{22} = \frac{2\pi(22)}{309} = 0.44735$,

$\omega_{25} = \frac{2\pi(25)}{309} = 0.50835$, $\omega_{26} = \frac{2\pi(26)}{309} = 0.52868$, $\omega_{44} = \frac{2\pi(44)}{309} = 0.89469$,

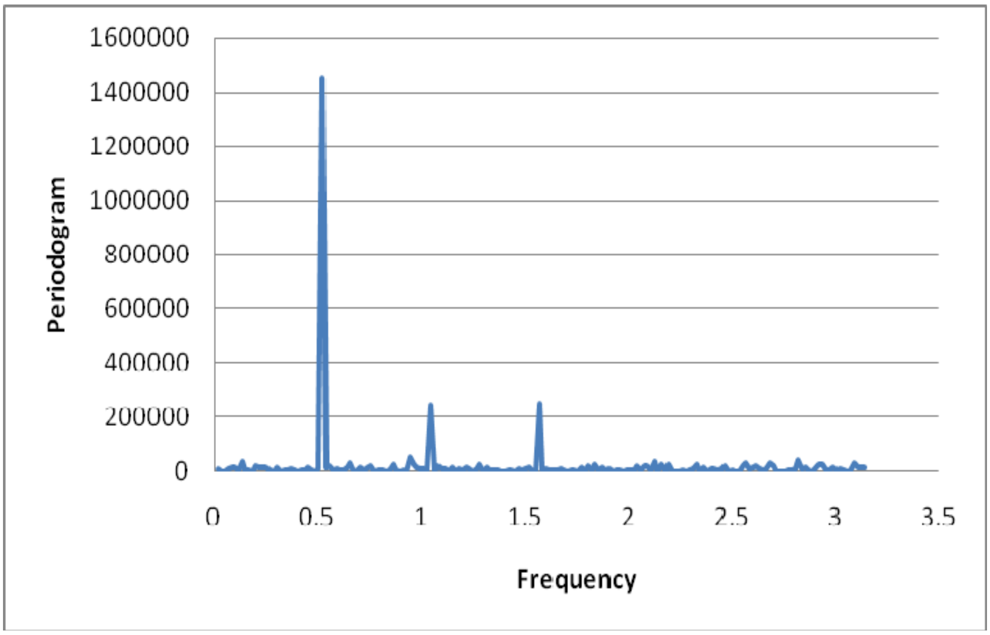
$\omega_{52} = \frac{2\pi(52)}{309} = 1.05737$, and $\omega_{53} = \frac{2\pi(53)}{309} = 1.07770$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_1} = 309$ months, $P = \frac{2\pi}{\omega_4} = 77.25$ months, $P = \frac{2\pi}{\omega_7} = 44.14$ months, $P = \frac{2\pi}{\omega_8} = 38.63$ months, $P = \frac{2\pi}{\omega_{10}} = 30.90$ months, $P = \frac{2\pi}{\omega_{18}} = 17.17$

months, $P = \frac{2\pi}{\omega_{22}} = 14.05$ months, $P = \frac{2\pi}{\omega_{25}} = 12.36$ months, $P = \frac{2\pi}{\omega_{26}} = 11.88$ months, $P = \frac{2\pi}{\omega_{44}} = 7.02$ months, $P = \frac{2\pi}{\omega_{52}} = 5.94$ months, and $P = \frac{2\pi}{\omega_{53}} = 5.83$ months.

It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,306) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.02033$, $\omega_4 = 0.08134$, $\omega_7 = 0.14234$, $\omega_8 = 0.16267$, $\omega_{10} = 0.20334$, $\omega_{18} = 0.36601$, $\omega_{22} = 0.44735$, $\omega_{25} = 0.50835$, $\omega_{26} = 0.52868$, $\omega_{44} = 0.89469$, $\omega_{52} = 1.05737$, and $\omega_{53} = 1.07770$.

Periodogram analysis of rainfall in Angthong

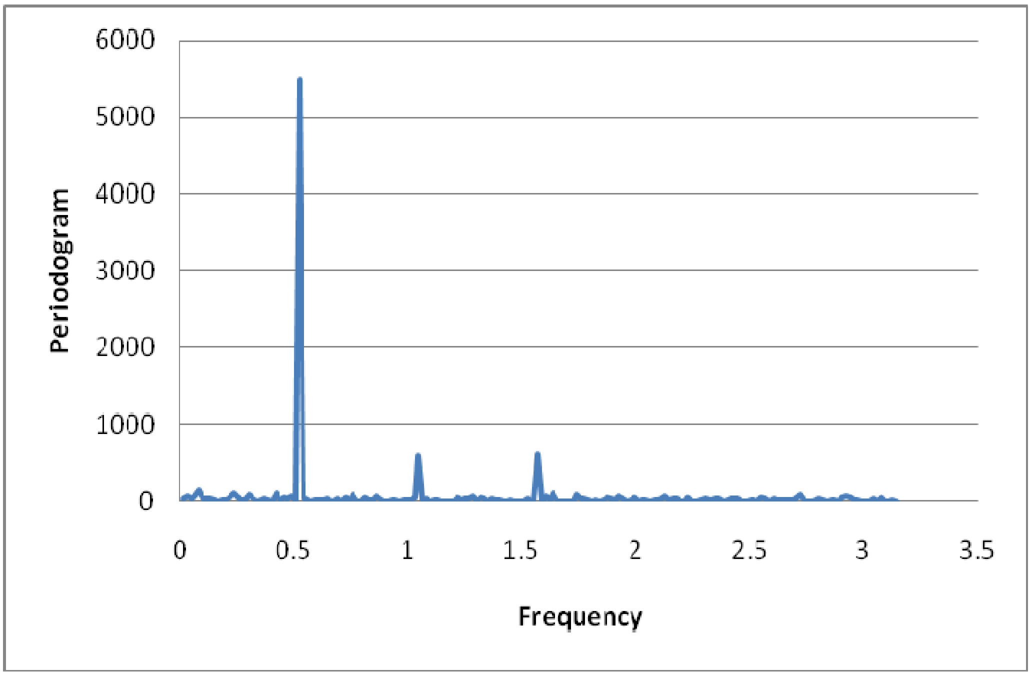


The periodogram is clearly dominated by a very large peak at frequency

$\omega_{31} = \frac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \frac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \frac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_{31}} = 12$ months, $P = \frac{2\pi}{\omega_{62}} = 6$ months, and $P = \frac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,333) = 3.00$, and the periodogram is significant only at frequency $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of rainfall in Ayutthaya

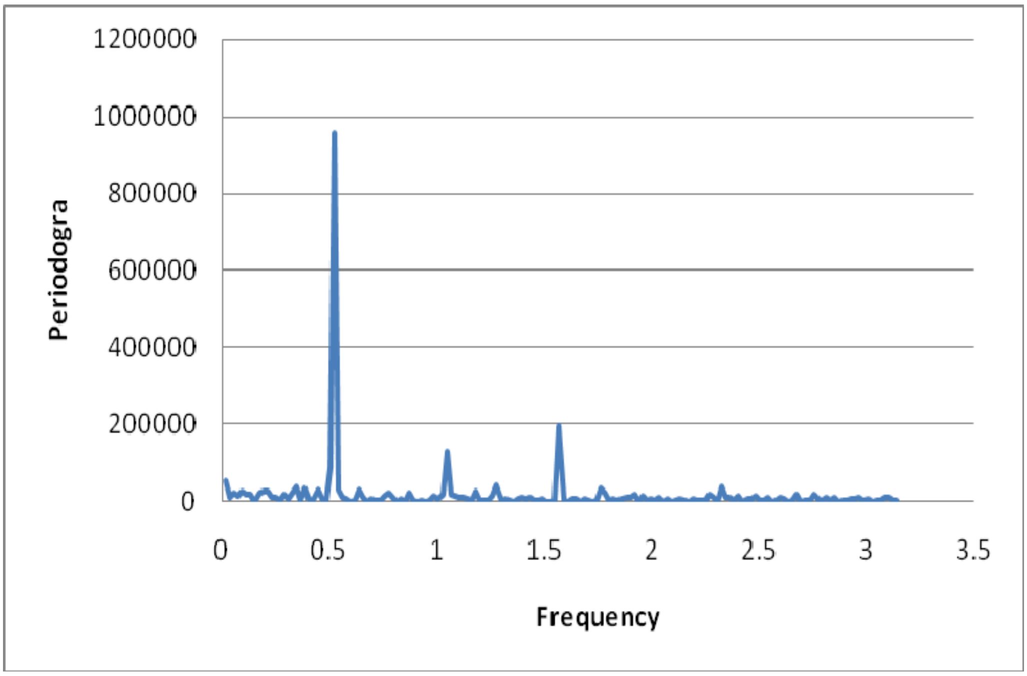


The periodogram is clearly dominated by a very large peak at frequency

$\omega_{31} = \frac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \frac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \frac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_{31}} = 12$ months, $P = \frac{2\pi}{\omega_{62}} = 6$ months, and $P = \frac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05,\ F_{0.05}(2,369) = 3.00$, and the periodogram is significant only at frequency $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of rainfall in Pathumthani



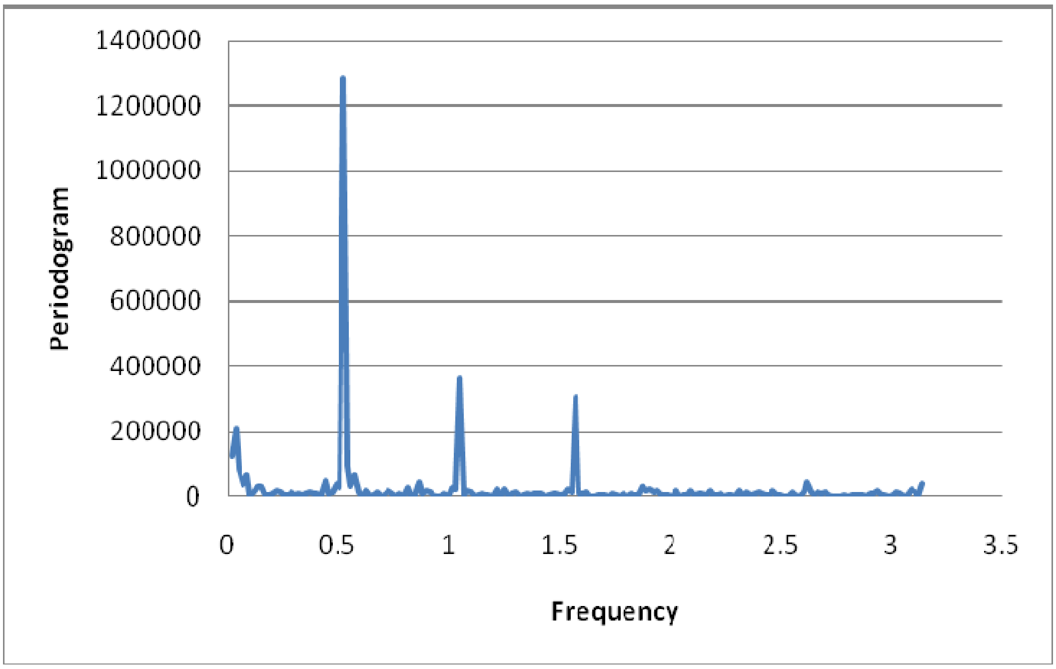
The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \dfrac{2\pi(1)}{324} = 0.01939$, $\omega_{27} = \dfrac{2\pi(27)}{324} = 0.52360$, $\omega_{54} = \dfrac{2\pi(54)}{324} = 1.04720$, and
$\omega_{81} = \dfrac{2\pi(81)}{324} = 1.57080$.

This frequency corresponds to a period of $P = \dfrac{2\pi}{\omega_1} = 324$ months, $P = \dfrac{2\pi}{\omega_{27}} = 12$ months, $P = \dfrac{2\pi}{\omega_{54}} = 6$ months, and $P = \dfrac{2\pi}{\omega_{81}} = 4$ months. It indicates that the data exhibit a 12-month cycle.

Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05,\ F_{0.05}(2,321) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.01939$, $\omega_{27} = 0.52360$, $\omega_{54} = 1.04720$, and $\omega_{81} = 1.57080$.

Periodogram analysis of rainfall in Nonthaburi

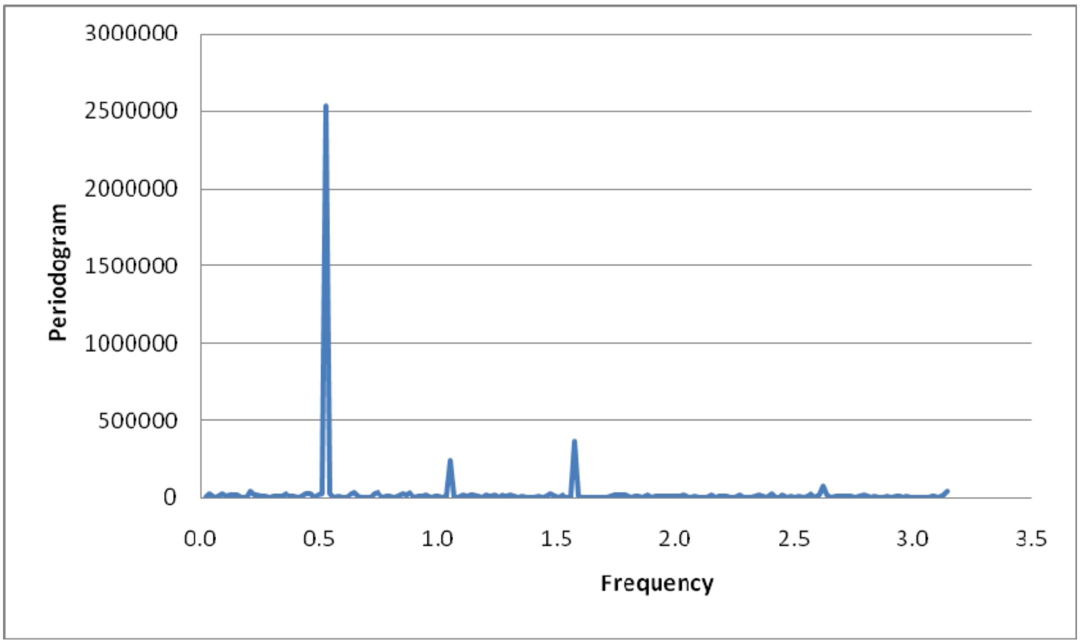


The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \frac{2\pi(1)}{372} = 0.01689$, $\omega_2 = \frac{2\pi(2)}{372} = 0.03378$, $\omega_3 = \frac{2\pi(3)}{372} = 0.05067$, $\omega_{31} = \frac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \frac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \frac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_1} = 372$ months, $P = \frac{2\pi}{\omega_2} = 186$ months, $P = \frac{2\pi}{\omega_3} = 124$ months, $P = \frac{2\pi}{\omega_{31}} = 12$ months, $P = \frac{2\pi}{\omega_{62}} = 6$ months, and $P = \frac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,369) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.01689$, $\omega_2 = 0.03378$, $\omega_3 = 0.05067$ $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of rainfall in Bangkok

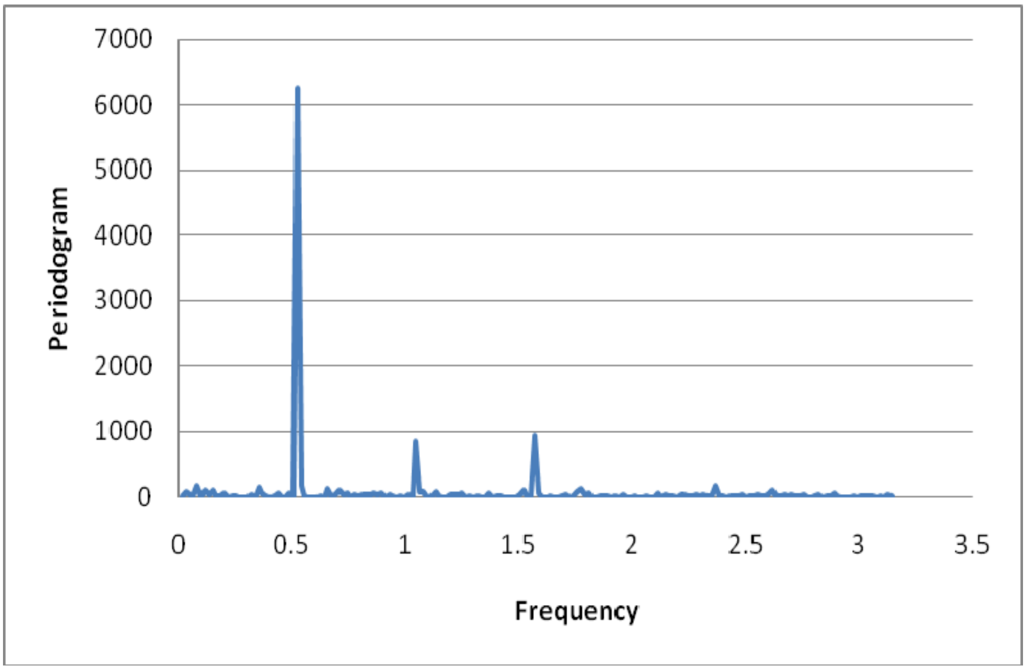


The periodogram is clearly dominated by a very large peak at frequency

$\omega_{31} = \dfrac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \dfrac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \dfrac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \dfrac{2\pi}{\omega_{31}} = 12$ months, $P = \dfrac{2\pi}{\omega_{62}} = 6$ months, and $P = \dfrac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,369) = 3.00$, and the periodogram is significant only at frequency $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of rainfall in Samutprakarn

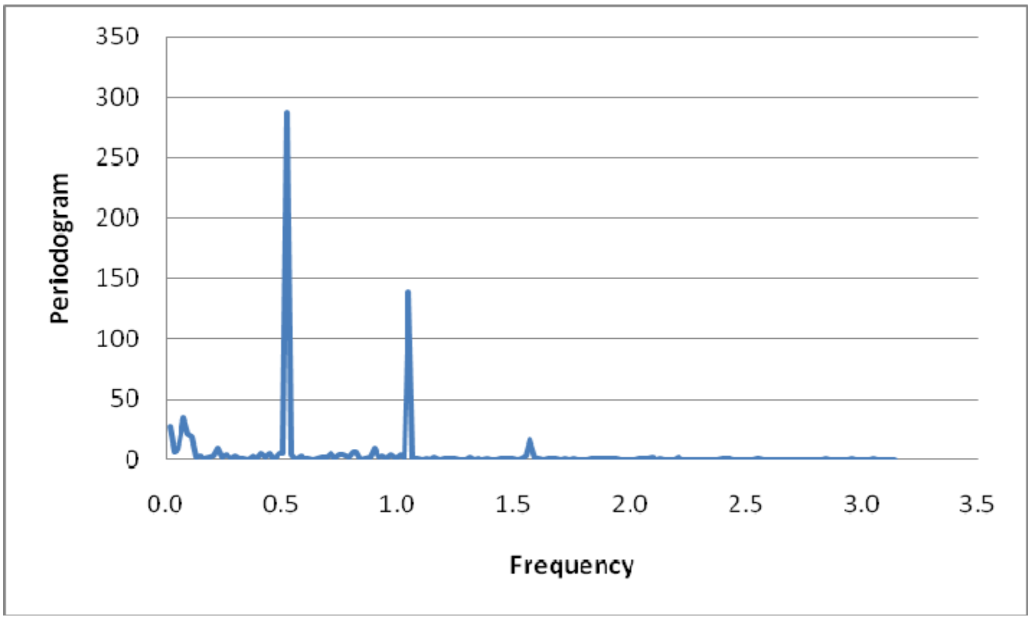


The periodogram is clearly dominated by a very large peak at frequency

$\omega_{31} = \frac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \frac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \frac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_{31}} = 12$ months, $P = \frac{2\pi}{\omega_{62}} = 6$ months, and $P = \frac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,369) = 3.00$, and the periodogram is significant only at frequency $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of water level in Nakhonsawan

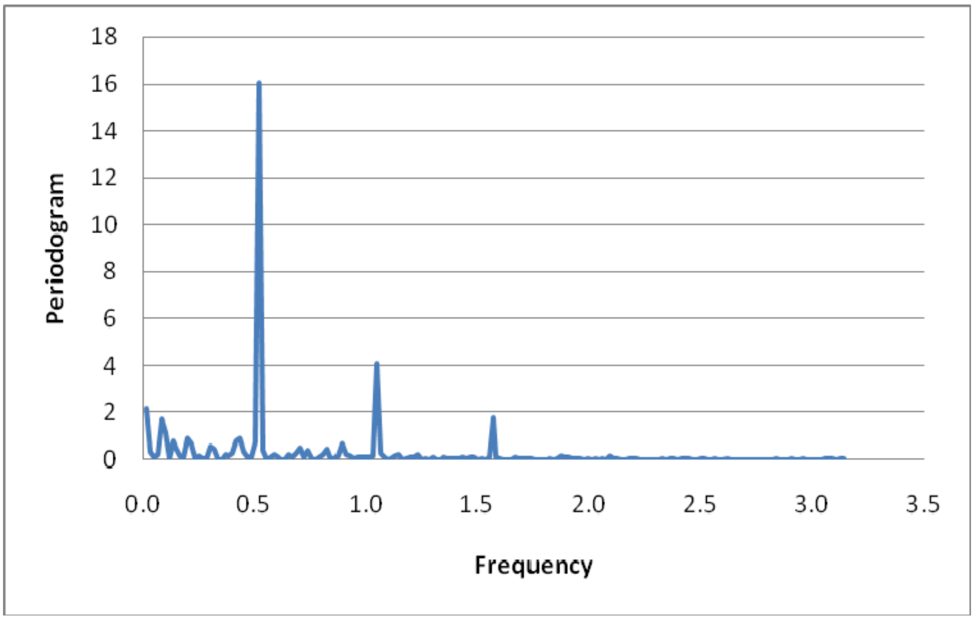


The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \dfrac{2\pi(1)}{336} = 0.01870$, $\omega_4 = \dfrac{2\pi(4)}{336} = 0.07480$, $\omega_5 = \dfrac{2\pi(5)}{336} = 67.2$, $\omega_6 = \dfrac{2\pi(6)}{336} = 0.11220$,
$\omega_{28} = \dfrac{2\pi(28)}{336} = 0.52360$, $\omega_{56} = \dfrac{2\pi(56)}{336} = 1.04720$, and $\omega_{84} = \dfrac{2\pi(84)}{336} = 1.57080$.

This frequency corresponds to a period of $P = \dfrac{2\pi}{\omega_1} = 336$ months, $P = \dfrac{2\pi}{\omega_4} = 84$ months, $P = \dfrac{2\pi}{\omega_5} = 67.2$ months, $P = \dfrac{2\pi}{\omega_6} = 56$ months, $P = \dfrac{2\pi}{\omega_{28}} = 12$ months, $P = \dfrac{2\pi}{\omega_{56}} = 6$ months, and $P = \dfrac{2\pi}{\omega_{84}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,333) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.01870$, $\omega_4 = 0.07480$, $\omega_5 = 0.09350$, $\omega_6 = 0.11220$, $\omega_{28} = 0.52360$, $\omega_{56} = 1.04720$, and $\omega_{84} = 1.57080$.

Periodogram analysis of water level in Chainat

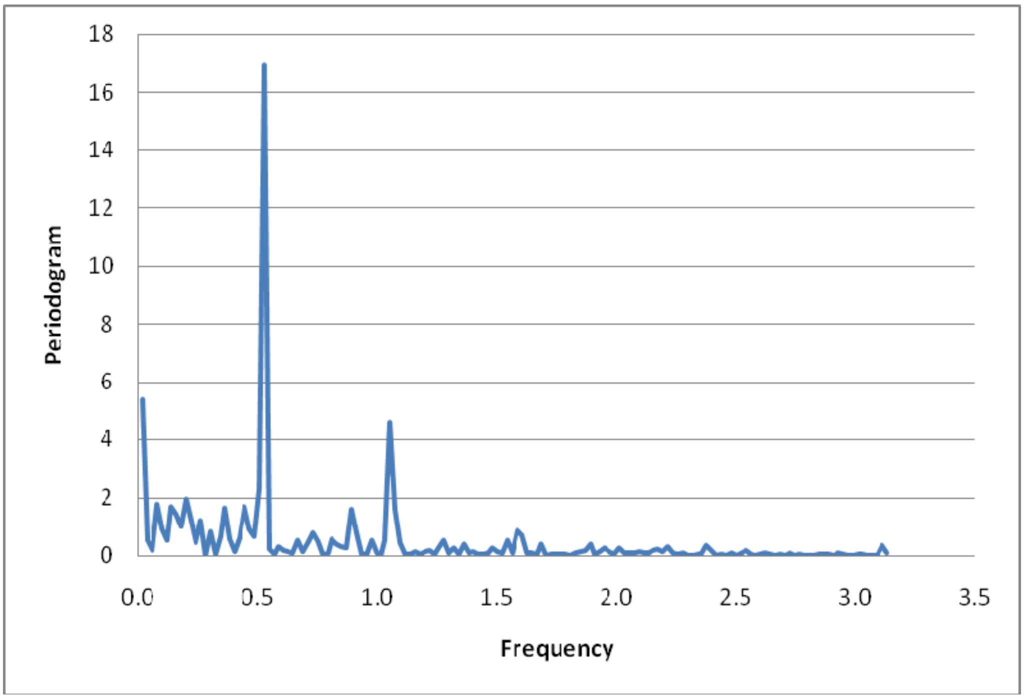


The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \frac{2\pi(1)}{372} = 0.01689$, $\omega_5 = \frac{2\pi(5)}{372} = 0.08445$, $\omega_6 = \frac{2\pi(6)}{372} = 0.10134$, $\omega_8 = \frac{2\pi(8)}{372} = 0.13512$, $\omega_{12} = \frac{2\pi(12)}{372} = 0.20268$, $\omega_{25} = \frac{2\pi(25)}{372} = 0.42226$, $\omega_{26} = \frac{2\pi(26)}{372} = 0.43915$, $\omega_{31} = \frac{2\pi(31)}{372} = 0.52360$, $\omega_{62} = \frac{2\pi(62)}{372} = 1.04720$, and $\omega_{93} = \frac{2\pi(93)}{372} = 1.57080$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_1} = 372$ months, $P = \frac{2\pi}{\omega_5} = 74.4$ months, $P = \frac{2\pi}{\omega_6} = 62$ months, $P = \frac{2\pi}{\omega_8} = 46.5$ months, $P = \frac{2\pi}{\omega_{12}} = 31$ months, $P = \frac{2\pi}{\omega_{25}} = 14.88$ months, $P = \frac{2\pi}{\omega_{26}} = 14.31$ months, $P = \frac{2\pi}{\omega_{31}} = 12$ months, $P = \frac{2\pi}{\omega_{62}} = 6$ months, and $P = \frac{2\pi}{\omega_{93}} = 4$ months. It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,369) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.01689$, $\omega_5 = 0.08445$, $\omega_6 = 0.10134$, $\omega_8 = 0.13512$, $\omega_{12} = 0.20268$, $\omega_{25} = 0.42226$, $\omega_{26} = 0.43915$, $\omega_{31} = 0.52360$, $\omega_{62} = 1.04720$, and $\omega_{93} = 1.57080$.

Periodogram analysis of water level in Singburi



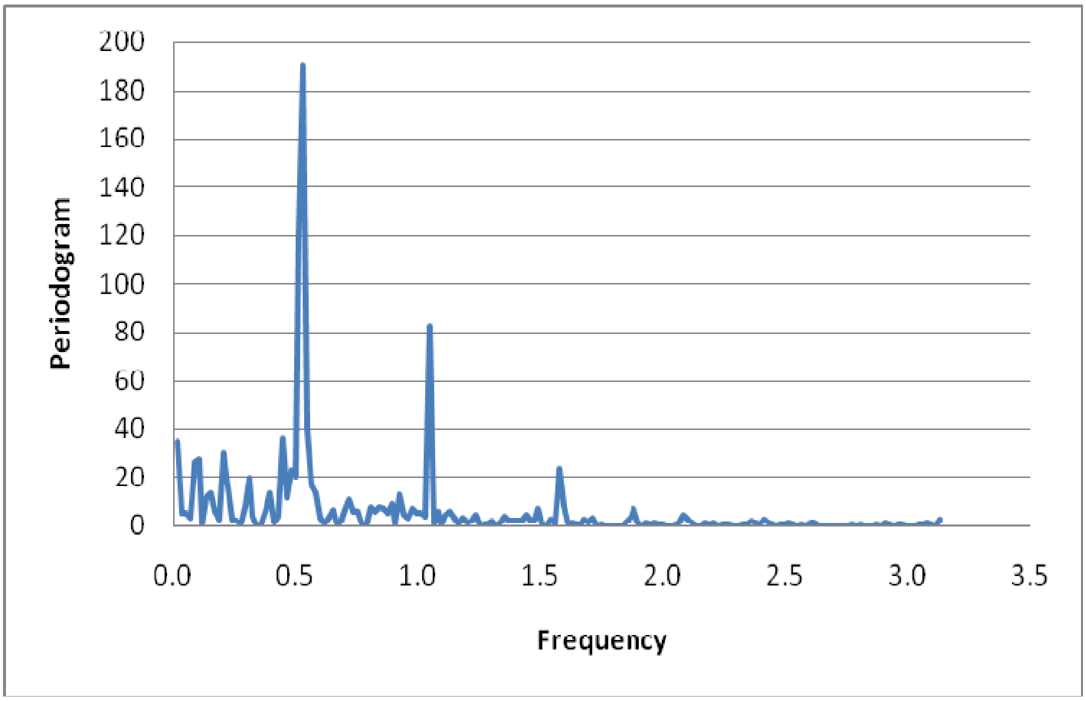
The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \frac{2\pi(1)}{309} = 0.02033$, $\omega_4 = \frac{2\pi(14)}{309} = 0.08134$, $\omega_7 = \frac{2\pi(7)}{309} = 0.14234$, $\omega_8 = \frac{2\pi(8)}{309} = 0.16267$, $\omega_{10} = \frac{2\pi(10)}{309} = 0.20334$, $\omega_{18} = \frac{2\pi(18)}{309} = 0.36601$, $\omega_{22} = \frac{2\pi(22)}{309} = 0.44735$, $\omega_{25} = \frac{2\pi(25)}{309} = 0.50835$, $\omega_{26} = \frac{2\pi(26)}{309} = 0.52868$, $\omega_{44} = \frac{2\pi(44)}{309} = 0.89469$, $\omega_{52} = \frac{2\pi(52)}{309} = 1.05737$, and $\omega_{53} = \frac{2\pi(53)}{309} = 1.07770$.

This frequency corresponds to a period of $P = \frac{2\pi}{\omega_1} = 309$ months, $P = \frac{2\pi}{\omega_4} = 77.25$ months, $P = \frac{2\pi}{\omega_7} = 44.14$ months, $P = \frac{2\pi}{\omega_8} = 38.63$ months, $P = \frac{2\pi}{\omega_{10}} = 30.90$ months, $P = \frac{2\pi}{\omega_{18}} = 17.17$ months, $P = \frac{2\pi}{\omega_{22}} = 14.05$ months, $P = \frac{2\pi}{\omega_{25}} = 12.36$ months, $P = \frac{2\pi}{\omega_{26}} = 11.88$ months, $P = \frac{2\pi}{\omega_{44}} = 7.02$ months, $P = \frac{2\pi}{\omega_{52}} = 5.94$ months, and $P = \frac{2\pi}{\omega_{53}} = 5.83$ months.

It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,306) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.02033$, $\omega_4 = 0.08134$, $\omega_7 = 0.14234$, $\omega_8 = 0.16267$, $\omega_{10} = 0.20334$, $\omega_{18} = 0.36601$, $\omega_{22} = 0.44735$, $\omega_{25} = 0.50835$, $\omega_{26} = 0.52868$, $\omega_{44} = 0.89469$, $\omega_{52} = 1.05737$, and $\omega_{53} = 1.07770$.

Periodogram analysis of water level in Angthong



The periodogram is clearly dominated by a very large peak at frequency

$\omega_1 = \dfrac{2\pi(1)}{367} = 0.01712$, $\omega_5 = \dfrac{2\pi(5)}{367} = 0.08560$, $\omega_6 = \dfrac{2\pi(6)}{367} = 0.10272$, $\omega_{12} = \dfrac{2\pi(12)}{367} = 0.20545$,
$\omega_{18} = \dfrac{2\pi(18)}{367} = 0.30817$, $\omega_{26} = \dfrac{2\pi(26)}{367} = 0.44513$, $\omega_{28} = \dfrac{2\pi(28)}{367} = 0.47937$,
$\omega_{29} = \dfrac{2\pi(29)}{367} = 0.49649$, $\omega_{30} = \dfrac{2\pi(30)}{367} = 0.51361$, $\omega_{31} = \dfrac{2\pi(31)}{367} = 0.53073$,
$\omega_{32} = \dfrac{2\pi(32)}{367} = 0.54785$, and $\omega_{92} = \dfrac{2\pi(92)}{367} = 1.57508$.

This frequency corresponds to a period of $P = \dfrac{2\pi}{\omega_1} = 367$ months, $P = \dfrac{2\pi}{\omega_5} = 73.40$ months, $P = \dfrac{2\pi}{\omega_6} = 61.17$ months, $P = \dfrac{2\pi}{\omega_{12}} = 30.58$ months, $P = \dfrac{2\pi}{\omega_{18}} = 20.39$ months, $P = \dfrac{2\pi}{\omega_{26}} = 14.12$

months, $P = \frac{2\pi}{\omega_{28}} = 13.11$ months, $P = \frac{2\pi}{\omega_{29}} = 12.66$ months, $P = \frac{2\pi}{\omega_{30}} = 12.23$ months, $P = \frac{2\pi}{\omega_{31}} = 11.84$ months, $P = \frac{2\pi}{\omega_{32}} = 11.47$ months, and $P = \frac{2\pi}{\omega_{92}} = 3.99$ months.

It indicates that the data exhibit a 12-month cycle. Also included in Table are the values of the F statistics to test the significance of the periodogram at each Fourier frequency. For significance level $\alpha = 0.05$, $F_{0.05}(2,364) = 3.00$, and the periodogram is significant only at frequency $\omega_1 = 0.01712$, $\omega_5 = 0.08560$, $\omega_6 = 0.10272$, $\omega_{12} = 0.20545$, $\omega_{18} = 0.30817$, $\omega_{26} = 0.44513$, $\omega_{28} = 0.47937$, $\omega_{29} = 0.49649$, $\omega_{30} = 0.51361$, $\omega_{31} = 0.53073$, $\omega_{32} = 0.54785$, and $\omega_{92} = 1.57508$.

According to our results, time series forecasting models are used to forecast the monthly rainfall and water level series of the locations along the Chao-Phraya River in Thailand. However, there are some remarks to discuss.

Rainfall and water level data are important information for food production plans, helping ensure the safety of all people and all activities. There are many factors which effect to the amount of rainfall. These are the monsoon, topography and season whereas the water level depends on the amount of rainfall, the season and the seawater level. Hence the accuracy of prediction of rainfall and water level series is important and useful with information to decide or plan related activities.

From the results, both of the rainfall and water level series have the same seasonal period at 12-months which are supported by the periodogram of Fourier analysis. This indicates that season is one of factors of rainfall and water level amounts. For some locations, rainfall and water level are nonstationary in the variance, but can be reduced to stationary series by proper transformation. The best models of rainfall series are divided into 4 classes while water level series are divided into 2 classes. For rainfall series, the first class (ARIMA (1, 1, 1) × (0, 1, 1)$_{12}$) consists of Nakhonsawan, Singburi, Nonthaburi and Bangkok location. The second class (ARIMA (5, 0, 0) × (0, 1, 1)$_{12}$) consists of Chainat alone. The third class (ARIMA (0, 1, 1) × (0, 1, 1)12) includes Angthong, Ayutthaya and Samutprakarn. The fourth class ((1, 0, 1) × (0, 1, 1)$_{12}$) consists of Pathumtani alone. For water level series, the first class (ARIMA (1, 0, 1) × (1, 1, 0)$_{12}$) consists of Nakhonsawan and Angthong while the second class (ARIMA (1, 0, 0) × (0, 1,

1) 12) consists of Chainat and Singburi. We see that, at the same location, the best model for rainfall and water level is different. It means that we can not use the same model to forecast both rainfall and water level at the same location. water level and rainfall depend on many factors which are different; the pattern of data is also different.

References

1. Akaike H, (1973). Information theory and an extension of the minimum likelihood principle, Proc. 2nd International Symposium on Information Theory, Akademiai Kiado, Budapest, 267-281.
2. Akaike H, (1974). A new look at the statistic model identification, IEEE Transactions on Automatic Control, 19, 716-723.
3. Boochabun K, Tych W, Chappell NA, Carling PA, Lorsirirat K and Pa-Obsaeng S (2004). Statistical Modelling of Rainfall and River in Thailand. Journal Geological Society of India, 4: 503-516.
4. Box GEP and Cox DR, (1964). An analysis of transformation, Journal of the Royal Statistical Society , 26, 211-252.
5. Box GEP, Jenkins GM, Reinsel GC.(1994). Time series analysis forecasting andcontrol. 3rd Edition. New Jersey: Prentice-Hall, USA.
6. Box GEP and Pierce DA, (1970). Distribution of residual autocorrelations in autoregressive-intergrated moving average time series models, J.Amer. Statist. Assoc., 65, 1509-1526.
7. Durbin J, (1960). The fitting of time series models, Review of the Institute of International Statistics, 28, 233-244.
8. Jung JS, Kim KS and Lee JH (2000). Short-term Prediction Model of Local Rainfall. 4th International Conference on Hydro-Science and Engineering *http://kfki.baw.de/conferences/ICHE/2000-Seoul/pdf/.../PAP_083.PDF*
9. Kalman RE, (1960). A new approach to linear filtering and prediction problems, Trans. ASME J. Basic Engrg., Ser. D, 82, 35-45.
10. Kolmogorov A, (1939). Sur Pinterpolation et l’extrapolation des suits stationnaires, C.R. Acad. Sci. Paris, 208, 2043-2045.
11. Kolmogorov A, (1941). Interpolation und extrapolation von stationaren Zufalligen Folgen, Bull. Acad. Sci. (Nauk), USSA, Ser. Math., 5, 3-14.

12. Lee S, Cho S and Wong M.P (1998). Rainfall Prediction using Artificial Neural Networks, Journal of Geographic Information and Decision Analysis 2(2): 233-242.
13. Ljung GM and Box GEP. (978).On a measure of lack of fit in the time series models, Biometrika, 65, 297-303.
14. Mebrhatu MT, Tsubo M and Walker S (2004). A Statistical Model for Seasonal Rainfall Forecasting over the Highlands of Eritrea. *4th* International Crop Science Congress.
15. Onodera S, and Sriratanatabuchanon M. (1986). Organochlorine Pesticide Residues in the Lower Chao Phraya River and Klongs along the River at Bangkok Metropolitan Area, J. Sci. Soc, 12, 225-238.
16. Panichkitkosolkul W. (2009). Forecasting of rainfall of Nakhonratchasima, Thammasat Int. J. Science Technology, 17, 1-12.
17. Schwartz G, (1978). Estimating the dimension of a model, The Annals of Statistic, 6, 461-464.
18. Sen N (2003).New forecast models for Indian south-west monsoon season rainfall. Journal of Current Science, 84(10) : 1290-1292.
19. Sosa J, Sosa C and Paz B (2000). ARIMA Models in the Rain Attenuation Prediction in a Mexican Tropical Area. IEEE Transactions on Antennas and Propagation Society International Symposium, 2 : 546-549.
20. Tingsanchali T (2000). Forecasting model of Chao Phraya river flood levels at Bangkok, International Conference on Chao Phraya Delta Bangkok Thailand.
21. Weesakul U and Lowanichchai S (2005). Rainfall Forecast for Agricultural Water Allocation Planning in Thailand. Thammasat Int. J. Science Technology, 10(3) : 18-27.
22. Wei WW, (1990). Time series analysis univariate and multivariate methods, Addison-Wesley, USA.
23. Whittle P, (1983). Prediction and Regulation by Linear Least-Square Methods, 2nd ed., University of Minnesota, Minneapolis.
24. Wiener N, (1949). The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications, Wiley, New York, USA.
25. Yaglom AM, (1952). An Introduction to the Theory of Stationary RandomFunctions, Pentice-Hall, Englewood Cliffs, NJ, USA.

26. Zaw W.T.and Naing T.T.(2008). Empirical Statistical Modeling of Rainfall Prediction over Myanmar, Proceedings of World Academy of Science, Engineering and Technology, 36, 569-572.

7.3 Nonlinear Mixed Effects Models

In the field of medical science, data usually consists of repeated measurements on individuals observed under various conditions. Repeated measurements arise in many diverse fields and possibly are even more common than single measurements. The term ‘repeated’ will be used here to describe measurements which are made of the same characteristic on the same observational unit but on more than one occasion. In longitudinal studies, individuals may be monitored over a period of time to record the developing pattern of their observed values. Over such a period the conditions may be deliberately changed, as in crossover trials, to study the effects on the individual. Even in the studies which are not intentionally longitudinal, once a sample of individual units has been assembled, or identified, it is often easier and more efficient in practice to monitor and observe them repeatedly rather than to discard each one after a single observation and start afresh with another sample (Crowder et al., 1990). For example, in longitudinal clinical studies, measurements are taken on each subject over time. Participants in pharmacokinetic experiments undergo serial blood sampling following administration of a test agent. Pharmacodynamic studies may involve repeated measurement of physiological effects in the same subject in response to differing doses of a drug. Data from clinical trials such as prostate-specific antigen (PSA), are clinically useful to know when PSA levels first begin to rise rapidly and to determine if the natural history of PSA progression is different in men with locally confined prostate cancers compared to men with metastatic tumors.

Prostate cancer is the most common cancer for male populations in many parts of the world. It is a slowly growing deadly cancer with very few signs and symptoms in the early stage. In the United States, an estimated 244,000 new cases and 40,400 deaths from prostate cancer occurred in Morrell, 1995. The PSA is made by prostate cells and is released into the bloodstream. Large prostates have more PSA, therefore a rise in PSA means that the gland is enlarging rapidly, which can either be a sign of cancer or irritation by infection (Phyllis, 1997).

Another clinical usefulness is that a rising Carcinoembryonic Antigen (CEA) level indicates progression or recurrence of the cancer. The CEA is a type of protein molecule that can be found in many different cells of the body, but it is typically associated with certain tumors and the developing fetus (Benchimol, 1989). The CEA is used as a tumor marker, especially for cancers of the gastrointestinal tract such as colon cancer or colorectal cancer.

When the CEA level is abnormally high before surgery or other treatment, it is expected to fall to a normal level after success in removing all of cancer. It is recommended by Bast *et al.*(2001) that the data were insufficient to recommend routine use of the serum CEA alone for monitoring response to treatment. If no other simple test is available to indicate a response, the CEA should be measured at the start of treatment for metastatic disease and every two to three months during active treatment. Two values above the baseline are adequate to document progressive disease even in the absence of corroborating radiographs. The CEA is regarded as the marker of choice for monitoring colorectal cancer. This recommendation is consistent with Zheng *et al.*(2001) who found that the measurement of preoperative serum levels of CEA will aid in predicting the prognosis of patients with colorectal cancer who have been treated surgically. Elevated serum levels of tumor markers indicate high risk of cancer recurrence and poor survival. Such patients should receive adjuvant therapy at the earlier stages of this disease. After surgery, the CEA level can be periodically monitored. If the level begins to rise above 6 ng/ml, there will be a high correlation of recurrence of the cancer. However, this does not always happen in every case as there might be other factors which can increase the CEA level, including diverticulitis, pancreatitis, hepatitis and smoking. If these other causes are excluded, then one must look for recurrence of cancer.

Local recurrence of colorectal cancer after ‘curative’ surgery is a major clinical problem. Typically, 50-70 per cent of patients presented to a surgical clinic will undergo apparently curative surgery for disease and about 10-25 per cent of those will develop local recurrence in either the tumor bed or bowel wall (Doccherty, 1994). In 1987, Claudio *et al.* made a comparative study of sixty-four consecutive patients who had undergone curative resection for colorectal carcinoma which was conducted to evaluate the roles of sequential CEA determinations and independent instrumental. The study was also a follow-up in the early detection of respective recurrences. They found that CEA was the best predictor of recurrence when compared with the other two markers (tissue polypeptide antigen (TPA), colon cancer antigen detected with a monoclonal antibody (Ca19-9)). It is in accordance with 2000 update of American Society of Clinical Oncology colorectal cancer surveillance guidelines by Benson *et al.*(2000) that follow up testing was done by protocol guidelines. Ninety-six of the 421 patients who developed recurrent disease underwent surgical resection with curative intent. For the subgroup of patients with resectable, the first test to detect recurrence was CEA, chest x-ray, colonoscopy, and other tests. The CEA was the most cost-effective approach to detect potentially metastases from colon cancer. Another study followed up patients with a specified testing strategy after curative colorectal surgery. Here, 64% of recurrences were detected first by CEA, far more than the other tests in the battery (Castells, 1998).

By definition, studies of growth and decay involve repeated measurements taken on sample units, which could be human or animal subjects, plants, or cultures. Modeling data of this kind usually involves characterization of the relationship between the measured response, y, and the repeated measurement factor, or covariate, x. In many applications, the proposed systematic relationship between y and x is nonlinear with unknown parameters of interest. The model has two types of parameters, that is global parameters that correspond to the fixed effects and parameters which vary among the population that correspond to the random effects.

We propose a nonlinear mixed-effects model to describe the longitudinal data in the CEA value of colorectal cancer patients. The model will be approached to find a recurrent time of disease after surgery. The parameters of the model will be estimated by using Lindstrom and Bates (LB) and Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithms. Therefore, the aims of this study are to propose the nonlinear mixed effects model of colorectal cancer patients, estimate the parameters of the proposed model by using LB and SAEM algorithms and compare the residual of the prediction model based on these algorithms. Further, the recurrent time of colorectal cancer patients will be presented and illustrated.

We address the LB and SAEM algorithms for estimating parameters of repeated measurements in the nonlinear model. We consider the general nonlinear mixed effects model for repeated data by using the notation of Linstrom and Bates (1990), and apply the nonlinear mixed effects model to CEA values which are obtained from colorectal cancer patients. Further, the concepts of LB and SAEM algorithms are illustrated.

Algorithms

Individual repeated measurement data often exhibit a relationship between response and measurement factors that is best characterized by a model nonlinear in its parameters. In some settings, an appropriate nonlinear model may be derived on the basis of theoretical considerations. In other situations, a nonlinear relationship may be employed to provide an empirical description of the data. Several authors have investigated the problem of parameter estimation for the nonlinear model as the following.

In 1990, Lindstrom and Bates proposed a general, nonlinear mixed effects model for repeated measures data and defined estimators for its parameters by a combination of least squares estimators for nonlinear fixed effects models and maximum likelihood (or restricted maximum likelihood, RML) estimators for linear mixed effects models. They applied Newton-Raphson estimation using previously developed computational methods for nonlinear fixed effects models and for linear mixed effects models.

In 1995, Morrell *et al.* used a nonlinear mixed effects model to describe longitudinal changes in PSA in men before their prostate cancers were detected clinically. The model was linear in PSA values a long time before the cancer was detected and becomes exponential after the cancer was detected. The time when the PSA levels changed from linear to exponential PSA progression was unknown but could be estimated by including random terms that allowed each subject to have his own transition time. They used software developed by Lindstrom and Bates (1990) to estimate the parameters of their model.

In 1996, Walker simulated data and obtained the following nonlinear population pharmacodynamic model, is

$$y_{ij} = \theta_{1i} - \frac{\theta_{2i} d_j}{\theta_{3i} + d_j} + \varepsilon_{ij} ,$$

y_{ij} is blood pressure for the *j*th response on the *i*th individual,

d_j is dose for the *j*th response,

θ_{1i} is absence of treatment for the *i*th individual,

θ_{2i} is maximum effect of the drug for the *i*th individual,

θ_{3i} is dose which gives 50% of the maximum effect for the *i*th individual, and

$$\varepsilon_{ij} \sim N(0,\sigma^2).$$

The exact maximum likelihood estimates (MLEs) obtained from the expectation maximization (EM) algorithm are compared, over ten simulated data sets, with the approximate MLEs from a first order conditional estimation (FOCE) and Laplacian methods of nonlinear mixed effects models. A summary of the simulations is given by the averages of the parameters estimates (mean) and also the averages of the mean squared errors (MSE). The MSE is a suitable criterion for judging the merit of the estimates when the variance is unsuitable as a result of the bias in the estimates. This suggests that the extra computing time is worthwhile in order to overcome the difficulties that the EM algorithm is more conservative than the FOCE and Laplacian methods.

In 1999, Delyon *et al*. suggested that the Expectation and Maximization (EM) algorithm was a powerful computational technique for locating maxima of functions. It is widely used in statistics for maximum likelihood estimation in incomplete data models but the expectation step cannot be performed in close form. To deal with these problems, a novel method, the Stochastic Approximation version of the standard Expectation and Maximization step (SAEM), is introduced in order to replace the expectation step of the EM algorithm by one iteration of a stochastic approximation procedure.

In 2003, Fieuws *et al*. suggested that a series of marker measurements turns out to be a better screening tool than a single measurement. Classical discriminated analysis can be applied whenever the data have a balance structure. Extensions have been proposed for unbalancedata by using linear mixed effects models or linearized versions of nonlinear models to describe the longitudinal profiles in each group.

In 2004, Liang and Sha investigated patterns of tumor response in mouse xenograft tumors. They applied a biexponential nonlinear mixed-effects model to an analysis of changes in tumor volume over a given period of treatment. The model gave a good fit to the data, even for small sample sizes.

In 2005, Kuhn and Lavielle developed an algorithm for parameter estimation of nonlinear mixed effects models which combined the SAEM algorithm and a Markov Chain Monte Carlo procedure. They showed good statistical convergence properties of this algorithm. Further, they showed that the SAEM algorithm was very efficient for computing the maximum likelihood estimates of parameters. The SAEM algorithm can be used for estimation homoscedastic models, but also heteroscedastic models. The main advantage of this algorithm was its ability to provide an estimator closed to the MLE in very few iterations. They reproduced the values from Walker (1996) and compared different popular methods of estimation, such as the FOCE and Laplacian methods, with the standard errors obtained by the SAEM algorithm in the pharmacodynamic model. They clearly saw that the EM algorithm of Walker and the SAEM algorithm showed similar results where the later had the smallest values not only of the MSE's but also of the variance-covariance matrix.

Theoretical Background

A nonlinear mixed effects framework is widely used in modeling repeated measurement data, where measures are obtained for a number of individuals under varying experimental conditions. Nonlinear mixed effects models incorporate the population (fixed), with parameters associated with an entire population or with certain repeatable levels of experimental factors as well as individual specifics (random), which are associated with individual experimental units drawn at random from population characteristics. This enables us to make an interest for both random and fixed effects. Mixed effects models for repeated measures data have become popular in part because their flexible variance covariance structure allows for non constant correlation among the observation and/or unbalanced data (designs that vary among individuals). Mixed effects models are also intuitively appealing. The notation that individuals' responses all follow a similar functional form with parameters that vary among individuals seems to be appropriate in many situations. Within the framework, much of the interest is focused on representing the mean function describing the dynamic relationship between the response and explanatory variables such as time.

Let us consider the following general nonlinear mixed effects model by using the notation of Linstrom and Bates (1990), the model for the jth observation on the ith individual in the study is written as

$$y_{ij} = f(x_{ij}, \phi_i) + e_{ij},$$

where y_{ij} is the jth response on the ith individual, $i = 1,\ldots, m$,

x_{ij} is the vector of predictor variables for the jth response on the individual i,

f is a nonlinear function of the predictor variables and parameter vector,

ϕ_i is a $r \times 1$ vector of parameters for the i th individual, and

e_{ij} ~ $N(0, \sigma^2 I_i)$.

The nonlinear function is at least one of the derivatives of the expectation function with respect to the parameters and depends on at least one of the parameters (Bates, et al., 1998). Random-effects terms may be included in the parameter vector to allow the parameter value to vary from individual to individual by writing

$$\phi_i = A_i \phi + B_i b_i,$$

$$b_i \sim N(0, D) \ ,$$

where ϕ is a $p \times 1$ vector of fixed population parameters,

b_i is a $q \times 1$ vector of random effects associated with individual i,

A_i is a design matrix of size $r \times p$ for the fixed effects,

B_i is a design matrix of size $r \times q$ for the random effects, and

D is a positive definite variance covariance matrix.

This is a general form of the nonlinear mixed effects model since any nonlinear function of fixed and random effects can be written as $f(A_i \phi + B_i \mathrm{b_i}, x_{ij})$. From the statistical model the intra-individual variation is written as

$$y_{ij} \ = f(x_{ij}, \phi_i) + e_{ij} \,,$$

$$e_{ij} \sim N(0, \sigma^2 I_i) \ ,$$

$$f(x_{ij}, \phi_i) = \exp(\phi_{0i}) \exp(\phi_{1i}(x_{ij} - \phi_{2i})) \,,$$

where e_{ij} is a random error term reflecting uncertainty in the response, given the i th individual, with E($e_{ij} \,|\phi_i$) = 0. Collecting the responses and errors for the i th individual into the $n_i \times 1$. This is accomplished by letting

$$\mathrm{y}_i = \begin{bmatrix} y_{i1} \\ \cdot \\ \cdot \\ \cdot \\ y_{in_i} \end{bmatrix}, \quad \mathrm{e_i} = \begin{bmatrix} e_{i1} \\ \cdot \\ \cdot \\ \cdot \\ e_{in_i} \end{bmatrix}, \text{ and } \quad \mathrm{f}_i(\phi_i) = \begin{bmatrix} f(x_{ij}, \phi_i) \\ \cdot \\ \cdot \\ \cdot \\ f(x_{in_i}, \phi_i) \end{bmatrix},$$

so that

$$\mathrm{y}_i = \mathrm{f}_i(\phi_i) + \mathrm{e}_i,$$

where $\mathrm{e}_i \sim N(0, \sigma^2 I_i)$ and I_i is an identity matrix for individual i. We may summarize the data for the i th individual as

$$\mathrm{y}|\mathrm{b} \sim N\left(\mathrm{f}_i(\phi_i), \sigma^2 I_i\right),$$

$$\mathrm{y}_i = \mathrm{f}_i(\phi_i) + \mathrm{e}_i,$$

$$\phi_i = A_i\phi + B_i\mathrm{b}_i, \quad \mathrm{b}_i \sim N(0, D) .$$

Let ϕ_i be a $r \times 1$ vector of regression parameters specific to the i th individual. Let a_i be an $a \times 1$ covariate vector corresponding to individual attributes for individual i, let b_i be a $k \times 1$ vector of random effects associated with the i th individual, and let ϕ be a $p \times 1$ vector of fixed parameters, or fixed effects. Then a general model for ϕ_i is given by

$$\phi_i = d(a_i, \phi, \mathrm{b}_i),$$

where d is a $p-$dimensional vector-valued function. Each element of d is associated with the corresponding element of ϕ_i, so that the functional relationship may be of a different form for each element. In a model of the form of the above equation, a complete characterization of inter-individual variation requires an assumption about the distribution of the random effects b_i. In this study, we assume that b_i has mean 0 and variance covariance matrix D.

With the nonlinear mixed effects model there are many methods to estimate the unknown parameter vector. One method is the algorithm proposed by Lindstrom and Bates (LB)(1990) to estimate the parameters of their model. This algorithm is used by Lewis and Stuart (1980) in a similar approximation but evaluated at the expectation of the random effects rather than at the current estimates ($\mathrm{b}=0$) by Taylor series expansion for the function $f(\theta, b_i, x_i)$ which is given by

$$f(\theta,\mathrm{b}_i,x_i) \;\approx\; f(\theta,\mathrm{b}_i=0,x_i)+\left.\frac{\partial f(\theta,\mathrm{b}_i=0,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0}(\mathrm{b}_i-0)+e_i \;,$$

$$\mathrm{b}_i=0,\; E(\mathrm{b}_i)=0 \;,$$

$$e_i \sim N(0,\sigma^2),$$

so that the expected values of $f(\theta,\mathrm{b}_i,x_i)$ are given as

$$E(f(\theta,\mathrm{b}_i,x_i)) = E\big(f(\theta,\mathrm{b}_i=0,x_i)\big)+E\left(\left.\frac{\partial f(\theta,\mathrm{b}_i=0,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0}(\mathrm{b}_i-0)\right)+E(e_i)$$

$$= \; f(\theta,\mathrm{b}_i=0,x_i)+\left.\frac{\partial f(x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0}E(\mathrm{b}_i-0)$$

$$= \; f(\theta,\mathrm{b}_i=0,x_i),$$

and the variance of $f(\theta,\mathrm{b}_i,x_i)$ is

$$Var(f(\theta,\mathrm{b}_i,x_i)) \;=\; \left.\frac{\partial f(\theta,\mathrm{b}_i,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0} Var(\mathrm{b}_i)\left.\frac{\partial f(\theta,\mathrm{b}_i,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0}^{T}+Var(e_i)\,.$$

Hence, the response variables are distributed as

$$y_i \sim N\left(f(\theta,\mathrm{b}_i=0,x_i),\left.\frac{\partial f(\theta,\mathrm{b}_i=0,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0} Var(\mathrm{b}_i)\left.\frac{\partial f(\theta,\mathrm{b}_i=0,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0}^{T}+Var(e_i)\right)$$

The probability density function of y_i is given as

$$p(y_i)=\frac{1}{\sqrt{2\pi}\,|V|^{1/2}}\exp\left(-\frac{1}{2}(y_i-f_i(\theta,b_i=0,x_i))^T(V)^{-1}(y_i-f_i(\theta,b_i=0,x_i))\right),$$

where $V=\left.\frac{\partial f(\theta,\mathrm{b}_i=0,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0} Var(\mathrm{b}_i)\left.\frac{\partial f(\theta,\mathrm{b}_i=0,x_i)}{\partial \mathrm{b}_i}\right|_{\mathrm{b}_i=0}^{T}+Var(e_i)\,.$

The likelihood function of y_i, is written as

$$L = \prod p(y, b_i = 0)$$
$$= \left(\frac{1}{2\pi}\right)^{m/2} \left(\frac{1}{|V|}\right)^{m/2} \exp\left(-\frac{1}{2}\sum (y_i - f_i(\theta, b_i = 0, x_i))^T (V)^{-1} (y_i - f_i(\theta, b_i = 0, x_i))\right),$$

and the log-likelihood function of y_i is given as

$$\log L = -\frac{m}{2}\log(2\pi) - \frac{m}{2}\log|V| - \frac{1}{2}\sum (y_i - f(\theta, b_i = 0, x_i))^T (V)^{-1} (y_i - f(\theta, b_i = 0, x_i)).$$

Now the objective function is given by

$$\sum_{i=1}^{m}\left(\log|V| + \left(y_i - f(\theta, b_i = 0, x_i)\right)^T (V)^{-1} \left(y_i - f(\theta, b_i = 0, x_i)\right)\right).$$

Lindstrom and Bates (1990) used a combination of least squares estimators for the nonlinear fixed-effects model and maximum likelihood (ML) estimators for a linear mixed-effects model. The linear mixed-effects model is used to estimate parameters from an approximation to the marginal distribution of the complete data vector, y. This combination provides for approximate maximum likelihood estimates for the nonlinear mixed-effects model. An orthogonality convergence criterion is used to determine convergence.

Another method is the Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithm developed by Kuhn and Lavielle (2005). They developed an estimation method which combines the SAEM algorithm with the Markov Chain Monte Carlo (MCMC) procedure for maximum likelihood estimation in nonlinear mixed effects models without linearization. This method is implemented in the MATLAB software, namely MONOLIX which is available at http://software.monolix.org. The details of using this software can be found either from this webpage or the online manual. One of the objectives of this software is to perform parameter estimation for nonlinear mixed effects models by the following steps.

1. Computing the maximum likelihood estimator of the population parameters without any approximation of the model (linearization, quadrature approximation, etc) using the Stochastic Approximation version of the standard Expectation Maximization (SAEM) algorithm,
2. Computing the standard error for the maximum likelihood estimator,

3. Computing the conditional modes, the conditional means and the conditional standard deviations of the individual parameters, using the Hastings-Metropolis algorithm.

In this study we modify MONOLIX to analyse the CEA level of the colorectal cancer patients. Standard errors derived from the Fisher information matrix and confidence intervals based on normality of the estimators are obtained.

Methodology

This section presents the concept of a nonlinear model which is used for this topic. The basic knowledge as well as Lindstrom and Bates (LB) and Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithms used for parameters estimation are also presented.

The nonlinear model

Nonlinear models are often mechanistic, that is based on a model for the mechanism producing the response. As a consequence, the model parameters in a nonlinear model generally have a natural physical interpretation. Even when derived empirically, nonlinear models usually incorporate known, theoretical characteristics of the data such as asymptotes and monotonic, and in these cases, can be considered as semi-mechanistic models. A nonlinear model such as a polynomial, generally uses fewer parameters than a competitor linear model, giving a more parsimonious description of the data. Nonlinear models also provide more reliable predictions for the response variable outside the observed range of the data than polynomial models would (Davidian, and Giltinan, 1995). From the statistical model the intra-individual variation is

$$y_{ij} = f(x_{ij}, \phi_i) + e_{ij},$$

$$e_{ij} \sim N(0, \sigma^2 I_{n_i}) .$$

We consider each of the elements of ϕ_i, assuming there is no systematic dependence on individual attributes for applying with carcinoembryonic antigen (CEA) data. Let us call the nonlinear model a CEA model which is written as

$$f(x_{ij}, \phi_i) = \exp(\phi_{0i}) \exp(\phi_{1i}(x_{ij} - \phi_{2i})) .$$

Since, $\phi_i = A_i\phi + B_i b_i$ then $\phi_{0i} = \phi_0 + b_{0i}$, $\phi_{1i} = \phi_1 + b_{1i}$, and $\phi_{2i} = \phi_2 + b_{2i}$.

Therefore we can rewrite the CEA model as follows

$$f(x_{ij},\phi_i) = \exp(\phi_0 + b_{0i})\exp\Big((\phi_1 + b_{1i})\big(x_{ij} - (\phi_2 + b_{2i})\big)\Big)$$
$$= \exp(\phi_0)\exp(b_{0i})\exp\Big((\phi_1 + b_{1i})\big(x_{ij} - (\phi_2 + b_{2i})\big)\Big).$$

Here, $\mathrm{b}_i = \begin{bmatrix} b_{0i} & b_{1i} & b_{2i} \end{bmatrix}'$ has mean zero and variance covariance matrix D.

Let ϕ_i be a $r \times 1$ vector of regression parameters specific to the i th individual,

a_i be an $a \times 1$ covariate vector corresponding to individual attributes for individual i,

b_i be a $k \times 1$ vector of random effects associated with the i th individual,

and ϕ be a $p \times 1$ vector of fixed parameters, or fixed effects.

Then a general model for ϕ_i is given by $\phi_i = d(a_i, \phi, \mathrm{b}_i)$, where d is a $p-$dimensional vector-valued function. Each element of d is associated with the corresponding element of ϕ_i, so that the functional relationship may be of a different form for each element. For our model may be expressed by defining the elements of d to be

$$d_0(a_i,\phi,\mathrm{b}_i) = \phi_0 + b_{0i},\ d_1(a_i,\phi,\mathrm{b}_i) = \phi_1 + b_{1i}, \text{ and } \quad d_2(a_i,\phi,\mathrm{b}_i) = \phi_2 + b_{2i}.$$

Let us transform the CEA level to a natural logarithm CEA level to fit the CEA model. We explain the parameters of CEA model as follows.

The ϕ_0 is the log(CEA) level at the exponential phase, ϕ_1 represents the exponential rate constant during the exponential phase, and ϕ_2 represents the time point (months after surgery date) between the flat and rapid increase in the exponential phase which we assume is the time when the recurrent disease is developed.

Starting values of parameters must be specified from the beginning. As in standard nonlinear estimation, poor starting values can lead to poor results. Recommendations of Davidian and Giltinan (1995) for selecting starting values, are required for all parameters at STEP 1 of the initial iterate in the LB algorithm. As the procedures already have been discussed, they suggested use of the OLS estimator as the initial values.

The Ordinary Least Squares (OLS) method

The ordinary least squares (OLS) estimator (Davidian, and Giltinan, 1995) for $\beta, \hat{\beta}_{OLS}$, minimizes

$$\sum_{j=1}^{n}\{y_j - f(x_j,\beta)\}^2;$$

equivalently, $\hat{\beta}_{OLS}$ are solved by the p estimating equations

$$\sum_{j=1}^{n}\{y_j - f(x_j,\beta)\} f_\beta(x_j,\beta) = 0,$$

where $f_\beta(x_j,\beta)$ is the $p\times 1$ vector of derivatives of f with respect to the elements of β, that is the vector with k^{th} element $\partial/\partial\beta_k\{f(x_j,\beta)\}, k = 1,..., p.$

The nonlinear mixed effects framework is widely used in modeling repeated measurement data, where measures are obtain for a number of individuals under varying experimental conditions. Within the framework of the nonlinear mixed effects model, most of the interest is focused on representing the mean function describing the dynamic relationship between the response and explanatory variables. There are many methods to estimate the unknown parameter vector. One method is the algorithm proposed by Lindstrom and Bates (LB)(1990) to estimate the parameters of their model. Another method is the Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithm developed by Kuhn and Lavielle (2005). These two algorithms are illustrated as follows.

The Lindstrom and Bates (LB) algorithm

Lindstrom and Bates (1990) used Taylor Series to expand the function $f(x)$ which is given by

$$f(x) \quad \approx f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}(x - x_0) + \varepsilon\,,$$

with $x_0 = E(x)$, so that the expected value of $f(x)$ is

$$E(f(x)) = E(f(x_0)) + E\left(\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}(x - x_0)\right) + E(\varepsilon)$$

$$= f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0} E(x - x_0)$$

$$= f(x_0)\ ,$$

and the variance of $f(x)$ is

$$Var(f(x)) = Var(f(x_0)) + Var\left(\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} (x - x_0) \right) + Var(\varepsilon)$$

$$= 0 + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} Var(x - x_0) \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0}^{T}$$

$$= \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} Var(x) \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0}^{T} .$$

Hence, the response variables are distributed as a normal distribution with

$$y_i \sim N\left(f(x_0),\ \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} Var(x) \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0}^{T} \right). \qquad (3.1)$$

The function $f(\theta, b_i, x)$ is given by

$$f(\theta, b_i, x) \approx f(\theta, b_i = 0, x) + \left. \frac{\partial f(\theta, b_i = 0, x)}{\partial b_i} \right|_{b_i=0} (b_i - 0) + \varepsilon \ ;\ b_i = 0, E(b_i) = 0\ ;$$

$$\varepsilon \sim N(0, \sigma^2),$$

so that the expected value of $f(\theta, b_i, x)$ is

$$E(f(\theta, b_i, x)) = E\big(f(\theta, b_i = 0, x) \big) + E\left(\left. \frac{\partial f(\theta, b_i = 0, x)}{\partial b_i} \right|_{b_i=0} (b_i - 0) \right) + E(\varepsilon)$$

$$= f(\theta, b_i = 0, x) + \left. \frac{\partial f(x)}{\partial b_i} \right|_{b_i=0} E(b_i - 0)$$

$$= f(\theta, b_i = 0, x),$$

and the variance of $f(\theta, b_i, x)$ is

$$Var(f(\theta, b_i, x)) = \left. \frac{\partial f}{\partial b_i} \right|_{b_i=0} Var(b_i) \left. \frac{\partial f(\theta, b_i, x)}{\partial b_i} \right|_{b_i=0}^{T} + Var(\varepsilon).$$

Hence, the response variables y_{ij}, are distributed as a normal distribution; that is

$$y_{ij} \sim N\left(f(\theta, b_i = 0, x)\ , \left. \frac{\partial f(\theta, b_i = 0, x)}{\partial b_i} \right|_{b_i=0} Var(b_i) \left. \frac{\partial f(\theta, b_i = 0, x)}{\partial b_i} \right|_{b_i=0}^{T} + Var(\varepsilon) \right)$$

(3.2)

Lindstrom and Bates (1990) showed that $\hat{\phi}$ is a maximum likelihood estimate relative to an approximate marginal distribution of y. As in the linear case these estimates can be calculated as the solution to a (nonlinear) least squares problem formed by augmenting the data vector with "pseudo-data" as

$$\mathrm{y}_i \quad = \quad f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i) + \mathrm{e}_i,$$

where $\mathrm{e}_i \sim N(0, \sigma^2 I_i)$. The probability density function of y_i is given by

$$p(\mathrm{y}_i) = \frac{1}{\sqrt{2\pi}\left|\sigma^2 I_i\right|^{1/2}}.\exp\left(-\frac{1}{2}\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)^T (\sigma^2 I_i)^{-1}\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)\right).$$

Since b_i has a normal distribution with mean zero and variance covariance matrix D that is $\mathrm{b}_i \sim N(0, D)$ the probability density function of b_i is

$$p(\mathrm{b}_i) = \frac{1}{\sqrt{2\pi}\left|D\right|^{1/2}}.\exp\left(-\frac{1}{2}\mathrm{b}_i^T D^{-1}\mathrm{b}_i\right).$$

Thus the joint probability density function of the response y_i and b_i is written as

$$p(\mathrm{y}_i, \mathrm{b}_i) = \frac{1}{\sqrt{2\pi}\left|\sigma^2 I_i\right|^{1/2}}.\exp\left(-\frac{1}{2}(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i))^T (\sigma^2 I_i)^{-1}(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i))\right)$$

$$\frac{1}{\sqrt{2\pi}\left|D\right|^{1/2}}\exp\left(-\frac{1}{2}\mathrm{b}_i^T (D)^{-1}\mathrm{b}_i\right),$$

the likelihood function of the response y_i, is given by

$$L = \prod p(\mathrm{y}, \mathrm{b})$$

$$= \left(\frac{1}{2\pi}\right)^{m/2}\left(\frac{1}{\left|\sigma^2 I_i\right|}\right)^{m/2}\exp\left(-\frac{1}{2}\sum\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)^T (\sigma^2 I_i)^{-1}\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)\right)$$

$$\left(\frac{1}{2\pi}\right)^{m/2}\left(\frac{1}{\left|D\right|}\right)^{m/2}\exp\left(-\frac{1}{2}\sum \mathrm{b}_i^T \hat{D}^{-1}\mathrm{b}_i\right). \qquad (3.3)$$

Then the log-likelihood function of the response y_i is written as

$$\log L = -\frac{m}{2}\log(2\pi) - \frac{m}{2}\log\left|\sigma^2 I_i\right| - \frac{1}{2}\sum\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)^T (\sigma^2 I_i)^{-1}\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)$$

$$-\frac{m}{2}\log(2\pi)-\frac{m}{2}\log|D|-\frac{1}{2}\sum \mathrm{b}_i^T\hat{D}^{-1}\mathrm{b}_i.$$

For known ω, that is $\hat{\omega}$, we get $\hat{\sigma}^2$ and $\hat{D}$. The values of ϕ and b_i that jointly minimize

$$\sum_{i=1}^{m}\left(\log\left|\hat{D}\right|+\mathrm{b}_i^T\hat{D}^{-1}\mathrm{b}_i+\log\left|\hat{\sigma}^2 I_i\right|+\left(\mathrm{y}_i-f_i(\mathrm{x}_i,\phi_i,\mathrm{b}_i)\right)^T(\hat{\sigma}^2 I_i)^{-1}\left(\mathrm{y}_i-f_i(\mathrm{x}_i,\phi_i,\mathrm{b}_i)\right)\right)$$

(STEP1)

are twice the negative log of the posterior density of b_i for fixed ϕ and twice the negative log-likelihood for ϕ for fixed b_i. Thus an obvious strategy for estimation of b_i is to minimize and evaluate at suitable estimates of $\hat{\sigma}^2$ and $\hat{D}$ (Davidian and Giltinan, 1995).

Lindstrom and Bates (1990) defined maximum likelihood estimators for ϕ with respect to the marginal density function of y, that is

$$p(\mathrm{y})=\int p(\mathrm{y}\,|\,\mathrm{b})\,p(\mathrm{b})d\mathrm{b}, \tag{3.4}$$

however, the expectation function $\mathrm{f}_i(\phi_i)$ is nonlinear in b, and there is no closed-form expression for this density function and the calculation of such estimates would be very difficult. Instead, they approximated the conditional distribution of y for b near $\hat{\mathrm{b}}$ by a multivariate normal with expectation that is linear in b. To accomplish this they approximated the residual $\mathrm{y}-f(\phi)$ near $\hat{\mathrm{b}}$ as Taylor series expansion about $\mathrm{b}_i=\hat{\mathrm{b}}_i$ of $f_i\{d(a_i,\phi,\mathrm{b}_i)\}$ so that

$$\mathrm{y_i} \;=\; f_i\{d(a_i,\phi,\mathrm{b}_i)\}\approx f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\}+Z_i(\hat{\phi},\hat{\mathrm{b}}_i).(\mathrm{b}_i-\hat{\mathrm{b}}_i)+\mathrm{e}_i\,, \tag{3.5}$$

$$\mathrm{b}_i\sim N(0,D),\ \mathrm{e}_i\sim N(0,\sigma^2 I_i),$$

where $Z_i(\hat{\phi},\hat{\mathrm{b}}_i)$ is the $n_i\times k$ matrix of derivatives of $f_i\{d(a_i,\phi,\mathrm{b}_i)\}$ with respect to b_i evaluated at $\phi_i=d(a_i,\hat{\phi},\hat{\mathrm{b}}_i)$ and $\mathrm{b}_i=\hat{\mathrm{b}}_i$. The expectation of $\mathrm{y_i}$ is given as

$$\begin{aligned}E(\mathrm{y}_i) &= E\left(f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\}+Z_i(\hat{\phi},\hat{\mathrm{b}}_i)E\left(\mathrm{b}_i-\hat{\mathrm{b}}_i\right)+\mathrm{e}_i\right)\\ &= E\left(f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\}\right)+Z_i(\hat{\phi},\hat{\mathrm{b}}_i)E\left(\mathrm{b}_i-\hat{\mathrm{b}}_i\right)+E(\mathrm{e}_i)\\ &= f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\}-Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\,,\end{aligned}$$

and the variance of y_i is given by

$$Var(\mathrm{y}_i) = Var\left(f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)E\left(\mathrm{b}_i - \hat{\mathrm{b}}_i\right) + \mathrm{e}_i \right)$$

$$= Var\left(f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\}\right) + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)Var\left(\mathrm{b}_i - \hat{\mathrm{b}}_i\right)Z_i^T(\hat{\phi},\hat{\mathrm{b}}_i) + Var(\mathrm{e}_i)$$

$$= Z_i(\hat{\phi},\hat{\mathrm{b}}_i)Var(\mathrm{b}_i)Z_i^T(\hat{\phi},\hat{\mathrm{b}}_i) + \sigma^2 I_i$$

$$= Z_i(\hat{\phi},\hat{\mathrm{b}}_i)DZ_i^T(\hat{\phi},\hat{\mathrm{b}}_i) + \sigma^2 I_i.$$

Then, the response variables, y_i are distributed as a normal distribution with

$$\mathrm{y}_i \sim N\left(f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} - Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i \ ,\ Z_i(\hat{\phi},\hat{\mathrm{b}}_i)DZ_i^T(\hat{\phi},\hat{\mathrm{b}}_i) + \sigma^2 I_i \right).$$

Therefore, the marginal distribution of y_i is written as

$$p(\mathrm{y}_i) = \frac{1}{\sqrt{2\pi}\,|V_i|^{1/2}} \exp\left(-\frac{1}{2}\left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)^T V_i^{-1} \left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right) \right),$$

where $V_i = Z_i(\hat{\phi},\hat{\mathrm{b}}_i)DZ_i^T(\hat{\phi},\hat{\mathrm{b}}_i) + \sigma^2 I_i$.

The likelihood function of y_i, is written as

$$L = \prod p(\mathrm{y}_i) = \left(\frac{1}{2\pi}\right)^{m/2} \left(\frac{1}{|V_i|}\right)^{m/2} \exp\left(-\frac{1}{2}\sum\left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)^T V_i^{-1} \left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right) \right).$$

The log-likelihood function of y_i is obtained by

$$\log L = -\frac{m}{2}\log(2\pi) - \frac{m}{2}\log(|V_i|) - \frac{1}{2}\sum\left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)^T V_i^{-1} \left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right).$$

Hence, the twice negative log-likelihood function is given as

$$\sum\left(\log|V_i| + \left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)^T V_i^{-1} \left(\mathrm{y}_i - f_i\{d(a_i,\phi,\hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)\right).$$

(STEP 2)

We can rewrite the LB process which consists of the following two-step estimation scheme as follows.

STEP 1. Given the current estimate of ω, $\hat{\omega}$ (and thus $\hat{\sigma}^2$ and $\hat{D}$), minimize in ϕ and b_i, $i = 1, \ldots, m$. The twice negative log-likelihood is given as

$$-2\log L = \sum_{i=1}^{m}\Big(\log\left|\hat{D}\right| + \mathrm{b}_i^T \hat{D}^{-1}\mathrm{b}_i + \log\left|\hat{\sigma}^2 I_i\right| + \left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)^T$$

$$(\hat{\sigma}^2 I_i)^{-1}\left(\mathrm{y}_i - f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)\right)\Big),$$

where $\hat{D}$ is variance covariance of random effects,

b_i is random effects,

y_i is observations, and

$f_i(\mathrm{x}_i, \phi_i, \mathrm{b}_i)$ is the nonlinear fixed effects model.

Denote the resulting estimates of b_i and ϕ_0 as $\hat{\mathrm{b}}_i$ and $\hat{\phi}_0$.

STEP 2. Estimate ϕ and ω as the values $\hat{\phi}$ and $\hat{\omega}$ that give twice the negative approximate marginal normal log-likelihood function is obtained as

$$-2\log L = \sum\Big(\log\left|V_i\right| + \left(\mathrm{y}_i - f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi}, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)^T V_i^{-1}$$

$$\left(\mathrm{y}_i - f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi}, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)\Big),$$

where $V_i = Z_i(\hat{\phi}, \hat{\mathrm{b}}_i) D Z_i^T(\hat{\phi}, \hat{\mathrm{b}}_i) + \sigma^2 I_i$.

Iterate this process to a convergence, denoting the final estimates as $\hat{\phi}_{LB}$, $\hat{\omega}_{LB}$ and $\hat{\mathrm{b}}_{i,LB}$. Lindstrom and Bates (1990) called STEP 1 the 'pseudo-data' step, because joint estimation of ϕ and b_i by minimization of STEP 1 may be accomplished simultaneously by specifying and augmenting the nonlinear least square problem, analogous to the linear case. The STEP 2 of the algorithm is referred to as the 'linear mixed effects' step for the following reason. A further approximation allows the minimization in STEP 2 to be expressed as a linear estimation problem. Specifically, they approximate $f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\}$ by a linear function of ϕ by expansion about the current estimate $\hat{\phi}_0$ from STEP 1. They obtain

$$\mathrm{y}_i - f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i \approx \mathrm{y}_i - \left(f_i\{d(a_i, \hat{\phi}_0, \hat{\mathrm{b}}_i)\} + X_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)(\phi - \hat{\phi}_0)\right)$$

$$+ Z_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i$$

$$= \mathrm{y}_i^* - X_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)\phi ,$$

where $\mathrm{y}_i^* = \mathrm{y}_i - f_i\{d(a_i, \hat{\phi}_0, \hat{\mathrm{b}}_i)\} + X_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)\hat{\phi}_0 + Z_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i$.

Substituting $\mathrm{y}_i^* - X_i(\hat{\phi}_0, \hat{\mathrm{b}}_i)\phi$ for $\mathrm{y}_i - f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\} + Z_i(\hat{\phi}, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i$ in STEP 2 yields a problem similar in the form of the linear case.

Lindstrom and Bates (1990) implemented Newton-Raphson estimation by using previously developed computational methods for nonlinear fixed effects and linear mixed effects models in the LB algorithm.

A Newton-Raphson method

Lindstrom and Bates (1988) developed an efficient and effective implementation of the Newton-Raphson algorithm for estimating the parameters in mixed effects models for repeated data measurement. The maximum likelihood (ML) estimators for σ and θ are obtained by maximizing the log-likelihood corresponding to the marginal density of

$$\mathrm{y}_i \sim N\left(f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\} - Z_i(\hat{\phi}, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i \ ,\ Z_i(\hat{\phi}, \hat{\mathrm{b}}_i) D Z_i^T(\hat{\phi}, \hat{\mathrm{b}}_i) + \sigma^2 I_i \right).$$

This log-likelihood, l_F is given as

$$l_F(\phi, \sigma, \theta \mid \mathrm{y}) = -\frac{1}{2}\sum_{i=1}^{m}\log\left|\sigma^2 V_i(\theta)\right| - \frac{1}{2}\sigma^2 \sum_{i=1}^{m} r_i^T V_i^{-1}(\theta) r_i ,$$

where $r_i = f_i\{d(a_i, \phi, \hat{\mathrm{b}}_i)\} - Z_i(\hat{\phi}, \hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i$, $V(\theta) = Z_i(\hat{\phi}, \hat{\mathrm{b}}_i) D Z_i^T(\hat{\phi}, \hat{\mathrm{b}}_i) + \sigma^2 I_i$.

They solved for the ML estimate of σ^2 as a function of ϕ and θ.

That is $\hat{\sigma}^2 = \dfrac{1}{N}\sum_{i=1}^{m} r_i^T V_i^{-1}(\theta) r_i$, $N = \sum_{i=1}^{m} n_i$.

They obtain the profile log-likelihood, p_F of ϕ and θ by substituting σ^2 into l_F. The profile log-likelihood is $p_F(\phi, \theta \mid \mathrm{y}) = -\dfrac{1}{2}\sum_{i=1}^{m}\log\left|V_i(\theta)\right| - \dfrac{N}{2}\log\left[\sum_{i=1}^{m} r_i^T V_i^{-1}(\theta) r_i\right]$.

The derivatives of the profile log-likelihood for the Newton-Raphson algorithm can be calculated as follows;

$$\frac{\partial r_i^T V_i^{-1} r_i}{\partial \phi} = -2 Z_i^T(\hat{\phi}, \hat{\mathrm{b}}_i) V_i^T r_i , \quad \frac{\partial r_i^T V_i^{-1} r_i}{\partial \theta_i} = -r_i^T A_{ij} r_i ,$$

$$\frac{\partial^2 r_i^T V_i^{-1} r_i}{\partial\theta_j \partial\phi} = Z_i^T(\hat{\phi}, \hat{\mathrm{b}}_i)(A_{ij} + A_{ij}^T) r_i, \quad \frac{\partial \log|V_i|}{\partial\theta_j} = tr\left[V_i^{-1}\frac{\partial V_i}{\partial\theta_j}\right],$$

where $A_{ij} = V_i^{-1}\frac{\partial V_i}{\partial\theta_j}V_i^{-1}, \quad \frac{\partial A_{ij}}{\partial\theta_k} = -V_i^{-1}\left(\frac{\partial V_i}{\partial\theta_k}V_i^{-1}\frac{\partial V_i}{\partial\theta_j} - \frac{\partial^2 V_i}{\partial\theta_k\partial\theta_j} + \frac{\partial V_i}{\partial\theta_j}V_i^{-1}\frac{\partial V_i}{\partial\theta_k}\right)V_i^{-1}.$

In the mixed effects model each individual's vector of responses is modeled as a parametric function, where some of the parameters of 'effects' are random variables with a multivariate normal distribution.

The Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithm

The Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithm which is proposed by Kuhn and Lavielle (2005) is an another algorithm to estimate parameters of nonlinear mixed effects models. The general form of the model can be written as

$$y_{ij} = f(x_{ij}, \phi_i) + g(x_{ij}, \phi_i)e_{ij} \;\; ; \;\; e_{ij} \sim N(0, \sigma^2), \qquad (3.6)$$

$$\phi_i = \phi + b_i \qquad (3.7)$$

where y_{ij} is the jth response on the ith individual, $i = 1,\ldots,m$,

x_{ij} is the vector of predictor variables for the jth response on the individual i,

$f(x_{ij}, \phi_i)$ and $g(x_{ij}, \phi_i)$ are nonlinear functions of the predictor variables,

ϕ_i is unknown random vector of dimension $r \times 1$ for the ith individual, and

e_{ij} are independent $N(0, \sigma^2)$ random error terms.

The expected value of y_{ij} is given by

$$\begin{aligned} E(y_{ij}) &= E\left(f(x_{ij}, \phi_i) + g(x_{ij}, \phi_i)e_{ij}\right) \\ &= E(f(x_{ij}, \phi_i)) + E(g(x_{ij}, \phi_i)e_{ij}) \\ &= f(x_{ij}, \phi_i) + g(x_{ij}, \phi_i)E(e_{ij}) \\ &= f(x_{ij}, \phi_i), \end{aligned}$$

and the variance of y_{ij} is given as

$$
\begin{aligned}
Var(y_{ij}) &= Var\left(f(x_{ij},\phi_i) + g(x_{ij},\phi_i)e_{ij} \right) \\
&= Var\left(f(x_{ij},\phi_i)\right) + Var\left(g(x_{ij},\phi_i)e_{ij}\right) \\
&= 0 + g^2(x_{ij},\phi_i)Var(e_{ij}) \\
&= 0 + g^2(x_{ij},\phi_i)\sigma^2 \\
&= g^2(x_{ij},\phi_i)\sigma^2.
\end{aligned}
$$

Hence, the probability density function of the model (3.6) is written as

$$
f(y_{ij}) \quad = \frac{1}{\sqrt{2\pi\sigma^2}\left|g^2(x_{ij},\phi_i)\right|^{1/2}} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{y_{ij} - f(x_{ij},\phi_i)}{g(x_{ij},\phi_i)} \right)^2 \right).
$$

The expected value of ϕ_i from (3.7) is given by

$$
\begin{aligned}
E(\phi_i) \quad &= E\left(\phi + b_i\right) \\
&= E(\phi) - E(b_i) \\
&= \phi,
\end{aligned}
$$

and the variance of ϕ_i is given as

$$
\begin{aligned}
Var(\phi_i) &= Var\left(\phi + b_i\right) \\
&= Var\left(\phi\right) + Var\left(b_i\right) \\
&= D.
\end{aligned}
$$

Hence, the probability density function of the model (3.7) is written as

$$
f(\phi_i) \quad = \frac{1}{\sqrt{2\pi D}} \exp\left(-\frac{1}{2}\left(\phi_i - \phi\right)^T D^{-1}\left(\phi_i - \phi\right) \right).
$$

The complete probability density function of the model which is defined by equations (3.6) and (3.7) can be written as

$$
f(y_{ij},\phi) = \frac{1}{\sqrt{2\pi\sigma^2}\left|g^2(x_{ij},\phi_i)\right|^{1/2}}.\exp\left(-\frac{1}{2\sigma^2}\left(\frac{y_{ij} - f(x_{ij},\phi_i)}{g(x_{ij},\phi_i)} \right)^2 \right)
$$

$$-\frac{1}{\sqrt{2\pi D}}\exp\left(-\frac{1}{2}(\phi_i-\phi)^T D^{-1}(\phi_i-\phi)\right).$$

Therefore, the log-likelihood of $f(y_{ij},\phi)$ is obtained by

$$\log f(y_{ij},\phi) = -\sum_i^m\sum_j^{n_i}\log(g(x_{ij},\phi_i))-\frac{1}{2\sigma^2}\sum_i^m\sum_j^{n_i}\left(\frac{y_{ij}-f(x_{ij},\phi_i)}{g(x_{ij},\phi_i)}\right)^2$$
$$-\frac{m}{2}\log(|D|)-\frac{1}{2}\sum_{i=1}^m(\phi_i-\phi)^T D^{-1}(\phi_i-\phi)-\frac{N_{tot}+N}{2}\log(2\pi),$$

where $N_{tot}=\sum_{i=1}^m n_i$ is the total number of observations.

The Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithm consists in replacing the usual Expectation step of Expectation and Maximization step by a stochastic procedure. They perform an iteration k of the SAEM algorithm, as the following

1. Simulation-step: draw $\phi^{(k)}$ from the conditional distribution $f(.|y;\theta_k)$.
2. Stochastic approximation: update S_k according to

$$S_k = S_{k-1}+\gamma_k(\tilde{S}(y,\phi^{(k)})-S_k),$$

where γ_k is a decreasing sequence of step sizes used in MONOLIX which decreases as k^{-a}. More precisely, for any sequence of integers $K_1,K_2,...,K_J$ and any sequence $a_1,a_2,...,a_J$ of real numbers such that $0\le a_1<a_2<...<a_J\le 1$, they define the sequence of step sizes (γ_k) as follows.

$$\gamma_k = \frac{1}{k^{a_1}} \text{ for any } 1\le k\le K_1 \text{ and for } 2\le j\le J,$$

$$\gamma_k = \frac{1}{\left(k-K_{j-1}+\gamma_{K_{j-1}}^{-1/a_j}\right)^{a_j}} \text{ for any } \sum_{i=1}^{j-1}K_i+1\le k\le\sum_{i=1}^{j}K_i.$$

Here, $K=\sum_{j=1}^J K_j$ is the total number of iteration. They recommend to use $a_1=0$ (that is $\gamma_k=1$) during the first iterations, and $a_J=1$ during the last iteration. Indeed, the initial guess θ_0 may be far from the maximum likelihood value they were looking for and the first iteration with

$\gamma_k = 1$ which is allowed to converge quickly to a neighborhood of the maximum likelihood estimator. Then smaller step sizes ensure that the algorithm converge to the maximum likelihood estimator. In the case where J =2 with $a_1 = 0$ and $a_2 = 1$, the sequence of step sizes is

$$\gamma_k = 1 \quad \text{for } 1 \le k \le K_1$$

$$\gamma_k = \frac{1}{k - K_1 + 1} \quad \text{for } K_1 + 1 \le k \le K_1 + K_2$$

3. Maximization-step: update θ_k according to

$$\theta_k = Arg \max_{\theta} L(S_k, \theta).$$

The model (3.6) is considered in the case of $g(x_{ij}, \phi_i)$ = 1. The set of parameters to estimate is $\theta = (\phi, D, \sigma^2)$, where the complete model belongs to the exponential family and the approximation step reduces to only updating the sufficient statistics of the complete model.

$$S_{1,i,k} = S_{1,i,k-1} + \gamma_k (\phi_i^{[k]} - S_{1,i,k-1}), \quad i = 1, \ldots, m,$$

$$S_{2,k} = S_{2,k-1} + \gamma_k (\sum_{i=1}^{m} \phi_i^{[k]} \phi_i^{[k]/} - S_{2,k-1}),$$

$$S_{3,k} = S_{3,k-1} + \gamma_k (\sum_{ij} (y_{ij} - f(x_{ij}, \phi_i^{[k]}))^2 - S_{3,k-1}).$$

Then, θ_k is obtained in the maximization step as follows

$$\phi^{[k+1]} = \left(\sum_{i=1}^{m} C_i^{/} D_k^{-1} C_i \right)^{-1} \sum_{i=1}^{m} C_i^{/} D_k^{-1} S_{1,i,k},$$

$$D^{[k+1]} = \frac{1}{m} \left(S_{2,k} - \sum_{i=1}^{m} (C_i \phi_{k+1}) S_{1,i,k}^{/} - \sum_{i=1}^{m} S_{1,i,k} (C_i \phi_{k+1})^{/} + \sum_{i=1}^{m} (C_i \phi_{k+1})(C_i \phi_{k+1})^{/} \right),$$

$$\sigma^{2^{[k+1]}} = \sqrt{\frac{S_{3,k}}{N_{tot}}},$$

where $\phi^{[k]}$ is simulated according to the conditional distribution $f(.\,|\, y, \theta^{[k-1]})$ either directly or using a Metropolis-Hastings algorithm.

Metropolis-Hastings algorithm

The Metropolis-Hasting algorithm is a method for creating a Markov chain that can be used to generate a sequence of samples directly. This sequence can be used in Markov Chain Monte Carlo simulation to approximate the distribution, or to compute an integral such as an expected value as the following (Chib and Greenberg, 1995).

For $i = 1, 2, ..., m$, let $\phi_{i,0} = \phi_i^{[k-1]}$ for $p = 1, 2, ..., m,$ then

1. Generate u from $U(0,1)$,
2. draw $\tilde{\phi}_{i,p}$ using the normal distribution and draw $\phi_{i,p-1}$ from OLS,
3. find $f(y, \tilde{\phi}_{i,p}; \theta_k)$ and $f(y, \phi_{i,p-1}; \theta_k)$ by substituting $\tilde{\phi}_{i,p}$ and $\phi_{i,p-1}$, respectively,
4. $\alpha(\phi_{i,p-1}, \tilde{\phi}_{i,p}) = \min\left(1, \dfrac{f(y \mid \tilde{\phi}_{i,p}; \theta_k)}{f(y \mid \phi_{i,p-1}; \theta_k)}\right),$
5. if $u \le \alpha(\phi_{i,p-1}, \tilde{\phi}_{i,p})$ then $\phi_{i,p} = \tilde{\phi}_{i,p}$ else $\phi_{i,p} = \phi_{i,p-1}$.

The SAEM algorithm uses the sufficient statistics in the process of estimating parameters. The sufficient statistics is illustrated in the next section.

Sufficient Statistics

A statistic $T(X)$ is a sufficient statistic for θ if the conditional distribution of the sample X given the value of $T(X)$ does not depend on θ. To verify that a statistic $T(X)$ is a sufficient statistic for θ, by verification for any fixed values of x and *t*, the conditional probability $P_\theta(X = \mathrm{x} | T(X) = T(\mathrm{x}))$ is the same for all values of θ. Now, this probability is equal to 0 for all values of θ if $T(X) \ne t$. So, we verify only that $P_\theta(X = \mathrm{x} | T(X) = T(\mathrm{x}))$ does not depend on θ. But since $\{X = \mathrm{x}\}$ is a subset of $\{T(X) = T(\mathrm{x})\}$, hence

$$P_\theta(X = \mathrm{x} \mid T(X) = T(\mathrm{x})) = \frac{P_\theta(X = \mathrm{x} \text{ and } T(X) = T(\mathrm{x}))}{P_\theta(T(X) = T(\mathrm{x}))}$$

$$= \frac{P_\theta(X = \mathrm{x})}{P_\theta(T(X) = T(\mathrm{x}))}$$

$$= \frac{p(\mathrm{x} \mid \theta)}{q(T(\mathrm{x}) \mid \theta)},$$

where $p(\mathrm{x} \mid \theta)$ is the joint probability density function of the sample X and $q(t \mid \theta)$ is the probability density function of $T(X)$. Thus, $T(X)$ is a sufficient statistic for θ if and only if, for every x, the above ratio of probability density functions is constant as a function of θ (Casella and Berger, 2002).

The standard error of parameters derived from the Fisher information matrix is in the following section.

Estimation of the Fisher Information matrix

Let θ^* be the true unknown value of θ, and let $\hat{\theta}$ be the maximum likelihood estimate of θ. If the observed likelihood function l is sufficiently smooth, the asymptotic theory for maximum likelihood estimation holds and

$$\sqrt{N}(\hat{\theta} - \theta^*) \xrightarrow[N \to \infty]{} N(0, I(\theta^*)^{-1}),$$

where $I(\theta^*) = -\partial_\theta^2 \log l(y; \theta^*)$ is the true Fisher information matrix. Thus, an estimate of the asymptotic covariance of $\hat{\theta}$ is the inverse of the Fisher information matrix $I(\hat{\theta}) = -\partial_\theta^2 \log l(y; \hat{\theta})$. The Fisher information matrix of the nonlinear mixed effects model cannot be computed in a closed-form. A stochastic approximation of this matrix is used in previous versions of MONOLIX. An alternative is to approximate this information matrix by the Fisher information matrix of the Gaussian model deduced from the nonlinear mixed effects model after linearization of the function f around the conditional expectation of the individual parameters ($E(\phi_i \mid y; \hat{\theta}),\ \ 1 \le i \le N$). The Fisher information matrix of this Gaussian model is a block matrix (no correlations between the estimated fixed effects and the estimated variances). The gradient of f is numerically computed (Davidian and Giltinan, 1995).

Applications and results

In this section, we consider the carcinoembryonic antigen (CEA) values of colorectal cancer patients. We propose the statistical model (CEA model) and obtain the coefficient of determination (R^2) of the predicted model. The numerical results for the parameters in the CEA model which are estimated by the ordinary least squares (OLS) method are also presented. The results from the OLS method for individuals will be averaged to be the initial value for estimate parameters by Lindstrom and Bates (LB) and the Stochastic Approximation version of the standard Expectation and Maximization step (SAEM) algorithms. The numerical results are shown in this chapter. Finally, the comparison between the numerical results of the LB and SAEM algorithms are presented, and show the observation values and predicted curves of the CEA model with parameters estimated by the LB and SAEM algorithms.

Carcinoembryonic Antigen (CEA) Data

In this study the subjects have participated in the National Cancer Institute of Thailand from July 1998 to April 2008. All of seven cancer cases have been detected clinically and confirmed histologically from biopsy tissue specimens. All subjects have undergone colorectal surgery after the colorectal cancer diagnosis and were recurrences appeared later. The follow ups started at the surgery date and are in the range negative two to 55 months with a median of 18 months over their follow up period. The follow ups started at the surgery date and then at least six times for measurement of CEA (ng/ml) values and had a median of 10 observations over their follow up period. The recurrent times are in the range three to 38 months after the surgery date and have a median of nine months over their follow up period. Table 3.1 contains descriptive statistics for times of the participants, the amount of follow-up, number of repeated CEA measurements and the recurrent time.

We plot the data for each group of participants. Figure 3.1 shows the scatter plot of the observed CEA levels for each subject. It shows the scatter plot of the CEA levels on the large scale. Therefore we transformed these CEA values with the logarithm to make it easier to visualize. We plot the logarithm of the CEA data for each group of participants in Figure 3.2. The logarithm of the CEA value increases slowly exponentially for each curve. The black dot for each curve is the recurrent time (month after surgery). The logarithms of the CEA values after the recurrent time are higher than the logarithms of the CEA values before the recurrent time.

Table 3.1 Description of study participants after surgery

	Colorectal cancer patients
No. of participants	7
Months of follow up	
- median	18
- range	-2 – 55
No. of repeated measurements	
- median	10
- range	6 – 20
Recurrent time (months)	
- median	9
- range	3 – 38

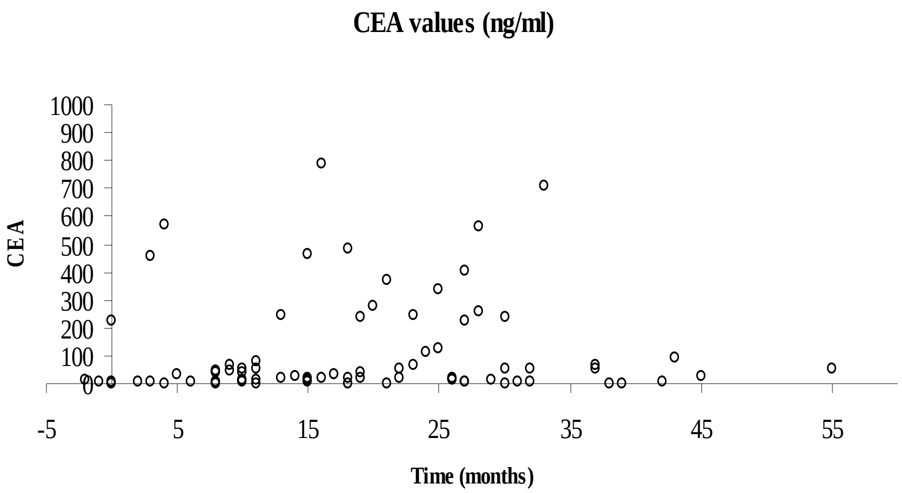


Figure 3.1 CEA (in nanograms per milliliter) versus time (months) of seven subjects.

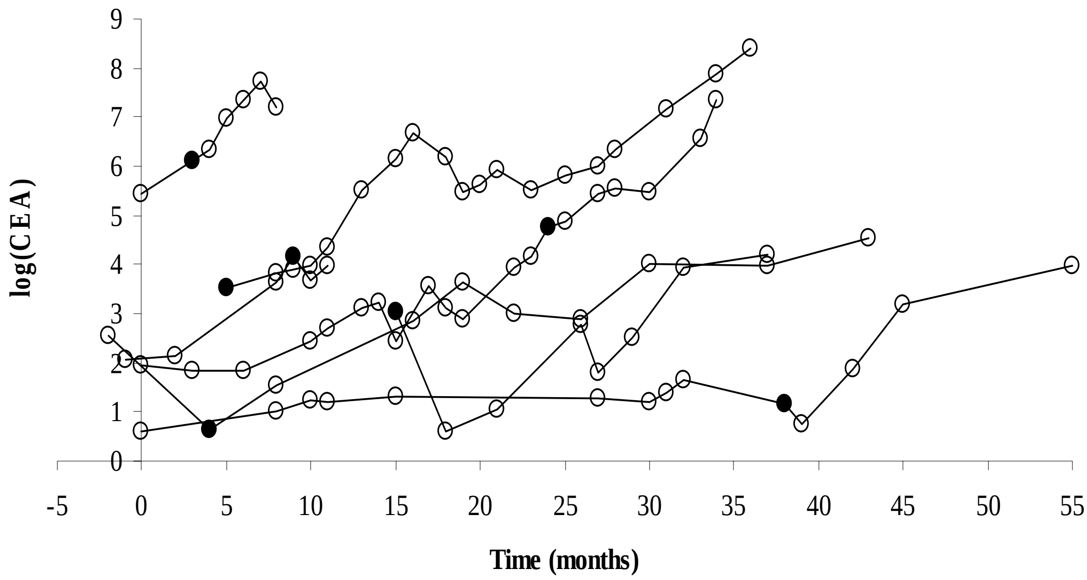


Figure 3.2 Log(CEA) (in nanograms per milliliter) versus time (months) seven subjects. The black dots are the recurrent time for each subject.

Statistical Modeling

From the logarithm of the CEA value curve (Figure 3.2) we see that trends in each subject case represent a period of slow exponential increase in the peripheral logarithm of CEA levels. Next we try to use the exponential model for these data. The exponential model will be approached to find a recurrent time after surgery, which is the time point between the flat and rapid increase in the exponential phase.

We used the LB and SAEM algorithms to estimate the parameters of the proposed model. Let y_{ij} is logarithm of CEA value for the jth response on the ith individual, the nonlinear model is $y_{ij} = f(x_{ij}, \phi_i) + e_{ij}$.

The proposed model is written as

$$f(x_{ij}, \phi_i) = \exp(\phi_{0i})\exp(\phi_{1i}(x_{ij} - \phi_{2i})).$$

Since, $\phi_i = A_i\phi + B_i b_i$, then $\phi_{0i} = \phi_0 + b_{0i}$,

$\phi_{1i} = \phi_1 + b_{1i}$, and

$\phi_{2i} = \phi_2 + b_{2i}$.

Therefore we can rewrite the CEA model as follows

$$f(x_{ij}, \phi_i) = \exp(\phi_0 + b_{0i})\exp\Big((\phi_1 + b_{1i})\big(x_{ij} - (\phi_2 + b_{2i})\big)\Big)$$
$$= \exp(\phi_0)\exp(b_{0i})\exp\Big((\phi_1 + b_{1i})\big(x_{ij} - (\phi_2 + b_{2i})\big)\Big). \quad (3.8)$$

Here, $\mathrm{b}_i = [b_{0i} \quad b_{1i} \quad b_{2i}]'$ has mean zero and variance-covariance matrix D.

x_{ij} is time(months) (at surgery date, x_{ij} = 0),

ϕ_0 is the log(CEA) level (ng/ml) at the exponential phase,

ϕ_1 is the exponential rate constant during the exponential log(CEA) phase,

ϕ_2 represents the time (months after surgery date) between the flat and rapid increase in the exponential log(CEA) phase,

b_{0i} is random and allows the exponential phase to vary among participants,

b_{1i} is random and allows the exponential rate constant to vary among participants, and

b_{2i} is random and allows the time between flat and rapid increase in the exponential log(CEA) phase to vary among participants.

Initial values

Initial values for parameters ϕ_0, ϕ_1, ϕ_2 and σ^2 must be specified for the first step. As in a standard nonlinear estimation, poor starting values can lead to poor results. We use the ordinary least square (OLS) method to find the initial values. Lindstrom and Bates (1990) discussed a number of computational issues regarding implementation of the algorithm, including recommendations for selecting starting values, which are required for all parameters at STEP1 of the initial iterate. As the procedures already have been discussed, they suggested the OLS estimator minimizing

$$\sum_{i=1}^{m}\left[\mathrm{y}_i - f_i\{d(\mathrm{a}_i, \phi, 0)\}\right]'\left[\mathrm{y}_i - f_i\{d(\mathrm{a}_i, \phi, 0)\}\right] \text{ for } \phi \text{ and } \mathrm{b}_i = 0,\ i = 1, \ldots, m.$$

We obtain the individual parameters ϕ_0, ϕ_1, ϕ_2 which are estimated by the OLS method. We fit the data for each subject by using the proposed model and compute the coefficient of determination (R^2). The R^2 of each subject was greater than 0.5. Table 3.2 contains the number of repeated measurements and individual parameters ϕ_0, ϕ_1, ϕ_2 and R^2 of proposed model for each subject.

Table 3.2 The individual parameters for each subject by the OLS method

Subject	No.of repeated measurements	ϕ_0	ϕ_1	ϕ_2	R^2
1	20	0.45573	0.044651	0.53271	0.95812
2	7	2.3821	0.042363	16.038	0.86748
3	10	0.70245	0.022613	5.3244	0.68099
4	6	0.96852	0.061193	3.4735	0.89942
5	19	1.3659	0.02237	3.6023	0.78796
6	14	-0.65292	0.035884	1.9065	0.65062
7	8	-0.429	0.053346	1.8518	0.50668
Mean		0.684683	0.040346	4.675601	

Table 3.2, shows that the coefficient of determination (R^2) of all subjects are greater than 0.5, and the average means of parameters ϕ_0, ϕ_1 and ϕ_2 are 0.68463, 0.040346 and 4.675601 respectively. Let $\overline{\phi}_0$, $\overline{\phi}_1$ and $\overline{\phi}_2$ be the average mean of parameters ϕ_0, ϕ_1 and ϕ_2 respectively. We find the estimates of random effects (b_i) by using

$$b_{0i} = \phi_{0i} - \overline{\phi}_0 ,\ b_{1i} = \phi_{1i} - \overline{\phi}_1 ,\ b_{2i} = \phi_{2i} - \overline{\phi}_2 ,$$

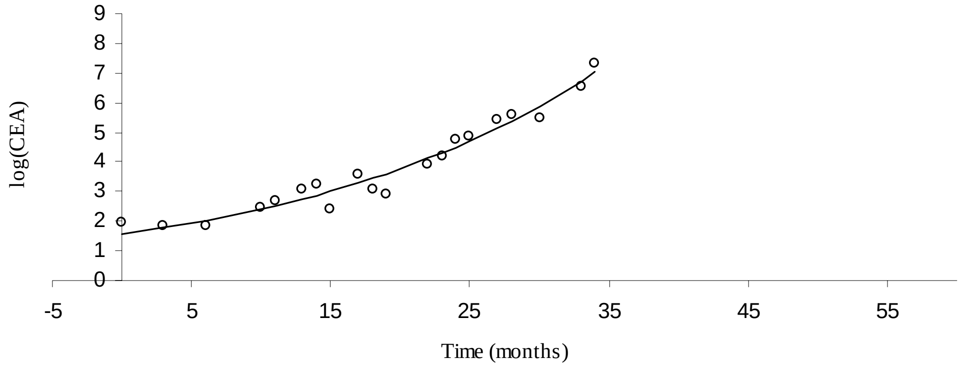
which are shown in Table 3.3.

Table 3.3 Estimates of random effects of the proposed model

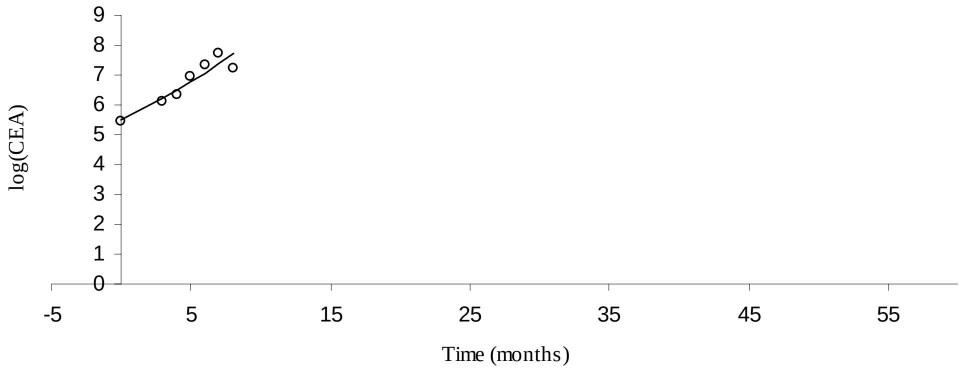
Subject	b_{0i}	b_{1i}	b_{2i}
1	-0.22895	0.004305	-4.14289
2	1.697417	0.002017	11.3624
3	0.017767	-0.01773	0.648799
4	0.283837	0.020847	-1.2021
5	0.681217	-0.01798	-1.0733
6	-1.3376	-0.00446	-2.7691
7	-1.11368	0.013	-2.8238

The trajectories of the change in log(CEA) volumes with time for the individual subjects as estimated by OLS are shown in Figure 3.2. The individual curves are indicated by solid lines, and the observed data are indicated by open circles. For each subject, the individual curve is much closer to the observed values. This reflects the benefit of mixed effects models in that subject variations are simultaneously accounted for.

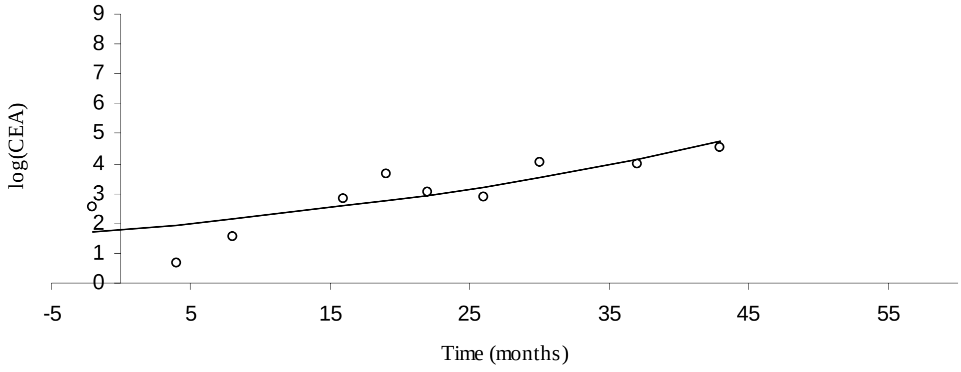
Subject 1



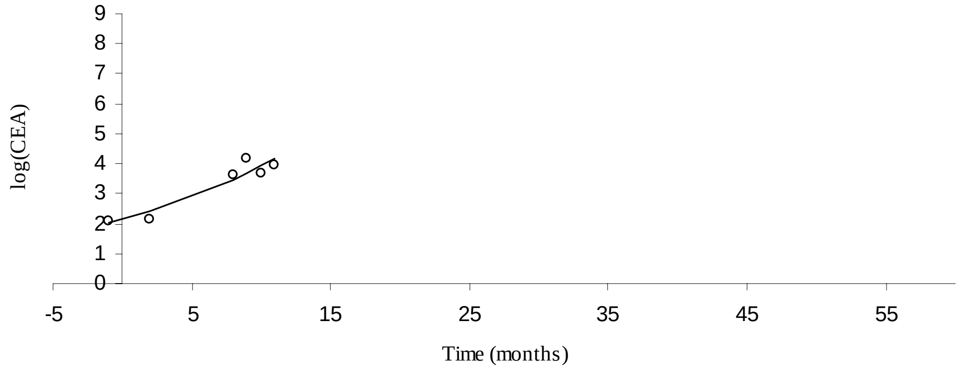
Subject 2



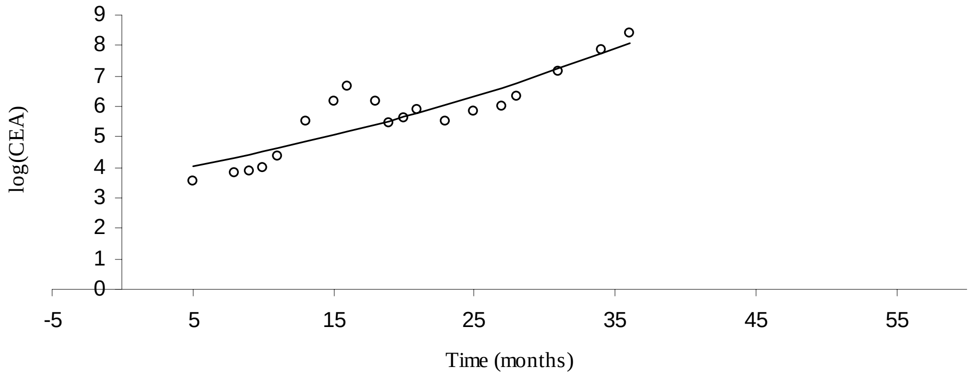
Subject 3

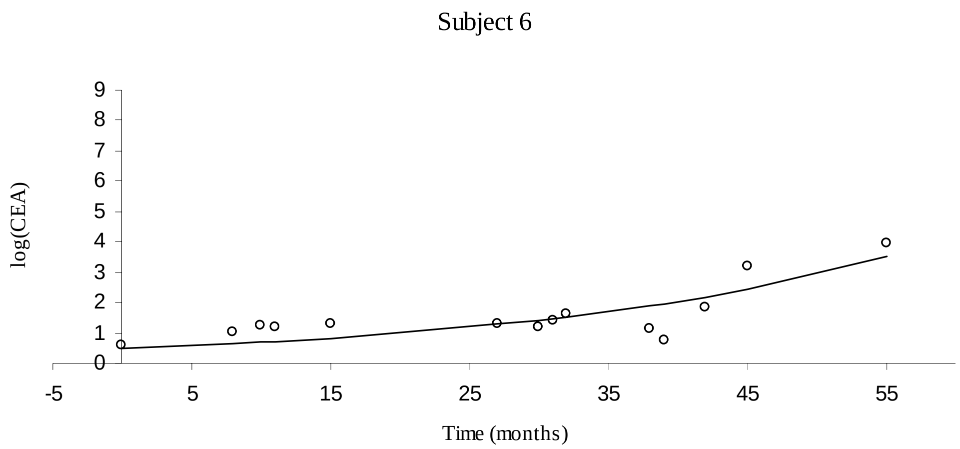


Subject 4



Subject 5





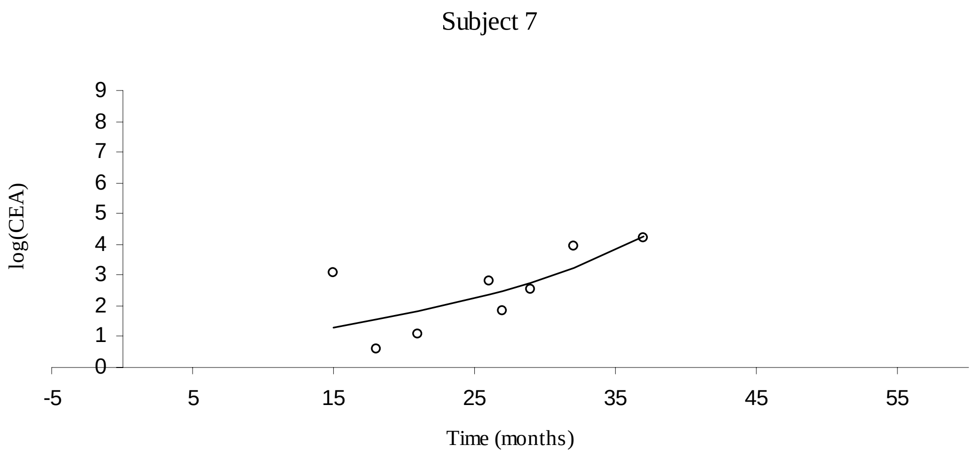


Figure 3.3 Estimated values by OLS method of the log(CEA) from model (3.8) for seven subjects. The solid lines are estimated individual curves and the observed values are indicated by the open circles.

The Lindstrom and Bates (LB) algorithm

In the above model there are many methods to estimate the unknown parameter vector. One method is a nonlinear mixed effects model algorithm as proposed by Lindstrom and Bates(1990). We also employed the LB algorithm to estimate the parameters of our proposed model. The algorithm uses a combination of least square estimators for nonlinear fixed effects models and maximum likelihood estimators for a linear mixed effects model. The linear mixed effects model is used to estimate parameters from an approximation to the marginal distribution of the complete data vector. This combination provides approximate maximum likelihood estimates for the nonlinear mixed effects model. The LB algorithm comprised of two steps of estimation schemes as follows.

STEP 1 Pseudo-data (PD)

Give the current estimate of ω, $\hat{\omega}$ (and thus $\hat{\sigma}^2$ and $\hat{D}$), and minimize in ϕ and b_i, $i=1,...,m$. The twice negative log-likelihood is

$$-2\log L = \sum_{i=1}^{m}\Big(\log\left|\hat{D}\right| + \mathrm{b}_i^T\hat{D}^{-1}\mathrm{b}_i + \log\left|\hat{\sigma}^2 I\right| + \left(\mathrm{y}_i - f_i(\mathrm{x}_i,\phi_i,\mathrm{b}_i)\right)^T$$
$$(\hat{\sigma}^2 I)^{-1}\left(\mathrm{y}_i - f_i(\mathrm{x}_i,\phi_i,\mathrm{b}_i)\right)\Big).$$

where

$\hat{D}$ is the variance covariance of random effects,

b_i represents random effects,

y_i is observations, and

$f_i(\mathrm{x}_i,\phi,\mathrm{b}_i)$ is the nonlinear fixed effects model.

We denote the resulting estimates as $\hat{\mathrm{b}}_i$ and $\hat{\phi}_0$.

STEP 2 Linear mixed effects (LME)

Estimate ϕ and ω as the values of $\hat{\phi}$ and $\hat{\omega}$. The log-likelihood is

$$L_{LB}(\beta,\omega) = \sum_{i=1}^{m}\left(\log\left|V_i(\hat{\beta}_0,\hat{b}_i,\omega)\right| + \left(y_i - f_i\left\{d(a_i,\beta,b_i)\right\} + \left.\frac{\partial f_i\left\{d(a_i,\beta,b_i)\right\}}{\partial b_i}\right|_{b_i=\hat{b}_i}\hat{b}_i\right)^T\right.$$
$$\left.V_i^{-1}(\hat{\beta}_0,\hat{b}_i,\omega)\left(y_i - f_i\left\{d(a_i,\beta,b_i)\right\} + \left.\frac{\partial f_i\left\{d(a_i,\beta,b_i)\right\}}{\partial b_i}\right|_{b_i=\hat{b}_i}\hat{b}_i\right)\right).$$

The twice negative approximate marginal normal log-likelihood is

$$-2\log L = \sum\Big(\log\left|V_i\right| + \left(\mathrm{y}_i - f_i(\mathrm{a}_i,\phi,\hat{\mathrm{b}}_i) + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)^T V_i^{-1}$$
$$\left(\mathrm{y}_i - f_i(\mathrm{a}_i,\phi,\hat{\mathrm{b}}_i) + Z_i(\hat{\phi},\hat{\mathrm{b}}_i)\hat{\mathrm{b}}_i\right)\Big),$$

where $V_i = Z_i(\hat{\phi},\hat{\text{b}}_i)DZ_i^T(\hat{\phi},\hat{\text{b}}_i)+\sigma^2 I$ and $Z_i(\hat{\phi},\hat{\text{b}}_i)$ is the $n_i \times k$ matrix of derivatives of $f_i\{d(a_i,\phi,\text{b}_i)\}$ with respect to b_i and evaluated at $\phi_i = d(\text{a}_i,\hat{\phi},\hat{\text{b}}_i)$ so that $\text{b}_i = \hat{\text{b}}_i$.

From the marginal distribution of response y_i

$$\text{y}_i = f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\} + Z_i(\hat{\phi},\hat{\text{b}}_i)(\text{b}_i - \hat{\text{b}}_i) + \text{e}_i\,,$$

we obtain the expected value and variance as the following.

The expectation of y_i is obtained as

$$\begin{aligned}
E(\text{y}_i) &= E\Big(f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\} + Z_i(\hat{\phi},\hat{\text{b}}_i).E\big(\text{b}_i - \hat{\text{b}}_i\big) + \text{e}_i\Big)\\
&= E\Big(f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\}\Big) + Z_i(\hat{\phi},\hat{\text{b}}_i)E\big(\text{b}_i - \hat{\text{b}}_i\big) + E(\text{e}_i)\\
&= f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\} - Z_i(\hat{\phi},\hat{\text{b}}_i)\hat{\text{b}}_i,
\end{aligned}$$

and the variance of y_i is given as

$$\begin{aligned}
Var(\text{y}_i) &= Var\Big(f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\} + Z_i(\hat{\phi},\hat{\text{b}}_i)E\big(\text{b}_i - \hat{\text{b}}_i\big) + \text{e}_i\Big)\\
&= Var\Big(f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\}\Big) + Z_i(\hat{\phi},\hat{\text{b}}_i)Var\big(\text{b}_i - \hat{\text{b}}_i\big)Z_i^T(\hat{\phi},\hat{\text{b}}_i) + Var(\text{e}_i)\\
&= Z_i(\hat{\phi},\hat{\text{b}}_i)Var(\text{b}_i)Z_i^T(\hat{\phi},\hat{\text{b}}_i) + \sigma^2 I_i\\
&= Z_i(\hat{\phi},\hat{\text{b}}_i)DZ_i^T(\hat{\phi},\hat{\text{b}}_i) + \sigma^2 I_i.
\end{aligned}$$

Hence, the response variables are distributed with

$$\text{y}_i \sim N\Big(f_i\{d(\text{a}_i,\phi,\hat{\text{b}}_i)\} - Z_i(\hat{\phi},\hat{\text{b}}_i)\hat{\text{b}}_i\ ,\ Z_i(\hat{\phi},\hat{\text{b}}_i)DZ_i^T(\hat{\phi},\hat{\text{b}}_i) + \sigma^2 I_i\Big).$$

We obtain the parameter estimates $\hat{\phi},\ \hat{\omega}$ and go to STEP1. We iterate this process until its convergence and denote the final estimates as $\hat{\phi}_{LB}, \hat{\omega}_{LB}$, and $\hat{\text{b}}_{i,LB}$.

The estimates of the fixed effects parameters (the ϕ's) as well as their asymptotic z values (z value = estimate/standard error) and estimates of the error variance and variance-covariance matrix for random effects (D) by the LB algorithm are given in Table 3.4.

Table 3.4. Estimates of the fixed effects, standard errors (SE) and approximate estimates of the error variance and variance-covariance matrix for the random effects by the LB algorithm

parameters	LB			
	Estimates	SE	z	CI
ϕ_0	1.1325	0.5138	2.204	(0.125 , 2.139)
ϕ_1	0.0283	0.0062	4.564	(0.016 , 0.040)
ϕ_2	17.489	8.584	2.037	(0.664 , 34.313)
σ^2	0.831			
D_{11}	0.9044			
D_{22}	0.0007			
D_{33}	0.1868			

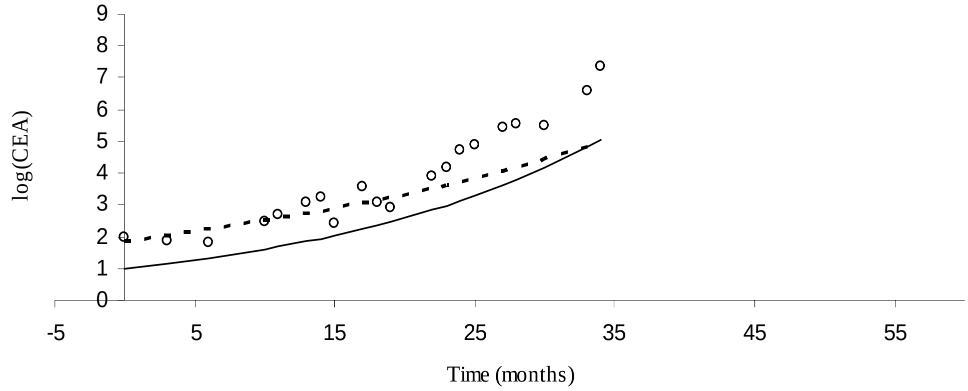
From Table 3.4 the fixed parameters ϕ_0, ϕ_1, and ϕ_2 of CEA model which are estimated by the LB algorithm as equal to 1.1325, 0.0283, and 17.489 respectively. The standard errors of each fixed parameters are 0.5138, 0.0062, and 8.584. The asymptotic z values (z value = estimate/standard error) of ϕ_0, ϕ_1, and ϕ_2 are 2.204, 4.564, and 2.037 respectively. Therefore the population and individual curves are computed, respectively as follows

$$f(x_{ij}, \phi_i) = \exp(1.1325)\exp\Big(0.0283\big(x_{ij} - 17.489\big)\Big),$$

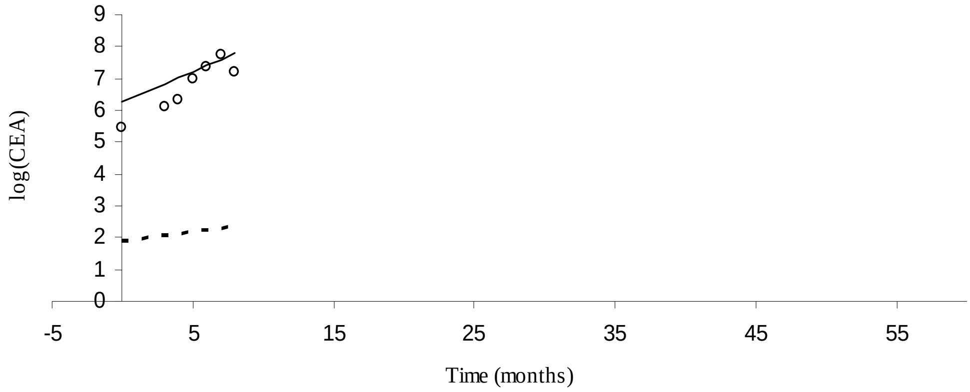
$$f(x_{ij}, \phi_i) = \exp(1.1325)\exp(b_{0i})\exp\Big((0.0283 + b_{1i})\big(x_{ij} - (17.489 + b_{2i})\big)\Big).$$

The trajectories of the log(CEA) volumes with time for the population and for the individual subjects which are estimated by the LB algorithm are shown in Figure 3.4. The population and individual curves are indicated by dashed and solid lines, respectively, and the observed data are indicated by open circles. For each subject, the individual curve is much closer to the observed values and also shows the slow exponential phase of log(CEA) values. All individual curves are similar to the population curve. This reflects the benefit of mixed effects models in that subject variations are simultaneously accounted for.

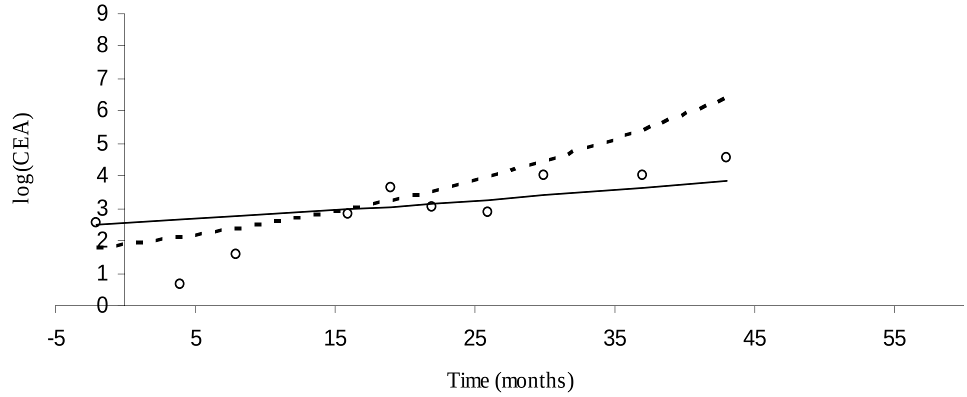
Subject 1



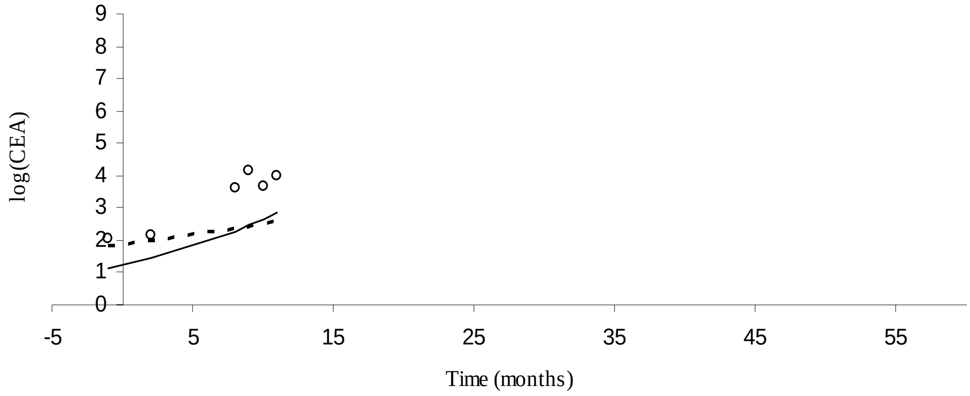
Subject 2



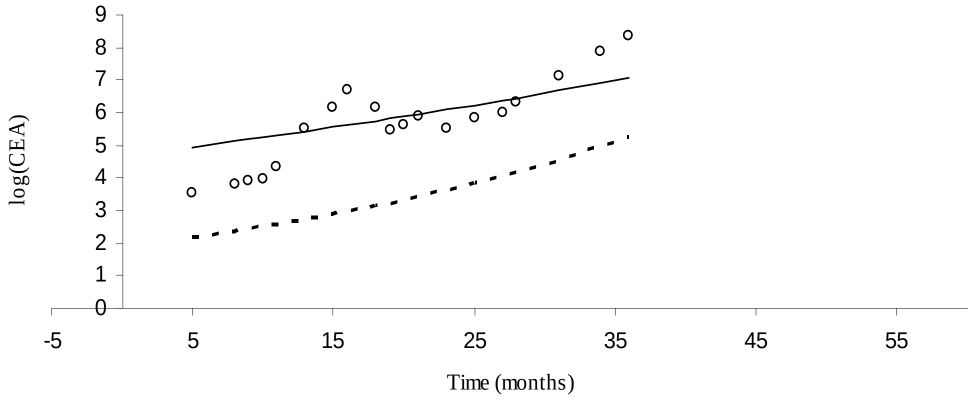
Subject 3



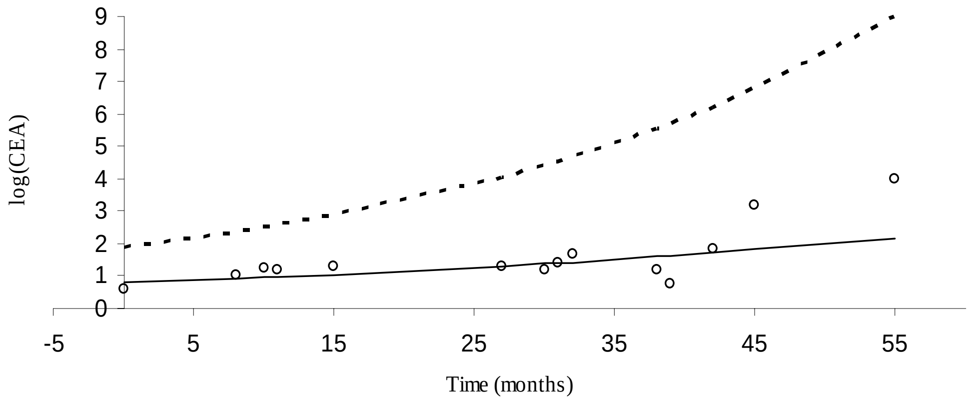
Subject 4



Subject 5



Subject 6



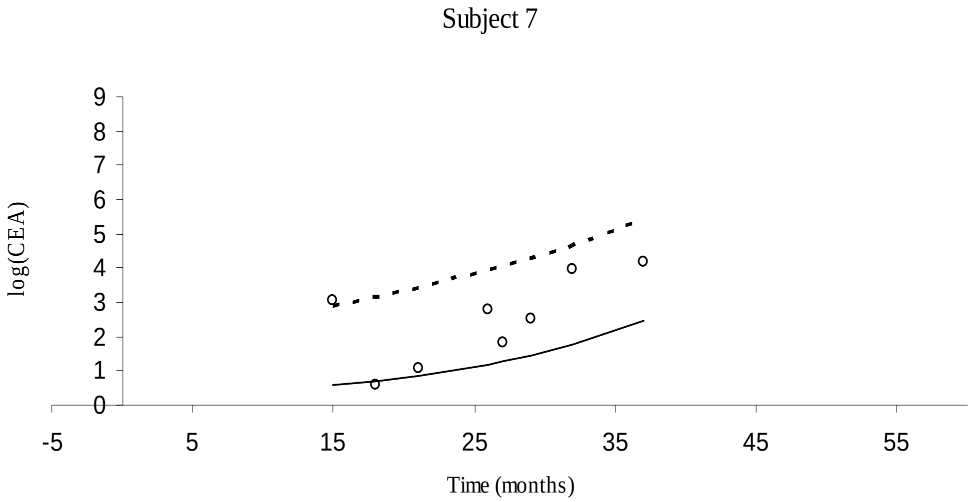


Figure 3.4 Curves of estimated values from the LB algorithm of the log(CEA) volumes from the CEA model for seven subjects. The solid and dashed lines represent estimated individual and population curves respectively. The observed values are indicated by the open circles.

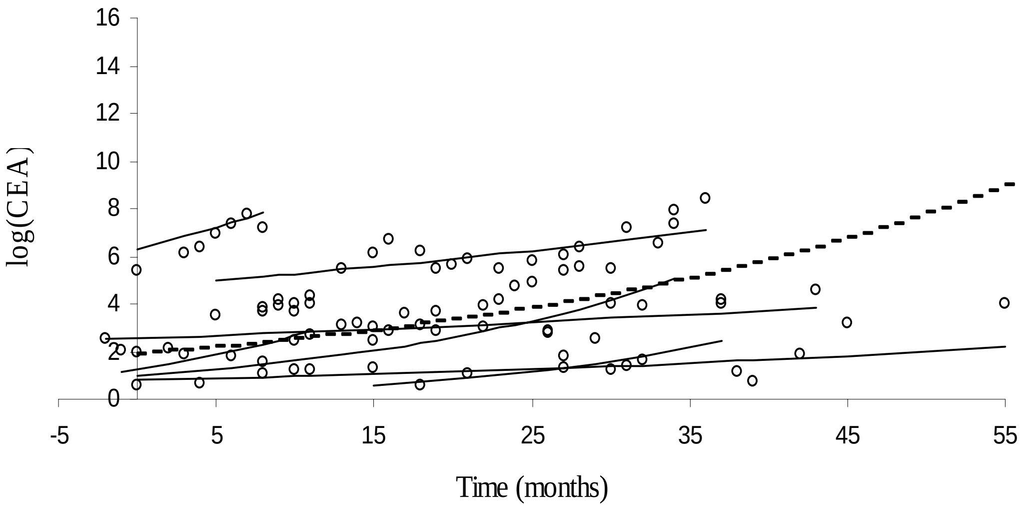


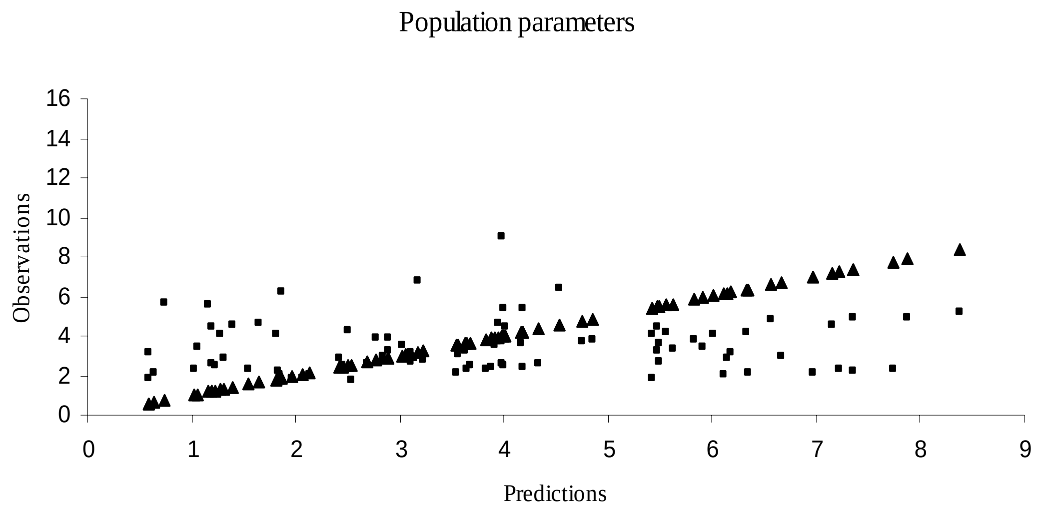
Figure 3.5 Log(CEA) level (in ng/ml) of seven patients: Data (open circles) and individual (solid lines) fitted curves from LB estimation. The dashed line represents the mean curve.

From Figure 3.4 we found that the estimated individual curves of the CEA model fit the observed values of colorectal cancer patients for each of the subjects very well; better than the population curves.

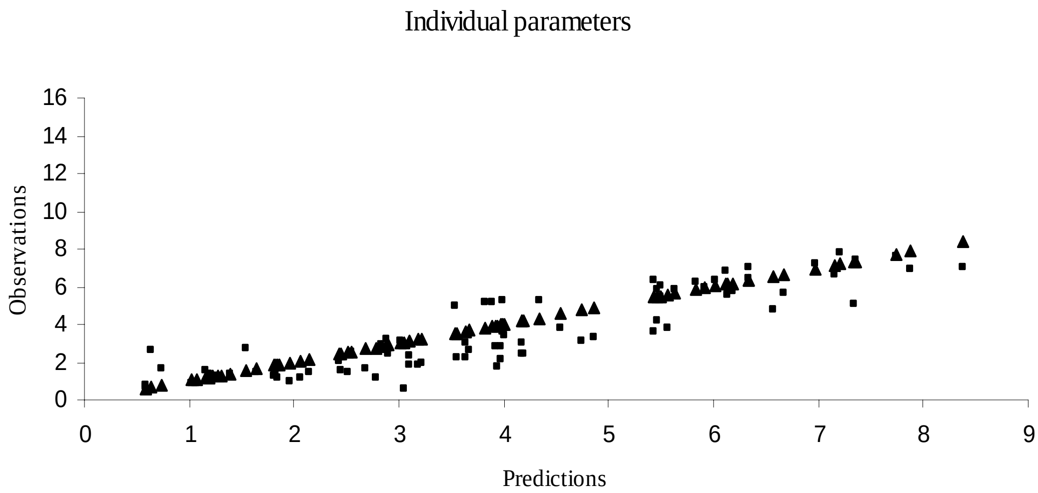
The predicted curves for the LB estimates are shown in Figure 3.5. The dashed line represents the mean curve, that is the curve fitted from model (3.8) with $b = 0$.

From Figure 3.5 we found that the individual curves of the CEA model, which is estimated by the LB algorithm, fit the observed values of colorectal cancer patients very well. While the mean curve of the CEA model fits the logarithm of the CEA of colorectal cancer patients less than individual curves.

The goodness of fit of the quality of CEA model with time for population and individual prediction plots by the LB algorithm are displayed in Figure 3.6.



(a)



(b)

Figure 3.6 Goodness of fit plots by the LB algorithm for the CEA model. Observation versus (a) population prediction (in ng/ml), (b) individual prediction (in ng/ml). The triangles represent observation and squares represent prediction.

References

1. Bast R.C., Ravdin Jr.P., Hayes D.F., Bates S., Fritsche H., Jessup J.M., Kemeny N., Locker G.Y., Mennel R.G., and Mark R. (2001). 2000 Update of Recommendations for the Use of Tumor Markers in Breast and Colorectal Cancer: Clinical Practice Guidelines of the American Society of Clinical Oncology. Journal of Clinical Oncology; 15:1865-1878.
2. Bates D.M. and Watts D.G. (1998). Nonlinear Regression Analysis and its Applications. Wiley, New York.
3. Benchimol S., Fuks A., Jothy S., Beauchemin N., Shirota K., and Stanners C.P. (1989). Carcinoembryonic Antigen, a human tumor marker, functions as an intercellular adhesion molecule. Cell; 57: 327-334.
4. Benson III Al.B., Desch C.E., Benson III Al.B., Smith T.J., Flynn P.J., Krause C., Loprinzi C.L., Minsky B.D., Petrelli N.J., Pfister D.G., and Somerfield M.R. (2000). 2000 Update of American Society of Clinical Oncology Colorectal Cancer surveillance guidelines. Journal of Clinical Oncology; 18(20):3586-3588.
5. Casella G. and Berger R.L.(2002). Statistical inference. California: Duxbury Press.
6. Castells A., Bessa X., Daniels M., Ascaso C., Lacy A.M., Garcia-Valdecasas J.C., Gargallo L., Novell F., Astudillo E., Filella X., and Piqu e′ J.M. (1998). Value of postoperative surveillance after radical surgery for colorectal cancer. Diseases of the Colon & Rectum; 41:714-724.
7 Chib S., and Greenberg E. (1995). Understanding the Metropolis-Hastings algorithm. American Statistical Association; 49(4): 327-335.
8. Claudio F., Silvia M.T., Stefano R., Giorgio M., Gaetano C., Sergio P., and Ugo B. (1987). Follow-up of colorectal cancer resected for cure an experience with CEA, TPA, Ca19-9 analysis and second-look surgery. Diseases of the Colon & Rectum; 30:273-277.
9. Crowder, M.J. and Hand, D.J.(1990). Analysis of Repeated Measures. London: Chapman and Hall.
10. Davidian, M. and Giltinan, D.M. (1995). Nonlinear models for repeated measurement data. London, Chapman and Hall.
11. Delyon B., Lavielle M. and Moulines E. (1999). Convergence of a stochastic approximation version of the EM algorithm. The annals of statistics, 27: 94-128.

12. Doccherty J.G., Mc Gregor J.R., O'Dwyer P.J., Galloway D.J. (1994). Local recurrence of colorectal cancer: the problem, mechanisms management and adjuvant therapy. British Journal of Surgery, Jan; 81(1): 7-19.

13. Fieuws, S., Verbeke, G. and L. J. Brant (2003). Classification of longitudinal profiles using nonlinear mixed-effects models. Technical Report.IAP Statistics Network.

14. Kuhn E. and Lavielle M. (2005). Maximum likelihood estimation in nonlinear mixed effects models. Computational Statistics and Data Analysis, 49: 1020-1038.

15. Lewis B.S. and Stuart L.B. (1980). Evaluation of Methods for Estimating Population Pharmacokinetic Parameters. I. Michaelis-Menten Model: Routine Clinical Pharmacokinetic Data. Journal of Pharmacokinetics and Biopharmaceutics; 8(6): 553-571.

16. Liang H. and Sha N. (2004). Modeling antitumor activity by using a non-linear mixed-effects model. Mathematical Biosciences 189: 61-73.

17. Lindstrom M.J. and Bates D.M. (1988). Newton-Raphson and EM algorithms for linear mixed effects models for repeated measures data. Journal of the American Statistical Association; 83: 1014-1022.

18. Lindstrom M.J. and Bates D.M. (1990). Nonlinear mixed effects model for repeated measures data. Biometrics, 46: 673-678.

19. Morrell C.H., Pearson J.D., Carter H.B. and Brant L.J.(1995). Estimating unknown transition times using a piecewise nonlinear mixed-effects model in men with prostate cancer. Journal of the American Statistical Association; 90: 45-53.

20. Mukta M.W., Anuradha W., and Diana B. (1995). Prostate-specific antigen, a serin protease, facilitates human prostate cancer cell invasion. Clinical cancer research; 1: 1089-1094.

21. Phyllis A.W, Sarah L., and Lynn A.G.R. (1997). An adjustment to the 1997 estimate for new prostate cancer cases. American cancer society; 80(9): 1810-1813.

22. Walker S. (1996). An EM algorithm for nonlinear random effects models. Biometrics; 52: 934-944.

23. Zheng C.X., Zhan W.H., Zhao J.Z., Wang D.P., He Y.L., and Zheng Z.Q. (2001).The prognostic value of preoperative serum levels of CEA, CA19-9 and CA72-4 in patients with colorectal cancer. World J of Gastroenterology; 7(3): 431-434.

24. http://software.monolix.org

25. http://www.cudahychamber.com/NutritionTheDigestiveSystem.htm

7.4 Dose Response Relationship with Applications

Dose response models are mathematical relationships (functions) that relate (predict) between the dose and the measure of observed effect. The observed effect can be extremely complex, depending on a variety of factors including the absorption, metabolism and elimination of the drug, and the presence of other drugs or disease.

Dose response models have been used to model quantal response data. Quantal response data may be classified by one of two possibilities, e.g. dead or alive, with or without a particular type of tumor, normal or abnormal level of a hormone. Researchers have considered the experiment about carcinogen with animal (mice). They would like to know levels of the dose of the drug that the mice will get cancer. Dose response models have been used to utilize about these problems.

The multistage Weibull model, logistic model and log-logistic model are three popular dose response models. There are many examples of research that utilize dose response models Guess, 1997; Feldstein, 1978). For example, first, these models have been centered on the problems to experimental animal-carcinogenesis data. Low-dose-rate extrapolations for DDT and chloroform are examples for using the models. Responders are animals with liver hepatomas and kidney epithelial tumors for DDT and chloroform, respectively. Second, there are 37 breast cancer patients who underwent chemotherapy. The laboratory procedures are used to measure their tumor enzyme profile. The problem of the clinician is that of evaluating relationship between a patient's chance to respond to cytotoxic chemotherapy and the knowledge of the patient's enzyme activity profile. Third, the Harrogate benzopyrene skin painting experiment in mice where the responses are for infiltrating carcinomas; the units of dose rate are μg of benzopyrene per week for the 69-week duration of experiment.

In Thailand, the most extreme cases of cancers that could occur in people are human papilloma, breast cancer, liver cancer and lung cancer (http://www.nci.go.th/Knowledge/thaicancer.html). There are many factors leading to further progression of cancers, i.e., tobacco smoking, drinking, radiation, chemicals or virus. If we can obtain the patient's chance who will get cancer after they got carcinogen or the decreasing chance of cancer after it can be treated of the patients, these information are useful to decrease the mortality rate of people with cancer.

In this topic we consider the multistage Weibull, logistic and log-logistic models involving two parameters θ_0 and θ_1. We classify the parameters of interest into three cases, that is Case I: θ_1 is known and θ_0 is unknown, Case II: θ_0 is known and θ_1 is unknown, and Case III: θ_0 and θ_1 are unknown.

Many authors have considered dose response models and estimated parameters of the models as the followings.

In 1944, Berkson considered the logistic function of mortality rate versus x corresponding to parameters α and β

$$Q = \frac{1}{1+e^{\alpha-\beta x}}$$

where x is dose in log unit. The logistic function is close to the integrated normal curve if the dosage of a drug is expressed in proportion to its logarithm. For the application of the logistic function, he estimated parameters α and β based on the least squares method. For the LD50, that is, the dose which is lethal to just 50 per cent of the population exposed is compared between the logistic and normal curves. He found that on the basis of the comparison the logistic and normal curves are the same.

In 1953, Berkson considered the logistic function

$$P = 1-Q = \frac{1}{1+e^{-(\alpha+\beta x)}}$$

or

$$\text{logit}P = \ln\left(P/Q\right) = \alpha+\beta x$$

where P is the probability of death at the dosage x and α, β are unknown parameters. The quantal response is the familiar situation of trials where the observation may take one of two forms, that is, death or survival. This function is assumed that the observation p at x can be considered a random variable binomially distributed around P at x. He estimated α and β by using the modified minimum logit χ^2 estimate based on the minimization of the following quantity

$$\chi^2 = \sum npq\left(l-\hat{l}\right)^2$$

where n is the number exposed at x, $p = 1-q$ is the observed proportion affected, $l = \ln\left(P/Q\right) = \alpha+\beta x$ is the logit of P, $\hat{l} = \ln\left(\hat{p}/\hat{q}\right)$ is the logit of $\hat{p}$, where $\hat{p}$ is the estimate of P at x.

In 1955, Berkson used the maximum likelihood and minimum χ^2 methods to estimate parameters α and β in the logistic function. He found that the equations to be solved for the maximum likelihood estimate of α and β are

$$\sum n_i \left(p_i - \hat{p}_i \right) = 0$$

$$\sum n_i x_i \left(p_i - \hat{p}_i \right) = 0 ,$$

where n_i is the number at x_i, p_i is the proportion of n_i observed to respond, and $\hat{p}_i$ is the estimate of P_i. For the minimum χ^2 estimate the equations to be solved are

$$\sum n_i \frac{\left(\hat{p}_i q_i + \hat{q}_i p_i \right)}{\hat{p}_i \hat{q}_i} \left(p_i - \hat{p}_i \right) = 0$$

$$\sum n_i \frac{\left(\hat{p}_i q_i + \hat{q}_i p_i \right)}{\hat{p}_i \hat{q}_i} x_i \left(p_i - \hat{p}_i \right) = 0 .$$

The minimum χ^2 and maximum likelihood estimates required iterative methods for solution because $\hat{p}_i$ was not linear in the parameters. He considered two cases to estimate α and β, one in which β was known and only α to be estimated, the other in which α and β were to be estimated. Finally, he obtained that for α to be estimated with β unknown, both the variance and mean square error of the estimate of α for the minimum χ^2 estimate were smaller than the maximum likelihood estimate and also when he estimated α and β, the results were the same with the latter case.

In 1962, Hitchcock considered the literatures of Berkon (1953) and Anscombe (1956). Berkson estimated α and β by using modified minimum χ^2 method in logistic form and chosen a and b to minimize

$$\chi^2 = \sum_{j=1}^{k} w_i \left(l_i - a - b x_i \right)^2 ,$$

where $l_i = \ln \dfrac{r_i}{n_i - r_i}$, $w_i = r_i \left(n_i - r_i \right) / n_i$ and $p_i = r_i / n_i$. He knew that l is a biased estimator of $\ln\left(P/Q \right)$. Anscombe has shown that the bias can be practically eliminated by taking

$$l = \ln \frac{r + 1/2}{n - r + 1/2} ,$$

provided that n is fairly large and neither P nor Q is close to zero. Hitchcock supposed that in general

$$l_i = \ln \frac{r_i + \varepsilon}{n - r_i + \varepsilon},$$

and found that for $k > 3$ it would be sufficient to take $l = \ln \frac{r}{n-r}$.

In 1980, Amemiya derived the mean squared error matrices to the order of n^{-2} for the maximum likelihood and Berkson's modified minimum chi-square estimator in the dichotomous logit regression model. He proposed the model as follow.

$$P\left(y_{tv} = 1\right) = \frac{1}{1 + e^{-x_t'\beta_0}} \equiv P_t,$$

where x_t is a K-dimensional vector of known constants and β_0 is a K-dimensional vector of known parameters. He defined the $\sum_{t=1}^{T} n_t$ observable dichotomous random variables y_{tv} with $t = 1, 2, \ldots, T$ and $v = 1, 2, \ldots, n_t$. He presented many numerical examples and found that the mean squared error of the minimum chi-square estimator has a smaller MSE than the maximum likelihood estimator. Further he had not been able to show theoretically that the mean square error of the minimum chi-square estimator is smaller than the maximum likelihood estimator to the order of n^{-2}.

In 1981, Ghosh and Sinha proved a theorem which gives a necessary and sufficient condition for improving the mean squared error of the maximum likelihood estimate. In the one unknown parameter case, the comparison of mean squared errors up to $o\left(n^{-2}\right)$ was considered. If there is no $d\left(\theta\right)$ (a continuously differentiable function) satisfying the condition

$$E_\theta \left\{ \hat{\theta} + \frac{d\left(\hat{\theta}\right)}{n} - \theta \right\}^2 \leq E_\theta \left(\hat{\theta} - \theta\right)^2$$

up to $o\left(n^{-2}\right)$ for all θ, then $\hat{\theta}$ is admissible up to $o\left(n^{-2}\right)$. They showed an application of the theorem to Berkson's problem of estimating θ on the basis of observations on independent random variables having the binomial distribution $B\left(n, \pi_i\right)$, $i = 1, \ldots, k$, with

$$\pi_i = \left[1 + \exp\left(-\theta - \beta d_i\right)\right]^{-1},$$

where β is known, n trials represent the number of responses at dose level d_i. They found that the maximum likelihood estimate $\hat{\theta}$ is inadmissible, and the Berkson's estimate $\tilde{\theta}$ is admissible for $k \geq 4$ and inadmissible for $k < 4$, respectively.

In 2007, Sinha considered the logistic model involving two parameters θ and β,

$$\pi_i = \left[1 + \exp\left(-\theta - \beta d_i\right)\right]^{-1}.$$

He used the maximum likelihood and Berkson's minimum chi-square methods to estimate the unknown parameter θ, assuming that β is known.

4.1 Theoretical Background

Model specification

We assume that there are $m+1$ dose groups with doses $d_0 = 0$ (control), $d_1, \ldots, d_m$; two unknown parameters and n_i subjects to the *i*th dose group, the number of respondents are independent and distributed as binomial distribution $X_i \sim B\left[n_i, \pi_i\left(d_i, \underline{\theta}\right)\right]$, $i = 0, \ldots, m$. The three popular dose response models that we consider in this study are given below.

The multistage Weibull model (MWM)(Portier, 1983), is

$$\pi\left(d, \underline{\theta}\right) = 1 - \mathrm{e}^{-\left(\theta_0 + \theta_1 d\right)},\ \theta_0, \theta_1 > 0.$$

The logistic model (LM) (Agresti, 2007), is

$$\pi\left(d, \underline{\theta}\right) = \frac{1}{1 + \mathrm{e}^{-\left(\theta_0 + \theta_1 d\right)}},\ -\infty < \theta_0, \theta_1 < \infty.$$

The log-logistic model (LLM) (Filipsson, 2003), is

$$\pi\left(d, \underline{\theta}\right) = \frac{1}{1 + \mathrm{e}^{-\left(\theta_0 + \theta_1 \ln\left(d\right)\right)}},\ -\infty < \theta_0, \theta_1 < \infty.$$

For each model, *d* is the average lifetime dose of the chemical, $\pi\left(d, \underline{\theta}\right)$ is the lifetime probability of effect from the dose level *d*. In these models, the parameters $\left(\underline{\theta}\right)$ are constrained to ensure that $0 < \pi\left(d, \underline{\theta}\right) < 1$.

Methods of finding estimators

Three methods of finding estimators and the method of evaluating estimators are explained in the following.

The method of maximum likelihood (Casella, 2001) is the popular technique for deriving estimators. Recall that if $X_1,\ldots,X_n$ are independent and identically distributed (iid) sample from a population with probability density function (pdf) or probability mass function (pmf) $f\left(x\,|\,\theta_1,\ldots,\theta_k\right)$, the likelihood function is defined by

$$L\left(\underline{\theta}\,|\,\underline{x}\right)=L\left(\theta_1,\ldots,\theta_k\,|\,x_1,\ldots,x_n\right)=\prod\nolimits_{i=1}^{n} f\left(x_i\,|\,\theta_1,\ldots,\theta_k\right).$$

If the likelihood function is differentiable (with respect to each θ_i), possible candidates for the MLE are the values of $\left(\theta_1,\ldots,\theta_k\right)$ that solve

$$\frac{\partial}{\partial\theta_i}L\left(\underline{\theta}\,|\,\underline{x}\right)=0\,,\; i=1,\ldots,k\,.$$

The maximum likelihood estimators of parameters are parameter values at which the likelihood function attains its maximum as a function of parameters.

In most cases, especially when differentiation is used, it is easier to work with the natural logarithm of $L\left(\underline{\theta}\,|\,\underline{x}\right)$, $\log L\left(\underline{\theta}\,|\,\underline{x}\right)$ (known as the log likelihood), than it is to work with $L\left(\underline{\theta}\,|\,\underline{x}\right)$ directly. This is possible because the log function is strictly increasing on $\left(0,\infty\right)$, which implies that the extremes of $L\left(\underline{\theta}\,|\,\underline{x}\right)$ and $\log L\left(\underline{\theta}\,|\,\underline{x}\right)$ coincide.

The Berkson's minimum chi-square estimation (Harris, 1983) is the method for minimizing a chi-square function. So the chi-square function is the summation of the squared different between the observed frequencies and their expectations and divided by their variances. Estimates which come from minimization of chi-squared function are chi-squared estimators. In the two extreme cases when $x=n$ or 0, we modify the observe proportion p as $p=1-1/2n$ if $x=n$ and, $p=1/2n$ if $x=0$.

Method of evaluating estimators

We first investigate finite sample measures of the quality of an estimator, beginning with its mean squared error (Casella, 2001). The mean squared error (MSE) of an estimator $\hat{\theta}$ of a parameter θ is a function of θ defined by $E\left(\hat{\theta}-\theta\right)^2$.

Notice that the MSE measures the average squared difference between the estimator $\hat{\theta}$ and the parameter θ. It is quite tractable analytically and, second, it has the interpretation

$$E\left(\hat{\theta}-\theta\right)^2 = Var\left(\hat{\theta}\right)+\left(E\left(\hat{\theta}\right)-\theta\right)^2 = Var\left(\hat{\theta}\right)+\left(\text{Bias}\left(\hat{\theta}\right)\right)^2,$$

where the bias of a point estimator $\hat{\theta}$ of a parameter θ is the difference between the expected value of $\hat{\theta}$ and θ; that is, $\text{Bias}\left(\hat{\theta}\right)=E\left(\hat{\theta}\right)-\theta$.

We consider MSEs of one and two parameters and derive asymptotically to the order of approximation n^{-2}.

The case of one parameter

Assume the general conditions of estimators (Ghosh and Sinha, 1981; Sinha, 2007):

$$E\left(\hat{\theta}\right)=\theta+\frac{b(\theta)}{n}+o\left(n^{-1}\right) \tag{4.1.1}$$

$$E\left(\hat{\theta}-\theta\right)^2=\frac{1}{nI(\theta)}+\frac{\varphi(\theta)}{n^2}+o\left(n^{-2}\right) \tag{4.1.2}$$

$$\varphi(\theta)=b^2(\theta)+\frac{2b'(\theta)}{I(\theta)} \tag{4.1.3}$$

where $b(\theta)$ is a leading bias term of estimator, $I(\theta)$ is the Fisher information and $\varphi(\theta)$ is a part of MSE of estimator. The above equation follows from the first order efficient (FOE) property of an estimator T whose asymptotic variance (via Rao-Cramer inequality) equals $b_T^2(\theta)+\frac{\left(1+b_T'(\theta)\right)^2}{nI(\theta)}$, along with an asymptotic expansion of $b_T(\theta)=\frac{b(\theta)}{n}+o\left(n^{-1}\right)$. Obviously, $I(\theta)$ depends only on the model, and is given by

$$I_{MW}(\theta)=\sum_{i=0}^{m}\frac{1-\pi_i(\theta)}{\pi_i(\theta)} \text{ for multistage Weibull model,} \tag{4.1.4}$$

$$I_L(\theta)=\sum_{i=0}^{m}\pi_i(\theta)\left(1-\pi_i(\theta)\right) \text{ for logistic model,} \tag{4.1.5}$$

$$I_{LL}(\theta)=\sum_{i=0}^{m}\pi_i(\theta)\left(1-\pi_i(\theta)\right) \text{ for log-logistic model.} \tag{4.1.6}$$

To compare $\varphi_{MLE}(\theta)$ and $\varphi_B(\theta)$, following Ghosh and Sinha (1981), we equivalently compare $\varphi_{MLE}(\theta)\times I^4(\theta)=\varphi^*_{MLE}(\theta)$ and $\varphi_B(\theta)\times I^4(\theta)=\varphi^*_B(\theta)$.

The case of two parameters

Suppose that $\underline{\theta}=(\theta_0 \quad \theta_1)'$ and $E(\hat{\theta}_0)=\theta_0+b_0(\underline{\theta})$, $E(\hat{\theta}_1)=\theta_1+b_1(\underline{\theta})$, $E\left[\left(\hat{\theta}_i-E(\hat{\theta}_i)\right)\left(\hat{\theta}_j-E(\hat{\theta}_j)\right)\right]=v_{ij}$, $i, j=0,1$ where variance-covariance matrix of $\hat{\theta}_0$ and $\hat{\theta}_1$ is $V=\begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11}\end{pmatrix}$ and the bias terms of estimates $b_0(\underline{\theta})$ and $b_1(\underline{\theta})$, $b_0(\underline{\theta})=\frac{b_{0MLE}(\underline{\theta})}{n}+o(n^{-1})$, $b_1(\underline{\theta})=\frac{b_{1MLE}(\underline{\theta})}{n}+o(n^{-1})$ for MLEs, and $b_0(\underline{\theta})=\frac{b_{0B}(\underline{\theta})}{n}+o(n^{-1})$, $b_1(\underline{\theta})=\frac{b_{1B}(\underline{\theta})}{n}+o(n^{-1})$ for Berkson's minimum chi-square estimates. We obtain the MSE of θ_0 and θ_1, $E(\hat{\theta}_0-\theta_0)^2=v_{00}+b_0^2(\underline{\theta})$, $E(\hat{\theta}_1-\theta_1)^2=v_{11}+b_1^2(\underline{\theta})$ and $E\left[\left(\hat{\theta}_0-E(\hat{\theta}_0)\right)\left(\hat{\theta}_1-E(\hat{\theta}_1)\right)\right]=v_{01}+b_0(\underline{\theta})b_1(\underline{\theta})$.

Then, the MSE matrix of θ_0 and θ_1 is

$$V+\begin{pmatrix} b_0^2(\underline{\theta}) & b_0(\underline{\theta})b_1(\underline{\theta}) \\ b_1(\underline{\theta})b_0(\underline{\theta}) & b_1^2(\underline{\theta})\end{pmatrix}.$$

From Rao-Cramer information inequality (Rao, 1973), $V\geq \Delta I^{-1}(\underline{\theta})\Delta'$ where $I(\underline{\theta})$ is Fisher information matrix and $\Delta=\begin{pmatrix} \frac{\partial g_0(\underline{\theta})}{\partial\theta_0} & \frac{\partial g_0(\underline{\theta})}{\partial\theta_1} \\ \frac{\partial g_1(\underline{\theta})}{\partial\theta_0} & \frac{\partial g_1(\underline{\theta})}{\partial\theta_1}\end{pmatrix}$ where $g_i(\underline{\theta})=E(\hat{\theta}_i)=\theta_i+b_i(\underline{\theta})$. Then,

$$\Delta=\begin{pmatrix} 1+\frac{\partial b_0(\underline{\theta})}{\partial\theta_0} & \frac{\partial b_0(\underline{\theta})}{\partial\theta_1} \\ \frac{\partial b_1(\underline{\theta})}{\partial\theta_0} & 1+\frac{\partial b_1(\underline{\theta})}{\partial\theta_1}\end{pmatrix}.$$

For MLEs, ignoring the terms of a smaller order than n^{-1} of $b_0(\underline{\theta})$ and $b_1(\underline{\theta})$, we obtain

$$\Delta = I + \frac{1}{n}\Lambda \text{ where } \Lambda = \begin{pmatrix} \frac{\partial b_{0MLE}(\underline{\theta})}{\partial \theta_0} & \frac{\partial b_{0MLE}(\underline{\theta})}{\partial \theta_1} \\ \frac{\partial b_{1MLE}(\underline{\theta})}{\partial \theta_0} & \frac{\partial b_{1MLE}(\underline{\theta})}{\partial \theta_1} \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have inverse of Fisher information matrix, $I^{-1}(\underline{\theta}) = \frac{1}{n}\Sigma = \frac{1}{n}\begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix}$, then

$$\Delta I^{-1}(\underline{\theta})\Delta' = \left[I + \frac{1}{n}\Lambda\right]\frac{\Sigma}{n}\left[I + \frac{1}{n}\Lambda'\right] = \frac{\Sigma}{n} + \frac{1}{n^2}\left[\Lambda\Sigma + \Sigma\Lambda'\right]. \qquad (4.1.7)$$

The MSE matrix of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$ is written as

$$\frac{\Sigma}{n} + \frac{1}{n^2}\left[\Lambda\Sigma + \Sigma\Lambda'\right] + \frac{1}{n^2}\begin{pmatrix} b_{0MLE}^2(\underline{\theta}) & b_{0MLE}(\underline{\theta})b_{1MLE}(\underline{\theta}) \\ b_{1MLE}(\underline{\theta})b_{0MLE}(\underline{\theta}) & b_{1MLE}^2(\underline{\theta}) \end{pmatrix}.$$

Therefore, we obtain the mean squared errors of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$,

$$E\left(\hat{\theta}_{0MLE} - \theta_0\right)^2 = \frac{\sigma_{00}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{0MLE}(\underline{\theta})}{\partial \theta_0}\sigma_{00} + \frac{\partial b_{0MLE}(\underline{\theta})}{\partial \theta_1}\sigma_{01}\right)\right] + \frac{b_{0MLE}^2(\underline{\theta})}{n^2}, \quad (4.1.8)$$

$$E\left(\hat{\theta}_{1MLE} - \theta_1\right)^2 = \frac{\sigma_{11}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{1MLE}(\underline{\theta})}{\partial \theta_0}\sigma_{10} + \frac{\partial b_{1MLE}(\underline{\theta})}{\partial \theta_1}\sigma_{11}\right)\right] + \frac{b_{1MLE}^2(\underline{\theta})}{n^2}. \quad (4.1.9)$$

For Berkson's minimum chi-square estimates, the MSE of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$ could be processed in the same manner with MLEs and we obtain

$$E\left(\hat{\theta}_{0B} - \theta_0\right)^2 = \frac{\sigma_{00}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{0B}(\underline{\theta})}{\partial \theta_0}\sigma_{00} + \frac{\partial b_{0B}(\underline{\theta})}{\partial \theta_1}\sigma_{01}\right)\right] + \frac{b_{0B}^2(\underline{\theta})}{n^2}, \quad (4.1.10)$$

$$E\left(\hat{\theta}_{1B} - \theta_1\right)^2 = \frac{\sigma_{11}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{1B}(\underline{\theta})}{\partial \theta_0}\sigma_{10} + \frac{\partial b_{1B}(\underline{\theta})}{\partial \theta_1}\sigma_{11}\right)\right] + \frac{b_{1B}^2(\underline{\theta})}{n^2}. \quad (4.1.11)$$

Remark:

1. The MSEs of one unknown (4.1.2) and two unknown (4.1.7) parameters are different depending on the number of unknown parameters.

2. For asymptotic comparison, we assume that subject in the *i*th dose group $n_0 = n_1 = \cdots = n_m = n$; $i = 0,1,\ldots,m$ are equal and *n* is large.

3. Our comparison depends on MSEs (4.1.2) and (4.1.7), the number of dose groups m and dosage amounts $d_0, d_1, \ldots, d_m$.

4.2 Case I: θ_0 is unknown and θ_1 is known

In this section, the maximum likelihood and Berkson's minimum chi-square estimations are employed in fitting dose response models. The multistage Weibull, logistic and log-logistic models with θ_0 is unknown and θ_1 is known are considered separately. The mean squared errors of estimators are derived asymptotically to the order of approximation n^{-2}. Therefore, there are two expressions of the part of MSEs that we derive in this chapter that is $\varphi^*_{0MLE}\left(\theta_0\right)$ for MLE and $\varphi^*_{0B}\left(\theta_0\right)$ for Berkson's minimum chi-square estimate. Moreover, these expressions could be applied to real data sets to compare between these two methods.

