CHAPTER 3 METHODOLOGY

3.1 Consider Variable Parameters of EWMA Control Chart

For each variable parameter, there are 2 parameters as follow:

n_1, n_2 are the size of samples of control chart plans 1 and 2.

h_1, h_2 are the interval of sampling of control chart plans 1 and 2.

k_1, k_2 are the width of control region of control chart plans 1 and 2.

w_1, w_2 are the width of warning region of control chart plan 1 and 2.

Covering regions under the condition of $n_1 < n_2$, $h_1 > h_2$, $k_1 > k_2$ and $w_1 > w_2$.

The central area is (LWL, UWL). The warning regions are (UWL, UCL) and (LCL, LWL). When production starts, the process will be in the control region. Given the number of sampling as n_1, the interval of sampling as h_1, the width of warning region as w_1, and the width of control region as k_1. When the samples are in the warning region, the number of sampling must increase $n_1 < n_2$, with the higher frequent interval $h_1 > h_2$, the increase warning region width $w_1 > w_2$, and the narrower control region $k_1 > k_2$. In order to inspect defect more accuracy and reduce the cost in inspection, variable parameters (n, h, w, k, r) are used. In the case of the samples in the central line, the use of parameters will be the same as previous. But if the sample is out of the action region, defect inspection must perform and problem must be solved immediately.

If the size of samples is small and the sample is in the action region, the probability will be p_0. If the size of samples is large, the probability will be $1 - p_0$.
Thus, in the chart 1, the probability will be:

$$p_0 = P\left(|Z| < w_1 \,\middle|\, |Z| < k_1\right) = \frac{\Phi(w_1) - \Phi(-w_1)}{\Phi(k_1) - \Phi(-k_1)} \tag{3.1}$$

And in the chart 2, the probability will be:

$$p_0 = P\left(|Z| < w_2 \,\middle|\, |Z| < k_2\right) = \frac{\Phi(w_2) - \Phi(-w_2)}{\Phi(k_2) - \Phi(-k_2)} \tag{3.2}$$

under the condition of $Z \sim N(0,1)$ (assuming that the data is in normal distribution).

3.1.1 Consider Cost Model of EWMA Control Chart

Find Expected Cost per Cycle

We develop Linderman (2000) and general cost function of Lorenzen and Vance (1986), then the expected cycle time and cycle cost for each of the six scenarios follows. In this model, the expected time in sampling is assigned as h_i, where each sample is independent. During the process is in control, the expected time of sampling will be exponentially distribution, with the mean at $\frac{1}{\lambda}$. When production starts, the process will be in statistical control, with the sample average at μ_0 and standard deviation at $\frac{\sigma}{\sqrt{n}}\sqrt{\frac{r}{2-r}}$. When defect occurs during production, the alarm will start warning. This results in the shift of sample mean from μ_0 to $\mu_0+\delta\sigma_{\bar{X}}$ or $\mu_0-\delta\sigma_{\bar{X}}$. At this stage, the production process would not inspect for defect by itself. Thus to control the production process, the symbol γ_1 will be used. If $\gamma_1=1$, the production will continuingly process during inspection for defect. But if $\gamma_1=0$, the production process will cease during defect inspection. And during repairing and fixing the defect, the symbol γ_2 will be used for continuing process or stop. If $\gamma_2=1$, the process will continue running. And if $\gamma_2=0$, the process will stop during repairing or defect fixing. Assuming that we know the value of μ,σ,δ and want to estimate the value of n_i,h_i,w_i,k_i,r_i ; $i=1,2$.

Magalhães and Epprecht (2001) established the economic cost model based on the cost model of Lorenzen and Vance (1986). If considering all cost per time unit, the function will be below:

$$ECTU=\frac{\text{Expected cost percycle}}{\text{Expected time percycle}}=\frac{E(C)}{E(T)}. \tag{3.3}$$

When $E(C)$ is the mean of all cost occurred per production cycle.

When $E(T)$ is the mean of the total expected time used per production cycle.

The cost in production can be classified into 5 groups, which are:

1. The cost per production hour that does not follow the condition when the process is in control (C_1).
2. The cost per production hour that does not follow the condition when the process is out of control (C_2).
3. The cost from random sampling for quality inspection (C_3).
4. The cost from random sampling for inspecting cause of alarm (C_4).
5. The cost from defect inspection where cause of defect can be assignable and fixed (C_5).

The expected time of production process in here can be classified into 5 groups, which are:

1. The expected time that production process is in control (T_1); which the process starts until the defect, which can be assignable the cause occurs (in control period).

2. The expected time that production process is out of control (T_2); which the mean of samples shifts from μ_0 to $\mu_0 + \delta\sigma_{\bar{X}}$ until the warning happens (out of control period).

3. The expected time to analyze sample and chart (T_3).

4. The expected time to inspect cause (T_4); which cause of defect, which can be assignable, is being inspected while the process is out of control.

5. The expected time to repair (T_5).

The means of each period compose of:

1. The mean of the expected time while production process is in the action region, because the expected time that the process is in control is in exponential distribution so the mean will be $\frac{1}{\lambda}$.