Multistage Weibull model

Maximum likelihood estimator

The likelihood function of multistage Weibull model is given by

$$L\left(\theta_0 \mid \text{data}\right) \propto \prod_{i=0}^{m} \left\{e^{\left(\theta_0+\theta_1 d_i\right)} - 1\right\}^{x_i} \left\{e^{-\left(\theta_0+\theta_1 d_i\right)}\right\}^{n_i},$$

and the log-likelihood function is written as

$$\log L\left(\theta_0 \mid \text{data}\right) = -\left(\theta_0 \sum_{i=0}^{m} n_i + \theta_1 \sum_{i=0}^{m} d_i n_i\right) + \sum_{i=0}^{m} x_i \log\left(e^{\left(\theta_0+\theta_1 d_i\right)} - 1\right).$$

Differentiating $\log L\left(\theta_0 \mid \text{data}\right)$ with respect to θ_0 and setting its derivative equal to zero, then the normal equation is

$$\sum_{i=0}^{m} n_i = \sum_{i=0}^{m} \frac{x_i}{1 - e^{-\left(\hat{\theta}_{0MLE} + \theta_1 d_i\right)}},$$

where $\hat{\theta}_{0MLE}$ is the maximum likelihood estimator of θ_0. We assume that $n_i = n$ for all i and $p_i = x_i / n$, then

$$m + 1 = \sum_{i=0}^{m} \frac{p_i}{1 - e^{-\left(\psi\left(p_0, \ldots, p_m\right) + \theta_1 d_i\right)}}, \tag{4.2.1}$$

where $\hat{\theta}_{0MLE} = \psi\left(p_0, \ldots, p_m\right)$. To compute the biased term of θ_0, $b_{0MLE}\left(\theta_0\right)$, we use Taylor series to express $\psi\left(p_0, \ldots, p_m\right)$, that is

$$\hat{\theta}_{0MLE} = \psi\left(p_0,\ldots,p_m\right)$$

$$= \psi\left(P_0,\ldots,P_m\right) + \sum_{i=0}^{m}\left(p_i - P_i\right)\left.\frac{\partial\psi}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m}\left(p_i - P_i\right)^2 \left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}$$

$$+\sum_{i\neq j}^{m}\left(p_i - P_i\right)\left(p_j - P_j\right)\left.\frac{\partial^2\psi}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots. \tag{4.2.2}$$

The mean of $\hat{\theta}_{0MLE}$ could be found by taking expectation on both sides of (4.2.2), that is

$$E\left(\hat{\theta}_{0MLE}\right) = \theta_0 + \sum_{i=0}^{m} E\left(p_i - P_i\right)\left.\frac{\partial\psi}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m} E\left(p_i - P_i\right)^2 \left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}$$

$$+\sum_{i\neq j}^{m} E\left\{\left(p_i - P_i\right)\left(p_j - P_j\right)\right\}\left.\frac{\partial^2\psi}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.$$

We note that $E\left(p_i - P_i\right)$ and $E\left\{\left(p_i - P_i\right)\left(p_j - P_j\right)\right\}$ are equal to zero. Also, $E\left(p_i - P_i\right)^2 = \pi_i\left(\theta_0\right)\left(1-\pi_i\left(\theta_0\right)\right)/n$. By using (1.1) and ignore the terms of a smaller order than n^{-1}, we have

$$\theta_0 + \frac{b_{0MLE}\left(\theta_0\right)}{n} = \theta_0 + \frac{1}{2}\sum_{i=0}^{m} E\left(p_i - P_i\right)^2 \left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}. \tag{4.2.3}$$

To find the second derivative of $\psi\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing by P_i, $\left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}$. First, the expression (4.2.1) is expressed as below,

$$m+1 = \frac{p_0}{1-e^{-(\psi+\theta_1 d_0)}} + \frac{p_1}{1-e^{-(\psi+\theta_1 d_1)}} + \cdots + \frac{p_m}{1-e^{-(\psi+\theta_1 d_m)}}.$$

The first derivative of (4.2.1) with respect to p_0 is

$$0 = \frac{1}{1-e^{-(\psi+\theta_1 d_0)}} - \frac{p_0 e^{-(\psi+\theta_1 d_0)}}{\left(1-e^{-(\psi+\theta_1 d_0)}\right)^2}\frac{\partial\psi}{\partial p_0} - \sum_{i=1}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial\psi}{\partial p_0}$$

$$= \frac{1}{1-e^{-(\psi+\theta_1 d_0)}} - \sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial\psi}{\partial p_0}.$$

The first derivative of (4.2.1) with respect to p_1 is

$$0=\frac{1}{1-e^{-(\psi+\theta_1 d_1)}}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial\psi}{\partial p_1}.$$

We proceed in the same process, then the first derivative of (4.2.1) with respect to p_m is

$$0=\frac{1}{1-e^{-(\psi+\theta_1 d_m)}}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial\psi}{\partial p_m}.$$

Therefore, we obtain the first derivative of (4.2.1) with respect to p_i for $i=0,1,\ldots,m$

$$0=\frac{1}{1-e^{-(\psi+\theta_1 d_i)}}-\frac{\partial\psi}{\partial p_i}\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}$$

or $$\frac{\partial\psi}{\partial p_i}=\frac{1}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)\left[\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\right]}. \tag{4.2.4}$$

After we obtain the first derivative of (4.2.1) with respect to p_i, then we find the second derivative of (4.2.1) with respect to p_i. The second derivative of (4.2.1) with respect to p_0 is

$$\begin{aligned}
0=&\frac{-e^{-(\psi+\theta_1 d_0)}}{\left(1-e^{-(\psi+\theta_1 d_0)}\right)^2}\frac{\partial\psi}{\partial p_0}-\frac{p_0 e^{-(\psi+\theta_1 d_0)}}{\left(1-e^{-(\psi+\theta_1 d_0)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}\\
&-\frac{\partial\psi}{\partial p_0}\left\{\frac{-p_0 e^{-(\psi+\theta_1 d_0)}\frac{\partial\psi}{\partial p_0}+e^{-(\psi+\theta_1 d_0)}}{\left(1-e^{-(\psi+\theta_1 d_0)}\right)^2}-\frac{2p_0 e^{-2(\psi+\theta_1 d_0)}}{\left(1-e^{-(\psi+\theta_1 d_0)}\right)^3}\frac{\partial\psi}{\partial p_0}\right\}-\frac{p_1 e^{-(\psi+\theta_1 d_1)}}{\left(1-e^{-(\psi+\theta_1 d_1)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}\\
&-\frac{\partial\psi}{\partial p_0}\left\{\frac{-p_1 e^{-(\psi+\theta_1 d_1)}}{\left(1-e^{-(\psi+\theta_1 d_1)}\right)^2}\frac{\partial\psi}{\partial p_0}-\frac{2p_1 e^{-2(\psi+\theta_1 d_1)}}{\left(1-e^{-(\psi+\theta_1 d_1)}\right)^3}\frac{\partial\psi}{\partial p_0}\right\}-\cdots-\frac{p_m e^{-(\psi+\theta_1 d_m)}}{\left(1-e^{-(\psi+\theta_1 d_m)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}\\
&-\frac{\partial\psi}{\partial p_0}\left\{\frac{-p_m e^{-(\psi+\theta_1 d_m)}}{\left(1-e^{-(\psi+\theta_1 d_m)}\right)^2}\frac{\partial\psi}{\partial p_0}-\frac{2p_m e^{-2(\psi+\theta_1 d_m)}}{\left(1-e^{-(\psi+\theta_1 d_m)}\right)^3}\frac{\partial\psi}{\partial p_0}\right\}.
\end{aligned}$$

$$0=\frac{-2e^{-(\psi+\theta_1 d_0)}}{\left(1-e^{-(\psi+\theta_1 d_0)}\right)^2}\frac{\partial\psi}{\partial p_0}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}$$

$$+\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\left(\frac{\partial\psi}{\partial p_0}\right)^2+\sum_{i=0}^{m}\frac{2p_i e^{-2(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^3}\left(\frac{\partial\psi}{\partial p_0}\right)^2 .$$

The second derivative of (4.2.1) with respect to p_1 is

$$0=\frac{-2e^{-(\psi+\theta_1 d_1)}}{\left(1-e^{-(\psi+\theta_1 d_1)}\right)^2}\frac{\partial\psi}{\partial p_1}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_1^2}$$

$$+\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\left(\frac{\partial\psi}{\partial p_1}\right)^2+\sum_{i=0}^{m}\frac{2p_i e^{-2(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^3}\left(\frac{\partial\psi}{\partial p_1}\right)^2 .$$

We proceed in the same process, then the second derivative of (4.2.1) with respect to p_m is

$$0=\frac{-2e^{-(\psi+\theta_1 d_m)}}{\left(1-e^{-(\psi+\theta_1 d_m)}\right)^2}\frac{\partial\psi}{\partial p_m}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_m^2}$$

$$+\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\left(\frac{\partial\psi}{\partial p_m}\right)^2+\sum_{i=0}^{m}\frac{2p_i e^{-2(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^3}\left(\frac{\partial\psi}{\partial p_m}\right)^2 .$$

Therefore, we obtain the second derivative of (4.2.1) with respect to p_i for $i=0,1,\dots,m$

$$0=\frac{-2e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\frac{\partial\psi}{\partial p_i}-\frac{\partial^2\psi}{\partial p_i^2}\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}$$

$$+\left(\frac{\partial\psi}{\partial p_i}\right)^2\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}+\left(\frac{\partial\psi}{\partial p_i}\right)^2\sum_{i=0}^{m}\frac{2p_i e^{-2(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^3}$$

or

$$\frac{\partial^2\psi}{\partial p_j^2}=\frac{\dfrac{-2e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}\dfrac{\partial\psi}{\partial p_i}+\left(\dfrac{\partial\psi}{\partial p_i}\right)^2\displaystyle\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}+\left(\dfrac{\partial\psi}{\partial p_i}\right)^2\displaystyle\sum_{i=0}^{m}\frac{2p_i e^{-2(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^3}}{\displaystyle\sum_{i=0}^{m}\frac{p_i e^{-(\psi+\theta_1 d_i)}}{\left(1-e^{-(\psi+\theta_1 d_i)}\right)^2}} .$$

(4.2.5)

Substituting (4.2.4) in (4.2.5), $\left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}$ is written as

$$\left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i} = \frac{-\frac{2\left(1-\pi_i(\theta_0)\right)}{\pi_i^2(\theta_0)} + \frac{1}{\pi_i(\theta_0) I_{MW}(\theta_0)}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)\left(2-\pi_i(\theta_0)\right)}{\pi_i^2(\theta_0)}\right]}{\pi_i(\theta_0) I_{MW}^2(\theta_0)}, \quad (4.2.6)$$

where $P_i = \pi_i(\theta_0)$ and $I_{MW}(\theta_0)$ is the Fisher information per observation, $I_{MW}(\theta_0) = \sum_{i=0}^{m}\left(1-\pi_i(\theta_0)\right)/\pi_i(\theta_0)$.

From (4.2.3) and (4.2.6), $b_{0MLE}(\theta_0)$ based on MLE can be simplified as

$$\theta_0 + \frac{b_{0MLE}(\theta_0)}{n} = \theta_0 + \frac{1}{2}\sum_{i=0}^{m} E\left(p_i - P_i\right)^2 \left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}$$

$$\frac{b_{0MLE}(\theta_0)}{n} = \frac{1}{2}\sum_{i=0}^{m}\frac{\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)}{n}\left\{-\frac{2\left(1-\pi_i(\theta_0)\right)}{\pi_i^3(\theta_0) I_{MW}^2(\theta_0)} + \frac{\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)\left(2-\pi_i(\theta_0)\right)}{\pi_i^2(\theta_0)}\right]}{\pi_i^2(\theta_0) I_{MW}^3(\theta_0)}\right\}$$

$$b_{0MLE}(\theta_0) = -\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)^2}{\pi_i^2(\theta_0) I_{MW}^2(\theta_0)}\right] + \left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)\left(2-\pi_i(\theta_0)\right)}{\pi_i^2(\theta_0)}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)}{2\pi_i(\theta_0) I_{MW}^3(\theta_0)}\right]$$

$$= -\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)^2}{\pi_i^2(\theta_0) I_{MW}^2(\theta_0)}\right] + \frac{1}{2I_{MW}^2(\theta_0)}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)\left(2-\pi_i(\theta_0)\right)}{\pi_i^2(\theta_0)}\right]$$

$$= \sum_{i=0}^{m}\frac{\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)}{2\pi_i^2(\theta_0) I_{MW}^2(\theta_0)}$$

$$= \frac{1}{2I_{MW}(\theta_0)}. \quad (4.2.7)$$

We find the first derivative of $b_{0MLE}(\theta_0)$ with respect to θ_0 for substituting in (4.1.3) and obtain

$$b'_{0MLE}(\theta_0) = \frac{\sum_{i=0}^{m}\left[\frac{1-\pi_i(\theta_0)}{\pi_i^2(\theta_0)}\right]}{2I_{MW}^2(\theta_0)}. \quad (4.2.8)$$

From (4.1.3), (4.2.7) and (4.2.8), we obtain the part of MSE of MLE, that is $\varphi_{0MLE}^{*}(\theta_0)$,

$$\varphi_{0MLE}^{*}\left(\theta_0\right)=\frac{1}{4}\left[\sum_{i=0}^{m}\frac{1-\pi_i\left(\theta_0\right)}{\pi_i\left(\theta_0\right)}\right]^2+\left[\sum_{i=0}^{m}\frac{1-\pi_i\left(\theta_0\right)}{\pi_i\left(\theta_0\right)}\right]\left[\sum_{i=0}^{m}\frac{1-\pi_i\left(\theta_0\right)}{\pi_i^2\left(\theta_0\right)}\right]. \qquad (4.2.9)$$

Therefore, we obtain the part of MSEs of MLE at the two extreme values,

$$\varphi_{0MLE}^{*}\left(\theta_0\right)\simeq\frac{1}{4}\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2+\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2$$

$$=\frac{5}{4}\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2 \text{ as } \theta_0\to\infty . \qquad (4.2.10)$$

$$\varphi_{0MLE}^{*}\left(\theta_0\right)\simeq\frac{1}{4}\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]^2+\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{\left(1-e^{-\theta_1 d_i}\right)^2}\right] \text{ as } \theta_0\to 0 \qquad (4.2.11)$$

These limiting values of $\varphi_{0MLE}^{*}\left(\theta_0\right)$ will be used later.

Berkson's minimum chi-square estimator

Let us consider

$$1-\pi\left(\theta_0\right)=e^{-\left(\theta_0+\theta_1 d\right)}$$

$$-\ln\left(1-\pi\left(\theta_0\right)\right)=\theta_0+\theta_1 d .$$

We denote the observed frequencies by $p_i=x_i/n_i$ where $i=0,\ldots,m$; $Var\left[-\ln\left(1-p_i\right)\right]=p_i/\left(n_i q_i\right)$ and when $x_i=n_i$ then $p_i=1-1/2n_i$ and $x_i=0$ then $p_i=1/2n_i$. Therefore a chi-square function $Q\left(\theta_0\right)$ is given by

$$Q\left(\theta_0\right)=\sum_{i=0}^{m}\frac{n_i\left(1-p_i\right)}{p_i}\left[\ln\left(1-p_i\right)+\theta_0+\theta_1 d_i\right]^2 .$$

The Berkson's minimum chi-square estimator of θ_0, $\hat{\theta}_{0B}$ is computed by minimizing $Q\left(\theta_0\right)$ with respect to θ_0. Differentiating $Q\left(\theta_0\right)$ with respect to θ_0 and setting its derivative equal to zero, then the normal equation is written as

$$\sum_{i=0}^{m}\frac{n_i\left(1-p_i\right)}{p_i}\left[\ln\left(1-p_i\right)+\theta_0+\theta_1 d_i\right]=0 . \qquad (4.2.12)$$

We obtain $\hat{\theta}_{0B}$ by solving (4.2.12),

$$\hat{\theta}_{0B} = \frac{\sum_{i=0}^{m} \frac{n_i(1-p_i)}{p_i}\left[-ln(1-p_i)-\theta_1 d_i\right]}{\sum_{i=0}^{m} \frac{n_i(1-p_i)}{p_i}}.$$

We suppose $n_i = n$ and write $\hat{\theta}_{0B} = \tilde{\psi}(p_0,\dots,p_m)$. By Taylor expansion (4.2.2), we obtain the mean of $\hat{\theta}_{0B}$,

$$E\left(\hat{\theta}_{0B}\right) = \theta_0 + \sum_{i=0}^{m} E(p_i - P_i)\left.\frac{\partial\tilde{\psi}}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m} E(p_i - P_i)^2 \left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}$$

$$+\sum_{i\neq j} E\left\{(p_i - P_i)(p_j - P_j)\right\}\left.\frac{\partial^2\tilde{\psi}}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.$$

Ignoring the terms of a smaller order than n^{-1}, we have

$$\theta_0 + \frac{b_{0B}(\theta_0)}{n} = \theta_0 + \frac{1}{2}\sum_{i=0}^{m} E(p_i - P_i)^2 \left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}. \tag{4.2.13}$$

We find the second derivative of $\tilde{\psi}(p_0,\dots,p_m)$ with respect to p_i replacing by P_i.

First, we find the first derivative of $\tilde{\psi}(p_0,\dots,p_m)$ with respect to p_i, and obtain

$$\frac{\partial\tilde{\psi}}{\partial p_i} = \frac{\frac{1}{p_i} + \left(-\ln(1-p_i)-\theta_1 d_i\right)\left(-\frac{1}{p_i^2}\right)}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}} - \frac{\left(-\frac{1}{p_i^2}\right)\left(\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\left(-\ln(1-p_i)-\theta_1 d_i\right)\right)}{\left[\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\right]^2}.$$

The second derivative of $\tilde{\psi}(p_0,\dots,p_m)$ with respect to p_i replacing by P_i is

$$\left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i} = \frac{-\frac{n}{P_i^2}\left(\frac{2-P_i}{1-P_i}\right) + \left(-\ln(1-P_i)-\theta_1 d_i\right)\left(\frac{2n}{P_i^3}\right) - \frac{2n\theta_0}{P_i^3}}{\sum_{i=0}^{m}\frac{n(1-P_i)}{P_i}}$$

$$+\frac{2\left(\frac{n}{P_i^2}\right) + \left[\frac{n}{P_i^2} + \left(-\ln(1-P_i)-\theta_1 d_i\right)\left(-\frac{n}{P_i^2}\right)\right] + 2\left(\frac{n}{P_i^2}\right)^2\theta_0}{\left[\sum_{i=0}^{m}\frac{n(1-P_i)}{P_i}\right]^2}. \tag{4.2.14}$$

From (4.2.13) and (4.2.14), the biased term of θ_0, $b_{0B}(\theta_0)$ based on Berkson's minimum chi-square estimator can be simplified as (using $P_i = \pi_i(\theta_0)$)

$$\theta_0 + \frac{b_{0B}(\theta_0)}{n} = \theta_0 + \frac{1}{2}\sum_{i=0}^{m} E(p_i - P_i)^2 \left.\frac{\partial^2 \tilde{\psi}}{\partial p_i^2}\right|_{P_i}$$

$$b_{0B}(\theta_0) = \frac{-\frac{1}{2}\sum_{i=0}^{m}\left(\frac{2-P_i}{P_i}\right) + \sum_{i=0}^{m}\left(\frac{1-P_i}{P_i^2}\right)\left(-\ln(1-P_i) - \theta_0 - \theta_1 d_i\right)}{\sum_{i=0}^{m}\frac{(1-P_i)}{P_i}}$$

$$+\frac{\sum_{i=0}^{m}\left(\frac{1-P_i}{P_i^2}\right) - \sum_{i=0}^{m}\left(\frac{1-P_i}{P_i^3}\right)\left(-\ln(1-P_i) - \theta_0 - \theta_1 d_i\right)}{\left(\sum_{i=0}^{m}\frac{(1-P_i)}{P_i}\right)^2}$$

$$= \frac{\sum_{i=0}^{m}\left(\frac{1-P_i}{P_i^2}\right)}{\left(\sum_{i=0}^{m}\frac{(1-P_i)}{P_i}\right)^2} - \frac{\frac{1}{2}\sum_{i=0}^{m}\left(\frac{2-P_i}{P_i}\right)}{\sum_{i=0}^{m}\frac{(1-P_i)}{P_i}}$$

$$= \frac{\sum_{i=0}^{m}\left(1-\pi_i(\theta_0)\right)/\pi_i^2(\theta_0)}{I_{MW}^2(\theta_0)} - \frac{\sum_{i=0}^{m}\left(2-\pi_i(\theta_0)\right)/\pi_i(\theta_0)}{2I_{MW}(\theta_0)}. \tag{4.2.15}$$

The first derivative of $b_{0B}(\theta_0)$ with respect to θ_0 for substituting in (4.1.3) is written as

$$b'_{0B}(\theta_0) = \frac{2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)}{\pi_i^2(\theta_0)}\right]^2}{I_{MW}^3(\theta_0)} - \frac{\sum_{i=0}^{m}\frac{\left(1-\pi_i(\theta_0)\right)\left(2-\pi_i(\theta_0)\right)}{\pi_i^3(\theta_0)}}{I_{MW}^2(\theta_0)}$$

$$-\frac{\left[\sum_{i=0}^{m}\frac{1-\pi_i(\theta_0)}{\pi_i^2(\theta_0)}\right]\left[\sum_{i=0}^{m}\frac{2-\pi_i(\theta_0)}{\pi_i(\theta_0)}\right]}{2I_{MW}^2(\theta_0)} + \frac{\sum_{i=0}^{m}\frac{1-\pi_i(\theta_0)}{\pi_i^2(\theta_0)}}{I_{MW}(\theta_0)}. \tag{4.2.16}$$

From (4.1.3), (4.2.15) and (4.2.16), we obtain the part of MSE of Berkson's minimum chi-square estimate, that is $\varphi_{0B}^*(\theta_0)$,

$$\varphi_{0B}^{*}\left(\theta_0\right)=5\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i^2\left(\theta_0\right)}\right]^2-2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i\left(\theta_0\right)}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i^2\left(\theta_0\right)}\right]\left[\sum_{i=0}^{m}\frac{\left(2-\pi_i\left(\theta_0\right)\right)}{\pi_i\left(\theta_0\right)}\right]$$
$$+\frac{1}{4}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i\left(\theta_0\right)}\right]^2\left[\sum_{i=0}^{m}\frac{\left(2-\pi_i\left(\theta_0\right)\right)}{\pi_i\left(\theta_0\right)}\right]^2$$
$$-2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i\left(\theta_0\right)}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)\left(2-\pi_i\left(\theta_0\right)\right)}{\pi_i^3\left(\theta_0\right)}\right]$$
$$+2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i\left(\theta_0\right)}\right]^2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\left(\theta_0\right)\right)}{\pi_i^2\left(\theta_0\right)}\right]. \qquad (4.2.17)$$

Therefore, we obtain the part of MSEs of Berkson's minimum chi-square estimate at two extreme values,

$$\varphi_{0B}^{*}\left(\theta_0\right)\simeq 5\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2-2\left(m+1\right)\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2+\frac{\left(m+1\right)^2}{4}\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2$$
$$-2\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2-2\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^3$$
$$=\left(\frac{\left(m+1\right)^2}{4}-2\left(m+1\right)+3\right)\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2 \text{ as } \theta_0\to\infty. \qquad (4.2.18)$$

$$\varphi_{0B}^{*}\left(\theta_0\right)\simeq 5\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{\left(1-e^{-\theta_1 d_i}\right)^2}\right]^2-2\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{\left(1-e^{-\theta_1 d_i}\right)^2}\right]\left[\sum_{i=0}^{m}\frac{1+e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]$$
$$+\frac{1}{4}\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]^2\left[\sum_{i=0}^{m}\frac{1+e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]^2-2\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}\left(1+e^{-\theta_1 d_i}\right)}{\left(1-e^{-\theta_1 d_i}\right)^3}\right]$$
$$+2\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{1-e^{-\theta_1 d_i}}\right]^2\left[\sum_{i=0}^{m}\frac{e^{-\theta_1 d_i}}{\left(1-e^{-\theta_1 d_i}\right)^2}\right] \text{ as } \theta_0\to 0 \qquad (4.2.19)$$

These limiting values of $\varphi_{0B}^{*}\left(\theta_0\right)$ will be use later.

Logistic model

Maximum likelihood estimator

The likelihood function of logistic model is given by

$$L\left(\theta_0 \mid \text{data}\right) \propto e^{-\sum_{i=0}^{m}\left(\theta_0+d_i\theta_1\right)\left(n_i-x_i\right)} \prod_{i=0}^{m}\left(\frac{1}{1+e^{-\left(\theta_0+\theta_1 d_i\right)}}\right)^{n_i},$$

and the log-likelihood function is written as

$$\log L\left(\theta_0 \mid \text{data}\right) = -\sum_{i=0}^{m}\left(\theta_0+\theta_1 d_i\right)\left(n_i-x_i\right)+\sum_{i=0}^{m} n_i \log\left(\frac{1}{1+e^{-\left(\theta_0+\theta_1 d_i\right)}}\right).$$

Differentiating $\log L\left(\theta_0 \mid \text{data}\right)$ with respect to θ_0 and setting its derivative equal to zero, then the normal equation is

$$\sum_{i=0}^{m} x_i = \sum_{i=0}^{m} n_i \pi_i\left(\hat{\theta}_{0MLE}\right), \tag{4.2.20}$$

where $\hat{\theta}_{0MLE}$ is the maximum likelihood estimator of θ_0. To study the properties of $\hat{\theta}_{0MLE}$, we use Taylor series to express $\pi_i\left(\hat{\theta}_{0MLE}\right)$, that is

$$\pi_i\left(\hat{\theta}_{0MLE}\right) = \pi_i\left(\theta_0\right)+\left(\hat{\theta}_{0MLE}-\theta_0\right)\left.\frac{\partial \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}}\right|_{\theta_0}+\frac{\left(\hat{\theta}_{0MLE}-\theta_0\right)^2}{2}\left.\frac{\partial^2 \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}^2}\right|_{\theta_0}+\cdots.$$

From (4.2.20), we assume $n_i = n$, hence

$$\sum_{i=0}^{m} x_i = \sum_{i=0}^{m} n\pi_i\left(\theta_0\right)+\left(\hat{\theta}_{0MLE}-\theta_0\right)\sum_{i=0}^{m}\left.\frac{n\partial \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}}\right|_{\theta_0}+\frac{\left(\hat{\theta}_{0MLE}-\theta_0\right)^2}{2}\sum_{i=0}^{m}\left.\frac{n\partial^2 \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}^2}\right|_{\theta_0}+\cdots \tag{4.2.21}$$

The mean of $\hat{\theta}_{0MLE}$ could be found by taking expectation on both sides of (4.2.21), that is

$$0 = E\left(\hat{\theta}_{0MLE}-\theta_0\right)\sum_{i=0}^{m}\left.\frac{n\partial \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}}\right|_{\theta_0}+\frac{E\left(\hat{\theta}_{0MLE}-\theta_0\right)^2}{2}\sum_{i=0}^{m}\left.\frac{n\partial^2 \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}^2}\right|_{\theta_0}+\cdots.$$

Ignoring the terms of a smaller order than n^{-1}, we have the biased term of θ_0, $b_{0MLE}\left(\theta_0\right)$,

$$b_{0MLE}\left(\theta_0\right) = \frac{-\left[\sum_{i=0}^{m}\left.\frac{\partial^2 \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}^2}\right|_{\theta_0}\right]}{2I_L\left(\theta_0\right)\left[\sum_{i=0}^{m}\left.\frac{\partial \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}}\right|_{\theta_0}\right]}, \tag{4.2.22}$$

where $I_L(\theta_0)$ is the Fisher information per observation, $I_L(\theta_0)=\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)$.

We find the first and the second derivative of $\pi_i\left(\hat{\theta}_{0MLE}\right)$ with respect to $\hat{\theta}_{0MLE}$ replacing by θ_0, and obtain

$$\left.\frac{\partial \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}_{0MLE}}\right|_{\theta_0}=\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right) \tag{4.2.23}$$

$$\left.\frac{\partial^2 \pi_i\left(\hat{\theta}_{0MLE}\right)}{\partial \hat{\theta}^2_{0MLE}}\right|_{\theta_0}=\pi_i(\theta_0)\left(\pi_i(\theta_0)-1\right)\left(2\pi_i(\theta_0)-1\right). \tag{4.2.24}$$

From (4.2.22) - (4.2.24), $b_{0MLE}(\theta_0)$ based on MLE can be simplified as

$$b_{0MLE}(\theta_0)=\frac{\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\left(2\pi_i(\theta_0)-1\right)}{2I_L(\theta_0)\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)}$$

$$=\frac{\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\left(2\pi_i(\theta_0)-1\right)}{2I_L^2(\theta_0)}. \tag{4.2.25}$$

The first derivative of $b_{0MLE}(\theta_0)$ with respect to θ_0 for substituting in (4.1.3) is written as

$$b'_{0MLE}(\theta_0)=\frac{\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\left(-6\pi_i^2(\theta_0)+6\pi_i(\theta_0)-1\right)}{2I_L^2(\theta_0)}$$

$$+\frac{\left[\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\left(2\pi_i(\theta_0)-1\right)\right]^2}{I_L^3(\theta_0)}. \tag{4.2.26}$$

From (4.1.3), (4.2.25) and (4.2.26), we obtain the part of MSE of MLE, that is $\varphi^*_{0MLE}(\theta_0)$,

$$\varphi^*_{0MLE}(\theta_0)=\frac{\left[\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\left(2\pi_i(\theta_0)-1\right)\right]^2}{4}$$

$$+\left[\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\right]\left[\sum_{i=0}^{m}\pi_i(\theta_0)\left(1-\pi_i(\theta_0)\right)\left(-6\pi_i^2(\theta_0)+6\pi_i(\theta_0)-1\right)\right]$$

$$+2\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\left(1-\pi_i\left(\theta_0\right)\right)\left(1-2\pi_i\left(\theta_0\right)\right)\right]^2. \quad (4.2.27)$$

Therefore, we obtain the following limiting values which will be used later.

$$\varphi_{0MLE}^{*}\left(\theta_0\right)\simeq\frac{1}{4}\left[\sum_{i=0}^{m}\left(\pi_i\left(\theta_0\right)-1\right)\right]^2-\left[\sum_{i=0}^{m}\left(\pi_i\left(\theta_0\right)-1\right)\right]^2+2\left[\sum_{i=0}^{m}\left(\pi_i\left(\theta_0\right)-1\right)\right]^2$$

$$=\frac{5}{4}\left[\sum_{i=0}^{m}\left(\pi_i\left(\theta_0\right)-1\right)\right]^2 \text{ as } \theta_0\to\infty. \quad (4.2.28)$$

$$\varphi_{0MLE}^{*}\left(\theta_0\right)\simeq\frac{1}{4}\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2-\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2+2\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2$$

$$=\frac{5}{4}\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2 \text{ as } \theta_0\to-\infty. \quad (4.2.29)$$

These are $\varphi_{0MLE}^{*}\left(\theta_0\right)$ at two extreme values.

Berkson's minimum chi-square estimator

Let us consider

$$1-\pi\left(\theta_0\right)=\frac{e^{-\left(\theta_0+\theta_1 d\right)}}{1+e^{-\left(\theta_0+\theta_1 d\right)}}$$

$$\frac{1-\pi\left(\theta_0\right)}{\pi\left(\theta_0\right)}=e^{-\left(\theta_0+\theta_1 d\right)}$$

$$\ln\left(\frac{\pi\left(\theta_0\right)}{1-\pi\left(\theta_0\right)}\right)=\theta_0+\theta_1 d.$$

We denote the obesrved frequencies by $p_i = x_i/n_i$ where $i=0,\ldots,m$ and $Var\left[\ln\left(\frac{p_i}{1-p_i}\right)\right]=n_i p_i q_i$. Therefore a chi-square function $Q\left(\theta_0\right)$ is given by

$$Q\left(\theta_0\right)=\sum_{i=0}^{m}n_i p_i\left(1-p_i\right)\left[\ln\left(\frac{p_i}{1-p_i}\right)-\theta_0-\theta_1 d_i\right]^2.$$

The Berkson's minimum chi-square estimator of θ_0, $\hat{\theta}_{0B}$ is computed by minimizing $Q(\theta_0)$ with respect to θ_0. Differentiating $Q(\theta_0)$ with respect to θ_0 and setting its derivative equal to zero, then the normal equation is written as

$$\sum_{i=0}^{m} n_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right)-\theta_0-\theta_1 d_i\right]=0. \qquad (4.2.30)$$

We obtain $\hat{\theta}_{0B}$ by solving (4.2.30),

$$\hat{\theta}_{0B}=\frac{\sum_{i=0}^{m} n_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right)-\theta_1 d_i\right]}{\sum_{i=0}^{m} n_i p_i (1-p_i)}.$$

We assume $n_i = n$ and write $\hat{\theta}_{0B}=\tilde{\psi}(p_0,\ldots,p_m)$. By Taylor expansion (4.2.2), we obtain the mean of $\hat{\theta}_{0B}$,

$$E\left(\hat{\theta}_{0B}\right)=\theta_0+\sum_{i=0}^{m} E(p_i-P_i)\left.\frac{\partial\tilde{\psi}}{\partial p_i}\right|_{P_i}+\frac{1}{2}\sum_{i=0}^{m} E(p_i-P_i)^2\left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}$$

$$+\sum_{i\neq j}^{m} E\left\{(p_i-P_i)(p_j-P_j)\right\}\left.\frac{\partial^2\tilde{\psi}}{\partial p_i \partial p_j}\right|_{P_i,P_j}+\cdots.$$

Ignoring the terms of a smaller order than n^{-1}, we have

$$\theta_0+\frac{b_{0B}(\theta_0)}{n}=\theta_0+\frac{1}{2}\sum_{i=0}^{m} E(p_i-P_i)^2\left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}. \qquad (4.2.31)$$

We find the second derivative of $\tilde{\psi}(p_0,\ldots,p_m)$ with respect to p_i replacing by P_i.

First, we find the first derivative of $\tilde{\psi}(p_0,\ldots,p_m)$ with respect to p_i, and obtain

$$\frac{\partial\tilde{\psi}}{\partial p_i}=\frac{\left[1+\left(\ln\left(\frac{p_i}{1-p_i}\right)-\theta_1 d_i\right)(1-2p_i)\right]}{\sum_{i=0}^{m} p_i(1-p_i)}-\frac{(1-2p_i)\left[\sum_{i=0}^{m} p_i(1-p_i)\left(\ln\left(\frac{p_i}{1-p_i}\right)-\theta_1 d_i\right)\right]}{\left[\sum_{i=0}^{m} p_i(1-p_i)\right]^2}.$$

The second derivative of $\tilde{\psi}(p_0,\ldots,p_m)$ with respect to p_i replacing by P_i is

$$\left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i} = \frac{-2\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_1 d_i\right)+\left(\frac{(1-2P_i)}{P_i(1-P_i)}\right)+2\theta_0}{\sum_{i=0}^{m} P_i(1-P_i)}$$

$$-\frac{2(1-2P_i)+2\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_1 d_i\right)(1-2P_i)^2-2(1-2P_i)^2\theta_0}{\left[\sum_{i=0}^{m} P_i(1-P_i)\right]^2}. \qquad (4.2.32)$$

From (4.2.31) and (4.2.32), the biased term of θ_0, $b_{0B}(\theta_0)$ based on Berkson's minimum chi-square estimator can be simplified as (using $P_i = \pi_i(\theta_0)$)

$$\theta_0 + \frac{b_{0B}(\theta_0)}{n} = \theta_0 + \frac{1}{2}\sum_{i=0}^{m} E(p_i - P_i)^2 \left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}$$

$$b_{0B}(\theta_0) = \frac{-\sum_{i=0}^{m} P_i(1-P_i)\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_1 d_i\right)+\frac{1}{2}\sum_{i=0}^{m}(1-2P_i)+\sum_{i=0}^{m} P_i(1-P_i)\theta_0}{\sum_{i=0}^{m} P_i(1-P_i)}$$

$$+\frac{\sum_{i=0}^{m} P_i(1-P_i)(2P_i-1)-\sum_{i=0}^{m} P_i(1-P_i)(2P_i-1)^2\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_1 d_i\right)}{\left[\sum_{i=0}^{m} P_i(1-P_i)\right]^2}$$

$$+\frac{\sum_{i=0}^{m} P_i(1-P_i)(2P_i-1)^2\theta_0}{\left[\sum_{i=0}^{m} P_i(1-P_i)\right]^2}$$

$$= \frac{\frac{1}{2}\sum_{i=0}^{m}(1-2P_i)}{\sum_{i=0}^{m} P_i(1-P_i)} + \frac{\sum_{i=0}^{m} P_i(1-P_i)(2P_i-1)}{\left[\sum_{i=0}^{m} P_i(1-P_i)\right]^2}$$

$$= \frac{2\sum_{i=0}^{m} P_i(1-P_i)(2P_i-1)}{2\left[\sum_{i=0}^{m} P_i(1-P_i)\right]^2} - \frac{\sum_{i=0}^{m}(2P_i-1)}{2\sum_{i=0}^{m} P_i(1-P_i)}$$

$$= \frac{2\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(2\pi_i(\theta_0)-1)}{2I_L^2(\theta_0)} - \frac{\sum_{i=0}^{m}(2\pi_i(\theta_0)-1)}{2I_L(\theta_0)}. \qquad (4.2.33)$$

The first derivative of $b_{0B}(\theta_0)$ with respect to θ_0 for substituting in (4.1.3) is written as

$$b'_{0B}(\theta_0) = \frac{\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(-6\pi_i^2(\theta_0)+6\pi_i(\theta_0)-1)}{I_L^2(\theta_0)}$$
$$+\frac{2\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(2\pi_i(\theta_0)-1)\right]^2}{I_L^3(\theta_0)} - 1$$
$$-\frac{\left[\sum_{i=0}^{m}(2\pi_i(\theta_0)-1)\right]\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(2\pi_i(\theta_0)-1)\right]}{2I_L^2(\theta_0)}. \qquad (4.2.34)$$

From (4.1.3), (4.2.33) and (4.2.34), we obtain the part of MSE of Berkson's minimum chi-square estimate, that is $\varphi_{0B}^*(\theta_0)$,

$$\varphi_{0B}^*(\theta_0) = 5\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(2\pi_i(\theta_0)-1)\right]^2$$
$$+\frac{1}{4}\left[\sum_{i=0}^{m}(2\pi_i(\theta_0)-1)\right]^2\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))\right]^2$$
$$+2\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(-6\pi_i^2(\theta_0)+6\pi_i(\theta_0)-1)\right]\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))\right]$$
$$-2\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))\right]^3$$
$$+2\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))(1-2\pi_i(\theta_0))\right]\left[\sum_{i=0}^{m}(2\pi_i(\theta_0)-1)\right]\left[\sum_{i=0}^{m}\pi_i(\theta_0)(1-\pi_i(\theta_0))\right].$$

(4.2.35)

Therefore, we obtain the following limiting values as $\theta_0 \to \pm\infty$. These limiting values will be used later.

$$\varphi_{0B}^*(\theta_0) \simeq 5\left[\sum_{i=0}^{m}(1-\pi_i(\theta_0))\right]^2 + \frac{(m+1)^2}{4}\left[\sum_{i=0}^{m}(1-\pi_i(\theta_0))\right]^2 - 2\left[\sum_{i=0}^{m}(1-\pi_i(\theta_0))\right]^2$$

$$+2\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^3-2\left(m+1\right)\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2$$

$$=\left(\frac{\left(m+1\right)^2}{4}-2\left(m+1\right)+3\right)\left[\sum_{i=0}^{m}\left(1-\pi_i\left(\theta_0\right)\right)\right]^2 \text{ as } \theta_0\to\infty. \qquad (4.2.36)$$

$$\varphi_{0B}^{*}\left(\theta_0\right)\simeq 5\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2+\frac{\left(m+1\right)^2}{4}\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2-2\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2$$

$$-2\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^3-2\left(m+1\right)\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2$$

$$=\left(\frac{\left(m+1\right)^2}{4}-2\left(m+1\right)+3\right)\left[\sum_{i=0}^{m}\pi_i\left(\theta_0\right)\right]^2 \text{ as } \theta_0\to-\infty. \qquad (4.2.37)$$

Log-logistic model

In this study we consider the one unknown parameter case, that is θ_0. Therefore, log-logistic model could be processed in the same manner with logistic model. We find that similar expressions of $\varphi_{0MLE}^{*}\left(\theta_0\right)$ and $\varphi_{0B}^{*}\left(\theta_0\right)$ hold with "*d*" replaced by "$\ln\left(d\right)$".

Comparison and conclusion

In all the models considered above, it turns out that both the maximum likelihood and Berkson's minimum chi-square estimators of θ_0 are FOE with their asymptotic variance equal to the inverse of Fisher informaion. Therefore, a comparison of the two takes us beyond the first order, and hence the necessity of our calculations for terms of order n^{-2} in their asymptotic variances. As expected none dominates the other over the entire parameter space, and our computations for θ_0 and θ_1, separately at the two extremes enable us to draw the following conclusions.

Table 4.2.1 Each data set with dose, number responding, and number in experiment

Data set	Dose	Number of respondents	Number of animals tested
1	0	0	50
	1	10	50
	3.1064	25	50
	8.2131	42	50
2	0	0	50
	1	10	50
	1.4591	25	50
	2.0176	42	50
3	0	0	50
	22.5	3	50
	45	10	50
	90	29	50
4	0	0	50
	6.25	0	50
	20	0	50
	62.5	1	49
	200	21	50

In this study, we use four real data sets in Table 4.2.1 for analyzing the difference between two MSEs in each model for some cases that they are complicated to derive mathematical relationship between two MSEs. Data sets 1 and 2 are the number of animals with tumors for four groups of 50 animals with the low doses and the numbers of respondents are the same, USEPA, 1985. Data set 3 is the number of male F344/N rats exhibiting squamous cell papilloma of the fore stomach after they were exposed to 2,4-hexadienal via gavages for four groups of 50 rats, Wheeler and Bailer, 2008. Data set 4 is the number of female B6C3F1 rats exhibiting heart hemangiosarcomas after they were exposed to 1,3-Butadiene, NTP, 1993.

Multistage Weibull model

We compare $\varphi_0^*\left(\theta_0\right)$ between $\varphi_{0MLE}^*\left(\theta_0\right)$ (4.2.10) and $\varphi_{0B}^*\left(\theta_0\right)$ (4.2.18) when $\theta_0 \to \infty$, and obtain that (4.2.10) is greater than (4.2.18) in the case $m \le 5$. For $m \ge 7$, (4.2.10) is less than (4.2.18). They are the same for $m = 6$. This shows that, for $m \le 5$, there is an interval I_0 of values of θ_0 within which the MLE is better than the Berkson's minimum chi-square estimate in the sense of second order asymtotic variance, and outside which it's the other way around. On the other hand, for $m \ge 7$, there is an interval I_0 of values of θ_0 within which the MLE is worse than the Berkson's minimum chi-square estimate in the sense of second order asymtotic variance, and outside which it's the other way around. For $m = 6$, the MLE and Berkson's minimum chi-square estimate are second order equivalent.

We compare $\varphi_{0MLE}^{*}\left(\theta_0\right)$ (4.2.11) and $\varphi_{0B}^{*}\left(\theta_0\right)$ (4.2.19) when $\theta_0 \to 0$ by using the number of dose groups and the dosage amounts in Table 4.2.1. We compute the expressions of $\varphi_{0MLE}^{*}\left(\theta_0\right)$ and $\varphi_{0B}^{*}\left(\theta_0\right)$ for a few cases in Table 4.2.2 with $\theta_1 =$0.0001, 0.001, 0.01, 0.1, 0.5 and 1.0 by assuming $d_0 = 0.00005$. Our observation is that $\varphi_{0B}^{*}\left(\theta_0\right)$ is less than $\varphi_{0MLE}^{*}\left(\theta_0\right)$. This shows that, Berkson's minimum chi-square estimate is better than MLE.

Table 4.2.2 Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ for θ_1 = 0.0001, 0.001, 0.01, 0.1, 0.5 and 1.0

θ_1	Data set	m	$\varphi^*_{0MLE}(\theta_0)$	$\varphi^*_{0B}(\theta_0)$
0.0001	1	1	8.000e+24	-3.199e+29
		2	8.001e+24	-4.229e+29
		3	8.001e+24	-4.619e+29
	2	1	8.000e+24	-3.199e+29
		2	8.001e+24	-5.392e+29
		3	8.001e+24	-6.978e+29
	3	1	8.000e+24	-1.421e+28
		2	8.000e+24	-2.131e+28
		3	8.000e+24	-2.486e+28
	4	1	8.000e+24	-5.118e+28
		2	8.000e+24	-6.717e+28
		3	8.000e+24	-7.229e+28
		4	8.000e+24	-7.388e+28
0.001	1	1	8.000e+21	-3.198e+25
		2	8.001e+21	-4.227e+25
		3	8.001e+21	-4.616e+25
	2	1	8.000e+21	-3.198e+25
		2	8.001e+21	-5.390e+25
		3	8.001e+21	-6.975e+25
	3	1	8.000e+21	-1.406e+24
		2	8.000e+21	-2.109e+24
		3	8.000e+21	-2.457e+24
	4	1	8.000e+21	-5.104e+24
		2	8.000e+21	-6.696e+24
		3	8.000e+21	-7.200e+24
		4	8.000e+21	-7.353e+24
0.01	1	1	8.000e+18	-3.184e+21
		2	8.001e+18	-4.206e+21
		3	8.001e+18	-4.587e+21
	2	1	8.000e+18	-3.184e+21
		2	8.001e+18	-5.368e+21
		3	8.001e+18	-6.946e+21
	3	1	8.000e+18	-1.268e+20
		2	8.000e+18	-1.911e+20
		3	8.000e+18	-2.211e+20
	4	1	8.000e+18	-4.962e+20
		2	8.000e+18	-6.487e+20
		3	8.000e+18	-6.935e+20
		4	8.000e+18	-7.065e+20

Table 4.2.2 Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ for θ_1 = 0.0001, 0.001, 0.01, 0.1, 0.5 and 1.0(cont.)

θ_1	Data set	m	$\varphi^*_{0MLE}(\theta_0)$	$\varphi^*_{0B}(\theta_0)$
0.1	1	1	8.000e+15	-3.042e+17
		2	8.000e+15	-4.000e+17
		3	8.001e+15	-4.332e+17
	2	1	8.000e+15	-3.042e+17
		2	8.001e+15	-5.159e+17
		3	8.001e+15	-6.670e+17
	3	1	8.000e+15	-3.770e+15
		2	8.000e+15	-1.213e+16
		3	8.000e+15	-2.013e+16
	4	1	8.000e+15	-3.686e+16
		2	8.000e+15	-4.986e+16
		3	8.000e+15	-5.793e+16
		4	8.000e+15	-6.593e+16
0.5	1	1	6.400e+13	-3.946e+14
		2	6.400e+13	-5.272e+14
		3	6.400e+13	-5.955e+14
	2	1	6.400e+13	-3.946e+14
		2	6.400e+13	-6.968e+14
		3	6.400e+13	-9.077e+14
	3	1	6.400e+13	-3.330e+09
		2	6.400e+13	-6.400e+13
		3	6.400e+13	-1.280e+14
	4	1	6.400e+13	-1.176e+13
		2	6.400e+13	-7.577e+13
		3	6.400e+13	-1.398e+14
		4	6.400e+13	-2.038e+14
1.0	1	1	8.000e+12	-1.862e+13
		2	8.000e+12	-2.812e+13
		3	8.000e+12	-3.613e+13
	2	1	8.000e+12	-1.862e+13
		2	8.000e+12	-3.631e+13
		3	8.000e+12	-3.613e+13
	3	1	8.000e+12	-5.312e+03
		2	8.000e+12	-8.000e+12
		3	8.000e+12	-1.600e+13
	4	1	8.000e+12	-6.189e+10
		2	8.000e+12	-8.062e+12
		3	8.000e+12	-1.606e+13
		4	8.000e+12	-2.406e+13

Logistic model

We compare $\varphi_0^*(\theta_0)$ between $\varphi_{0MLE}^*(\theta_0)$ (4.2.28) and $\varphi_{0B}^*(\theta_0)$ (4.2.36) when $\theta_0 \to \infty$, and obtain that (4.2.28) is greater than (4.2.36) in the case $m \leq 5$. For $m \geq 7$, (4.2.28) is less than (4.2.36). They are the same for $m = 6$. We compare $\varphi_0^*(\theta_0)$ between $\varphi_{0MLE}^*(\theta_0)$ (4.2.29) and $\varphi_{0B}^*(\theta_0)$ (4.2.37) when $\theta_0 \to -\infty$, and obtain that (4.2.29) is greater than (4.2.37) in the case $m \leq 5$. In the case for $m \geq 7$, (4.2.29) is less than (4.2.37). They are the same for $m = 6$. This shows that, for $m \leq 5$, there is an interval I_0 of values of θ_0 within which the MLE is better than the Berkson's minimum chi-square estimate in the sense of second order asymtotic variance, and outside which it's the other way around. On the other hand, for $m \geq 7$, there is an interval I_0 of values of θ_0 within which the MLE is worse than the Berkson's minimum chi-square estimate in the sense of second order asymtotic variance, and outside which it's the other way around. For $m = 6$, the MLE and Berkson's minimum chi-square estimate are second order equivalent.

Log-logistic model

The log-logistic model could be processed in the same manner with the logistic model, we find that similar expressions of $\varphi_{0MLE}^*(\theta_0)$ and $\varphi_{0B}^*(\theta_0)$ hold with "*d*" replaced by "$\ln(d)$". Therefore, the results are the same as the results of the logistic model .

4.3 Case II: θ_0 is known and θ_1 is unknown

In this section, the multistage Weibull, logistic and log-logistic models with θ_1 is unknown and θ_0 is known could be processed in the same manner with section 4.2, that is $\varphi_{1MLE}^*(\theta_1)$ for MLE and $\varphi_{1B}^*(\theta_1)$ for Berkson's minimum chi-square estimates are derived.

4.3.1 Multistage Weibull model

Maximum likelihood estimator

The likelihood function of multistage Weibull model is given by

$$L(\theta_1 \mid \text{data}) \propto \prod_{i=0}^{m} \left\{ e^{(\theta_0 + d_i\theta_1)} - 1 \right\}^{x_i} \left\{ e^{-(\theta_0 + d_i\theta_1)} \right\},$$

and the log-likelihood function is given by

$$\log L\left(\theta_1 \mid \text{data}\right) = -\left(\theta_0 \sum_{i=0}^{m} n_i + \theta_1 \sum_{i=0}^{m} d_i n_i\right) + \sum_{i=0}^{m} x_i \log\left(e^{(\theta_0+\theta_1 d_i)} - 1\right).$$

Differentiating $\log L\left(\theta_1 \mid \text{data}\right)$ with respect to θ_1 and setting its derivative equal to zero, then the normal equation is

$$\sum_{i=0}^{m} n_i d_i = \sum_{i=0}^{m} \frac{x_i d_i}{1 - e^{-\left(\theta_0 + \hat{\theta}_{1MLE} d_i\right)}},$$

where $\hat{\theta}_{1MLE}$ is the maximum likelihood estimator of θ_1. We assume that $n_i = n$ for all i and $p_i = x_i / n$, then

$$\sum_{i=0}^{m} d_i = \sum_{i=0}^{m} \frac{d_i p_i}{1 - e^{-\left[\theta_0 + \psi\left(p_0, \ldots, p_m\right) d_i\right]}}, \qquad (4.3.1)$$

where $\hat{\theta}_{1MLE} = \psi\left(p_0, \ldots, p_m\right)$. To compute the biased term of θ_1, $b_{1MLE}\left(\theta_1\right)$, we use Taylor series to express $\psi\left(p_0, \ldots, p_m\right)$, that is

$$\begin{aligned}
\hat{\theta}_{1MLE} &= \psi\left(p_0, \ldots, p_m\right) \\
&= \psi\left(P_0, \ldots, P_m\right) + \sum_{i=0}^{m} \left(p_i - P_i\right) \left.\frac{\partial \psi}{\partial p_i}\right|_{P_i} + \frac{1}{2} \sum_{i=0}^{m} \left(p_i - P_i\right)^2 \left.\frac{\partial^2 \psi}{\partial p_i^2}\right|_{P_i} \\
&\quad + \sum_{i \neq j}^{m} \left(p_i - P_i\right)\left(p_j - P_j\right) \left.\frac{\partial^2 \psi}{\partial p_i \partial p_j}\right|_{P_i, P_j} + \cdots. \qquad (4.3.2)
\end{aligned}$$

The mean of $\hat{\theta}_{1MLE}$ is obtained by taking expectation on both sides of (4.3.2), that is

$$\begin{aligned}
E\left(\hat{\theta}_{1MLE}\right) &= \theta_1 + \sum_{i=0}^{m} E\left(p_i - P_i\right) \left.\frac{\partial \psi}{\partial p_i}\right|_{P_i} + \frac{1}{2} \sum_{i=0}^{m} E\left(p_i - P_i\right)^2 \left.\frac{\partial^2 \psi}{\partial p_i^2}\right|_{P_i} \\
&\quad + \sum_{i \neq j}^{m} E\left\{\left(p_i - P_i\right)\left(p_j - P_j\right)\right\} \left.\frac{\partial^2 \psi}{\partial p_i \partial p_j}\right|_{P_i, P_j} + \cdots.
\end{aligned}$$

We note that $E\left(p_i - P_i\right)$ and $E\left\{\left(p_i - P_i\right)\left(p_j - P_j\right)\right\}$, $i \neq j$, are equal to zero. Also, $E\left(p_i - P_i\right)^2 = \pi_i\left(\theta_1\right)\left(1 - \pi_i\left(\theta_1\right)\right)/n$. By using (1.1) and ignoring the terms of smaller order than n^{-1}, we have

$$\theta_1 + \frac{b_{1MLE}\left(\theta_1\right)}{n} = \theta_1 + \frac{1}{2}\sum_{i=0}^{m} E\left(p_i - P_i\right)^2 \frac{\partial^2 \psi}{\partial p_i^2}\bigg|_{P_i} . \tag{4.3.3}$$

We find the second derivative of $\psi\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing by P_i.

First, we would like to find the first derivative of (4.3.1) with respect to p_i,

$$\sum_{i=0}^{m} d_i = \frac{d_0 p_0}{1-e^{-(\theta_0+\psi d_0)}} + \frac{d_1 p_1}{1-e^{-(\theta_0+\psi d_1)}} + \cdots + \frac{d_m p_m}{1-e^{-(\theta_0+\psi d_m)}} .$$

The first derivative of (4.3.1) with respect to p_0 is

$$0 = \frac{d_0}{1-e^{-(\theta_0+\psi d_0)}} - \frac{d_0^2 p_0 e^{-(\theta_0+\psi d_0)}}{\left(1-e^{-(\theta_0+\psi d_0)}\right)^2}\frac{\partial \psi}{\partial p_0} - \sum_{i=1}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial \psi}{\partial p_0}$$

$$= \frac{d_0}{1-e^{-(\theta_0+\psi d_0)}} - \sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial \psi}{\partial p_0} .$$

The first derivative of (4.3.1) with respect to p_1 is

$$0 = \frac{d_1}{1-e^{-(\theta_0+\psi d_1)}} - \sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial \psi}{\partial p_1} .$$

We proceed in the same process, then the first derivative of (4.3.1) with respect to p_m is

$$0 = \frac{d_m}{1-e^{-(\theta_0+\psi d_m)}} - \sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial \psi}{\partial p_m} .$$

We obtain the first derivative of (3.1) with respect to p_i for $i = 0,1,\ldots,m$

$$0 = \frac{d_i}{1-e^{-(\theta_0+\psi d_i)}} - \frac{\partial \psi}{\partial p_i}\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}$$

or

$$\frac{\partial \psi}{\partial p_i} = \frac{d_i}{\left(1-e^{-(\theta_0+\psi d_i)}\right)\left[\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\right]} . \tag{4.3.4}$$

After we obtain the first derivative of (4.3.1) with respect to p_i, then we find the second derivative of (4.2.1) with respect to p_i. The second derivative of (4.3.1) with respect to p_0 is

$$0=\frac{-d_0^2e^{-(\theta_0+\psi d_0)}}{\left(1-e^{-(\theta_0+\psi d_0)}\right)^2}\frac{\partial\psi}{\partial p_0}-\frac{d_0^2\,p_0e^{-(\theta_0+\psi d_0)}}{\left(1-e^{-(\theta_0+\psi d_0)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}$$

$$-\frac{\partial\psi}{\partial p_0}\left\{\frac{-d_0^3\,p_0e^{-(\theta_0+\psi d_0)}\dfrac{\partial\psi}{\partial p_0}+d_0^2e^{-(\theta_0+\psi d_0)}}{\left(1-e^{-(\theta_0+\psi d_0)}\right)^2}-\frac{2d_0^3\,p_0e^{-2(\theta_0+\psi d_0)}}{\left(1-e^{-(\theta_0+\psi d_0)}\right)^3}\frac{\partial\psi}{\partial p_0}\right\}-\frac{d_1^2\,p_1e^{-(\theta_0+\psi d_1)}}{\left(1-e^{-(\theta_0+\psi d_1)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}$$

$$-\frac{\partial\psi}{\partial p_0}\left\{\frac{-d_1^3\,p_1e^{-(\theta_0+\psi d_1)}}{\left(1-e^{-(\theta_0+\psi d_1)}\right)^2}\frac{\partial\psi}{\partial p_0}-\frac{2d_1^3\,p_1e^{-2(\theta_0+\psi d_1)}}{\left(1-e^{-(\theta_0+\psi d_1)}\right)^3}\frac{\partial\psi}{\partial p_0}\right\}-\cdots-\frac{d_m^2\,p_me^{-(\theta_0+\psi d_m)}}{\left(1-e^{-(\theta_0+\psi d_m)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}$$

$$-\frac{\partial\psi}{\partial p_0}\left\{\frac{-d_m^3\,p_me^{-(\theta_0+\psi d_m)}}{\left(1-e^{-(\theta_0+\psi d_m)}\right)^2}\frac{\partial\psi}{\partial p_0}-\frac{2d_m^3\,p_me^{-2(\theta_0+\psi d_m)}}{\left(1-e^{-(\theta_0+\psi d_m)}\right)^3}\frac{\partial\psi}{\partial p_0}\right\}.$$

$$0=\frac{-2d_0^2e^{-(\theta_0+\psi d_0)}}{\left(1-e^{-(\theta_0+\psi d_0)}\right)^2}\frac{\partial\psi}{\partial p_0}-\sum_{i=0}^{m}\frac{d_i^2\,p_ie^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_0^2}$$

$$+\sum_{i=0}^{m}\frac{d_i^3\,p_ie^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\left(\frac{\partial\psi}{\partial p_0}\right)^2+\sum_{i=0}^{m}\frac{2d_i^3\,p_ie^{-2(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^3}\left(\frac{\partial\psi}{\partial p_0}\right)^2.$$

The second derivative of (4.3.1) with respect to p_1 is

$$0=\frac{-2d_1^2e^{-(\theta_0+\psi d_1)}}{\left(1-e^{-(\theta_0+\psi d_1)}\right)^2}\frac{\partial\psi}{\partial p_1}-\sum_{i=0}^{m}\frac{d_i^2\,p_ie^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_1^2}$$

$$+\sum_{i=0}^{m}\frac{d_i^3\,p_ie^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\left(\frac{\partial\psi}{\partial p_1}\right)^2+\sum_{i=0}^{m}\frac{2d_i^3\,p_ie^{-2(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^3}\left(\frac{\partial\psi}{\partial p_1}\right)^2.$$

We proceed in the same process, then the second derivative of (4.3.1) with respect to p_m is

$$0=\frac{-2d_m^2e^{-(\theta_0+\psi d_m)}}{\left(1-e^{-(\theta_0+\psi d_m)}\right)^2}\frac{\partial\psi}{\partial p_m}-\sum_{i=0}^{m}\frac{d_i^2\,p_ie^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_m^2}$$

$$+\sum_{i=0}^{m}\frac{d_i^3\,p_ie^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\left(\frac{\partial\psi}{\partial p_m}\right)^2+\sum_{i=0}^{m}\frac{2d_i^3\,p_ie^{-2(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^3}\left(\frac{\partial\psi}{\partial p_m}\right)^2.$$

We obtain the second derivative of (4.3.1) with respect to p_i for $i = 0,1,\ldots,m$

$$0 = \frac{-2d_i^2 e^{-(\theta_0+\psi d_j)}}{\left(1-e^{-(\theta_0+\psi d_j)}\right)^2}\frac{\partial\psi}{\partial p_i} - \sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\frac{\partial^2\psi}{\partial p_i^2}$$

$$+\left(\frac{\partial\psi}{\partial p_i}\right)^2\sum_{i=0}^{m}\frac{d_i^3 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2} + \left(\frac{\partial\psi}{\partial p_i}\right)^2\sum_{i=0}^{m}\frac{2d_i^3 p_i e^{-2(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^3}$$

or

$$\frac{\partial^2\psi}{\partial p_i^2} = \frac{\dfrac{-2d_j^2 e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}\dfrac{\partial\psi}{\partial p_i} + \left(\dfrac{\partial\psi}{\partial p_i}\right)^2\displaystyle\sum_{i=0}^{m}\dfrac{d_i^3 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2} + \left(\dfrac{\partial\psi}{\partial p_i}\right)^2\displaystyle\sum_{i=0}^{m}\dfrac{2d_i^3 p_i e^{-2(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^3}}{\displaystyle\sum_{i=0}^{m}\dfrac{d_i^2 p_i e^{-(\theta_0+\psi d_i)}}{\left(1-e^{-(\theta_0+\psi d_i)}\right)^2}}. \quad (4.3.5)$$

We substitute (4.3.4) in (4.3.5) and obtain

$$\left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i} = \frac{-\dfrac{2d_i^3\left(1-\pi_i(\theta_1)\right)}{\pi_i^2(\theta_1)} + \dfrac{d_i^2}{\pi_i(\theta_1)I_{MW}(\theta_1)}\left[\displaystyle\sum_{i=0}^{m}\dfrac{d_i^3\left(1-\pi_i(\theta_1)\right)\left(2-\pi_i(\theta_1)\right)}{\pi_i^2(\theta_1)}\right]}{\pi_i(\theta_1)I_{MW}^2(\theta_1)}, \quad (4.3.6)$$

where $P_i = \pi_i(\theta_1)$ and $I_{MW}(\theta_1)$ is the Fisher information per observation,

$$I_{MW}(\theta_1) = \sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i(\theta_1)\right)}{\pi_i(\theta_1)}.$$

From (4.3.3) and (4.3.6), $b_{1MLE}(\theta_1)$ based on MLE can be simplified as

$$\theta_1 + \frac{b_{1MLE}(\theta_1)}{n} = \theta_1 + \frac{1}{2}\sum_{i=0}^{m}E\left(p_i - P_i\right)^2\left.\frac{\partial^2\psi}{\partial p_i^2}\right|_{P_i}$$

$$\frac{b_{1MLE}\left(\theta_1\right)}{n}=\frac{1}{2}\sum_{i=0}^{m}\frac{\pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)}{n}\left\{-\frac{2d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^3\left(\theta_1\right)I_{MW}^2\left(\theta_1\right)}+\frac{d_i^2\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]}{\pi_i^2\left(\theta_1\right)I_{MW}^3\left(\theta_1\right)}\right\}$$

$$b_{1MLE}\left(\theta_1\right)=-\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)^2}{\pi_i^2\left(\theta_1\right)I_{MW}^2\left(\theta_1\right)}\right]+\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{2\pi_i\left(\theta_1\right)I_{MW}^3\left(\theta_1\right)}\right]$$

$$=-\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)^2}{\pi_i^2\left(\theta_1\right)I_{MW}^2\left(\theta_1\right)}\right]+\frac{1}{2I_{MW}^2\left(\theta_1\right)}\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]$$

$$=\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{2\pi_i\left(\theta_1\right)I_{MW}^2\left(\theta_1\right)}. \qquad (4.3.7)$$

We find the first derivative of $b_{1MLE}\left(\theta_1\right)$ with respect to θ_1 for substituting in (4.1.3) and obtain

$$b'_{1MLE}\left(\theta_1\right)=\frac{\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]\sum_{i=0}^{m}\left[\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]}{I_{MW}^3\left(\theta_1\right)}-\frac{\sum_{i=0}^{m}\left[\frac{d_i^4\left(1-\pi_i\left(\theta_1\right)\right)}{2\pi_i^2\left(\theta_1\right)}\right]}{I_{MW}^2\left(\theta_1\right)}. \qquad (4.3.8)$$

From (4.1.3), (4.3.7) and (4.3.8), we obtain the part of MSE of MLE, that is $\varphi_{1MLE}^*\left(\theta_1\right)$,

$$\varphi_{1MLE}^*\left(\theta_1\right)=\frac{1}{4}\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]^2-\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i^4\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]$$

$$+2\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]. \qquad (4.3.9)$$

Therefore, we obtain the part of MSEs of MLE at the two extreme values,

$$\varphi_{1MLE}^*\left(\theta_1\right)\simeq\frac{1}{4}\left[\sum_{i=0}^{m}d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]^2-\left[\sum_{i=0}^{m}d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m}d_i^4\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

$$+2\left[\sum_{i=0}^{m}d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]^2$$

$$=\frac{9}{4}\left[\sum_{i=0}^{m}d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]^2-\left[\sum_{i=0}^{m}d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m}d_i^4\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

as $\theta_1\rightarrow\infty$ (4.3.10)

$$\varphi^*_{1MLE}\left(\theta_1\right) \simeq \frac{1}{4}\left[\sum_{i=0}^{m}\frac{d_i^3 e^{-\theta_0}}{1-e^{-\theta_0}}\right]^2 - \left[\sum_{i=0}^{m}\frac{d_i^2 e^{-\theta_0}}{1-e^{-\theta_0}}\right]\left[\sum_{i=0}^{m}\frac{d_i^4 e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\right] + 2\left[\sum_{i=0}^{m}\frac{d_i^3 e^{-\theta_0}}{1-e^{-\theta_0}}\right]\left[\sum_{i=0}^{m}\frac{d_i^3 e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\right]$$

as $\theta_1 \to 0$ (4.3.11)

The limiting values of $\varphi^*_{1MLE}\left(\theta_1\right)$ as $\theta_1 \to \infty$ and $\theta_1 \to 0$ will be used later.

Berkson's minimum chi-square estimator

Let us consider

$$1-\pi\left(\theta_1\right) = e^{-(\theta_0+\theta_1 d)}$$

$$-\ln\left(1-\pi\left(\theta_1\right)\right) = \theta_0 + \theta_1 d .$$

We denote the observed frequencies by $p_i = x_i / n_i$ where $i = 0,\ldots,m$; $Var\left[-\ln\left(1-p_i\right)\right] = p_i/\left(n_i q_i\right)$, when $x_i = n_i$ then $p_i = 1-1/2n_i$ and $x_i = 0$ then $p_i = 1/2n_i$. Therefore a chi-square function $Q\left(\theta_1\right)$ is given by

$$Q\left(\theta_1\right) = \sum_{i=0}^{m}\frac{n_i\left(1-p_i\right)}{p_i}\left[-ln\left(1-p_i\right)-\theta_0-\theta_1 d_i\right]^2 .$$

We compute the Berkson's minimum chi-square estimator of θ_1, $\hat{\theta}_{1B}$, based on minimizing $Q\left(\theta_1\right)$ with respect to θ_1. Differentiating $Q\left(\theta_1\right)$ with respect to θ_1 and setting its derivative equal to zero, then the normal equation is written as

$$\sum_{i=0}^{m}\frac{n_i d_i\left(1-p_i\right)}{p_i}\left[\ln\left(1-p_i\right)+\theta_0+\theta_1 d_i\right] = 0 . \qquad (4.3.12)$$

We obtain $\hat{\theta}_{1B}$ by solving (4.3.12),

$$\hat{\theta}_{1B} = \frac{\displaystyle\sum_{i=0}^{m}\frac{n_i d_i\left(1-p_i\right)}{p_i}\left[-ln\left(1-p_i\right)-\theta_0\right]}{\displaystyle\sum_{i=0}^{m}\frac{n_i d_i^2\left(1-p_i\right)}{p_i}} .$$

We assume $n_i = n$ and write $\hat{\theta}_{1B} = \tilde{\psi}\left(p_0,\ldots,p_m\right)$. By Taylor expansion (4.3.2), we obtain the mean of $\hat{\theta}_{1B}$,

$$E\left(\hat{\theta}_{1B}\right) = \theta_1 + \sum_{i=0}^{m}E\left(p_i-P_i\right)\frac{\partial\tilde{\psi}}{\partial p_i}\bigg|_{P_i} + \frac{1}{2}\sum_{i=0}^{m}E\left(p_i-P_i\right)^2\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\bigg|_{P_i}$$

$$+\sum_{i\neq j}^{m}E\left\{\left(p_i-P_i\right)\left(p_j-P_j\right)\right\}\frac{\partial^2\tilde{\psi}}{\partial p_i\partial p_j}\bigg|_{P_i,P_j}+\cdots.$$

Ignoring the terms of a smaller order than n^{-1}, we have

$$\theta_1+\frac{b_{1B}\left(\theta_1\right)}{n}=\theta_1+\frac{1}{2}\sum_{i=0}^{m}E\left(p_i-P_i\right)^2\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\bigg|_{P_i}. \qquad (4.3.13)$$

We find the second derivative of $\tilde{\psi}\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing by P_i.

First, we find the first derivative of $\tilde{\psi}\left(p_0,\ldots,p_m\right)$ with respect to p_i, and obtain

$$\frac{\partial\tilde{\psi}}{\partial p_i}=\frac{\left[\frac{d_i}{p_i}+\left(-\ln\left(1-p_i\right)-\theta_0\right)\left(-\frac{d_i}{p_i^2}\right)\right]}{\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}}-\frac{\left(-\frac{d_i^2}{p_i^2}\right)\left(\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\left(-\ln\left(1-p_i\right)-\theta_0\right)\right)}{\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}\right]^2}.$$

The second derivative of $\tilde{\psi}\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing by P_i is

$$\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\bigg|_{P_i}=\frac{-\frac{d_i}{P_i^2}\left(\frac{2-P_i}{1-P_i}\right)+\left(-\ln\left(1-P_i\right)-\theta_0\right)\left(\frac{2d_i}{P_i^3}\right)-\frac{2d_i^2\theta_1}{P_i^3}}{\sum_{i=0}^{m}\frac{\left(1-P_i\right)d_i^2}{P_i}}$$
$$+\frac{\frac{2d_i^3}{P_i^3}+\left[2\left(-\frac{d_i^3}{P_i^4}\right)\left(-\ln\left(1-P_i\right)-\theta_0\right)\right]+\frac{2d_i^4\theta_1}{P_i^4}}{\left(\sum_{i=0}^{m}\frac{\left(1-P_i\right)d_i^2}{P_i}\right)^2}. \qquad (4.3.14)$$

From (4.3.13) and (4.3.14), the bias term of θ_1, $b_{1B}\left(\theta_1\right)$ based on Berkson's minimum chi-square estimator can be simplified as (using $P_i=\pi_i\left(\theta_1\right)$)

$$\theta_1+\frac{b_{1B}\left(\theta_1\right)}{n}=\theta_1+\frac{1}{2}\sum_{i=0}^{m}E\left(p_i-P_i\right)^2\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\bigg|_{P_i}$$

$$b_{1B}\left(\theta_1\right)=\frac{-\frac{1}{2}\sum_{i=0}^{m}d_i\left(\frac{2-P_i}{P_i}\right)+\sum_{i=0}^{m}d_i\left(\frac{1-P_i}{P_i^2}\right)\left(-\ln\left(1-P_i\right)-\theta_0\right)-\sum_{i=0}^{m}d_i^2\left(\frac{1-P_i}{P_i^2}\right)\theta_1}{\sum_{i=0}^{m}\frac{d_i^2\left(1-P_i\right)}{P_i}}$$

$$+\frac{\sum_{i=0}^{m} d_i^3\left(\frac{1-P_i}{P_i^2}\right)-\sum_{i=0}^{m} d_i^3\left(\frac{1-P_i}{P_i^3}\right)\left(-\ln\left(1-P_i\right)-\theta_0\right)+\sum_{i=0}^{m} d_i^4\left(\frac{1-P_i}{P_i^3}\right)\theta_1}{\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-P_i\right)}{P_i}\right]^2}$$

$$=\frac{-\frac{1}{2}\sum_{i=0}^{m} d_i\left(\frac{2-P_i}{P_i}\right)+\sum_{i=0}^{m} d_i\left(\frac{1-P_i}{P_i^2}\right)\left(-\ln\left(1-P_i\right)-\theta_0-\theta_1 d_i\right)}{\sum_{i=0}^{m}\frac{d_i^2\left(1-P_i\right)}{P_i}}$$

$$+\frac{\sum_{i=0}^{m} d_i^3\left(\frac{1-P_i}{P_i^2}\right)-\sum_{i=0}^{m} d_i^3\left(\frac{1-P_i}{P_i^3}\right)\left(-\ln\left(1-P_i\right)-\theta_0-\theta_1 d_i\right)}{\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-P_i\right)}{P_i}\right]^2}$$

$$=\frac{\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}}{I_{MW}^2\left(\theta_1\right)}-\frac{\sum_{i=0}^{m}\frac{d_i\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}}{2I_{MW}\left(\theta_1\right)}. \qquad (4.3.15)$$

We find the first derivative of $b_{1B}\left(\theta_1\right)$ with respect to θ_1 for substituting in (4.1.3) and obtain

$$b_{1B}'\left(\theta_1\right)=\frac{2\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]^2}{I_{MW}^3\left(\theta_1\right)}-\frac{\sum_{i=0}^{m}\frac{d_i^4\left(1-\pi_i\left(\theta_1\right)\right)\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i^3\left(\theta_1\right)}}{I_{MW}^2\left(\theta_1\right)}$$

$$-\frac{\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]}{2I_{MW}^2\left(\theta_1\right)}+\frac{\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}}{I_{MW}\left(\theta_1\right)}. \qquad (4.3.16)$$

From (4.1.3), (4.3.15) and (4.3.16), we obtain the part of MSE of Berkson's minimum chi-square estimate, that is $\varphi_{1B}^*\left(\theta_1\right)$,

$$\varphi_{1B}^*\left(\theta_1\right)=5\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]^2+\frac{1}{4}\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]^2$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]\left[\sum_{i=0}^{m}\frac{d_i^4\left(1-\pi_i\left(\theta_1\right)\right)\left(2-\pi_i\left(\theta_1\right)\right)}{\pi_i^3\left(\theta_1\right)}\right]$$

$$+2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i\left(\theta_1\right)}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\left(\theta_1\right)\right)}{\pi_i^2\left(\theta_1\right)}\right]. \qquad (4.3.17)$$

Therefore, we obtain the part of MSEs of Berkson's minimum chi-square estimate at two extreme values,

$$\varphi_{1B}^*\left(\theta_1\right)\simeq 5\left[\sum_{i=0}^{m}d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]^2-2\left[\sum_{i=0}^{m}d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m}d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m}d_i\right]$$

$$+\frac{1}{4}\left[\sum_{i=0}^{m}d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]^2\left[\sum_{i=0}^{m}d_i\right]^2-2\left[\sum_{i=0}^{m}d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m}d_i^4\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

$$+2\left[\sum_{i=0}^{m}d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]^3 \text{ as } \theta_1\to\infty \qquad (4.3.18)$$

$$\varphi_{1B}^*\left(\theta_1\right)\simeq 5\left[\sum_{i=0}^{m}\frac{d_i^3e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\right]^2+\frac{1}{4}\left[\sum_{i=0}^{m}\frac{d_i^2e^{-\theta_0}}{1-e^{-\theta_0}}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1+e^{-\theta_0}\right)}{1-e^{-\theta_0}}\right]^2$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i^3e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\right]\left[\sum_{i=0}^{m}\frac{d_i^2e^{-\theta_0}}{1-e^{-\theta_0}}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1+e^{-\theta_0}\right)}{1-e^{-\theta_0}}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i^2e^{-\theta_0}}{1-e^{-\theta_0}}\right]\left[\sum_{i=0}^{m}\frac{d_i^4e^{-\theta_0}\left(1+e^{-\theta_0}\right)}{\left(1-e^{-\theta_0}\right)^3}\right]+2\left[\sum_{i=0}^{m}\frac{d_i^2e^{-\theta_0}}{1-e^{-\theta_0}}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\right]$$

$$\text{as } \theta_1\to 0 \qquad (4.3.19)$$

These limiting values of $\varphi_{1B}^*\left(\theta_1\right)$ will be used later.

4.3.2 Logistic model

Maximum likelihood estimator

The likelihood function of logistic model is given by

$$L\left(\theta_1\mid\text{data}\right)\propto e^{-\sum_{i=0}^{m}\left(\theta_0+d_i\theta_1\right)\left(n_i-x_i\right)}\prod_{i=0}^{m}\left(\frac{1}{1+e^{-\left(\theta_0+d_i\theta_1\right)}}\right)^{n_i},$$

and the log-likelihood function is given by

$$\log L\left(\theta_1 \mid \text{data}\right) = -\sum_{i=0}^{m}\left(\theta_0 + \theta_1 d_i\right)\left(n_i - x_i\right) + \sum_{i=0}^{m} n_i \log\left(\frac{1}{1+e^{-(\theta_0 + d_i\theta_1)}}\right).$$

Differentiating $\log L\left(\theta_1 \mid \text{data}\right)$ with respect to θ_1 and setting its derivative equal to zero, then the normal equation is

$$\sum_{i=0}^{m} d_i x_i = \sum_{i=0}^{m} n_i d_i \pi_i\left(\hat{\theta}_{1MLE}\right), \tag{4.3.20}$$

where $\hat{\theta}_{1MLE}$ is the maximum likelihood estimator of θ_1. To study the properties of $\hat{\theta}_{1MLE}$, we use Taylor series to express $\pi_i\left(\hat{\theta}_{1MLE}\right)$, that is

$$\pi_i\left(\hat{\theta}_{1MLE}\right) = \pi_i\left(\theta_1\right) + \left(\hat{\theta}_{1MLE} - \theta_1\right)\left.\frac{\partial \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}}\right|_{\theta_1} + \frac{\left(\hat{\theta}_{1MLE} - \theta_1\right)^2}{2}\left.\frac{\partial^2 \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}^2}\right|_{\theta_1} + \cdots.$$

From (4.3.20), we assume $n_i = n$,

$$\sum_{i=0}^{m} d_i x_i = \sum_{i=0}^{m} n d_i \pi_i\left(\theta_1\right) + \left(\hat{\theta}_{1MLE} - \theta_1\right)\sum_{i=0}^{m}\left.\frac{n d_i \partial \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}}\right|_{\theta_1} + \frac{\left(\hat{\theta}_{1MLE} - \theta_1\right)^2}{2}\sum_{i=0}^{m}\left.\frac{n d_i \partial^2 \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}^2}\right|_{\theta_1} + \cdots. \tag{4.3.21}$$

We find the mean of $\hat{\theta}_{1MLE}$ by taking expectation on both sides of (4.3.21), that is

$$0 = E\left(\hat{\theta}_{1MLE} - \theta_1\right)\sum_{i=0}^{m}\left.\frac{n d_i \partial \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}}\right|_{\theta_1} + \frac{E\left(\hat{\theta}_{1MLE} - \theta_1\right)^2}{2}\sum_{i=0}^{m}\left.\frac{n d_i \partial^2 \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}^2}\right|_{\theta_1} + \cdots.$$

Ignoring the terms of a smaller order than n^{-1}, we have

$$b_{1MLE}\left(\theta_1\right) = \frac{-\left[\sum_{i=0}^{m}\left.\frac{d_i \partial^2 \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}^2}\right|_{\theta_1}\right]}{2 I_L\left(\theta_1\right)\left[\sum_{i=0}^{m}\left.\frac{d_i \partial \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}}\right|_{\theta_1}\right]}, \tag{4.3.22}$$

where $I_L\left(\theta_1\right)$ is the Fisher information per observation, $I_L\left(\theta_1\right) = \sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1 - \pi_i\left(\theta_1\right)\right)$.

We find the first and the second derivative of $\pi_i\left(\hat{\theta}_{1MLE}\right)$ with respect to $\hat{\theta}_{1MLE}$ replacing by θ_1, and obtain

$$\left.\frac{\partial \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}_{1MLE}}\right|_{\theta_1} = d_i \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right) \tag{4.3.23}$$

$$\left.\frac{\partial^2 \pi_i\left(\hat{\theta}_{1MLE}\right)}{\partial \hat{\theta}^2_{1MLE}}\right|_{\theta_1} = d_i^2 \pi_i\left(\theta_1\right)\left(\pi_i\left(\theta_1\right)-1\right)\left(2\pi_i\left(\theta_1\right)-1\right). \tag{4.3.24}$$

From (4.3.22) - (4.3.24), the bias term of θ_1, $b_{1MLE}\left(\theta_1\right)$ based on MLE can be simplified as

$$\begin{aligned} b_{1MLE}\left(\theta_1\right) &= \frac{\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)}{2I_L\left(\theta_1\right)\sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)} \\ &= \frac{\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)}{2I_L^2\left(\theta_1\right)}. \end{aligned} \tag{4.3.25}$$

We find the first derivative of $b_{1MLE}\left(\theta_1\right)$ with respect to θ_1 for substituting in (4.1.3) and obtain

$$\begin{aligned} b'_{1MLE}\left(\theta_1\right) &= \frac{\sum_{i=0}^{m} d_i^4 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(-6\pi_i^2\left(\theta_1\right)+6\pi_i\left(\theta_1\right)-1\right)}{2I_L^2\left(\theta_1\right)} \\ &\quad + \frac{\left[\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)\right]^2}{I_L^3\left(\theta_1\right)}. \end{aligned} \tag{4.3.26}$$

From (4.1.3), (4.3.25) and (4.3.26), we obtain the part of MSE of MLE, that is $\varphi^*_{1MLE}\left(\theta_1\right)$,

$$\begin{aligned} \varphi^*_{1MLE}\left(\theta_1\right) &= \frac{9}{4}\left[\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)\right]^2 \\ &\quad - \left[\sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m} d_i^4 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(6\pi_i^2\left(\theta_1\right)-6\pi_i\left(\theta_1\right)+1\right)\right]. \end{aligned} \tag{4.3.27}$$

Therefore, we obtain the following limiting values which will be used later.

$$\varphi^*_{1MLE}(\theta_1) \simeq \frac{9}{4}\left[\sum_{i=0}^{m} d_i^3\left(1-\pi_i(\theta_1)\right)\right]^2 - \left[\sum_{i=0}^{m} d_i^4\left(1-\pi_i(\theta_1)\right)\right]\left[\sum_{i=0}^{m} d_i^2\left(1-\pi_i(\theta_1)\right)\right]$$

$$\text{as } \theta_1 \to \infty. \qquad (4.3.28)$$

$$\varphi^*_{1MLE}(\theta_1) \simeq \frac{9}{4}\left[\sum_{i=0}^{m} d_i^3\pi_i(\theta_1)\right]^2 - \left[\sum_{i=0}^{m} d_i^4\pi_i(\theta_1)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i(\theta_1)\right] \text{ as } \theta_1 \to -\infty. \qquad (4.3.29)$$

These are $\varphi^*_{1MLE}(\theta_1)$ at two extreme values.

Berkson's minimum chi-square estimator

Let us consider

$$1-\pi(\theta_1) = \frac{e^{-(\theta_0+\theta_1 d)}}{1+e^{-(\theta_0+\theta_1 d)}}$$

$$\frac{1-\pi(\theta_1)}{\pi(\theta_1)} = e^{-(\theta_0+\theta_1 d)}$$

$$\ln\left(\frac{\pi(\theta_1)}{1-\pi(\theta_1)}\right) = \theta_0 + \theta_1 d \ .$$

We denote the observed frequencies by $p_i = x_i/n_i$ where $i = 0,\ldots,m$ and $Var\left[\ln\left(\frac{p_i}{1-p_i}\right)\right] = n_i p_i q_i$. Therefore a chi-square function $Q(\theta_1)$ is given by

$$Q(\theta_1) = \sum_{i=0}^{m} n_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right) - \theta_0 - \theta_1 d_i\right]^2 .$$

We compute the Berkson's minimum chi-square estimator of θ_1, $\hat{\theta}_{1B}$, based on minimizing $Q(\theta_1)$ with respect to θ_1. Differentiating $Q(\theta_1)$ with respect to θ_1 and setting its derivative equal to zero, then the normal equation is written as

$$\sum_{i=0}^{m} n_i d_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right) - \theta_0 - \theta_1 d_i\right] = 0 . \qquad (4.3.30)$$

We obtain $\hat{\theta}_{1B}$ by solving (4.3.30),

$$\hat{\theta}_{1B} = \frac{\sum_{i=0}^{m} n_i d_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right)-\theta_0\right]}{\sum_{i=0}^{m} n_i d_i^2 p_i (1-p_i)}.$$

We suppose $n_i = n$ and write $\hat{\theta}_{1B} = \tilde{\psi}(p_0,\dots,p_m)$. By Taylor expansion (4.3.2), we obtain the mean of $\hat{\theta}_{1B}$,

$$E\left(\hat{\theta}_{1B}\right) = \theta_1 + \sum_{i=0}^{m} E(p_i - P_i)\left.\frac{\partial\tilde{\psi}}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m} E(p_i - P_i)^2 \left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}$$

$$+\sum_{i\neq j}^{m} E\left\{(p_i - P_i)(p_j - P_j)\right\}\left.\frac{\partial^2\tilde{\psi}}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.$$

Ignoring the terms of a smaller order than n^{-1}, we have

$$\theta_1 + \frac{b_{1B}(\theta_1)}{n} = \theta_1 + \frac{1}{2}\sum_{i=0}^{m} E(p_i - P_i)^2 \left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}. \qquad (4.3.31)$$

We find the second derivative of $\tilde{\psi}(p_0,\dots,p_m)$ with respect to p_i replacing by P_i.

First, we find the first derivative of $\tilde{\psi}(p_0,\dots,p_m)$ with respect to p_i, and obtain

$$\frac{\partial\tilde{\psi}}{\partial p_i} = \frac{\left[d_i + \left(\ln\left(\frac{p_i}{1-p_i}\right)-\theta_0\right)\left(d_i(1-2p_i)\right)\right]}{\sum_{i=0}^{m} d_i^2 p_i(1-p_i)}$$

$$-\frac{\left[d_i^2(1-2p_i)\right]\left[\sum_{i=0}^{m} d_i p_i (1-p_i)\left(\ln\left(\frac{p_i}{1-p_i}\right)-\theta_0\right)\right]}{\left[\sum_{i=0}^{m} d_i^2 p_i (1-p_i)\right]^2}.$$

The second derivative of $\tilde{\psi}(p_0,\dots,p_m)$ with respect to p_i replacing by P_i is

$$\left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i} = \frac{-2d_i\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_0\right)+\left(\frac{d_i(1-2P_i)}{P_i(1-P_i)}\right)+2d_i^2\theta_1}{\sum_{i=0}^{m} d_i^2 P_i(1-P_i)}$$

$$+\frac{-2d_i^3\left(1-2P_i\right)-2d_i^3\left(1-2P_i\right)^2\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_0\right)+2\left(d_i^2\left(1-2P_i\right)\right)^2\theta_1}{\left[\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)\right]^2}. \quad (4.3.32)$$

From (4.3.31) and (4.3.32), the bias term of θ_1, $b_{1B}\left(\theta_1\right)$ based on Berkson's minimum chi-square estimator can be simplified as (using $P_i=\pi_i\left(\theta_1\right)$)

$$\theta_1+\frac{b_{1B}\left(\theta_1\right)}{n}=\theta_1+\frac{1}{2}\sum_{i=0}^{m}E\left(p_i-P_i\right)^2\left.\frac{\partial^2\tilde{\psi}}{\partial p_i^2}\right|_{P_i}$$

$$b_{1B}\left(\theta_1\right)=\frac{-\sum_{i=0}^{m}d_iP_i\left(1-P_i\right)\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_0\right)+\frac{1}{2}\sum_{i=0}^{m}d_i\left(1-2P_i\right)+\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)\theta_1}{\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)}$$

$$+\frac{\sum_{i=0}^{m}d_i^3P_i\left(1-P_i\right)\left(2P_i-1\right)-\sum_{i=0}^{m}d_i^3P_i\left(1-P_i\right)\left(2P_i-1\right)^2\left(\ln\left(\frac{P_i}{1-P_i}\right)-\theta_0\right)}{\left[\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)\right]^2}$$

$$+\frac{\sum_{i=0}^{m}d_i^4P_i\left(1-P_i\right)\left(2P_i-1\right)^2\theta_1}{\left[\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)\right]^2}$$

$$=\frac{\sum_{i=0}^{m}d_i^3P_i\left(1-P_i\right)\left(2P_i-1\right)}{\left[\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)\right]^2}-\frac{\sum_{i=0}^{m}d_i\left(2P_i-1\right)}{2\sum_{i=0}^{m}d_i^2P_i\left(1-P_i\right)}$$

$$=\frac{\sum_{i=0}^{m}d_i^3\pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)}{\left[\sum_{i=0}^{m}d_i^2\pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\right]^2}-\frac{\sum_{i=0}^{m}d_i\left(2\pi_i\left(\theta_1\right)-1\right)}{2\sum_{i=0}^{m}d_i^2\pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)}$$

$$=\frac{\sum_{i=0}^{m}d_i^3\pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)}{I_L^2\left(\theta_1\right)}-\frac{\sum_{i=0}^{m}d_i\left(2\pi_i\left(\theta_1\right)-1\right)}{2I_L\left(\theta_1\right)}. \quad (4.3.33)$$

We find the first derivative of $b_{1B}\left(\theta_1\right)$ with respect to θ_1 for substituting in (4.1.3) and obtain

$$b'_{1B}\left(\theta_1\right)=\frac{\sum_{i=0}^{m} d_i^4 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(-6\pi_i^2\left(\theta_1\right)+6\pi_i\left(\theta_1\right)-1\right)}{I_L^2\left(\theta_1\right)}$$

$$+\frac{2\left[\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)\right]^2}{I_L^3\left(\theta_1\right)}-1$$

$$-\frac{\left[\sum_{i=0}^{m} d_i\left(2\pi_i\left(\theta_1\right)-1\right)\right]\left[\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)\right]}{2I_L^2\left(\theta_1\right)}. \qquad (4.3.34)$$

From (4.1.3) and (4.3.31), we obtain the part of MSE of Berkson's minimum chi-square estimate, that is $\varphi_{1B}^*\left(\theta_1\right)$,

$$\varphi_{1B}^*\left(\theta_1\right)=5\left[\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)\right]^2$$

$$-2\left[\sum_{i=0}^{m} d_i^3 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(2\pi_i\left(\theta_1\right)-1\right)\right]\left[\sum_{i=0}^{m} d_i\left(2\pi_i\left(\theta_1\right)-1\right)\right]$$

$$\times\left[\sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

$$+\frac{1}{4}\left[\sum_{i=0}^{m} d_i\left(2\pi_i\left(\theta_1\right)-1\right)\right]^2\left[\sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\right]^2$$

$$-2\left[\sum_{i=0}^{m} d_i^4 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\left(6\pi_i^2\left(\theta_1\right)-6\pi_i\left(\theta_1\right)+1\right)\right]\left[\sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

$$-2\left[\sum_{i=0}^{m} d_i^2 \pi_i\left(\theta_1\right)\left(1-\pi_i\left(\theta_1\right)\right)\right]^3. \qquad (4.3.35)$$

Therefore, we obtain the following limiting values as $\theta_1 \to \pm\infty$. These limiting values will be used later.

$$\varphi_{1B}^*\left(\theta_1\right)\simeq 5\left[\sum_{i=0}^{m} d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]^2-2\left[\sum_{i=0}^{m} d_i^3\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m} d_i\right]\left[\sum_{i=0}^{m} d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

$$+\frac{1}{4}\left[\sum_{i=0}^{m} d_i\right]^2\left[\sum_{i=0}^{m} d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]^2-2\left[\sum_{i=0}^{m} d_i^4\left(1-\pi_i\left(\theta_1\right)\right)\right]\left[\sum_{i=0}^{m} d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]$$

$$-2\left[\sum_{i=0}^{m} d_i^2\left(1-\pi_i\left(\theta_1\right)\right)\right]^3 \text{ as } \theta_1 \to \infty . \tag{4.3.36}$$

$$\varphi_{1B}^*\left(\theta_1\right) \simeq 5\left[\sum_{i=0}^{m} d_i^3\pi_i\left(\theta_1\right)\right]^2-2\left[\sum_{i=0}^{m} d_i^3\pi_i\left(\theta_1\right)\right]\left[\sum_{i=0}^{m} d_i\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(\theta_1\right)\right]$$

$$+\frac{1}{4}\left[\sum_{i=0}^{m} d_i\right]^2\left[\sum_{i=0}^{m} d_i^2\pi_i\left(\theta_1\right)\right]^2-2\left[\sum_{i=0}^{m} d_i^4\pi_i\left(\theta_1\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(\theta_1\right)\right]$$

$$-2\left[\sum_{i=0}^{m} d_i^2\pi_i\left(\theta_1\right)\right]^3 \text{ as } \theta_1 \to -\infty . \tag{4.3.37}$$

4.3.3 Log-logistic model

The log-logistic model could be processed in the same manner with the logistic model, we find that similar expressions of $\varphi_{1MLE}^*\left(\theta_1\right)$ and $\varphi_{1B}^*\left(\theta_1\right)$ hold with "*d*" replaced by "$\ln\left(d\right)$".

4.3.4 Comparison and conclusion

In this section we provide an asymptotic comparison of $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ under the three dose response models. For the multistage Weibull model, considering (4.3.9) (MLE) and (4.3.17) (Berkson), we find that a direct comparison of the asymptotic MSEs would involve computing these expressions for various values of θ_1. To examine what happens at the two extremes as $\theta_1 \to \infty$ or 0, [(4.3.10),(4.3.18)] and [(4.3.11),(4.3.19)] are helpful. A similar idea holds in the other two models.

4.3.4.1 Comparison of MSEs for simpler cases of *m* = 1 and 2

Multistage Weibull model

Comparing (4.3.10) and (4.3.18), we assume $1-\pi_i\left(\theta_1\right)=\varepsilon$ for $\theta_1 \to \infty$ and obtain

$$\varphi_{1MLE}^*\left(\theta_1\right) \simeq \varepsilon^2\left[\frac{9}{4}\left(\sum_{i=0}^{m} d_i^3\right)^2-\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)\right]+o\left(\varepsilon^2\right)=\varepsilon^2 A_1\left(d\right)+o\left(\varepsilon^2\right),$$

$$\varphi_{1B}^*\left(\theta_1\right) \simeq \varepsilon^2\left[5\left(\sum_{i=0}^{m} d_i^3\right)^2-2\left(\sum_{i=0}^{m} d_i^3\right)\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i\right)+\frac{1}{4}\left(\sum_{i=0}^{m} d_i^2\right)^2\left(\sum_{i=0}^{m} d_i\right)^2\right.$$

$$-2\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)+2\varepsilon\left(\sum_{i=0}^{m} d_i^2\right)^3\Bigg]+o\left(\varepsilon^2\right)=\varepsilon^2 A_2\left(d\right)+o\left(\varepsilon^2\right).$$

We compare two MSEs between $A_1\left(d\right)$ and $A_2\left(d\right)$ for m = 1 and 2.

Case $\boldsymbol{m}=1$ and $d_0=0$, we obtain

$A_1\left(d\right)=\frac{9}{4}d_1^6-d_1^6=\frac{5}{4}d_1^6$ and $A_2\left(d\right)=5d_1^6-2d_1^6+\frac{1}{4}d_1^6-2d_1^6+2\varepsilon d_1^6=\frac{5}{4}d_1^6+2\varepsilon d_1^6$.

Then, $A_1\left(d\right)\simeq A_2\left(d\right)$ when $\varepsilon\to 0$. We conclude that $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ are quite the same.

Case $\boldsymbol{m}=2$ and $d_0=0$, we obtain

$A_1\left(d\right)=\frac{9}{4}\left(d_1^3+d_2^3\right)^2-\left(d_1^2+d_2^2\right)\left(d_1^4+d_2^4\right)=\frac{5}{4}d_1^6-d_1^4d_2^2+\frac{9}{2}d_1^3d_2^3-d_1^2d_2^4+\frac{5}{4}d_2^6$, and

$$A_2\left(d\right)=5\left(d_1^3+d_2^3\right)^2-2\left(d_1^3+d_2^3\right)\left(d_1^2+d_2^2\right)\left(d_1+d_2\right)+\frac{1}{4}\left(d_1^2+d_2^2\right)^2\left(d_1+d_2\right)^2$$

$$-2\left(d_1^2+d_2^2\right)\left(d_1^4+d_2^4\right)+2\varepsilon\left(d_1^2+d_2^2\right)^3$$

$$=\frac{5}{4}d_1^6-\frac{3}{2}d_1^5d_2-\frac{13}{4}d_1^4d_2^2+7d_1^3d_2^3-\frac{13}{4}d_1^2d_2^4-\frac{3}{2}d_1d_2^5+\frac{5}{4}d_2^6+2\varepsilon\left(d_1^2+d_2^2\right)^3.$$

To consider,

$A_1\left(d\right)-A_2\left(d\right)=\frac{3}{2}d_1^5d_2+\frac{9}{4}d_1^4d_2^2-\frac{5}{2}d_1^3d_2^3+\frac{9}{4}d_1^2d_2^4+\frac{3}{2}d_1d_2^5-2\varepsilon\left(d_1^2+d_2^2\right)^3>0$, then $A_1\left(d\right)$ is greater than $A_2\left(d\right)$ when $\varepsilon\to 0$. Therefore, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$.

Comparing (4.3.11) and (4.3.19), we assume $\pi_i\left(\theta_1\right)\approx 1-e^{-\theta_0}$ for $\theta_1\to 0$ and obtain

$$\varphi_{1MLE}^*\left(\theta_1\right)\simeq\frac{e^{-2\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\left[\frac{1}{4}\left(\sum_{i=0}^{m} d_i^3\right)^2-\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)\frac{1}{1-e^{-\theta_0}}+2\left(\sum_{i=0}^{m} d_i^3\right)^2\frac{1}{1-e^{-\theta_0}}\right]$$

$$=\frac{e^{-2\theta_0}}{\left(1-e^{-\theta_0}\right)^2}B_1\left(d\right),$$

$$\varphi_{1B}^*\left(\theta_1\right)\simeq\frac{e^{-2\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\left[5\left(\sum_{i=0}^{m} d_i^3\right)^2\frac{1}{\left(1-e^{-\theta_0}\right)^2}+\frac{1}{4}\left(\sum_{i=0}^{m} d_i^2\right)^2\left(\sum_{i=0}^{m} d_i\right)^2\frac{\left(1+e^{-\theta_0}\right)^2}{\left(1-e^{-\theta_0}\right)^2}\right.$$

$$-2\left(\sum_{i=0}^{m} d_i^3\right)\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i\right)\frac{1+e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}$$

$$\left.-2\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)\frac{1+e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}+2\left(\sum_{i=0}^{m} d_i^2\right)^3\frac{e^{-\theta_0}}{\left(1-e^{-\theta_0}\right)^2}\right]$$

$$=\frac{e^{-2\theta_0}}{\left(1-e^{-\theta_0}\right)^2}B_2\left(d\right).$$

We consider $\theta_0 \to 0$, and assume $N=\frac{1}{1-e^{-\theta_0}}$, then

$$B_1\left(d\right)=\frac{1}{4}\left(\sum_{i=0}^{m} d_i^3\right)^2-\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)N+2\left(\sum_{i=0}^{m} d_i^3\right)^2 N,$$

$$B_2\left(d\right)=\left[5\left(\sum_{i=0}^{m} d_i^3\right)^2+\left(\sum_{i=0}^{m} d_i^2\right)^2\left(\sum_{i=0}^{m} d_i\right)^2-4\left(\sum_{i=0}^{m} d_i^3\right)\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i\right)\right.$$

$$\left.-4\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)+2\left(\sum_{i=0}^{m} d_i^2\right)^3\right]N^2.$$

We compare two MSEs between $B_1\left(d\right)$ and $B_2\left(d\right)$ for m = 1 and 2.

Case $m=1$ and $d_0=0$, we obtain

$B_1\left(d\right)=\frac{1}{4}d_1^6-d_1^6N+2d_1^6N=\frac{1}{4}d_1^6+d_1^6N$ and

$B_2\left(d\right)=\left(5d_1^6+d_1^6-4d_1^6-4d_1^6+2d_1^6\right)N^2=0$.

We obtain that $B_1\left(d\right)$ is greater than $B_2\left(d\right)$ when $N\to\infty$. Therefore, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$.

Case $m=2$ and $d_0=0$, it is complicated to derive mathematical relationship between the two MSEs. Therefore, we will show the comparison based on numerical computation with real data in Section 4.3.4.2.

Logistic model

Comparing (4.3.28) and (4.3.36), we assume $1-\pi_i\left(\theta_1\right)=\varepsilon$ for $\theta_1\to\infty$ and obtain

$$\varphi_{1MLE}^{*}\left(\theta_1\right)\simeq\varepsilon^2\left[\frac{9}{4}\left(\sum_{i=0}^{m} d_i^3\right)^2-\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right)\right]+o\left(\varepsilon^2\right)=\varepsilon^2A_2\left(d\right)+o\left(\varepsilon^2\right),$$

$$\varphi_{1B}^{*}(\theta_1) \simeq \varepsilon^2 \left[5\left(\sum_{i=0}^{m} d_i^3\right)^2 - 2\left(\sum_{i=0}^{m} d_i^3\right)\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i\right) + \frac{1}{4}\left(\sum_{i=0}^{m} d_i^2\right)^2\left(\sum_{i=0}^{m} d_i\right)^2 \right.$$
$$\left. -2\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right) - 2\varepsilon\left(\sum_{i=0}^{m} d_i^2\right)^3 \right] + o(\varepsilon^2) = \varepsilon^2 A_2(d) + o(\varepsilon^2).$$

Comparing (3.29) and (3.37), we assume $\pi_i(\theta_1) = \varepsilon$ for $\theta_1 \to -\infty$ and obtain

$$\varphi_{1MLE}^{*}(\theta_1) \simeq \varepsilon^2 \left[\frac{9}{4}\left(\sum_{i=0}^{m} d_i^3\right)^2 - \left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right) \right] + o(\varepsilon^2) = \varepsilon^2 A_1(d) + o(\varepsilon^2),$$

$$\varphi_{1B}^{*}(\theta_1) \simeq \varepsilon^2 \left[5\left(\sum_{i=0}^{m} d_i^3\right)^2 - 2\left(\sum_{i=0}^{m} d_i^3\right)\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i\right) + \frac{1}{4}\left(\sum_{i=0}^{m} d_i^2\right)^2\left(\sum_{i=0}^{m} d_i\right)^2 \right.$$
$$\left. -2\left(\sum_{i=0}^{m} d_i^2\right)\left(\sum_{i=0}^{m} d_i^4\right) - 2\varepsilon\left(\sum_{i=0}^{m} d_i^2\right)^3 \right] + o(\varepsilon^2) = \varepsilon^2 A_2(d) + o(\varepsilon^2).$$

We compare two MSEs between $A_1(d)$ and $A_2(d)$ for m = 1 and 2.

Case $m = 1$ and $d_0 = 0$, we obtain

$A_1(d) = \frac{9}{4}d_1^6 - d_1^6 = \frac{5}{4}d_1^6$ and $A_2(d) = 5d_1^6 - 2d_1^6 + \frac{1}{4}d_1^6 - 2d_1^6 - 2\varepsilon d_1^6 = \frac{5}{4}d_1^6 - 2\varepsilon d_1^6$.

Therefore, $A_1(d) \simeq A_2(d)$ when $\varepsilon \to 0$. We conclude that $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ are quite the same.

Case $m = 2$ and $d_0 = 0$, we obtain

$A_1(d) = \frac{9}{4}\left(d_1^3 + d_2^3\right)^2 - \left(d_1^2 + d_2^2\right)\left(d_1^4 + d_2^4\right) = \frac{5}{4}d_1^6 - d_1^4 d_2^2 + \frac{9}{2}d_1^3 d_2^3 - d_1^2 d_2^4 + \frac{5}{4}d_2^6$, and

$$A_2(d) = 5\left(d_1^3 + d_2^3\right)^2 - 2\left(d_1^3 + d_2^3\right)\left(d_1^2 + d_2^2\right)\left(d_1 + d_2\right) + \frac{1}{4}\left(d_1^2 + d_2^2\right)^2\left(d_1 + d_2\right)^2$$
$$-2\left(d_1^2 + d_2^2\right)\left(d_1^4 + d_2^4\right) - 2\varepsilon\left(d_1^2 + d_2^2\right)^3$$
$$= \frac{5}{4}d_1^6 - \frac{3}{2}d_1^5 d_2 - \frac{13}{4}d_1^4 d_2^2 + 7d_1^3 d_2^3 - \frac{13}{4}d_1^2 d_2^4 - \frac{3}{2}d_1 d_2^5 + \frac{5}{4}d_2^6 - 2\varepsilon\left(d_1^2 + d_2^2\right)^3.$$

We see that,

$A_1(d) - A_2(d) = \frac{3}{2}d_1^5 d_2 + \frac{9}{4}d_1^4 d_2^2 - \frac{5}{2}d_1^3 d_2^3 + \frac{9}{4}d_1^2 d_2^4 + \frac{3}{2}d_1 d_2^5 + 2\varepsilon\left(d_1^2 + d_2^2\right)^3 > 0$, then $A_1(d)$ is greater than $A_2(d)$ when $\varepsilon \to 0$. Therefore, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$.

Log-logistic model

For the log-logistic model, we can derive $A_1(d)$ and $A_2(d)$ in the same manner with "*d*" replaced by "$\ln(d)$" in the logistic model, taking $d_0 = 0.00005$ for m = 1 and 2.

Case $m = 1$, we consider these expressions,

$$A_1(d) - A_2(d) = \frac{3}{2}\ln(d_0)^5 \ln(d_1) + \frac{9}{4}\ln(d_0)^4 \ln(d_1)^2 - \frac{5}{2}\ln(d_0)^3 \ln(d_1)^3 + \frac{9}{4}\ln(d_0)^2 \ln(d_1)^4$$

$$+\frac{3}{2}\ln(d_0)\ln(d_1)^5 + 2\varepsilon\left[\ln(d_0)^2 + \ln(d_1)^2\right]^3.$$

If $d_1 = 1$, we obtain $A_1(d) - A_2(d) = 2\varepsilon \ln(d_0)^6 \to 0$ when $\varepsilon \to 0$, then $A_1(d) \simeq A_2(d)$ that is $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ are quite the same.

If $d_1 < 1$ then $A_1(d)$ is greater than $A_2(d)$ when $\varepsilon \to 0$. We conclude that $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$.

In case of $d_1 > 1$, it is complicated to analyze the difference between $A_1(d)$ and $A_2(d)$, then it could be considered with numerical computation in section 4.3.4.2.

Case $m = 2$, it is complicated to derive mathematical relationship between two MSEs. Therefore, we will show the comparison based on numerical computation with real data in section 4.3.4.2.

4.3.4.2 Comparison of MSEs for complicated cases

In this section, we use four real data sets in Table 4.2.1 for analyzing the difference between two MSEs in each model for some cases that they are complicated to derive mathematical relationship between two MSEs.

Multistage Weibull model

To consider $\theta_1 \to \infty$, we compute the values of $A_1(d)$ and $A_2(d)$ when $m = 3$ for Data sets 4.1-3, and $m = 3$, 4 for Data set 4 in Table 4.3.1 by using data in Table 4.2.1. Our observation is that, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ for large values of θ_1.

Table 4.3.1: Values of $\boldsymbol{A}_1(\boldsymbol{d})$ and $\boldsymbol{A}_2(\boldsymbol{d})$ where $\boldsymbol{\varepsilon} =$ 0.01 for MWM

Data set	***m***	$\boldsymbol{A}_1(\boldsymbol{d})$	$\boldsymbol{A}_2(\boldsymbol{d})$
1	3	4.072e+05	1.008e+05
2	3	1.823e+02	-8.640e+01
3	3	8.119e+11	-9.024e+10
4	3	7.632e+10	2.864e+10
	4	8.159e+13	2.863e+13

To consider $\theta_1 \to 0$, we would like to consider the relationship of $\lim_{N\to\infty} \frac{\varphi^*_{1MLE}(\theta_1)}{\varphi^*_{1B}(\theta_1)} = \frac{B_1(d)}{B_2(d)} = 0$.

We compare $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ when $\theta_1 \to 0$ for $m = 2$, 3 for Data sets 4.1-3, and $m = 2$, 3 and 4 for Data set 4, we use data in Table 4.2.1 to compute the ratio $\frac{B_1(d)}{B_2(d)}$. The results are shown in Table 4.3.2.

Table 4.3.2: Values of $\boldsymbol{B}_1(\boldsymbol{d})/\boldsymbol{B}_2(\boldsymbol{d})$ where $\theta_0 =$ 0.00000005 for MWM

Data set	***m***	$\boldsymbol{B}_1(\boldsymbol{d})$	$\boldsymbol{B}_2(\boldsymbol{d})$	$\boldsymbol{B}_1(\boldsymbol{d})/\boldsymbol{B}_2(\boldsymbol{d})$
1	2	1.833e+10	-1.208e+17	-1.517e-07
	3	6.434e+12	-4.506e+19	-1.428e-07
2	2	3.283e+08	3.662e+15	8.965e-08
	3	2.888e+09	1.277e+17	2.262e-08
3	2	1.998e+17	0	-
	3	1.278e+19	4.775e+25	2.677e-07
4	2	1.300e+15	-8.721e+21	-1.491e-07
	3	1.208e+18	-1.042e+25	-1.160e-07
	4	1.291e+21	-1.214e+28	-1.064e-07

We conclude that, for $\theta_1 \to 0$, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ when $m \geq 1$ for small values of θ_0. On the other hand, $\hat{\theta}_{1MLE}$ is preferred to $\hat{\theta}_{1B}$ in Data set 2 and Data set 3 for $m = 3$.

We also consider some other values of θ_0, i.e. $\theta_0 = 1.5$, 2, 2.5 and 3 for case $m = 2$, 3 for Data sets 4.1-3, and $m = 2$, 3 and 4 for Data set 4.4, and compute below the values of $B_1(d)$ and $B_2(d)$ for a few cases in Table 4.3.3 by using data in Table 4.2.1. Our observation is that, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ for moderate values of θ_0 except Data set 2 for $m = 3$.

Table 4.3.3: Values of $\boldsymbol{B}_1(\boldsymbol{d})$ and $\boldsymbol{B}_2(\boldsymbol{d})$ where $\theta_0 =$ 1.5, 2, 2.5 and 3 for MWM

θ_0	**Data set**	$\boldsymbol{m}$	$\boldsymbol{B}_1(\boldsymbol{d})$	$\boldsymbol{B}_2(\boldsymbol{d})$
1.5	1	2	1.420e+03	-5.006e+02
		3	4.996e+05	-1.867e+05
	2	2	2.534e+01	1.517e+01
		3	2.238e+02	5.290e+02
	3	2	1.549e+10	0
		3	9.954e+11	1.978e+11
	4	2	1.007e+08	-3.613e+07
		3	9.366e+10	-4.315e+10
		4	1.001e+14	-5.028e+13
2	1	2	1.300e+03	-4.041e+02
		3	4.576e+05	-1.507e+05
	2	2	2.320e+01	1.224e+01
		3	2.049e+02	4.270e+02
	3	2	1.418e+10	0
		3	9.119e+11	1.597e+11
	4	2	9.218e+07	-2.916e+07
		3	8.577e+10	-3.483e+10
		4	9.170e+13	-4.058e+13
2.5	1	2	1.239e+03	-3.586e+02
		3	4.360e+05	-1.337e+05
	2	2	2.210e+01	1.087e+01
		3	1.953e+02	3.789e+02
	3	2	1.351e+10	0
		3	8.690e+11	1.417e+11
	4	2	8.782e+07	-2.588e+07
		3	8.172e+10	-3.091e+10
		4	8.737e+13	-3.601e+13
3	1	2	1.205e+03	-3.346e+02
		3	4.241e+05	-1.248e+05
	2	2	2.149e+01	1.014e+01
		3	1.899e+02	3.536e+02
	3	2	1.314e+10	0
		3	8.453e+11	1.322e+11
	4	2	8.541e+07	-2.415e+07
		3	7.948e+10	-2.884e+10
		4	8.498e+13	-3.361e+13

Logistic model

We compute the values of $A_1(d)$ and $A_2(d)$ when $m=3$ for Data sets 1-3, and $m=3$ and 4 for Data set 4 in Table 4.3.4 by using data in Table 4.2.1. Our observation is that, $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$.

Table 4.3.4: Values of $\boldsymbol{A_1(d)}$ and $\boldsymbol{A_2(d)}$ where $\varepsilon = 0.01$ for LM

Data set	***m***	$\boldsymbol{A_1(d)}$	$\boldsymbol{A_2(d)}$
1	3	4.072e+05	8.175e+04
2	3	1.823e+02	-1.013e+02
3	3	8.119e+11	-1.383e+11
4	3	8.159e+13	2.514e+13
	4	7.632e+10	2.536e+10

Log-logistic model

We compute the expressions of $A_1(d)$ and $A_2(d)$ in Table 4.3.5 with taking $d_0 = 0.00005$ and $\varepsilon = 0.010$, 0.015, 0.020, and 0.025 by using data in Table 4.2.1. For Data sets 1 and 2, we have $d_1 = 1$ and we have had the conclusion that $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ are quite the same. Therefore, we only compute the expressions of $A_1(d)$ and $A_2(d)$ for $m=2$ and 3. For Data sets 3 and 4, we have $d_1 > 1$, then we would like to compute the expressions of $A_1(d)$ and $A_2(d)$ when $m=1$, 2 and 3 for Data set 3, and also compute them when $m=1$, 2, 3 and 4 for Data set 4. Our observation based on several other choices of d_0 and ε (not reported here) is that, $\hat{\theta}_{1MLE}$ is preferred to $\hat{\theta}_{1B}$.

Table 4.3.5: Values of $A_1(d)$ and $A_2(d)$ where $\varepsilon =$ 0.010, 0.015, 0.020 and 0.025 for LLM

ε	**Data set**	m	$A_1(d)$	$A_2(d)$
0.010	1	2	1.161e+06	1.271e+06
		3	1.075e+06	1.380e+06
	2	2	1.178e+06	1.210e+06
		3	1.172e+06	1.293e+06
	3	1	9.461e+05	1.067e+06
		2	5.529e+05	8.486e+05
		3	-5.106e+04	5.112e+05
	4	1	1.119e+06	1.270e+06
		2	9.086e+05	1.257e+06
		3	4.180e+05	9.664e+05
		4	-5.164e+05	2.401e+05
0.015	1	2	1.161e+06	1.261e+06
		3	1.075e+06	1.369e+06
	2	2	1.178e+06	1.200e+06
		3	1.172e+06	1.284e+06
	3	1	9.461e+05	1.054e+06
		2	5.529e+05	8.304e+05
		3	-5.106e+04	4.822e+05
	4	1	1.119e+06	1.260e+06
		2	9.086e+05	1.244e+06
		3	4.180e+05	9.457e+05
		4	-5.164e+05	2.025e+05
0.020	1	2	1.161e+06	1.252e+06
		3	1.075e+06	1.358e+06
	2	2	1.178e+06	1.191e+06
		3	1.172e+06	1.274e+06
	3	1	9.461e+05	1.041e+06
		2	5.529e+05	8.121e+05
		3	-5.106e+04	4.533e+05
	4	1	1.119e+06	1.249e+06
		2	9.086e+05	1.231e+06
		3	4.180e+05	9.250e+05
		4	-5.164e+05	1.648e+05

Table 4.3.5: Values of $\boldsymbol{A}_1(\boldsymbol{d})$ and $\boldsymbol{A}_2(\boldsymbol{d})$ where $\boldsymbol{\varepsilon} =$ 0.010, 0.015, 0.020 and 0.025 for LLM (cont.)

$\boldsymbol{\varepsilon}$	**Data set**	$\boldsymbol{m}$	$\boldsymbol{A}_1(\boldsymbol{d})$	$\boldsymbol{A}_2(\boldsymbol{d})$
0.025	1	2	1.161e+06	1.242e+06
		3	1.075e+06	1.347e+06
	2	2	1.178e+06	1.181e+06
		3	1.172e+06	1.264e+06
	3	1	9.461e+05	1.029e+06
		2	5.529e+05	7.938e+05
		3	-5.106e+04	4.243e+05
	4	1	1.119e+06	1.240e+06
		2	9.086e+05	1.217e+06
		3	4.180e+05	9.042e+05
		4	-5.164e+05	1.271e+05

4.3.4.3 Numerical example

For the four real data sets in Table 4.2.1, we assume some known values of θ_0 when $m = 1$, 2 and 3 for Data sets 1-3 and, m = 3 and 4 for Data set 4 to estimate parameter (θ_1) . Since, there are no the number of respondents when m = 1 and 2 for Data set 4, then m = 3 and 4 are considered. $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ with θ_1 replaced by respective estimates are computed based on MLE and Berkson. The comparison of two MSEs [(4.3.9),(4.3.17)] for multistage Weibull model, [(4.3.27),(4.3.35)] for logistic model and "d" replaced by "$\ln(d)$" in logistic model for log-logistic model are shown in Tables 4.3.6-3.8, respectively.

Table 4.3.6: Values of $\hat{\theta}_{1MLE}$, $\hat{\theta}_{1B}$, $\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$, and $\varphi_1^*\left(\hat{\theta}_{1B}\right)$ for MWM assuming $\theta_0 = \mathbf{5.0e\text{-}06}$, 5.0e-05, 5.0e-04, and 5.0e-03

Data set	θ_0	***m***	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$	$\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$	$\varphi_1^*\left(\hat{\theta}_{1B}\right)$	**Conclusion**
1	5.0e-06	1	2.231e-01	2.231e-01	8.404e+01	8.404e+01	Same
		2	2.231e-01	2.231e-01	2.908e+03	2.089e+03	Berkson
		3	2.231e-01	2.231e-01	2.943e+04	4.614e+03	Berkson
	5.0e-05	1	2.231e-01	2.231e-01	8.399e+01	8.399e+01	Same
		2	2.231e-01	2.231e-01	2.907e+03	2.087e+03	Berkson
		3	2.231e-01	2.231e-01	2.943e+04	4.602e+03	Berkson
	5.0e-04	1	2.226e-01	2.226e-01	8.405e+01	8.405e+01	Same
		2	2.229e-01	2.229e-01	2.907e+03	2.078e+03	Berkson
		3	2.230e-01	2.230e-01	2.943e+04	4.521e+03	Berkson
	5.0e-03	1	2.181e-01	2.181e-01	8.405e+01	8.405e+01	Same
		2	2.205e-01	2.205e-01	2.921e+03	2.014e+03	Berkson
		3	2.215e-01	2.215e-01	2.976e+04	3.883e+03	Berkson
2	5.0e-06	1	2.231e-01	2.232e-01	8.404e+01	8.393e+01	Berkson
		2	3.577e-01	3.107e-01	1.404e+02	2.873e+02	MLE
		3	5.194e-01	3.778e-01	1.503e+02	1.199e+03	MLE
	5.0e-05	1	2.231e-01	2.231e-01	8.399e+01	8.399e+01	Same
		2	3.577e-01	3.106e-01	1.403e+02	2.876e+02	MLE
		3	5.194e-01	3.777e-01	1.503e+02	1.200e+03	MLE
	5.0e-04	1	2.226e-01	2.226e-01	8.405e+01	8.405e+01	Same
		2	3.573e-01	3.102e-01	1.404e+02	2.876e+02	MLE
		3	5.191e-01	3.773e-01	1.503e+02	1.201e+03	MLE
	5.0e-03	1	2.181e-01	2.181e-01	8.405e+01	8.405e+01	Same
		2	3.539e-01	3.062e-01	1.401e+02	2.879e+02	MLE
		3	5.163e-01	3.735e-01	1.502e+02	1.209e+03	MLE
3	5.0e-06	1	2.750e-03	2.750e-03	5.386e+11	5.386e+11	Same
		2	4.182e-03	3.866e-03	1.432e+12	1.522e+13	MLE
		3	6.810e-03	5.412e-03	4.151e+12	5.907e+13	MLE
	5.0e-05	1	2.749e-03	2.748e-03	5.380e+11	5.386e+11	Same
		2	4.180e-03	3.864e-03	2.325e+12	1.523e+13	MLE
		3	6.810e-03	5.411e-03	4.149e+12	5.906e+13	MLE
	5.0e-04	1	2.729e-03	2.728e-03	5.380e+11	5.386e+11	Same
		2	4.168e-03	3.849e-03	2.202e+12	1.520e+13	MLE
		3	6.803e-03	5.399e-03	4.147e+12	5.905e+13	MLE
	5.0e-03	1	2.528e-03	2.528e-03	5.386e+11	5.386e+11	Same
		2	4.046e-03	3.700e-03	2.187e+12	1.489e+13	MLE
		3	6.730e-03	5.276e-03	4.130e+12	5.916e+13	MLE

Table 4.3.6: Values of $\hat{\theta}_{1MLE}$, $\hat{\theta}_{1B}$, $\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$, and $\varphi_1^*\left(\hat{\theta}_{1B}\right)$ for MWM assuming $\theta_0 = \mathbf{5.0e\text{-}06}$, 5.0e-05, 5.0e-04, and 5.0e-03 (cont.)

Data set	θ_0	m	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$	$\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$	$\varphi_1^*\left(\hat{\theta}_{1B}\right)$	Conclusion
4	5.0e-06	3	2.294e-04	3.808e-04	3.172e+16	1.444e+18	MLE
		4	1.806e-03	8.330e-04	1.868e+15	1.273e+18	MLE
	5.0e-05	3	2.261e-04	3.797e-04	3.248e+16	1.406e+18	MLE
		4	1.806e-03	8.320e-04	1.866e+15	1.264e+18	MLE
	5.0e-04	3	2.189e-04	3.688e-04	2.996e+16	1.101e+18	MLE
		4	1.804e-03	8.228e-04	1.854e+15	1.181e+18	MLE
	5.0e-03	3	1.469e-04	2.597e-04	2.307e+16	1.875e+17	MLE
		4	1.779e-03	7.304e-04	1.775e+15	7.854e+17	MLE

Table 4.3.7: Values of $\hat{\theta}_{1MLE}$, $\hat{\theta}_{1B}$, $\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$, and $\varphi_1^*\left(\hat{\theta}_{1B}\right)$ for LM assuming θ_0 = -8.0, -7.0, -6.0, and -5.0

Data set	θ_0	m	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$	$\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$	$\varphi_1^*\left(\hat{\theta}_{1B}\right)$	Conclusion
1	-8.0	1	6.614e+00	6.614e+00	1.972e-02	1.972e-02	Same
		2	2.657e+00	2.827e+00	2.786e+01	3.340e+01	MLE
		3	1.994e+00	1.541e+00	8.972e+00	-3.418e+01	Berkson
	-7.0	1	5.614e+00	5.614e+00	1.972e-02	1.972e-02	Same
		2	2.333e+00	2.462e+00	2.790e+01	3.342e+01	MLE
		3	1.677e+00	1.365e+00	-1.374e+00	-7.296e+01	Berkson
	-6.0	1	4.614e+00	4.614e+00	1.972e-02	1.972e-02	Same
		2	2.007e+00	2.098e+00	2.797e+01	3.324e+01	MLE
		3	1.376e+00	1.190e+00	-3.881e+01	-1.439e+02	Berkson
	-5.0	1	3.614e+00	3.614e+00	1.972e-02	1.972e-02	Same
		2	1.678e+00	1.734e+00	2.812e+01	3.206e+01	MLE
		3	1.113e+00	1.015e+00	-9.443e+01	-2.439e+02	Berkson
2	-8.0	1	6.614e+00	6.614e+00	1.972e-02	1.972e-02	Same
		2	5.690e+00	5.744e+00	2.909e-01	2.023e-01	Berkson
		3	5.347e+00	5.322e+00	7.742e-02	-1.165e+00	Berkson
	-7.0	1	5.614e+00	5.614e+00	1.972e-02	1.972e-02	Same
		2	4.958e+00	4.986e+00	3.114e-01	1.691e-01	Berkson
		3	4.693e+00	4.679e+00	1.626e-01	-1.512e+00	Berkson
	-6.0	1	4.614e+00	4.614e+00	1.972e-02	1.972e-02	Same
		2	4.218e+00	4.228e+00	3.417e-01	1.612e-01	Berkson
		3	4.042e+00	4.037e+00	3.669e-01	-1.873e+00	Berkson
	-5.0	1	3.614e+00	3.614e+00	1.972e-02	1.972e-02	Same
		2	3.469e+00	3.470e+00	3.772e-01	1.265e-01	Berkson
		3	3.396e+00	3.395e+00	7.492e-01	-2.212e+00	Berkson

Table 4.3.7: Values of $\hat{\theta}_{1MLE}$, $\hat{\theta}_{1B}$, $\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$, and $\varphi_1^*\left(\hat{\theta}_{1B}\right)$ for LM assuming θ_0 = -8.0, -7.0, -6.0, and -5.0 (cont.)

Data set	θ_0	m	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$	$\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$	$\varphi_1^*\left(\hat{\theta}_{1B}\right)$	Conclusion
3	-8.0	1	2.333e-01	2.333e-01	4.466e+05	4.466e+05	Same
		2	1.503e-01	1.540e-01	1.872e+08	1.126e+08	Berkson
		3	9.733e-02	1.018e-01	1.450e+10	2.163e+10	MLE
	-7.0	1	1.888e-01	1.888e-01	4.457e+05	4.457e+05	Same
		2	1.277e-01	1.299e-01	1.871e+08	1.036e+08	Berkson
		3	8.578e-02	8.873e-02	1.452e+10	2.224e+10	MLE
	-6.0	1	1.444e-01	1.444e-01	4.465e+05	4.465e+05	Same
		2	1.048e-01	1.059e-01	1.864e+08	9.569e+07	Berkson
		3	7.401e-02	7.566e-02	1.469e+10	2.168e+10	MLE
	-5.0	1	9.993e-02	9.993e-02	4.460e+05	4.460e+05	Same
		2	8.163e-02	8.189e-02	1.854e+08	8.815e+07	Berkson
		3	6.191e-02	6.259e-02	1.521e+10	1.991e+10	MLE
4	-8.0	3	6.569e-02	7.331e-02	2.720e+07	3.406e+07	MLE
		4	3.849e-02	3.868e-02	1.979e+12	1.830e+12	Berkson
	-7.0	3	4.930e-02	5.500e-02	2.604e+07	2.588e+07	Berkson
		4	3.346e-02	3.357e-02	1.981e+12	1.797e+12	Berkson
	-6.0	3	3.251e-02	3.669e-02	2.382e+07	1.923e+07	Berkson
		4	2.841e-02	2.846e-02	1.986e+12	1.761e+12	Berkson
	-5.0	3	1.482e-02	1.838e-02	1.966e+07	1.362e+07	Berkson
		4	2.331e-02	2.334e-02	1.995e+12	1.718e+12	Berkson

Table 4.3.8: Values of $\hat{\theta}_{1MLE}$, $\hat{\theta}_{1B}$, $\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$, and $\varphi_1^*\left(\hat{\theta}_{1B}\right)$ for LLM assuming θ_0 = -4.25, -4.00, -3.75, and -3.50

Data set	θ_0	m	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$	$\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$	$\varphi_1^*\left(\hat{\theta}_{1B}\right)$	Conclusion
1	-4.25	1	6.957e+00	3.485e-02	3.432e-58	1.128e+02	MLE
		2	3.750e+00	9.582e-01	6.627e-02	1.013e-01	MLE
		3	3.193e+00	1.541e+00	7.827e-02	3.271e+01	MLE
	-4.00	1	8.898e+00	6.009e-02	1.138e-74	1.128e+02	MLE
		2	3.529e+00	9.223e-01	6.627e-02	1.454e-01	MLE
		3	3.026e+00	1.479e+00	1.577e-01	3.575e+01	MLE
	-3.75	1	11.367e+00	8.534e-02	1.083e-95	1.128e+02	MLE
		2	3.308e+00	8.865e-01	6.627e-02	2.054e-01	MLE
		3	2.861e+00	1.417e+00	2.520e-01	3.860e+01	MLE
	-3.50	1	14.501e+00	1.106e-01	1.964e-122	1.128e+02	MLE
		2	3.088e+00	8.506e-01	6.627e-02	2.836e-01	MLE
		3	2.698e+00	1.355e+00	3.613e-01	4.116e+01	MLE

Table 4.3.8: Values of $\hat{\theta}_{1MLE}$, $\hat{\theta}_{1B}$, $\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$, and $\varphi_1^*\left(\hat{\theta}_{1B}\right)$ for LLM assuming θ_0 = -4.25, -4.00, -3.75, and -3.50 (cont.)

Data set	θ_0	m	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$	$\varphi_1^*\left(\hat{\theta}_{1MLE}\right)$	$\varphi_1^*\left(\hat{\theta}_{1B}\right)$	Conclusion
2	-4.25	1	6.957e+00	3.485e-02	3.432e-58	1.128e+02	MLE
		2	11.249e+00	4.324e-01	9.090e-05	3.139e-02	MLE
		3	9.578e+00	9.252e-01	1.077e-04	6.755e-03	MLE
	-4.00	1	8.898e+00	6.009e-02	1.138e-74	1.128e+02	MLE
		2	10.587e+00	4.333e-01	9.090e-05	5.081e-02	MLE
		3	9.079e+00	9.041e-01	2.158e-04	1.015e-02	MLE
	-3.75	1	11.367e+00	8.534e-02	1.083e-95	1.128e+02	MLE
		2	9.925e+00	4.342e-01	9.090e-05	8.234e-02	MLE
		3	8.584e+00	8.829e-01	3.452e-04	1.495e-02	MLE
	-3.50	1	14.501e+00	1.106e-01	1.964e-122	1.128e+02	MLE
		2	9.264e+00	4.351e-01	9.090e-05	1.336e-01	MLE
		3	8.094e+00	8.618e-01	4.956e-04	2.144e-02	MLE
3	-4.25	1	4.834e-01	1.957e-01	1.731e+00	-1.651e+01	Berkson
		2	6.838e-01	5.321e-01	9.326e+01	1.916e+02	MLE
		3	8.767e-01	8.044e-01	7.845e+02	1.749e+03	MLE
	-4.00	1	4.068e-01	1.829e-01	-6.607e-01	-2.806e+01	Berkson
		2	6.135e-01	4.873e-01	9.164e+01	2.119e+02	MLE
		3	8.141e-01	7.536e-01	7.999e+02	1.814e+03	MLE
	-3.75	1	3.357e-01	1.701e-01	-6.404e+00	-4.611e+01	Berkson
		2	5.432e-01	4.426e-01	8.754e+01	2.260e+02	MLE
		3	7.513e-01	7.028e-01	8.137e+02	1.878e+03	MLE
	-3.50	1	2.747e-01	1.573e-01	-1.766e+01	-7.246e+01	Berkson
		2	4.731e-01	3.978e-01	7.725e+01	2.229e+02	MLE
		3	6.885e-01	6.519e-01	8.248e+02	1.936e+03	MLE
4	-4.25	3	6.672e-02	3.391e-02	-2.881e+01	1.044e+01	MLE
		4	6.064e-01	6.185e-01	1.170e+03	1.556e+03	MLE
	-4.00	3	5.258e-02	2.826e-02	-2.748e+01	5.046e+01	MLE
		4	5.513e-01	5.786e-01	1.166e+03	1.636e+03	MLE
	-3.75	3	4.223e-02	2.261e-02	-1.084e+01	1.421e+02	MLE
		4	4.958e-01	5.386e-01	1.149e+03	1.708e+03	MLE
	-3.50	3	3.456e-02	1.696e-02	3.561e+01	3.349e+02	MLE
		4	4.400e-01	4.986e-01	1.108e+03	1.761e+03	MLE

We observe the following from the above data analysis.

For multistage Weibull model (Table 4.3.6): For Data set 1, the MLE and Berkson estimate perform equally when m = 1 and Berkson estimate is preferred for m = 2 and 3. For

Data sets 2 and 3, the MLE and Berkson estimate perform the same for *m* = 1 while for *m* = 2 and 3, the MLE performs better. For Data set 4, the MLE is preferred for *m* = 3 and 4.

For logistic model (Table 4.3.7): For Data set 1, the MLE and Berkson are equivalence for *m* = 1 and MLE is preferred for *m* = 2 while Berkson estimate is preferred for *m* = 3. For Data set 2, the MLE and Berkson estimate perform equally *m* = 1 while Berkson estimate is preferred for *m* = 2 and 3. For Data set 3, the MLE and Berkson are equivalence for *m* = 1 and Berkson estimate is preferred for *m* = 2 while MLE is preferred for *m* = 3. For Data set 4, often Berkson estimate is preferred for *m* = 3 and 4.

For log-logistic model (Table 4.3.8): For Data sets 1 and 2, the MLE performs better for *m* = 1, 2 and 3. For Data set 3, Berkson estimate is preferred for *m* = 1 while MLE is preferred for *m* = 2 and 3. For Data set 4, the MLE performs better for *m* = 3 and 4.

4.3.4.4 Overall conclusion

The above results show that in general unlike in the case when θ_0 is unknown and θ_1 is known, here (when θ_1 is unknown and θ_0 is known) the mean squared error of the maximum likelihood and Berkson's minimum chi-square estimators behave differently depending on the number of dose groups and also on the dosage amounts. It turns out that the Berkson's minimum chi-square estimates often have preference over MLEs in large samples. Otherwise, unless $m=1$ and $d_1 \le 1$, MLEs have preference over Berkson's minimum chi-square estimates for the log-logistic model.

4.4 Case III: θ_0 and θ_1 are unknown

In this chapter, the multistage Weibull, logistic and log-logistic models with θ_0 and θ_1 are unknown that could be processed in the same manner as Chapters 2 and 3. For the case of two unknown parameters, we examine the mean squared errors in (1.8) and (1.9) for maximum likelihood estimates, and (1.10) and (1.11) for Berkson's minimum chi-square estimates. We refer to the mean squared errors of maximum likelihood estimates, that is

$$E\left(\hat{\theta}_{0MLE}-\theta_0\right)^2=\frac{\sigma_{00}}{n}+\frac{1}{n^2}\left[2\left(\frac{\partial b_{0MLE}\left(\underline{\theta}\right)}{\partial\theta_0}\sigma_{00}+\frac{\partial b_{0MLE}\left(\underline{\theta}\right)}{\partial\theta_1}\sigma_{01}\right)\right]+\frac{b_{0MLE}^2\left(\underline{\theta}\right)}{n^2}$$

$$= \frac{\sigma_{00}}{n} + \varphi^*_{0MLE}\left(\theta_0\right),$$

$$E\left(\hat{\theta}_{1MLE} - \theta_1\right)^2 = \frac{\sigma_{11}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{1MLE}\left(\underline{\theta}\right)}{\partial \theta_0}\sigma_{10} + \frac{\partial b_{1MLE}\left(\underline{\theta}\right)}{\partial \theta_1}\sigma_{11}\right)\right] + \frac{b^2_{1MLE}\left(\underline{\theta}\right)}{n^2}$$

$$= \frac{\sigma_{11}}{n} + \varphi^*_{1MLE}\left(\theta_1\right),$$

and the mean squared errors of Berkson's minimum chi-square estimates,

$$E\left(\hat{\theta}_{0B} - \theta_0\right)^2 = \frac{\sigma_{00}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{0B}\left(\underline{\theta}\right)}{\partial \theta_0}\sigma_{00} + \frac{\partial b_{0B}\left(\underline{\theta}\right)}{\partial \theta_1}\sigma_{01}\right)\right] + \frac{b^2_{0B}\left(\underline{\theta}\right)}{n^2}$$

$$= \frac{\sigma_{00}}{n} + \varphi^*_{0B}\left(\theta_0\right),$$

$$E\left(\hat{\theta}_{1B} - \theta_1\right)^2 = \frac{\sigma_{11}}{n} + \frac{1}{n^2}\left[2\left(\frac{\partial b_{1B}\left(\underline{\theta}\right)}{\partial \theta_0}\sigma_{10} + \frac{\partial b_{1B}\left(\underline{\theta}\right)}{\partial \theta_1}\sigma_{11}\right)\right] + \frac{b^2_{1B}\left(\underline{\theta}\right)}{n^2}$$

$$= \frac{\sigma_{11}}{n} + \varphi^*_{1B}\left(\theta_1\right).$$

The elements of variance-covariance matrix, that is σ_{00}, σ_{01}, σ_{10} and σ_{11} are considered in each model.

The multistage Weibull model, the expressions of σ_{00}, σ_{01}, σ_{10} and σ_{11} are given as

$$\sigma_{00} = \frac{\frac{1}{n}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}}{\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i} - \left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2},$$

$$\sigma_{01} = \sigma_{10} = -\frac{\frac{1}{n}\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}}{\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i} - \left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2}, \text{ and}$$

$$\sigma_{11} = \frac{\frac{1}{n}\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}}{\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i} - \left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2}, \text{ where } \pi = 1-\mathrm{e}^{-\left(\theta_0+\theta_1 d\right)}.$$

The logistic model, the expressions of σ_{00}, σ_{01}, σ_{10} and σ_{11} are given as

$$\sigma_{00}=\frac{\frac{1}{n}\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2},$$

$$\sigma_{01}=\sigma_{10}=-\frac{\frac{1}{n}\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}, \text{ and}$$

$$\sigma_{11}=\frac{\frac{1}{n}\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}, \text{ where } \pi=\frac{1}{1+\mathrm{e}^{-\left(\theta_0+\theta_1 d\right)}}.$$

The log-logistic model, the expressions of σ_{00}, σ_{01}, σ_{10} and σ_{11} could be replaced d by $\ln\left(d\right)$ in logistic model. Therefore, $\pi=\frac{1}{1+\mathrm{e}^{-\left(\theta_0+\theta_1\ln\left(d\right)\right)}}$.

The mean squared errors are used to compare the estimates for these three models. Therefore, we find the bias terms of estimates, $b_{0MLE}\left(\underline{\theta}\right)$, $b_{1MLE}\left(\underline{\theta}\right)$, $b_{0B}\left(\underline{\theta}\right)$ and $b_{1B}\left(\underline{\theta}\right)$ for each model and compare MSEs between $\varphi_{0MLE}^{*}\left(\theta_0\right)$ and $\varphi_{0B}^{*}\left(\theta_0\right)$ and also compare MSEs between $\varphi_{1MLE}^{*}\left(\theta_1\right)$ and $\varphi_{1B}^{*}\left(\theta_1\right)$.

4.4.1 Multistage Weibull model

4.4.1.1 Maximum likelihood estimator

The likelihood function of multistage Weibull model is given by

$$L\left(\theta_0,\theta_1\,|\,\text{data}\right)=\prod_{i=0}^{m}\binom{n_i}{x_i}\left[1-e^{-\left(\theta_0+\theta_1 d_i\right)}\right]^{x_i}\left[e^{-\left(\theta_0+\theta_1 d_i\right)}\right]^{n_i-x_i}$$

$$\propto\prod_{i=0}^{m}\left[e^{\left(\theta_0+\theta_1 d_i\right)}-1\right]^{x_i}\left[e^{-\left(\theta_0+\theta_1 d_i\right)}\right]^{n_i},$$

and the log-likelihood function is written as

$$\ln L\left(\theta_0,\theta_1\,|\,\text{data}\right)=-\left(\theta_0\sum_{i=0}^{m}n_i+\theta_1\sum_{i=0}^{m}d_i n_i\right)+\sum_{i=0}^{m}x_i\ln\left(e^{\left(\theta_0+\theta_1 d_i\right)}-1\right).$$

Differentiating $\log L\left(\theta_0,\theta_1 \,|\, \text{data}\right)$ with respect to θ_0 and θ_1, and setting its derivative equal to zero, then the normal equations are

$$\sum_{i=0}^{m} n_i = \sum_{i=0}^{m} \frac{x_i}{1-e^{-\left(\hat{\theta}_{0MLE}+\hat{\theta}_{1MLE}d_i\right)}} \tag{4.4.1}$$

$$\sum_{i=0}^{m} n_i d_i = \sum_{i=0}^{m} \frac{x_i d_i}{1-e^{-\left(\hat{\theta}_{0MLE}+\hat{\theta}_{1MLE}d_i\right)}} \tag{4.4.2}$$

where $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$ are the maximum likelihood estimators of θ_0 and θ_1. We assume that $n_i = n$ for all i and $p_i = x_i / n$, then

$$m+1 = \sum_{i=0}^{m} \frac{p_i}{1-e^{-\left(\psi_0\left(p_0,\ldots,p_m\right)+\psi_1\left(p_0,\ldots,p_m\right)d_i\right)}} \tag{4.4.3}$$

$$\sum_{i=0}^{m} d_i = \sum_{i=0}^{m} \frac{d_i p_i}{1-e^{-\left(\psi_0\left(p_0,\ldots,p_m\right)+\psi_1\left(p_0,\ldots,p_m\right)d_i\right)}} \tag{4.4.4}$$

where $\hat{\theta}_{0MLE} = \psi_0\left(p_0,\ldots,p_m\right)$ and $\hat{\theta}_{1MLE} = \psi_1\left(p_0,\ldots,p_m\right)$. Taylor series are used to express $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$, that is

$$\begin{aligned}\hat{\theta}_{0MLE} &= \psi_0\left(p_0,\ldots,p_m\right)\\ &= \psi_0\left(P_0,\ldots,P_m\right) + \sum_{i=0}^{m}\left(p_i - P_i\right)\left.\frac{\partial\psi_0}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m}\left(p_i - P_i\right)^2 \left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i}\\ &\qquad + \sum_{i\neq j}^{m}\left(p_i - P_i\right)\left(p_j - P_j\right)\left.\frac{\partial^2\psi_0}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.\end{aligned} \tag{4.4.5}$$

$$\begin{aligned}\hat{\theta}_{1MLE} &= \psi_1\left(p_0,\ldots,p_m\right)\\ &= \psi_1\left(P_0,\ldots,P_m\right) + \sum_{i=0}^{m}\left(p_i - P_i\right)\left.\frac{\partial\psi_1}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m}\left(p_i - P_i\right)^2 \left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}\\ &\qquad + \sum_{i\neq j}^{m}\left(p_i - P_i\right)\left(p_j - P_j\right)\left.\frac{\partial^2\psi_1}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.\end{aligned} \tag{4.4.6}$$

The mean of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$ are obtained by taking expectation on both sides of (4.5) and (4.6), that is

$$E\left(\hat{\theta}_{0MLE}\right)=\theta_0+\sum_{i=0}^{m}E\left(p_i-P_i\right)\frac{\partial\psi_0}{\partial p_i}\bigg|_{P_i}+\frac{1}{2}\sum_{i=0}^{m}E\left(p_i-P_i\right)^2\frac{\partial^2\psi_0}{\partial p_i^2}\bigg|_{P_i}$$

$$+\sum_{i\neq j}^{m}E\left\{\left(p_i-P_i\right)\left(p_j-P_j\right)\right\}\frac{\partial^2\psi_0}{\partial p_i\partial p_j}\bigg|_{P_i,P_j}+\cdots$$

$$E\left(\hat{\theta}_{1MLE}\right)=\theta_1+\sum_{i=0}^{m}E\left(p_i-P_i\right)\frac{\partial\psi_1}{\partial p_i}\bigg|_{P_i}+\frac{1}{2}\sum_{i=0}^{m}E\left(p_i-P_i\right)^2\frac{\partial^2\psi_1}{\partial p_i^2}\bigg|_{P_i}$$

$$+\sum_{i\neq j}^{m}E\left\{\left(p_i-P_i\right)\left(p_j-P_j\right)\right\}\frac{\partial^2\psi_1}{\partial p_i\partial p_j}\bigg|_{P_i,P_j}+\cdots.$$

We note that $E\left(p_i-P_i\right)$ and $E\left\{\left(p_i-P_i\right)\left(p_j-P_j\right)\right\}$ are equal to zero.

Ignoring the terms of a smaller order than n^{-2}, the mean of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$ are obtained as follows

$$E\left(\hat{\theta}_{0MLE}\right)=\theta_0+\frac{1}{2}\sum_{i=0}^{m}\frac{P_i\left(1-P_i\right)}{n}\frac{\partial^2\psi_0}{\partial p_i^2}\bigg|_{P_i},$$

$$E\left(\hat{\theta}_{1MLE}\right)=\theta_1+\frac{1}{2}\sum_{i=0}^{m}\frac{P_i\left(1-P_i\right)}{n}\frac{\partial^2\psi_1}{\partial p_i^2}\bigg|_{P_i}$$

where $E\left(p_i-P_i\right)^2=P_i\left(1-P_i\right)/n$. Therefore, the bias terms of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$, that is $b_{0MLE}\left(\underline{\theta}\right)$ and $b_{1MLE}\left(\underline{\theta}\right)$, are written as

$$b_{0MLE}\left(\underline{\theta}\right)=\frac{1}{2n}\sum_{i=0}^{m}P_i\left(1-P_i\right)\frac{\partial^2\psi_0}{\partial p_i^2}\bigg|_{P_i} \tag{4.4.7}$$

$$b_{1MLE}\left(\underline{\theta}\right)=\frac{1}{2n}\sum_{i=0}^{m}P_i\left(1-P_i\right)\frac{\partial^2\psi_1}{\partial p_i^2}\bigg|_{P_i}. \tag{4.4.8}$$

We find the second derivative of $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing by P_i for $i=0,1,\ldots,m$, that is $\frac{\partial^2\psi_0}{\partial p_i^2}\Big|_{P_i}$ and $\frac{\partial^2\psi_1}{\partial p_i^2}\Big|_{P_i}$.

To find the first derivative of $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$ with respect to p_j for $j=0,1,\ldots,m$, we differentiate (4.4.3) and (4.4.4) with respect to p_j on both sides and obtain

$$0=\frac{1}{1-e^{-(\psi_0+\psi_1 d_j)}}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right)-\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right), \tag{4.4.9}$$

$$0=\frac{d_j}{1-e^{-(\psi_0+\psi_1 d_j)}}-\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right)-\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right). \tag{4.4.10}$$

To find the second derivative of $\psi_0(p_0,\ldots,p_m)$ and $\psi_1(p_0,\ldots,p_m)$ with respect to p_j, we differentiate (4.4.9) and (4.4.10) with respect to p_j on both sides and obtain

$$\begin{aligned}0=&-\frac{2e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_0}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\frac{2d_j e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_1}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)\\&-\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2\\&+\sum_{i=0}^{m}\frac{2p_i e^{-2(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2.\end{aligned} \tag{4.4.11}$$

$$\begin{aligned}0=&-\frac{2d_j e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_0}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\frac{2d_j^2 e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_1}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)\\&-\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2\\&+\sum_{i=0}^{m}\frac{2d_i p_i e^{-2(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2.\end{aligned} \tag{4.4.12}$$

From (4.4.9) and (4.4.10), we find $\frac{\partial\psi_0}{\partial p_j}$ and $\frac{\partial\psi_1}{\partial p_j}$ by solving two linear equations,

$$\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right)+\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right)=\frac{1}{1-e^{-(\psi_0+\psi_1 d_j)}}$$

$$\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right)+\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right)=\frac{d_j}{1-e^{-(\psi_0+\psi_1 d_j)}},$$

and obtain

$$\frac{\partial\psi_0}{\partial p_j}=\frac{\dfrac{1}{1-e^{-(\psi_0+\psi_1 d_j)}}\displaystyle\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\dfrac{d_j}{1-e^{-(\psi_0+\psi_1 d_j)}}\displaystyle\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{\displaystyle\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\left[\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\right]^2} \tag{4.4.13}$$

$$\frac{\partial\psi_1}{\partial p_j}=\frac{\dfrac{d_j}{1-e^{-(\psi_0+\psi_1 d_j)}}\displaystyle\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\dfrac{1}{1-e^{-(\psi_0+\psi_1 d_j)}}\displaystyle\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{\displaystyle\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\left[\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\right]^2}. \tag{4.4.14}$$

From (4.4.11) and (4.4.12), we find $\dfrac{\partial^2\psi_0}{\partial p_j^2}$ and $\dfrac{\partial^2\psi_1}{\partial p_j^2}$ by solving two linear equations,

$$\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)$$

$$=-\frac{2e^{-(\psi_0+\psi_1 d_j)}\left(\dfrac{\partial\psi_0}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\frac{2d_j e^{-(\psi_0+\psi_1 d_j)}\left(\dfrac{\partial\psi_1}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}$$

$$+\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2,$$

$$\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)$$

$$=-\frac{2d_j e^{-(\psi_0+\psi_1 d_j)}\left(\dfrac{\partial\psi_0}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\frac{2d_j^2 e^{-(\psi_0+\psi_1 d_j)}\left(\dfrac{\partial\psi_1}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}$$

$$+\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2 ,$$

and obtain

$$\frac{\partial^2\psi_0}{\partial p_j^2}=\frac{b_1\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-b_2\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{B} \tag{4.4.15}$$

$$\frac{\partial^2\psi_1}{\partial p_j^2}=\frac{b_2\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-b_1\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{B} \tag{4.4.16}$$

where

$$B=\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\sum_{i=0}^{m}\frac{d_i^2 p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\left[\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}\right]^2 ,$$

$$b_1=-\frac{2e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_0}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\frac{2d_j e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_1}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}$$

$$+\sum_{i=0}^{m}\frac{p_i e^{-(\psi_0+\psi_1 d_i)}\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2 ,$$

$$b_2=-\frac{2d_j e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_0}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}-\frac{2d_j^2 e^{-(\psi_0+\psi_1 d_j)}\left(\frac{\partial\psi_1}{\partial p_j}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_j)}\right)^2}$$

$$+\sum_{i=0}^{m}\frac{d_i p_i e^{-(\psi_0+\psi_1 d_i)}\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2 .$$

Therefore, the second derivative of $\psi_0(p_0,\ldots,p_m)$ and $\psi_1(p_0,\ldots,p_m)$ with respect to p_i and substitute p_i by P_i for $i=0,1,\ldots,m$, that is $\left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}$ are given as below.

Remark: $\pi_i = 1 - e^{-(\psi_0 + \psi_1 d_i)}$ when p_i is replaced by P_i.

$$
\left.\frac{\partial^2 \psi_0}{\partial p_i^2}\right|_{P_i} = \left\{ \frac{-2(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]^2 + \frac{2d_i(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right] \right.
$$

$$
- \frac{2d_i^2(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]
$$

$$
+ \frac{2d_i(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]
$$

$$
+ \frac{2d_i(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right] - \frac{2d_i^2(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right]^2
$$

$$
+ \frac{2d_i^3(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right]
$$

$$
\left. - \frac{2d_i^2(1-\pi_i)}{\pi_i^3} \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right]^2 \right\} \Bigg/
$$

$$
\left[\sum_{i=0}^{m} \frac{(1-\pi_i)}{\pi_i} \sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} - \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right]^2 \right]^2
$$

$$
+ \left\{ \frac{1}{\pi_i^2} \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]^3 \left[\sum_{i=0}^{m} \frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2} \right] \right.
$$

$$
- \frac{2d_i}{\pi_i^2} \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]^2 \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2} \right]
$$

$$
+ \frac{d_i^2}{\pi_i^2} \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right]^2 \left[\sum_{i=0}^{m} \frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2} \right]
$$

$$
+ \frac{d_i}{\pi_i^2} \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]^2 \left[\sum_{i=0}^{m} \frac{(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2} \right]
$$

$$
- \frac{1}{\pi_i^2} \left[\sum_{i=0}^{m} \frac{d_i^2(1-\pi_i)}{\pi_i} \right]^2 \left[\sum_{i=0}^{m} \frac{d_i(1-\pi_i)}{\pi_i} \right] \left[\sum_{i=0}^{m} \frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2} \right]
$$

$$-\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]\Bigg\}\Bigg/$$

$$\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\right]^3. \tag{4.4.17}$$

$$\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}=\Bigg\{\frac{-2d_i\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]$$

$$+\frac{2d_i^2\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]-\frac{2d_i^3\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^2$$

$$+\frac{2d_i^2\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]$$

$$+\frac{2\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]-\frac{2d_i\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2$$

$$+\frac{2d_i^2\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]$$

$$-\frac{2d_i\left(1-\pi_i\right)}{\pi_i^3}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\Bigg\}\Bigg/$$

$$\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\right]^2$$

$$+\Bigg\{\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{2d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{2d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{2d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{d_i^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{2d_i}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$\left.-\frac{1}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]\right\}\Bigg/$$

$$\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\right]^3. \tag{4.4.18}$$

The bias terms of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$, that is $b_{0MLE}\left(\underline{\theta}\right)$ and $b_{1MLE}\left(\underline{\theta}\right)$ are obtained by substituting (4.4.17) in (4.4.7) and substituting (4.4.18) in (4.4.8), respectively.

$$b_{0MLE}\left(\underline{\theta}\right)=\left\{-\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]^2\right.$$

$$+3\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]$$

$$-\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)^2}{\pi_i^2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2$$

$$\left.+\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\right\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\right]^2$$

$$+\left\{-\frac{3}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]\right.$$

$$+\frac{3}{2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^4\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)(2-\pi_i)}{\pi_i^2}\right]\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\right]^3. \qquad (4.4.19)$$

$$b_{1MLE}(\underline{\theta})=\Bigg\{-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]$$

$$+3\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2$$

$$+\left[\sum_{i=0}^{m}\frac{(1-\pi_i)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)^2}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\right]^2$$

$$+\Bigg\{\frac{1}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^4\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{3}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{3}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^3\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)\left(2-\pi_i\right)}{\pi_i^2}\right]\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2\right]^3. \qquad (4.4.20)$$

To find $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{1MLE}(\theta_1)$, the expressions of $\frac{\partial b_{0MLE}(\underline{\theta})}{\partial\theta_0}$, $\frac{\partial b_{0MLE}(\underline{\theta})}{\partial\theta_1}$, $\frac{\partial b_{1MLE}(\underline{\theta})}{\partial\theta_0}$ and $\frac{\partial b_{1MLE}(\underline{\theta})}{\partial\theta_1}$ are obtained by using Maple and the values of $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{1MLE}(\theta_1)$ in (4.4.21) and (4.4.22) are respectively computed by using MATLAB with real data sets from Table 4.2.1,

$$\varphi^{*}_{0MLE}(\theta_0)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{0MLE}(\underline{\theta})}{\partial\theta_0}\sigma_{00}+\frac{\partial b_{0MLE}(\underline{\theta})}{\partial\theta_1}\sigma_{01}\right)\right]+\frac{b^2_{0MLE}(\underline{\theta})}{n^2} \qquad (4.4.21)$$

$$\varphi^{*}_{1MLE}(\theta_1)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{1MLE}(\underline{\theta})}{\partial\theta_0}\sigma_{10}+\frac{\partial b_{1MLE}(\underline{\theta})}{\partial\theta_1}\sigma_{11}\right)\right]+\frac{b^2_{1MLE}(\underline{\theta})}{n^2}. \qquad (4.4.22)$$

The values of these expressions $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{1MLE}(\theta_1)$ are shown in Section 4.4.4.1.

4.4.1.2 Berkson’s minimum chi-square estimator

Let us consider

$$1-\pi(\theta_0,\theta_1)=e^{-(\theta_0+\theta_1 d_i)}$$

$$-\ln\left(1-\pi(\theta_0,\theta_1)\right)=\theta_0+\theta_1 d_i.$$

We denote the observed frequencies by $p_i = x_i/n_i$ where $i=0,\ldots,m$; $Var\left[-\ln(1-p_i)\right]=\pi_i/\left[n_i(1-\pi_i)\right]$, when $x_i=n_i$ then $p_i=1-1/2n_i$ and $x_i=0$ then $p_i=1/2n_i$. Therefore a chi-square function $Q(\theta_0,\theta_1)$ is given by

$$Q(\theta_0,\theta_1)=\sum_{i=0}^{m}\frac{n_i(1-p_i)}{p_i}\left[\ln(1-p_i)+\theta_0+\theta_1 d_i\right]^2.$$

Berkson’s minimum chi-square estimator of θ_0 and θ_1, that is $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$, are respectively estimated based on minimizing $Q(\theta_0,\theta_1)$ with respect to θ_0 and θ_1. Differentiating $Q(\theta_0,\theta_1)$ with respect to θ_0 and θ_1 and setting its derivative equal to zero, then the normal equations are written as

$$\sum_{i=0}^{m}\frac{n_i(1-p_i)}{p_i}\left[\ln(1-p_i)+\theta_0+\theta_1 d_i\right]=0 \qquad (4.4.23)$$

$$\sum_{i=0}^{m}\frac{n_i d_i(1-p_i)}{p_i}\left[\ln(1-p_i)+\theta_0+\theta_1 d_i\right]=0. \qquad (4.4.24)$$

We assume that $n_i = n$ for all *i*, from (4.4.23) and (4.4.24) and obtain

$$\theta_0 \sum_{i=0}^{m} \frac{(1-p_i)}{p_i} + \theta_1 \sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} = -\sum_{i=0}^{m} \frac{(1-p_i)}{p_i} \ln(1-p_i) \tag{4.4.25}$$

$$\theta_0 \sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} + \theta_1 \sum_{i=0}^{m} \frac{d_i^2(1-p_i)}{p_i} = -\sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} \ln(1-p_i). \tag{4.4.26}$$

Berkson's minimum chi-square estimates, $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$ are obtained by solving (4.4.25) and (4.4.26),

$$\hat{\theta}_{0B} = \frac{\sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} \sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} \ln(1-p_i) - \sum_{i=0}^{m} \frac{d_i^2(1-p_i)}{p_i} \sum_{i=0}^{m} \frac{(1-p_i)}{p_i} \ln(1-p_i)}{\sum_{i=0}^{m} \frac{(1-p_i)}{p_i} \sum_{i=0}^{m} \frac{d_i^2(1-p_i)}{p_i} - \left[\sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i}\right]^2} \tag{4.4.27}$$

$$\hat{\theta}_{1B} = \frac{\sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} \sum_{i=0}^{m} \frac{(1-p_i)}{p_i} \ln(1-p_i) - \sum_{i=0}^{m} \frac{(1-p_i)}{p_i} \sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i} \ln(1-p_i)}{\sum_{i=0}^{m} \frac{(1-p_i)}{p_i} \sum_{i=0}^{m} \frac{d_i^2(1-p_i)}{p_i} - \left[\sum_{i=0}^{m} \frac{d_i(1-p_i)}{p_i}\right]^2}. \tag{4.4.28}$$

By Taylor expansion as (4.4.5) and (4.4.6), we write $\hat{\theta}_{0B} = \tilde{\psi}_0(p_0, \ldots, p_m)$ and $\hat{\theta}_{1B} = \tilde{\psi}_1(p_0, \ldots, p_m)$ and obtain the mean of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$ by ignoring the terms of a smaller order than n^{-2},

$$E(\hat{\theta}_{0B}) = \theta_0 + \frac{1}{2} \sum_{i=0}^{m} \frac{P_i(1-P_i)}{n} \left.\frac{\partial^2 \tilde{\psi}_0}{\partial p_i^2}\right|_{P_i},$$

$$E(\hat{\theta}_{1B}) = \theta_1 + \frac{1}{2} \sum_{i=0}^{m} \frac{P_i(1-P_i)}{n} \left.\frac{\partial^2 \tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}$$

where $E(p_i - P_i)^2 = P_i(1-P_i)/n$. Therefore, the bias terms of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$, that is $b_{0B}(\underline{\theta})$ and $b_{1B}(\underline{\theta})$, are written as

$$b_{0B}(\underline{\theta}) = \frac{1}{2n} \sum_{i=0}^{m} P_i(1-P_i) \left.\frac{\partial^2 \tilde{\psi}_0}{\partial p_i^2}\right|_{P_i} \tag{4.4.29}$$

$$b_{1B}(\underline{\theta}) = \frac{1}{2n} \sum_{i=0}^{m} P_i(1-P_i) \left.\frac{\partial^2 \tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}. \tag{4.4.30}$$

We find the second derivative of $\tilde{\psi}_0\left(p_0,\ldots,p_m\right)$ and $\tilde{\psi}_1\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing p_i by P_i for $i=0,1,\ldots,m$, that is $\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}$.

To find the first derivative of $\tilde{\psi}_0\left(p_0,\ldots,p_m\right)$ and $\tilde{\psi}_1\left(p_0,\ldots,p_m\right)$ with respect to p_j for $j=0,1,\ldots,m$, we differentiate (4.4.27) and (4.4.28) with respect to p_j and obtain

$$
\begin{aligned}
\frac{\partial\tilde{\psi}_0}{\partial p_i} &= \frac{\left(-\frac{d_i}{p_i}-\frac{d_i\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}-\frac{d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2} \\
&+\frac{-\left(-\frac{1}{p_i}-\frac{\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}+\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2} \\
&+\left\{\frac{\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)-\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right. \\
&\qquad\left.\times\frac{\left[\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right\},
\end{aligned}
\tag{4.4.29}
$$

$$
\begin{aligned}
\frac{\partial\tilde{\psi}_1}{\partial p_i} &= \frac{\left(-\frac{1}{p_i}-\frac{\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}-\frac{d_i}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2} \\
&+\frac{\left(\frac{d_i}{p_i}+\frac{d_i\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}
\end{aligned}
$$

$$+\left\{\frac{\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)-\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right.$$

$$\left.\times\frac{\left[\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right\}. \tag{4.4.30}$$

To find the second derivative of $\tilde{\psi}_0\left(p_0,\ldots,p_m\right)$ and $\tilde{\psi}_1\left(p_0,\ldots,p_m\right)$ with respect to p_j, we differentiate (4.4.29) and (4.4.30) with respect to p_j and obtain

$$\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}=\frac{\left(\frac{d_i}{p_i^2}+\frac{d_i}{p_i^2\left(1-p_i\right)}+\frac{2d_i\ln\left(1-p_i\right)}{p_i^3}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$+\frac{\left(\frac{2d_i}{p_i^3}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$-\frac{\left(\frac{1}{p_i^2}+\frac{1}{p_i^2\left(1-p_i\right)}+\frac{2\ln\left(1-p_i\right)}{p_i^3}\right)\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$-\frac{\left(\frac{2d_i^2}{p_i^3}\right)\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$-\left\{\frac{\left(-\frac{d_i}{p_i}-\frac{d_i\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}-\left(\frac{d_i}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right.$$

$$
\left. + \frac{-\left(-\frac{1}{p_i}-\frac{\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}+\left(\frac{d_i^2}{p_i^2}\right)\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right\}
$$

$$
\times\left\{\frac{\left(-\frac{d_i^2}{p_i^2}\right)\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}+\left(-\frac{1}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left(-\frac{2d_i}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right\}
$$

$$
+\left\{\frac{\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)-\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right.
$$

$$
\times\frac{\left[-\frac{2d_i^2}{p_i^3}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}-\frac{2}{p_i^3}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}+\frac{4d_i}{p_i^3}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}
$$

$$
+\frac{\left[\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}
$$

$$
\times\frac{\left[\left(\frac{-d_i}{p_i}-\frac{d_i\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}-\frac{d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)\right.}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}
$$

$$
+\frac{\left.-\left(\frac{-1}{p_i}-\frac{\ln\left(1-p_i\right)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}+\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}
$$

$$-\frac{2\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)-\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$\times\frac{\left[\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$\left.\times\frac{\left[-\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}-\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}+\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}\right\}, \qquad (4.4.31)$$

$$\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}=\frac{\left(\frac{1}{p_i^2}+\frac{1}{p_i^2\left(1-p_i\right)}+\frac{2\ln\left(1-p_i\right)}{p_i^3}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$+\frac{\left(\frac{2d_i}{p_i^3}\right)\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$+\frac{\left(-\frac{d_i}{p_i^2}-\frac{d_i}{p_i^2\left(1-p_i\right)}-\frac{2d_i\ln\left(1-p_i\right)}{p_i^3}\right)\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$-\frac{\left(\frac{2}{p_i^3}\right)\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\ln\left(1-p_i\right)}{\sum_{i=0}^{m}\frac{\left(1-p_i\right)}{p_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-p_i\right)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-p_i\right)}{p_i}\right]^2}$$

$$-\left\{\frac{\left(-\frac{1}{p_i}-\frac{\ln(1-p_i)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}-\left(\frac{d_i}{p_i^2}\right)\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\ln(1-p_i)}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}\right.$$

$$\left.+\frac{\left(\frac{d_i}{p_i}+\frac{d_i\ln(1-p_i)}{p_i^2}\right)\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}+\left(\frac{1}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\ln(1-p_i)}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}\right\}$$

$$\times\left\{\frac{\left(-\frac{d_i^2}{p_i^2}\right)\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}+\left(-\frac{1}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left(-\frac{2d_i}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}\right\}$$

$$+\left\{\frac{\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\ln(1-p_i)-\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\ln(1-p_i)\right]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}\right.$$

$$\times\frac{\left[-\frac{2d_i^2}{p_i^3}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}-\frac{2}{p_i^3}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}+\frac{4d_i}{p_i^3}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}$$

$$+\frac{\left[\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}$$

$$\times\frac{\left[\left(-\frac{1}{p_i}-\frac{\ln(1-p_i)}{p_i^2}\right)\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}-\frac{d_i}{p_i^2}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\ln(1-p_i)\right.}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}$$

$$+\frac{\left(\frac{d_i}{p_i}+\frac{d_i\ln(1-p_i)}{p_i^2}\right)\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\ln(1-p_i)\Bigg]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}$$

$$-\frac{2\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\ln(1-p_i)-\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\ln(1-p_i)\right]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}$$

$$\times\frac{\left[\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}+\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}$$

$$\left.\times\frac{\left[-\frac{d_i^2}{p_i^2}\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}-\frac{1}{p_i^2}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}+\frac{2d_i}{p_i^2}\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]}{\sum_{i=0}^{m}\frac{(1-p_i)}{p_i}\sum_{i=0}^{m}\frac{d_i^2(1-p_i)}{p_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-p_i)}{p_i}\right]^2}\right\}. \qquad (4.4.32)$$

Therefore, the second derivative of $\tilde{\psi}_0(p_0,\ldots,p_m)$ and $\tilde{\psi}_1(p_0,\ldots,p_m)$ with respect to p_i replacing p_i by P_i for $i=0,1,\ldots,m$, that is $\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}$ are given as below.

Remark: $\pi_i = 1-e^{-(\psi_0+\psi_1 d_i)}$ when p_i is replaced by P_i.

$$\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i}=\frac{\left(\frac{d_i}{\pi_i^2}+\frac{d_i}{(1-\pi_i)\pi_i^2}+\frac{2d_i\ln(1-\pi_i)}{\pi_i^3}+\frac{4d_i\theta_0}{\pi_i^3}\right)\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}$$

$$+\frac{\frac{2d_i}{\pi_i^3}\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}$$

$$
-\frac{\left(\frac{1}{\pi_i^2}+\frac{1}{(1-\pi_i)\pi_i^2}+\frac{2\ln(1-\pi_i)}{\pi_i^3}+\frac{2\theta_0}{\pi_i^3}\right)\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}
$$

$$
+\frac{-\frac{2d_i^2}{\pi_i^3}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)-\frac{2d_i^2\theta_0}{\pi_i^3}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}
$$

$$
+\left\{\frac{\left(-\frac{d_i}{\pi_i}-\frac{d_i\ln(1-\pi_i)}{\pi_i^2}-\frac{2d_i\theta_0}{\pi_i^2}\right)\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}-\frac{d_i}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)}{\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\right]^2}\right.
$$

$$
+\frac{-\left(-\frac{1}{\pi_i}-\frac{\ln(1-\pi_i)}{\pi_i^2}-\frac{\theta_0}{\pi_i^2}\right)\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}+\frac{d_i^2}{\pi_i^2}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}
$$

$$
\left.+\frac{\frac{d_i^2\theta_0}{\pi_i^2}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}\right\}
$$

$$
\times\left\{\frac{\frac{2d_i^2}{\pi_i^2}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}+\frac{2}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\frac{4d_i}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}\right\}, \tag{4.4.33}
$$

$$
\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}=\frac{\left(\frac{1}{\pi_i^2}+\frac{1}{(1-\pi_i)\pi_i^2}+\frac{2\ln(1-\pi_i)}{\pi_i^3}+\frac{4d_i\theta_1}{\pi_i^3}\right)\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}
$$

$$+\frac{\frac{2d_i}{\pi_i^3}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}$$

$$+\frac{\left(-\frac{d_i}{\pi_i^2}-\frac{d_i}{(1-\pi_i)\pi_i^2}-\frac{2d_i\ln(1-\pi_i)}{\pi_i^3}-\frac{2d_i^2\theta_1}{\pi_i^3}\right)\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}$$

$$+\frac{-\frac{2}{\pi_i^3}\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)-\frac{2\theta_1}{\pi_i^3}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}$$

$$+\left\{\frac{\left(-\frac{1}{\pi_i}-\frac{\ln(1-\pi_i)}{\pi_i^2}-\frac{2d_i\theta_1}{\pi_i^2}\right)\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}-\frac{d_i}{\pi_i^2}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}\right.$$

$$+\frac{\left(\frac{d_i}{\pi_i}+\frac{d_i\ln(1-\pi_i)}{\pi_i^2}+\frac{d_i^2\theta_1}{\pi_i^2}\right)\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}+\frac{1}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}$$

$$\left.+\frac{\frac{\theta_1}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}\right\}$$

$$\times\left\{\frac{\frac{2d_i^2}{\pi_i^2}\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}+\frac{2}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\frac{4d_i}{\pi_i^2}\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}}{\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2}\right\}. \qquad (4.4.34)$$

The bias terms of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$, that is $b_{0B}(\underline{\theta})$ and $b_{1B}(\underline{\theta})$ are obtained by substituting (4.4.33) in (4.4.29) and substituting (4.4.34) in (4.4.30), respectively.

$$b_{0B}\left(\underline{\theta}\right)=\left\{\frac{1}{2}\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)+d_i}{\pi_i}+\frac{2d_i\left(1-\pi_i\right)\ln\left(1-\pi_i\right)+4d_i\left(1-\pi_i\right)\theta_0}{\pi_i^2}\right]\right.$$

$$+\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\ln\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i^2}\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)+1}{\pi_i}+\frac{2\left(1-\pi_i\right)\ln\left(1-\pi_i\right)+2\left(1-\pi_i\right)\theta_0}{\pi_i^2}\right]$$

$$\left.-\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\ln\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i^2}\right]-\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)\theta_0}{\pi_i^2}\right]\right\}\Bigg/$$

$$n\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]^2$$

$$+\left\{-\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)}{\pi_i^2}+\frac{d_i^3\left(1-\pi_i\right)\ln\left(1-\pi_i\right)+2d_i^3\left(1-\pi_i\right)\theta_0}{\pi_i^3}\right]\right.$$

$$-\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\ln\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\frac{d_i^3\left(1-\pi_i\right)}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i^2}+\frac{d_i^2\left(1-\pi_i\right)\ln\left(1-\pi_i\right)+d_i^2\left(1-\pi_i\right)\theta_0}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\ln\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^4\left(1-\pi_i\right)}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^4\left(1-\pi_i\right)\theta_0}{\pi_i^3}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i^2}+\frac{d_i\left(1-\pi_i\right)\ln\left(1-\pi_i\right)+2d_i\left(1-\pi_i\right)\theta_0}{\pi_i^3}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i}\ln\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i\left(1-\pi_i\right)}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{d_i^2\left(1-\pi_i\right)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{\left(1-\pi_i\right)}{\pi_i^2}+\frac{\left(1-\pi_i\right)\ln\left(1-\pi_i\right)+\left(1-\pi_i\right)\theta_0}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)\theta_0}{\pi_i^3}\right]$$

$$+2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^2}+\frac{d_i^2(1-\pi_i)\ln(1-\pi_i)+2d_i^2(1-\pi_i)\theta_0}{\pi_i^3}\right]$$

$$+2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^3}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i^2}+\frac{d_i(1-\pi_i)\ln(1-\pi_i)+d_i(1-\pi_i)\theta_0}{\pi_i^3}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)}{\pi_i^3}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)\theta_0}{\pi_i^3}\right]\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\right]^2, \tag{4.4.35}$$

$$b_{1B}(\underline{\theta})=+\left\{\frac{1}{2}\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)+1}{\pi_i}+\frac{2(1-\pi_i)\ln(1-\pi_i)+4d_i(1-\pi_i)\theta_1}{\pi_i^2}\right]\right.$$

$$+\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i^2}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)+d_i}{\pi_i}+\frac{2d_i(1-\pi_i)\ln(1-\pi_i)+2d_i^2(1-\pi_i)\theta_1}{\pi_i^2}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i^2}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)\theta_1}{\pi_i^2}\right]\Bigg\}\Bigg/$$

$$n\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2$$

$$+\left\{-\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^2}+\frac{d_i^2(1-\pi_i)\ln(1-\pi_i)+2d_i^3(1-\pi_i)\theta_1}{\pi_i^3}\right]\right.$$

$$-\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i^3(1-\pi_i)}{\pi_i^2}+\frac{d_i^3(1-\pi_i)\ln(1-\pi_i)+d_i^4(1-\pi_i)\theta_1}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^3}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)\theta_1}{\pi_i^3}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}+\frac{(1-\pi_i)\ln(1-\pi_i)+2d_i(1-\pi_i)\theta_1}{\pi_i^2}\right]$$

$$-\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]$$

$$+\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}+\frac{d_i(1-\pi_i)\ln(1-\pi_i)+d_i^2(1-\pi_i)\theta_1}{\pi_i^2}\right]$$

$$+\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i^2}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]$$

$$+\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)\theta_1}{\pi_i^2}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i^2}+\frac{d_i(1-\pi_i)\ln(1-\pi_i)+2d_i^2(1-\pi_i)\theta_1}{\pi_i^3}\right]$$

$$+2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^3}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i^2}+\frac{d_i^2(1-\pi_i)\ln(1-\pi_i)+d_i^3(1-\pi_i)\theta_1}{\pi_i^3}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\ln(1-\pi_i)\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i^3}\right]$$

$$-2\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}\right]\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)\theta_1}{\pi_i^3}\right]\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\frac{(1-\pi_i)}{\pi_i}\sum_{i=0}^{m}\frac{d_i^2(1-\pi_i)}{\pi_i}-\left[\sum_{i=0}^{m}\frac{d_i(1-\pi_i)}{\pi_i}\right]^2\right]^2. \tag{4.4.36}$$

To find $\varphi_{0B}^*(\theta_0)$ and $\varphi_{1B}^*(\theta_1)$, the expressions of $\frac{\partial b_{0B}(\underline{\theta})}{\partial\theta_0}$, $\frac{\partial b_{0B}(\underline{\theta})}{\partial\theta_1}$, $\frac{\partial b_{1B}(\underline{\theta})}{\partial\theta_0}$ and $\frac{\partial b_{1B}(\underline{\theta})}{\partial\theta_1}$ are obtained by using Maple. The values of $\varphi_{0B}^*(\theta_0)$ and $\varphi_{1B}^*(\theta_1)$ in (4.4.37) and (4.4.38) are respectively computed by using MATLAB with real data sets from Table 4.2.1,

$$\varphi_{0B}^*(\theta_0)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{0B}(\underline{\theta})}{\partial\theta_0}\sigma_{00}+\frac{\partial b_{0B}(\underline{\theta})}{\partial\theta_1}\sigma_{01}\right)\right]+\frac{b_{0B}^2(\underline{\theta})}{n^2} \tag{4.4.37}$$

$$\varphi_{1B}^*(\theta_1)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{1B}(\underline{\theta})}{\partial\theta_0}\sigma_{10}+\frac{\partial b_{1B}(\underline{\theta})}{\partial\theta_1}\sigma_{11}\right)\right]+\frac{b_{1B}^2(\underline{\theta})}{n^2}. \tag{4.4.38}$$

The values of these expressions $\varphi_{0B}^*(\theta_0)$ and $\varphi_{1B}^*(\theta_1)$ are shown in Section 4.4.4.1.