When
$$E(T_1) = \frac{1}{\lambda} + (1-\gamma_1)s\frac{t_0}{ARL_0} \tag{3.4}$$

$\gamma_1 = 0$, when the process is continuing during the warning

$\gamma_1 = 1$, when the process cease during the warning

ARL_0 is average run length while the process is in the action region

t_0 is the mean of the expected time to assignable cause of alarm

s is the mean of sample size when the sampling is performed during the process is in control. That is:

$$\begin{aligned}
s = \sum_{j=0}^{\infty} jP(N=j) &= \sum_{j=0}^{\infty} jP(t_j < T < t_{j+1}) \\
&= \sum_{j=0}^{\infty} j \int_{jh}^{(j+1)h} \lambda e^{-\lambda t} dt \\
&= \sum_{j=0}^{\infty} j\left(e^{-j\lambda h} - e^{-(j+1)\lambda h}\right) \\
&= e^{-\lambda h} + e^{-2\lambda h} + e^{-3\lambda h} + \ldots \\
&= \frac{e^{-\lambda h}}{1-e^{-\lambda h}}
\end{aligned}$$

If the mean of samples is in the action region but in the central, we will get:

$$P(h=h_1)=p_0$$

And if the mean of sample is in the action region but in the warning region, we will get:

$$P(h=h_2)=1-p_0$$

Thus $$E(N)=s=\frac{e^{-\lambda h_1}p_0+e^{-\lambda h_2}(1-p_0)}{1-e^{-\lambda h_1}p_0-e^{-\lambda h_2}(1-p_0)} \qquad (3.5)$$

t_0 is the mean of the expected time to assignable cause of alarm
p_0 is the probability of small-size samples while the process is in control
$1-p_0$ is the probability of large-size samples while the process is out of control
$\Phi(\cdot)$ is the cumulative normal distribution.

2. The mean of the expected time that the process is out of the action region $E(T_2)$; if assigned t as the mean of expected time that starts since the cause of alarm occurs until the warning, with location between samples j and $j+1$

$$t=\frac{\int_{jh}^{(j+1)h}\lambda(t-jh_1)e^{-\lambda h}dt}{\int_{jh}^{(j+1)h}\lambda e^{-\lambda h}dt}=\frac{1-e^{-\lambda h}(1+\lambda h)}{\lambda(1-e^{-\lambda h})}. \qquad (3.6)$$

The mean of the expected time since the cause of alarm occurs to the first sample to the new shift is produced $E(R)$ will be:

$$E(R)=E(E(R|A)) \quad \text{and} \quad E(E(R|A))=E(E((h_i-t_i)|A)),$$

$$E(R)=\sum_{i=1}^{2}(h_i-t_i)\cdot P(A=h_i),$$

$$E(R)=\left\{h_1-\frac{1-e^{-\lambda h_1}(1+\lambda h_1)}{\lambda(1-e^{-\lambda h_1})}\right\}P(A=h_1)+\left\{h_2-\frac{1-e^{-\lambda h_2}(1+\lambda h_2)}{\lambda(1-e^{-\lambda h_2})}\right\}P(A=h_2). \qquad (3.7)$$

As Reynolds (1988) hypothesized that $P(A=h_i)$ is proportionate to the length of the expected time A, thus, the chance of incidence will be:

$$P(A=h_1)=\frac{p_0 h_1}{p_0 h_1+(1-p_0)h_2}, \tag{3.8}$$

$$P(A=h_2)=\frac{(1-p_0)h_2}{p_0 h_1+(1-p_0)h_2}. \tag{3.9}$$

$E(S)$ is the expected time between the first sample after the shift and the next warning, which $E(S)$ value depends on the position of B

$$E(S)=E(S|B=B_1)P(B=B_1)+E(S|B=B_2)P(B=B_2),$$

$$E(S)=E(T_1)P(B=B_1)+E(T_2)P(B=B_2). \tag{3.10}$$

When
$$E(S|B=B_1)=E(T_1), \tag{3.11}$$

$$E(S|B=B_2)=E(T_2). \tag{3.12}$$

T_1 is the expected time since the first sample falls into the central region after the shift
T_2 is the expected time since the first sample falls into the warning region after the shift.

When B is the position of the first sample falling after the shift; if it falls in the central region, the symbol B_1 will be used; if it falls in the warning region, the symbol B_2 will be used; and if it falls in the action region (out of control), the symbol B_3 will be used. B value will depend on the length of the expected time h_i when shift occurs. Thus the probability of B_i will be:

$$\begin{aligned}P(B=B_1)&=P(B=B_1|A=h_1)P(A=h_1)+P(B=B_1|A=h_2)P(A=h_2)\\&=p_{11}P(A=h_1)+p_{21}P(A=h_2),\end{aligned} \tag{3.13}$$

$$\begin{aligned}P(B=B_2)&=P(B=B_2|A=h_1)P(A=h_1)+P(B=B_2|A=h_2)P(A=h_2)\\&=p_{12}P(A=h_1)+p_{22}P(A=h_2),\end{aligned} \tag{3.14}$$

and
$$P(B=B_3)=1-P(B=B_1)-P(B=B_2), \tag{3.15}$$

when
$$p_{11}=P(B=B_1|A=h_1)=P\left(LWL_1<U<UWL_1 \middle| U\sim N\left(\delta\sqrt{n_1},1\right)\right),$$

$$p_{21}=P(B=B_1|A=h_2)=P\left(LWL_2<U<UWL_2 \middle| U\sim N\left(\delta\sqrt{n_2},1\right)\right),$$

$$p_{12} = P\left(B = B_2 \middle| A = h_1\right)$$
$$= P\left(\left(LCL_1 < U < LWL_1\right) \cup \left(UWL_1 < U < UCL_1\right) \middle| U \sim N\left(\delta\sqrt{n_1}, 1\right)\right),$$

$$p_{22} = P\left(B = B_2 \middle| A = h_2\right)$$
$$= P\left(\left(LCL_2 < U < LWL_2\right) \cup \left(UWL_2 < U < UCL_2\right) \middle| U \sim N\left(\delta\sqrt{n_2}, 1\right)\right),$$