4.4.2 Logistic model

4.4.2.1 Maximum likelihood estimator

The likelihood function of logistic model is given by

$$L(\theta_0,\theta_1\,|\,\text{data})=\prod_{i=0}^{m}\binom{n_i}{x_i}\left[\frac{1}{1+e^{-(\theta_0+\theta_1 d_i)}}\right]^{x_i}\left[\frac{e^{-(\theta_0+\theta_1 d_i)}}{1+e^{-(\theta_0+\theta_1 d_i)}}\right]^{n_i-x_i}$$

$$\propto\prod_{i=0}^{m}\left[e^{-(\theta_0+\theta_1 d_i)}\right]^{n_i-x_i}\left[\frac{1}{1+e^{-(\theta_0+\theta_1 d_i)}}\right]^{n_i}$$

$$\propto e^{-\sum_{i=0}^{m}(\theta_0+\theta_1 d_i)(n_i-x_i)}\prod_{i=0}^{m}\left[\frac{1}{1+e^{-(\theta_0+\theta_1 d_i)}}\right]^{n_i},$$

and the log-likelihood function is written as

$$\ln L\left(\theta_0,\theta_1 \mid \text{data}\right) = -\sum_{i=0}^{m}\left(\theta_0+\theta_1 d_i\right)\left(n_i - x_i\right) + \sum_{i=0}^{m} n_i \ln\left(\frac{1}{1+e^{-(\theta_0+\theta_1 d_i)}}\right).$$

Differentiating $\ln L\left(\theta_0,\theta_1 \mid \text{data}\right)$ with respect to θ_0 and θ_1, and setting its derivative equal to zero, then the normal equation are

$$\sum_{i=0}^{m} x_i = \sum_{i=0}^{m} \frac{n_i}{1+e^{-\left(\hat{\theta}_{0MLE}+\hat{\theta}_{1MLE} d_i\right)}} \tag{4.4.39}$$

$$\sum_{i=0}^{m} d_i x_i = \sum_{i=0}^{m} \frac{n_i d_i}{1+e^{-\left(\hat{\theta}_{0MLE}+\hat{\theta}_{1MLE} d_i\right)}} \tag{4.4.40}$$

where $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$ are the maximum likelihood estimators of θ_0 and θ_1. We assume that $n_i = n$ for all *i* and $p_i = x_i/n$, then

$$\sum_{i=0}^{m} p_i = \sum_{i=0}^{m} \frac{1}{1+e^{-\left(\psi_0\left(p_{0,\ldots,}p_m\right)+\psi_1\left(p_{0,\ldots,}p_m\right)d_i\right)}} \tag{4.4.41}$$

$$\sum_{i=0}^{m} d_i p_i = \sum_{i=0}^{m} \frac{d_i}{1+e^{-\left(\psi_0\left(p_{0,\ldots,}p_m\right)+\psi_1\left(p_{0,\ldots,}p_m\right)d_i\right)}} \tag{4.4.42}$$

where $\hat{\theta}_{0MLE} = \psi_0\left(p_0,\ldots,p_m\right)$ and $\hat{\theta}_{1MLE} = \psi_1\left(p_0,\ldots,p_m\right)$. Taylor series are used to express $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$, that is

$$\begin{aligned}\hat{\theta}_{0MLE} &= \psi_0\left(p_0,\ldots,p_m\right)\\ &= \psi_0\left(P_0,\ldots,P_m\right) + \sum_{i=0}^{m}\left(p_i - P_i\right)\left.\frac{\partial \psi_0}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m}\left(p_i - P_i\right)^2 \left.\frac{\partial^2 \psi_0}{\partial p_i^2}\right|_{P_i}\\ &\qquad + \sum_{i\neq j}^{m}\left(p_i - P_i\right)\left(p_j - P_j\right)\left.\frac{\partial^2 \psi_0}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.\end{aligned} \tag{4.4.43}$$

$$\begin{aligned}\hat{\theta}_{1MLE} &= \psi_1\left(p_0,\ldots,p_m\right)\\ &= \psi_1\left(P_0,\ldots,P_m\right) + \sum_{i=0}^{m}\left(p_i - P_i\right)\left.\frac{\partial \psi_1}{\partial p_i}\right|_{P_i} + \frac{1}{2}\sum_{i=0}^{m}\left(p_i - P_i\right)^2 \left.\frac{\partial^2 \psi_1}{\partial p_i^2}\right|_{P_i}\\ &\qquad + \sum_{i\neq j}^{m}\left(p_i - P_i\right)\left(p_j - P_j\right)\left.\frac{\partial^2 \psi_1}{\partial p_i \partial p_j}\right|_{P_i,P_j} + \cdots.\end{aligned} \tag{4.4.44}$$

The mean of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$ are obtained by taking expectation on both sides of (4.4.43) and (4.4.44), and ignoring the terms of a smaller order than n^{-2}, we obtain

$$E\left(\hat{\theta}_{0MLE}\right) = \theta_0 + \frac{1}{2}\sum_{i=0}^{m}\frac{P_i\left(1-P_i\right)}{n}\left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i},$$

$$E\left(\hat{\theta}_{1MLE}\right) = \theta_1 + \frac{1}{2}\sum_{i=0}^{m}\frac{P_i\left(1-P_i\right)}{n}\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i},$$

where $E\left(p_i - P_i\right)^2 = P_i\left(1-P_i\right)/n$. Therefore, the bias terms of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$, that is $b_{0MLE}\left(\underline{\theta}\right)$ and $b_{1MLE}\left(\underline{\theta}\right)$, are written as

$$b_{0MLE}\left(\underline{\theta}\right) = \frac{1}{2n}\sum_{i=0}^{m}P_i\left(1-P_i\right)\left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i} \tag{4.4.45}$$

$$b_{1MLE}\left(\underline{\theta}\right) = \frac{1}{2n}\sum_{i=0}^{m}P_i\left(1-P_i\right)\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}. \tag{4.4.46}$$

We find the second derivative of $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$ with respect to p_i and substitute p_i by P_i for $i = 0,1,\ldots,m$, that is $\left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}$.

To find the first derivative of $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$ with respect to p_j for $j = 0,1,\ldots,m$, we differentiate (4.4.41) and (4.4.42) with respect to p_j on both sides and obtain

$$1 = \sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\frac{\partial\psi_0}{\partial p_j} + \sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\frac{\partial\psi_1}{\partial p_j}, \tag{4.4.47}$$

$$d_j = \sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right) + \sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right). \tag{4.4.48}$$

To find the second derivative of $\psi_0\left(p_0,\ldots,p_m\right)$ and $\psi_1\left(p_0,\ldots,p_m\right)$ with respect to p_j, we differentiate (4.4.47) and (4.4.48) with respect to p_j on both sides and obtain

$$0 = \sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right) + \sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)$$

$$-\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2+\sum_{i=0}^{m}\frac{2e^{-2(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2, \quad (4.4.49)$$

$$0=\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)$$

$$-\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2+\sum_{i=0}^{m}\frac{2d_i e^{-2(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2. \quad (4.4.50)$$

From (4.4.47) and (4.4.48), we find $\frac{\partial\psi_0}{\partial p_j}$ and $\frac{\partial\psi_1}{\partial p_j}$ by solving two linear equations,

$$1=\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right)+\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right)$$

$$d_j=\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}\right)+\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_1}{\partial p_j}\right),$$

and obtain

$$\frac{\partial\psi_0}{\partial p_j}=\frac{\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}-d_j\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\left[\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\right]^2} \quad (4.4.51)$$

$$\frac{\partial\psi_1}{\partial p_j}=\frac{d_j\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\left[\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\right]^2}. \quad (4.4.52)$$

From (4.4.49) and (4.4.50), we find $\frac{\partial^2\psi_0}{\partial p_j^2}$ and $\frac{\partial^2\psi_1}{\partial p_j^2}$ by solving two linear equations,

$$\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)=$$

$$\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2-\sum_{i=0}^{m}\frac{2e^{-2(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2,$$

$$\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_0}{\partial p_j^2}\right)+\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial^2\psi_1}{\partial p_j^2}\right)=$$

$$\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2-\sum_{i=0}^{m}\frac{2d_i e^{-2(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2,$$

and obtain

$$\frac{\partial^2\psi_0}{\partial p_j^2}=\frac{\tilde{b}_1\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\tilde{b}_2\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{\tilde{B}} \tag{4.4.53}$$

$$\frac{\partial^2\psi_1}{\partial p_j^2}=\frac{\tilde{b}_2\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\tilde{b}_1\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}}{\tilde{B}} \tag{4.4.54}$$

where

$$\tilde{B}=\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\sum_{i=0}^{m}\frac{d_i^2 e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}-\left[\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^2}\right]^2,$$

$$\tilde{b}_1=\sum_{i=0}^{m}\frac{e^{-(\psi_0+\psi_1 d_i)}\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2,$$

$$\tilde{b}_2=\sum_{i=0}^{m}\frac{d_i e^{-(\psi_0+\psi_1 d_i)}\left(1-e^{-(\psi_0+\psi_1 d_i)}\right)}{\left(1+e^{-(\psi_0+\psi_1 d_i)}\right)^3}\left(\frac{\partial\psi_0}{\partial p_j}+\frac{d_i\partial\psi_1}{\partial p_j}\right)^2.$$

Therefore, the second derivative of $\psi_0(p_0,\ldots,p_m)$ and $\psi_1(p_0,\ldots,p_m)$ with respect to p_i and substitute p_i by P_i for $i=0,1,\ldots,m$, that is $\left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}$ are given as below.

Remark: $\pi_i=\frac{1}{1+e^{-(\psi_0+\psi_1 d_i)}}$ when p_i is replaced by P_i.

$$
\begin{aligned}
\left.\frac{\partial^2\psi_0}{\partial p_i^2}\right|_{P_i} = \Bigg\{ & \left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^3 \left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -2d_i\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +d_i^2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +2d_i\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -2d_i^2\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +2d_i\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +d_i^2\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -2d_i\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +2d_i\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2 \left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -d_i^2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^3 \left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -2d_i\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]
\end{aligned}
$$

$$+\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+2d_i^2\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-2d_i\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-d_i^2\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+2d_i\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m} d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\Bigg\}\Bigg/$$

$$\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\right]^3. \qquad (4.4.55)$$

$$\left.\frac{\partial^2\psi_1}{\partial p_i^2}\right|_{P_i}=\Bigg\{\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-2d_i\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+d_i^2\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+d_i\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-d_i^2\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+d_i\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$
\begin{aligned}
&+d_i^2\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-2d_i\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&+\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&+2d_i\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-d_i^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-d_i\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&+\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&+d_i^2\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-d_i\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-d_i^2\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&+2d_i\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\\
&-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\Bigg\}\Bigg/\\
&\qquad\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\right]^3. \qquad (4.4.56)
\end{aligned}
$$

The bias terms of $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$, that is $b_{0MLE}\left(\underline{\theta}\right)$ and $b_{1MLE}\left(\underline{\theta}\right)$ are obtained by substituting (4.4.55) in (4.4.45) and substituting (4.4.56) in (4.4.46), respectively.

$$
\begin{aligned}
b_{0MLE}\left(\underline{\theta}\right) = \Bigg\{ & \frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -\frac{1}{2}\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -\frac{3}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +\frac{3}{2}\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^4\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\Bigg\} \Bigg/ \\
& n\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\right]^3 . \qquad (4.4.57)
\end{aligned}
$$

$$
\begin{aligned}
b_{1MLE}\left(\underline{\theta}\right) = \Bigg\{ & \frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& +\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right] \\
& -\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^4\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]
\end{aligned}
$$

$$-\frac{3}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+\frac{3}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}d_i^3\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]^2\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^3\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(2\pi_i-1\right)\right]\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\right]^3. \tag{4.4.58}$$

To find $\varphi_{0MLE}^{*}\left(\theta_0\right)$ and $\varphi_{1MLE}^{*}\left(\theta_1\right)$, the expressions of $\frac{\partial b_{0MLE}\left(\underline{\theta}\right)}{\partial\theta_0}$, $\frac{\partial b_{0MLE}\left(\underline{\theta}\right)}{\partial\theta_1}$, $\frac{\partial b_{1MLE}\left(\underline{\theta}\right)}{\partial\theta_0}$ and $\frac{\partial b_{1MLE}\left(\underline{\theta}\right)}{\partial\theta_1}$ are obtained by using Maple. The values of $\varphi_{0MLE}^{*}\left(\theta_0\right)$ and $\varphi_{1MLE}^{*}\left(\theta_1\right)$ in (4.4.59) and (4.4.60) are respectively computed by using MATLAB with real data sets from Table 4.2.1,

$$\varphi_{0MLE}^{*}\left(\theta_0\right)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{0MLE}\left(\underline{\theta}\right)}{\partial\theta_0}\sigma_{00}+\frac{\partial b_{0MLE}\left(\underline{\theta}\right)}{\partial\theta_1}\sigma_{01}\right)\right]+\frac{b_{0MLE}^2\left(\underline{\theta}\right)}{n^2} \tag{4.4.59}$$

$$\varphi_{1MLE}^{*}\left(\theta_1\right)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{1MLE}\left(\underline{\theta}\right)}{\partial\theta_0}\sigma_{10}+\frac{\partial b_{1MLE}\left(\underline{\theta}\right)}{\partial\theta_1}\sigma_{11}\right)\right]+\frac{b_{1MLE}^2\left(\underline{\theta}\right)}{n^2}. \tag{4.4.60}$$

The values of these expressions $\varphi_{0MLE}^{*}\left(\theta_0\right)$ and $\varphi_{1MLE}^{*}\left(\theta_1\right)$ are shown in Section 4.4.4.2.

4.4.2.2 Berkson's minimum chi-square estimator

Let us consider

$$1-\pi\left(\theta_0,\theta_1\right)=\frac{e^{-\left(\theta_0+\theta_1 d_i\right)}}{1+e^{-\left(\theta_0+\theta_1 d_i\right)}}$$

$$\frac{1-\pi(\theta_0,\theta_1)}{\pi(\theta_0,\theta_1)} = e^{-(\theta_0+\theta_1 d_i)}$$

$$\ln\left(\frac{\pi(\theta_0,\theta_1)}{1-\pi(\theta_0,\theta_1)}\right) = \theta_0 + \theta_1 d_i \, .$$

We denote the observed frequencies by $p_i = x_i / n_i$ where $i = 0,\ldots,m$; $Var\left[\ln\left(p_i/(1-p_i)\right)\right] = n_i\pi_i(1-\pi_i)$, when $x_i = n_i$ then $p_i = 1-1/2n_i$ and $x_i = 0$ then $p_i = 1/2n_i$. Therefore a chi-square function $Q(\theta_0,\theta_1)$ is given by

$$Q(\theta_0,\theta_1) = \sum_{i=0}^{m} n_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right) - \theta_0 - \theta_1 d_i\right]^2 .$$

Berkson's minimum chi-square estimator of θ_0 and θ_1, that is $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$, are respectively estimated based on minimizing $Q(\theta_0,\theta_1)$ with respect to θ_0 and θ_1. Differentiating $Q(\theta_0,\theta_1)$ with respect to θ_0 and θ_1 and setting its derivative equal to zero, then the normal equations are written as

$$\sum_{i=0}^{m} n_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right) - \theta_0 - \theta_1 d_i\right] = 0 \qquad (4.4.61)$$

$$\sum_{i=0}^{m} n_i d_i p_i (1-p_i)\left[\ln\left(\frac{p_i}{1-p_i}\right) - \theta_0 - \theta_1 d_i\right] = 0 \, . \qquad (4.4.62)$$

Suppose that $n_i = n$ for all *i*, from (4.4.61) and (4.4.62) and obtain

$$\theta_0 \sum_{i=0}^{m} p_i (1-p_i) + \theta_1 \sum_{i=0}^{m} d_i p_i (1-p_i) = \sum_{i=0}^{m} p_i (1-p_i) \ln\left(\frac{p_i}{1-p_i}\right) \qquad (4.4.63)$$

$$\theta_0 \sum_{i=0}^{m} d_i p_i (1-p_i) + \theta_1 \sum_{i=0}^{m} d_i^2 p_i (1-p_i) = \sum_{i=0}^{m} d_i p_i (1-p_i) \ln\left(\frac{p_i}{1-p_i}\right) . \qquad (4.4.64)$$

Berkson's minimum chi-square estimates, $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$ are obtained by solving (4.4.63) and (4.4.64),

$$\hat{\theta}_{0B}=\frac{\sum_{i=0}^{m}d_i^2 p_i(1-p_i)\sum_{i=0}^{m}p_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_i p_i(1-p_i)\sum_{i=0}^{m}d_i p_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2 p_i(1-p_i)-\left[\sum_{i=0}^{m}d_i p_i(1-p_i)\right]^2} \quad (4.4.65)$$

$$\hat{\theta}_{1B}=\frac{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i p_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_i p_i(1-p_i)\sum_{i=0}^{m}p_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2 p_i(1-p_i)-\left[\sum_{i=0}^{m}d_i p_i(1-p_i)\right]^2}.$$

(4.4.66)

By Taylor expansion as (4.4.5) and (4.4.6), we write $\hat{\theta}_{0B}=\tilde{\psi}_0(p_0,\ldots,p_m)$ and $\hat{\theta}_{1B}=\tilde{\psi}_1(p_0,\ldots,p_m)$ and obtain the mean of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$ by ignoring the terms of a smaller order than n^{-2},

$$E\left(\hat{\theta}_{0B}\right)=\theta_0+\frac{1}{2}\sum_{i=0}^{m}\frac{P_i(1-P_i)}{n}\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i},$$

$$E\left(\hat{\theta}_{1B}\right)=\theta_1+\frac{1}{2}\sum_{i=0}^{m}\frac{P_i(1-P_i)}{n}\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}$$

where $E(p_i-P_i)^2=P_i(1-P_i)/n$. Therefore, the bias terms of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$, that is $b_{0B}(\underline{\theta})$ and $b_{1B}(\underline{\theta})$, are written as

$$b_{0B}(\underline{\theta})=\frac{1}{2n}\sum_{i=0}^{m}P_i(1-P_i)\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i} \quad (4.4.67)$$

$$b_{1B}(\underline{\theta})=\frac{1}{2n}\sum_{i=0}^{m}P_i(1-P_i)\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}. \quad (4.4.68)$$

We find the second derivative of $\tilde{\psi}_0(p_0,\ldots,p_m)$ and $\tilde{\psi}_1(p_0,\ldots,p_m)$ with respect to p_i and substitute p_i by P_i for $i=0,1,\ldots,m$, that is $\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}$.

To find the first derivative of $\tilde{\psi}_0(p_0,\ldots,p_m)$ and $\tilde{\psi}_1(p_0,\ldots,p_m)$ with respect to p_j for $j=0,1,\ldots,m$, we differentiate (4.4.65) and (4.4.66) with respect to p_j and obtain

$$\frac{\partial\tilde{\psi}_0}{\partial p_i}=\frac{\left(1+\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)+d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}$$

$$+\frac{-\left(d_i+d_i\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)-d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}$$

$$-\left\{\frac{\left[\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right]}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right.$$

$$\left.\times\frac{\left[d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\},$$

(4.4.69)

$$\frac{\partial\tilde{\psi}_1}{\partial p_i}=\frac{\left(d_i-d_i\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}$$

$$+\frac{-\left(1+\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)-d_i\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}$$

$$-\left\{\frac{\left[\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right]}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right.$$

$$\left.\times\frac{\left[d_i^2(1-2p_i)\sum_{i=0}^{m}p_i(1-p_i)+(1-2p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-2d_i(1-2p_i)\sum_{i=0}^{m}d_ip_i(1-p_i)\right]}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-\left[\sum_{i=0}^{m}d_ip_i(1-p_i)\right]^2}\right\}. \tag{4.4.70}$$

To find the second derivative of $\tilde{\psi}_0(p_0,\ldots,p_m)$ and $\tilde{\psi}_1(p_0,\ldots,p_m)$ with respect to p_j, we differentiate (4.4.69) and (4.4.70) with respect to p_j and obtain

$$\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}=\frac{\left(\left(\frac{1-2p_i}{p_i(1-p_i)}\right)-2\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_i^2p_i(1-p_i)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-\left[\sum_{i=0}^{m}d_ip_i(1-p_i)\right]^2}$$

$$-\frac{2d_i^2\sum_{i=0}^{m}p_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-\left[\sum_{i=0}^{m}d_ip_i(1-p_i)\right]^2}$$

$$-\frac{d_i\left(\left(\frac{1-2p_i}{p_i(1-p_i)}\right)-2\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_ip_i(1-p_i)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-\left[\sum_{i=0}^{m}d_ip_i(1-p_i)\right]^2}$$

$$+\frac{2d_i\sum_{i=0}^{m}d_ip_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-\left[\sum_{i=0}^{m}d_ip_i(1-p_i)\right]^2}$$

$$-\left\{\frac{\left(1+(1-2p_i)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_i^2p_i(1-p_i)+d_i^2(1-2p_i)\sum_{i=0}^{m}p_i(1-p_i)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i(1-p_i)\sum_{i=0}^{m}d_i^2p_i(1-p_i)-\left[\sum_{i=0}^{m}d_ip_i(1-p_i)\right]^2}\right.$$

$$+\frac{-\left(d_i+d_i\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)-d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\Bigg\}$$

$$\times\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$-\left\{\frac{\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{-2d_i^2\sum_{i=0}^{m}p_i\left(1-p_i\right)-2\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)+4d_i\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$-\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{\left(1+\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)+d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right.$$

$$
\left.+\frac{-\left(d_i+d_i\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)-d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}\right\}
$$

$$
+2\left\{\frac{\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}\right\}
$$

$$
\times\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}\right\}
$$

$$
\times\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}\right\},
$$

(4.4.71)

$$
\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}=\frac{d_i\left(\left(\frac{1-2p_i}{p_i\left(1-p_i\right)}\right)-2\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}
$$

$$
-\frac{2\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}
$$

$$
-\frac{\left(\left(\frac{1-2p_i}{p_i\left(1-p_i\right)}\right)-2\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2 p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_i p_i\left(1-p_i\right)\right]^2}
$$

$$+\frac{2d_i\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}$$

$$-\left\{\frac{\left(d_i+d_i\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right.$$

$$\left.+\frac{-\left(1+\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)-d_i\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$-\left\{\frac{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{-2d_i^2\sum_{i=0}^{m}p_i\left(1-p_i\right)-2\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)+4d_i\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$-\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{\left(d_i+d_i\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right.$$

$$\left.+\frac{-\left(1+\left(1-2p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)-d_i\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$+2\left\{\frac{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)-\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)\ln\left(\frac{p_i}{1-p_i}\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}p_i\left(1-p_i\right)+\left(1-2p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-2d_i\left(1-2p_i\right)\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)}{\sum_{i=0}^{m}p_i\left(1-p_i\right)\sum_{i=0}^{m}d_i^2p_i\left(1-p_i\right)-\left[\sum_{i=0}^{m}d_ip_i\left(1-p_i\right)\right]^2}\right\}. \tag{4.4.72}$$

Therefore, the second derivative of $\tilde{\psi}_0\left(p_0,\ldots,p_m\right)$ and $\tilde{\psi}_1\left(p_0,\ldots,p_m\right)$ with respect to p_i replacing p_i by P_i for $i=0,1,\ldots,m$, that is $\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i}$ and $\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}$ are given as below.

Remark: $\pi_i=\frac{1}{1+e^{-\left(\psi_0+\psi_1 d_i\right)}}$ when p_i is replaced by P_i.

$$\left.\frac{\partial^2\tilde{\psi}_0}{\partial p_i^2}\right|_{P_i}=\frac{\left(\left(\frac{1-2\pi_i}{\pi_i\left(1-\pi_i\right)}\right)-2\ln\left(\frac{\pi_i}{1-\pi_i}\right)+2\theta_0\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{2d_i^2\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{d_i\left(\left(\frac{1-2\pi_i}{\pi_i\left(1-\pi_i\right)}\right)-2\ln\left(\frac{\pi_i}{1-\pi_i}\right)+4\theta_0\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$+\frac{2d_i\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$+\frac{2d_i^2\theta_0\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-2\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)+\left(1-2\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-2d_i\left(1-2\pi_i\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{\left(1+\left(1-2\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)-\left(1-2\pi_i\right)\theta_0\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}\right.$$

$$+\frac{d_i^2\left(1-2\pi_i\right)\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{\left(d_i+d_i\left(1-2\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)-2d_i\left(1-2\pi_i\right)\theta_0\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{d_i\left(1-2\pi_i\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$\left.-\frac{d_i^2\left(1-2\pi_i\right)\theta_0\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}\right\}, \tag{4.4.73}$$

$$\left.\frac{\partial^2\tilde{\psi}_1}{\partial p_i^2}\right|_{P_i}=\frac{d_i\left(\left(\frac{1-2\pi_i}{\pi_i\left(1-\pi_i\right)}\right)-2\ln\left(\frac{\pi_i}{1-\pi_i}\right)+2\theta_1\right)\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{2\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{\left(\left(\frac{1-2\pi_i}{\pi_i\left(1-\pi_i\right)}\right)-2\ln\left(\frac{\pi_i}{1-\pi_i}\right)+4d_i\theta_1\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$+\frac{2d_i\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$+\frac{2\theta_1\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-2\left\{\frac{d_i^2\left(1-2p_i\right)\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)+\left(1-2\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-2d_i\left(1-2\pi_i\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}\right\}$$

$$\times\left\{\frac{\left(d_i+d_i\left(1-2\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)-d_i^2\left(1-2\pi_i\right)\theta_1\right)\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}\right.$$

$$+\frac{\left(1-2\pi_i\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{\left(1+\left(1-2\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)-2d_i\left(1-2\pi_i\right)\theta_1\right)\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$-\frac{d_i\left(1-2\pi_i\right)\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}$$

$$\left.+\frac{\left(1-2\pi_i\right)\theta_1\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)}{\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2}\right\}. \tag{4.4.74}$$

The bias terms of $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$, that is $b_{0B}\left(\underline{\theta}\right)$ and $b_{1B}\left(\underline{\theta}\right)$ are obtained by substituting (4.4.73) in (4.4.67) and substituting (4.4.74) in (4.4.68), respectively.

$$b_{0B}\left(\underline{\theta}\right)=\left\{\frac{1}{2}\left[\sum_{i=0}^{m}\left(1-2\pi_i\right)-2\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)+2\pi_i\left(1-\pi_i\right)\theta_0\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\right.$$

$$-\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}d_i\left(1-2\pi_i\right)-2d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)+4d_i\pi_i\left(1-\pi_i\right)\theta_0\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)\theta_0\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\Bigg\}\Bigg/$$

$$n\sum_{i=0}^{m} \pi_i (1-\pi_i)\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i) - \left[\sum_{i=0}^{m} d_i \pi_i (1-\pi_i)\right]^2$$

$$+\Bigg\{\left[\sum_{i=0}^{m} d_i^3 \pi_i (1-\pi_i)(1-2\pi_i)\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i \pi_i (1-\pi_i)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)(1-2\pi_i)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)\right]$$

$$-\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)(1-2\pi_i)^2 \theta_0\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)\right]$$

$$-\left[\sum_{i=0}^{m} d_i^4 \pi_i (1-\pi_i)(1-2\pi_i)^2\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^3 \pi_i (1-\pi_i)(1-2\pi_i)\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i \pi_i (1-\pi_i)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^3 \pi_i (1-\pi_i)(1-2\pi_i)^2 \ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i \pi_i (1-\pi_i)\right]$$

$$-\left[\sum_{i=0}^{m} 2d_i^3 \pi_i (1-\pi_i)(1-2\pi_i)^2 \theta_0\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i \pi_i (1-\pi_i)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^3 \pi_i (1-\pi_i)(1-2\pi_i)^2\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]\left[\sum_{i=0}^{m} d_i \pi_i (1-\pi_i)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^4 \pi_i (1-\pi_i)(1-2\pi_i)^2 \theta_0\right]\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)\right]^2$$

$$-\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)(1-2\pi_i)\right]\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)\right]^2$$

$$-\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)(1-2\pi_i)^2 \ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)\right]^2$$

$$+\left[\sum_{i=0}^{m} \pi_i (1-\pi_i)(1-2\pi_i)^2 \theta_0\right]\left[\sum_{i=0}^{m} d_i^2 \pi_i (1-\pi_i)\right]^2$$

$$-\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m} 2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_0\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_0\right]\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} 2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} 2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m} 2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_0\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} 2d_i^3\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} \pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$-\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$-\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$+\left[\sum_{i=0}^{m} 4d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_0\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$-\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$-\left[\sum_{i=0}^{m} 2d_i^3\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_0\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\Bigg\} \Bigg/$$

$$n\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2\right]^2, \tag{4.4.75}$$

$$b_{1B}\left(\underline{\theta}\right)=\left\{\frac{1}{2}\left[\sum_{i=0}^{m} d_i\left(1-2\pi_i\right)-2d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)+2d_i^2\pi_i\left(1-\pi_i\right)\theta_1\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\right.$$

$$-\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$-\frac{1}{2}\left[\sum_{i=0}^{m}\left(1-2\pi_i\right)-2\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)+4d_i\pi_i\left(1-\pi_i\right)\theta_1\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$\left.+\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\theta_1\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\right\} \Bigg/$$

$$n\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$+\left\{-\left[\sum_{i=0}^{m} d_i^3\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2\right.$$

$$-\left[\sum_{i=0}^{m} d_i^3\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2$$

$$+\left[\sum_{i=0}^{m} d_i^4\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]^2$$

$$-\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m} 2d^{3}\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^3\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m} 2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$+\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m} d_i^2\pi_i\left(1-\pi_i\right)\right]^2$$

$$+\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]$$

$$+\left[\sum_{i=0}^{m} 2d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m} d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m}2d_i^3\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\right]$$

$$-\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$-\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$-\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$+\left[\sum_{i=0}^{m}4d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2$$

$$-\left[\sum_{i=0}^{m}2d_i^2\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\ln\left(\frac{\pi_i}{1-\pi_i}\right)\right]$$

$$-\left[\sum_{i=0}^{m}2d_i\pi_i\left(1-\pi_i\right)\left(1-2\pi_i\right)^2\theta_1\right]\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]\left[\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)\right]\Bigg\}\Bigg/$$

$$n\left[\sum_{i=0}^{m}\pi_i\left(1-\pi_i\right)\sum_{i=0}^{m}d_i^2\pi_i\left(1-\pi_i\right)-\left[\sum_{i=0}^{m}d_i\pi_i\left(1-\pi_i\right)\right]^2\right]^2. \tag{4.4.76}$$

To find $\varphi_{0B}^*\left(\theta_0\right)$ and $\varphi_{1B}^*\left(\theta_1\right)$, the expressions of $\frac{\partial b_{0B}\left(\underline{\theta}\right)}{\partial\theta_0}$, $\frac{\partial b_{0B}\left(\underline{\theta}\right)}{\partial\theta_1}$, $\frac{\partial b_{1B}\left(\underline{\theta}\right)}{\partial\theta_0}$ and $\frac{\partial b_{1B}\left(\underline{\theta}\right)}{\partial\theta_1}$ are obtained by using Maple. The values of $\varphi_{0B}^*\left(\theta_0\right)$ and $\varphi_{1B}^*\left(\theta_1\right)$ in (4.4.77) and (4.4.78) are respectively computed by using MATLAB with real data sets from Table 2.1,

$$\varphi_{0B}^*\left(\theta_0\right)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{0B}\left(\underline{\theta}\right)}{\partial\theta_0}\sigma_{00}+\frac{\partial b_{0B}\left(\underline{\theta}\right)}{\partial\theta_1}\sigma_{01}\right)\right]+\frac{b_{0B}^2\left(\underline{\theta}\right)}{n^2} \tag{4.4.77}$$

$$\varphi_{1B}^*\left(\theta_1\right)=\frac{1}{n^2}\left[2\left(\frac{\partial b_{1B}\left(\underline{\theta}\right)}{\partial\theta_0}\sigma_{10}+\frac{\partial b_{1B}\left(\underline{\theta}\right)}{\partial\theta_1}\sigma_{11}\right)\right]+\frac{b_{1B}^2\left(\underline{\theta}\right)}{n^2}. \tag{4.4.78}$$

The values of these expressions $\varphi_{0B}^*\left(\theta_0\right)$ and $\varphi_{1B}^*\left(\theta_1\right)$ are shown in Section 4.4.4.2.

4.4.3 Log-logistic model

The log-logistic model could be processed in the same manner as the logistic model, we find that similar expressions of $\varphi^*_{0MLE}(\theta_0)$, $\varphi^*_{1MLE}(\theta_1)$, $\varphi^*_{0B}(\theta_0)$ and $\varphi^*_{1B}(\theta_1)$ hold with "*d*" replaced by "$\ln(d)$". Therefore, the comparison of MSEs of estimates between MLEs and Berkson's minimum chi-square estimates are shown in Section 4.4.4.3.

4.4.4 Comparison and conclusion

In this section we provide the comparison of MLEs and Berkson's minimum chi-square estimates under the three dose response models. For the multistage Weibull model, considering (4.4.21) (MLE) and (4.4.37) (Berkson) for $\hat{\theta}_0$ and, considering (4.4.22) (MLE) and (4.4.38) (Berkson) for $\hat{\theta}_1$, we find a direct comparison of the asymptotic MSEs would involve computing these expressions for various values of θ_0 and θ_1. The real data sets in Table 4.2.1 are used to estimate parameters which based on the two methods. We can compare the estimates between MLE and Berkson by choosing the values of θ_0 and θ_1 around the estimates $\hat{\theta}_0$ and $\hat{\theta}_1$ then compute and compare the expressions of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ for $\hat{\theta}_0$, and the expressions of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ for $\hat{\theta}_1$. The appropriate estimates are preferred when $\varphi^*_0(\theta_0)$ or $\varphi^*_1(\theta_1)$ are smaller. A similar idea holds for the other two models.

4.4.4.1 Multistage Weibull model

We estimate parameters, θ_0 and θ_1 by using MATLAB based on the two methods for multistage Weibull model (MWM). The estimates are shown in Table 4.4.1.

Table 4.4.1: Values of $\hat{\theta}_{0MLE}$, $\hat{\theta}_{0B}$, $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ for MWM

Data Set	$\hat{\theta}_{0MLE}$	$\hat{\theta}_{0B}$	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$
1	7.039e-10	9.817e-03	2.231e-01	2.199e-01
2	1.677e-07	7.278e-03	5.194e-01	3.716e-01
3	3.493e-08	2.428e-03	6.811e-03	5.346e-03
4	1.485e-10	1.000e-10	1.816e-03	8.397e-04

We consider the values of θ_0 and θ_1 around the estimates $\hat{\theta}_0$ and $\hat{\theta}_1$, and then compute the expressions of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ for $\hat{\theta}_0$, and the expressions of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ for $\hat{\theta}_1$. These values are shown in Tables 4.4.2-4.4.15.

Table 4.4.2: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of MWM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		**5.000e-10**	**6.000e-10**	**7.000e-10**	**8.000e-10**	**9.000e-10**
0.150	$\varphi^*_{0MLE}(\theta_0)$	2.631e-17	3.157e-17	3.683e-17	4.209e-17	4.735e-17
	$\varphi^*_{0B}(\theta_0)$	1.057e-13	8.776e-14	-7.737e-14	-6.803e-14	-6.081e-14
0.175	$\varphi^*_{0MLE}(\theta_0)$	2.477e-17	2.973e-17	3.468e-17	3.963e-17	4.459e-17
	$\varphi^*_{0B}(\theta_0)$	1.059e-13	8.800e-14	-7.709e-14	-6.771e-14	-6.044e-14
0.200	$\varphi^*_{0MLE}(\theta_0)$	2.336e-17	2.803e-17	3.270e-17	3.737e-17	4.204e-17
	$\varphi^*_{0B}(\theta_0)$	1.061e-13	8.818e-14	-7.687e-14	-6.746e-14	-6.016e-14
0.225	$\varphi^*_{0MLE}(\theta_0)$	2.210e-17	2.651e-17	3.093e-17	3.535e-17	3.977e-17
	$\varphi^*_{0B}(\theta_0)$	1.062e-13	8.834e-14	-7.669e-14	-6.725e-14	-5.993e-14
0.250	$\varphi^*_{0MLE}(\theta_0)$	2.101e-17	2.522e-17	2.942e-17	3.362e-17	3.783e-17
	$\varphi^*_{0B}(\theta_0)$	1.063e-13	8.846e-14	-7.654e-14	-6.709e-14	-5.974e-14

Table 4.4.3: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of MWM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and **$\hat{\theta}_{1B}$**

θ_1 \ θ_0		**7.000e-03**	**8.000e-03**	**9.000e-03**	**1.000e-02**	**1.100e-02**
0.150	$\varphi^*_{0MLE}(\theta_0)$	3.305e-10	3.728e-10	4.140e-10	4.544e-10	4.939e-10
	$\varphi^*_{0B}(\theta_0)$	-5.489e-09	-5.789e-09	-5.991e-09	-6.100e-09	-6.124e-09
0.175	$\varphi^*_{0MLE}(\theta_0)$	3.155e-10	3.563e-10	3.962e-10	4.353e-10	4.737e-10
	$\varphi^*_{0B}(\theta_0)$	-3.652e-09	-3.813e-09	-3.899e-09	-3.913e-09	-3.859e-09
0.200	$\varphi^*_{0MLE}(\theta_0)$	3.003e-10	3.395e-10	3.779e-10	4.156e-10	4.526e-10
	$\varphi^*_{0B}(\theta_0)$	-2.089e-09	-2.109e-09	-2.069e-09	-1.972e-09	-1.820e-09
0.225	$\varphi^*_{0MLE}(\theta_0)$	2.860e-10	3.236e-10	3.604e-10	3.967e-10	4.322e-10
	$\varphi^*_{0B}(\theta_0)$	-7.348e-10	-6.182e-10	-4.526e-10	-2.399e-10	1.812e-11
0.250	$\varphi^*_{0MLE}(\theta_0)$	2.734e-10	3.094e-10	3.448e-10	3.797e-10	4.139e-10
	$\varphi^*_{0B}(\theta_0)$	4.553e-10	7.024e-10	9.903e-10	1.318e-09	1.684e-09

Table 4.4.4: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of MWM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		5.000e-10	6.000e-10	7.000e-10	8.000e-10	9.000e-10
0.150	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	3.776e-09 1.495e-10	3.776e-09 1.495e-10	3.776e-09 1.495e-10	3.776e-09 1.495e-10	3.776e-09 1.495e-10
0.175	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	5.328e-09 -6.248e-10	5.328e-09 -6.248e-10	5.328e-09 -6.248e-10	5.328e-09 -6.248e-10	5.328e-09 -6.248e-10
0.200	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	7.297e-09 -1.850e-09	7.297e-09 -1.850e-09	7.297e-09 -1.850e-09	7.297e-09 -1.850e-09	7.297e-09 -1.850e-09
0.225	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	9.747e-09 -3.712e-09	9.747e-09 -3.712e-09	9.747e-09 -3.712e-09	9.747e-09 -3.712e-09	9.747e-09 -3.712e-09
0.250	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.274e-08 -6.444e-09	1.274e-08 -6.444e-09	1.274e-08 -6.444e-09	1.274e-08 -6.444e-09	1.274e-08 -6.444e-09

Table 4.4.5: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of MWM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and **$\hat{\theta}_{1B}$**

θ_1 \ θ_0		7.000e-03	8.000e-03	9.000e-03	1.000e-02	1.100e-02
0.150	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	3.870e-09 2.777e-09	3.884e-09 3.018e-09	3.899e-09 3.231e-09	3.915e-09 3.419e-09	3.930e-09 3.583e-09
0.175	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	5.460e-09 1.780e-09	5.480e-09 2.013e-09	5.501e-09 2.223e-09	5.522e-09 2.411e-09	5.543e-09 2.578e-09
0.200	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	7.478e-09 3.501e-10	7.505e-09 5.708e-10	7.533e-09 7.713e-10	7.560e-09 9.526e-10	7.589e-09 1.116e-09
0.225	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	9.987e-09 -1.712e-09	1.002e-08 -1.508e-09	1.006e-08 -1.321e-09	1.010e-08 -1.152e-09	1.013e-08 -9.995e-10
0.250	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.305e-08 -4.652e-09	1.310e-08 -4.469e-09	1.314e-08 -4.302e-09	1.319e-08 -4.150e-09	1.324e-08 -4.013e-09

For Data set 1, we see that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\varphi^*_{0B}(\theta_0)$ is less than $\varphi^*_{0MLE}(\theta_0)$ in Tables 4.4.2-4.4.3 and also $\varphi^*_{1B}(\theta_1)$ is less than $\varphi^*_{1MLE}(\theta_1)$ in Tables 4.4.4-4.4.5. Therefore, we conclude that $\hat{\theta}_{0B}$ is preferred to $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ while outside they are the other way around.

Table 4.4.6: Values of $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{0B}(\theta_0)$ of MWM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		**9.000e-08**	**1.000e-07**	**2.000e-07**	**3.000e-07**	**4.000e-07**
0.25	$\varphi^{*}_{0MLE}(\theta_0)$	1.326e-14	1.473e-14	2.946e-14	4.419e-14	5.891e-14
	$\varphi^{*}_{0B}(\theta_0)$	9.267e-14	1.030e-13	2.074e-13	3.107e-13	4.144e-13
0.35	$\varphi^{*}_{0MLE}(\theta_0)$	1.325e-14	1.472e-14	2.944e-14	4.415e-14	5.887e-14
	$\varphi^{*}_{0B}(\theta_0)$	8.422e-14	8.971e-14	1.808e-13	3.611e-13	3.768e-13
0.45	$\varphi^{*}_{0MLE}(\theta_0)$	1.324e-14	1.471e-14	2.942e-14	4.413e-14	5.884e-14
	$\varphi^{*}_{0B}(\theta_0)$	8.069e-14	8.971e-14	1.808e-13	2.707e-13	3.611e-13
0.55	$\varphi^{*}_{0MLE}(\theta_0)$	1.323e-14	1.470e-14	2.940e-14	4.410e-14	5.880e-14
	$\varphi^{*}_{0B}(\theta_0)$	7.945e-14	8.834e-14	1.780e-13	2.666e-13	3.556e-13
0.65	$\varphi^{*}_{0MLE}(\theta_0)$	1.322e-14	1.469e-14	2.939e-14	4.408e-14	5.877e-14
	$\varphi^{*}_{0B}(\theta_0)$	7.949e-14	8.838e-14	1.781e-13	2.668e-13	3.558e-13

Table 4.4.7: Values of $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{0B}(\theta_0)$ of MWM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and **$\hat{\theta}_{1B}$**

θ_1 \ θ_0		**5.000e-03**	**6.000e-03**	**7.000e-03**	**8.000e-03**	**9.000e-03**
0.25	$\varphi^{*}_{0MLE}(\theta_0)$	7.327e-10	8.784e-10	1.024e-09	1.169e-09	1.314e-09
	$\varphi^{*}_{0B}(\theta_0)$	6.739e-09	8.451e-09	1.028e-08	1.223e-08	1.429e-08
0.35	$\varphi^{*}_{0MLE}(\theta_0)$	7.355e-10	8.825e-10	1.030e-09	1.177e-09	1.324e-09
	$\varphi^{*}_{0B}(\theta_0)$	5.611e-09	6.947e-09	8.354e-09	9.830e-09	1.138e-08
0.45	$\varphi^{*}_{0MLE}(\theta_0)$	7.369e-10	8.846e-10	1.033e-09	1.181e-09	1.329e-09
	$\varphi^{*}_{0B}(\theta_0)$	5.130e-09	6.303e-09	7.525e-09	8.796e-09	1.012e-08
0.55	$\varphi^{*}_{0MLE}(\theta_0)$	7.376e-10	8.858e-10	1.034e-09	1.183e-09	1.332e-09
	$\varphi^{*}_{0B}(\theta_0)$	4.911e-09	6.005e-09	7.136e-09	8.305e-09	9.511e-09
0.65	$\varphi^{*}_{0MLE}(\theta_0)$	7.381e-10	8.865e-10	1.035e-09	1.184e-09	1.334e-09
	$\varphi^{*}_{0B}(\theta_0)$	4.824e-09	5.879e-09	6.965e-09	8.081e-09	9.227e-09

Table 4.4.8: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of MWM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		9.000e-08	1.000e-07	2.000e-07	3.000e-07	4.000e-07
0.35	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.122e-08 -4.846e-09	1.122e-08 -4.846e-09	1.122e-08 -4.846e-09	1.122e-08 -4.846e-09	1.122e-08 -4.846e-09
0.40	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.454e-08 -8.135e-09	1.454e-08 -8.135e-09	1.454e-08 -8.135e-09	1.454e-08 -8.135e-09	1.454e-08 -8.135e-09
0.45	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.855e-08 -1.199e-08	1.855e-08 -1.199e-08	1.855e-08 -1.199e-08	1.855e-08 -1.199e-08	1.855e-08 -1.199e-08
0.50	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	2.335e-08 -1.651e-08	2.335e-08 -1.651e-08	2.335e-08 -1.651e-08	2.335e-08 -1.651e-08	2.335e-08 -1.651e-08
0.55	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	2.907e-08 -2.181e-08	2.907e-08 -2.181e-08	2.907e-08 -2.181e-08	2.907e-08 -2.181e-08	2.907e-08 -2.181e-08

Table 4.4.9: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of MWM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		5.000e-03	6.000e-03	7.000e-03	8.000e-03	9.000e-03
0.35	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.169e-08 5.275e-09	1.178e-08 7.192e-09	1.188e-08 9.074e-09	1.197e-08 1.092e-08	1.206e-08 1.274e-08
0.40	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.505e-08 7.554e-10	1.516e-08 2.452e-09	1.526e-08 4.121e-09	1.536e-08 5.765e-09	1.546e-08 7.382e-09
0.45	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.911e-08 -4.085e-09	1.922e-08 -2.568e-09	1.933e-08 -1.073e-09	1.944e-08 4.023e-10	1.955e-08 1.857e-09
0.50	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	2.396e-08 -9.417e-09	2.408e-08 -8.051e-09	2.420e-08 -6.701e-09	2.433e-08 -5.368e-09	2.445e-08 -4.052e-09
0.55	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	2.975e-08 -1.294e-08	2.988e-08 -1.417e-08	3.002e-08 -1.294e-08	3.016e-08 -1.173e-08	3.029e-08 -1.054e-08

For Data set 2, we see that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\varphi^*_{0MLE}(\theta_0)$ is less than $\varphi^*_{0B}(\theta_0)$ in Tables 4.4.6-4.4.7 and also $\varphi^*_{1B}(\theta_1)$ is less than $\varphi^*_{1MLE}(\theta_1)$ in Tables 4.4.8-4.4.9.Therefore, we conclude that $\hat{\theta}_{0MLE}$ is preferred to $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ while outside they are the other way around.

Table 4.4.10: Values of $\varphi^{*}_{0MLE}\left(\theta_0\right)$ and $\varphi^{*}_{0B}\left(\theta_0\right)$ of MWM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		1.000e-08	2.000e-08	3.000e-08	4.000e-08	5.000e-08
0.0035	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.128e-15	2.255e-15	3.383e-15	4.510e-15	5.638e-15
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-1.184e-14	-2.635e-14	-3.552e-14	-5.003e-14	-6.240e-14
0.0045	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.123e-15	2.247e-15	3.370e-15	4.493e-15	5.617e-15
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-7.205e-15	-1.708e-14	-2.162e-14	-3.149e-14	-3.922e-14
0.0055	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.119e-15	2.238e-15	3.358e-15	4.477e-15	5.596e-15
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-4.209e-15	-1.108e-14	-1.263e-14	-1.950e-14	-2.424e-14
0.0065	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.115e-15	2.230e-15	3.345e-15	4.460e-15	5.575e-15
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-2.095e-15	-6.854e-15	-6.284e-15	-1.104e-14	-1.367e-14
0.0075	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.111e-15	2.222e-15	3.333e-15	4.444e-15	5.555e-15
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-5.085e-16	-3.682e-15	-1.525e-15	-4.698e-15	-5.740e-15

Table 4.4.11: Values of $\varphi^{*}_{0MLE}\left(\theta_0\right)$ and $\varphi^{*}_{0B}\left(\theta_0\right)$ of MWM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		1.000e-03	2.000e-03	3.000e-03	4.000e-03	5.000e-03
0.0035	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.100e-10	2.150e-10	3.152e-10	4.112e-10	5.032e-10
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-9.310e-10	-1.296e-09	-1.156e-09	-5.663e-10	4.252e-10
0.0045	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.103e-10	2.167e-10	3.195e-10	4.188e-10	5.150e-10
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-5.854e-10	-8.162e-10	-7.223e-10	-3.307e-10	3.343e-10
0.0055	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.103e-10	2.175e-10	3.218e-10	4.233e-10	5.222e-10
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-3.460e-10	-4.482e-10	-3.229e-10	1.458e-11	5.504e-10
0.0065	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.102e-10	2.179e-10	3.231e-10	4.261e-10	5.268e-10
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-1.694e-10	-1.599e-10	1.856e-11	3.567e-10	8.460e-10
0.0075	$\varphi^{*}_{0MLE}\left(\theta_0\right)$	1.100e-10	2.179e-10	3.238e-10	4.278e-10	5.298e-10
	$\varphi^{*}_{0B}\left(\theta_0\right)$	-3.271e-11	7.221e-11	3.084e-10	6.697e-10	1.151e-09

Table 4.4.12: Values of $\varphi^{*}_{1MLE}\left(\theta_1\right)$ and $\varphi^{*}_{1B}\left(\theta_1\right)$ of MWM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		**1.000e-08**	**2.000e-08**	**3.000e-08**	**4.000e-08**	**5.000e-08**
0.0035	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	2.197e-12	2.197e-12	2.197e-12	2.197e-12	2.197e-12
	$\varphi^{*}_{1B}\left(\theta_1\right)$	5.042e-12	5.042e-12	5.042e-12	5.042e-12	5.042e-12
0.0045	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	3.126e-12	3.126e-12	3.126e-12	3.126e-12	3.126e-12
	$\varphi^{*}_{1B}\left(\theta_1\right)$	4.451e-12	4.452e-12	4.452e-12	4.452e-12	4.452e-12
0.0055	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	4.222e-12	4.222e-12	4.222e-12	4.222e-12	4.222e-12
	$\varphi^{*}_{1B}\left(\theta_1\right)$	3.749e-12	3.749e-12	3.749e-12	3.749e-12	3.749e-12
0.0065	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	5.507e-12	5.507e-12	5.507e-12	5.507e-12	5.507e-12
	$\varphi^{*}_{1B}\left(\theta_1\right)$	2.912e-12	2.913e-12	2.913e-12	2.913e-12	2.913e-12
0.0075	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	7.002e-12	7.002e-12	7.002e-12	7.002e-12	7.002e-12
	$\varphi^{*}_{1B}\left(\theta_1\right)$	1.920e-12	1.920e-12	1.920e-12	1.920e-12	1.920e-12

Table 4.4.13: Values of $\varphi^{*}_{1MLE}\left(\theta_1\right)$ and $\varphi^{*}_{1B}\left(\theta_1\right)$ of MWM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and **$\hat{\theta}_{1B}$**

θ_1 \ θ_0		**1.000e-03**	**2.000e-03**	**3.000e-03**	**4.000e-03**	**5.000e-03**
0.0035	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	2.232e-012	2.265e-012	2.296e-012	2.327e-012	2.356e-012
	$\varphi^{*}_{1B}\left(\theta_1\right)$	1.074e-011	1.576e-011	2.016e-011	2.401e-011	2.735e-011
0.0045	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	3.164e-012	3.200e-012	3.235e-012	3.270e-012	3.304e-012
	$\varphi^{*}_{1B}\left(\theta_1\right)$	8.998e-012	1.312e-011	1.685e-011	2.022e-011	2.325e-011
0.0055	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	4.263e-012	4.303e-012	4.343e-012	4.381e-012	4.419e-012
	$\varphi^{*}_{1B}\left(\theta_1\right)$	7.545e-012	1.105e-011	1.428e-011	1.726e-011	2.000e-011
0.0065	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	5.551e-012	5.596e-012	5.639e-012	5.682e-012	5.725e-012
	$\varphi^{*}_{1B}\left(\theta_1\right)$	6.180e-012	9.235e-012	1.209e-011	1.475e-011	1.723e-011
0.0075	$\varphi^{*}_{1MLE}\left(\theta_1\right)$	7.052e-012	7.101e-012	7.149e-012	7.197e-012	7.244e-012
	$\varphi^{*}_{1B}\left(\theta_1\right)$	4.794e-012	7.506e-012	1.006e-011	1.247e-011	1.474e-011

For Data set 3, we see that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\varphi^{*}_{0MLE}\left(\theta_0\right)$ is less than $\varphi^{*}_{0B}\left(\theta_0\right)$ in Tables 4.10-4.11 then, $\hat{\theta}_{0MLE}$ is preferred to $\hat{\theta}_{0B}$ while outside it is the other way around. In Table 4.4.12, $\varphi^{*}_{1B}\left(\theta_1\right)$ is less than $\varphi^{*}_{1MLE}\left(\theta_1\right)$ but $\varphi^{*}_{1MLE}\left(\theta_1\right)$ is less than $\varphi_{1B}^{*}(\theta_1)$ in Table 4.4.13. Moreover, $\varphi_{1MLE}^{*}(\theta_1)$ in Table 4.4.12 is less than $\varphi_{1B}^{*}(\theta_1)$ in Table 4.4.13. Therefore, we obtain that $\hat{\theta}_{1MLE}$ is preferred to $\hat{\theta}_{1B}$ while outside it is the other way around.

Table 4.4.14: Values of $\varphi_{0MLE}^{*}(\theta_0)$ and $\varphi_{0B}^{*}(\theta_0)$ of MWM for Data set 4 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		1.000e-12	1.000e-11	1.000e-10	1.000e-09	1.000e-08
0.0001	$\varphi_{0MLE}^{*}(\theta_0)$	-2.125e-20	-2.124e-19	-2.123e-18	-2.123e-17	-2.123e-16
	$\varphi_{0B}^{*}(\theta_0)$	-1.078e-15	-1.078e-14	-1.078e-13	-1.131e-12	-1.077e-11
0.0010	$\varphi_{0MLE}^{*}(\theta_0)$	-2.900e-20	-2.883e-19	-2.884e-18	-2.884e-17	-2.884e-16
	$\varphi_{0B}^{*}(\theta_0)$	-1.064e-16	-1.064e-15	-1.064e-14	-1.597e-13	-1.059e-12
0.0015	$\varphi_{0MLE}^{*}(\theta_0)$	-3.309e-20	-3.316e-19	-3.315e-18	-3.315e-17	-3.315e-16
	$\varphi_{0B}^{*}(\theta_0)$	-7.042e-17	-7.042e-16	-7.042e-15	-1.237e-13	-6.989e-13
0.0020	$\varphi_{0MLE}^{*}(\theta_0)$	-3.754e-20	-3.750e-19	-3.751e-18	-3.751e-17	-3.751e-16
	$\varphi_{0B}^{*}(\theta_0)$	-5.239e-17	-5.240e-16	-5.240e-15	-1.057e-13	-5.186e-13
0.0025	$\varphi_{0MLE}^{*}(\theta_0)$	-4.195e-20	-4.189e-19	-4.189e-18	-4.189e-17	-4.189e-16
	$\varphi_{0B}^{*}(\theta_0)$	-4.157e-17	-4.157e-16	-4.157e-15	-9.486e-14	-4.103e-13

Table 4.4.15: Values of $\varphi_{1MLE}^{*}(\theta_1)$ and $\varphi_{1B}^{*}(\theta_1)$ of MWM for Data set 4 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		1.000e-12	1.000e-11	1.000e-10	1.000e-09	1.000e-08
0.0001	$\varphi_{1MLE}^{*}(\theta_1)$	3.017e-14	3.017e-14	3.017e-14	3.017e-14	3.017e-14
	$\varphi_{1B}^{*}(\theta_1)$	1.922e-12	1.926e-12	1.926e-12	1.926e-12	1.929e-12
0.0010	$\varphi_{1MLE}^{*}(\theta_1)$	3.713e-13	3.713e-13	3.713e-13	3.713e-13	3.713e-13
	$\varphi_{1B}^{*}(\theta_1)$	1.983e-12	1.983e-12	1.983e-12	1.983e-12	1.983e-12
0.0015	$\varphi_{1MLE}^{*}(\theta_1)$	6.227e-13	6.227e-13	6.227e-13	6.227e-13	6.227e-13
	$\varphi_{1B}^{*}(\theta_1)$	2.003e-12	2.006e-12	2.006e-12	2.006e-12	2.006e-12
0.0020	$\varphi_{1MLE}^{*}(\theta_1)$	9.260e-13	9.260e-13	9.260e-13	9.260e-13	9.260e-13
	$\varphi_{1B}^{*}(\theta_1)$	2.016e-12	2.018e-12	2.018e-12	2.018e-12	2.018e-12
0.0025	$\varphi_{1MLE}^{*}(\theta_1)$	1.288e-12	1.288e-12	1.288e-12	1.288e-12	1.288e-12
	$\varphi_{1B}^{*}(\theta_1)$	2.005e-12	2.013e-12	2.013e-12	2.013e-12	2.013e-12

For Data set 4, we see that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\varphi^*_{0MLE}(\theta_0)$ is greater than $\varphi^*_{0B}(\theta_0)$ in Table 4.4.14, then $\hat{\theta}_{0B}$ is preferred to $\hat{\theta}_{0MLE}$ while outside it is the other way around. In Table 4.4.15, we see that $\varphi^*_{1B}(\theta_1)$ is greater than $\varphi^*_{1MLE}(\theta_1)$, then $\hat{\theta}_{1MLE}$ is preferred to $\hat{\theta}_{1B}$ while outside it is the other way around.

For multistage Weibull model, the results show that MLEs are preferred to Berkson estimates or Berkson estimates are preferred to MLEs depending on each data set. Therefore, there are intervals I_0 of θ_0 and I_1 of θ_1 within which the Berkson's minimum chi-square estimates or MLEs are better based on each data set.

4.4.4.2 Logistic model

For data sets in Table 4.2.1, parameters θ_0 and θ_1 are estimated by using MATLAB which based on the two methods for logistic model (LM). The estimates are shown in Table 4.4.16.

Table 4.4.16: Values of $\hat{\theta}_{0MLE}$, $\hat{\theta}_{0B}$, $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ for LM

Data Set	$\hat{\theta}_{0MLE}$	$\hat{\theta}_{0B}$	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$
1	-2.118	-1.661	5.149e-01	4.294e-01
2	-4.662	-4.423	3.178	3.025
3	-3.798	-3.579	4.680e-02	4.402e-02
4	-6.401	-4.959	3.044e-02	2.314e-02

We consider the values of θ_0 and θ_1 around the estimates $\hat{\theta}_0$ and $\hat{\theta}_1$, and then compute the expressions of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ for $\hat{\theta}_0$ and, the expressions of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ for $\hat{\theta}_1$. These values are shown in Tables 4.4.17-4.4.24.

Table 4.4.17: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LM for Data set 1 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		-4	-3	-2	-1	-0.15
0.2	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	6.009e-05 -4.144e-05	8.918e-06 -6.228e-06	1.579e-06 -1.049e-06	4.615e-07 -1.831e-07	3.398e-07 -3.632e-08
0.3	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	4.579e-05 -3.312e-05	7.010e-06 -5.215e-06	1.401e-06 -9.118e-07	5.502e-07 -9.872e-08	4.636e-07 -7.917e-09
0.4	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	3.376e-05 -2.592e-05	5.516e-06 -4.337e-06	1.432e-06 -7.109e-07	7.811e-07 -6.220e-09	6.141e-07 -1.982e-07
0.5	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	2.437e-05 -1.994e-05	4.647e-06 -3.555e-06	1.891e-06 -3.875e-07	1.079e-06 -2.312e-07	6.369e-07 -8.807e-07
0.6	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	1.776e-05 -1.524e-05	4.917e-06 -2.667e-06	2.873e-06 -3.499e-07	1.136e-06 -1.196e-06	4.828e-07 -1.770e-06

Table 4.4.18: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LM for Data set 1 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		-4	-3	-2	-1	-0.15
0.2	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.109e-06 -2.358e-07	1.829e-07 -3.318e-08	4.709e-08 2.193e-09	3.631e-08 1.919e-08	6.876e-08 4.086e-08
0.3	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	7.152e-07 -3.836e-07	1.289e-07 -5.925e-08	5.257e-08 4.423e-09	7.801e-08 4.132e-08	1.748e-07 5.889e-08
0.4	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	5.188e-07 -3.598e-07	1.162e-07 -5.595e-08	9.682e-08 2.811e-08	1.863e-07 5.743e-08	3.752e-07 -8.492e-08
0.5	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	3.965e-07 -3.001e-07	1.518e-07 -3.225e-08	2.307e-07 7.025e-08	3.526e-07 -1.118e-07	5.824e-07 -7.764e-07
0.6	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	3.510e-07 -2.382e-07	3.175e-07 3.218e-08	4.793e-07 -3.568e-08	4.378e-07 -7.696e-07	6.808e-07 -2.126e-06

Table 4.4.19: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LM for Data set 2 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		**-6**	**-5**	**-4**	**-3**	**-2**
1	$\varphi^*_{0MLE}(\theta_0)$	3.561e-03	5.041e-04	7.693e-05	1.411e-05	3.671e-06
	$\varphi^*_{0B}(\theta_0)$	-3.466e-03	-4.684e-04	-6.225e-05	-7.165e-06	1.373e-07
2	$\varphi^*_{0MLE}(\theta_0)$	4.155e-04	6.778e-05	1.637e-05	7.856e-06	4.803e-06
	$\varphi^*_{0B}(\theta_0)$	-8.583e-04	-1.447e-04	-2.950e-05	-5.446e-06	7.172e-07
3	$\varphi^*_{0MLE}(\theta_0)$	4.581e-05	1.841e-05	1.592e-05	1.660e-05	9.214e-06
	$\varphi^*_{0B}(\theta_0)$	-1.097e-04	-4.132e-05	-2.269e-05	-7.723e-06	3.872e-07
4	$\varphi^*_{0MLE}(\theta_0)$	3.804e-05	3.030e-05	4.581e-05	4.539e-05	1.256e-05
	$\varphi^*_{0B}(\theta_0)$	-5.300e-05	-6.315e-05	-6.259e-05	-1.366e-05	-1.395e-06
5	$\varphi^*_{0MLE}(\theta_0)$	4.141e-05	9.806e-05	2.084e-04	7.425e-05	1.353e-05
	$\varphi^*_{0B}(\theta_0)$	-1.149e-04	-2.286e-04	-1.387e-04	-1.896e-05	-2.522e-06

Table 4.4.20: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LM for Data set 2 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		**-6**	**-5**	**-4**	**-3**	**-2**
1	$\varphi^*_{1MLE}(\theta_1)$	8.631e-04	1.244e-04	1.997e-05	4.184e-06	1.428e-06
	$\varphi^*_{1B}(\theta_1)$	-1.298e-03	-1.769e-04	-2.396e-05	-2.819e-06	1.366e-07
2	$\varphi^*_{1MLE}(\theta_1)$	9.663e-05	1.654e-05	4.944e-06	3.505e-06	3.109e-06
	$\varphi^*_{1B}(\theta_1)$	-2.411e-04	-4.305e-05	-9.732e-06	-1.941e-06	2.696e-07
3	$\varphi^*_{1MLE}(\theta_1)$	1.197e-05	6.983e-06	8.759e-06	1.219e-05	9.284e-06
	$\varphi^*_{1B}(\theta_1)$	-3.051e-05	-1.443e-05	-1.049e-05	-4.895e-06	-2.789e-06
4	$\varphi^*_{1MLE}(\theta_1)$	1.785e-05	1.908e-05	3.833e-05	4.329e-05	2.282e-05
	$\varphi^*_{1B}(\theta_1)$	-2.309e-05	-3.653e-05	-4.293e-05	-1.467e-05	-1.406e-05
5	$\varphi^*_{1MLE}(\theta_1)$	2.922e-05	9.294e-05	2.008e-04	8.870e-05	8.839e-05
	$\varphi^*_{1B}(\theta_1)$	-7.780e-05	-1.763e-04	-1.148e-04	-2.949e-05	-6.041e-05

Table 4.4.21: Values of $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{0B}(\theta_0)$ of LM for Data set 3 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		-5	-4	-3	-2	-1
0.02	$\varphi^{*}_{0MLE}(\theta_0)$	4.001e-04	5.689e-05	8.859e-06	1.797e-06	6.640e-07
	$\varphi^{*}_{0B}(\theta_0)$	-3.227e-04	-4.579e-05	-6.872e-06	-1.042e-06	-4.136e-08
0.03	$\varphi^{*}_{0MLE}(\theta_0)$	2.482e-04	3.615e-05	6.139e-06	1.642e-06	8.533e-07
	$\varphi^{*}_{0B}(\theta_0)$	-2.121e-04	-3.166e-05	-5.231e-06	-7.904e-07	6.451e-08
0.04	$\varphi^{*}_{0MLE}(\theta_0)$	1.442e-04	2.209e-05	4.621e-06	1.974e-06	1.174e-06
	$\varphi^{*}_{0B}(\theta_0)$	-1.328e-04	-2.144e-05	-3.972e-06	-4.647e-07	-4.028e-08
0.05	$\varphi^{*}_{0MLE}(\theta_0)$	7.986e-05	1.384e-05	4.681e-06	2.843e-06	1.443e-06
	$\varphi^{*}_{0B}(\theta_0)$	-7.961e-05	-1.459e-05	-2.899e-06	-4.729e-07	-6.718e-07
0.06	$\varphi^{*}_{0MLE}(\theta_0)$	4.380e-05	1.083e-05	6.884e-06	3.672e-06	1.606e-06
	$\varphi^{*}_{0B}(\theta_0)$	-4.701e-05	-1.034e-05	-2.332e-06	-1.821e-06	-1.522e-06

Table 4.4.22: Values of $\varphi^{*}_{1MLE}(\theta_1)$ and $\varphi^{*}_{1B}(\theta_1)$ of LM for Data set 3 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		-5	-4	-3	-2	-1
0.02	$\varphi^{*}_{1MLE}(\theta_1)$	5.539e-08	8.154e-09	1.427e-09	4.280e-10	3.464e-10
	$\varphi^{*}_{1B}(\theta_1)$	-3.596e-08	-5.230e-09	-7.860e-10	-4.423e-11	1.510e-10
0.03	$\varphi^{*}_{1MLE}(\theta_1)$	3.092e-08	4.728e-09	9.989e-10	5.434e-10	7.044e-10
	$\varphi^{*}_{1B}(\theta_1)$	-2.488e-08	-3.901e-09	-6.610e-10	3.240e-11	1.948e-10
0.04	$\varphi^{*}_{1MLE}(\theta_1)$	1.772e-08	2.969e-09	1.004e-09	1.060e-09	1.395e-09
	$\varphi^{*}_{1B}(\theta_1)$	-1.618e-08	-2.811e-09	-4.940e-10	1.590e-10	-3.063e-10
0.05	$\varphi^{*}_{1MLE}(\theta_1)$	1.002e-08	2.213e-09	1.720e-09	2.132e-09	2.215e-09
	$\varphi^{*}_{1B}(\theta_1)$	-1.003e-08	-2.044e-09	-2.655e-10	-2.805e-10	-2.344e-09
0.06	$\varphi^{*}_{1MLE}(\theta_1)$	5.949e-09	2.769e-09	3.760e-09	3.306e-09	3.055e-09
	$\varphi^{*}_{1B}(\theta_1)$	-6.249e-09	-1.546e-09	-5.753e-10	-2.695e-09	-5.760e-09

Table 4.4.23: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LM for Data set 4 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		-7	-6	-5	-4	-3
0.01	$\varphi^*_{0MLE}(\theta_0)$	1.230e-02	1.671e-03	2.286e-04	3.186e-05	4.687e-06
	$\varphi^*_{0B}(\theta_0)$	-1.038e-02	-1.418e-03	-1.967e-04	-2.843e-05	-4.560e-06
0.02	$\varphi^*_{0MLE}(\theta_0)$	7.243e-03	9.864e-04	1.359e-04	1.945e-05	3.245e-06
	$\varphi^*_{0B}(\theta_0)$	-6.166e-03	-8.470e-04	-1.195e-04	-1.817e-05	-3.304e-06
0.03	$\varphi^*_{0MLE}(\theta_0)$	3.631e-03	4.982e-04	7.093e-05	1.246e-05	5.247e-06
	$\varphi^*_{0B}(\theta_0)$	-3.073e-03	-4.269e-04	-6.319e-05	-1.124e-05	-2.068e-06
0.04	$\varphi^*_{0MLE}(\theta_0)$	1.581e-03	2.270e-04	4.684e-05	3.069e-05	8.534e-06
	$\varphi^*_{0B}(\theta_0)$	-1.291e-03	-1.888e-04	-3.375e-05	-9.269e-06	-1.483e-05
0.05	$\varphi^*_{0MLE}(\theta_0)$	6.552e-04	1.904e-04	1.834e-04	3.859e-05	4.992e-06
	$\varphi^*_{0B}(\theta_0)$	-4.941e-04	-8.922e-05	-7.731e-05	-8.057e-05	-7.910e-06

Table 4.4.24: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LM for Data set 4 when values of θ_0 and θ_1 around the estimates from MLE and Berkson

θ_1 \ θ_0		-7	-6	-5	-4	-3
0.01	$\varphi^*_{1MLE}(\theta_1)$	3.585e-07	4.900e-08	6.812e-09	9.967e-10	1.732e-10
	$\varphi^*_{1B}(\theta_1)$	-2.842e-08	-4.027e-09	-6.132e-00	-1.088e-10	-2.225e-11
0.02	$\varphi^*_{1MLE}(\theta_1)$	1.763e-07	2.409e-08	3.369e-09	5.316e-10	1.862e-10
	$\varphi^*_{1B}(\theta_1)$	-1.355e-07	-1.866e-08	-2.669e-09	-4.311e-10	-6.007e-11
0.03	$\varphi^*_{1MLE}(\theta_1)$	9.026e-08	1.251e-08	1.971e-09	8.729e-10	1.672e-09
	$\varphi^*_{1B}(\theta_1)$	-7.480e-08	-1.052e-08	-1.682e-09	-2.534e-10	9.671e-11
0.04	$\varphi^*_{1MLE}(\theta_1)$	4.008e-08	6.725e-09	4.693e-09	8.941e-09	1.754e-09
	$\varphi^*_{1B}(\theta_1)$	-3.251e-08	-5.243e-09	-7.899e-10	-2.921e-09	-1.081e-08
0.05	$\varphi^*_{1MLE}(\theta_1)$	2.237e-08	2.819e-08	4.947e-08	6.749e-09	6.441e-10
	$\varphi^*_{1B}(\theta_1)$	-1.410e-08	-2.935e-09	-3.395e-08	-3.628e-08	-1.193e-09

For logistic model in Tables 4.4.17-4.4.24, the results show that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\varphi^*_{0MLE}(\theta_0)$ is greater than $\varphi^*_{0B}(\theta_0)$ and also $\varphi^*_{1MLE}(\theta_1)$ is greater than $\varphi^*_{1B}(\theta_1)$ for all data sets. Therefore, the Berkson's minimum chi-square estimates are better than the MLEs while outside they are the other way around.

4.4.4.3 Log-logistic model

For data sets in Table 4.2.1, parameters, θ_0 and θ_1 are estimated by using MATLAB which based on the two methods for log-logistic model (LLM). The estimates are shown in Table 4.4.25.

Table 4.4.25: Values of $\hat{\theta}_{0MLE}$, $\hat{\theta}_{0B}$, $\hat{\theta}_{1MLE}$ and $\hat{\theta}_{1B}$ for LLM

Data Set	$\hat{\theta}_{0MLE}$	$\hat{\theta}_{0B}$	$\hat{\theta}_{1MLE}$	$\hat{\theta}_{1B}$
1	-1.487	-6.222e-01	1.432	6.415e-01
2	-1.487	-1.752e-01	4.296	5.805e-01
3	-10.033	-2.210	2.295	3.897e-01
4	-16.774	-2.586	3.105	3.529e-01

We consider the values of θ_0 and θ_1 around the estimates $\hat{\theta}_0$ and $\hat{\theta}_1$, and then compute the expressions of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ for $\hat{\theta}_0$ and, the expressions of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ for $\hat{\theta}_1$. These values are shown in Tables 4.4.26-4.4.41.

Table 4.4.26: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LLM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-1.9	-1.7	-1.5	-1.3	-1.1
1.1	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	3.980e-06 4.110e-05	3.169e-06 3.424e-05	2.557e-06 2.934e-05	2.083e-06 2.589e-05	1.707e-06 2.356e-05
1.3	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	4.198e-06 4.332e-05	3.480e-06 3.667e-05	2.909e-06 3.190e-05	2.443e-06 2.857e-05	2.058e-06 2.637e-05
1.5	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	4.513e-06 4.624e-05	3.841e-06 3.964e-05	3.259e-06 3.486e-05	2.749e-06 3.150e-05	2.309e-06 2.927e-05
1.7	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	5.091e-06 5.079e-05	4.371e-06 4.396e-05	3.697e-06 3.895e-05	3.084e-06 3.535e-05	2.547e-06 3.292e-05
1.9	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	5.936e-06 5.737e-05	5.043e-06 4.995e-05	4.187e-06 4.438e-05	3.414e-06 4.028e-05	2.755e-06 3.740e-05

Table 4.4.27: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LLM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		-0.9	-0.7	-0.5	-0.3	-0.1
0.4	$\varphi^*_{0MLE}(\theta_0)$	5.671e-06	4.615e-06	3.747e-06	3.020e-06	2.408e-06
	$\varphi^*_{0B}(\theta_0)$	-2.389e-06	-9.326e-07	-1.787e-08	5.053e-07	7.494e-07
0.5	$\varphi^*_{0MLE}(\theta_0)$	1.917e-06	2.420e-06	2.860e-06	3.208e-06	3.424e-06
	$\varphi^*_{0B}(\theta_0)$	-2.003e-05	-1.504e-05	-1.093e-05	-7.501e-06	-4.682e-06
0.6	$\varphi^*_{0MLE}(\theta_0)$	-4.349e-06	-3.697e-06	-2.977e-06	-2.116e-06	-1.062e-06
	$\varphi^*_{0B}(\theta_0)$	-2.663e-05	-2.464e-05	-2.292e-05	-2.114e-05	-1.903e-05
0.7	$\varphi^*_{0MLE}(\theta_0)$	-4.047e-06	-4.288e-06	-4.587e-06	-4.907e-06	-5.164e-06
	$\varphi^*_{0B}(\theta_0)$	-1.058e-05	-1.226e-05	-1.433e-05	-1.685e-05	-1.972e-05
0.8	$\varphi^*_{0MLE}(\theta_0)$	-1.511e-06	-1.959e-06	-2.503e-06	-3.189e-06	-4.065e-06
	$\varphi^*_{0B}(\theta_0)$	5.885e-06	3.847e-06	1.652e-06	-9.731e-07	-4.350e-06

Table 4.4.28: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-1.9	-1.7	-1.5	-1.3	-1.1
1.1	$\varphi^*_{1MLE}(\theta_1)$	1.477e-06	1.315e-06	1.198e-06	1.112e-06	1.048e-06
	$\varphi^*_{1B}(\theta_1)$	2.729e-05	2.406e-05	2.190e-05	2.059e-05	2.001e-05
1.3	$\varphi^*_{1MLE}(\theta_1)$	1.853e-06	1.739e-06	1.656e-06	1.598e-06	1.565e-06
	$\varphi^*_{1B}(\theta_1)$	2.942e-05	2.690e-05	2.540e-05	2.477e-05	2.497e-05
1.5	$\varphi^*_{1MLE}(\theta_1)$	2.366e-06	2.280e-06	2.204e-06	2.144e-06	2.114e-06
	$\varphi^*_{1B}(\theta_1)$	3.349e-05	3.163e-05	3.083e-05	3.100e-05	3.217e-05
1.7	$\varphi^*_{1MLE}(\theta_1)$	3.148e-06	3.041e-06	2.923e-06	2.825e-06	2.784e-06
	$\varphi^*_{1B}(\theta_1)$	4.060e-05	3.950e-05	3.958e-05	4.083e-05	4.335e-05
1.9	$\varphi^*_{1MLE}(\theta_1)$	4.230e-06	4.016e-06	3.795e-06	3.634e-06	3.597e-06
	$\varphi^*_{1B}(\theta_1)$	5.212e-05	5.199e-05	5.326e-05	5.602e-05	6.050e-05

Table 4.4.29: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 1 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		**-0.9**	**-0.7**	**-0.5**	**-0.3**	**-0.1**
0.4	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	3.234e-06 -1.931e-06	2.892e-06 -8.098e-07	2.570e-06 -4.318e-08	2.255e-06 4.469e-07	1.943e-06 7.201e-07
0.5	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	4.834e-07 -1.393e-05	1.238e-06 -1.100e-05	1.956e-06 -8.368e-06	2.615e-06 -5.945e-06	3.168e-06 -3.735e-06
0.6	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	-3.697e-06 -1.807e-05	-3.161e-06 -1.808e-05	-2.507e-06 -1.812e-05	-1.625e-06 -1.792e-05	-4.131e-07 -1.715e-05
0.7	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	-3.334e-06 -6.581e-06	-3.586e-06 -9.009e-06	-3.910e-06 -1.199e-05	-4.251e-06 -1.567e-05	-4.486e-06 -2.005e-05
0.8	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	-1.439e-06 5.466e-06	-1.798e-06 3.491e-06	-2.299e-06 1.087e-06	-2.995e-06 -2.142e-06	-3.941e-06 -6.740e-06

For Data set 1, we see that $\varphi^*_{0MLE}(\theta_0)$ is less than $\varphi^*_{0B}(\theta_0)$ when we consider the values of θ_0 and θ_1 around MLEs in Table 4.4.26. We also see that $\varphi^*_{0B}(\theta_0)$ is less than $\varphi^*_{0MLE}(\theta_0)$ when we consider the values of θ_0 and θ_1 around Berkson estimates in Table 4.4.27. Moreover, there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\varphi^*_{0B}(\theta_0)$ in Table 4.4.27 is less than $\varphi^*_{0MLE}(\theta_0)$ in Table 4.4.26. For $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ in Tables 4.4.28-4.4.29, we proceed the same as $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ and obtain that $\varphi^*_{1B}(\theta_1)$ is less than $\varphi^*_{1MLE}(\theta_1)$. Therefore, there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\hat{\theta}_{0B}$ is preferred to $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ while outside they are the other way around.

Table 4.4.30: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LLM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-1.9	-1.7	-1.5	-1.3	-1.1
3.0	$\varphi^*_{0MLE}(\theta_0)$	4.244e-06	3.353e-06	2.700e-06	2.213e-06	1.846e-06
	$\varphi^*_{0B}(\theta_0)$	4.554e-04	3.689e-04	3.063e-04	2.611e-04	2.289e-04
3.5	$\varphi^*_{0MLE}(\theta_0)$	4.176e-06	3.393e-06	2.796e-06	2.333e-06	1.967e-06
	$\varphi^*_{0B}(\theta_0)$	4.612e-04	3.794e-04	3.194e-04	2.755e-04	2.439e-04
4.0	$\varphi^*_{0MLE}(\theta_0)$	4.258e-06	3.552e-06	2.984e-06	2.516e-06	2.128e-06
	$\varphi^*_{0B}(\theta_0)$	4.789e-04	3.999e-04	3.408e-04	2.968e-04	2.645e-04
4.5	$\varphi^*_{0MLE}(\theta_0)$	4.518e-06	3.846e-06	3.264e-06	2.755e-06	2.315e-06
	$\varphi^*_{0B}(\theta_0)$	5.106e-04	4.318e-04	3.715e-04	3.255e-04	2.909e-04
5.0	$\varphi^*_{0MLE}(\theta_0)$	4.976e-06	4.272e-06	3.620e-06	3.028e-06	2.510e-06
	$\varphi^*_{0B}(\theta_0)$	5.583e-04	4.764e-04	4.119e-04	3.615e-04	3.228e-04

Table 4.4.31: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LLM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		-0.5	-0.4	-0.3	-0.2	-0.1
0.3	$\varphi^*_{0MLE}(\theta_0)$	4.481e-07	3.838e-07	3.321e-07	2.911e-07	2.592e-07
	$\varphi^*_{0B}(\theta_0)$	9.909e-08	6.596e-08	3.784e-08	1.362e-08	-7.623e-09
0.4	$\varphi^*_{0MLE}(\theta_0)$	1.855e-06	1.543e-06	1.285e-06	1.074e-06	9.015e-07
	$\varphi^*_{0B}(\theta_0)$	1.303e-06	1.076e-06	8.856e-07	7.253e-07	5.900e-07
0.5	$\varphi^*_{0MLE}(\theta_0)$	8.401e-06	7.110e-06	6.012e-06	5.077e-06	4.283e-06
	$\varphi^*_{0B}(\theta_0)$	5.609e-06	4.914e-06	4.289e-06	3.728e-06	3.227e-06
0.6	$\varphi^*_{0MLE}(\theta_0)$	2.357e-05	2.129e-05	1.918e-05	1.722e-05	1.540e-05
	$\varphi^*_{0B}(\theta_0)$	1.581e-06	3.404e-06	4.723e-06	5.617e-06	6.160e-06
0.7	$\varphi^*_{0MLE}(\theta_0)$	1.517e-05	1.716e-05	1.887e-05	2.031e-05	2.144e-05
	$\varphi^*_{0B}(\theta_0)$	-7.582e-05	-6.308e-05	-5.158e-05	-4.121e-05	-3.190e-05

Table 4.4.32: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-1.9	-1.7	-1.5	-1.3	-1.1
3.0	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.371e-05 2.419e-03	1.202e-05 2.106e-03	1.086e-05 1.894e-03	1.009e-05 1.760e-03	9.634e-06 1.694e-03
3.5	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.501e-05 2.520e-03	1.371e-05 2.263e-03	1.282e-05 2.098e-03	1.224e-05 2.008e-03	1.193e-05 1.987e-03
4.0	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.743e-05 2.764e-03	1.648e-05 2.558e-03	1.578e-05 2.439e-03	1.530e-05 2.398e-03	1.507e-05 2.431e-03
4.5	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	2.132e-05 3.186e-03	2.056e-05 3.030e-03	1.988e-05 2.960e-03	1.934e-05 2.973e-03	1.909e-05 3.076e-03
5.0	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	2.698e-05 3.838e-03	2.609e-05 3.730e-03	2.514e-05 3.714e-03	2.433e-05 3.795e-03	2.399e-05 3.986e-03

Table 4.4.33: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 2 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		-0.5	-0.4	-0.3	-0.2	-0.1
0.3	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.881e-06 1.810e-06	1.561e-06 1.505e-06	1.295e-06 1.251e-06	1.076e-06 1.040e-06	8.940e-07 8.641e-07
0.4	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.147e-05 1.026e-05	9.587e-06 8.653e-06	8.013e-06 7.295e-06	6.696e-06 6.144e-06	5.593e-06 5.169e-06
0.5	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	5.662e-05 3.986e-05	4.868e-05 3.554e-05	4.181e-05 3.156e-05	3.586e-05 2.790e-05	3.070e-05 2.456e-05
0.6	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	1.606e-04 7.608e-06	1.483e-04 2.233e-05	1.366e-04 3.342e-05	1.253e-04 4.140e-05	1.144e-04 4.670e-05
0.7	$\varphi^*_{1MLE}(\theta_1)$ $\varphi^*_{1B}(\theta_1)$	9.592e-05 -5.453e-04	1.155e-04 -4.591e-00	1.332e-04 -3.796e-04	1.490e-04 -3.061e-04	1.625e-04 -2.386e-04

For Data set 2 in Tables 4.4.30-4.4.33, we proceed the same as case of Data set 1 and obtain that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\hat{\theta}_{0B}$ is preferred to $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ while outside they are the other way around.

Table 4.4.34: Values of $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{0B}(\theta_0)$ of LLM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		**-13.5**	**-12.0**	**-10.5**	**-9.0**	**-7.5**
1.60	$\varphi^{*}_{0MLE}(\theta_0)$	9.049e+00	4.525e-01	2.299e-02	1.262e-03	1.058e-04
	$\varphi^{*}_{0B}(\theta_0)$	4.822e+02	2.424e+01	1.259e+00	7.520e-02	7.585e-03
1.85	$\varphi^{*}_{0MLE}(\theta_0)$	1.539e+00	7.749e-02	4.065e-03	2.677e-04	5.613e-05
	$\varphi^{*}_{0B}(\theta_0)$	8.057e+01	4.115e+00	2.293e-01	1.808e-02	3.939e-03
2.10	$\varphi^{*}_{0MLE}(\theta_0)$	2.587e-01	1.325e-02	7.621e-04	9.385e-05	5.934e-05
	$\varphi^{*}_{0B}(\theta_0)$	1.359e+01	7.248e-01	4.842e-02	6.888e-03	4.207e-03
2.35	$\varphi^{*}_{0MLE}(\theta_0)$	4.316e-02	2.315e-03	1.867e-04	8.191e-05	9.636e-05
	$\varphi^{*}_{0B}(\theta_0)$	2.338e+00	1.400e-01	1.438e-02	5.115e-03	8.570e-03
2.60	$\varphi^{*}_{0MLE}(\theta_0)$	7.237e-03	4.547e-04	1.136e-04	9.088e-05	3.696e-04
	$\varphi^{*}_{0B}(\theta_0)$	4.250e-01	3.420e-02	7.794e-03	7.101e-03	2.953e-02

Table 4.4.35: Values of $\varphi^{*}_{0MLE}(\theta_0)$ and $\varphi^{*}_{0B}(\theta_0)$ of LLM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		**-6.0**	**-4.5**	**-3.0**	**-1.5**	**-0.1**
0.10	$\varphi^{*}_{0MLE}(\theta_0)$	1.958e-02	9.815e-04	5.030e-05	2.828e-06	3.062e-07
	$\varphi^{*}_{0B}(\theta_0)$	1.428e-02	7.162e-04	3.680e-05	2.074e-06	1.613e-07
0.35	$\varphi^{*}_{0MLE}(\theta_0)$	2.480e-01	1.405e-02	1.113e-03	1.456e-04	1.668e-05
	$\varphi^{*}_{0B}(\theta_0)$	-2.378e-01	-1.086e-02	-2.372e-04	8.420e-05	1.595e-05
0.60	$\varphi^{*}_{0MLE}(\theta_0)$	-3.684e-02	-2.859e-03	-6.415e-04	-2.614e-04	1.074e-03
	$\varphi^{*}_{0B}(\theta_0)$	2.203e-02	-2.063e-04	-1.185e-03	-2.364e-03	5.591e-04
0.85	$\varphi^{*}_{0MLE}(\theta_0)$	2.522e-04	-1.053e-06	-5.517e-05	-1.390e-03	5.778e-03
	$\varphi^{*}_{0B}(\theta_0)$	4.100e-02	4.261e-03	1.736e-03	-5.603e-04	-3.547e-02
1.10	$\varphi^{*}_{0MLE}(\theta_0)$	1.367e-04	2.635e-05	2.284e-05	-1.269e-03	-1.043e-01
	$\varphi^{*}_{0B}(\theta_0)$	1.007e-02	2.402e-03	3.808e-03	2.322e-02	-3.476e-01

Table 4.4.36: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-13.5	-12.0	-10.5	-9.0	-7.5
1.60	$\varphi^*_{1MLE}(\theta_1)$	4.476e-01	2.239e-02	1.140e-03	6.321e-05	5.710e-06
	$\varphi^*_{1B}(\theta_1)$	2.771e+01	1.394e+00	7.263e-02	4.403e-03	4.674e-04
1.85	$\varphi^*_{1MLE}(\theta_1)$	7.576e-02	3.817e-03	2.011e-04	1.371e-05	3.520e-06
	$\varphi^*_{1B}(\theta_1)$	4.519e+00	2.313e-01	1.302e-02	1.067e-03	2.577e-04
2.10	$\varphi^*_{1MLE}(\theta_1)$	1.270e-02	6.517e-04	3.802e-05	5.355e-06	4.446e-06
	$\varphi^*_{1B}(\theta_1)$	7.469e-01	4.011e-02	2.753e-03	4.266e-04	3.031e-04
2.35	$\varphi^*_{1MLE}(\theta_1)$	2.118e-03	1.142e-04	9.847e-06	5.558e-06	8.763e-06
	$\varphi^*_{1B}(\theta_1)$	1.267e-01	7.725e-03	8.463e-04	3.476e-04	6.836e-04
2.60	$\varphi^*_{1MLE}(\theta_1)$	3.556e-04	2.292e-05	7.095e-06	7.276e-06	3.741e-05
	$\varphi^*_{1B}(\theta_1)$	2.290e-02	1.929e-03	4.976e-04	5.400e-04	2.555e-03

Table 4.4.37: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 3 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		-6.0	-4.5	-3.0	-1.5	-0.1
0.10	$\varphi^*_{1MLE}(\theta_1)$	1.094e-03	5.490e-05	2.829e-06	1.626e-07	1.711e-08
	$\varphi^*_{1B}(\theta_1)$	1.085e-03	5.450e-05	2.817e-06	1.619e-07	1.489e-08
0.35	$\varphi^*_{1MLE}(\theta_1)$	1.486e-02	8.523e-04	7.005e-05	9.831e-06	1.229e-06
	$\varphi^*_{1B}(\theta_1)$	-1.621e-02	-7.469e-04	-1.770e-05	5.699e-06	1.101e-06
0.60	$\varphi^*_{1MLE}(\theta_1)$	-2.395e-03	-1.871e-04	-4.301e-05	-1.717e-05	8.253e-05
	$\varphi^*_{1B}(\theta_1)$	1.953e-03	1.925e-05	-7.648e-05	-1.667e-04	4.290e-05
0.85	$\varphi^*_{1MLE}(\theta_1)$	1.019e-05	-4.456e-07	-3.778e-06	-1.010e-04	5.292e-04
	$\varphi^*_{1B}(\theta_1)$	2.622e-03	2.793e-04	1.193e-04	-5.615e-05	-2.694e-03
1.10	$\varphi^*_{1MLE}(\theta_1)$	7.309e-06	1.692e-06	2.058e-06	-9.301e-05	-7.969e-03
	$\varphi^*_{1B}(\theta_1)$	6.351e-04	1.611e-04	2.758e-04	1.732e-03	-2.860e-02

For Data set 3 in Tables 4.4.34-4.4.37, we proceed the same as case of Data set 1 and obtain that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\hat{\theta}_{0B}$ is preferred to $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ while outside they are the other way around.

Table 4.4.38: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LLM for Data set 4 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-19	-17	-15	-13	-11
2.5	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	1.137e+02 3.745e+03	2.086e+00 6.910e+01	3.859e-02 1.338e+00	7.854e-04 3.786e-02	2.088e-04 6.543e-03
2.8	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	1.007e+01 3.633e+02	1.854e-01 6.843e+00	3.542e-03 1.566e-01	2.186e-04 1.483e-02	3.361e-04 7.841e-03
3.1	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	8.794e-01 3.407e+01	1.642e-02 6.955e-01	4.318e-04 3.345e-02	8.897e-04 2.245e-02	1.704e-04 8.057e-03
3.4	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	7.631e-02 3.201e+00	1.569e-03 9.526e-02	9.319e-04 3.907e-02	6.714e-04 2.227e-02	5.145e-04 1.568e-02
3.7	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	6.777e-03 3.419e-01	7.085e-04 5.142e-02	3.037e-03 7.981e-02	3.272e-04 1.929e-02	1.784e-03 3.892e-02

Table 4.4.39: Values of $\varphi^*_{0MLE}(\theta_0)$ and $\varphi^*_{0B}(\theta_0)$ of LLM for Data set 4 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		-5	-4	-3	-2	-1
0.1	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	1.703e-03 7.049e-04	2.343e-04 9.892e-05	3.312e-05 1.465e-05	4.978e-06 2.410e-06	8.489e-07 4.540e-07
0.2	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	3.244e-03 -1.341e-03	4.667e-04 -1.613e-04	7.341e-05 -1.372e-05	1.367e-05 1.542e-06	3.034e-06 1.489e-06
0.3	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	5.220e-04 -7.627e-03	9.425e-05 -1.116e-03	2.446e-05 -1.797e-04	1.086e-05 -3.115e-05	6.710e-06 -2.176e-06
0.4	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	-4.765e-04 -3.340e-03	-8.310e-05 -5.798e-04	-1.912e-05 -1.391e-04	-5.192e-06 -5.480e-05	3.964e-06 -2.511e-05
0.5	$\varphi^*_{0MLE}(\theta_0)$ $\varphi^*_{0B}(\theta_0)$	-4.084e-05 3.003e-04	-1.462e-05 2.116e-05	-7.834e-06 -1.422e-05	-7.195e-06 -2.680e-05	-6.472e-06 -5.108e-05

Table 4.4.40: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 4 when values of θ_0 and θ_1 around $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1MLE}$

θ_1 \ θ_0		-19	-17	-15	-13	-11
2.5	$\varphi^*_{1MLE}(\theta_1)$	4.027e+00	7.386e-02	1.367e-03	2.818e-05	1.150e-05
	$\varphi^*_{1B}(\theta_1)$	1.378e+02	2.546e+00	4.971e-02	1.501e-03	3.378e-04
2.8	$\varphi^*_{1MLE}(\theta_1)$	3.574e-01	6.578e-03	1.260e-04	1.062e-05	2.022e-05
	$\varphi^*_{1B}(\theta_1)$	1.324e+01	2.504e-01	5.921e-03	6.889e-04	4.979e-04
3.1	$\varphi^*_{1MLE}(\theta_1)$	3.124e-02	5.837e-04	1.707e-05	5.180e-05	1.346e-05
	$\varphi^*_{1B}(\theta_1)$	1.235e+00	2.560e-02	1.412e-03	1.270e-03	6.332e-04
3.4	$\varphi^*_{1MLE}(\theta_1)$	2.714e-03	5.705e-05	5.318e-05	4.060e-05	5.559e-05
	$\varphi^*_{1B}(\theta_1)$	1.162e-01	3.726e-03	2.038e-03	1.466e-03	1.531e-03
3.7	$\varphi^*_{1MLE}(\theta_1)$	2.423e-04	3.711e-05	1.780e-04	2.780e-05	1.986e-04
	$\varphi^*_{1B}(\theta_1)$	1.276e-02	2.456e-03	4.746e-03	1.567e-03	4.355e-03

Table 4.4.41: Values of $\varphi^*_{1MLE}(\theta_1)$ and $\varphi^*_{1B}(\theta_1)$ of LLM for Data set 4 when values of θ_0 and θ_1 around $\hat{\theta}_{0B}$ and $\hat{\theta}_{1B}$

θ_1 \ θ_0		-5	-4	-3	-2	-1
0.1	$\varphi^*_{1MLE}(\theta_1)$	8.655e-05	1.202e-05	1.737e-06	2.751e-07	5.190e-08
	$\varphi^*_{1B}(\theta_1)$	5.103e-05	7.275e-06	1.118e-06	1.983e-07	4.227e-08
0.2	$\varphi^*_{1MLE}(\theta_1)$	1.545e-04	2.276e-05	3.787e-06	7.891e-07	2.078e-07
	$\varphi^*_{1B}(\theta_1)$	-1.126e-04	-1.396e-05	-1.352e-06	6.337e-08	1.127e-07
0.3	$\varphi^*_{1MLE}(\theta_1)$	-1.048e-05	-3.948e-07	5.477e-07	5.604e-07	5.006e-07
	$\varphi^*_{1B}(\theta_1)$	-4.614e-04	-6.864e-05	-1.149e-05	-2.140e-06	-1.531e-07
0.4	$\varphi^*_{1MLE}(\theta_1)$	-4.137e-05	-7.104e-06	-1.612e-06	-4.469e-07	3.738e-07
	$\varphi^*_{1B}(\theta_1)$	-1.663e-04	-3.037e-05	-8.054e-06	-3.682e-06	-1.946e-06
0.5	$\varphi^*_{1MLE}(\theta_1)$	-6.717e-06	-1.586e-06	-6.429e-07	-5.375e-07	-3.984e-07
	$\varphi^*_{1B}(\theta_1)$	3.421e-05	4.051e-06	-3.492e-07	-1.841e-06	-4.382e-06

For Data set 4 in Tables 4.4.38-4.4.41, we proceed the same as case of Data set 1 and obtain that there are intervals I_0 of θ_0 and I_1 of θ_1 within which $\hat{\theta}_{0B}$ is preferred to $\hat{\theta}_{0MLE}$ and $\hat{\theta}_{1B}$ is preferred to $\hat{\theta}_{1MLE}$ while outside they are the other way around.

For log-logistic model, the results show that there are intervals I_0 of θ_0 and I_1 of θ_1 within which the Berkson's minimum chi-square estimates are better than the MLEs while outside they are the other way around.

4.4.4.4 Model selection

In this thesis there are two criterions which are used to compare models with the same data set as follows.

1. Akaike's information criterion (AIC) [21] is computed as

$$\text{AIC} = 2k - 2\ln(L),$$

where k is the number of parameters in the model, and L is the maximized value of the likelihood function for the estimated model.

2. Log-likelihood ratio statistic (Deviance) [22] is defined as

$$D = 2\sum_{i=0}^{m}\left[y_i \log\left(\frac{y_i}{\hat{y}_i}\right) + (n_i - y_i)\log\left(\frac{n_i - y_i}{n_i - \hat{y}_i}\right)\right],$$

where y_i is the observed value, $\hat{y}_i$ denotes the fitted value and n_i is the number of subjects or cases tested.

We apply these criteria to choose the appropriate model for these four data sets. We have the estimates which are preferred. For the logistic and log-logistic models, Berkson's minimum chi-square estimates are better than MLEs for all data sets. Otherwise, the multistage Weibull model, the estimates [$\hat{\theta}_{0B}, \hat{\theta}_{1B}$], [$\hat{\theta}_{0MLE}, \hat{\theta}_{1B}$], [$\hat{\theta}_{0MLE}, \hat{\theta}_{1MLE}$] and [$\hat{\theta}_{0B}, \hat{\theta}_{1MLE}$] are used for Data sets 1, 2, 3 and 4, respectively. The value of AIC and deviance (D) are respectively presented for each model in Tables 4.4.42-4.4.43, with the one having the lowest AIC or deviance being the best model.

Table 4.4.42: AIC values for Data sets 1 - 4 of MWM, LM and LLM

Data set	MWM	LM	LLM
1	168.315	186.518	179.993
2	201.324	168.520	200.139
3	152.273	147.984	171.151
4	95.127	87.765	136.916

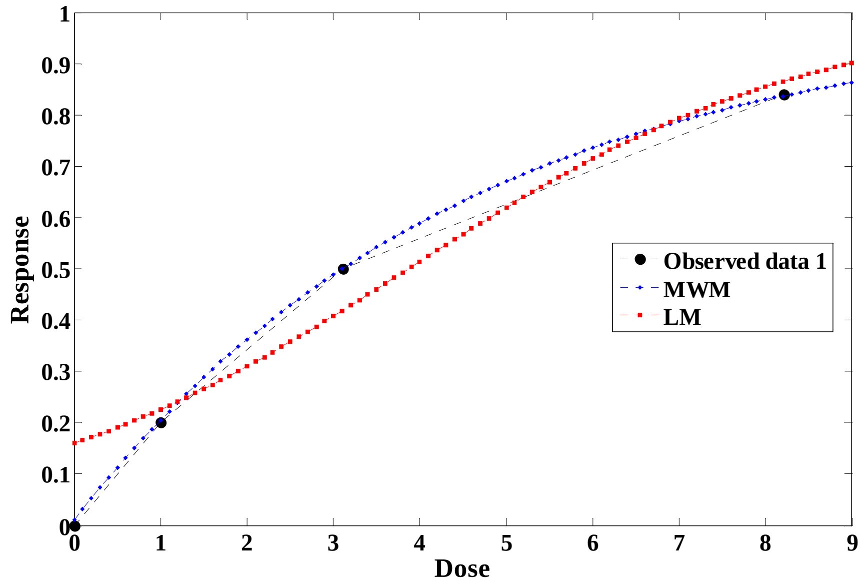
Table 4.4.43: Deviance values for Data sets 1 - 4 of MWM, LM and LLM

Data set	MWM	LM	LLM
1	0.019	0.338	0.117
2	0.219	0.023	0.314
3	0.109	0.061	0.384
4	0.239	0.053	0.983

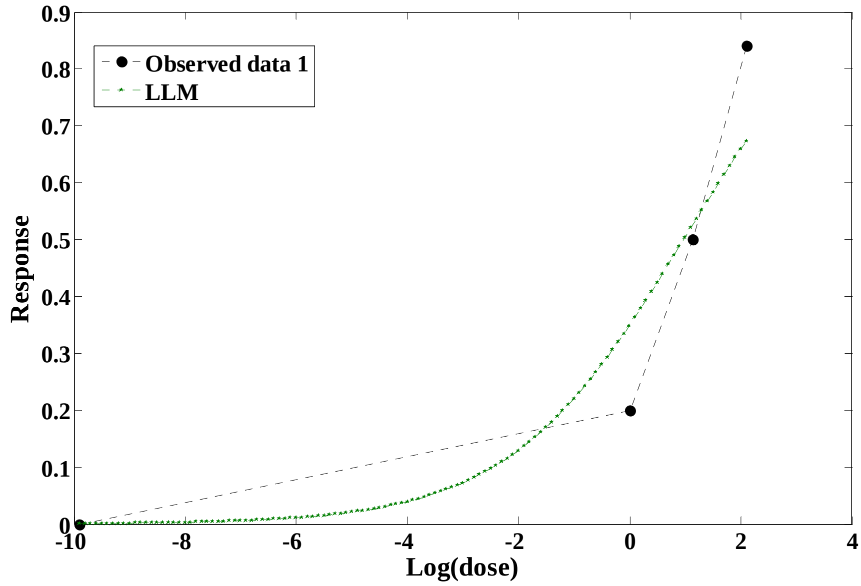
From Tables 4.4.42-4.4.43, the results show that the preference model for Data sets 2-4 is logistic model, and multistage Weibull model is the preference model for Data set 1 according to AIC and deviance criteria. Moreover, these results are illustrated to fit Data sets 1-4 in Figures 4.4.1-4.4.4, respectively.

Figure 4.4.1(a) shows the proportional to the number of respondents and the number of animals tested against dose. The proportions increase sharply as dose increases before leveling off at high dose. The multistage Weibull model fits the data perfectly for Data set 1. As is shown in Figure 4.4.1(b), the proportions of the observed data against logarithm of dose are plotted by a logistic curve (S-shaped) and log-logistic model does not fit the data particularly well. Therefore, the multistage Weibull model is adequate to Data set 1.

Figures 4.4.2-4.4.4 show the proportions of the observed data against dose and logarithm of dose in (a) and (b), respectively by the S-shaped. The results show that the logistic model fits the data perfectly that is this model is adequate to Data sets 2-4.

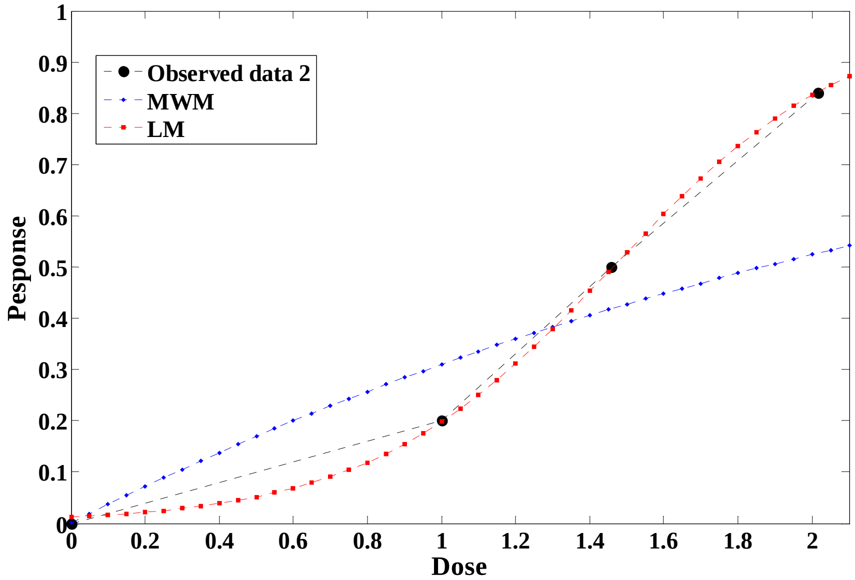


(a)

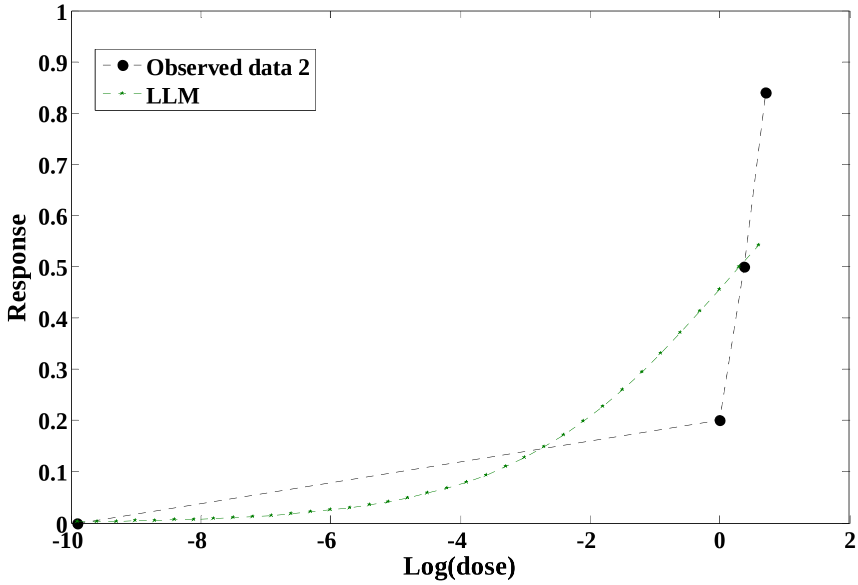


(b)

Figure 4.4.1: Dose response data for Data set 1 is fitted with the multistage Weibull, logistic models (a) and log-logistic model (b)

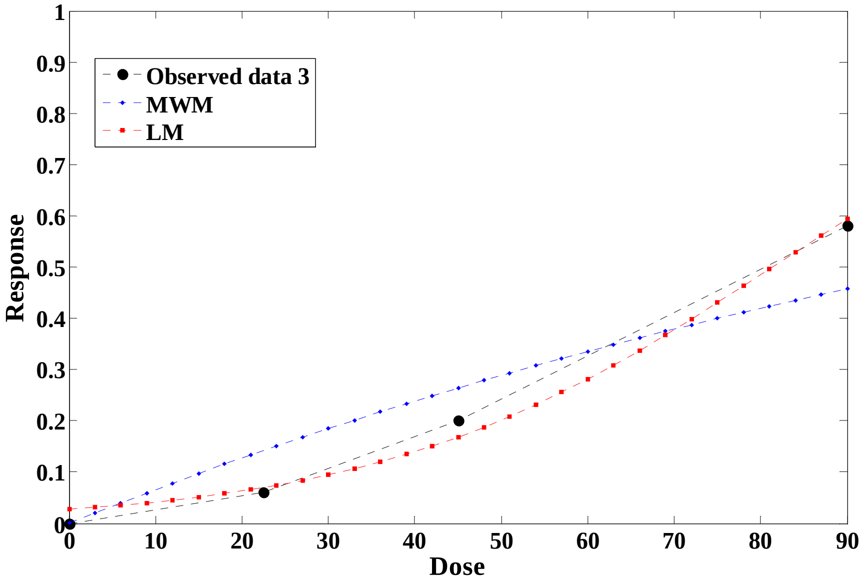


(a)

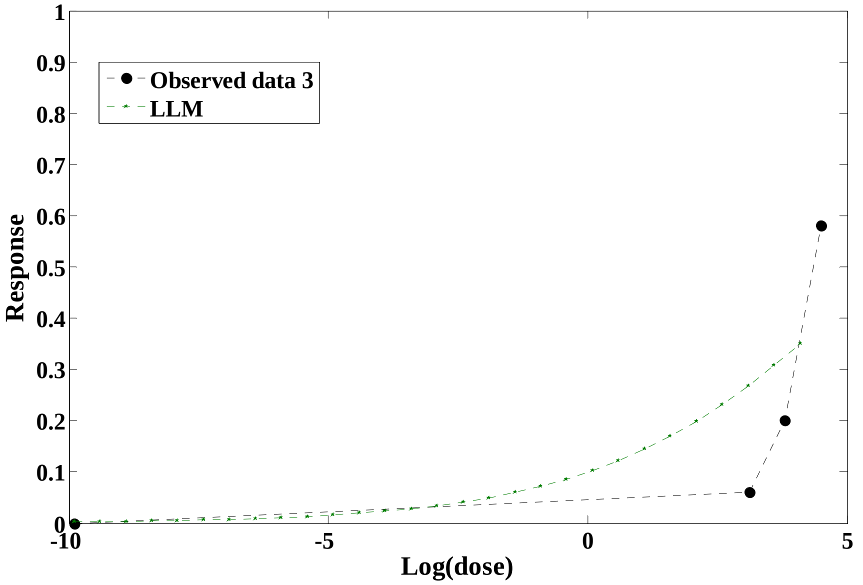


(b)

Figure 4.4.2: Dose response data for Data set 2 is fitted with the multistage Weibull, logistic models (a) and log-logistic model (b)

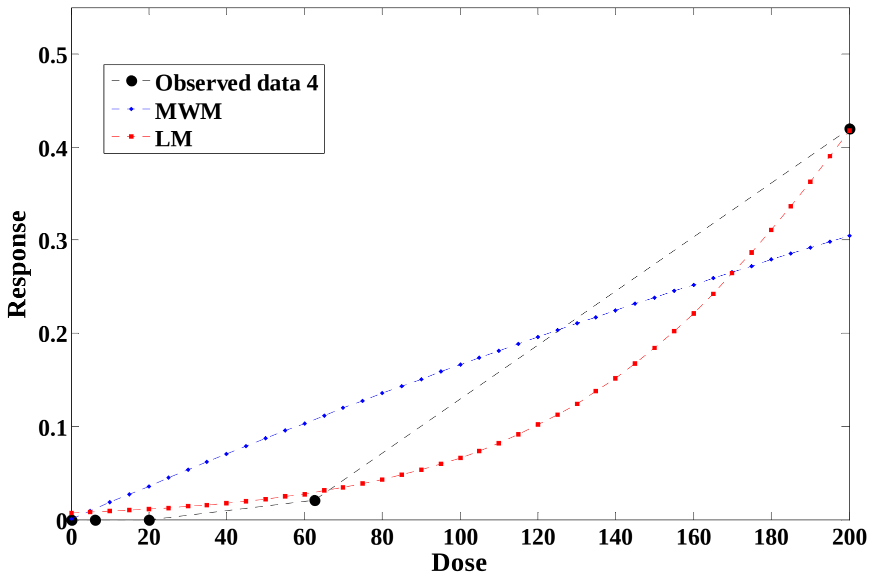


(a)

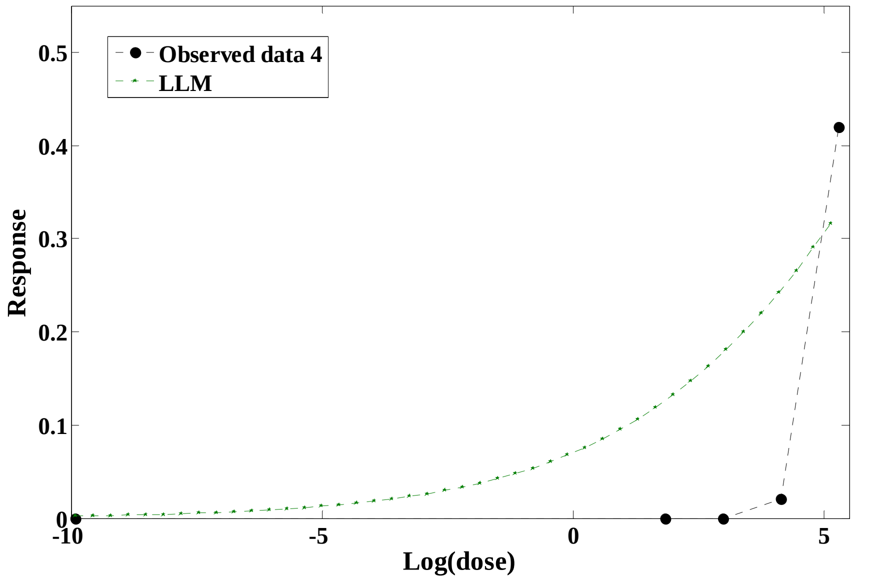


(b)

Figure 4.4.3: Dose response data for Data set 3 is fitted with the multistage Weibull, logistic models (a) and log-logistic model (b)



(a)



(b)

Figure 4.4.4: Dose response data for Data set 4 is fitted with the multistage Weibull, logistic models (a) and log-logistic model (b)

References

1. Agresti A. (2007). An introduction to categorical data analysis. John Wiley and Sons, Inc., New Jersey, USA.
2. Anscombe FJ. (1956). On estimating binomial response relations. Biometrika. 43(3/4), 461-464.
3. Amemiya T. (1980). The n^{-2}-order mean squared errors of the maximum likelihood and the minimum logit chi-square estimator. The Annals of Statistics. 8(3), 488-505.
4. Berkson J. (1944). Application of the logistic function to bio-assay. Journal of the American Statistical Association. 39(227), 357-365.
5. Berkson J. (1953). A statistically precise and relatively simple method of estimating the bio-assay with quantal response, based on the logistic function. Journal of the American Statistical Association. 48(263), 565-599.
6. Berkson J. (1955). Maximum likelihood and minimum χ^2 estimates of the logistic function. Journal of the American Statistical Association. 50(269), 130-162.
7. Casella G., Berger R.L. (2001). Statistical Inference. Duxbury Press, California, USA.
8. Feldstein ML., Savlov ED., Hilf R. (1978). A statistical model for predicting response of breast cancer patients to cytotoxic chemotherapy. Cancer Research. 38, 2544-2548.
9. Filipsson AF., Sand S., Nilsson J. (2003). The benchmark dose method-review of available models, and recommendations for application in health risk assessment. Critical Reviews in Toxicology. 33(5), 505-542.
10 Ghosh JK., Sinha BK. (1981). A necessary and sufficient condition for second order admissibility with application to Berkson's bioassay problem. The Annals of Statistics. 9(6), 1334-1338.
11. Guess H., Crump K., Peto R. (1977). Uncertainty estimates for low-dose-rate extrapolations of animal carcinogenicity data. Cancer Research. 37, 3475-3483.
12. Harris RR., Kanji GK. (1983). On the use of minimum chi-square estimation. The Statistician. 32, 379-394.
13. Hitchcock SE. (1962). A note on the estimation of the logistic function, using the minimum logit χ^2 method. Biometrika. 49(1/2), 250-252.
14. http://www.nci.go.th/Knowledge/thaicancer.html

15. Portier C., Hoel D. Low-dose-rate extrapolation using the multistage model. Biometrics. 1983; 39(4), 897-906.
16. Rao CR. (1973). Linear statistical inference and its applications. John Wiley and Sons, Inc., New York, USA.
17. Sinha BK. (2007). On some aspects of Berkson's bioassay problem: issues and controversies. International Conference on Advances in Interdisciplinary Statistics and Combinatorics. October 12-14, 2007.
18. U.S. Environmental Protection Agency (USEPA), (1985). Estimation of lifetime cancer risks using a multistage theory of carcinogenesis. Internal review draft. U.S. Environmental Protection Agency, Washington, DC.
19. Wheeler MW, Bailer AJ, (2008). Model averaging software for dichotomous dose response risk estimation. Journal of statistical software, 26(5), 1-14.
20. National Toxicology Program (NTP), (1993). Toxicology and carcinogenesis studies of 1,3-Butadiene (CAS NO. 106-99-0) in B6C3F1 mice (Inhalation Studies), NTP TR 434.
21. Wei WS. Time series analysis "Univariate and Multivariate methods". Addison-Wesley Publishing Company, California; 1990.
22. Dobson AJ. An introduction to generalized linear models. Chapman & Hall, London; 1990.

7.5 Comparison of Experiments

We often define 'Statistics' as the 'Science' of collection, classification, summarization, organization, analysis and interpretation of data. Our focus in this study will be on the summarization aspect, the reduction of large amount of data into a set of 'summary measures' to get the effective results of analysis. These summary results are referred to as 'statistics'. The intriguing questions faced by statisticians relate to choice of appropriate summary statistics so as to summarize a given data set in a given context.

A statistic that summarizes an entire data set without losing any information about the family of distributions or the model is often called a sufficient statistic. Generally we like to use the statistic that provides the most information about the model. A very large data set can be rather unwieldy to work with. In this case, we can use sufficient statistic(s) that will provide the same information about the model.

A 'Statistical Experiment' results in a sample as a consequence of performing the experiment and, subsequently, we form a statistic based on the sample data. Two different experiments can yield two different statistics. Under certain conditions, we can compare these statistics by comparing the experiments.

The concept of sufficient experiment provides a method of comparing the effectiveness of certain experiments. Suppose we have two kinds of experiments $\mathcal{E}$ and $\mathcal{F}$. The outputs of these experiments have two different laws of distribution depending on the same parameter set. Then the experiment $\mathcal{E}$ is said to be 'Sufficient' for the other experiment $\mathcal{F}$ iff given an outcome on '$\mathcal{E}$', we can generate an outcome of '$\mathcal{F}$', *without actually performing the experiment* $\mathcal{F}$ (Blackwell, 1951).

Some examples of comparison of experiments are given as follows.

1. $\mathcal{E} \sim \text{Binomial}(n, \alpha p)$ VS $\mathcal{F} \sim \text{Binomial}(n, \beta p)$ where $\alpha > \beta > 0$ are known and p is an unknown parameter.
2. $\mathcal{E} \sim \text{Poisson}(\alpha\lambda)$ VS $\mathcal{F} \sim \text{Poisson}(\beta\lambda)$ where $\alpha > \beta > 0$ are known and λ is an unknown parameter.
3. $\mathcal{E} \sim N(\theta, \sigma^2)$ VS $\mathcal{F} \sim N(\theta, \sigma^2(1+\delta))$ when $\sigma^2 > 0$ and $\delta > 0$ are both known and θ is an unknown parameter. Under normality with the same mean, sufficiency of '$\mathcal{E}$' for '$\mathcal{F}$' works only when population variance under $\mathcal{E}$ < population variance under $\mathcal{F}$ and the difference is a known positive quantity.

   This subtopic is exclusively based on the concept of sufficiency as applied to statistical experiments. This is an old concept with many applications in the statistical literature. Yet, there are fascinating research problems and we have attempted to solve some of the problems in different areas or applications. The main focus of our research is towards:

1. Sufficiency in bivariate and trivariate normal populations.
2. Sufficiency in linear and quadratic regression models with/without intercept term.
3. Sufficiency in one-way random effects analysis of variance model.

Some important remarks concerning the models are also given which give us more insightful information about our research. This also provides some open questions which will provide guidance for further research.

Many useful results have been obtained by researchers on the comparison of various types of experiments. An overview of the results is given below.

In 1951 and 1953, Blackwell established a method for comparing two experiments which states that an experiment *X* is sufficient for another experiment *Y* with the same parameter space as *X* if there exists a stochastic transformation of *X* to a random variable *Z*(*X*) such that, all parameters belonging to the same parameter space, the random variables *Z*(*X*) and *Y* have identical distributions. He also mentioned that if the sufficiency holds when only one observation is taken in each experiment, then the same relation holds when a random sample of '*n*' observations is taken from each experiment. Further, if *X* is sufficient for *Y* and also the reverse sufficiency holds, we can say that *X* and *Y* are equivalent.

In 1954, Blackwell and Girshick showed that if $X = \left(X_1,...,X_r\right)$ and $Y = \left(Y_1,...,Y_r\right)$ are experiments such that X_i is sufficient for $Y_i\,(i = 1,...,r)$, then $(X_1,...,X_r)$ is sufficient for $(Y_1,...,Y_r)$.

In 1966, DeGroot defined the term 'allocation' and found some results that are useful in determining whether one of the two extreme allocations, in which one observation is made on '*n*' randomly selected *Z*'s or all '*n'* observations are made on a single randomly selected *Z*, is optimal. He also explained how the existence of complete, sufficient statistics under various allocations can be used to simplify the search for an optimal allocation. Further he presented some problems on the optimal allocation of Bernoulli observations.

In 1973, Sinha generalized the concept of comparison of experiments under the heading of 'Restricted Experiments' with restriction on k, the number of classes, and/or restriction on the sample space X to a proper subspace X_0 of X. He presented various results on the comparison of experiments based on the full space as against the subspaces. For populations which are symmetric about zero, it was pointed that only the transformations of the form $\left\{X' = \pm X \text{ with probability one}\right\}$ establish equivalence of the experiments $\mathcal{E}_i$'s considered. Further, he investigated that the converse of above statement also holds under certain mild restrictions on the family of distributions. Next, he established a connection between simple random sampling with replacement as against without replacement by showing that simple random sampling without replacement is sufficient for simple random sampling with replacement. This result is then used to show that stratified random sampling without replacement from each stratum is sufficient for the stratified random sampling with replacement.

In 1974, Hansen and Torgersen introduced the comparison of linear normal experiments. They considered independent and normally distributed random variables $X_1,...,X_n$ such that $0 < \text{Var}\left(X_i\right)=\sigma^2;\ i=1,...,n$ and $E\left(X_1,...,X_n\right)' = A'\beta$ where A' is a known $n\times k$ matrix and $\beta=\left(\beta_1,...,\beta_k\right)'$ is unknown column matrix. Further they considered the cases of known and totally unknown σ^2. Denoting the experiment obtained by observing $X_1,...,X_n$ as ε_A, then taking A and B matrices of, respectively, dimensions $n_A\times k$ and $n_B\times k$, they proved that if σ^2 is known, (if σ^2 is unknown) ε_A is more informative than ε_B if and only if $AA'-BB'$ is nonnegative definite (and $n_A \ge n_B + \text{rank}(AA'-BB')$).

In 1979, Goel and DeGroot showed that for a scale parameter θ, and $k_1 > k_2 > 0$, the experiment with parameter θ^{k_1} is proved to be sufficient for the experiment with parameter θ^{k_2} on a class of distributions including gamma and normal distributions with known mean. They proposed a concept of sufficiency in which ε_X is more informative than ε_Y for a fixed prior distribution of θ if the expected Bayes risk from ε_X is not greater than ε_Y for every decision problems involving θ. He used this concept to develop a definition of marginal Bayesian sufficiency in the presence of nuisance parameters.

In 1982, Stepniak presented some complements to the result of DeGroot (1966). One of them concerns optimal allocation of the observations when the number of subclasses is fixed. He showed that the most uniform allocation of the observations in subclasses is optimal. The second problem involved the case when one of the variance components is unknown. He gave some results connected with the choice of an allocation in this situation. However, an optimal allocation exists. He included some remarks on allocations of the observations when both variance components are unknown. It appears that there are no two essentially different allocations, i.e. such allocations that one of them cannot be obtained from the other by a denumeration of subclasses, are comparable.

In 1988, Lehmann reviewed the classical theory of the comparison of two experiments with particular reference to the comparison of two location experiments. It is shown that the requirement of domination of one experiment by another for all decision problems is too strong to provide a reasonable basis for comparison. For one - parameter problems with monotone likelihood ratio, it is therefore proposed to restrict the comparison to decision problems that are monotone in the sense of Karlin and Rubin (1956). Application of this weaker definition to the location problem is shown to give satisfactory results. It is also explored that a scale – free comparison of this type leads to a new tail – ordering of distributions.

In 1992, Shaked and Tong presented the comparison of experiments via the positive dependence of normal variables with common univariate marginal distribution. They showed that positive dependence has an adverse effect on the information concerning the common mean θ, and give a partial ordering of the information via a majorization ordering of the correlation matrices. In a special case when the random variables are equally correlated, the main theorem yields a result for the comparison of experiments for permutation symmetric normal variables.

In 1993, Shaked and Tong reviewed some works on comparison of experiments of some multivariate distributions. They showed that positively dependent random variables contain less information than independent random variables. They described some results regarding the comparison of experiments of exchangeable and nonexchangeable normal random vectors. In particular, they showed how the majorization ordering can be used to identify various information orderings of multivariate normal random vectors which have a common marginal density.

In 1996, Greenshtein described a generalization for the theory of comparison of experiments to the case of sequential experiments. He investigated the case of ‘0’ deficiency and some applications are given to the case of exponential experiments.

In 1997, Stepniak considered X and Y as observation vectors in normal linear experiments, $\mathcal{E} = N\left(A\beta, \sigma V\right)$ and $\mathcal{F} = N\left(B\beta, \sigma W\right)$, respectively. He proved that $\mathcal{E}$ is sufficient for $\mathcal{F}$, i.e. $\mathcal{E} \succ \mathcal{F}$ if for any quadratic form $Y'GY$ there exists a quadratic form $X'HX$ such that $E\left(X'HX\right) = E\left(Y'GY\right)$ and $\mathrm{Var}\left(X'HX\right) \le \mathrm{Var}\left(Y'GY\right)$. The relation $\succ$ is characterized by the matrices *A*, *B*, *V* and *W*. Moreover he introduced some connections with known orderings of linear experiments.

Many concepts about comparison of experiments are given in Le Cam (1996) and Goel and Ginebra (2003).

5.1 Theoretical Background

Suppose $X = \{\mathcal{X}, \mathcal{A}\ ; P_\theta\ , \theta \in \Omega\}$ and $Y = \{\mathcal{Y}, \mathcal{B}\ ; Q_\theta\ , \theta \in \Omega\}$ are two experiments with the same parameter space Ω. The concept of *X* being sufficient for *Y* has been precisely defined by Blackwell (1951, 1953) and redefined by DeGroot (1966), Sinha (1973). Roughly speaking, an experiment involving the observation of the random variable *X* is sufficient for another experiment involving the observation of the random variable *Y* if it is possible, from an observation on *X* and an auxiliary known randomization, to generate a random variable with the same distribution as *Y* for all possible values of any unknown parameters.

We restate the concept of sufficient experiments in the following two definitions due to Blackwell (1951, 1953).

Suppose we have two experiments $X = \{\mathcal{X}, \mathcal{A}\ ; P_\theta\ , \theta \in \Omega\}$ and $Y = \{\mathcal{Y}, \mathcal{B};\ Q_\theta\ , \theta \in \Omega\}$ with the same parameter space Ω.

(i) A stochastic transformation from *X* to *Y* is a non-negative function $\pi(.|.)$ defined on $\mathcal{B} \times \mathcal{X}$ such that for each fixed $x \in \mathcal{X}$, $\pi(.|x)$ is a probability measure on $\mathcal{B}$ and for each fixed $B \in \mathcal{B}$, $\pi(B|.)$ is an $\mathcal{A}$-measurable function on $\mathcal{X}$.

(ii) The experiment *X* is sufficient for the experiment *Y* if there exists a stochastic transformation $\pi(.|.)$ from *X* to *Y* such that

$$Q_\theta(B) = \int_{\mathcal{X}} \pi(B|x)\,dP_\theta(x),\ \text{ for all } B \in \mathcal{B} \text{ and } \theta \in \Omega.$$

Information about an unknown correlation parameter (ρ) from bivariate normal distribution, $(X,Y) \sim N_2(0,0,1,1,\rho)$ is given by

$$I(\rho)|(X,Y) = E\left[\left\{\frac{d}{d\rho}\left(\log f(x,y)\right)\right\}^2\right] = \frac{1+\rho^2}{\left(1-\rho^2\right)^2},$$

where probability density function (p.d.f.) of (*X*, *Y*) is given by (Casella and Berger, 2001)

$$f(x,y)= \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(y^2-2\rho xy+x^2\right)\right].$$

Model specification

Some bivariate normal experiments:

Firstly we assume a pair of bivariate normal experiments, $\varepsilon_1:\left(X_1,X_2\right)\sim\ N_2\left(0,0,1,1,\rho\right)$ and $\varepsilon_2:\left(Y_1,Y_2\right)\ \sim\ N_2\left(0,0,1,1,c\rho\right)$ for all $c>1$, where ρ is an unknown parameter.

Then we assume another pair of bivariate normal experiments, $\varepsilon_1:\left(X_1,X_2\right)$ $\sim N_2\left(0,0,1,1,c_1\rho\right)$ and $\varepsilon_2:\left(Y_1,Y_2\right)\sim\ N_2\left(0,0,1,1,c_2\rho\right)$ for all $c_2>c_1>0$, where ρ is an unknown parameter.

Next we assume a third pair of bivariate normal experiments $\varepsilon_1:\left(X_1,X_2\right)\sim$ $N_2\left(m_1,m_2,\sigma_1^2\ ,\sigma_2^2\ ,c_1\rho\right)$ and $\varepsilon_2:\left(Y_1,Y_2\right)\sim\ N_2\left(\theta_1,\theta_2,\delta_1^{\,2},\delta_2^{\,2},c_2\rho\right)$ where all the parameters are known except ρ with $c_2>c_1>0$.

Finally we assume a fourth pair of bivariate normal experiments $\varepsilon_1:\left(X,Y\right)\sim$ $N_2\left(0,0,1,1,\rho\right)$ and $\varepsilon_2:\left(X^*,Y^*\right)\sim N_2\left(0,0,1,1,h(\rho)\right)$ with an unknown parameter ρ. Here we have to characterize the function of correlation parameter (ρ), i.e. $h(\rho)$ such that $I\left(\rho\right)|\,\varepsilon_2\geq I\left(\rho\right)|\,\varepsilon_1$ for all ρ where $I\left(\rho\right)|\,\varepsilon_1$ is an information about ρ from ε_1 and $I\left(\rho\right)|\,\varepsilon_2$ is information about ρ from ε_2. Some particular cases are given to address the above condition through characterization of $h(\rho)$.

Some trivariate normal experiments:

First, we consider a pair of trivariate normal experiments,

$$\varepsilon_1:\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}\sim N_3\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1&\rho&\rho\\\rho&1&\rho\\\rho&\rho&1\end{pmatrix}\right)$$

and $$\varepsilon_2:\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}\sim N_3\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1&c\rho&c\rho\\c\rho&1&c\rho\\c\rho&c\rho&1\end{pmatrix}\right)$$

for all $c > 1$, where ρ is an unknown parameter.

Then we study another pair of trivariate normal experiments with equal pair-wise correlation,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho & c_1\rho \\ c_1\rho & 1 & c_1\rho \\ c_1\rho & c_1\rho & 1 \end{pmatrix}\right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho & c_2\rho \\ c_2\rho & 1 & c_2\rho \\ c_2\rho & c_2\rho & 1 \end{pmatrix}\right),$$

for all $c_2 > c_1 > 0$, where ρ is an unknown parameter.

Next we assume another model of two trivariate normal experiments with equal pair-wise correlation,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & c_1\rho\sigma_1\sigma_2 & c_1\rho\sigma_1\sigma_3 \\ c_1\rho\sigma_1\sigma_2 & \sigma_2^2 & c_1\rho\sigma_2\sigma_3 \\ c_1\rho\sigma_1\sigma_3 & c_1\rho\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}\right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \begin{pmatrix} \delta_1^2 & c_2\rho\delta_1\delta_2 & c_2\rho\delta_1\delta_3 \\ c_2\rho\delta_1\delta_2 & \delta_2^2 & c_2\rho\delta_2\delta_3 \\ c_2\rho\delta_1\delta_3 & c_2\rho\delta_2\delta_3 & \delta_3^2 \end{pmatrix}\right),$$

where all the parameters are known except ρ with $c_2 > c_1 > 0$.

Then we assume different models of two trivariate normal experiments with different pair-wise correlation,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}\right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c\rho_1 & c\rho_2 \\ c\rho_1 & 1 & c\rho_3 \\ c\rho_2 & c\rho_3 & 1 \end{pmatrix}\right),$$

for all $c > 1$, where all ρ_i's are unknown parameters.

Next we assume another model of two trivariate normal experiments,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & c_1\rho_1\sigma_1\sigma_2 & c_1\rho_2\sigma_1\sigma_3 \\ c_1\rho_1\sigma_1\sigma_2 & \sigma_2^2 & c_1\rho_3\sigma_2\sigma_3 \\ c_1\rho_2\sigma_1\sigma_3 & c_1\rho_3\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix} \right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \begin{pmatrix} \delta_1^2 & c_2\rho_1\delta_1\delta_2 & c_2\rho_2\delta_1\delta_3 \\ c_2\rho_1\delta_1\delta_2 & \delta_2^2 & c_2\rho_3\delta_2\delta_3 \\ c_2\rho_2\delta_1\delta_3 & c_2\rho_3\delta_2\delta_3 & \delta_3^2 \end{pmatrix} \right),$$

where all the parameters are known except all ρ_i's with $c_2 > c_1 > 0$.

Finally we assume a model of two trivariate normal experiments,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix} \right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho_1 & c_2\rho_2 \\ c_1\rho_1 & 1 & c_3\rho_3 \\ c_2\rho_2 & c_3\rho_3 & 1 \end{pmatrix} \right),$$

where ρ_1, ρ_2, ρ_3 are unknown but $c_1, c_2, c_3 > 1$ are all known with $c_1 < c_2 c_3$, $c_2 < c_1 c_3$ and $c_3 < c_1 c_2$.

Simple linear regression model

We assume a simple linear regression model, $Y_x = \alpha + \beta x + e_x$, where Y_x is dependent variable, $x \in [-1,1]$ is independent variable(non-stochastic), $e_x \sim N\left(0, \sigma^2\right)$ is error term, σ^2 is known and α, β are unknown parameters. If $x \in [a,b]$ with $a < b$, then $u = 2\left(\frac{x-a}{b-a}\right) - 1 \in [-1,1]$. So, without any loss of generality, we have assumed all throughout this chapter that $x \in [-1,1]$.

Quadratic regression model with intercept term

We assume a quadratic regression model, $Y_x = \alpha + \beta x + \gamma x^2 + e_x$, where Y_x is dependent variable, $x \in [-1,1]$ is independent variable(non-stochastic), $e_x \sim N\left(0, \sigma^2\right)$, σ^2 is known and α, β, γ are unknown parameters.

Quadratic regression model without intercept term

We assume a quadratic regression model, $Y_x = \beta x + \gamma x^2 + e_x$, where Y_x is dependent variable, $x \in [-1,1]$ is independent variable(non-stochastic), $e_x \sim N\left(0,\sigma^2\right)$, σ^2 is known and β, γ are unknown parameters.

One-way analysis of variance random effects model:

We assume a one-way ANOVA random effects model, $Y_{ij} = \mu + d_i + e_{ij}$, $i = 1,...,k;\ j = 1,...,n_i; \sum_i n_i = n$, where Y_{ij} is variable of our study with parameters μ, σ_d^2, $\sigma_e{}^2$, and μ is the population mean, $d_i \sim N(0,\sigma_d^2)$ represents variability between treatments and $e_{ij} \sim N(0,\sigma_e{}^2)$ represents variability within treatments and d_i and e_{ij} are independent of each other for all *i* and *j*.

5.2 Sufficiency in bivariate and trivariate normal populations

Bivariate normal populations

A random sample (*X*, *Y*) has the bivariate normal distribution in the form $N_2\left(0,0,1,1,\rho\right)$ if p.d.f. is written as

$$f\left(x,y\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(y^2 - 2\rho xy + x^2\right)\right]. \tag{5.2.1}$$

Comparison by sufficiency

We can compare bivariate normal experiments in terms of sufficiency. A bivariate normal experiment generating a random vector $\left(Y_1,Y_2\right)$ is sufficient for another bivariate normal experiment generating a different random vector $\left(X_1, X_2\right)$ if with random vector $\left(Y_1,Y_2\right)$ and an auxiliary randomization, we can generate another random vector $Z\left(Y_1,Y_2\right)$ which has the same distribution as the random vector $\left(X_1, X_2\right)$. Some comparisons of bivariate normal experiments with theorems and remarks are given as follows:

Theorem 5.2.1 Given two bivariate normal experiments, $\varepsilon_1 : (X_1,\ X_2) \sim N_2(0,0,1,1,\rho)$ and $\varepsilon_2 : (Y_1, Y_2) \sim N_2(0,0,1,1,c\rho)$ where $c > 1$ is a known scalar and ρ is an unknown parameter, it follows that ε_2 is sufficient for ε_1.

Proof: Given a bivariate normal experiment $\varepsilon_2 : (Y_1, Y_2) \sim N_2(0,0,1,1,c\rho)$.

Let $Y_1^* = \frac{Y_1 + \sqrt{c-1}\,U}{\sqrt{c}}$ and $Y_2^* = \frac{Y_2 + \sqrt{c-1}V}{\sqrt{c}}$, where $U \sim N(0,1)$ and $V \sim N(0,1)$ are independent and identically distributed(i.i.d.) and independent of Y_1, Y_2.

Here $E(Y_1^*) = 0$, $E(Y_2^*) = 0$, $\mathrm{Var}(Y_1^*) = 1$ and $\mathrm{Var}(Y_2^*) = 1$. The correlation between Y_1^* and Y_2^* is obtained by

$$
\begin{aligned}
\mathrm{Corr}(Y_1^*, Y_2^*) &= \frac{\mathrm{Cov}(Y_1^*, Y_2^*)}{\sqrt{\mathrm{Var}(Y_1^*)}\sqrt{\mathrm{Var}(Y_2^*)}} \\
&= \mathrm{Cov}(Y_1^*, Y_2^*) \\
&= \mathrm{Cov}\left(\frac{Y_1 + \sqrt{c-1}\,U}{\sqrt{c}}, \frac{Y_2 + \sqrt{c-1}\,V}{\sqrt{c}}\right) \\
&= \frac{1}{\mathrm{c}}\mathrm{Cov}(Y_1, Y_2) \\
&= \frac{1}{\mathrm{c}}\mathrm{Corr}(Y_1, Y_2) \\
&= \rho .
\end{aligned}
$$

Hence $(Y_1^*, Y_2^*) \sim N_2(0,0,1,1,\rho)$.

Thus from the given bivariate normal experiment $\varepsilon_2 : (Y_1, Y_2) \sim N_2(0,0,1,1,c\rho)$, we can generate the output of another bivariate normal experiments (Y_1^*, Y_2^*) which has the same distribution as ε_1 of (X_1, X_2). Hence ε_2 is sufficient for ε_1.

The next theorem compares two bivariate normal experiments with different correlation parameters.

Theorem 5.2.2 Given two bivariate normal experiments, $\varepsilon_1 : (X_1, X_2) \sim N_2(0,0,1,1,c_1\rho)$ and $\varepsilon_2 : (Y_1, Y_2) \sim N_2(0,0,1,1,c_2\rho)$, where $c_2 > c_1 > 0$ are two known scalars, and ρ is an unknown parameter, it follows that ε_2 is sufficient for ε_1.

Proof: Start with a bivariate normal experiment $\varepsilon_2 : (Y_1, Y_2) \sim N_2(0,0,1,1,c_2\rho)$.

Let $Y_1^* = \dfrac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}$ and $Y_2^* = \dfrac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}\,V}{\sqrt{c_2}}$, where $U \sim N(0,1)$ and $V \sim N(0,1)$ are i.i.d. and independent of Y_1, Y_2.

Here $E(Y_1^*) = 0$, $E(Y_2^*) = 0$, $\mathrm{Var}(Y_1^*) = 1$ and $\mathrm{Var}(Y_2^*) = 1$. The correlation between Y_1^* and Y_2^* is obtained by

$$
\begin{aligned}
\mathrm{Corr}(Y_1^*, Y_2^*) &= \frac{\mathrm{Cov}(Y_1^*, Y_2^*)}{\sqrt{\mathrm{Var}(Y_1^*)}\sqrt{\mathrm{Var}(Y_2^*)}} \\
&= \mathrm{Cov}(Y_1^*, Y_2^*) \\
&= \mathrm{Cov}\left(\frac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}, \frac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}\,V}{\sqrt{c_2}}\right) \\
&= \frac{c_1}{\mathrm{c}_2}\mathrm{Cov}(Y_1, Y_2) \\
&= \frac{c_1}{\mathrm{c}_2}\mathrm{Corr}(Y_1, Y_2) \\
&= c_1\rho .
\end{aligned}
$$

Therefore, $(Y_1^*, Y_2^*) \sim N_2(0,0,1,1,c_1\rho)$.

Thus from the given bivariate normal experiment, $\varepsilon_2 : (Y_1, Y_2) \sim N_2(0,0,1,1,c_2\rho)$, we can generate the output of another bivariate normal experiment (Y_1^*, Y_2^*) which has the same distribution as ε_1 of (X_1, X_2). Hence ε_2 is sufficient for ε_1.

Corollary 5.2.2.1 Given two bivariate normal experiments, $\varepsilon_1 : (X_1, X_2) \sim N_2(m_1, m_2, \sigma_1^2, \sigma_2^2, c_1\rho)$ and $\varepsilon_2 : (Y_1, Y_2) \sim N_2(\theta_1, \theta_2, \delta_1^2, \delta_2^2, c_2\rho)$, where all the parameters are known except ρ with $c_2 > c_1 > 0$, it follows that ε_2 is sufficient for ε_1.

Proof: Given a bivariate normal experiment $\varepsilon_1 : (X_1, X_2) \sim N_2(m_1, m_2, \sigma_1^2, \sigma_2^2, c_1\rho)$.

Let $X_1^* = \dfrac{X_1 - m_1}{\sigma_1}$ and $X_2^* = \dfrac{X_2 - m_2}{\sigma_2}$.

Here $E(X_1^*) = 0$, $E(X_2^*) = 0$, $\mathrm{Var}(X_1^*) = 1$ and $\mathrm{Var}(X_2^*) = 1$. The correlation between X_1^* and X_2^* is obtained by

$\mathrm{Corr}(X_1^*, X_2^*) = \mathrm{Corr}(X_1, X_2) = c_1\rho$.

Hence $\varepsilon_3 : (X_1^*, X_2^*) \sim N_2(0,0,1,1,c_1\rho)$.

Therefore, ε_3 is equivalent to ε_1.

Given a bivariate normal experiment $\varepsilon_2 : (Y_1, Y_2) \sim N_2(\theta_1, \theta_2, \delta_1^2, \delta_2^2, c_2\rho)$.

Let $Y_1^* = \dfrac{Y_1 - \theta_1}{\delta_1}$ and $Y_2^* = \dfrac{Y_2 - \theta_2}{\delta_2}$.

Here $E(Y_1^*) = 0$, $E(Y_2^*) = 0$, $\mathrm{Var}(Y_1^*) = 1$ and $\mathrm{Var}(Y_2^*) = 1$. The correlation between Y_1^* and Y_2^* is obtained by

$\mathrm{Corr}(Y_1^*, Y_2^*) = \mathrm{Corr}(Y_1, Y_2) = c_2\rho$.

Hence $\varepsilon_4 : (Y_1^*, Y_2^*) \sim N_2(0,0,1,1,c_2\rho)$.

Therefore, ε_4 is equivalent to ε_2.

But from Theorem 5.2.2, we have $\varepsilon_4 \succ \varepsilon_3$.

Hence we can say that ε_2 is sufficient for ε_1.

Comparison in terms of information about ρ

The Fisher information or simply called information, $I(\ldots)$ is a way of measuring the amount of information that an observable random vector (*X*, *Y*) carries about an unknown parameter θ upon which the likelihood function of θ, $L(\theta) = f((X,Y);\theta)$, depends.

Information about an unknown parameter correlation (ρ) from bivariate normal random vector (*X*, *Y*) having bivariate normal distribution with the form $N_2(0,0,1,1,\rho)$ is given by

$$I(\rho)|(X,Y) = E\left[\left\{\frac{d}{d\rho}\left(\log f(x,y)\right)\right\}^2\right], \qquad (5.2.2)$$

where $f(x,y)$ is p.d.f. of bivariate normal random variable (*X*, *Y*). Some comparisons of bivariate normal experiments in terms of information about rho (ρ) with some particular cases are given as the following theorem.

Theorem 5.2.3 Given two bivariate normal experiments, $\varepsilon_1 : (X,Y) \sim N_2(0,0,1,1,\rho)$ and $\varepsilon_2 : (X^*,Y^*) \sim N_2(0,0,1,1,h(\rho))$, where $h(\rho)$ is the function of unknown parameter rho (ρ). The following characterization of the function $h(\rho)$ ensures $I(\rho)|\varepsilon_2 \geq I(\rho)|\varepsilon_1$, where $I(\rho)|\varepsilon_2$ is an information about correlation parameter (ρ) from ε_2 and $I(\rho)|\varepsilon_1$ is an information about the correlation parameter (ρ) from ε_1,

$$\frac{\left(1+h^2(\rho)\right)}{\left(1-h^2(\rho)\right)^2}\left\{h'(\rho)\right\}^2 \geq \frac{\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2}.$$

Proof: Let $I(\rho)|\varepsilon_1$ be an information about correlation parameter (ρ) from $\varepsilon_1 : (X,Y) \sim N_2(0,0,1,1,\rho)$.

The p.d.f. of (X, Y) is given by

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} Exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(y^2 - 2\rho xy + x^2\right)\right]$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} Exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\left(y-\rho x\right)^2 + x^2\left(1-\rho^2\right)\right)\right]$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} Exp\left[-\frac{1}{2}\left(\left\{\frac{y-\rho x}{\sqrt{\left(1-\rho^2\right)}}\right\}^2 + x^2\right)\right].$$

The logarithm of $f(x,y)$ is given as

$$\log f(x,y) = -\frac{1}{2}\log\left(1-\rho^2\right) - \frac{\left(y-\rho x\right)^2}{2\left(1-\rho^2\right)} - \log\left(2\pi\right) - \frac{1}{2}x^2.$$

The first derivative of $\log f(x,y)$ with respect to ρ is obtained by

$$\frac{d}{d\rho}\{\log f(x,y)\} = -\frac{(-2\rho)}{2\left(1-\rho^2\right)} - \frac{1}{2}\left[\frac{\left(1-\rho^2\right).2\left(y-\rho x\right).(-x)}{\left(1-\rho^2\right)^2} - \frac{\left(y-\rho x\right)^2.(-2\rho)}{\left(1-\rho^2\right)^2}\right]$$

$$= \frac{\rho}{\left(1-\rho^2\right)} + \frac{x\left(y-\rho x\right)}{\left(1-\rho^2\right)} - \frac{\rho\left(y-\rho x\right)^2}{\left(1-\rho^2\right)^2}$$

$$= \frac{\rho}{\left(1-\rho^2\right)} + \frac{\left(y-\rho x\right)}{\left(1-\rho^2\right)}\left[x - \frac{\rho\left(y-\rho x\right)}{\left(1-\rho^2\right)}\right]$$

$$= \frac{\rho}{\left(1-\rho^2\right)} + \frac{\left(y-\rho x\right)}{\left(1-\rho^2\right)}\left[\frac{x - x\rho^2 - \rho y + x\rho^2}{\left(1-\rho^2\right)}\right]$$

$$= \frac{\rho}{\left(1-\rho^2\right)} + \frac{\left(y-\rho x\right)\left(x-\rho y\right)}{\left(1-\rho^2\right)^2}.$$

Squaring both sides, we get

$$\left[\frac{d}{d\rho}\{\log f(x,y)\}\right]^2$$

$$= \left[\frac{\rho}{\left(1-\rho^2\right)} + \frac{x\left(y-\rho x\right)}{\left(1-\rho^2\right)} - \frac{\rho\left(y-\rho x\right)^2}{\left(1-\rho^2\right)^2}\right]^2$$

$$= \frac{\rho^2}{\left(1-\rho^2\right)^2} + \frac{\rho^2\left(y-\rho x\right)^4}{\left(1-\rho^2\right)^4} + \frac{x^2\left(y-\rho x\right)^2}{\left(1-\rho^2\right)^2} - 2\frac{\rho^2\left(y-\rho x\right)^3}{\left(1-\rho^2\right)^3} - 2\frac{\rho x\left(y-\rho x\right)^3}{\left(1-\rho^2\right)^3} + 2\frac{\rho x\left(y-\rho x\right)}{\left(1-\rho^2\right)^2}.$$

Taking the expectation on both sides, we get

$$E\left[\frac{d}{d\rho}\{\log f(x,y)\}\right]^2$$

$$= E\left[\frac{\rho^2}{\left(1-\rho^2\right)^2} + \frac{\rho^2\left(y-\rho x\right)^4}{\left(1-\rho^2\right)^4} + \frac{x^2\left(y-\rho x\right)^2}{\left(1-\rho^2\right)^2} - 2\frac{\rho^2\left(y-\rho x\right)^2}{\left(1-\rho^2\right)^3} - 2\frac{\rho x\left(y-\rho x\right)^3}{\left(1-\rho^2\right)^3} + 2\frac{\rho x\left(y-\rho x\right)}{\left(1-\rho^2\right)^2}\right]$$

$$= \frac{\rho^2}{\left(1-\rho^2\right)^2} + \frac{\rho^2}{\left(1-\rho^2\right)^4}.3\left(1-\rho^2\right)^2 + \frac{1}{\left(1-\rho^2\right)} - 2\frac{\rho^2\left(1-\rho^2\right)}{\left(1-\rho^2\right)^3} - 0 + 0$$

$$= \frac{2\rho^2}{\left(1-\rho^2\right)^2} + \frac{1}{\left(1-\rho^2\right)}$$

$$= \frac{2\rho^2 + 1 - \rho^2}{\left(1-\rho^2\right)^2}$$

$$= \frac{\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2}.$$

Hence, $I(\rho)\,|\,\varepsilon_1$, an information about correlation parameter (ρ) from

$\varepsilon_1 : (X, Y) \sim N_2(0,0,1,1,\rho)$ is $\frac{\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2}$.

Similarly, $I(\rho)\,|\,\varepsilon_2$ is an information about correlation parameter (ρ) from

$\varepsilon_2 : \left(X^*, Y^*\right) \sim N_2\left(0,0,1,1,h(\rho)\right)$.

Let $\rho^* = h(\rho)$.

Differentiating both sides with respect to ρ, we get $\frac{d\rho^*}{d\rho} = h^{'}(\rho)$, i.e.

$$\frac{d}{d\rho}\left\{\log f\left(x^*, y^*\right)\right\} = \frac{d}{d\rho^*}\left\{\log f\left(x^*, y^*\right)\right\}.\frac{d\rho^*}{d\rho} = h^{'}(\rho)\frac{d}{d\rho^*}\left\{\log f\left(x^*, y^*\right)\right\}.$$

From $\varepsilon_2 : \left(X^*, Y^*\right) \sim N_2\left(0,0,1,1,\rho^*\right)$, the information about ρ^* is obtained as

$$I(\rho^*) = E\left[\frac{d}{d\rho^*}\left\{\log f\left(x^*, y^*\right)\right\}\right]^2$$

$$= \frac{\left\{1+\left(\rho^*\right)^2\right\}}{\left\{1-\left(\rho^*\right)^2\right\}^2}.$$

Therefore, $I(\rho)\,|\,\varepsilon_2 \quad = E\left[\frac{d}{d\rho}\left\{\log f\left(x^*, y^*\right)\right\}\right]^2$

$$= E\left[h'(\rho)\frac{d}{d\rho^*}\left\{\log f\left(x^*, y^*\right)\right\}\right]^2$$

$$= \frac{\left\{1+\left(\rho^*\right)^2\right\}}{\left\{1-\left(\rho^*\right)^2\right\}^2}\left\{h^{'}(\rho)\right\}^2$$

$$= \frac{\left(1+h^2(\rho)\right)}{\left(1-h^2(\rho)\right)^2}\left\{h^{'}(\rho)\right\}^2.$$

Hence, we have to characterize a function $h(\rho)$ such that

$$I\left(\rho\right)|\,\varepsilon_2 \geq I\left(\rho\right)|\,\varepsilon_1$$

$$\frac{\left(1+h^2(\rho)\right)}{\left(1-h^2(\rho)\right)^2}\left\{h^{'}(\rho)\right\}^2 \geq \frac{\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2}. \tag{5.2.3}$$

Next we consider some particular cases of the function of unknown correlation parameter (ρ), i.e. $h(\rho)$ which follows the above condition in equation (5.2.3).

1. $h(\rho) = c\rho, c \geq 1$

Applying $h(\rho)$ in equation (5.2.3), we get

$$\frac{\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2} \leq \frac{\left(1+c^2\rho^2\right)}{\left(1-c^2\rho^2\right)^2}.c^2$$

$$\left(1+\rho^2\right)\left(1-c^2\rho^2\right)^2 \leq \left(1-\rho^2\right)^2\left(1+c^2\rho^2\right)c^2.$$

Let $\rho^2 = t$, then we obtain

$$\left(1+t\right)\left(1-c^2t\right)^2 \leq \left(1-t\right)^2\left(1+c^2t\right)c^2$$

$$\left(1+t\right)\left(1-2c^2t+c^4t^2\right) \leq \left(1-2t+t^2\right)\left(c^2+c^4t\right)$$

$$1-2c^2t+c^4t^2+t-2c^2t^2+c^4t^3 \leq c^2-2c^2t+c^2t^2+c^4t-2c^4t^2+c^4t^3$$

$$1+t+3c^4t^2 \leq c^2+c^4t+3c^2t^2$$

$$c^2\left(1+c^2t+3t^2\right) \geq 1+t+3c^4t^2$$

$$c^2\left(1+c^2t+3t^2\right)-\left(1+t+3c^4t^2\right) \geq 0$$

$$\left(c^2-1\right)\left\{1+t\left(c^2-1\right)-3c^2t^2\right\} \geq 0$$

$$1+t\left(c^2-1\right)-3c^2t^2 \geq 0$$

Let $\psi(t) = 1 + t(c^2 - 1) - 3c^2t^2$.

For maxima and minima, we take the first derivative of $\psi(t)$ equal to zero.

Therefore, $\psi'(t) = 0$ which gives us $t = \frac{c^2+1}{6c^2}$.

Taking second derivative of $\psi(t)$, we get $\psi''(t) < 0$.

Hence, $\psi(t)$ has maximum value at $t = \frac{c^2+1}{6c^2}$.

But the value of 't' lies between 0 and $\frac{1}{c^2}$.

Then, $\psi(t=0) = 1 > 0$ and $\psi\left(t = \frac{1}{c^2}\right) = \frac{2(c^2-1)}{c^2} > 0$, since $c > 1$.

Therefore, $\psi(t) \geq 0,\ \forall\ t \in \left(0, \frac{1}{c^2}\right)$ and $\forall\, c \geq 1$.

Hence we can conclude that (5.2.3) holds true for $h(\rho) = c\rho,\quad c \geq 1$ only for $t \in \left(0, \frac{1}{c^2}\right)$, where $\rho^2 = t$. Therefore $I(\rho)\,|\,N_2(0,0,1,1,\rho) \leq I(\rho)\,|\,N_2(0,0,1,1,c\rho),\quad c \geq 1,$ only for $t \in \left(0, \frac{1}{c^2}\right)$, where $\rho^2 = t$.

2. $h(\rho) = \sqrt{\rho}$

Applying $h(\rho)$ in equation (5.2.3), we get

$$\frac{(1+\rho^2)}{(1-\rho^2)^2} \leq \frac{(1+\rho)}{(1-\rho)^2}.\frac{1}{4\rho}$$

$$4\rho(1+\rho^2) \leq (1+\rho)^3$$

$$4\rho + 4\rho^3 \leq 1 + 3\rho + 3\rho^2 + \rho^3$$

$$3\rho^3 + \rho \leq 1 + 3\rho^2$$

$$\rho(1+3\rho^2) \leq (1+3\rho^2)$$

$\rho \leq 1$, which is true. Therefore, we can say that

$I(\rho) \mid N_2(0,0,1,1,\rho) \le I(\rho) \mid N_2\left(0,0,1,1,\sqrt{\rho}\right).$

3. $h(\rho) = \rho^s, 0 < s \le 1$

Applying $h(\rho)$ in equation (5.2.3), we get

$$\frac{\left(1+\rho^2\right)}{\left(1-\rho^2\right)^2} \le \frac{\left(1+\rho^{2s}\right)}{\left(1-\rho^{2s}\right)^2}.s^2.\rho^{2(s-1)}$$

Let $\rho^2 = t$, then

$$\frac{\left(1+t\right)}{\left(1-t\right)^2} \le \frac{\left(1+t^s\right)}{\left(1-t^s\right)^2}.s^2.t^{(s-1)}$$

$$\frac{\left(1+t\right)t}{\left(1-t\right)^2} \le \frac{\left(1+t^s\right)}{\left(1-t^s\right)^2}.s^2.t^s$$

$$\frac{\left(1+t\right)t}{\left(1-t\right)^2} \le \left(\frac{2}{1-t^s}-1\right)\left(\frac{1}{1-t^s}-1\right).s^2. \tag{5.2.4}$$

Here for s =1, LHS is equal to RHS, and also for fixed ‘t’, if we show the above right function is decreasing in s, the above inequality will be true.

Let $g(s) = \left(\frac{2}{1-t^s}-1\right)\left(\frac{1}{1-t^s}-1\right).s^2$.

Then, the first derivative of $g(s)$ with respect to s is given as

$$g'(s) = \frac{2t^s \log t}{\left(1-t^s\right)^2}\left(\frac{1}{1-t^s}-1\right).s^2 + \frac{t^s \log t}{\left(1-t^s\right)^2}\left(\frac{2}{1-t^s}-1\right).s^2 + 2s\left(\frac{2}{1-t^s}-1\right)\left(\frac{1}{1-t^s}-1\right).$$

If $g(s)$ is decreasing in s, we must have $g'(s) \le 0$, hence we get

$$\frac{2t^s \log t}{\left(1-t^s\right)^2}\left(\frac{1}{1-t^s}-1\right).s^2 + \frac{t^s \log t}{\left(1-t^s\right)^2}\left(\frac{2}{1-t^s}-1\right).s^2 + 2s\left(\frac{2}{1-t^s}-1\right)\left(\frac{1}{1-t^s}-1\right) \le 0$$

$$\frac{2t^s \log t}{\left(1-t^s\right)^2}\left(\frac{1}{1-t^s}-1\right).s^2 + \frac{t^s \log t}{\left(1-t^s\right)^2}\left(\frac{2}{1-t^s}-1\right).s^2 + 2s\left(\frac{2}{1-t^s}-1\right)\left(\frac{1}{1-t^s}-1\right) \le 0$$

$$\frac{2st^s\left(1+t^s\right)}{\left(1-t^s\right)^2} \le \left(-\log t\right)t^s s^2\left[\frac{2t^s}{\left(1-t^s\right)^3} + \frac{1+t^s}{\left(1-t^s\right)^3}\right]$$

$2\left(1-t^{2s}\right) \le \left(-\log t\right) s\left(1+3t^{s}\right)$.

Let $t^s = e^{\lambda}$, $-\infty \le \lambda < 0$ so that $s\log t = \lambda$.

Hence from above inequality, we get

$2\left(1-e^{2\lambda}\right) \le -\lambda\left(1+3e^{\lambda}\right)$.

Let $-\lambda = u,\ 0 \le u < \infty$, then we obtain

$$2 \le 2e^{-2u} + u\left(1+3e^{-u}\right)$$

$$2e^{2u} \le 2 + u\left(e^{2u} + 3e^{u}\right)$$

$$\frac{2\left(e^{2u}-1\right)}{u} \le \left(e^{2u} + 3e^{u}\right)$$

$$\frac{2}{u}\left(1+2u+\frac{(2u)^2}{2!}+\cdots-1\right) \le 1+2u+\frac{(2u)^2}{2!}+\cdots+3\left(1+u+\frac{u^2}{2!}+\cdots\right)$$

$$4+4u+\frac{8}{3}u^2+\cdots+\frac{2^{n+1}}{n!}u^{n-1}+\cdots \le 4+5u+\frac{7}{2}u^2+\cdots+\frac{\left(2^{n-1}+3\right)}{(n-1)!}u^{n-1}+\cdots \quad . \qquad (5.2.5)$$

Here comparing the coefficients of u, u^2, …, u^{n-1}, we see that the above inequality is true. Hence, we get

$I(\rho)\,|\,N_2(0,0,1,1,\rho) \le I(\rho)\,|\,N_2\left(0,0,1,1,\rho^s\right),\ 0 < s \le 1$.

Trivariate normal populations

A multivariate normal distribution in three variables is called trivariate normal distribution. So a random variable $\left(X_1, X_2, X_3\right)$ follows trivariate normal distribution if it has <u>probability density function</u> as

$$f\left(x_1,x_2,x_3\right) = \frac{exp\left[-w\Big/\left\{2\left(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}\rho_{23}-1\right)\right\}\right]}{2\sqrt{2}\,\pi^{3/2}\sqrt{1-\left(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2\right)+2\rho_{12}\rho_{13}\rho_{23}}}, \qquad (5.2.6)$$

where $w = x_1^2\left(\rho_{23}^2-1\right)+x_2^2\left(\rho_{13}^2-1\right)+x_3^2\left(\rho_{12}^2-1\right)$

$$+2\left[x_1x_2\left(\rho_{12}-\rho_{13}\rho_{23}\right)+x_1x_3\left(\rho_{13}-\rho_{12}\rho_{23}\right)+x_2x_3\left(\rho_{23}-\rho_{12}\rho_{13}\right)\right].$$

We denote a random variable $\left(X_1, X_2, X_3\right)$ following trivariate normal distribution as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & c_1\rho_1\sigma_1\sigma_2 & c_1\rho_2\sigma_1\sigma_3 \\ c_1\rho_1\sigma_1\sigma_2 & \sigma_2^2 & c_1\rho_3\sigma_2\sigma_3 \\ c_1\rho_2\sigma_1\sigma_3 & c_1\rho_3\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}\right),$$

where $X_1 \sim N\left(\mu_1,\sigma_1^2\right)$, $X_2 \sim N\left(\mu_2,\sigma_2^2\right)$, $X_3 \sim N\left(\mu_3,\sigma_3^2\right)$, $Cor\left(X_1, X_2\right) = c_1\rho_1$, $Cor\left(X_1, X_3\right) = c_1\rho_2$ and $Cor\left(X_2, X_3\right) = c_1\rho_3$.

With same pair-wise correlation

Here we compare different trivariate normal experiments having the same pair-wise correlation, i.e. $\mathrm{Corr}\left(X_1, X_2\right) = \mathrm{Corr}\left(X_1, X_3\right) = \mathrm{Corr}\left(X_2, X_3\right)$ and $\mathrm{Corr}\left(Y_1,Y_2\right) = \mathrm{Corr}\left(Y_1,Y_3\right) = \mathrm{Corr}\left(Y_2,Y_3\right)$. Some comparisons with these structures are given as follows.

Theorem 5. 2.4 Given two trivariate normal experiments,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}\right) \text{ and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c\rho & c\rho \\ c\rho & 1 & c\rho \\ c\rho & c\rho & 1 \end{pmatrix}\right)$$

for all $c > 1$, where ρ is an unknown parameter. Then ε_2 is sufficient for ε_1.

Proof: Given a trivariate normal experiment $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c\rho & c\rho \\ c\rho & 1 & c\rho \\ c\rho & c\rho & 1 \end{pmatrix}\right)$.

Let $Y_1^* = \dfrac{Y_1 + \sqrt{c-1}U}{\sqrt{c}}$, $Y_2^* = \dfrac{Y_2 + \sqrt{c-1}V}{\sqrt{c}}$ and $Y_3^* = \dfrac{Y_3 + \sqrt{c-1}W}{\sqrt{c}}$,

where $U \sim N\left(0,1\right)$, $V \sim N\left(0,1\right)$ and $W \sim N\left(0,1\right)$ are i.i.d. and independent of Y_1, Y_2, Y_3.

Here $E\left(Y_1^*\right) = 0$, $E\left(Y_2^*\right) = 0$, $E\left(Y_3^*\right) = 0$, $\mathrm{Var}\left(Y_1^*\right) = 1$, $\mathrm{Var}\left(Y_2^*\right) = 1$ and $\mathrm{Var}\left(Y_3^*\right) = 1$. Then,

$$\begin{aligned} \mathrm{Cov}\left(Y_1^*, Y_2^*\right) &= \mathrm{Cov}\left(\frac{Y_1 + \sqrt{c-1}U}{\sqrt{c}}, \frac{Y_2 + \sqrt{c-1}V}{\sqrt{c}}\right) \\ &= \frac{1}{\mathrm{c}}\mathrm{Cov}\left(Y_1,Y_2\right) \\ &= \frac{1}{\mathrm{c}}\mathrm{Corr}\left(Y_1,Y_2\right) \\ &= \rho, \end{aligned}$$

$$\begin{aligned}
\operatorname{Cov}\left(Y_1^*, Y_3^*\right) &= \operatorname{Cov}\left(\frac{Y_1+\sqrt{c-1}U}{\sqrt{c}}, \frac{Y_3+\sqrt{c-1}W}{\sqrt{c}}\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Cov}\left(Y_1, Y_3\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Corr}\left(Y_1, Y_3\right) \\
&= \rho ,
\end{aligned}$$

and
$$\begin{aligned}
\operatorname{Cov}\left(Y_2^*, Y_3^*\right) &= \operatorname{Cov}\left(\frac{Y_2+\sqrt{c-1}V}{\sqrt{c}}, \frac{Y_3+\sqrt{c-1}W}{\sqrt{c}}\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Cov}\left(Y_2, Y_3\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Corr}\left(Y_2, Y_3\right) \\
&= \rho .
\end{aligned}$$

Therefore, $\begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}\right)$.

Thus from the trivariate normal variables $\left(Y_1, Y_2, Y_3\right)$, we can generate other trivariate normal variables $\left(Y_1^*, Y_2^*, Y_3^*\right)$ which has the same distribution as $\left(X_1, X_2, X_3\right)$.

Hence the trivariate normal experiment ε_2 is sufficient for another trivariate normal experiment ε_1.

Next some corollaries are given as follows.

Corollary 5.2.4.1 Given two trivariate normal experiments,

$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho & c_1\rho \\ c_1\rho & 1 & c_1\rho \\ c_1\rho & c_1\rho & 1 \end{pmatrix}\right)$ and $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho & c_2\rho \\ c_2\rho & 1 & c_2\rho \\ c_2\rho & c_2\rho & 1 \end{pmatrix}\right)$ for all $c_2 > c_1 > 0$, where ρ is an unknown parameter. Then ε_2 is sufficient for ε_1.

Proof: Given a trivariate normal experiment $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho & c_2\rho \\ c_2\rho & 1 & c_2\rho \\ c_2\rho & c_2\rho & 1 \end{pmatrix}\right)$.

Let $Y_1^* = \dfrac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}$, $Y_2^* = \dfrac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}\,V}{\sqrt{c_2}}$ and $Y_3^* = \dfrac{\sqrt{c_1}Y_3 + \sqrt{c_2 - c_1}\,W}{\sqrt{c_2}}$, where $U \sim N(0,1)$, $V \sim N(0,1)$ and $W \sim N(0,1)$ are i.i.d. and independent of Y_1, Y_2, Y_3 .

Here $E(Y_1^*) = 0$, $E(Y_2^*) = 0$, $E(Y_3^*) = 0$, $\mathrm{Var}(Y_1^*) = 1$, $\mathrm{Var}(Y_2^*) = 1$ and $\mathrm{Var}(Y_3^*) = 1$.

$$\begin{aligned}
\text{Then, } \mathrm{Cov}(Y_1^*, Y_2^*) &= \mathrm{Cov}\left(\frac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}, \frac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}\,V}{\sqrt{c_2}}\right) \\
&= \frac{c_1}{c_2}\mathrm{Cov}(Y_1, Y_2) \\
&= \frac{c_1}{c_2}\mathrm{Corr}(Y_1, Y_2) \\
&= c_1\rho, \\
\mathrm{Cov}(Y_1^*, Y_3^*) &= \mathrm{Cov}\left(\frac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}, \frac{\sqrt{c_1}Y_3 + \sqrt{c_2 - c_1}\,W}{\sqrt{c_2}}\right) \\
&= \frac{c_1}{c_2}\mathrm{Cov}(Y_1, Y_3) \\
&= \frac{c_1}{c_2}\mathrm{Corr}(Y_1, Y_3) \\
&= c_1\rho, \\
\text{and } \mathrm{Cov}(Y_2^*, Y_3^*) &= \mathrm{Cov}\left(\frac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}\,V}{\sqrt{c_2}}, \frac{\sqrt{c_1}Y_3 + \sqrt{c_2 - c_1}\,W}{\sqrt{c_2}}\right) \\
&= \frac{c_1}{c_2}\mathrm{Cov}(Y_2, Y_3) \\
&= \frac{c_1}{c_2}\mathrm{Corr}(Y_2, Y_3) \\
&= c_1\rho.
\end{aligned}$$

Therefore, $\begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho & c_1\rho \\ c_1\rho & 1 & c_1\rho \\ c_1\rho & c_1\rho & 1 \end{pmatrix}\right)$.

Thus from the trivariate normal variables (Y_1, Y_2, Y_3), we can generate other trivariate normal variables (Y_1^*, Y_2^*, Y_3^*) which has the same distribution as (X_1, X_2, X_3).

Hence the trivariate normal experiment ε_2 is sufficient for another trivariate normal experiment ε_1.

Corollary 5.2.4.2 Given two trivariate normal experiments,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & c_1\rho\sigma_1\sigma_2 & c_1\rho\sigma_1\sigma_3 \\ c_1\rho\sigma_1\sigma_2 & \sigma_2^2 & c_1\rho\sigma_2\sigma_3 \\ c_1\rho\sigma_1\sigma_3 & c_1\rho\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}\right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \begin{pmatrix} \delta_1^2 & c_2\rho\delta_1\delta_2 & c_2\rho\delta_1\delta_3 \\ c_2\rho\delta_1\delta_2 & \delta_2^2 & c_2\rho\delta_2\delta_3 \\ c_2\rho\delta_1\delta_3 & c_2\rho\delta_2\delta_3 & \delta_3^2 \end{pmatrix}\right),$$

where all the parameters are known except ρ with $c_2 > c_1 > 0$. Then ε_2 is sufficient for ε_1.

Proof: Given a trivariate normal experiment

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & c_1\rho\sigma_1\sigma_2 & c_1\rho\sigma_1\sigma_3 \\ c_1\rho\sigma_1\sigma_2 & \sigma_2^2 & c_1\rho\sigma_2\sigma_3 \\ c_1\rho\sigma_1\sigma_3 & c_1\rho\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}\right).$$

Let $X_1^* = \dfrac{X_1 - m_1}{\sigma_1}$, $X_2^* = \dfrac{X_2 - m_2}{\sigma_2}$ and $X_3^* = \dfrac{X_3 - m_3}{\sigma_3}$.

Here $E(X_1^*) = 0$, $E(X_2^*) = 0$, $E(X_3^*) = 0$, $\text{Var}(X_1^*) = 1$, $\text{Var}(X_2^*) = 1$ and $\text{Var}(X_3^*) = 1$.

Further, $\text{Cov}(X_1^*, X_2^*) = \text{Corr}(X_1^*, X_2^*) = \text{Corr}(X_1, X_2) = c_1\rho$,

$\text{Cov}(X_2^*, X_3^*) = \text{Corr}(X_2^*, X_3^*) = \text{Corr}(X_2, X_3) = c_1\rho$,

and $\text{Cov}(X_1^*, X_3^*) = \text{Corr}(X_1^*, X_3^*) = \text{Corr}(X_1, \text{X}_3) = c_1\rho$.

Hence, $\varepsilon_3 : \begin{pmatrix} X_1^* \\ X_2^* \\ X_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho & c_1\rho \\ c_1\rho & 1 & c_1\rho \\ c_1\rho & c_1\rho & 1 \end{pmatrix}\right)$.

Therefore, ε_3 is equivalent to ε_1.

Given another trivariate normal experiment $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \begin{pmatrix} \delta_1^2 & c_2\rho\delta_1\delta_2 & c_2\rho\delta_1\delta_3 \\ c_2\rho\delta_1\delta_2 & \delta_2^2 & c_2\rho\delta_2\delta_3 \\ c_2\rho\delta_1\delta_3 & c_2\rho\delta_2\delta_3 & \delta_3^2 \end{pmatrix}\right)$.

Let $Y_1^* = \dfrac{Y_1 - \theta_1}{\delta_1}$, $Y_2^* = \dfrac{Y_2 - \theta_2}{\delta_2}$ and $Y_3^* = \dfrac{Y_3 - \theta_3}{\delta_3}$.

Here $E\left(Y_1^*\right) = 0$, $E\left(Y_2^*\right) = 0$, $E\left(Y_3^*\right) = 0$, $\text{Var}\left(Y_1^*\right) = 1$, $\text{Var}\left(Y_2^*\right) = 1$ and $\text{Var}\left(Y_3^*\right) = 1$.

Further, $\text{Cov}\left(Y_1^*, Y_2^*\right) = \text{Corr}\left(Y_1^*, Y_2^*\right) = \text{Corr}\left(Y_1, Y_2\right) = c_2\rho$,

$\text{Cov}\left(Y_2^*, Y_3^*\right) = \text{Corr}\left(Y_2^*, Y_3^*\right) = \text{Corr}\left(Y_2, Y_3\right) = c_2\rho$,

and $\text{Cov}\left(Y_1^*, Y_3^*\right) = \text{Corr}\left(Y_1^*, Y_3^*\right) = \text{Corr}\left(Y_1, Y_3\right) = c_2\rho$.

Hence $\varepsilon_4 : \begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho & c_2\rho \\ c_2\rho & 1 & c_2\rho \\ c_2\rho & c_2\rho & 1 \end{pmatrix}\right)$.

Therefore, ε_4 is equivalent to ε_2.

But from Corollary 2.4.1, we have ε_4 is sufficient for ε_3 .

Hence we can say that the trivariate normal experiment ε_2 is sufficient for another trivariate normal experiment ε_1.

With different pair-wise correlation

Here we compare different trivariate normal experiments having different pair-wise correlation, i.e. $\text{Corr}\left(X_1, X_2\right) \neq \text{Corr}\left(X_1, X_3\right) \neq \text{Corr}\left(X_2, X_3\right)$ and $\text{Corr}\left(Y_1, Y_2\right) \neq \text{Corr}\left(Y_1, Y_3\right) \neq \text{Corr}\left(Y_2, Y_3\right)$. Some comparisons with these structures are given as follows.

Theorem 5.2.5 Given two trivariate normal experiments, $\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}\right)$

and $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c\rho_1 & c\rho_2 \\ c\rho_1 & 1 & c\rho_3 \\ c\rho_2 & c\rho_3 & 1 \end{pmatrix}\right)$

for all $c > 1$, where ρ_1, ρ_2, ρ_3 are unknown parameters. Then ε_2 is sufficient for ε_1.

Proof: Given a trivariate normal experiment $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c\rho_1 & c\rho_2 \\ c\rho_1 & 1 & c\rho_3 \\ c\rho_2 & c\rho_3 & 1 \end{pmatrix}\right)$.