$$p_{i1} = P\left(\mu_0 - w_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}} < \bar{X}_i < \mu_0 + w_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}\right)$$

$$p_{i1} = P\left(\frac{\mu_0 - w_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}} - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}} < \frac{\bar{X}_i - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}} < \frac{\bar{X} + w_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}} - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}}\right)$$

$$= P\left(-w_i - \delta\sqrt{n_i}\sqrt{\frac{2-r}{r}} < \frac{\bar{X}_i - \mu'}{\frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}} < w_i - \delta\sqrt{n_i}\sqrt{\frac{r}{2-r}}\right)$$

$$= P\left(-w_i - \delta\sqrt{n_i}\sqrt{\frac{2-r}{r}} < Z < w_i - \delta\sqrt{n_i}\sqrt{\frac{2-r}{r}}\right)$$

$$p_{i1} = \Phi\left(w_i - \sqrt{\frac{2-r}{r}}\,\delta\sqrt{n_i}\right) - \Phi\left(-w_i - \sqrt{\frac{2-r}{r}}\,\delta\sqrt{n_i}\right), \tag{3.16}$$

$$p_{i2} = P\left(\mu_0 - k_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}} < \bar{X}_i < \mu_0 - w_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}\right)$$
$$+ P\left(\mu_0 + w_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}} < \bar{X}_i < \mu_0 + k_i \frac{\sigma}{\sqrt{n_i}}\sqrt{\frac{r}{2-r}}\right)$$

$$= P\left(-k_i - \delta\sqrt{n_i}\sqrt{\frac{2-r}{r}} < Z < -w_i - \delta\sqrt{n_i}\sqrt{\frac{r}{2-r}}\right)$$
$$+ P\left(w_i - \delta\sqrt{n_i}\sqrt{\frac{2-r}{r}} < Z < k_i - \delta\sqrt{n_i}\sqrt{\frac{2-r}{r}}\right)$$

$$p_{i2} = \Phi\left(-w_i - \sqrt{\frac{2-r}{r}}\,\delta\sqrt{n_i}\right) - \Phi\left(-k_i - \sqrt{\frac{2-r}{r}}\,\delta\sqrt{n_i}\right)$$
$$+ \Phi\left(k_i - \sqrt{\frac{2-r}{r}}\,\delta\sqrt{n_i}\right) - \Phi\left(w_i - \sqrt{\frac{2-r}{r}}\,\delta\sqrt{n_i}\right), \tag{3.17}$$

$$E(T_1) = E(M_1)E(V). \tag{3.18}$$

M_1 is a variable parameter of sample number that falls in the central region until warning, where M_1 has geometric distribution with the parameter $(1-p_1)$, when p_1 is the probability of sample falling in the central region, so:

$$p_1 = p_{11} + p_{12}\sum_{i=1}^{\infty} p_{22}^{i-1} p_{21} \tag{3.19}$$

and

$$\sum_{i=1}^{\infty} p_{22}^{i-1} p_{21} = \frac{p_{21}}{1-p_{22}} \tag{3.20}$$

when substitute $p_{ij}'s$ in a double.

$$E(M_1) = \frac{1}{1-p_1},$$

when substitute p_1 in a double, so we will get:

$$E(M_1) = \frac{1-p_{22}}{1-p_{22}-p_{11}+p_{11}p_{22}-p_{12}p_{21}}, \tag{3.21}$$

V is the length of the expected time during sample falling out of warning region, starting from the last sample in the central region, which the probability will be:

$$P(V = h_1) = p_{11} + p_{13} = 1 - p_{12}$$

$$\begin{aligned} P(V = h_1 + ih_2) &= p_{12}p_{22}^{i-1}p_{21} + p_{12}p_{22}^{i-1}p_{23} \\ &= p_{12}p_{22}^{i-1}(p_{21}+p_{23}) \\ &= p_{12}p_{22}^{i-1}(1-p_{22}) \quad ;i = 1,2,\ldots \end{aligned} \tag{3.22}$$

$$E(V) = h_1 + h_2\frac{p_{12}}{1-p_{22}} \tag{3.23}$$

If substitute $E(M_1), E(V)$ in $E(T_1)$, we will get:

$$E(T_1) = \frac{\left[h_1(1-p_{22}) + h_2 p_{12}\right]}{1-p_{11}-p_{22}+p_{11}p_{22}-p_{12}p_{21}} \tag{3.24}$$

In the same way, it will be:

$$E(T_2) = \frac{\left[h_2(1-p_{11}) + h_1 p_{21}\right]}{1-p_{11}-p_{22}+p_{11}p_{22}-p_{12}p_{21}} \tag{3.25}$$

$$E(T_2) = AATS = E(T_{out}) = E(R) + E(S) \tag{3.26}$$

3. The mean of the expected time to analyze sample and chart $E(T_3)$

when n is the mean of sample during the process is in control

n' is the mean of sample during the process is out of control

n_1 is the small size sample number which is used in analysis

n_2 is the large size sample number which is used in analysis

p_0 is the probability that the sample size will be small

$(1-p_0)$ is the probability that the sample size will be large

G is the mean of the expected time that uses in sample and chart analysis

$$p_0(\delta)=P\left(|Z|<w_i+\sqrt{\frac{2-r}{r}}\delta\sqrt{n_i}\;\middle|\;|Z|<k_i+\sqrt{\frac{2-r}{r}}\delta\sqrt{n_i}\right) \tag{3.27}$$