Let $Y_1^* = \dfrac{Y_1 + \sqrt{c-1}U}{\sqrt{c}}$, $Y_2^* = \dfrac{Y_2 + \sqrt{c-1}V}{\sqrt{c}}$ and $Y_3^* = \dfrac{Y_3 + \sqrt{c-1}W}{\sqrt{c}}$,

where $U \sim N(0,1)$, $V \sim N(0,1)$ and $W \sim N(0,1)$ are i.i.d. and independent of Y_1, Y_2, Y_3 .

Here $E\left(Y_1^*\right) = 0$, $E\left(Y_2^*\right) = 0$, $E\left(Y_3^*\right) = 0$, $\operatorname{Var}\left(Y_1^*\right) = 1$, $\operatorname{Var}\left(Y_2^*\right) = 1$ and $\operatorname{Var}\left(Y_3^*\right) = 1$.

$$
\begin{aligned}
\text{Then, } \operatorname{Cov}\left(Y_1^*, Y_2^*\right) &= \operatorname{Cov}\left(\frac{Y_1 + \sqrt{c-1}U}{\sqrt{c}}, \frac{Y_2 + \sqrt{c-1}V}{\sqrt{c}}\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Cov}\left(Y_1, Y_2\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Corr}\left(Y_1, Y_2\right) \\
&= \rho_1,
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Cov}\left(Y_1^*, Y_3^*\right) &= \operatorname{Cov}\left(\frac{Y_1 + \sqrt{c-1}U}{\sqrt{c}}, \frac{Y_3 + \sqrt{c-1}W}{\sqrt{c}}\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Cov}\left(Y_1, Y_3\right) \\
&= \frac{1}{\mathrm{c}}\operatorname{Corr}\left(Y_1, Y_3\right) \\
&= \rho_2,
\end{aligned}
$$

and $\quad \operatorname{Cov}\left(Y_2^*, Y_3^*\right) = \operatorname{Cov}\left(\frac{Y_2+\sqrt{c-1}V}{\sqrt{c}}, \frac{Y_3+\sqrt{c-1}W}{\sqrt{c}}\right)$

$$= \frac{1}{\mathrm{c}}\operatorname{Cov}\left(Y_2, Y_3\right)$$

$$= \frac{1}{\mathrm{c}}\operatorname{Corr}\left(Y_2, Y_3\right)$$

$$= \rho_3 .$$

Therefore, $\begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}\right)$.

Thus from the trivariate normal variables $\left(Y_1, Y_2, Y_3\right)$, we can generate next trivariate normal variables $\left(Y_1^*, Y_2^*, Y_3^*\right)$ which has the same distribution as $\left(X_1, X_2, X_3\right)$.

Hence the trivariate normal experiment ε_2 is sufficient for another trivariate normal experiment ε_1.

Corollary 5.2.5.1 Given two trivariate normal experiments,

$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho_1 & c_1\rho_2 \\ c_1\rho_1 & 1 & c_1\rho_3 \\ c_1\rho_2 & c_1\rho_3 & 1 \end{pmatrix}\right)$ and $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho_1 & c_2\rho_2 \\ c_2\rho_1 & 1 & c_2\rho_3 \\ c_2\rho_2 & c_2\rho_3 & 1 \end{pmatrix}\right)$ for all $c_2 > c_1 > 0$, where ρ_1, ρ_2, ρ_3 are unknown parameters. Then ε_2 is sufficient for ε_1.

Proof: Given a trivariate normal experiment $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho_1 & c_2\rho_2 \\ c_2\rho_1 & 1 & c_2\rho_3 \\ c_2\rho_2 & c_2\rho_3 & 1 \end{pmatrix}\right)$.

Let $Y_1^* = \dfrac{\sqrt{c_1}Y_1 + \sqrt{c_2-c_1}\,U}{\sqrt{c_2}}$, $Y_2^* = \dfrac{\sqrt{c_1}Y_2 + \sqrt{c_2-c_1}\,V}{\sqrt{c_2}}$ and $Y_3^* = \dfrac{\sqrt{c_1}Y_3 + \sqrt{c_2-c_1}\,W}{\sqrt{c_2}}$,

where $U \sim N\left(0,1\right)$, $V \sim N\left(0,1\right)$ and $W \sim N\left(0,1\right)$ are i.i.d. and independent of Y_1, Y_2, Y_3.

Here $E\left(Y_1^*\right)=0$, $E\left(Y_2^*\right)=0$, $E\left(Y_3^*\right)=0$, $\operatorname{Var}\left(Y_1^*\right)=1$, $\operatorname{Var}\left(Y_2^*\right)=1$ and $\operatorname{Var}\left(Y_3^*\right)=1$.

Then, $\mathrm{Cov}\left(Y_1^*, Y_2^*\right) = \mathrm{Cov}\left(\dfrac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}, \dfrac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}V}{\sqrt{c_2}}\right)$

$$= \frac{\mathrm{c}_1}{\mathrm{c}_2}\mathrm{Cov}\left(Y_1, Y_2\right)$$

$$= \frac{\mathrm{c}_1}{\mathrm{c}_2}\mathrm{Corr}\left(Y_1, Y_2\right)$$

$$= \mathrm{c}_1\rho_1,$$

$$\mathrm{Cov}\left(Y_1^*, Y_3^*\right) = \mathrm{Cov}\left(\frac{\sqrt{c_1}Y_1 + \sqrt{c_2 - c_1}\,U}{\sqrt{c_2}}, \frac{\sqrt{c_1}Y_3 + \sqrt{c_2 - c_1}\,W}{\sqrt{c_2}}\right)$$

$$= \frac{\mathrm{c}_1}{\mathrm{c}_2}\mathrm{Cov}\left(Y_1, Y_3\right)$$

$$= \frac{\mathrm{c}_1}{\mathrm{c}_2}\mathrm{Corr}\left(Y_1, Y_3\right)$$

$$= \mathrm{c}_1\rho_2,$$

and $\mathrm{Cov}\left(Y_2^*, Y_3^*\right) = \mathrm{Cov}\left(\dfrac{\sqrt{c_1}Y_2 + \sqrt{c_2 - c_1}V}{\sqrt{c_2}}, \dfrac{\sqrt{c_1}Y_3 + \sqrt{c_2 - c_1}W}{\sqrt{c_2}}\right)$

$$= \frac{\mathrm{c}_1}{\mathrm{c}_2}\mathrm{Cov}\left(Y_2, Y_3\right)$$

$$= \frac{\mathrm{c}_1}{\mathrm{c}_2}\mathrm{Corr}\left(Y_2, Y_3\right)$$

$$= \mathrm{c}_1\rho_3.$$

Therefore, $\begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho_1 & c_1\rho_2 \\ c_1\rho_1 & 1 & c_1\rho_3 \\ c_1\rho_2 & c_1\rho_3 & 1 \end{pmatrix}\right)$.

Thus from the trivariate normal variables $\left(Y_1, Y_2, Y_3\right)$, we can generate next trivariate normal variables $\left(Y_1^*, Y_2^*, Y_3^*\right)$ which has the same distribution as $\left(X_1, X_2, X_3\right)$.

Hence ε_2 is sufficient for ε_1.

Corollary 5.2.5.2 Given two trivariate normal experiments,

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^{\;2} & c_1\rho_1\sigma_1\sigma_2 & c_1\rho_2\sigma_1\sigma_3 \\ c_1\rho_1\sigma_1\sigma_2 & \sigma_2^{\;2} & c_1\rho_3\sigma_2\sigma_3 \\ c_1\rho_2\sigma_1\sigma_3 & c_1\rho_3\sigma_2\sigma_3 & \sigma_3^{\;2} \end{pmatrix} \right)$$

$$\text{and } \varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \begin{pmatrix} \delta_1^{\;2} & c_2\rho_1\delta_1\delta_2 & c_2\rho_2\delta_1\delta_3 \\ c_2\rho_1\delta_1\delta_2 & \delta_2^{\;2} & c_2\rho_3\delta_2\delta_3 \\ c_2\rho_2\delta_1\delta_3 & c_2\rho_3\delta_2\delta_3 & \delta_3^{\;2} \end{pmatrix} \right),$$

where all the parameters are known except ρ_1, ρ_2, ρ_3 with $c_2 > c_1 > 0$. Then ε_2 is sufficient for ε_1.

Proof: Given a trivariate normal experiment

$$\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} \sigma_1^{\;2} & c_1\rho_1\sigma_1\sigma_2 & c_1\rho_2\sigma_1\sigma_3 \\ c_1\rho_1\sigma_1\sigma_2 & \sigma_2^{\;2} & c_1\rho_3\sigma_2\sigma_3 \\ c_1\rho_2\sigma_1\sigma_3 & c_1\rho_3\sigma_2\sigma_3 & \sigma_3^{\;2} \end{pmatrix} \right).$$

Let $X_1^* = \dfrac{X_1 - m_1}{\sigma_1}$, $X_2^* = \dfrac{X_2 - m_2}{\sigma_2}$ and $X_3^* = \dfrac{X_3 - m_3}{\sigma_3}$.

Here $E\left(X_1^*\right) = 0$, $E\left(X_2^*\right) = 0$, $E\left(X_3^*\right) = 0$, $\mathrm{Var}\left(X_1^*\right) = 1$, $\mathrm{Var}\left(X_2^*\right) = 1$ and $\mathrm{Var}\left(X_3^*\right) = 1$.

Further, $\mathrm{Cov}\left(X_1^*, X_2^*\right) = \mathrm{Corr}\left(X_1^*, X_2^*\right) = \mathrm{Corr}\left(X_1, X_2\right) = c_1\rho_1$,

$\mathrm{Cov}\left(X_2^*, X_3^*\right) = \mathrm{Corr}\left(X_2^*, X_3^*\right) = \mathrm{Corr}\left(X_2, X_3\right) = c_1\rho_3$,

and $\mathrm{Cov}\left(X_1^*, X_3^*\right) = \mathrm{Corr}\left(X_1^*, X_3^*\right) = \mathrm{Corr}\left(X_1, X_3\right) = c_1\rho_2$.

$$\text{Hence, } \varepsilon_3 : \begin{pmatrix} X_1^* \\ X_2^* \\ X_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho_1 & c_1\rho_2 \\ c_1\rho_1 & 1 & c_1\rho_3 \\ c_1\rho_2 & c_1\rho_3 & 1 \end{pmatrix} \right).$$

Therefore, ε_3 is equivalent to ε_1.

$$\text{Given } \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \begin{pmatrix} \delta_1^{\;2} & c_2\rho_1\delta_1\delta_2 & c_2\rho_2\delta_1\delta_3 \\ c_2\rho_1\delta_1\delta_2 & \delta_2^{\;2} & c_2\rho_3\delta_2\delta_3 \\ c_2\rho_2\delta_1\delta_3 & c_2\rho_3\delta_2\delta_3 & \delta_3^{\;2} \end{pmatrix} \right).$$

Let $Y_1^* = \dfrac{Y_1 - \theta_1}{\delta_1}$, $Y_2^* = \dfrac{Y_2 - \theta_2}{\delta_2}$ and $Y_3^* = \dfrac{Y_3 - \theta_3}{\delta_3}$.

Here $E\left(Y_1^*\right) = 0$, $E\left(Y_2^*\right) = 0$, $E\left(Y_3^*\right) = 0$, $\mathrm{Var}\left(Y_1^*\right) = 1$, $\mathrm{Var}\left(Y_2^*\right) = 1$ and $\mathrm{Var}\left(Y_3^*\right) = 1$.

Further, $\text{Cov}\left(Y_1^*,Y_2^*\right) = \text{Corr}\left(Y_1^*,Y_2^*\right) = \text{Corr}\left(Y_1,Y_2\right) = c_2\rho_1$,

$\text{Cov}\left(Y_2^*,Y_3^*\right) = \text{Corr}\left(Y_2^*,Y_3^*\right) = \text{Corr}\left(Y_2,Y_3\right) = c_2\rho_3$,

and $\text{Cov}\left(Y_1^*,Y_3^*\right) = \text{Corr}\left(Y_1^*,Y_3^*\right) = \text{Corr}\left(Y_1,Y_3\right) = c_2\rho_2$.

Hence, $\varepsilon_4 : \begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_2\rho_1 & c_2\rho_2 \\ c_2\rho_1 & 1 & c_2\rho_3 \\ c_2\rho_2 & c_2\rho_3 & 1 \end{pmatrix}\right)$.

Therefore, ε_4 is equivalent to ε_2.

But from Corollary 2.5.1, we have ε_4 is sufficient for ε_3.

Hence we can say that ε_2 is sufficient for ε_1.

Theorem 5.2.6 Given two trivariate normal experiments, $\varepsilon_1 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}\right)$ and

$\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho_1 & c_2\rho_2 \\ c_1\rho_1 & 1 & c_3\rho_3 \\ c_2\rho_2 & c_3\rho_3 & 1 \end{pmatrix}\right)$,

where ρ_1, ρ_2, ρ_3 are unknown but $c_1, c_2, c_3 > 1$ are all known. Then ε_2 is sufficient for ε_1 whenever $c_1 < c_2c_3$, $c_2 < c_1c_3$ and $c_3 < c_1c_2$.

Proof: Given a trivariate normal experiment $\varepsilon_2 : \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & c_1\rho_1 & c_2\rho_2 \\ c_1\rho_1 & 1 & c_3\rho_3 \\ c_2\rho_2 & c_3\rho_3 & 1 \end{pmatrix}\right)$.

Let $Y_1^* = \dfrac{\sqrt{a_1}Y_1 + U}{\sqrt{a_1+1}}$, $Y_2^* = \dfrac{\sqrt{a_2}Y_2 + V}{\sqrt{a_2+1}}$ and $Y_3^* = \dfrac{\sqrt{a_3}Y_3 + W}{\sqrt{a_3+1}}$, where $U \sim N(0,1)$, $V \sim N(0,1)$ and $W \sim N(0,1)$ are i.i.d. and independent of Y_1, Y_2, Y_3.

Here $E\left(Y_1^*\right)=0$, $E\left(Y_2^*\right)=0$, $E\left(Y_3^*\right)=0$, $\text{Var}\left(Y_1^*\right)=1$, $\text{Var}\left(Y_2^*\right)=1$ and $\text{Var}\left(Y_3^*\right)=1$.

Then, $\text{Cov}\left(Y_1^*,Y_2^*\right) = \text{Cov}\left(\dfrac{\sqrt{a_1}Y_1 + U}{\sqrt{a_1+1}}, \dfrac{\sqrt{a_2}Y_2 + V}{\sqrt{a_2+1}}\right)$

$$= \frac{\sqrt{a_1 a_2}}{\sqrt{a_1+1}\sqrt{a_2+1}} \mathrm{Cov}(Y_1, Y_2)$$

$$= \frac{\sqrt{a_1 a_2}}{\sqrt{a_1+1}\sqrt{a_2+1}} c_1\rho_1,$$

$$\mathrm{Cov}(Y_1^*, Y_3^*) = \mathrm{Cov}\left(\frac{\sqrt{a_1}Y_1+U}{\sqrt{a_1+1}}, \frac{\sqrt{a_3}Y_3+W}{\sqrt{a_3+1}}\right)$$

$$= \frac{\sqrt{a_1 a_3}}{\sqrt{a_1+1}\sqrt{a_3+1}} \mathrm{Cov}(Y_1, Y_3)$$

$$= \frac{\sqrt{a_1 a_3}}{\sqrt{a_1+1}\sqrt{a_3+1}} c_2\rho_2,$$

and $\mathrm{Cov}(Y_2^*, Y_3^*) = \mathrm{Cov}\left(\frac{\sqrt{a_2}Y_2+V}{\sqrt{a_2+1}}, \frac{\sqrt{a_3}Y_3+W}{\sqrt{a_3+1}}\right)$

$$= \frac{\sqrt{a_2 a_3}}{\sqrt{a_2+1}\sqrt{a_3+1}} \mathrm{Cov}(Y_2, Y_3)$$

$$= \frac{\sqrt{a_2 a_3}}{\sqrt{a_2+1}\sqrt{a_3+1}} c_3\rho_3.$$

Therefore,

$$\varepsilon_3 : \begin{pmatrix} Y_1^* \\ Y_2^* \\ Y_3^* \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{\sqrt{a_1 a_2}}{\sqrt{a_1+1}\sqrt{a_2+1}} c_1\rho_1 & \frac{\sqrt{a_1 a_3}}{\sqrt{a_1+1}\sqrt{a_3+1}} c_2\rho_2 \\ \frac{\sqrt{a_1 a_2}}{\sqrt{a_1+1}\sqrt{a_2+1}} c_1\rho_1 & 1 & \frac{\sqrt{a_2 a_3}}{\sqrt{a_2+1}\sqrt{a_3+1}} c_3\rho_3 \\ \frac{\sqrt{a_1 a_3}}{\sqrt{a_1+1}\sqrt{a_3+1}} c_2\rho_2 & \frac{\sqrt{a_2 a_3}}{\sqrt{a_2+1}\sqrt{a_3+1}} c_3\rho_3 & 1 \end{pmatrix}\right).$$

Hence ε_2 is sufficient for ε_3. To make ε_2 sufficient for ε_1, we must have ε_3 which is equivalent to ε_1. Further for ε_3 equivalent with ε_1, we must have

$$\frac{\sqrt{a_1 a_2}}{\sqrt{a_1+1}\sqrt{a_2+1}} c_1\rho_1 = \rho_1$$

$$\frac{a_1a_2}{(a_1+1)(a_2+1)} = \frac{1}{c_1^{\,2}} \tag{5.2.7}$$

$$\frac{\sqrt{a_1a_3}}{\sqrt{a_1+1}\sqrt{a_3+1}}c_2\rho_2 = \rho_2$$

$$\frac{a_1a_3}{(a_1+1)(a_3+1)} = \frac{1}{c_2^{\,2}} \tag{5.2.8}$$

$$\frac{\sqrt{a_2a_3}}{\sqrt{a_2+1}\sqrt{a_3+1}}c_3\rho_3 = \rho_3$$

$$\frac{a_2a_3}{(a_2+1)(a_3+1)} = \frac{1}{c_3^{\,2}} \tag{5.2.9}$$

Multiplying (5.2.7) and (5.2.8) and using (5.2.9), we get

$$\frac{a_1^{\,2}}{(a_1+1)^2}\cdot\frac{a_2a_3}{(a_2+1)(a_3+1)} = \frac{1}{c_1^{\,2}c_2^{\,2}}$$

$$\frac{a_1}{a_1+1} = \frac{c_3}{c_1c_2} < 1$$

$$a_1 = \frac{c_3}{c_1c_2 - c_3} \text{ and } c_3 < c_1c_2.$$

Similarly multiplying (5.2.7) and (5.2.9) and using (5.2.8) we get

$$\frac{a_2^{\,2}}{(a_2+1)^2}\cdot\frac{a_1a_3}{(a_1+1)(a_3+1)} = \frac{1}{c_1^{\,2}c_3^{\,2}}$$

$$\frac{a_2}{a_2+1} = \frac{c_2}{c_1c_3} < 1$$

$$a_2 = \frac{c_2}{c_1c_3 - c_2} \text{ and } c_2 < c_1c_3.$$

Similarly multiplying (5.2.8) and (5.2.9) and using (5.2.7) we get

$$\frac{a_3^{\,2}}{(a_3+1)^2}\cdot\frac{a_1a_2}{(a_1+1)(a_2+1)} = \frac{1}{c_2^{\,2}c_3^{\,2}}$$

$$\frac{a_3}{a_3+1} = \frac{c_1}{c_2c_3} < 1$$

$a_3 = \dfrac{c_1}{c_2 c_3 - c_1}$ and $c_1 < c_2 c_3$.

Here with the values of a_1, a_2 and a_3, equations (5.2.7), (5.2.8) and (5.2.9) are satisfied.

Hence ε_2 sufficient for ε_1 whenever $c_1 < c_2 c_3$, $c_2 < c_1 c_3$ and $c_3 < c_1 c_2$.

Remark 5.2.1 For the bivariate and trivariate normal experiments, we have two and three different correlation parameters. Further we also have two and three variables to transform. So, we can make transformation of these two and three variables to cover two and three correlation parameters under the stated conditions. Therefore, it is possible to claim and establish sufficiency under the stated conditions.

Remark 5.2.2 For four and more variables in normal experiments, we have less number of variables to transform than the number of different correlation parameters. In general, we can not make transformation from these variables to cover all different correlation parameters. Hence we can not establish sufficiency in general. However, in the special case of equicorrelation, it is possible to establish sufficiency, by making suitable transformation, generalizing the result mentioned in Theorem 5.2.4.

5.3 Sufficiency in linear and quadratic regression models

In this topic, we first consider a simple linear regression model with intercept term. Next we consider quadratic regression model with/without intercept term. Then we apply the concept of sufficiency in these models for generating the unobservable random variables. Moreover, some important theorems and remarks are presented in order to clear our pictures towards the application of the concept of sufficiency in these models.

5.3.1 Simple linear regression model with intercept term

Here we consider a simple linear regression model, $Y_x = \alpha + \beta x + e_x$ where Y_x is dependent variable, $x \in [-1,1]$ is independent variable (non-stochastic), $e_x \sim N\left(0, \sigma^2\right)$ is error term, σ^2 is known and α, β are unknown parameters. Then we apply the concept of sufficiency in this model to generate a random variable which has the same distribution as Y_x. Further we present some important remarks giving the conditions for the independence of the generated random variables Y_x at different points of $x \in [-1,1]$. This we do based on observations made at 'x = 1' and 'x = -1'.

Theorem 5.3.1 Given a simple linear model $Y_x = \alpha + \beta x + e_x$ where $e_x \sim N\left(0, \sigma^2\right)$, $x \in [-1,1]$, σ^2 is known and α , β are unknown parameter, we can claim that data points Y_{+1} and Y_{-1} are jointly sufficient for generating Y_x, $\forall\, x \in [-1,1]$.

Proof: From the given model, we have $Y_x \sim N\left(\alpha + \beta x, \sigma^2\right)$.

For x = +1, $Y_{+1} = \alpha + \beta + e_{+1}$ and $Y_{+1} \sim N\left(\alpha + \beta, \sigma^2\right)$. For x = -1, $Y_{-1} = \alpha - \beta + e_{-1}$ and $Y_{-1} \sim N\left(\alpha - \beta, \sigma^2\right)$.

Let $Y_{+1} = Y_+$ and $Y_{-1} = Y_-$. Taking expectation of $\frac{Y_+ + Y_-}{2}$, we get

$E\left(\frac{Y_+ + Y_-}{2}\right) = \alpha$. Similarly taking expectation of $\frac{Y_+ - Y_-}{2}$, we get $E\left(\frac{Y_+ - Y_-}{2}\right) = \beta$.

Then the fitted value of Y_x is given by

$$
\begin{aligned}
\hat{Y}_x &= \hat{\alpha} + \hat{\beta} x \\
&= \left(\frac{Y_+ + Y_-}{2}\right) + \left(\frac{Y_+ - Y_-}{2}\right) x \\
&= \frac{Y_+}{2}(1+x) + \frac{Y_-}{2}(1-x) .
\end{aligned}
$$

Here, expectation of $\hat{Y}_x$ is $E(\hat{Y}_x)$ = $\alpha + \beta x$ and the variance is

$$
\begin{aligned}
\mathrm{Var}\left(\hat{Y}_x\right) &= \mathrm{Var}\left[\frac{Y_+}{2}(1+x) + \frac{Y_-}{2}(1-x)\right] \\
&= \frac{(1+x)^2}{4}\mathrm{Var(Y_+)} + \frac{(1-x)^2}{4}\mathrm{Var(Y_-)} \\
&= \frac{(1+x)^2}{4}\sigma^2 + \frac{(1-x)^2}{4}\sigma^2 \\
&= \frac{\sigma^2}{2}\left(1+x^2\right) \le \sigma^2 \text{ for } x \in [-1,1] .
\end{aligned}
$$

Therefore, $\hat{Y}_x \sim N\left(\alpha+\beta x, \frac{\sigma^2}{2}\left(1+x^2\right)\right)$.

This settles the part of sufficiency of the observations corresponding to the given data points for generating a single observation at any data point $x \in [-1,1]$. For a given 'x', one would add another independent random variable Z_x to $\hat{Y}_x$, where Z_x is normally distributed with mean zero and variance $\frac{\sigma^2}{2}\left(1-x^2\right)$. Then $\hat{Y}_x + Z_x = \tilde{Y}_x$ would have distributional equivalence with Y_x.

Remark 5.3.1 We can generate $\tilde{Y}_{x_1}$ and $\tilde{Y}_{x_2}$ at two points x_1 and x_2 such that $-1 \le x_1, x_2 \le 1$ but they will not form a pair of independent random variables unless $x_1 + x_2 = 0$, i.e. $x_2 = -x_1$.

Solution: We have $\left(\frac{Y_+ + Y_-}{2}\right) + \left(\frac{Y_+ - Y_-}{2}\right)x_1 + Z_{x_1} = \frac{Y_+}{2}\left(1+x_1\right) + \frac{Y_-}{2}\left(1-x_1\right) + Z_{x_1} \equiv \tilde{Y}_{x_1}$ and

$\left(\frac{Y_+ + Y_-}{2}\right) + \left(\frac{Y_+ - Y_-}{2}\right)x_2 + Z_{x_2} = \frac{Y_+}{2}\left(1+x_2\right) + \frac{Y_-}{2}\left(1-x_2\right) + Z_{x_2} \equiv \tilde{Y}_{x_2}$, where

$Z_{x_i} \sim N\left(0, \frac{\sigma^2}{2}\left(1-x_i^2\right)\right), i = 1, 2$.

Here from $\tilde{Y}_{x_1}$, we can generate a random variable which has the same distribution as Y_{x_1} and from $\tilde{Y}_{x_2}$, we can generate a random variable which has the same distribution as Y_{x_2}. Now our problem is how to generate two independent random variables, i.e. $\tilde{Y}_{x_1}$ and $\tilde{Y}_{x_2}$. For this problem, we must have $\mathrm{Cov}(\tilde{Y}_{x_1}, \tilde{Y}_{x_2}) = 0$.

Let $x_2 = -x_1$. Then taking $Z_{x_2} = -Z_{x_1}$, we get

$$\mathrm{Cov}(\tilde{Y}_{x_1}, \tilde{Y}_{x_2}) = \mathrm{Cov}\left(\frac{Y_+}{2}\left(1+x_1\right) + \frac{Y_-}{2}\left(1-x_1\right) + Z_{x_1}, \frac{Y_+}{2}\left(1-x_1\right) + \frac{Y_-}{2}\left(1+x_1\right) - Z_{x_1}\right)$$

$$= \frac{1}{4}\left(1-x_1^2\right)\mathrm{Var}\left(\mathrm{Y}_+\right) + \frac{1}{4}\left(1-x_1^2\right)\mathrm{Var}\left(Y_-\right) - \mathrm{Var}\left(\mathrm{Z}_{\mathrm{x}_1}\right)$$

$$= \frac{\sigma^2}{4}\left(1-x_1^2\right) + \frac{\sigma^2}{4}\left(1-x_1^2\right) - \frac{\sigma^2}{2}\left(1-x_1^2\right) = 0.$$

Also it can be shown that this condition viz., $x_2 = -x_1$ is necessary for independence of $\tilde{Y}_{x_1}$ and $\tilde{Y}_{x_2}$. Therefore, we conclude that we can generate two independent random variables $\tilde{Y}_{x_1}$ and $\tilde{Y}_{x_2}$ at two points x_1 and x_2 if and only if $x_1 + x_2 = 0$, i.e. $x_2 = -x_1$. Generalizing this, we can conclude that we can generate any pair of two random variables $\tilde{Y}_{\pm x}$ using Y_{+1} and Y_{-1}. We can see that we can generate many pairs of independent random variables like $\tilde{Y}_{\pm 0.3}$, $\tilde{Y}_{\pm 0.9}$, …. However, we cannot claim that $\tilde{Y}_{0.3}$ and $\tilde{Y}_{0.9}$ are independent because $0.9 \neq -0.3$.

Remark 5.3.2 Taking three data points, i.e. one data point at – 1 to yield $Y_{-}^{(1)}$ at $x = -1$ and two data points at 1 so that $Y_{+}^{(1)}, Y_{+}^{(2)}$ at x = +1 are independently obtained, we can generate $\tilde{Y}_{x_1}, \tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ at three points x_1, x_2 and x_3 such that $-1 \le x_1, x_2, x_3 \le 1$ but they will *not* be independent random variables unless $x_1 + x_2 + x_3 = 1$.

This is summarized in the following Theorem 5.3.2.

Theorem 5.3.2 Based on $Y_{+}^{(1)}$, $Y_{+}^{(2)}$ and $Y_{-}^{(1)}$, we can generate three independent random variables $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ under the condition $x_1 + x_2 + x_3 = 1$ such that $\tilde{Y}_{x_i}$'s are independently distributed as in the linear regression model.

Solution: We are given three data points, i.e. $Y_{-}^{(1)}$ at $x = -1$ and $Y_{+}^{(1)}, Y_{+}^{(2)}$ at $x = +1$. From $Y_x = \alpha + \beta x + e_x$, we have $Y_+ \sim N\left(\alpha + \beta, \sigma^2\right)$ and $Y_- \sim N\left(\alpha - \beta, \sigma^2\right)$.

Let $\bar{Y}_+ = \dfrac{Y_{+}^{(1)} + Y_{+}^{(2)}}{2}$, then $\bar{Y}_+ \sim N\left(\alpha + \beta, \dfrac{\sigma^2}{2}\right)$.

Taking expectation of $\dfrac{\bar{Y}_+ + Y_-}{2}$, we obtain $E\left(\dfrac{\bar{Y}_+ + Y_-}{2}\right) = \alpha$ and taking expectation of $\dfrac{\bar{Y}_+ - Y_-}{2}$, we obtain $E\left(\dfrac{\bar{Y}_+ - Y_-}{2}\right) = \beta$.

Then the fitted value of Y_x is given by

$$\hat{Y}_x = \hat{\alpha} + \hat{\beta} x$$

$$= \left(\frac{\bar{Y}_+ + Y_-}{2}\right) + \left(\frac{\bar{Y}_+ - Y_-}{2}\right) x$$

$$= \frac{\overline{Y}_+}{2}(1+x)+\frac{Y_-}{2}(1-x).$$

Here, the expectation of $\hat{Y}_x$ is $E\left(\hat{Y}_x\right)=\alpha+\beta x$ and the variance is

$$\begin{aligned}
\mathrm{Var}\left(\hat{Y}_x\right) &= \mathrm{Var}\left[\left(\frac{\overline{Y}_+ + Y_-}{2}\right)+\left(\frac{\overline{Y}_+ - Y_-}{2}\right)x\right] \\
&= \mathrm{Var}\left[\frac{\overline{Y}_+}{2}(1+x)+\frac{Y_-}{2}(1-x)\right] \\
&= \frac{(1+x)^2}{4}\mathrm{Var}\left(\overline{Y}_+\right)+\frac{(1-x)^2}{4}\mathrm{Var}\left(Y_-\right) \\
&= \frac{(1+x)^2}{8}\sigma^2+\frac{(1-x)^2}{4}\sigma^2 \\
&= \frac{\sigma^2}{8}\left(3-2x+3x^2\right)\le\sigma^2 \text{ for } x\in[-1,1].
\end{aligned}$$

This settles the part of sufficiency of the observations corresponding to the given data points for generating a single observation at any data point $x\in[-1,1]$. For a given ‘x’, one would add another independent random variable Z_x to $\hat{Y}_x$, where Z_x is normally distributed with mean zero and variance $\frac{\sigma^2}{8}\left(5+2x-3x^2\right)$. Then $\hat{Y}_x+Z_x=\tilde{Y}_x$ would have distributional equivalence with Y_x.

Further using the transformation $\tilde{Y}_{x_i}=\left(\frac{\overline{Y}_+ + Y_-}{2}\right)+\left(\frac{\overline{Y}_+ - Y_-}{2}\right)x_i+Z_{x_i}$,

where $Z_{x_i}\sim N\left(0,\frac{\sigma^2}{8}\left(5+2x_i-3x_i^{\,2}\right)\right)$, we can generate $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ at three points x_1, x_2 and x_3 such that $-1\le x_1,x_2,x_3\le 1$ with the same distribution as Y_{x_1}, Y_{x_2} and Y_{x_3}. Now we can generate three independent random variables, i.e. $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ at three points x_1, x_2 and x_3 such that $-1\le x_1,x_2,x_3\le 1$, if $x_1+x_2+x_3=-1$ which we can proceed to establish as follows.

To make $\tilde{Y}_{x_1},\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ independent, we must have $\mathrm{Cov}(\tilde{Y}_{x_i},\tilde{Y}_{x_j})=0$.

Thus, we consider

$$\begin{aligned}\mathrm{Cov}(\tilde{Y}_{x_i},\tilde{Y}_{x_j}) &= \mathrm{Cov}\left(\frac{\bar{Y}_+}{2}(1+x_i)+\frac{Y_-}{2}(1-x_i)+Z_{x_i},\frac{\bar{Y}_+}{2}(1+x_j)+\frac{Y_-}{2}(1-x_j)+Z_{x_j}\right)\\
&= \frac{(1+x_i)(1+x_j)}{4}\mathrm{Var}(\bar{Y}_+)+\frac{(1-x_i)(1-x_j)}{4}\mathrm{Var}(Y_-)+\mathrm{Cov}(\mathrm{Z}_{x_i},\mathrm{Z}_{x_j})\\
&= \frac{\sigma^2}{8}(1+x_i)(1+x_j)+\frac{\sigma^2}{4}(1-x_i)(1-x_j)+\mathrm{Cov}(\mathrm{Z}_{x_i},\mathrm{Z}_{x_j})\\
&= \frac{\sigma^2}{8}(3-x_i-x_j+3x_ix_j)+\mathrm{Cov}(\mathrm{Z}_{x_i},\mathrm{Z}_{x_j}) \;=0.\end{aligned}$$

Therefore, $\mathrm{Cov}(\mathrm{Z}_{x_i},\mathrm{Z}_{x_j})=-\frac{\sigma^2}{8}(3-x_i-x_j+3x_ix_j)$ and

$$\begin{pmatrix}Z_1\\Z_2\\Z_3\end{pmatrix}\sim N_3\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\sigma^2\Lambda\right),\text{ where}$$

$$\Lambda=\begin{pmatrix}\frac{(5+2x_1-3x_1^2)}{8} & -\frac{(3-x_1-x_2+3x_1x_2)}{8} & -\frac{(3-x_1-x_3+3x_1x_3)}{8}\\ -\frac{(3-x_1-x_2+3x_1x_2)}{8} & \frac{(5+2x_2-3x_2^2)}{8} & -\frac{(3-x_2-x_3+3x_2x_3)}{8}\\ -\frac{(3-x_1-x_3+3x_1x_3)}{8} & -\frac{(3-x_2-x_3+3x_2x_3)}{8} & \frac{(5+2x_3-3x_3^2)}{8}\end{pmatrix}.$$

For the generation of three independent random variables, i.e. $\tilde{Y}_{x_1},\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$, we need to ensure that Λ is non-negative definite (nnd) under the condition $x_1+x_2+x_3=1$. Here for Λ, each row total is zero when $x_1+x_2+x_3=1$.

After simplification, we get

$$\Lambda=\frac{1}{8}\left\{8I_{3\times3}-\left[\begin{bmatrix}1+x_1\\1+x_2\\1+x_3\end{bmatrix}\begin{bmatrix}1+x_1 & 1+x_2 & 1+x_3\end{bmatrix}+2\begin{bmatrix}1-x_1\\1-x_2\\1-x_3\end{bmatrix}\begin{bmatrix}1-x_1 & 1-x_2 & 1-x_3\end{bmatrix}\right]\right\}$$

$$=\frac{1}{8}(8I_{3\times3}-A)=\frac{1}{8}B,$$

where $B=\left(8I_{3\times3}-A\right)$ and $A=\left[\begin{bmatrix}1+x_1\\1+x_2\\1+x_3\end{bmatrix}\begin{bmatrix}1+x_1 & 1+x_2 & 1+x_3\end{bmatrix}+2\begin{bmatrix}1-x_1\\1-x_2\\1-x_3\end{bmatrix}\begin{bmatrix}1-x_1 & 1-x_2 & 1-x_3\end{bmatrix}\right]$.

Further, Λ is nnd if $\frac{1}{8}B$ has non-negative eigenvalues.

This will be proved if we can show that $\left(8I_{3\times3}-A\right)$ has non-negative eigenvalues.

To show all eigenvalues of $\left(8I_{3\times3}-A\right)$ non-negative, we proceed as the following steps.

- $B\underset{\sim}{1}=\vec{0}$, $\underset{\sim}{1}$ is 3×1 column matrix with each element one. So 0 is one eigenvalue of B.
- $8I_{3\times3}-B=A$ has rank 2. So, B has one eigenvalue 8.
- Sum of eigenvalues of B is obtained as

$$\sum_{i=1}^{3}\lambda_i = \text{trace}\left(B\right) = 8\times3-\left[\sum_{i=1}^{3}(1+x_i)^2+2\sum_{i=1}^{3}(1-x_i)^2\right]$$

$$= 24-\left[3+2\sum_{i=1}^{3}x_i+\sum_{i=1}^{3}x_i^2+6-2\sum_{i=1}^{3}x_i+2\sum_{i=1}^{3}x_i^2\right]$$

$$= 17-3\sum_{i=1}^{3}x_i^2 .$$

The third root is $17-3\sum_{i=1}^{3}x_i^2-8 = 9-3\sum_{i=1}^{3}x_i^2$

$$= 3\sum_{i=1}^{3}\left(1-x_i^2\right)\geq 0, \text{ since } -1\leq x_i\leq 1 .$$

Thus, ultimately, the theorem is proved.

Remark 5.3.3 Taking 3 data points, i.e. $Y_-^{(1)}$, $Y_-^{(2)}$ at x = - 1 and $Y_+^{(1)}$ at x = + 1, we can generate random variables having the same distribution as Y_x for every single value x in [-1, 1]. Further we can generate $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ at three points x_1, x_2 and x_3 such that $-1\leq x_1, x_2, x_3\leq 1$ but they will not be independent random variables unless $x_1+x_2+x_3=-1$.

Solution: We are given three data points, i.e. $Y_-^{(1)}$, $Y_-^{(2)}$ at x = - 1 and $Y_+^{(1)}$ at x = +1.

From $Y_x=\alpha+\beta x+e_x$, we have $Y_-\sim N\left(\alpha-\beta,\sigma^2\right)$ and $Y_+\sim N\left(\alpha+\beta,\sigma^2\right)$.

Let $\bar{Y}_{-} = \dfrac{Y^{(1)}{}_{-} + Y_{-}^{(2)}}{2}$. Then, $\bar{Y}_{-} \sim N\left(\alpha - \beta, \dfrac{\sigma^2}{2}\right)$.

Taking the expectation of $\dfrac{Y_{+} + \bar{Y}_{-}}{2}$, we get $E\left(\dfrac{Y_{+} + \bar{Y}_{-}}{2}\right) = \alpha$ and taking the expectation of $\dfrac{Y_{+} - \bar{Y}_{-}}{2}$, we get $E\left(\dfrac{Y_{+} - \bar{Y}_{-}}{2}\right) = \beta$. Then the fitted value of Y_x is given by

$$\begin{aligned}
\hat{Y}_x &= \hat{\alpha} + \hat{\beta} x \\
&= \left(\frac{Y_{+} + \bar{Y}_{-}}{2}\right) + \left(\frac{Y_{+} - \bar{Y}_{-}}{2}\right) x \\
&= \frac{Y_{+}}{2}(1+x) + \frac{\bar{Y}_{-}}{2}(1-x).
\end{aligned}$$

The expectation of $\hat{Y}_x$ is $E\left(\hat{Y}_x\right) = \alpha + \beta x$ and the variance is

$$\begin{aligned}
\mathrm{Var}\left(\hat{Y}_x\right) &= \mathrm{Var}\left[\frac{Y_{+}}{2}(1+x) + \frac{\bar{Y}_{-}}{2}(1-x)\right] \\
&= \frac{(1+x)^2}{4}\mathrm{Var}\left(Y_{+}\right) + \frac{(1-x)^2}{8}\mathrm{Var}\left(\bar{Y}_{-}\right) \\
&= \frac{(1+x)^2}{4}\sigma^2 + \frac{(1-x)^2}{8}\sigma^2 \\
&= \frac{\sigma^2}{8}\left(3 + 2x + 3x^2\right) \le \sigma^2 \text{ for } x \in [-1,1].
\end{aligned}$$

This settles the part of sufficiency of the observations corresponding to the given data points for generating a single observation at any data point $x \in [-1,1]$. For a given 'x', one would add another independent random variable Z_x to $\hat{Y}_x$, where Z_x is normally distributed with mean zero and variance $\dfrac{\sigma^2}{8}\left(5 - 2x - 3x^2\right)$. Then $\hat{Y}_x + Z_x = \tilde{Y}_x$ would have distributional equivalence with Y_x.

Further using the transformation $\tilde{Y}_{x_i} = \left(\dfrac{Y_{+} + \bar{Y}_{-}}{2}\right) + \left(\dfrac{Y_{+} - \bar{Y}_{-}}{2}\right) x_i + Z_{x_i}$, where $Z_{x_i} \sim N\left(0, \frac{\sigma^2}{8}\left(5-2x_i-3x_i^2\right)\right)$, we can generate $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ at three points x_1, x_2 and x_3 such that $-1 \le x_1, x_2, x_3 \le 1$ with the same distribution as Y_{x_1}, Y_{x_2} and Y_{x_3}. Now we can generate three independent random variables, i.e. $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ at three points x_1, x_2 and x_3 such that $-1 \le x_1, x_2, x_3 \le 1$, if $x_1 + x_2 + x_3 = -1$ which we can proceed to establish as follows.

To make $\tilde{Y}_{x_1}, \tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$ independent, we must have $\mathrm{Cov}(\tilde{Y}_{x_i}, \tilde{Y}_{x_j}) = 0$. Hence,

$$
\begin{aligned}
\mathrm{Cov}(\tilde{Y}_{x_i}, \tilde{Y}_{x_j}) &= \mathrm{Cov}\left(\frac{Y_+}{2}(1+x_i)+\frac{\bar{Y}_-}{2}(1-x_i)+Z_{x_i}, \frac{Y_+}{2}(1+x_j)+\frac{\bar{Y}_-}{2}(1-x_j)+Z_{x_j}\right) \\
&= \frac{(1-x_i)(1-x_j)}{4}\mathrm{Var}\left(\bar{Y}_-\right)+\frac{(1+x_i)(1+x_j)}{4}\mathrm{Var}\left(Y_+\right)+\mathrm{Cov}\left(Z_{x_i}, Z_{x_j}\right) \\
&= \frac{\sigma^2}{8}(1-x_i)(1-x_j)+\frac{\sigma^2}{4}(1+x_i)(1+x_j)+\mathrm{Cov}\left(Z_{x_i}, Z_{x_j}\right) \\
&= \frac{\sigma^2}{8}\left(3+x_i+x_j+3x_ix_j\right)+\mathrm{Cov}\left(Z_{x_i}, Z_{x_j}\right) = 0.
\end{aligned}
$$

Therefore, $\mathrm{Cov}\left(Z_{x_i}, Z_{x_j}\right) = -\frac{\sigma^2}{8}\left(3+x_i+x_j+3x_ix_j\right)$ and

$$
\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma^2\Lambda\right), \text{ where}
$$

$$
\Lambda = \begin{pmatrix}
\frac{\left(5-2x_1-3x_1^2\right)}{8} & -\frac{\left(3+x_1+x_2+3x_1x_2\right)}{8} & -\frac{\left(3+x_1+x_3+3x_1x_3\right)}{8} \\
-\frac{\left(3+x_1+x_2+3x_1x_2\right)}{8} & \frac{\left(5-2x_2-3x_2^2\right)}{8} & -\frac{\left(3+x_2+x_3+3x_2x_3\right)}{8} \\
-\frac{\left(3+x_1+x_3+3x_1x_3\right)}{8} & -\frac{\left(3+x_2+x_3+3x_2x_3\right)}{8} & \frac{\left(5-2x_3-3x_3^2\right)}{8}
\end{pmatrix}.
$$

For the generation of three independent random variables, i.e. $\tilde{Y}_{x_1}, \tilde{Y}_{x_2}$ and $\tilde{Y}_{x_3}$, we need to ensure that Λ is nnd under the condition $x_1 + x_2 + x_3 = -1$. Here for Λ, each row total is zero when $x_1 + x_2 + x_3 = -1$.

After simplification, we obtain

$$\Lambda = \frac{1}{8}\left\{8I_{3\times3} - \left[2\begin{bmatrix}1+x_1\\1+x_2\\1+x_3\end{bmatrix}\begin{bmatrix}1+x_1 & 1+x_2 & 1+x_3\end{bmatrix} + \begin{bmatrix}1-x_1\\1-x_2\\1-x_3\end{bmatrix}\begin{bmatrix}1-x_1 & 1-x_2 & 1-x_3\end{bmatrix}\right]\right\}$$

$$= \frac{1}{8}\left(8I_{3\times3} - A\right) = \frac{1}{8}B,$$

where $B = \left(8I_{3\times3} - A\right)$ and

$$A = \left[2\begin{bmatrix}1+x_1\\1+x_2\\1+x_3\end{bmatrix}\begin{bmatrix}1+x_1 & 1+x_2 & 1+x_3\end{bmatrix} + \begin{bmatrix}1-x_1\\1-x_2\\1-x_3\end{bmatrix}\begin{bmatrix}1-x_1 & 1-x_2 & 1-x_3\end{bmatrix}\right].$$

Further, Λ is nnd if $\frac{1}{8}B$ has non-negative eigenvalues.

This will be proved if we can show that $\left(8I_{3\times3} - A\right)$ has non-negative eigenvalues.

To show all eigenvalues of $\left(8I_{3\times3} - A\right)$ non-negative, we proceed as the following steps.

- $B\underset{\sim}{1} = \vec{0}$, $\underset{\sim}{1}$ is 3×1 column matrix with each element one. So 0 is one eigenvalue of B.
- $8I_{3\times3} - B = A$ has rank 2. So, B has one eigenvalue 8.
- Sum of eigenvalues of B is obtained as

$$\sum_{i=1}^{3}\lambda_i = \text{trace}(B) = 8\times3 - \left[2\sum_{i=1}^{3}(1+x_i)^2 + \sum_{i=1}^{3}(1-x_i)^2\right]$$

$$= 24 - \left[6 + 4\sum_{i=1}^{3}x_i + 2\sum_{i=1}^{3}x_i^2 + 3 - 2\sum_{i=1}^{3}x_i + \sum_{i=1}^{3}x_i^2\right]$$

$$= 17 - 3\sum_{i=1}^{3}x_i^2 .$$

The third root is $17 - 3\sum_{i=1}^{3}x_i^2 - 8 = 9 - 3\sum_{i=1}^{3}x_i^2$

$$= 3\sum_{i=1}^{3}\left(1 - x_i^2\right) \geq 0, \text{ since } -1 \leq x_i \leq 1 .$$

Thus, ultimately, the theorem is proved.

Remark 5.3.4 Taking four data points, i.e. two viz., $Y_{-}^{(1)}, Y_{-}^{(2)}$ at x = - 1 and other two viz., $Y_{+}^{(1)}, Y_{+}^{(2)}$ at x = + 1, we can generate random variables which have the same distribution as Y_x

for each x in [-1, 1]. Further we can generate $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$, $\tilde{Y}_{x_3}$ and $\tilde{Y}_{x_4}$ at four points x_1, x_2, x_3 and x_4 such that $-1 \le x_1, x_2, x_3, x_4 \le 1$ but they will not be independent unless $x_1 + x_2 + x_3 + x_4 = 0$.

Solution: Given $Y_+^{(1)}$, $Y_+^{(2)}$, $Y_-^{(1)}$ and $Y_-^{(2)}$.

Let $\bar{Y}_+ = \dfrac{Y_+^{(1)} + Y_+^{(2)}}{2}$ and $\bar{Y}_- = \dfrac{Y_-^{(1)} + Y_-^{(2)}}{2}$.

Then, $\bar{Y}_+ \sim N\left(\alpha + \beta, \dfrac{\sigma^2}{2}\right)$ and $\bar{Y}_- \sim N\left(\alpha - \beta, \dfrac{\sigma^2}{2}\right)$.

Taking the expectation of $\dfrac{\bar{Y}_+ + \bar{Y}_-}{2}$, we obtain $E\left(\dfrac{\bar{Y}_+ + \bar{Y}_-}{2}\right) = \alpha$ and the variance of $\dfrac{\bar{Y}_+ + \bar{Y}_-}{2}$ is

$\mathrm{Var}\left(\dfrac{\bar{Y}_+ + \bar{Y}_-}{2}\right) = \dfrac{\sigma^2}{4}$. Hence, $\dfrac{\bar{Y}_+ + \bar{Y}_-}{2} \sim N\left(\alpha, \dfrac{\sigma^2}{4}\right)$.

Similarly, taking the expectation of $\dfrac{\bar{Y}_+ - \bar{Y}_-}{2}$, we obtain $E\left(\dfrac{\bar{Y}_+ - \bar{Y}_-}{2}\right) = \beta$ and the variance of $\dfrac{\bar{Y}_+ - \bar{Y}_-}{2}$ is $\mathrm{Var}\left(\dfrac{\bar{Y}_+ - \bar{Y}_-}{2}\right) = \dfrac{\sigma^2}{4}$.

Hence, $\dfrac{\bar{Y}_+ - \bar{Y}_-}{2} \sim N\left(\beta, \dfrac{\sigma^2}{4}\right)$. Then the fitted value of Y_x is given by

$$\begin{aligned}
\hat{Y}_x &= \hat{\alpha} + \hat{\beta}x \\
&= \left(\frac{\bar{Y}_+ + \bar{Y}_-}{2}\right) + \left(\frac{\bar{Y}_+ - \bar{Y}_-}{2}\right)x \\
&= \frac{\bar{Y}_+}{2}(1+x) + \frac{\bar{Y}_-}{2}(1-x).
\end{aligned}$$

Here, the expected value of $\hat{Y}_x$ is $E\left(\hat{Y}_x\right) = \alpha + \beta x$ and the variance is

$$\begin{aligned}
\mathrm{Var}\left(\hat{Y}_x\right) &= \mathrm{Var}\left[\frac{\bar{Y}_+}{2}(1+x) + \frac{\bar{Y}_-}{2}(1-x)\right] \\
&= \frac{(1+x)^2}{8}\sigma^2 + \frac{(1-x)^2}{8}\sigma^2 \\
&= \frac{\sigma^2}{4}\left(1+x^2\right) \le \sigma^2 \text{ for } x \in [-1,1].
\end{aligned}$$

As before, for a given 'x', one would add another independent random variable Z_x to $\hat{Y}_x$, where Z_x is normally distributed with mean zero and variance $\frac{\sigma^2}{4}\left(3-x^2\right)$. Then $\hat{Y}_x + Z_x = \tilde{Y}_x$ would have distributional equivalence with Y_x. This settles the part of sufficiency of the observations corresponding to the given data points for generating a single observation at any data point $x \in [-1,1]$.

Further using the transformation $\tilde{Y}_{x_i} = \left(\frac{\bar{Y}_+ + \bar{Y}_-}{2}\right) + \left(\frac{\bar{Y}_+ - \bar{Y}_-}{2}\right) x_i + Z_{x_i}$ where

$Z_{x_i} \sim N\left(0, \frac{\sigma^2}{4}\left(3 - x_i^{\,2}\right)\right)$, we can generate $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$, $\tilde{Y}_{x_3}$ and $\tilde{Y}_{x_4}$ at four points x_1, x_2, x_3 and x_4 such that $-1 \le x_1, x_2, x_3, x_4 \le 1$ with the same distribution as Y_{x_1}, Y_{x_2}, Y_{x_3} and Y_{x_4}. Now we can generate four independent random variables, i.e. $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$, $\tilde{Y}_{x_3}$ and $\tilde{Y}_{x_4}$ at four points x_1, x_2, x_3 and x_4 such that $-1 \le x_1, x_2, x_3, x_4 \le 1$, if $x_1 + x_2 + x_3 + x_4 = 0$ which we can proceed to establish as follows.

To make $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$, $\tilde{Y}_{x_3}$ and $\tilde{Y}_{x_4}$ independent, we must have

$\mathrm{Cov}(\tilde{Y}_{x_i}, \tilde{Y}_{x_j}) = 0$. Then, we obtain

$$\begin{aligned}
\mathrm{Cov}(\tilde{Y}_{x_i}, \tilde{Y}_{x_j}) &= \mathrm{Cov}\left(\frac{\bar{Y}_+}{2}(1+x_i) + \frac{\bar{Y}_-}{2}(1-x_i) + Z_{x_i}, \frac{\bar{Y}_+}{2}\left(1+x_j\right) + \frac{\bar{Y}_-}{2}\left(1-x_j\right) + Z_{x_j}\right) \\
&= \frac{(1-x_i)(1-x_j)}{4}\mathrm{Var}\left(\bar{Y}_-\right) + \frac{(1+x_i)(1+x_j)}{4}\mathrm{Var}\left(\bar{Y}_+\right) + \mathrm{Cov}\left(\mathrm{Z}_{x_i}, \mathrm{Z}_{x_j}\right) \\
&= \frac{\sigma^2}{8}(1-x_i)(1-x_j) + \frac{\sigma^2}{8}(1+x_i)(1+x_j) + \mathrm{Cov}\left(\mathrm{Z}_{x_i}, \mathrm{Z}_{x_j}\right) \\
&= \frac{\sigma^2}{4}\left(1 + x_i x_j\right) + \mathrm{Cov}\left(\mathrm{Z}_{x_i}, \mathrm{Z}_{x_j}\right) = 0.
\end{aligned}$$

Therefore, $\mathrm{Cov}\left(\mathrm{Z}_{x_i}, \mathrm{Z}_{x_j}\right) = -\frac{\sigma^2}{4}\left(1 + x_i x_j\right)$ and

$$\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix} \sim N_4\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \sigma^2\Lambda\right), \text{ where}$$

$$\Lambda = \begin{pmatrix} \frac{(3-x_1^2)}{4} & -\frac{(1+x_1x_2)}{4} & -\frac{(1+x_1x_3)}{4} & -\frac{(1+x_1x_4)}{4} \\ -\frac{(1+x_1x_2)}{4} & \frac{(3-x_2^2)}{4} & -\frac{(1+x_2x_3)}{4} & -\frac{(1+x_2x_4)}{4} \\ -\frac{(1+x_1x_3)}{4} & -\frac{(1+x_2x_3)}{4} & \frac{(3-x_3^2)}{4} & -\frac{(1+x_3x_4)}{4} \\ -\frac{(1+x_1x_4)}{4} & -\frac{(1+x_2x_4)}{4} & -\frac{(1+x_3x_4)}{4} & \frac{(3-x_4^2)}{4} \end{pmatrix}.$$

Here for the generation of four independent random variables, i.e. $\tilde{Y}_{x_1}$, $\tilde{Y}_{x_2}$, $\tilde{Y}_{x_3}$ and $\tilde{Y}_{x_4}$, we need to ensure that Λ is nnd. Here for each row of Λ, row total is zero when $x_1+x_2+x_3+x_4=0$ such that $-1 \le x_1, x_2, x_3, x_4 \le 1$.

After simplification, we obtain

$$\Lambda = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

Further Λ is nnd if it has non-negative eigenvalues.

Here $\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$ has eigenvalues 0, 4, 4 and 4

and the eigenvectors are respectively given by $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$

Similarly $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$ has eigenvalues 0, 0, 0 and $x_1^{\ 2}+x_2^{\ 2}+x_3^{\ 2}+x_4^{\ 2}$

and the corresponding eigenvectors are given by

$\gamma_1=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \gamma_2, \gamma_3$ linear in $\alpha_2, \alpha_3, \alpha_4$, and $\gamma_4=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

But $x_1^{\ 2}+x_2^{\ 2}+x_3^{\ 2}+x_4^{\ 2}\le 4$ since $-1\le x_i \le 1$.

Hence, the eigenvalues of Λ are given by 0, 1, 1, $1-\dfrac{\sum_{i=1}^{k} x_i^{\ 2}}{4}\ge 0$.

Therefore, Λ has non-negative eigenvalues. This proves Remark 5.3.4.

5.3.2 Quadratic regression model with intercept term

Here we consider a quadratic regression model, $Y_x=\alpha+\beta x+\gamma x^2+e_x$, where Y_x is dependent variable, $x\in[-1,1]$ is independent variable (non-stochastic), $e_x\sim N\left(0,\sigma^2\right)$, σ^2 is known and α, β, γ are unknown parameters. Then we apply the concept of sufficiency in this model to generate a random variable which has the same distribution as Y_x. Further we present some important remarks, special case and general case related to the sufficient problems of our model.

Theorem 5.3.3 Given a quadratic regression model with intercept term, $Y_x=\alpha+\beta x+\gamma x^2+e_x$**,** where $e_x\sim N\left(0,\sigma^2\right)$, $x\in[-1,1]$, σ^2 is known and α, β, γ are unknown parameters, we claim that data points Y_{+1}, Y_{-1} and Y_0 are jointly sufficient for generating Y_x for any given x only inside [-1, 1]. Further, for any two given x' s, say x_1 and x_2 inside [-1, 1], Y_{x_1} and Y_{x_2} can not always be simultaneously independently generated. However,

(i) assuming $x_1+x_2=0$, this is possible iff $x_1\left(\text{assumed positive}\right)\ge\dfrac{1}{\sqrt{3}}$

(ii) assuming $x_1^2 + x_2^2 = 1$, this is possible iff $x_1\left(\text{assumed positive}\right)$ satisfies the condition: $0.1691 < x_1 < 0.9856$ and x_2 is negative.

Proof: From the given model, we have $Y_x \sim N\left(\alpha+\beta x+\gamma x^2,\sigma^2\right)$.

Further, $Y_{+1} = \alpha+\beta+\gamma+e_{+1}$ and $Y_{+1} \sim N\left(\alpha+\beta+\gamma,\sigma^2\right)$,

$Y_{-1} = \alpha-\beta+\gamma+e_{-1}$ and $Y_{-1} \sim N\left(\alpha-\beta+\gamma,\sigma^2\right)$.

At $x = 0$, $Y_0 = \alpha+e_0$ and $Y_0 \sim N\left(\alpha,\sigma^2\right)$.

Taking the expectation of $\dfrac{Y_+ + Y_-}{2}$, we obtain $E\left(\dfrac{Y_+ + Y_-}{2}\right) = \alpha+\gamma$ and taking the expectation of $\dfrac{Y_+ - Y_-}{2}$, we obtain $E\left(\dfrac{Y_+ - Y_-}{2}\right) = \beta$. Then, the fitted value of Y_x is given by $\hat{Y}_x$

$$= \hat{\alpha}+\hat{\beta}x+\hat{\gamma}x^2$$

$$= Y_0\left(1-x^2\right)+\left(\frac{Y_+ - Y_-}{2}\right)x+\left(\frac{Y_+ + Y_-}{2}\right)x^2$$

$$= Y_0\left(1-x^2\right)+\frac{Y_+}{2}\left(x^2+x\right)+\frac{Y_-}{2}\left(x^2-x\right).$$

Here, the expected value of $\hat{Y}_x$ is $E(\hat{Y}_x) = \alpha+\beta x+\gamma x^2$ and the variance is

$$\begin{aligned}
\mathrm{Var}\left(\hat{Y}_x\right) &= \mathrm{Var}\left[Y_0\left(1-x^2\right)+\frac{Y_+}{2}\left(x^2+x\right)+\frac{Y_-}{2}\left(x^2-x\right)\right] \\
&= \left(1-x^2\right)^2\sigma^2+\frac{\left(x^2+x\right)^2}{4}\sigma^2+\frac{\left(x^2-x\right)^2}{4}\sigma^2 \\
&= \frac{\sigma^2}{2}\left(2-3x^2+3x^4\right) \le \sigma^2 \text{ for } x \in [-1,1].
\end{aligned}$$

As before, for a given 'x', one would add another independent random variable Z_x to $\hat{Y}_x$, where Z_x is normally distributed with mean zero and variance $\dfrac{3\sigma^2}{2}x^2\left(1-x^2\right)$. Then $\hat{Y}_x + Z_x = \tilde{Y}_x$ would have distributional equivalence with Y_x. This settles the part of sufficiency of the observations corresponding to the given data points for generating a single observation at any data point $x \in [-1,1]$.

For generating simultaneously two independent observations corresponding to two data points, say x_1 and x_2; different from one another and each lying inside [-1, 1], we proceed as follows. First we compute an expression for the covariance between $\hat{Y}_{x_1}$ and $\hat{Y}_{x_2}$.

$$Cov\left(\hat{Y}_{x_1},\hat{Y}_{x_2}\right)$$

$$= Cov\left(Y_0\left(1-x_1^2\right)+\frac{Y_+}{2}\left(x_1^2+x_1\right)+\frac{Y_-}{2}\left(x_1^2-x_1\right),Y_0\left(1-x_2^2\right)+\frac{Y_+}{2}\left(x_2^2+x_2\right)+\frac{Y_-}{2}\left(x_2^2-x_2\right)\right)$$

$$= \sigma^2\left[\left(1-x_1^2\right)\left(1-x_2^2\right)+\frac{\left(x_1+x_1^2\right)\left(x_2+x_2^2\right)}{4}+\frac{\left(x_1-x_1^2\right)\left(x_2-x_2^2\right)}{4}\right]$$

$$=\sigma^2\left[1-x_1^2-x_2^2+\frac{x_1x_2}{2}+\frac{3x_1^2x_2^2}{2}\right].$$

Next we define the 2×2 matrix $\sigma^2 W=\sigma^2\left(w_{ij}\right)$ as the variance-covariance matrix of $[\hat{Y}_{x_1},\hat{Y}_{x_2}]$ where

$$W=\begin{pmatrix} 1-\frac{3x_1^2}{2}+\frac{3x_1^4}{2} & 1-x_1^2-x_2^2+\frac{x_1x_2}{2}+\frac{3x_1^2x_2^2}{2} \\ 1-x_1^2-x_2^2+\frac{x_1x_2}{2}+\frac{3x_1^2x_2^2}{2} & 1-\frac{3x_2^2}{2}+\frac{3x_2^4}{2} \end{pmatrix}.$$

Then for simultaneous independent generation of Y_{x_1} and Y_{x_2}, we need to ensure that the matrix $A\left(x_1,x_2\right)=I_{2\times 2}-W$ is nnd.

Once this is achieved, we can define two auxiliary random variables Z_1 and Z_2 having jointly bivariate normal distribution with means zeroes and variance -covariance matrix given by the above matrix *A* multiplied by σ^2. This we do independently of the observations realized at the data points $\left[\pm 1,0\right]$. Clearly, this condition is also necessary.

The 2×2 matrix *A* above, after simplification, is given by

$$\begin{pmatrix} \frac{3x_1^2\left(1-x_1^2\right)}{2} & -\left(1-x_1^2-x_2^2+\frac{x_1x_2}{2}+\frac{3x_1^2x_2^2}{2}\right) \\ -\left(1-x_1^2-x_2^2+\frac{x_1x_2}{2}+\frac{3x_1^2x_2^2}{2}\right) & \frac{3x_2^2\left(1-x_2^2\right)}{2} \end{pmatrix}.$$

We now examine the nnd – ness of ‘*A*’. Note that the diagonal elements of *A* are always non-negative since $-1 \le x_1 \ne x_2 \le 1$. Here the determinant of 'A' simplifies to $\frac{1}{4}\left[-4+8x_1^2+8x_2^2-4x_1x_2-4x_1^4-4x_2^4-12x_1^2x_2^2+4x_1^3x_2+4x_1x_2^3+3x_1^4x_2^2+3x_1^2x_2^4-6x_1^3x_2^3\right]$.**Special Case:** Set $x_1 = -x_2 = t$, so that $x_1 + x_2 = 0$. Then the determinant of A is given by

$$\phi(t) = \left[\frac{3t^2\left(1-t^2\right)}{2}\right]^2 - \left[1-2t^2-\frac{t^2}{2}+\frac{3t^4}{2}\right]^2 = F_1.F_2, \text{ where}$$

$$F_1 = \frac{3t^2\left(1-t^2\right)}{2}+1-2t^2-\frac{t^2}{2}+\frac{3t^4}{2} = 1-t^2 \text{ and}$$

$$\begin{aligned} F_2 &= \frac{3t^2\left(1-t^2\right)}{2}-1+2t^2+\frac{t^2}{2}-\frac{3t^4}{2} \\ &= -1+4t^2-3t^4 \\ &= (-1)\left(1-t^2\right)\left(1-3t^2\right). \end{aligned}$$

Hence $\phi(t) = (-1)\left(1-t^2\right)^2\left(1-3t^2\right) > 0$ iff $t^2 > \frac{1}{3}$, i.e. $t > \frac{1}{\sqrt{3}}$.

General Case: In the most general case, results may be uninteresting. However, we have an interesting result in the case: $x_1^2 + x_2^2 = 1$, $x_2 < 0 < x_1$.

Set $x_1^2 = t$. Clearly, in this case,

$$a_{11} = a_{22} = \frac{3t(1-t)}{2} \text{ and } a_{12} = a_{21} = (-1)\frac{\left[x_1x_2+3t(1-t)\right]}{2}.$$

The matrix A will be nnd iff

$$\left[3t(1-t)\right]^2 > \left[x_1x_2+3t(1-t)\right]^2,$$

$$\left[6t(1-t)+x_1x_2\right]\left[(-1)x_1x_2\right] > 0.$$

Since $x_1x_2 < 0$, the above holds iff

$6t(1-t) > (-1)x_1x_2$, i.e. $36t(1-t) > 1$, i.e. $t(1-t) > \frac{1}{36}$ and this yields $0.0286 < t < 0.9714$.

These in turn lead to: $0.1691 < x_1 < 0.9856$, $x_2^2 = 1 - x_1^2$ and x_2 is negative.

Other reciprocal choice corresponds to x_1 being negative and x_2 being positive.

With this we conclude the proof of Theorem 5.3.3.

5.3.3 Quadratic regression model without intercept term

Here we consider a quadratic regression model, $Y_x = \beta x + \gamma x^2 + e_x$, where Y_x is dependent variable, $x \in [-1,1]$ is independent variable(non-stochastic), $e_x \sim N\left(0, \sigma^2\right)$, σ^2 is known and β, γ are unknown parameters. Then we apply the concept of sufficiency in this model to generate a random variable which has the same distribution as Y_x. Further we present some important generalizations.

Theorem 5.3.4 Given a quadratic regression model without the intercept term, $Y_x = \beta x + \gamma x^2 + e_x$, where $e_x \sim N\left(0, \sigma^2\right)$, $x \in [-1,1]$, σ^2 is known and β, γ are unknown parameters, we claim that data points Y_{+1} and Y_{-1} are jointly sufficient for generating Y_x for any given *x* only inside [-1, 1].

Proof: From the given model, we have $Y_x \sim N\left(\beta x + \gamma x^2, \sigma^2\right)$.

Further $Y_{+1} = \beta + \gamma + e_{+1}$ and $Y_{+1} \sim N\left(\beta + \gamma, \sigma^2\right)$.

$Y_{-1} = -\beta + \gamma + e_{-1}$ and $Y_{-1} \sim N\left(-\beta + \gamma, \sigma^2\right)$.

Expected value of $\dfrac{Y_+ + Y_-}{2}$ is $E\left(\dfrac{Y_+ + Y_-}{2}\right) = \gamma$ and expected value of $\dfrac{Y_+ - Y_-}{2}$ is $E\left(\dfrac{Y_+ - Y_-}{2}\right) = \beta$. Then, the fitted value of Y_x is given by

$$\begin{aligned}
\hat{Y}_x &= \hat{\beta} x + \hat{\gamma} x^2 \\
&= \left(\frac{Y_+ - Y_-}{2}\right) x + \left(\frac{Y_+ + Y_-}{2}\right) x^2 \\
&= \frac{Y_+}{2}\left(x^2 + x\right) + \frac{Y_-}{2}\left(x^2 - x\right).
\end{aligned}$$

Expectation of $\hat{Y}_x$ is $E(\hat{Y}_x) = \beta x + \gamma x^2$ and the variance is

$$\begin{aligned}
\operatorname{Var}\left(\hat{Y}_x\right) &= \operatorname{Var}\left[\frac{Y_+}{2}\left(x^2 + x\right) + \frac{Y_-}{2}\left(x^2 - x\right)\right] \\
&= \frac{\sigma^2}{4}\left(x^2 + x\right)^2 + \frac{\sigma^2}{4}\left(x^2 - x\right)^2
\end{aligned}$$

$$= \frac{\sigma^2}{2}\left(x^2 + x^4\right) \le \sigma^2 \text{ for } x \in [-1,1].$$

As before, for a given 'x', one would add another independent random variable Z_x to $\hat{Y}_x$, where Z_x is normally distributed with mean zero and variance $\frac{\sigma^2}{2}\left(2 - x^2 - x^4\right)$. Then $\hat{Y}_x + Z_x = \tilde{Y}_x$ would have distributional equivalence with Y_x. This settles the part of sufficiency of the observations corresponding to the given data points for generating a single observation at any data point $x \in [-1,1]$.

Remark 5.3.5 We can generate Y_{x_1} and Y_{x_2} at two points x_1 and x_2 such that $-1 \le x_1, x_2 \le 1$ and they will form a pair of independent random variables when $x_1 + x_2 = 0$, i.e. $x_2 = -x_1$.

Proof: We have $\frac{Y_+}{2}\left(x_1^{\,2} + x_1\right) + \frac{Y_-}{2}\left(x_1^{\,2} - x_1\right) + Z_{x_1} \equiv \tilde{Y}_{x_1}$ and

$\frac{Y_+}{2}\left(x_2^{\,2} + x_2\right) + \frac{Y_-}{2}\left(x_2^{\,2} - x_2\right) + Z_{x_2} \equiv \tilde{Y}_{x_2}$, where $Z_{x_i} \sim N\left(0, \frac{\sigma^2}{2}\left(2 - x_i^{\,2} - x_i^{\,4}\right)\right), i = 1, 2$.

Here from $\tilde{Y}_{x_1}$, we can generate a random variable which has the same distribution as Y_{x_1} and from $\tilde{Y}_{x_2}$, we can generate a random variable which has the same distribution as Y_{x_2}.

For generating simultaneously two independent observations corresponding to the two data points, say x_1 and x_2; different from one another and each lying inside [-1, 1], we proceed as follows. First we compute an expression for the covariance between $\hat{Y}_{x_1}$ and $\hat{Y}_{x_2}$ under the condition $x_2 = -x_1$.

$$Cov\left(\hat{Y}_{x_1}, \hat{Y}_{x_2}\right) = Cov\left(\hat{Y}_{x_1}, \hat{Y}_{-x_1}\right)$$

$$= Cov\left(\frac{Y_+}{2}\left(x_1^{\,2} + x_1\right) + \frac{Y_-}{2}\left(x_1^{\,2} - x_1\right), \frac{Y_+}{2}\left(x_1^{\,2} - x_1\right) + \frac{Y_-}{2}\left(x_1^{\,2} + x_1\right)\right)$$

$$= (-1)\frac{\sigma^2 x_1^{\,2}\left(1 - x_1^{\,2}\right)}{2}.$$

Next we define the 2×2 matrix $\sigma^2 W = \sigma^2\left(w_{ij}\right)$ as the variance-covariance matrix of $[\hat{Y}_{x_1}, \hat{Y}_{x_2}]$ under the condition $x_2 = -x_1$ where

$$W = \begin{pmatrix} \dfrac{x_1^2\left(1+x_1^2\right)}{2} & \dfrac{-x_1^2\left(1-x_1^2\right)}{2} \\ \dfrac{-x_1^2\left(1-x_1^2\right)}{2} & \dfrac{x_1^2\left(1+x_1^2\right)}{2} \end{pmatrix}.$$

Then for simultaneous independent generation of Y_{x_1} and Y_{x_2} under the condition $x_2 = -x_1$, we need to ensure that the matrix $A\left(x_1, x_2\right) = I_{2\times 2} - W$ is nnd.

Once this is achieved, we can define two auxiliary random variables Z_1 and Z_2 having jointly bivariate normal distribution with means zeros and variance -covariance matrix given by the above matrix *A* multiplied by σ^2. This we do independently of the observations realized at the data points $\left[\pm 1\right]$. Clearly, this condition is also necessary.

The 2×2 matrix *A* above, after simplification, is given by

$$\begin{pmatrix} \dfrac{2-x_1^2-x_1^4}{2} & \dfrac{x_1^2\left(1-x_1^2\right)}{2} \\ \dfrac{x_1^2\left(1-x_1^2\right)}{2} & \dfrac{2-x_1^2-x_1^4}{2} \end{pmatrix}.$$

We now examine the nnd – ness of '*A*'. Note that the diagonal elements of *A* are always non-negative since $-1 \le x_1 \ne x_2 \le 1$. Here the determinant of '*A*' simplifies to

$$\left[\frac{2-x_1^2-x_1^4}{2}\right]^2 - \left[\frac{x_1^2\left(1-x_1^2\right)}{2}\right]^2 = \frac{1}{4}\left[\left(2-x_1^2-x_1^4\right)^2 - \left\{x_1^2\left(1-x_1^2\right)\right\}^2\right]$$

$$= \left(1-x_1^2\right)\left(1-x_1^4\right) \ge 0 \text{ for } x \in [-1,1].$$

Hence we can generate simultaneously two independent observations corresponding to two data points, say x_1 and x_2; different from one another and each lying inside [-1, 1] when $x_1 + x_2 = 0$, i.e. $x_2 = -x_1$.

It is interesting to note that $\hat{Y}_x$ and $\hat{Y}_{-x}$ are themselves orthogonal iff $x = \pm 1$.

Remark 5.3.6 We can generate simultaneously two independent random variables, i.e. Y_{x_1} and Y_{x_2} at two points x_1 and x_2 such that $-1 \le x_1, x_2 \le 1$ even when $x_1 + x_2 \ne 0$.

Proof: We have $Cov\left(\hat{Y}_{x_1}, \hat{Y}_{x_2}\right) = \dfrac{\sigma^2 x_1 x_2 \left(1 + x_1 x_2\right)}{2}$.

Next we define the 2×2 variance-covariance matrix of $\hat{Y}_{x_1}$ and $\hat{Y}_{x_2}$ and let it be $\sigma^2 W$.

Then for simultaneous and independent generation of Y_{x_1} and Y_{x_2}, we need to ensure that the matrix $A = I_{2\times 2} - W$ is nnd. Hence, we have to show that

$$A^* = \begin{pmatrix} 2 - x_1^{\ 2} - x_1^{\ 4} & -x_1 x_2 \left(1 + x_1 x_2\right) \\ -x_1 x_2 \left(1 + x_1 x_2\right) & 2 - x_2^{\ 2} - x_2^{\ 4} \end{pmatrix} \text{ is nnd.}$$

The diagonal elements of A^* are non-negative. For the determinant, it simplifies to

$$4 - 2\left(x_1^{\ 2} + x_2^{\ 2}\right) - 2\left(x_1^{\ 4} + x_2^{\ 4}\right) + x_1^{\ 2} x_2^{\ 2} \left(x_1 - x_2\right)^2. \qquad (5.3.1)$$

Set $x_2 = x_1 . p$ and take $p = \dfrac{1}{2}$.

Then the above expression simplifies to $\dfrac{64 - 40x_1^{\ 2} - 34x_1^{\ 4} + x_1^{\ 6}}{16}$. (5.3.2)

The limiting value of the above [as $x_1 \to 1$] is $-\dfrac{9}{16}$ which is negative.

There are plenty of pair-wise values of x_1 and x_2 for which simultaneous and independent generation of $\left[Y_{x_1}, Y_{x_2}\right]$ is not possible.

For example, with $x_1^{\ 2} = 0.95$, the above simplifies to $-\dfrac{3.8273}{16} < 0$.

However, taking $x_1^{\ 2} = 0.81$, the above is positive.

That means, for $\left[Y_{0.9}, Y_{0.45}\right]$ simultaneous and independent generation is possible. Indeed our choice is also extended to the following pairs: $\left[(0.9, 0.45); (0.9, -0.45); (-0.9, 0.45); (-0.9, -0.45)\right]$.

Table 5.3.1 shows various values of p [0.1, 0.2,…, 0.9, 0.95, 0.99] and for each such value of p, we indicate the critical value of x_1 for which the quantity above in (5.3.1) is zero. This will specify the range of values of x_1 [and hence, those of x_2 as well] for simultaneous independent generation of Y_{x_1} and Y_{x_2}.

Table 5.3.1 Various values of p for the critical value of x_1

p	$t_0 = x_1^2$	Range of x_1
0.1	0.998	(0, 0.9989)
0.2	0.9904	(0, 0.9952)
0.3	0.9748	(0, 0.9873)
0.4	0.949	(0, 0.9742)
0.5	0.912	(0, 0.955)
0.6	0.8641	(0, 0.9295)
0.7	0.8074	(0, 0.8985)
0.8	0.7453	(0, 0.8633)
0.9	0.6812	(0, 0.8253)
0.95	0.6493	(0, 0.8058)
0.99	0.6242	(0, 0.7901)

Note: Our work is different from ‘fitting’ the linear model and estimating the value of the linear parametric function corresponding to a given value of *x*. This estimation problem is a standard exercise in the context of linear models. For data ‘generation’, we need to assure that eventually, we have an observation having the same distributional property as ‘unobserved’ data. In our study, $\tilde{Y}_x$ served this purpose and this was possible only because $\mathrm{Var}\left(\hat{Y}_x\right) < \sigma^2$.

In this study, we can see that for the fitted value of Y_x, we can use the range of *x* outside [-1,1]. But for the generation of random variable which has the same distribution as Y_x, the range of *x* must lie inside [-1, 1]. If not, then the variance of fitted value will be greater than σ^2, i.e. variance of Y_x.

5.4 Sufficiency in one-way ANOVA random effects model

In this section, we consider one-way ANOVA random effects model. Then we apply the concept of sufficiency in different pairs of allocation of the observations under this model. In order to do this, we take some pairs of generalized allocations and compare these allocations in terms of sufficient experiments.

5.4.1 One-way ANOVA random effects model

In fixed effects ANOVA model, we take observations on all treatments of interest. If we have a large number or a population of treatments and we have to test all of them, it will not be easy. Therefore, if we randomly select some, i.e. k of these, we can call it as random effects ANOVA model. We want our conclusions to apply to the whole population, not just to those treatments tested. We are not interested in the particular treatment effects differences but in the variability among all treatment effects. Hence random effects models are sometimes called "components of variance" models.

The model

The one-way ANOVA random effects model is given by

$$Y_{ij} = \mu + d_i + e_{ij},\ i = 1,...,k;\ j = 1,...,n_i;\ \sum_{i}^{k} n_i = n, \qquad (5.4.1)$$

where Y_{ij}'s form a set of random realizations out of the experiment with parameters μ, σ_d^2 and σ_e^2, μ is the population mean, $d_i \sim N(0,\sigma_d^2)$ represents variability between treatments, $e_{ij} \sim N(0,\sigma_e^2)$ represents variability within treatments and d_i and e_{ij} are independent of each other for all i and j.

5.4.2 Sufficient allocation of the observations

A specific choice of values for k and $n_1, n_2, ..., n_k$ is called an allocation and denoted by $A(n;k;n_1,n_2,...,n_k)$ where $n = \sum_{i=1}^{k} n_i$. Suppose we have two allocations $A(n;k;n_1,n_2,...,n_k)$ and $A(n;k;n_1^*, n_2^*, ..., n_k^*)$ under one-way ANOVA random effects model (4.1) such that $n = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} n_i^*$. Then allocation $A(n;k;n_1,n_2,...,n_k)$ is said to be sufficient for the other allocation $A(n;k;n_1^*, n_2^*, ..., n_k^*)$ in the sense of sufficient experiments if we can generate the observations under allocation $A(n;k;n_1^*, n_2^*, ..., n_k^*)$ using the observations under allocation $A(n;k;n_1,n_2,...,n_k)$. Some important theorems regarding these allocations along with some remarks and examples are given in this chapter.

5.4.3 Application and Results

Theorem 5.4.1 Under one-way random effects model with unknown mean (μ) and known variance components $(\sigma_d^{\,2}$ and $\sigma_e^{\,2})$, for $n = mk$, the allocation $A(n;k;m,m,...,m)$ is sufficient for any other allocation $A(n;k;n_1,n_2,...,n_k)$ in the sense of sufficient experiment, where $n=\sum_{i=1}^{k} n_i$ and 'k' stands for the number of 'treatments'.

Proof: Let $\mathcal{E} \equiv A(n;k;m,m,...,m)$ result in the outcome vector $\underset{\sim}{X} \equiv \left(\underset{\sim}{X}^{(1)},\underset{\sim}{X}^{(2)},...,\underset{\sim}{X}^{(k)}\right)$ of dimensions $(m,m,...,m)$, respectively and $\mathcal{F} \equiv A(n;k;n_1,n_2,...,n_k)$ result in the outcome vector $\underset{\sim}{Y} \equiv \left(\underset{\sim}{Y}^{(1)},\underset{\sim}{Y}^{(2)},...,\underset{\sim}{Y}^{(k)}\right)$ of dimensions $(n_1,n_2,...,n_k)$, respectively. It is claimed that $\mathcal{E}$ is sufficient for $\mathcal{F}$, i.e. from $\underset{\sim}{X}$, we can 'create' $\underset{\sim}{Y}$ using known random variables, without actually performing the experiment $\mathcal{F}$.

To establish this claim, we proceed through the following steps.

Step 1: Define t = grand mean of all 'n' observations on $\left(\underset{\sim}{x}^{(1)},\underset{\sim}{x}^{(2)},...,\underset{\sim}{x}^{(k)}\right)$. Then,

$$t \sim N\left(\mu, \frac{\sigma_d^{\,2}}{k}+\frac{\sigma_e^{\,2}}{n}\right). \tag{5.4.2}$$

Step 2: Define $\underset{\sim}{z}_{k\times 1} \sim N\left(\underset{\sim}{0}_{k\times 1}, \sigma_d^{\,2}\left(I_{k\times k}-\frac{J_{k\times k}}{k}\right)\right)$, (5.4.3)

where I is an identity matrix and J is a matrix having all elements one. Note that $\underset{\sim}{z}$ has a singular distribution.

Step 3: Define $\underset{\sim}{S} \sim N_n\left(\underset{\sim}{0}_{n\times 1}, \sigma_e^{\,2}\left(I_{n\times n}-\frac{J_{n\times n}}{n}\right)\right)$. (5.4.4)

Decompose $\underset{\sim}{S}$ as $\left\{\underset{\sim}{S}^{(i)}, i=1,...k\right\}$ of dimensions $(n_1,n_2,...,n_k)$, respectively. Note that $\underset{\sim}{S}$ has a singular distribution as well.

Step 4: Set $\tilde{\underset{\sim}{Y}}^{(p)}_{n_p\times 1} = t\underset{\sim}{1}_{n_p\times 1} + z_p\underset{\sim}{1}_{n_p\times 1} + \underset{\sim}{S}^{(p)}_{n_p\times 1}$; $1 \le p \le k$, (5.4.5)

where $\underset{\sim}{1}$ is a column vector with all elements one.

Then $\underset{\sim}{\tilde{Y}} \equiv \underset{\sim}{Y} \equiv A(n;k;n_1,n_2,...,n_k)$, where $\underset{\sim}{\tilde{Y}} = \left(\underset{\sim}{\tilde{Y}}^{(1)},\underset{\sim}{\tilde{Y}}^{(2)},...,\underset{\sim}{\tilde{Y}}^{(k)}\right)$ and

$\underset{\sim}{Y} = \left(\underset{\sim}{Y}^{(1)},\underset{\sim}{Y}^{(2)},...,\underset{\sim}{Y}^{(k)}\right) \equiv A(n;k;n_1,n_2,...,n_k)$.

We can prove above theorem in detail as follows.

First we compute 't' which is a grand mean of all 'n' observations on

$\left(\underset{\sim}{x}^{(1)},\underset{\sim}{x}^{(2)},...,\underset{\sim}{x}^{(k)}\right) \equiv A(n;k;m,m,...,m)$.

Under one-way ANOVA random effects model,

$$t = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij}}{n}$$

$$= \mu + \left[\frac{md_1 + ... + md_k}{n}\right] + \bar{e} \text{ .}$$

Here the expected value of 't', $E(t) = \mu$ and the variance is

$$\text{Var}(t) = \frac{\sigma_e^{\ 2}}{n} + \frac{km^2}{n^2}\sigma_d^{\ 2}$$

$$= \frac{\sigma_d^{\ 2}}{k} + \frac{\sigma_e^{\ 2}}{n}.$$

Therefore, $t \sim N\left(\mu, \frac{\sigma_d^{\ 2}}{k} + \frac{\sigma_e^{\ 2}}{n}\right)$.

Recall $\underset{\sim}{z}_{k\times 1} \sim N_k\left(\underset{\sim}{0}_{k\times 1}, \sigma_d^{\ 2}\left(I_{k\times k} - \frac{J_{k\times k}}{k}\right)\right)$ and

$$\underset{\sim}{S} \sim N_n\left(\underset{\sim}{0}_{n\times 1}, \sigma_e^{\ 2}\left(I_{n\times n} - \frac{J_{n\times n}}{n}\right)\right).$$

Then we make a stochastic transformation to generate $\tilde{Y}_i^{(p)}$ as

$$t + z_p + S_i^{(p)},\ 1 \le i \le n_p,\ 1 \le p \le k. \qquad (5.4.6)$$

Here, the variance of $\tilde{Y}_i^{(p)}$ is given as

$$\text{Var}\left(\tilde{Y}_i^{(p)}\right) = \frac{\sigma_d^{\ 2}}{k} + \frac{\sigma_e^{\ 2}}{n} + \sigma_d^{\ 2}\left(1 - \frac{1}{k}\right) + \sigma_e^{\ 2}\left(1 - \frac{1}{n}\right)$$

$$= \sigma_d^{\ 2} + \sigma_e^{\ 2}, \qquad (5.4.7)$$

which is the same as $\mathrm{Var}\left(Y_i^{(p)}\right)$.

This is true for all '*i*' and '*p*'.

The covariance within the same group, i.e. *p* is given by

$$\begin{aligned}
Cov\left(\tilde{Y}_i^{(p)}, \tilde{Y}_j^{(p)}\right) &= Cov\left(t+z_p+S_i^{(p)}, t+z_p+S_j^{(p)}\right) \\
&= Var(t)+Var\left(z_p\right)+Cov\left(S_i^{(p)}, S_j^{(p)}\right) \\
&= \frac{\sigma_d^{\,2}}{k}+\frac{\sigma_e^{\,2}}{n}+\left(1-\frac{1}{k}\right)\sigma_d^{\,2}-\frac{\sigma_e^{\,2}}{n} \\
&= \sigma_d^{\,2}, \qquad (5.4.8)
\end{aligned}$$

which is the same as $Cov\left(Y_i^{(p)}, Y_j^{(p)}\right)$.

This is true for all $i \neq j$ and all $1 \leq p \leq k$.

The covariance between two different groups, i.e. *p* and *q* is given by

$$\begin{aligned}
Cov\left(\tilde{Y}_i^{(p)}, \tilde{Y}_j^{(q)}\right) &= Cov\left(t+z_p+S_i^{(p)}, t+z_q+S_j^{(q)}\right) \\
&= Var(t)+Cov\left(z_p, z_q\right)+Cov\left(S_i^{(p)}, S_j^{(q)}\right) \\
&= \frac{\sigma_d^{\,2}}{k}+\frac{\sigma_e^{\,2}}{n}-\frac{\sigma_d^{\,2}}{k}-\frac{\sigma_e^{\,2}}{n} \\
&= 0, \qquad (5.4.9)
\end{aligned}$$

which is the same as $Cov\left(Y_i^{(p)}, Y_j^{(q)}\right)$.

This is true for all $p \neq q$ and all '*i*' within group '*p*' and all '*j*' within group '*q*'. Hence $\underset{\sim}{\tilde{Y}} \equiv \underset{\sim}{Y}$,

where $\underset{\sim}{\tilde{Y}} = \left(\underset{\sim}{\tilde{Y}}^{(1)}, \underset{\sim}{\tilde{Y}}^{(2)}, \ldots, \underset{\sim}{\tilde{Y}}^{(k)}\right)$

and $\underset{\sim}{Y} = \left(\underset{\sim}{Y}^{(1)}, \underset{\sim}{Y}^{(2)}, \ldots, \underset{\sim}{Y}^{(k)}\right) \equiv A\left(n; k; n_1, n_2, \ldots, n_k\right)$.

Using '*t*' which is the grand mean together with $\underset{\sim}{z}$ and $\underset{\sim}{S}$ which are independent of unknown parameter μ, we can generate all '*n*' observations $\left(\underset{\sim}{\tilde{Y}}^{(1)}, \underset{\sim}{\tilde{Y}}^{(2)}, \ldots, \underset{\sim}{\tilde{Y}}^{(k)}\right)$.

Hence the allocation $A(n; k; m, m, \ldots, m)$ is sufficient for any other allocation $A\left(n; k; n_1, n_2, \ldots, n_k\right)$ in the sense of sufficient experiment.

Theorem 5.4.2 Under one-way random effects model with unknown mean (μ) and known variance components $\left(\sigma_d^{\,2} \text{ and } \sigma_e^{\,2}\right)$, for n = r (mod k), the allocation $A(n;k;m,m,...,m,m+1,m+1,...,m+1)$ where 'm' appears 'k-r' times and '$m+1$' appears 'r' times is sufficient for any allocation $A(n;k;n_1,n_2,...,n_k)$ in the sense of sufficient experiment, where $n=\sum_{i=1}^{k} n_i$ and 'k' stands for the number of 'treatments'.

Proof: Let $\mathcal{E} \equiv A(n;k;m,...m,m+1,...,m+1)$ result in the outcome vector $\underset{\sim}{X} \equiv \left(\underset{\sim}{X}^{(1)},\underset{\sim}{X}^{(2)},...,\underset{\sim}{X}^{(k)}\right)$ of dimensions $(m,m,...,m,m+1,m+1,...,m+1)$, respectively and $\mathcal{F} \equiv A(n;k;n_1,n_2,...,n_k)$ result in the outcome vector $\underset{\sim}{Y} \equiv \left(\underset{\sim}{Y}^{(1)},\underset{\sim}{Y}^{(2)},...,\underset{\sim}{Y}^{(k)}\right)$ of dimensions $(n_1,n_2,...,n_k)$, respectively. It is claimed that $\mathcal{E}$ is sufficient for $\mathcal{F}$, i.e. from $\underset{\sim}{X}$, we can 'create' $\underset{\sim}{Y}$ using known random variables, without actually performing the experiment $\mathcal{F}$.

To establish this claim, we proceed as follows.

First we compute 't' which is weighted mean of all 'n' observations on $\left(\underset{\sim}{x}^{(1)},\underset{\sim}{x}^{(2)},...,\underset{\sim}{x}^{(k)}\right) \equiv A(n;k;m,m,...,m,m+1,m+1,...,m+1)$.

Under one-way ANOVA random effects model,

$$\overline{x}^{(i)} \sim N\left(\mu,\sigma_d^{\,2}+\frac{\sigma_e^{\,2}}{m_i}=v_i\right);\ 1\le i\le k\ ,$$

where $m_i=m$ for $1\le i\le k-r$ and $m_i=m+1$ for $k-r+1\le i\le k$. Then, we get

$$t=\frac{\sum_{i=1}^{k}\overline{x}^{(i)}/v_i}{\sum_{i=1}^{k}1/v_i}\ .$$

$$\text{Therefore, } t\sim N\left(\mu,\frac{1}{\sum_{i=1}^{k}1/v_i}=V(t)\right). \tag{5.4.10}$$

Our purpose is to establish that based on the available data, i.e. $\left(\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)},..., \underset{\sim}{X}^{(k)}\right)$, we can *generate* observations on the other allocation $\left(\underset{\sim}{Y}^{(1)}, \underset{\sim}{Y}^{(2)},..., \underset{\sim}{Y}^{(k)}\right)$.

For this purpose, we generate $\underset{\sim}{S} \sim N_n\left(\underset{\sim}{0}_{n\times 1}, \Sigma_{n\times n}\right)$ and decompose $\underset{\sim}{S}$ as $\left\{\underset{\sim}{S}^{(i)}, i = 1,...k\right\}$ of dimensions $(n_1, n_2,..., n_k)$ respectively, where

$$\Sigma_{n\times n} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}$$

with $A_{ii} = A_{n_i\times n_i} = \sigma_e{}^2 I_{n_i\times n_i} + \sigma_d{}^2 J_{n_i\times n_i} - V\left(t\right) J_{n_i\times n_i}$; $i = 1, 2,..., k$;

$$A_{ij} = A_{n_i\times n_j} = -V\left(t\right) J_{n_i\times n_j}; 1 \le i \ne j \le k \tag{5.4.11}$$

and ‘ I ’ is an identity matrix and ‘ J ’ is a matrix having all elements one.

We first deduce that $\Sigma_{n\times n}$ is nnd.

Once this is done, in the above, using ‘t’ and $\underset{\sim}{S}$, we can generate observations $\left\{\left(t+S_i \,|1\le i \le n_1; t+S_{n_1+j} \,|1\le j \le n_2;...; t+S_{n_1+n_2+...+k} \,|1\le k \le n_k\right)\right\}$ which we claim to be equivalent to the observations $\left(\underset{\sim}{Y}^{(1)}, \underset{\sim}{Y}^{(2)},..., \underset{\sim}{Y}^{(k)}\right)$.

Essentially, then, we have to prove:

(I) $\Sigma_{n\times n}$ is nnd,

(II) The within and between properties of the generated observations $\left\{\left(t+S_i \,|1\le i \le n_1; t+S_{n_1+j} \,|1\le j \le n_2;...; t+S_{n_1+n_2+...+k} \,|1\le k \le n_k\right)\right\}$ will be equivalent to these properties of $\left(\underset{\sim}{Y}^{(1)}, \underset{\sim}{Y}^{(2)},..., \underset{\sim}{Y}^{(k)}\right)$.

However, it may be noted that our choice of $\Sigma_{n\times n}$ has the built-in property of satisfying the condition (II) laid down above. This follows essentially from the steps outlined in Theorem 5.4.1. Therefore, we are done once we settle (I) above.

Set $\sum_{n\times n} = \begin{pmatrix} aI+bJ & cJ & \cdots & cJ \\ cJ & aI+bJ & \cdots & cJ \\ \vdots & \vdots & \ddots & \vdots \\ cJ & cJ & \cdots & aI+bJ \end{pmatrix}$,

where $a=\sigma_e^2$, $b=\sigma_d^2 - V(t)$ and $c=-V(t)$. (5.4.12)

Consider the quadratic form as below.

$$Q\left(\underset{\sim}{x}^{(1)}_{n_1\times 1}, \underset{\sim}{x}^{(2)}_{n_2\times 1}, ..., \underset{\sim}{x}^{(k)}_{n_k\times 1}\right)$$

$$= \left(\underset{\sim}{x}^{(1)'}\ \underset{\sim}{x}^{(2)'} \cdots \underset{\sim}{x}^{(k)'}\right) \begin{pmatrix} aI+bJ & cJ & \cdots & cJ \\ cJ & aI+bJ & \cdots & cJ \\ \vdots & \vdots & \ddots & \vdots \\ cJ & cJ & \cdots & aI+bJ \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}^{(1)} \\ \underset{\sim}{x}^{(2)} \\ \vdots \\ \underset{\sim}{x}^{(k)} \end{pmatrix}$$

$$= a\sum_{i=1}^{k} \underset{\sim}{x}^{(i)'}\underset{\sim}{x}^{(i)} + b\sum_{i=1}^{k}\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right)^2 + c\sum\sum_{i\neq j=1}^{k}\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right)\left(\underset{\sim}{x}^{(j)'}\underset{\sim}{1}\right)$$

$$\geq a\sum_{i=1}^{k}\frac{\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right)^2}{n_i} + b\sum_{i=1}^{k}\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right)^2 + c\sum\sum_{i\neq j=1}^{k}\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right)\left(\underset{\sim}{x}^{(j)'}\underset{\sim}{1}\right), \quad (5.4.13)$$

since $a=\sigma_e^2$ and $\sigma_e^2 > 0$.

The last expression in (5.4.13) is a quadratic form involving $\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right)$ and $\left(\underset{\sim}{x}^{(j)'}\underset{\sim}{1}\right)$ and is equal to

$$\left(\underset{\sim}{x}^{(1)'}\underset{\sim}{1}\ \ \underset{\sim}{x}^{(2)'}\underset{\sim}{1}\cdots\underset{\sim}{x}^{(k)'}\underset{\sim}{1}\right) \begin{pmatrix} \frac{a}{n_1}+b & c & \cdots & c \\ c & \frac{a}{n_2}+b & \cdots & c \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \cdots & \frac{a}{n_k}+b \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}^{(1)'}\underset{\sim}{1} \\ \underset{\sim}{x}^{(2)'}\underset{\sim}{1} \\ \vdots \\ \underset{\sim}{x}^{(k)'}\underset{\sim}{1} \end{pmatrix}. \quad (5.4.14)$$

Let $\left(\underset{\sim}{x}^{(i)'}\underset{\sim}{1}\right) = u_i\,; i=1,2,...k$ in above expression, we get

$$\begin{pmatrix} u_1 & u_2 \cdots & u_k \end{pmatrix} \begin{pmatrix} \frac{a}{n_1}+b & c & \cdots & c \\ c & \frac{a}{n_2}+b & \cdots & c \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \cdots & \frac{a}{n_k}+b \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix} \tag{5.4.15}$$

$$= a\sum_{i=1}^{k}\frac{u_i^{\,2}}{n_i} + b\sum_{i=1}^{k}u_i^{\,2} + c\sum\sum_{i\neq j=1}^{k}u_iu_j \ .$$

Now we have to prove

$$a\sum_{i=1}^{k}\frac{u_i^{\,2}}{n_i} + b\sum_{i=1}^{k}u_i^{\,2} + c\sum\sum_{i\neq j=1}^{k}u_iu_j > 0$$

$$a\sum_{i=1}^{k}\frac{u_i^{\,2}}{n_i} + b\sum_{i=1}^{k}u_i^{\,2} + c\left[\left(\sum_{i=1}^{k}u_i\right)^2 - \sum_{i=1}^{k}u_i^{\,2}\right] > 0$$

$$\sum_{i=1}^{k}u_i^{\,2}\left(\frac{a}{n_i}+b-c\right) + c\left(\sum_{i=1}^{k}u_i\right)^2 > 0$$

$$\sum_{i=1}^{k}u_i^{\,2}\left(\frac{a}{n_i}+b-c\right) > (-c)\left(\sum_{i=1}^{k}u_i\right)^2 . \tag{5.4.16}$$

Using Cauchy-Schwarz inequality,

$$\sum_{i=1}^{k}u_i^{\,2}\left(\frac{a}{n_i}+b-c\right)\left[\frac{1}{\sum_{i=1}^{k}\left(\frac{a}{n_i}+b-c\right)}\right] \geq \left(\sum_{i=1}^{k}u_i\right)^2 \tag{5.4.17}$$

$$\sum_{i=1}^{k}u_i^{\,2}\left(\frac{a}{n_i}+b-c\right) \geq \frac{\left(\sum_{i=1}^{k}u_i\right)^2}{\dfrac{1}{\sum_{i=1}^{k}\left(\frac{a}{n_i}+b-c\right)}} . \tag{5.4.18}$$

Applying (5.4.12), (5.4.18) in (5.4.16), we get

$$\frac{\left(\sum_{i=1}^{k} u_i\right)^2}{\sum_{i=1}^{k} \frac{1}{\left(\frac{\sigma_e^2}{n_i}+\sigma_d^2\right)}} \geq V(t)\left(\sum_{i=1}^{k} u_i\right)^2$$

$$\frac{1}{\sum_{i=1}^{k}\left(\frac{\sigma_e^2}{n_i}+\sigma_d^2\right)^{-1}} \geq V(t)$$

$$\frac{1}{\sum_{i=1}^{k}\left(\frac{1}{v(n_i)}\right)} \geq V(t) \qquad (5.4.19)$$

$$\frac{1}{V(t)} \geq \sum_{i=1}^{k}\left(\frac{1}{v(n_i)}\right),$$

which is true that follows from Stepniak [8].

Hence the claim is established.

Theorem 5.4.3 Under one-way random effects model with unknown mean (μ) and known variance components $(\sigma_d^2 \text{ and } \sigma_e^2)$, the allocation $A(n;2;n_1,n_2)$ is sufficient for any other allocation $A(n;2;m_1,m_2)$ in the sense of sufficient experiment, where $n = n_1 + n_2 = m_1 + m_2; m_1 < n_1 < n_2 < m_2$.

Proof: From allocation $A(n;2;n_1,n_2)$, we obtain observations $x_1, x_2, ..., x_{n_1}$ on the first treatment and observations $y_1, y_2, ..., y_{n_2}$ on the second treatment.

Clearly, $x_i = \mu + d_1 + e_i$, $1 \leq i \leq n_1$,

$y_j = \mu + d_2 + f_j$, $1 \leq j \leq n_2$,

where $d_1, d_2 \sim iid\, N(0, \sigma_d^2)$ and d_1, d_2 are independent of e_i's and f_j's which are iid as $N(0, \sigma_e^2)$.

Then, $\bar{x} \sim N\left(\mu, \sigma_d^2 + \frac{\sigma_e^2}{n_1} = v(n_1)\right)$ and

$$\bar{y} \sim N\left(\mu, \sigma_d^2 + \frac{\sigma_e^2}{n_2} = v(n_2)\right).$$

First we compute 't' which is weighted mean of all 'n' observations from allocation $A(n;2;n_1,n_2)$. We get

$$t = \frac{\left[\bar{x}/v(n_1)\right]+\left[\bar{y}/v(n_2)\right]}{\left[1/v(n_1)\right]+\left[1/v(n_2)\right]}.$$

Then, $t \sim N\left(\mu, \frac{1}{\{1/v(n_1)\}+\{1/v(n_2)\}} = V(t)\right)$.

Our purpose is to establish the relation that based on the available data, i.e. $\left\{(x_1,x_2,...,x_{n_1});(y_1,y_2,...,y_{n_2})\right\}$, we can *generate* observations on the other allocation $A(n;2;m_1,m_2)$ whenever $n = n_1+n_2 = m_1+m_2; m_1 < n_1 < n_2 < m_2$.

Denote by $\left\{(U_1,U_2,...,U_{m_1});(V_1,V_2,...,V_{m_2})\right\}$ the *potential* observations from $A(n;2;m_1,m_2)$.

Note that $U_i = \mu + D_1 + E_i$, $1 \le i \le m_1$,

while $V_j = \mu + D_2 + F_j$, $1 \le j \le m_2$,

where $D_1, D_2 \sim iid\ N(0,\sigma_d^2)$ and D_1, D_2 are independent of E_i's and F_j's which are iid as $N(0,\sigma_e^2)$.

We describe below a method for generating the U_i's and V_j's from the x_i's and y_j's.

For this purpose, we generate $\underset{\sim}{W} \sim N_n\left(\underset{\sim}{0}_{n\times 1}, \Sigma_{n\times n}\right)$, where

$$\Sigma_{n\times n} = \begin{pmatrix} A_{m_1\times m_1} & B_{m_1\times m_2} \\ B'_{m_2\times m_1} & C_{m_2\times m_2} \end{pmatrix}$$

with $A_{m_1\times m_1} = \sigma_e^2 I_{m_1\times m_1} + \sigma_d^2 J_{m_1\times m_1} - V(t) J_{m_1\times m_1}$;

$$C_{m_2\times m_2} = \sigma_e^2 I_{m_2\times m_2} + \sigma_d^2 J_{m_2\times m_2} - V(t) J_{m_2\times m_2};\ B_{m_1\times m_2} = -V(t) J_{m_1\times m_2} \qquad (5.4.20)$$

and 'I' is an identity matrix and 'J' is a matrix having all elements one.

We first deduce that $\Sigma_{n\times n}$ is nnd.

Once this is done, in the above, using 't' and $\underset{\sim}{W}$, we can generate observations $\left\{\left(t+W_i \mid 1 \le i \le m_1; t+W_{m_1+j} \mid 1 \le j \le m_2\right)\right\}$ which we claim to be equivalent to the observations $\left\{\left(U_1, U_2, ..., U_{m_1}\right); \left(V_1, V_2, ..., V_{m_2}\right)\right\}$.

Essentially, then, we have to prove:

(I) $\sum_{n\times n}$ is nnd,

(II) $\left(t+W_i \mid 1 \le i \le m_1\right)$ follow the joint distribution of $\left(U_i, 1 \le i \le m_1\right)$,

(III) $\left(t+W_{m_1+j} \mid 1 \le j \le m_2\right)$ follow the joint distribution of $\left(V_j, 1 \le j \le m_2\right)$,

(IV) $\left(t+W_i\right), \left(t+W_{m_1+j}\right)$ are independently distributed for $1 \le i \le m_1; 1 \le j \le m_2$.

However, it may be noted that our choice of $\sum_{n\times n}$ has the built-in property of satisfying the conditions (II), (III) and (IV) laid down above. This follows essentially from the steps outlined in Theorem 5.4.1. Therefore, we are done once we settle (I) above.

Set $\sum_{n\times n} = \begin{pmatrix} aI+bJ & cJ \\ cJ & dI+eJ \end{pmatrix}$,

where $a=d=\sigma_e^2, b=e=\sigma_d^2-V(t)$ and $c=-V(t)$. (5.4.21)

Consider the quadratic form as below.

$$Q\left(\underset{\sim}{x}_{m_1\times 1}, \underset{\sim}{y}_{m_2\times 1}\right) = \left(\underset{\sim}{x}'\ \underset{\sim}{y}'\right) \begin{pmatrix} aI+bJ & cJ \\ cJ & dI+eJ \end{pmatrix} \begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix}$$

$$= \left(\underset{\sim}{x}'\ \underset{\sim}{y}'\right) \begin{pmatrix} a\underset{\sim}{x}+bJ\underset{\sim}{x}+cJ\underset{\sim}{y} \\ cJ\underset{\sim}{x}+d\underset{\sim}{y}+eJ\underset{\sim}{y} \end{pmatrix}$$

$$= \underset{\sim}{x}'a\underset{\sim}{x}+b\underset{\sim}{x}'J\underset{\sim}{x}+c\underset{\sim}{x}'J\underset{\sim}{y}+c\underset{\sim}{y}'J\underset{\sim}{x}+d\underset{\sim}{y}'\underset{\sim}{y}+e\underset{\sim}{y}'J\underset{\sim}{y}$$

$$= a\underset{\sim}{x}'\underset{\sim}{x}+b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+d\underset{\sim}{y}'\underset{\sim}{y}+e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$

$$\ge \frac{a}{m_1}\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+\frac{d}{m_2}\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2+e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2 \qquad (5.4.22)$$

since $a=d=\sigma_e^2$ and $\sigma_e^2>0$.

The last expression in (5.4.22) is a quadratic form involving $\underset{\sim}{x}'\underset{\sim}{1}$ and $\underset{\sim}{y}'\underset{\sim}{1}$ and the underlying matrix is given by

$$Q = \begin{pmatrix} \frac{a}{m_1}+b & c \\ c & \frac{d}{m_2}+e \end{pmatrix}.$$

To ensure that Q is non-negative definite (nnd), we have to show the followings:

(I) $\frac{a}{m_1}+b>0$, $\frac{d}{m_2}+e>0$. (5.4.23)

(II) $\begin{vmatrix} \frac{a}{m_1}+b & c \\ c & \frac{d}{m_2}+e \end{vmatrix} > 0$. (5.4.24)

Recall $v(n_i)=\sigma_d{}^2+\frac{\sigma_e{}^2}{n_i}$.

Set $v(m_i)=\sigma_d{}^2+\frac{\sigma_e{}^2}{m_i}$, $i=1,2$. (5.4.25)

It follows from Stepniak [8] that

$$\frac{1}{v(m_1)}+\frac{1}{v(m_2)}<\frac{1}{v(n_1)}+\frac{1}{v(n_2)}=\frac{1}{V(t)}$$

$$\frac{1}{\frac{1}{v(m_1)}+\frac{1}{v(m_2)}}>\frac{1}{\frac{1}{v(n_1)}+\frac{1}{v(n_2)}}=V(t) \tag{5.4.26}$$

$$\frac{1}{\frac{1}{v(m_i)}}>\frac{1}{\frac{1}{v(n_1)}+\frac{1}{v(n_2)}}=V(t) \text{ for } i=1,2. \tag{5.4.27}$$

It now follows from (5.4.21), (5.4.25) and (5.4.27) that (I) holds, i.e. (5.4.23) is true.

For (II), we note that (5.4.24) is equivalent to

$$v(m_1)\,v(m_2) - V(t)\left[v(m_1)+v(m_2)\right]>0$$

$$\frac{1}{\frac{1}{v(m_1)}+\frac{1}{v(m_2)}}>\frac{1}{\frac{1}{v(n_1)}+\frac{1}{v(n_2)}}=V(t),$$

which is the expression (5.4.26) stated earlier.

Hence the claim is established.

We can see the following example to know about the above concepts of sufficiency under one-way random effects model with unknown mean (μ) and known variance components $\left(\sigma_d^{\,2} \text{ and } \sigma_e^{\,2}\right)$.

Example 5.4.1 Under one-way random effects model with unknown mean (μ) and known variance components $\left(\sigma_d^{\,2} \text{ and } \sigma_e^{\,2}\right)$, the allocation $A(5;2;2,3)$ is sufficient for another allocation $A(5;2;1,4)$ in the sense of sufficient experiment.

Proof: From allocation $A(5;2;2,3)$, we obtain observations x_1, x_2 on the first treatment and observations y_1, y_2, y_3 on the second treatment.

Clearly, $x_i = \mu + d_1 + e_i$, $i = 1,2$,

$$y_j = \mu + d_2 + f_j, \; 1 \le j \le 3,$$

where $d_1, d_2 \sim iid\, N\left(0, \sigma_d^{\,2}\right)$ and d_1, d_2 are independent of e_i's and f_j's which are iid as $N\left(0, \sigma_e^{\,2}\right)$.

Then, $\bar{x} \sim N\left(\mu, \sigma_d^{\,2} + \dfrac{\sigma_e^{\,2}}{2} = v(2)\right)$ and

$$\bar{y} \sim N\left(\mu, \sigma_d^{\,2} + \frac{\sigma_e^{\,2}}{3} = v(3)\right).$$

First we compute '*t*' which is weighted mean of all observations from allocation $A(5;2;2,3)$. We get

$$t = \frac{\left[\bar{x}/v(2)\right] + \left[\bar{y}/v(3)\right]}{\left[1/v(2)\right] + \left[1/v(3)\right]}.$$

Then, $t \sim N\left(\mu, \dfrac{1}{\dfrac{1}{v(2)} + \dfrac{1}{v(3)}} = V(t)\right)$.

Our purpose is to establish the relation that based on the available data, i.e. $\{(x_1, x_2);(y_1, y_2, y_3)\}$, we can *generate* observations on the other allocation $A(5;2;1,4)$. Denote by $\{(U_1);(V_1,V_2,V_3,V_4)\}$ the *potential* observations from $A(5;2;1,4)$.

Note that $U_1 = \mu + D_1 + E_1$,

while $V_j = \mu + D_2 + F_j$, $1 \le j \le 4$,

where $D_1, D_2 \sim iid\ N\left(0,\sigma_d^{\ 2}\right)$ and D_1, D_2 are independent of E_1 and F_j's which are iid as $N\left(0,\sigma_e^{\ 2}\right)$.

We describe below a method for generating the U_1's and V_j's.

We generate $\underset{\sim}{W} \sim N_5\left(\underset{\sim}{0}_{5\times1},\Sigma_{5\times5}\right)$, where

$$\Sigma_{5\times5} = \begin{pmatrix} A & B_{1\times4} \\ B'_{4\times1} & C_{4\times4} \end{pmatrix}$$

with $A = \sigma_e^{\ 2} + \sigma_d^{\ 2} - V(t)$;

$$C_{4\times4} = \sigma_e^{\ 2} I_{4\times4} + \sigma_d^{\ 2} J_{4\times4} - V(t) J_{4\times4};$$

$$B_{1\times4} = -V(t) J_{1\times4}\ , \tag{5.4.28}$$

'I' is an identity matrix and 'J' is a matrix having all elements one.

We first deduce that $\Sigma_{5\times5}$ is nnd.

Once this is done, in the above, using 't' and $\underset{\sim}{W}$, we can generate observations $\{(t+W_1;t+W_i \mid 2\le i\le 5)\}$ which we claim to be equivalent to the observations $\{(U_1);(V_1,V_2,V_3,V_4)\}$.

Essentially, then, we have to prove:

(I) $\Sigma_{n\times n}$ is nnd.

(II) $(t+W_1)$ follow the distribution of U_1.

(III) $(t+W_i \mid 2\le i\le 5)$ follow the joint distribution of $(V_j, 1\le j\le 4)$.

(IV) $(t+W_1),(t+W_i)$ are independently distributed for $2\le i\le 5$.

However, it may be noted that our choice of $\sum_{n\times n}$ has the built-in property of satisfying the conditions (II), (III) and (IV) laid down above. This follows essentially from the steps outlined in Theorem 4.1. Therefore, we are done once we settle (I) above.

Set $\sum_{n\times n} = \begin{pmatrix} a+b & cJ \\ cJ & dI+eJ \end{pmatrix}$,

where $a=d=\sigma_e^2, b=e=\sigma_d^2-V(t)$ and $c=-V(t)$. (5.4.29)

Consider the quadratic form as below.

$$Q\left(\underset{\sim}{x}_{n_1\times 1}, \underset{\sim}{y}_{n_2\times 1}\right) = \left(\underset{\sim}{x}' \ \underset{\sim}{y}'\right)\begin{pmatrix} a+b & cJ \\ cJ & dI+eJ \end{pmatrix}\begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix}$$

$$= \left(\underset{\sim}{x}' \ \underset{\sim}{y}'\right)\begin{pmatrix} a\underset{\sim}{x}+b\underset{\sim}{x}+cJ\underset{\sim}{y} \\ cJ\underset{\sim}{x}+d\underset{\sim}{y}+eJ\underset{\sim}{y} \end{pmatrix}$$

$$= \underset{\sim}{x}'a\underset{\sim}{x}+b\underset{\sim}{x}'\underset{\sim}{x}+c\underset{\sim}{x}'J\underset{\sim}{y}+c\underset{\sim}{y}'J\underset{\sim}{x}+d\underset{\sim}{y}'\underset{\sim}{y}+e\underset{\sim}{y}'J\underset{\sim}{y}$$

$$= a\underset{\sim}{x}'\underset{\sim}{x}+b\underset{\sim}{x}'\underset{\sim}{x}+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+d\underset{\sim}{y}'\underset{\sim}{y}+e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$

$$= (a+b)\underset{\sim}{x}'\underset{\sim}{x}+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+d\underset{\sim}{y}'\underset{\sim}{y}+e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$

$$\geq (a+b)\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+\frac{d}{4}\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2+e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2 \qquad (5.4.30)$$

since $d>0; a+b>0$.

The last expression in (5.4.30) is a quadratic form involving $\underset{\sim}{x}'\underset{\sim}{1}$ and $\underset{\sim}{y}'\underset{\sim}{1}$ and the underlying matrix is given by

$$Q = \begin{pmatrix} a+b & c \\ c & \frac{d}{4}+e \end{pmatrix}.$$

To ensure that Q is non-negative definite (nnd), we have to show the followings.

(I) $a+b>0$, $\frac{d}{4}+e>0$. (5.4.31)

(II) $\begin{vmatrix} a+b & c \\ c & \frac{d}{4}+e \end{vmatrix} > 0$. (5.4.32)

Recall for allocation $A(5;2;1,4)$,

$v(1) = \sigma_d{}^2 + \sigma_e{}^2$ and $v(4) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{4}$. (5.4.33)

We have, $v(2) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{2}$ and $v(3) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{3}$. (5.4.34)

It follows from Stepniak [8] that

$$\frac{1}{v(1)} + \frac{1}{v(4)} < \frac{1}{v(2)} + \frac{1}{v(3)} = \frac{1}{V(t)}$$

$$\frac{1}{\frac{1}{v(1)} + \frac{1}{v(4)}} > \frac{1}{\frac{1}{v(2)} + \frac{1}{v(3)}} = V(t) \qquad (5.4.35)$$

$$\frac{1}{\frac{1}{v(1)}} > \frac{1}{\frac{1}{v(2)} + \frac{1}{v(3)}} = V(t) \;;\; \frac{1}{\frac{1}{v(4)}} > \frac{1}{\frac{1}{v(2)} + \frac{1}{v(3)}} = V(t). \qquad (5.4.36)$$

It now follows from (5.4.29), (5.4.33) and (5.4.36) that (I) holds, i.e. (5.4.31) is true.

For (II), we note that (5.4.32) is equivalent to

$$\left[v(1) - V(t)\right]\left[v(4) - V(t)\right] - V^2(t) > 0$$

$$v(1)\, v(4) - V(t)\left[v(1) + v(4)\right] > 0$$

$$\frac{1}{\frac{1}{v(1)} + \frac{1}{v(4)}} > V(t),$$

which is the expression (5.4.35) stated earlier.

Hence the claim is established.

Remark 5.4.1 Under one-way random effects model with unknown mean (μ) and known variance components $(\sigma_d{}^2 \text{ and } \sigma_e{}^2)$, the allocation $A(n;2;m,m+1)$ is sufficient for any other allocation $A(n;2;n_1,n_2)$ in the sense of sufficient experiment, where $n = 2m+1 = n_1 + n_2$.

Proof: From allocation $A(n;2;m,m+1)$, we obtain observations $x_1, x_2, ..., x_m$ on the first treatment and observations $y_1, y_2, ..., y_{m+1}$ on the second treatment.

Clearly, $x_i = \mu + d_1 + e_i$, $1 \le i \le m$,

$y_j = \mu + d_2 + f_j$, $1 \le j \le m+1$, where $d_1, d_2 \sim iid\, N\left(0, \sigma_d^{\,2}\right)$ and d_1, d_2 are independent of e_i's and f_j's which are iid as $N\left(0, \sigma_e^{\,2}\right)$.

Then, $\bar{x} \sim N\left(\mu, \sigma_d^{\,2} + \dfrac{\sigma_e^{\,2}}{m} = v(m)\right)$ and

$$\bar{y} \sim N\left(\mu, \sigma_d^{\,2} + \frac{\sigma_e^{\,2}}{m+1} = v(m+1)\right) .$$

First we compute 't' which is weighted mean of all 'n' observations from allocation $A(n;2;m,m+1)$. Hence, we get

$$t = \frac{\left[\bar{x}/v(m)\right] + \left[\bar{y}/v(m+1)\right]}{\left[1/v(m)\right] + \left[1/v(m+1)\right]} .$$

Then, $t \sim N\left(\mu, \dfrac{1}{\left\{1/v(m)\right\} + \left\{1/v(m+1)\right\}} = V(t)\right)$.

Our purpose is to establish the relation that based on the available data, i.e. $\left\{\left(x_1, x_2, ..., x_m\right); \left(y_1, y_2, ..., y_{m+1}\right)\right\}$, we can *generate* observations on the other allocation $A\left(n;2;n_1,n_2\right)$. Denote by $\left\{\left(U_1, U_2, ..., U_{n_1}\right); \left(V_1, V_2, ..., V_{n_2}\right)\right\}$ the *potential* observations from $A\left(n;2;n_1,n_2\right)$.

Note that $U_i = \mu + D_1 + E_i$, $1 \le i \le n_1$,

while $V_j = \mu + D_2 + F_j$, $1 \le j \le n_2$,

where $D_1, D_2 \sim iid\, N\left(0, \sigma_d^{\,2}\right)$ and D_1, D_2 are independent of E_i's and F_j's which are iid as $N\left(0, \sigma_e^{\,2}\right)$.

We describe below a method for generating the U_i's and V_j's from the x_i's and y_j's.

For this purpose, we generate $\underset{\sim}{W} \sim N_n\left(\underset{\sim}{0}_{n\times 1}, \Sigma_{n\times n}\right)$, where

$$\Sigma_{n\times n} = \begin{pmatrix} A_{n_1\times n_1} & B_{n_1\times n_2} \\ B'_{n_2\times n_1} & C_{n_2\times n_2} \end{pmatrix}$$

with $A_{n_1\times n_1}=\sigma_e{}^2 I_{n_1\times n_1}+\sigma_d{}^2 J_{n_1\times n_1}-V(t)J_{n_1\times n_1}$; $C_{n_2\times n_2}=\sigma_e{}^2 I_{n_2\times n_2}+\sigma_d{}^2 J_{n_2\times n_2}-V(t)J_{n_2\times n_2}$;

$B_{n_1\times n_2}=-V(t)J_{n_1\times n_2}$ (5.4.37)

and 'I' is an identity matrix and 'J' is a matrix having all elements one.

We first deduce that $\sum_{n\times n}$ is nnd.

Once this is done, in the above, using 't' and $\underset{\sim}{W}$, we can generate observations $\left\{\left(t+W_i\,|1\le i\le n_1;t+W_{n_1+j}\,|1\le j\le n_2\right)\right\}$ which we claim to be equivalent to the observations $\{\left(U_1,U_2,...,U_{n_1}\right);\left(V_1,V_2,...,V_{n_2}\right)\}$.

Essentially, then, we have to prove:

(I) $\sum_{n\times n}$ is nnd,

(II) $\left(t+W_i\,|1\le i\le n_1\right)$ follow the joint distribution of $\left(U_i,1\le i\le n_1\right)$,

(III) $\left(t+W_{n_1+j}\,|1\le j\le n_2\right)$ follow the joint distribution of $\left(V_j,1\le j\le n_2\right)$,

(IV) $\left(t+W_i\right),\left(t+W_{n_1+j}\right)$ are independently distributed for $1\le i\le n_1;1\le j\le n_2$.

However, it may be noted that our choice of $\sum_{n\times n}$ has the built-in property of satisfying the conditions (II), (III) and (IV) laid down above. This follows essentially from the steps outlined in Theorem 5.4.1. Therefore, we are done once we settle (I) above.

Set $\sum_{n\times n}=\begin{pmatrix} aI+bJ & cJ \\ cJ & dI+eJ \end{pmatrix}$,

where $a=d=\sigma_e^2, b=e=\sigma_d^2-V(t)$ and $c=-V(t)$. (5.4.38)

Consider the quadratic form as below.

$$Q\left(\underset{\sim}{x}_{n_1\times 1},\underset{\sim}{y}_{n_2\times 1}\right)=\left(\underset{\sim}{x}'\ \underset{\sim}{y}'\right)\begin{pmatrix} aI+bJ & cJ \\ cJ & dI+eJ \end{pmatrix}\begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix}$$

$$=\left(\underset{\sim}{x}'\ \underset{\sim}{y}'\right)\begin{pmatrix} a\underset{\sim}{x}+bJ\underset{\sim}{x}+cJ\underset{\sim}{y} \\ cJ\underset{\sim}{x}+d\underset{\sim}{y}+eJ\underset{\sim}{y} \end{pmatrix}$$

$$=\underset{\sim}{x}'a\underset{\sim}{x}+b\underset{\sim}{x}'J\underset{\sim}{x}+c\underset{\sim}{x}'J\underset{\sim}{y}+c\underset{\sim}{y}'J\underset{\sim}{x}+d\underset{\sim}{y}'\underset{\sim}{y}+e\underset{\sim}{y}'J\underset{\sim}{y}$$

$$=a\underset{\sim}{x}'\underset{\sim}{x}+b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+d\underset{\sim}{y}'\underset{\sim}{y}+e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$

$$\geq \frac{a}{n_1}\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2 + b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2 + 2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right) + \frac{d}{n_2}\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2 + e\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2 \qquad (5.4.39)$$

since $a = d = \sigma_e^2$ and $\sigma_e^2 > 0$.

The last expression in (4.39) is a quadratic function involving $\underset{\sim}{x}'\underset{\sim}{1}$ and $\underset{\sim}{y}'\underset{\sim}{1}$ and the underlying matrix is given by

$$Q = \begin{pmatrix} \frac{a}{n_1} + b & c \\ c & \frac{d}{n_2} + e \end{pmatrix}.$$

To ensure that Q is non-negative definite (nnd), we have to show the followings.

(I) $\frac{a}{n_1} + b > 0$, $\frac{d}{n_2} + e > 0$. (5.4.40)

(II) $\begin{vmatrix} \frac{a}{n_1} + b & c \\ c & \frac{d}{n_2} + e \end{vmatrix} > 0$. (5.4.41)

Recall $v(m) = \sigma_d^2 + \frac{\sigma_e^2}{m}$ and $v(m+1) = \sigma_d^2 + \frac{\sigma_e^2}{m+1}$.

Set $v(n_i) = \sigma_d^2 + \frac{\sigma_e^2}{n_i}$ for $i = 1, 2$. (5.4.42)

It follows from Stepniak [8] that

$$\frac{1}{v(n_1)} + \frac{1}{v(n_2)} < \frac{1}{v(m)} + \frac{1}{v(m+1)} = \frac{1}{V(t)}$$

$$\frac{1}{\frac{1}{v(n_1)} + \frac{1}{v(n_2)}} > \frac{1}{\frac{1}{v(m)} + \frac{1}{v(m+1)}} = V(t) \qquad (5.4.43)$$

$$\frac{1}{\frac{1}{v(n_i)}} > \frac{1}{\frac{1}{v(m)} + \frac{1}{v(m+1)}} = V(t) \text{ for } i = 1, 2. \qquad (5.4.44)$$

It now follows from (5.4.38), (5.4.42) and (5.4.44) that (I) holds, i.e. (5.4.40) is true.

For (II), we note that (5.4.41) is equivalent to

$$\left[v(n_1)-V(t)\right]\left[v(n_2)-V(t)\right]-V^2(t)>0$$

$$v(n_1)\,v(n_2)-V(t)\left[v(n_1)+v(n_2)\right]>0$$

$$\frac{1}{\frac{1}{v(n_1)}+\frac{1}{v(n_2)}}>\frac{1}{\frac{1}{v(m)}+\frac{1}{v(m+1)}}=V(t),$$

which is the expression (5.4.43) stated earlier.

Hence the claim is established.

Remark 5.4.2 Under one-way random effects model with unknown mean (μ) and known variance components $\left(\sigma_d^{\ 2} \text{ and } \sigma_e^{\ 2}\right)$, the allocation $A(n;3;m,m,m+1)$ is sufficient for any other allocation $A(n;3;n_1,n_2,n_3)$ in the sense of sufficient experiment, where $n=3m+1=n_1+n_2+n_3$.

Proof: From allocation $A(n;3;m,m,m+1)$, we obtain observations $x_1,x_2,...,x_m$ on the first treatment, observations $y_1,y_2,...,y_m$ on the second treatment and observations $z_1,z_2,...,z_{m+1}$ on the third treatment.

Clearly, $x_i=\mu+d_1+e_i$, $1\le i\le m$,

$y_j=\mu+d_2+f_j$, $1\le j\le m$,

$z_k=\mu+d_3+q_k$, $1\le k\le m+1$,

where $d_1,d_2,d_3\sim iid\ N\left(0,\sigma_d^{\ 2}\right)$ and d_1,d_2,d_3 are independent of e_i's, f_j's and q_k's which are iid as $N\left(0,\sigma_e^{\ 2}\right)$. Then,

$$\bar{x}\sim N\left(\mu,\sigma_d^{\ 2}+\frac{\sigma_e^{\ 2}}{m}=v_1(m)\right),$$

$$\bar{y}\sim N\left(\mu,\sigma_d^{\ 2}+\frac{\sigma_e^{\ 2}}{m}=v_2(m)\right) \text{ and}$$

$$\bar{z}\sim N\left(\mu,\sigma_d^{\ 2}+\frac{\sigma_e^{\ 2}}{m+1}=v_3(m+1)\right).$$

First we compute 't' which is weighted mean of all observations from allocation $A(n;3;m,m,m+1)$. Therefore,

$$t = \frac{\left[\bar{x}/v_1(m)\right]+\left[\bar{y}/v_2(m)+\left[\bar{z}/v_3(m+1)\right]\right]}{\left[1/v_1(m)\right]+\left[1/v_2(m)\right]+\left[1/v_3(m+1)\right]}.$$

Then, $t \sim N\left(\mu, \frac{1}{\{1/v_1(m)\}+\{1/v_2(m)\}+\{1/v_3(m+1)\}} = V(t)\right)$.

Our purpose is to establish the relation that based on the available data, i.e. $\{(x_1, x_2, ..., x_m); (y_1, y_2, ..., y_m); (z_1, z_2, ..., z_{m+1})\}$, we can *generate* observations on the other allocation $A(n;3;n_1,n_2,n_3)$.

Denote by $\{(U_1,U_2,...,U_{n_1}); (V_1,V_2,...,V_{n_2}); (W_1,W_2,...,W_{n_3})\}$ the *potential* observations from $A(n;3;n_1,n_2,n_3)$.

Note that $U_i = \mu + D_1 + E_i$, $1 \le i \le n_1$,

while $V_j = \mu + D_2 + F_j$, $1 \le j \le n_2$,

and $W_k = \mu + D_2 + Q_k$, $1 \le k \le n_3$,

where $D_1, D_2, D_3 \sim iid\ N(0, \sigma_d^2)$ and D_1, D_2, D_3 are independent of E_i's, F_j's and Q_k's which are iid as $N(0, \sigma_e^2)$.

We describe below a method for generating the U_i's, V_j's and W_k's from the x_i's, y_j's and z_k's.

We generate $\underset{\sim}{S} \sim N_n(\underset{\sim}{0}_{n\times 1}, \Sigma_{n\times n})$, where

$$\Sigma_{n\times n} = \begin{pmatrix} A_{n_1\times n_1} & B_{n_1\times n_2} & C_{n_1\times n_3} \\ B'_{n_2\times n_1} & D_{n_2\times n_2} & E_{n_2\times n_3} \\ C'_{n_3\times n_1} & E'_{n_3\times n_2} & F_{n_3\times n_3} \end{pmatrix}$$

with $A_{n_1\times n_1} = \sigma_e^2 I_{n_1\times n_1} + \sigma_d^2 J_{n_1\times n_1} - V(t) J_{n_1\times n_1}$; $D_{n_2\times n_2} = \sigma_e^2 I_{n_2\times n_2} + \sigma_d^2 J_{n_2\times n_2} - V(t) J_{n_2\times n_2}$;

$F_{n_3\times n_3} = \sigma_e^2 I_{n_3\times n_3} + \sigma_d^2 J_{n_3\times n_3} - V(t) J_{n_3\times n_3}$;

$$B_{n_1\times n_2} = -V(t) J_{n_1\times n_2};\ C_{n_1\times n_3} = -V(t) J_{n_1\times n_3};\ E_{n_2\times n_3} = -V(t) J_{n_2\times n_3}, \qquad (5.4.45)$$

'I' is an identity matrix and 'J' is a matrix having all elements one.

We first deduce that $\Sigma_{n\times n}$ is nnd.

Once this is done, in the above, using 't' and $\underset{\sim}{S}$, we can generate observations $\left\{\left(t+S_i \mid 1\le i\le n_1; t+S_{n_1+j} \mid 1\le j\le n_2; t+S_{n_1+n_2+k} \mid 1\le k\le n_3\right)\right\}$ which we claim to be equivalent to the observations $\left\{\left(U_1,U_2,...,U_{n_1}\right);\left(V_1,V_2,...,V_{n_2}\right);\left(W_1,W_2,...,W_{n_3}\right)\right\}$.

Essentially, then, we have to prove:

(I) $\sum_{n\times n}$ is nnd.

(II) $\left(t+S_i \mid 1\le i\le n_1\right)$ follow the joint distribution of $\left(U_i, 1\le i\le n_1\right)$.

(III) $\left(t+S_{n_1+j} \mid 1\le j\le n_2\right)$ follow the joint distribution of $\left(V_j, 1\le j\le n_2\right)$.

(IV) $\left(t+S_{n_1+n_2+k} \mid 1\le k\le n_3\right)$ follow the joint distribution of $\left(W_k, 1\le k\le n_3\right)$.

(V) $\left(t+S_i\right),\left(t+S_{n_1+j}\right)$ are independently distributed for $1\le i\le n_1; 1\le j\le n_2$.

(VI) $\left(t+S_i\right),\left(t+S_{n_1+n_2+k}\right)$ are independently distributed for $1\le i\le n_1; 1\le k\le n_3$.

(VII) $\left(t+S_{n_1+n_2+k}\right),\left(t+S_{n_1+j}\right)$ are independently distributed for $1\le k\le n_3; 1\le j\le n_2$.

However, it may be noted that our choice of $\sum_{n\times n}$ has the built-in property of satisfying the conditions (II) - (VII) laid down above. This follows essentially from the steps outlined in Theorem 4.1. Therefore, we are done once we settle (I) above.

Set $\sum_{n\times n} = \begin{pmatrix} aI+bJ & cJ & dJ \\ cJ & eI+fJ & pJ \\ dJ & pJ & pI+hJ \end{pmatrix}$,

where $a=e=q=\sigma_e^2, b=f=h=\sigma_d^{\ 2}-V(t)$ and $c=d=p=-V(t)$. (5.4.46)

Consider the quadratic form as below.

$$Q\left(\underset{\sim}{x}_{n_1\times 1}, \underset{\sim}{y}_{n_2\times 1}, \underset{\sim}{z}_{n_3\times 1}\right)$$

$$= \left(\underset{\sim}{x}'\ \underset{\sim}{y}'\ \underset{\sim}{z}'\right)\begin{pmatrix} aI+bJ & cJ & dJ \\ cJ & eI+fJ & pJ \\ dJ & pJ & pI+hJ \end{pmatrix}\begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \\ \underset{\sim}{z} \end{pmatrix}$$

$$= \left(\underset{\sim}{x}'\ \underset{\sim}{y}'\ \underset{\sim}{z}'\right)\begin{pmatrix} a\underset{\sim}{x} + b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\underset{\sim}{1} + cJ\underset{\sim}{y} + dJ\underset{\sim}{z} \\ cJ\underset{\sim}{x} + e\underset{\sim}{y} + fJ\underset{\sim}{y} + pJ\underset{\sim}{z} \\ dJ\underset{\sim}{x} + pJ\underset{\sim}{y} + q\underset{\sim}{z} + hJ\underset{\sim}{z} \end{pmatrix}$$

$$= a\left(\underset{\sim}{x}'\underset{\sim}{x}\right) + b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2 + 2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right) + 2d\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right) + e\left(\underset{\sim}{y}'\underset{\sim}{y}\right) + f\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$

$$+ 2p\left(\underset{\sim}{y}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right) + q\left(\underset{\sim}{z}'\underset{\sim}{z}\right) + h\left(\underset{\sim}{z}'\underset{\sim}{1}\right)^2$$

$$\geq a\frac{\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2}{n_1} + b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2 + 2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right) + 2d\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right) + e\frac{\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2}{n_2} + f\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$

$$+ 2p\left(\underset{\sim}{y}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right) + q\frac{\left(\underset{\sim}{z}'\underset{\sim}{1}\right)^2}{n_3} + h\left(\underset{\sim}{z}'\underset{\sim}{1}\right)^2 \qquad (5.4.47)$$

since $a=e=q=\sigma_e^2$ and $\sigma_e^2 > 0$.

The last expression in (5.4.47) is a quadratic form involving $\underset{\sim}{x}'\underset{\sim}{1}$, $\underset{\sim}{y}'\underset{\sim}{1}$ and $\underset{\sim}{z}'\underset{\sim}{1}$ and the underlying matrix is given by

$$Q = \begin{pmatrix} \frac{a}{n_1} + b & c & d \\ c & \frac{e}{n_2} + f & p \\ d & p & \frac{q}{n_3} + h \end{pmatrix}.$$

To ensure that Q is non-negative definite(nnd), we have to show the followings.

(I) $\frac{a}{n_1} + b > 0$, $\frac{e}{n_2} + f > 0$, $\frac{q}{n_3} + h > 0$. (5.4.48)

(II) All 2×2 "principal" minors > 0. (5.4.49)

(III) $\begin{vmatrix} \frac{a}{n_1} + b & c & d \\ c & \frac{e}{n_2} + f & p \\ d & p & \frac{q}{n_3} + h \end{vmatrix} > 0$. (5.4.50)

Recall

$v_1(m) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{m}$,

$v_2(m) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{m}$ and

$v_3(m+1) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{m+1}$.

Set $v(n_i) = \sigma_d{}^2 + \frac{\sigma_e{}^2}{n_i}$, $i = 1,2,3$. (5.4.51)

It follows from Stepniak [8] that

$$\frac{1}{v(n_1)} + \frac{1}{v(n_2)} + \frac{1}{v(n_3)} < \frac{1}{v_1(m)} + \frac{1}{v_2(m)} + \frac{1}{v_3(m+1)}$$

$$\frac{1}{\frac{1}{v(n_1)} + \frac{1}{v(n_2)} + \frac{1}{v(n_3)}} > \frac{1}{\frac{1}{v_1(m)} + \frac{1}{v_2(m)} + \frac{1}{v_3(m+1)}} = V(t). \tag{5.4.52}$$

Hence, $\frac{1}{1/v(n_i)} > V(t)$ for $1 \le i \le 3$ (5.4.53)

and also $\frac{1}{1/v(n_i) + 1/v(n_j)} > V(t)$ for $1 \le i \ne j \le 3$. (5.4.54)

It now follows from (5.4.46), (5.4.51) and (5.4.53) that (I) holds, i.e. (5.4.48) is true.

Also it follows from (5.4.46), (5.4.51) and (5.4.54) that (II) holds, i.e. (5.4.49) is true.

For (III), we note that (5.4.50) is equivalent to (Schott [18])

$$\begin{vmatrix} \frac{a}{n_1}+b & c \\ c & \frac{e}{n_2}+f \end{vmatrix} \left[\left(\frac{q}{n_3}+h\right) - \begin{pmatrix} d & p \end{pmatrix} \begin{pmatrix} \frac{a}{n_1}+b & c \\ c & \frac{e}{n_2}+f \end{pmatrix}^{-1} \begin{pmatrix} d \\ p \end{pmatrix} \right] > 0$$

$$\left(\frac{q}{n_3}+h\right) \begin{vmatrix} \frac{a}{n_1}+b & c \\ c & \frac{e}{n_2}+f \end{vmatrix} - \begin{pmatrix} d & p \end{pmatrix} \begin{pmatrix} \frac{e}{n_2}+f & -c \\ -c & \frac{a}{n_1}+b \end{pmatrix} \begin{pmatrix} d \\ p \end{pmatrix} > 0$$

$$\left[v(n_3) - V(t)\right]\left[\{v(n_1) - V(t)\}\{v(n_2) - V(t)\} - V^2(t)\right] - \begin{pmatrix} d & p \end{pmatrix} \begin{pmatrix} -v(n_2)V(t) \\ -v(n_1)V(t) \end{pmatrix} > 0$$

$$\left[v(n_3)-V(t)\right]\left[\{v(n_1)-V(t)\}\{v(n_2)-V(t)\}-V^2(t)\right]\text{-}V^2(t)\left[v(n_2)+v(n_1)\right]>0$$

$$v(n_1)\,v(n_2)\,v(n_3)>V(t)\left[v(n_1)v(n_2)+v(n_1)v(n_3)+v(n_2)v(n_3)\right]$$

$$\frac{1}{\dfrac{1}{v(n_1)}+\dfrac{1}{v(n_2)}+\dfrac{1}{v(n_3)}}>V(t),$$

which is the expression (5.4.52) stated earlier.

Hence the claim is established.

Remark 5.4.3 Under one-way random effects model with unknown mean (μ) and known variance components $(\sigma_d{}^2 \text{ and } \sigma_e{}^2)$, the allocation $A(n;3;m,m+1,m+1)$ is sufficient for any other allocation $A(n;3;n_1,n_2,n_3)$ in the sense of sufficient experiment, where $n=3m+2=n_1+n_2+n_3$.

Proof: From allocation $A(n;3;m,m+1,m+1)$, we obtain observations $x_1,x_2,...,x_m$ on the first treatment, observations $y_1,y_2,...,y_{m+1}$ on the second treatment and observations $z_1,z_2,...,z_{m+1}$ on the third treatment.

Clearly, $x_i=\mu+d_1+e_i$, $1\le i\le m$,

$y_j=\mu+d_2+f_j$, $1\le j\le m+1$,

$z_k=\mu+d_3+q_k$, $1\le k\le m+1$,

where $d_1,d_2,d_3\sim iid\ N(0,\sigma_d{}^2)$ and d_1,d_2,d_3 are independent of e_i's, f_j's and q_k's which are iid as $N(0,\sigma_e{}^2)$. Then,

$$\overline{x}\sim N\left(\mu,\sigma_d{}^2+\frac{\sigma_e{}^2}{m}=v_1(m)\right),$$

$$\overline{y}\sim N\left(\mu,\sigma_d{}^2+\frac{\sigma_e{}^2}{m+1}=v_2(m+1)\right) \text{ and}$$

$$\overline{z}\sim N\left(\mu,\sigma_d{}^2+\frac{\sigma_e{}^2}{m+1}=v_3(m+1)\right).$$

First we compute 't' which is weighted mean of all observations from allocation $A(n;3;m,m,m+1)$. Hence,

$$t=\frac{\left[\bar{x}/v_1(m)\right]+\left[\bar{y}/v_2(m+1)+\left[\bar{z}/v_3(m+1)\right]\right]}{\left[1/v_1(m)\right]+\left[1/v_2(m+1)\right]+\left[1/v_3(m+1)\right]}.$$

Then, $t\sim N\left(\mu,\frac{1}{\{1/v_1(m)\}+\{1/v_2(m+1)\}+\{1/v_3(m+1)\}}=V(t)\right)$.

Our purpose is to establish the relation that based on the available data, i.e. $\{(x_1,x_2,...,x_m);(y_1,y_2,...,y_{m+1});(z_1,z_2,...,z_{m+1})\}$, we can *generate* observations on the other allocation $A(n;3;n_1,n_2,n_3)$.

Denote by $\{(U_1,U_2,...,U_{n_1});(V_1,V_2,...,V_{n_2});(W_1,W_2,...,W_{n_3})\}$ the *potential* observations from $A(n;3;n_1,n_2,n_3)$.

Note that $U_i=\mu+D_1+E_i$, $1\le i\le n_1$,

while $V_j=\mu+D_2+F_j$, $1\le j\le n_2$,

and $W_k=\mu+D_2+Q_k$, $1\le k\le n_3$

where $D_1,D_2,D_3\sim iid\ N(0,\sigma_d^2)$ and D_1,D_2,D_3 are independent of E_i's, F_j's and Q_k's which are iid as $N(0,\sigma_e^2)$.

We describe below a method for generating the U_i's, V_j's and W_k's from the x_i's, y_j's and z_k's.

We generate $\underset{\sim}{S}\sim N_n\left(\underset{\sim}{0}_{n\times1},\Sigma_{n\times n}\right)$, where

$$\Sigma_{n\times n}=\begin{pmatrix} A_{n_1\times n_1} & B_{n_1\times n_2} & C_{n_1\times n_3}\\ B'_{n_2\times n_1} & D_{n_2\times n_2} & E_{n_2\times n_3}\\ C'_{n_3\times n_1} & E'_{n_3\times n_2} & F_{n_3\times n_3}\end{pmatrix}$$

With $A_{n_1\times n_1}=\sigma_e^2 I_{n_1\times n_1}+\sigma_d^2 J_{n_1\times n_1}-V(t)J_{n_1\times n_1}$;

$D_{n_2\times n_2}=\sigma_e^2 I_{n_2\times n_2}+\sigma_d^2 J_{n_2\times n_2}-V(t)J_{n_2\times n_2}$;

$F_{n_3\times n_3}=\sigma_e^2 I_{n_3\times n_3}+\sigma_d^2 J_{n_3\times n_3}-V(t)J_{n_3\times n_3}$;

$$B_{n_1\times n_2} = -V(t)J_{n_1\times n_2}\ ;\ C_{n_1\times n_3} = -V(t)J_{n_1\times n_3}\ ; E_{n_2\times n_3} = -V(t)J_{n_2\times n_3}\ , \qquad (5.4.55)$$

' I ' is an identity matrix and ' J ' is a matrix having all elements one.

We first deduce that $\sum_{n\times n}$ is nnd.

Once this is done, in the above, using 't' and $\underset{\sim}{S}$, we can generate observations $\left\{\left(t+S_i \mid 1\le i\le n_1; t+S_{n_1+j} \mid 1\le j\le n_2; t+S_{n_1+n_2+k} \mid 1\le k\le n_3\right)\right\}$ which we claim to be equivalent to the observations $\left\{\left(U_1,U_2,...,U_{n_1}\right);\left(V_1,V_2,...,V_{n_2}\right);\left(W_1,W_2,...,W_{n_3}\right)\right\}$.

Essentially, then, we have to prove:

(I) $\sum_{n\times n}$ is nnd.

(II) $\left(t+S_i \mid 1\le i\le n_1\right)$ follow the joint distribution of $\left(U_i, 1\le i\le n_1\right)$.

(III) $\left(t+S_{n_1+j} \mid 1\le j\le n_2\right)$ follow the joint distribution of $\left(V_j, 1\le j\le n_2\right)$.

(IV) $\left(t+S_{n_1+n_2+k} \mid 1\le k\le n_3\right)$ follow the joint distribution of $\left(W_k, 1\le k\le n_3\right)$.

(V) $\left(t+S_i\right),\left(t+S_{n_1+j}\right)$ are independently distributed for $1\le i\le n_1; 1\le j\le n_2$.

(VI) $\left(t+S_i\right),\left(t+S_{n_1+n_2+k}\right)$ are independently distributed for $1\le i\le n_1; 1\le k\le n_3$.

(VII) $\left(t+S_{n_1+n_2+k}\right),\left(t+S_{n_1+j}\right)$ are independently distributed for $1\le k\le n_3; 1\le j\le n_2$.

However, it may be noted that our choice of $\sum_{n\times n}$ has the built-in property of satisfying the conditions (II) - (VII) laid down above. This follows essentially from the steps outlined in Theorem 5.4.1. Therefore, we are done once we settle (I) above.

Set $\sum_{n\times n} = \begin{pmatrix} aI+bJ & cJ & dJ \\ cJ & eI+fJ & pJ \\ dJ & pJ & pI+hJ \end{pmatrix}$,

where $a=e=q=\sigma_e^2, b=f=h=\sigma_d{}^2-V(t)$ and $c=d=p=-V(t)$. (5.4.56)

Consider the quadratic form as below.

$Q\left(\underset{\sim}{x}_{n_1\times 1}, \underset{\sim}{y}_{n_2\times 1}, \underset{\sim}{z}_{n_3\times 1}\right)$

$$= \left(\underset{\sim}{x}'\ \underset{\sim}{y}'\ \underset{\sim}{z}'\right)\begin{pmatrix} aI+bJ & cJ & dJ \\ cJ & eI+fJ & pJ \\ dJ & pJ & pI+hJ \end{pmatrix}\begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \\ \underset{\sim}{z} \end{pmatrix}$$

$$= \left(\underset{\sim}{x}'\ \underset{\sim}{y}'\ \underset{\sim}{z}'\right)\begin{pmatrix} a\underset{\sim}{x}+b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\underset{\sim}{1}+cJ\underset{\sim}{y}+dJ\underset{\sim}{z} \\ cJ\underset{\sim}{x}+e\underset{\sim}{y}+fJ\underset{\sim}{y}+pJ\underset{\sim}{z} \\ dJ\underset{\sim}{x}+pJ\underset{\sim}{y}+q\underset{\sim}{z}+hJ\underset{\sim}{z} \end{pmatrix}$$

$$= a\left(\underset{\sim}{x}'\underset{\sim}{x}\right)+b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+2d\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right)+e\left(\underset{\sim}{y}'\underset{\sim}{y}\right)+f\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$
$$+2p\left(\underset{\sim}{y}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right)+q\left(\underset{\sim}{z}'\underset{\sim}{z}\right)+h\left(\underset{\sim}{z}'\underset{\sim}{1}\right)^2$$

$$\geq a\frac{\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2}{n_1}+b\left(\underset{\sim}{x}'\underset{\sim}{1}\right)^2+2c\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{y}'\underset{\sim}{1}\right)+2d\left(\underset{\sim}{x}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right)+e\frac{\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2}{n_2}+f\left(\underset{\sim}{y}'\underset{\sim}{1}\right)^2$$
$$+2p\left(\underset{\sim}{y}'\underset{\sim}{1}\right)\left(\underset{\sim}{z}'\underset{\sim}{1}\right)+q\frac{\left(\underset{\sim}{z}'\underset{\sim}{1}\right)^2}{n_3}+h\left(\underset{\sim}{z}'\underset{\sim}{1}\right)^2, \tag{5.4.57}$$

since $a=e=q=\sigma_e^2$ and $\sigma_e^2>0$.

The last expression in (5.4.57) is a quadratic function involving $\underset{\sim}{x}'\underset{\sim}{1}$, $\underset{\sim}{y}'\underset{\sim}{1}$ and $\underset{\sim}{z}'\underset{\sim}{1}$ and the underlying matrix is given by

$$Q=\begin{pmatrix} \frac{a}{n_1}+b & c & d \\ c & \frac{e}{n_2}+f & p \\ d & p & \frac{q}{n_3}+h \end{pmatrix}.$$

To ensure that Q is non-negative definite(nnd), we have to show the followings.

(I) $\frac{a}{n_1}+b>0$, $\frac{e}{n_2}+f>0$, $\frac{q}{n_3}+h>0$. (5.4.58)

(II) All 2×2 “principal” minors >0. (5.4.59)

(III) $$\begin{vmatrix} \frac{a}{n_1}+b & c & d \\ c & \frac{e}{n_2}+f & p \\ d & p & \frac{q}{n_3}+h \end{vmatrix} > 0. \tag{5.4.60}$$

Recall $v_1(m)=\sigma_d{}^2+\frac{\sigma_e{}^2}{m}$, $v_2(m+1)=\sigma_d{}^2+\frac{\sigma_e{}^2}{m+1}$ and $v_3(m+1)=\sigma_d{}^2+\frac{\sigma_e{}^2}{m+1}$.

Set $v(n_i)=\sigma_d{}^2+\frac{\sigma_e{}^2}{n_i}$, $i=1,2,3$. (5.4.61)

It follows from Stepniak [8] that

$$\frac{1}{v(n_1)}+\frac{1}{v(n_2)}+\frac{1}{v(n_3)}<\frac{1}{v_1(m)}+\frac{1}{v_2(m+1)}+\frac{1}{v_3(m+1)}$$

$$\frac{1}{\frac{1}{v(n_1)}+\frac{1}{v(n_2)}+\frac{1}{v(n_3)}}>\frac{1}{\frac{1}{v_1(m)}+\frac{1}{v_2(m+1)}+\frac{1}{v_3(m+1)}}=V(t) \tag{5.4.62}$$

Hence, $\frac{1}{1/v(n_i)}>V(t)$ for $1\le i\le 3$, (5.4.63)

and also $\frac{1}{1/v(n_i)+1/v(n_j)}>V(t)$ for $1\le i\ne j\le 3$. (5.4.64)

It now follows from (5.4.56), (5.4.61) and (5.4.63) that (I) holds, i.e. (5.4.58) is true. Also it follows from (5.4.56), (5.4.61) and (5.4.64) that (II) holds, i.e. (5.4.59) is true. For (III), we note that (5.4.60) is equivalent to

$$\begin{vmatrix} \frac{a}{n_1}+b & c \\ c & \frac{e}{n_2}+f \end{vmatrix}\left[\left(\frac{q}{n_3}+h\right)-\begin{pmatrix} d & p \end{pmatrix}\begin{pmatrix} \frac{a}{n_1}+b & c \\ c & \frac{e}{n_2}+f \end{pmatrix}^{-1}\begin{pmatrix} d \\ p \end{pmatrix}\right]>0$$

$$\left(\frac{q}{n_3}+h\right)\begin{vmatrix} \frac{a}{n_1}+b & c \\ c & \frac{e}{n_2}+f \end{vmatrix}-\begin{pmatrix} d & p \end{pmatrix}\begin{pmatrix} \frac{e}{n_2}+f & -c \\ -c & \frac{a}{n_1}+b \end{pmatrix}\begin{pmatrix} d \\ p \end{pmatrix}>0$$

$$\left[v(n_3)-V(t)\right]\left[\{v(n_1)-V(t)\}\{v(n_2)-V(t)\}-V^2(t)\right]\text{-}(d\ \ p)\begin{pmatrix}-v(n_2)V(t)\\-v(n_1)V(t)\end{pmatrix}>0$$

$$\left[v(n_3)-V(t)\right]\left[\{v(n_1)-V(t)\}\{v(n_2)-V(t)\}-V^2(t)\right]\text{-}V^2(t)\left[v(n_2)+v(n_1)\right]>0$$

$$v(n_1)\,v(n_2)\,v(n_3)>V(t)\left[v(n_1)v(n_2)+v(n_1)v(n_3)+v(n_2)v(n_3)\right]>0$$

$$\frac{1}{\frac{1}{v(n_1)}+\frac{1}{v(n_2)}+\frac{1}{v(n_3)}}>V(t),$$

which is the expression (5.4.62) stated earlier.

Hence the claim is established.

Theorem 5.4.4 Under one-way random effects model with unknown mean (μ) and known variance components $(\sigma_d{}^2 \text{ and } \sigma_e{}^2)$, the allocation $A(n;n;1,1,...,1)$ is sufficient for any other allocation $A(n;k;n_1,n_2,...,n_k)$ in the sense of sufficient experiment, where $n=\sum_{i=1}^{k}n_i$ and 'k' stands for the number of 'treatments'.

Proof: It is enough to prove that the allocation $A(n;n;1,1,...,1)$ is sufficient for other allocation $A(n;1;n)$.

From allocation $A(n;n;1,1,...,1)$, we obtain observations $x_1,x_2,...,x_n$.

Clearly, $x_i=\mu+d_i+e_i$, $1\le i\le n$,

where $d_1,d_2,...,d_n\sim iid\ N(0,\sigma_d{}^2)$ and $e_1,e_2,...,e_n\sim iid\ N(0,\sigma_e{}^2)$ and all d_i's are independent of all e_i's.

We compute 't' = $\bar{x}$ = grand mean of all observations from allocation $A(n;n;1,1,...,1)$.

Then, $t\sim N\left(\mu,\frac{\sigma_d{}^2+\sigma_e{}^2}{n}\right)$.

Our purpose is to establish the relation that based on the available data, i.e. $(x_1,x_2,...,x_n)$, we can *generate* observations on the other allocation $A(n;1;n)$. Denote by $(Y_1,Y_2,...,Y_n)$ the observations from $A(n;1;n)$.

Note that $Y_i = \mu + D_1 + E_i$, $1 \le i \le n$.

Then, we define $z = N\left(0, \left(1-\frac{1}{n}\right)\sigma_d^{\,2}\right)$

and $\underset{\sim}{S}_{n\times 1} \sim N_n\left(\underset{\sim}{0}_{n\times 1}, \sigma_e^{\,2}\left(I_{n\times n} - \frac{J_{n\times n}}{n}\right)\right)$,

where I is an identity matrix and J is a matrix having all elements one.

Then we make a stochastic transformation to generate $\tilde{Y}_i$ as

$t + z + S_i, 1 \le i \le n$.

$$\text{Here } \operatorname{Var}\left(\tilde{Y}_i\right) = \frac{\sigma_d^{\,2} + \sigma_e^{\,2}}{n} + \sigma_d^{\,2}\left(1-\frac{1}{n}\right) + \sigma_e^{\,2}\left(1-\frac{1}{n}\right)$$

$$= \sigma_d^{\,2} + \sigma_e^{\,2}, \qquad (5.4.65)$$

which is the same as $\operatorname{Var}\left(Y_i\right)$. This is true for all '$i$'. The covariance between $\tilde{Y}_i$ and $\tilde{Y}_j$ is given as

$$Cov\left(\tilde{Y}_i, \tilde{Y}_j\right) = Cov\left(t+z+S_i, t+z+S_j\right)$$

$$= \operatorname{Var}(t) + \operatorname{Var}(z) + Cov\left(S_i, S_j\right)$$

$$= \frac{\sigma_d^{\,2} + \sigma_e^{\,2}}{n} + \left(1-\frac{1}{n}\right)\sigma_d^{\,2} - \frac{\sigma_e^{\,2}}{n}$$

$$= \sigma_d^{\,2}, \qquad (5.4.66)$$

which is the same as $Cov\left(Y_i, Y_j\right)$. This is true for all $1 \le i \ne j \le n$.

Hence $\underset{\sim}{\tilde{Y}} \equiv \underset{\sim}{Y}$ where $\underset{\sim}{\tilde{Y}} = \left(\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_n\right)$ and $\underset{\sim}{Y} = \left(Y_1, Y_2, ..., Y_n\right)$.

Hence, using 't' which is the grand mean together with $\underset{\sim}{z}$ and $\underset{\sim}{S}$ which are independent of unknown parameter μ, we can generate all 'n' observations $\underset{\sim}{Y} = \left(Y_1, Y_2, ..., Y_n\right)$.

Therefore, the allocation $A(n;n;1,1,...,1)$ is sufficient for other allocation $A(n;1;n)$.

Here $A\left(n;k;n_1,n_2,...,n_k\right) \equiv A\left(n_1;1;n_1\right) \times A\left(n_2;1;n_2\right) \times \cdots \times A\left(n_k;1;n_k\right)$

Using above result, $A(n;n;1,1,...,1) \equiv A\left(n_1;n_1;1,1,...,1\right) \times A\left(n_2;n_2;1,1,...,1\right) \times \cdots \times A\left(n_k;n_k;1,1,...,1\right)$ is sufficient for $A\left(n_1;1;n_1\right) \times A\left(n_2;1;n_2\right) \times \cdots \times A\left(n_k;1;n_k\right)$.

Hence, the allocation $A(n;n;1,1,...,1)$ is sufficient for any other allocation $A(n;k;n_1,n_2,...,n_k)$ in the sense of sufficient experiment.

References

1. Blackwell, D. (1951). Comparison of experiments. Proc. Second Berkeley Symp. Math. Statist. Prob. University of california Press. 93-102.
2. Blackwell, D. Equivalent comparisons of experiments. Ann. Math. Statist. 1953; 24, 265-272.
3. Blackwell D. and Girshick M. A. (1954). Theory of Games and Statistical Decisions. Wiley, New York, USA.
4. Casella, G. and Berger, R.L. (2001). Statistical Inference (2nd ed.), Duxbury Press, California, USA.
5. DeGroot, M. H. (1966). Optimal allocation of observations. Ann. Inst. Statist. Math. 18, 13-28.
6. Goel, P.K. and DeGroot, M.H. (1979). Comparison of experiments and information measures. Ann. Statist. 7, 1066-1077.
7. Goel, P. K. and Ginebra, J. (2003). When is one experiment always better than Another?. J. Royal Statist. Soc. 52, 515-537.
8. Greenshtein, E. (1996). Comparison of sequential experiments. Ann. Statist. 24, 436-448.
9. Hansen, O.H. and Torgersen, E. N. (1974). Comparison of linear normal experiments. Ann. Statist. 2, 367-373.
10. Karlin, S. and Rubin, H. (1956). The theory of decision procedures for distributions with monotone likelihood ratio. Ann. Math. Statist. 27, 72-299.
11. Le Cam, L. (1996). Comparison of experiments: a short review. *IMS Lect. Notes* Monogr. Ser. 30, 127-138.
12. Lehmann, E.L. (1988). Comparing location experiments. Ann. Statist. 16, 521- 533.
13. Schott, J. R. *Matrix Analysis For Statistics*, John Wiley and Sons, Inc., Hoboken, New Jersey; 2005.
14. Shaked, M. and Tong, Y.L. (1992). Comparison of experiments via dependence of normal variables with a common marginal distribution. Ann. Statist. 20, 614- 618.

15. Shaked, M. and Tong, Y.L. (1993). Comparison of experiments of some Multivariate distributions with a common marginal. Stochastic Inequalities. 22, 388 – 398.
16. Sinha, B.K. (1973). Comparison of experiments with sufficiency consideration. *Ann.* Inst. Statist. Math. 25, 501-520.
17. Stepniak, C. (1982). Optimal allocation of observations in one – way random normal model. Ann. Inst. Statist. Math. 34, 175-180.
18. Stepniak, C. (1997). Comparison of normal linear experiments by quadratic forms. Ann. Inst. Statist. Math. 1997; 49, 569-584.

8. Overall Output

8.1 Summary Table

Table: Outputs

Outputs	Proposed No. / Actual No.
1. Papers submitted in international journals*	6 / 8
2. Papers accepted for publication*	6 / 7
3. Papers appeared in international journals*	6 / 7
4. Papers presented in international conferences*	3 /6
5. Graduated Students M.Sc.: Two students graduated Ph.D.: Three students graduated	4/5
6. Invited speaker and give seminars on 6.1 Topic: A statistical study of dose response relationship with applications at the Statistics Unit, Tampere School of Public Health, University of Tampere, Tampere, Finland, October 5, 2010 6.2 Topic: Nonlinear mixed effects model of colorectal cancer patients at Department of Statistics, Faculty of Science, Khon Kaen University, June 6, 2011 6.3 Topic: Estimating recurrent time using nonlinear mixed effects model in patients with colorectal cancer at the Statistics Unit, Tampere School of Public Health, University of Tampere, Tampere, Finland, September 26, 2011	0/3
7. Research Award 2008: The Consolation Research Award from the National Research Council of Thailand (NRCT) "Statistical Modeling of Malaria and Cancer Patients in Thailand"	-

(* see Appendices)

8.2 Papers appeared in international journals

8.2.1 Dhungana, A.R., Tiensuwan, M[*]., Sinha, B.K. (2010). Sufficiency in bivariate and trivariate normal populations. *Far East Journal of Theoretical Statistics*, 32(1), pp.59-80.
(**Indexed/Abstract** in MathSciNet, Zentralblatt MATH)

8.2.2 Tiensuwan, M., Dhungana, A.R Sinha, B.K. (2010). Sufficiency in linear and quadratic regression models. *Journal of Statistical Theory and Applications*, Vol. 9(3), pp.387-404.
(**Indexed/Abstract** in MathSciNet, Zentralblatt MATH)

8.2.3 Sontimoon N, Tiensuwan M, Panunzi S, Sumetchotimaytha W. (2010). Estimating recurrent time using nonlinear mixed effects model in patients with colorectal cancer, *Advances and Applications in Statistics*, Vol. 17(1), pp. 41-60.
(**Indexed/Abstract** in MathSciNet, Zentralblatt MATH)

8.2.4 Srisodaphol, W., Tiensuwan, M[*]., Sinha, B.K. (2011). On an asymptotic comparison of the maximum likelihood and Berkson's minimum chi-square estimators in some standard dose response models with one unknown parameter, *Model Assisted Statistics and Applications*, Vol.6(1), pp.21-38.
(**Indexed/Abstract** in MathSciNet, Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 3 / SNIP : 0.226 / SJR : 0.028 / Category : Applied Mathematics Q4

8.2.5 Tiensuwan, M. and Dhungana, A.R (2011). Understanding sufficiency in one-way random effects model, *Sankhya B*, Vol. 73, Number 2, pp. 263-275.
(**Indexed/Abstract** in Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 6 / SNIP : 0.377 / SJR : 0.029 / Category : Statistics and Probability Q4

8.2.6 Thoppradid R and Tiensuwan M. (2012). On comparison of vegetabbles nutrition by multiple criteria decision making, East-West J. of Mathematics, Special Vol (2012), pp. 361-373.
(**Indexed/Abstract** in MathSciNet, Zentralblatt MATH)

8.2.7 Srisodaphol W, Tiensuwan M and Sinha BK.(2013). On an asymptotic comparison of maximum likelihood and Berkson's modified minimum chi-square estimates in the two parameter dose response models, *Journal of Applied Statistical Science*, Vol.19(4), pp.49-69.
(**Indexed/Abstract** in MathSciNet, Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 1 / SNIP : 0.103 / SJR : 0.100/ Category : Statistics and Probability Q4

8.3 Papers submitted for publication in international journals

8.3.1 Tusto P, Tiensuwan M. Applications of Box-Jenkins Models on Rainfall and Water Level Predictions along the Cha Phraya River in Thailand (Submitted).

8.4 Papers presented in the international conferences

8.4.1 Tusto P, Tiensuwan M (2009). Applications of Box-Jenkins models on rainfall prediction for the chao phraya river in Thailand, Proceeding of ICMA-MU 2009: International Conference in Mathematics and Applications, Decmer 17-19, 2009, Twin Tower Hotel, Bangkok, Thailand, pp.455-463.

8.4.2 Srisodaphol W, Tiensuwan M and Sinha BK. (2009). On an asymptotic comparison of maximum likelihood and Berkson's minimum chi-square estimates in the dose response models with one unknown parameter, Proceeding of ICMA-MU 2009: International Conference in Mathematics and Applications, December 17-19, 2009, Twin Tower Hotel, Bangkok, Thailand, p.125.

8.4.3 Dhungana AR, Sinha BK, Tiensuwan M (2009). Comparison of experiment in the framework of bivariate normal populations, Proceeding of ICMA-MU 2009: International Conference in Mathematics and Applications, December 17-19, 2009, Twin Tower Hotel, Bangkok, Thailand, p.90.

8.4.4 Thoppradid, R. and Tiensuwan, M. (2011) On comparison of vegetables nutrition by multiple criteria decision making. Proceeding of the International Conference in Mathematics and Applications (ICMA – MU 2011), December 17-19, 2011, Twin Towers Hotel, Bangkok Thailand, pp. 441-450.

8.4.5 Tiensuwan, M. and Sontimoon, N. (2012) Recurrence of Colorectal Cancer Using Nonlinear Mixed-Effects Models. Abstracts for the XXVIth International Biometric Conference, 26-31 August, 2012, Kobe, Japan. International Biometric Society (http://secretariat.ne.jp/ibc2012/programme.html). ISBN 978-0-9821919-2-7.

8.4.6 Srisodaphol, W., Tiensuwan, M*, Sinha, B.K. (2012) On An Asymptotic Comparison of the Maximum Likelihood and Berkson's Minimum Chi-square Estimators in Some Standard Dose Response Models with One Unknown Parameter, การประชุม "นักวิจัยรุ่นใหม่...พบ...เมธีวิจัยอาวุโส สกว. "ครั้งที่ 12 : 10-12 ตุลาคม 2555 ณ โรงแรมฮอลิเดย์อินน์ รีสอร์ทรีเจนท์ บีชชะอำ จังหวัดเพชรบุรี

8.4.7 Tiensuwan, M. and Sontimoon, N. (2012) Recurrence of Colorectal Cancer Using Nonlinear Mixed-Effects Models. การประชุม "นักวิจัยรุ่นใหม่...พบ...เมธีวิจัยอาวุโส สกว. "ครั้งที่ 13 : 16-18 ตุลาคม 2556 ณ โรงแรมฮอลิเดย์อินน์ รีสอร์ทรีเจนท์ บีชชะอำ จังหวัดเพชรบุรี

8.5 Papers appeared in international journals (BRG4980012)

8.5.1 Tiensuwan, M. and Sarikavanich, S. (2010). Applications of Box-Jenkins models to rainfall data of Chiang Mai province in Thailand. *Advances and Applications in Statistics*, 15(1), pp.27-48.
(**Indexed/Abstract** in MathSciNet, Zentralblatt MATH)

8.5.2 Lertprapai, S., Tiensuwan, M. (2010). An application of multiple criteria decision making to water quality data in Thailand. *Far East Journal of Applied Mathematics*, 40(2), pp.103-124.
(**Indexed/Abstract** in MathSciNet, Zentralblatt MATH)

8.6 Other research papers appeared in international journals

8.6.1 Arom KV, Ruengsakulrach P, Belkin M, **Tiensuwan M**.(2009). Efficacy of intramyocardial injection of angiogenic cell precursors for non-ischemic dilated cardiomyopathy: a case match study, *Asian Cardiovascular & Thoracic Annals*;17(4), pp. 382-388.
(**Indexed/Abstracted** in Scopus, PubMed)
Indexed in : Scopus : h index = 10 / SJR = 0.060 / Category = Cardiology and Cardiovascular Medicine Q3

8.6.2 Pattarapanitchai, N., Tiensuwan, M*. and Riengrojpitak, S. (2010). A Retrospective Study on Homicidal Autopsy Cases at Ramathibodi Hospital in Bangkok Thailand. Chiang Mai Journal of Science, 37(2), pp. 282-292.
(Indexed/Abstracted **in MathSciNet, ISI, Scopus, Zentralblatt MATH)**
Indexed in Scopus : h indexed = 3 / SNIP : 0.584 / SJR : 0.032 / Category : Materials Science (miscellaneous) Q3

8.6.3 Tangprasittipap A, Tiensuwan M, Withyachumnarkul B. (2010). Characterization of candidate genes involved in growth of black tiger shrimp Penaeus monodon, Aquaculture, 307(1-2), pp.150-156.
(**Indexed/Abstracted** in ISI-WOS, Scopus)
Indexed in WOS : Impact Factor = 1.925 Category : Fisheries (with median IF = 1.227)
Indexed in Scopus : h indexed = 81 / SNIP : 1.482 / SJR : 0.096 / Category : Aquatic Science Q1

8.6.4 Chantaren P, Ruamviboonsuk P, Ponglikitmongkol M, Tiensuwan M, and Promso S. (2012). Major single nucleotide polymorphisms in polypoidal choroidal vasculopathy: a comparative analysis between Thai and other Asian populations, Clinical Ophthalmology, Vol. 6, pp.465–471.
(**Indexed/Abstracted** in Scopus)
Indexed in Scopus : h indexed = 9 / SNIP : 0.592 / SJR : 0.361 / Category : Ophthalmology Q2

8.6.5 Sawasdee K, Tiensuwan M, Siripinyanond A, Chirachariyavejd C, and Meejoo S.S. (2012). Elemental analysis of burnt human bone for classifying sex and age at death by logistic regression, Analytical Methods, Jun;4(6):1769-75.
(**Indexed/Abstracted in** Scopus**)**
Indexed in WOS : Impact Factor = 1.547 / Category : Food Science & Technology (with median IF = 1.165) - Q2
Indexed in Scopus : h indexed = 10 / SNIP : 0.667 / SJR : 0.486 / Category : Engineering (miscellaneous) Q1

8.6.6 Lertprapai, S. and Tiensuwan, M. On a Comparison of the Variance Estimates of Exponential Distribution by Multiple Criteria Decision Making Method, *Model Assisted Statistics and Application*, 7(2012): 251-260.
(**Indexed/Abstract** in MathSciNet, Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 4 / SNIP : 0.226 / SJR : 0.195 / Category : Applied Mathematics Q4

8.6.7 Sarikavanij, S., Kasala, S., Sinha, B. And Tiensuwan, M. Estimation of Location and Scale Parameters in Two-Parameter Exponential Distribution based on Ranked Set Sample, *Communications in Statistics: Simulation and Computation* (In press).
(Indexed/Abstracted in ISI-WOS, SCOPUS)
(Impact factor = 1.831 (from JCR: Science Edition 2008 – ISI Database) **Subject**
Indexed in Scopus : h indexed = 21 / SNIP : 0.525 / SJR : 0.421 / Category : Modeling and Simulation Q3

8.6.8 Pongsumpun, P. and Tiensuwan, M. Applications of log-linear models to dengue virus infection patients in Thailand, *Model Assisted Statistics and Application* (In Press).
(**Indexed/Abstract** in MathSciNet, Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 4 / SNIP : 0.226 / SJR : 0.195 / Category : Applied Mathematics Q4, Modeling and Simulation Q4, Statistics and Probability Q4

8.6.9 Tiensuwan, M. and O'Brien, T. Modeling dengue virus infection patients for each severity of dengue disease in Thailand, *Far East Journal of Mathematical Science*, Special Volume Part I (Mathematical aspects in Computer Sciences, Information Sciences, Financial Management and Biological Sciences), 2013, pp.1-20.
(**Indexed/Abstract** in MathSciNet, Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 5 / SNIP : 0.159 / SJR : 0.227 / Category : Mathematics (miscellaneous) Q4

8.6.10 Lertprapai, S. and Tiensuwan, M. (2013). On a Comparison of the Variance Estimates of Two-Parameter Exponential Distribution by Multiple Criteria Decision Making Method, *Applied Mathematical Sciences*, Vol. 7(36), pp.1785 – 1795.
(**Indexed/Abstract** in MathSciNet, Scopus, Zentralblatt MATH)
Indexed in Scopus : h indexed = 6 / SNIP : 0.468 / SJR : 0.245 / Category : Applied Mathematics Q4

8.7 Graduated Students

8.7.1 Ph.D. Students

(1) Mr.Wuttichai Srisodaphol

(2) Mrs. Nuengruithai Sontimool

(3) Mr. Ananta Raj Dhungana

8.7.2 M.Sc. Students

(1) Mrs. Pattaraporn Tusto

(2) Mrs. Ratchaneekorn Thoppradid

9. Appendix

9.1 Papers appeared in international journal

Signature…………………………………….

(Assoc.Prof. MONTIP TIENSUWAN, Ph.D.)

Principal Investigator

Date....10........June.....2013...............