when $i=1.2$; so

$$p_0(\delta)=\frac{\Phi\left(w_i-\sqrt{\frac{2-r}{r}}\delta\sqrt{n_i}\right)-\Phi\left(-w_i-\sqrt{\frac{2-r}{r}}\delta\sqrt{n_i}\right)}{\Phi\left(k_i-\sqrt{\frac{2-r}{r}}\delta\sqrt{n_i}\right)-\Phi\left(-k_i-\sqrt{\frac{2-r}{r}}\delta\sqrt{n_i}\right)} \tag{3.28}$$

when $i=1,2$

where

$$n=n_1p_0+n_2(1-p_0) \tag{3.29}$$

$$n'=n_1p_0(\delta)+n_2(1-p_0(\delta)) \tag{3.30}$$

$$E(T_3)=n'G \tag{3.31}$$

4. $E(T_4)$ is the expected time to assignable cause while the process is out of the action region;

let t_4 is the expected time to assignable cause, so:

$$E(T_4)=t_4 \tag{3.32}$$

5. $E(T_5)$ is the expected time to repair.

let t_5 is the expected time to repair; so:

$$E(T_5)=t_5 \tag{3.33}$$

Thus the mean of all expected time used per production cycle is:

$$E(T)=E(T_1)+E(T_2)+E(T_3)+E(T_4)+E(T_5) \tag{3.34}$$

$$E(T)=\frac{1}{\lambda}+(1-\gamma_1)s\frac{t_0}{ARL_0}+AATS+n'G+t_4+t_5 \qquad (3.35)$$

The Expected Cost per Cycle *E(C)* composes of:

1. The cost mean per production cycle that does not follow the condition while the process is in control $E(C_{in})$ and out of control $E(C_{out})$ thus:

$$E(C_{in})+E(C_{out})=\frac{1}{\lambda}c_1+c_2\left[AATS+E(T_3)+\gamma_1 t_4+\gamma_2 t_5\right] \qquad (3.36)$$

When c_1 is cost per production hour that does not follow the condition during the process is in control.

c_2 is cost per production hour that does not follow the condition during the process is out of control.

$\frac{1}{\lambda}$ is the mean of the expected time that the process is in control.

$AATS$ is the mean time used in improving the cause of alarm.

$E(T_3)$ is the mean of the expected time to analyze sample and chart.

t_4 is the expected time to assign the cause.

t_5 is the mean of the expected time to repair.

2. The mean of the cost to inspect the warning $E(C_3)$

$$E(C_3)=c_3E(F) \qquad (3.37)$$

When c_3 is the cost to inspect the warning.

$E(F)$ is the mean number of warning which warning is independent to each other.

Thus

$$E(F)=\left[\alpha_1 p_0+\alpha_2(1-p_0)\right]s \qquad (3.38)$$

When $\alpha_i=P(|Z|>k_i)=2\Phi(-k_i)$ is the probability of the error type I in each chart.

p_0 is the probability of small size sample during the process is in control.

$1-p_0$ is the probability of large size sample during the process is in control.

s is the mean of the sample number where sampling is performed during the process is in control.

3. The mean of the cost to assign and repair the cause of warning by using the symbol $E(C_4)$ where:

$$E(C_4)=c_4 \qquad (3.39)$$

4. The mean of the cost to sampling and inspect by using the symbol $E(C_5)$ where:

$$E(C_5) = (a + bn)s + (a + bn')s' \tag{3.40}$$

when

$$s' = \frac{AATS + n'G + \gamma_1 t_4 + \gamma_2 t_5}{h'}, \tag{3.41}$$

$$h' = h_1 p_0(\delta) + h_2(1 - p_0(\delta)). \tag{3.42}$$

- a is the constant cost in sampling.
- b is the variable cost per one sample.
- n is the mean of the sample size which is in the action region.
- n' is the mean of the sample size which is out of control.
- s is the mean of sample number which is sampling during the process is in control.
- s' is the mean of the sample number which is sampling during the process is out of control. Thus:

$$E(C) = \frac{1}{\lambda}c_1 + c_2\left[AATS + nG' + \gamma_1 t_4 + \gamma_2 t_5\right] + c_3 E(F) + c_4 + (a + bn)s + (a + bn')s' \tag{3.43}$$

When $E(C)$ is the mean of all cost occurred per production cycle.

$E(T)$ is the mean of all expected time used in production.

$$ECTU = \frac{E(C)}{E(T)} \tag{3.44}$$

$$ECTU = \frac{\frac{1}{\lambda}c_1 + c_2\left[AATS + nG' + \gamma_1 t_4 + \gamma_2 t_5\right] + c_3 E(F) + c_4 + (a + bn)s + (a + bn')s'}{\frac{1}{\lambda} + (1 - \gamma_1)s\frac{t_0}{ARL_0} + AATS + n'G + t_4 + t_5} \tag{3.45}$$

3.2 Consider VP MEWMA Control Chart and VPSC MEWMA Control Chart

3.2.1 VP MEWMA Control Chart

Lowry, et al. (1992) extended the EWMA control chart to the multivariate case. The MEWMA chart monitors p quality characteristics, through a sequence of independent multivariate normal random $p \times 1$ vector: $X_1, \ldots, X_n$ where X_i has mean vector, μ and known covariance matrix, Σ_X, the in-control process mean vector is assumed to be μ_0.

For each of the $j = 1,2,3,\ldots, p$ quality characteristics to be examined, we assign to past observations exponential weighted based on $r_j \left(0 \le r \le 1\right)$ as follows:

Let $R = diag\left(r_1,\ldots,r_p\right)$ denote the diagonal matrix of exponential weights.
Let the multivariate exponentially weighted moving average vectors are:

$$Z_i = RX_i + \left(I - R\right)Z_{i-1} \tag{3.46}$$

where $i = 1,2,3,\ldots$ and $Z_0 = \mu_0$

The VP MEWMA control chart signals that the process is out of control whenever:

$$T_i^2 = Z_i' \sum\nolimits_{Z_i}^{-1} Z_i > UCL, \quad UCL = k \tag{3.47}$$

and VP MEWMA control chart warns when

$$UWL < T_i^2 < UCL, \quad UWL = w,$$

where w and k are chosen to achieve a specified in-control ARL_0 and Σ_{Z_i} is the covariance matrix of Z_i. Often, there is reason to apply different exponential weights to past observations of the p different quality characteristics. In this situation, we assume equal weights across characteristics so that $r_j = r$ for $j = 1,2,3,\ldots, p$.

The MEWMA vectors in Equation (3.46) can thus be written (for $i = 1,2,\ldots$) under the assumption of equal weights across characteristics as:

$$Z_i = rX_i + \left(1 - r\right)Z_{i-1}, \tag{3.48}$$

under the assumption of equal weights, Lowry, et al. (1992) have shown that the covariance matrix of the MEWMA vectors in Equation (3.49) is:

$$\sum\nolimits_{Z_i} = \left\{ \frac{r\left[1-\left(1-r\right)^{2i}\right]}{2-r} \right\} \sum\nolimits_X . \tag{3.49}$$

Note that if $r = 1$, the MEWMA chart is equivalent to Hotelling's T^2 chart.
When the process appears likely to fail quickly, limiting our information about the in-control state, we might proceed with a MEWMA chart based on the covariance matrix above. Instead, we make the assumption that the expected time to failure for our process is fairly long, so that we may use the following asymptotic approximation to the covariance matrix:

$$\sum\nolimits_{Z_i} = \frac{r}{2-r} \sum\nolimits_X . \tag{3.50}$$

A reviewer pointed out that the exact covariance matrix and the asymptotic covariance matrix lead to two different procedures. This work is concerned solely with the MEWMA chart using the asymptotic covariance matrix.

Suppose that we plan to use a MEWMA chart to study the p quality characteristics associated with a process. The process begins in the in-control state with knowing mean vector μ_0 and covariance matrix Σ_{Z_i}.

3.2.2 VPSC MEWMA Control Chart

Assume that there are p characteristics to measured in a given process and denote the measurements by $\underset{\sim}{X} = \left(x_1, x_2, \ldots, x_p\right)'$, $\underset{\sim}{X} \sim N_p\left(\underset{\sim}{\mu}, \sum\right)$ which a multivariate normal with mean vector $\underset{\sim}{\mu}$ and covariance matrix Σ (See Appendix A.5).

If we take a random sample of size n from the process and find Z_i then we define the static a SC MEWMA (Chen, et al. 2004-2005)

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[(Z_i - \mu)' \sum\nolimits_Z^{-1} (Z_i - \mu)\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\underset{\sim}{Z}_i - \underset{\sim}{\mu}\right)' \sum\nolimits_Z^{-1} \left(\underset{\sim}{Z}_i - \underset{\sim}{\mu}\right)\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\underset{\sim}{Z}_i - \underset{\sim}{\mu}\right)' \sum\nolimits_Z^{-1} \left(\underset{\sim}{Z}_i - \underset{\sim}{\mu}\right)\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)' + \left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)'\right\} \sum\nolimits_Z^{-1} \left\{\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right) + \left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)\right\}\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)' + \left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)'\right\} \left\{\sum\nolimits_Z^{-1}\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right) + \sum\nolimits_Z^{-1}\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)\right\}\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)\right\} + \left\{\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)\right\}\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)\right]$$

$$T_p = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)\right] + \frac{1}{n}\cdot n\left[\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)\right]$$

$$T_p = \left[\left(\underset{\sim}{\bar{Z}} - \mu\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{\bar{Z}} - \underset{\sim}{\mu}\right)\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)' \sum\nolimits_Z^{-1}\left(\underset{\sim}{Z}_i - \underset{\sim}{\bar{Z}}\right)\right]$$

$$T_p = \left(T_{\bar{Z}} + T_s\right) \tag{3.51}$$

$T_{\bar{Z}}$ and T_s can be used to measure the shift of the location and variability. It can easily be shown that $T_{\bar{Z}} \sim \chi_p^2$ a Chi-square distribution with "p" degree of freedom and $T_s \sim \chi_{np-p}^2$ a Chi-square distribution with "$np - p$" degree of freedom (See Appendix A.6). So VPSC MEWMA control chart has a Chi-square distribution with "np" degree of freedom. On an ordinary graph paper, if we let $T_{\bar{Z}}$ be the x-axis and T_s be the y-axis, then the relation $T_p = (T_{\bar{z}} + T_s)$ defines a straight line with a slop of -1. Since both $T_{\bar{Z}}$ and T_s are greater than 0, the 'Control Region" of T_p is formed by an isosceles right angle as shown below in Figure 3.1.

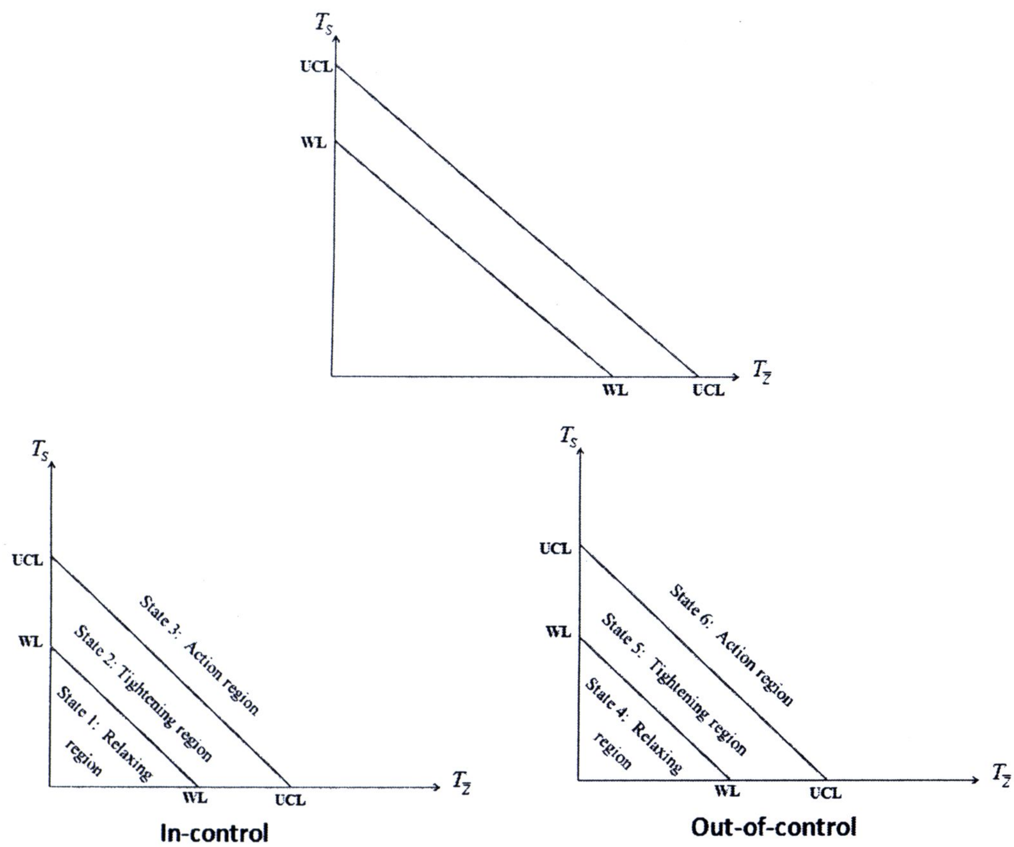

Figure 3.1 Show Control Region, In control state and Out of control state

Hence we can determine the size of the "Control Region" as

$$T_p = (T_{\bar{z}} + T_s) < UCL \tag{3.52}$$

and "Warning Region" as

$$WL \le T_p \le UCL$$

where $UCL = C$ will satisfy

$$P(T_p \le C) = P(nT_p \le nC) = P\left(\chi^2_{np} \le \chi^2_{np,(1-\alpha)}\right) = 1-\alpha \tag{3.53}$$

and $\chi^2_{np,(1-\alpha)}$ is the $100(1-\alpha)^{th}$ percentile of χ^2_{np} distribution, hence

$$UCL = \frac{\chi^2_{np,(1-\alpha)}}{n}$$

3.2.3 The Procedure

Suppose that the process is subject to a single assignable cause which shifts the process mean from μ_0 to μ and let the magnitude of the process shift is reflected in the noncentrality parameter:

$$d = \left(\mu' \sum\nolimits_Z^{-1} \mu\right)^{\frac{1}{2}}. \tag{3.54}$$

We denote in-control state with $d = 0$. The VP MEWMA control chart and VPSC MEWMA control chart are a modification of the traditional MEWMA control chart. Let (n_1, h_1) be a pair of minimum sample size and longest sampling interval, and (n_2, h_2) be a pair of maximum sample size and shortest sampling interval. These pairs are chosen such that $n_1 < n_2$ and $h_2 < h_1$. The decision to switch between pairs (n_1, h_1) and (n_2, h_2) depends on the prior sample point on the control chart. That is, the position of the prior sample points i. On the other hand, if the prior sample point $i-1$ falls in the relaxing region, the pair (n_1, h_1) should be used for the current sample point i. Here the tightening region is given by (w, k) and relaxing region is given by $(0, w)$, where w is called the warning limit.

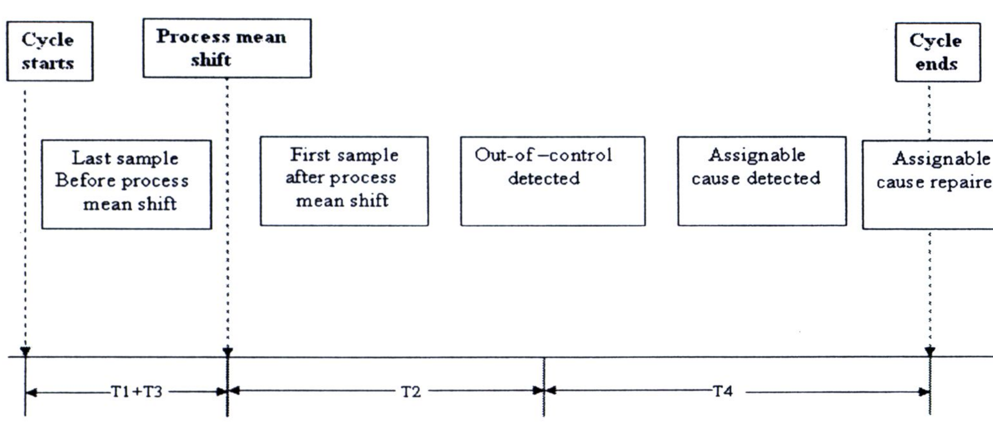

Figure 3.2 Production cycle considered in the cost model

The in control period is denoted by **T1**.
The out-of-control period $(AATS)$ is denoted by **T2**.
The searching period due to false alarm is denoted by **T3**.
The time period for identifying and correcting assignable cause is denoted by **T4**.
The following function defines the switch principle of the VP MEWMA control and VPSC MEWMA control scheme:

The Cost Models for Economic Design

The following cost model is an extension of Costa' model (2001), which was employed in an unvaried case. To simplify the mathematical analysis in cost, a number of assumptions are made.

Model assumptions

1. The $p-$dimensional random vector X, which represents p quality characteristics, is normal distributed with the mean vector μ and known covariance matrix Σ_X.
2. The process assumed to start with an in-control state $(\mu = \mu_0)$ but after a random time of in-control operation it will be disturbed by a single assignable cause that causes a fixed shift in the process mean vector $(\mu = \mu_1)$
3. The process after the shift remains out-of-control until the assignable cause is eliminated (if possible).
4. The inter-arrival time of the assignable cause disturbing the process is assumed following an exponential distribution with a mean $\frac{1}{\lambda}$ hours.
5. For the VP MEWMA control chart, the process is stopped if the T_i^2 falls outside the action limit, and then a search starts to find the assignable cause and adjust the process.
6. For the VPSC MEWMA control chart, the process is stopped if T_p value falls outside the action limit, and then a search starts to find the assignable cause and adjust the process.
7. During each sampling interval, there exists at most one assignable cause which makes the process out of control, the assignable cause will not occur at sampling time.
8. All the process cost (including sampling costs, in-control and out-of-control production cost, warning, false alarm and repair costs) are known.
9. $r_1 = r_2 = ... = r_p = r$ (weight past observation similarly for the p quality characteristics).

The cost function

The economic design of VP MEWMA control chart and VPSC MEWMA control chart are implemented by specifying a cost function, and searching the optimal design parameters for minimizing the hourly loss cost function over a production cycle. The production cycle length is defined as the average time from the start (or restart) of production until the assignable cause identified and eliminated. Once the expected cycle length is determined, the cost over the production cycle can be converted to an index-long run expected hourly loss cost per hour (Ross, 1970).

Figure 3.2 show the production cycle, which is divided into four time intervals of: in-control period, out-of-control period, searching period due to false alarm, and the time period for identifying and correcting the assignable cause. Individuals are now illustrated before they are grouped together. The expected length of in-control period

(T1) is $\frac{1}{\lambda}$. The expected length of out-of-control period (T2) represents the average time needed for the control chart to produce a signal after the process mean shift. This average time is called the Adjusted Average Time to Signal $(AATS)$, which is the most widely used statistical measure for comparing the efficiencies of different adaptive control chart. The memoryless property of the exponential distribution allows the computation of $AATS$ using the Markov chain approach. The fundamental concepts used in the following paragraphs can be found in Cinlar (1975).

Table 3.1 The states of the Markov chain

State	i th sampling	
	Position of T_i^2 or T_p	Process status
1	Relaxing region	In-control
2	Tightening region	In-control
3	Action region	In-control
4	Relaxing region	Out-of-control
5	Tighten region	Out-of-control
6	Action region	Out-of-control

Let M be the average time from the cycle start to the time the chart signals after the process shift. Then

$$AATS = M - \frac{1}{\lambda}. \tag{3.55}$$

At each sampling time during the period M, one of the five transient states is reached according to the status of the process (in or out-of-control) and the position of T_i^2 or T_p (relaxing, tightening, action region) (see Table 3.1):

State 1: the process is in-control and T_i^2 or T_p falls into the relaxing region

State 2: the process is in-control and T_i^2 or T_p falls into the tightening region

State 3: the process is in-control and T_i^2 or T_p falls into the action region

State 4: the process is out-of-control and T_i^2 or T_p falls into the relaxing region

State 5: the process is out-of -control and T_i^2 or T_p falls into the tightening region

When state 3 is reached, the signal the chart produces is false alarm. If T_i^2 or T_p falls in to the action region at some sampling time while the process status is out-of-control, then the signal is a true alarm and the absorbing state, state 6, is reached.

The transition probability matrix is given by:

$$P=\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{54} & p_{55} & p_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.56}$$

where p_{ij} denotes the transition probability that i is the prior state , j is the current state and $\sum_{j=1}^{6} p_{1j}=1,\ \sum_{j=1}^{6} p_{2j}=1,\dots,\ \sum_{j=1}^{6} p_{6j}=1$. In what follows, $\eta_1 = n_1 d^2,\ \eta_2 = n_2 d^2$ and $F(x,p,n)$ will denote cumulative probability distribution function of a non-central chi-square distribution with p degrees of freedom and non-centrality parameter η.

$p_{ij}'s$ of VP MEWMA Control Chart

The $p_{ij}'s$ of VP MEWMA control chart, which are probabilities conditional on the prior states are:

$$p_{11} = \frac{F(w,p,\eta=0)}{F(k,p,\eta=0)} \times e^{-\lambda h_1}$$

$$p_{21} = p_{31} = \frac{F(w,p,\eta=0)}{F(k,p,\eta=0)} \times e^{-\lambda h_2}$$

$$p_{12} = \frac{F(k,p,\eta=0) - F(w,p,\eta=0)}{F(k,p,\eta=0)} \times e^{-\lambda h_1}$$

$$p_{22} = p_{32} = \frac{F(k,p,\eta=0) - F(w,p,\eta=0)}{F(k,p,\eta=0)} \times e^{-\lambda h_2}$$

$$p_{13} = \frac{1 - F(k,p,\eta=0)}{F(k,p,\eta=0)} \times e^{-\lambda h_1}$$

$$p_{23} = p_{33} = \frac{1 - F(k,p,\eta=0)}{F(k,p,\eta=0)} \times e^{-\lambda h_2}$$

$$p_{14} = F(k,p,\eta_1) \times \left(1 - e^{-\lambda h_1}\right)$$

$$p_{24} = p_{34} = F(k,p,\eta_2) \times \left(1 - e^{-\lambda h_2}\right)$$

$$p_{15} = \left[F(k,p,\eta_1) - F(w,p,\eta_1)\right] \times \left(1 - e^{-\lambda h_1}\right)$$

$$p_{25} = p_{35} = \left[F(k,p,\eta_2) - F(w,p,\eta_2)\right] \times \left(1 - e^{-\lambda h_2}\right)$$

$$p_{16} = \left[1 - F\left(k, p, \eta_1\right)\right] \times \left(1 - e^{-\lambda h_1}\right)$$
$$p_{26} = p_{36} = \left[1 - F\left(k, p, \eta_2\right)\right] \times \left(1 - e^{-\lambda h_2}\right)$$
$$p_{44} = p_{54} = F\left(w, p, \eta_1\right)$$
$$p_{45} = p_{55} = \left[F\left(k, p, \eta_1\right) - F\left(w, p, \eta_1\right)\right]$$
$$p_{46} = p_{56} = 1 - F\left(k, p, \eta_1\right)$$

$p_{ij}{}'s$ of VPSC MEWMA Control Chart

The $p_{ij}{}'s$ of VPSC MEWMA control chart, which are probabilities conditional on the prior states are:

$$p_{11} = \frac{F\left(w_1, p, \eta = 0\right)}{F\left(k_1, p, \eta = 0\right)} \times e^{-\lambda h_1}$$
$$p_{21} = p_{31} = \frac{F\left(w_2, p, \eta = 0\right)}{F\left(k_2, p, \eta = 0\right)} \times e^{-\lambda h_2}$$
$$p_{12} = \frac{F\left(k_1, p, \eta = 0\right) - F\left(w_1, p, \eta = 0\right)}{F\left(k_1, p, \eta = 0\right)} \times e^{-\lambda h_1}$$
$$p_{22} = p_{32} = \frac{F\left(k_2, p, \eta = 0\right) - F\left(w_2, p, \eta = 0\right)}{F\left(k_2, p, \eta = 0\right)} \times e^{-\lambda h_2}$$
$$p_{13} = \frac{1 - F\left(k_1, p, \eta = 0\right)}{F\left(k_1, p, \eta = 0\right)} \times e^{-\lambda h_1}$$
$$p_{23} = p_{33} = \frac{1 - F\left(k_2, p, \eta = 0\right)}{F\left(k_2, p, \eta = 0\right)} \times e^{-\lambda h_2}$$
$$p_{14} = F\left(w_1, p, \eta_1\right) \times \left(1 - e^{-\lambda h_1}\right)$$
$$p_{24} = p_{34} = F\left(w_2, p, \eta_2\right) \times \left(1 - e^{-\lambda h_2}\right)$$
$$p_{15} = \left[F\left(k_1, p, \eta_1\right) - F\left(w_1, p, \eta_1\right)\right] \times \left(1 - e^{-\lambda h_1}\right)$$
$$p_{25} = p_{35} = \left[F\left(k_2, p, \eta_2\right) - F\left(w_2, p, \eta_2\right)\right] \times \left(1 - e^{-\lambda h_2}\right)$$
$$p_{16} = \left[1 - F\left(k_1, p, \eta_1\right)\right] \times \left(1 - e^{-\lambda h_1}\right)$$
$$p_{26} = p_{36} = \left[1 - F\left(k_2, p, \eta_2\right)\right] \times \left(1 - e^{-\lambda h_2}\right)$$
$$p_{44} = p_{54} = F\left(w_1, p, \eta_1\right)$$
$$p_{45} = p_{55} = F\left(k_1, p, \eta_1\right) - F\left(w_1, p, \eta_1\right)$$
$$p_{46} = p_{56} = 1 - F\left(k_1, p, \eta_1\right)$$

Once the transition probability matrix is identified, the average number of transitions in each transient state before the true alarm signals would be calculated by $B'(I-Q)^{-1}$ Cinlar (1975).where $B'=(b_1,b_2,b_3,b_4,b_5)$ is a vector of initial probability, with $\sum_{i=1}^{5} b_i = 1$ I is the identity matrix of order 5; Q is the 5×5 matrix obtained from P on deleting the elements corresponding to the absorbing state. Finally, the product of the average number of visiting the transient state and the corresponding sampling interval determines M. That is:

$$M = B'(I-Q)^{-1} t, \tag{3.57}$$

where t is the vector of the sampling intervals corresponding to five transient states used for next sampling. Here we set the vector $B'=(0,1,0,0,0)$ and $t'=(h_1,h_2,h_2,h_1,h_2)$ respectively (Chen, 2009). The third element in t' is placed by h_2 in order to provide an additional protection to prevent problems that arise during start-up.

Let t_0 denote the average amount of time exhausted searching for the assignable cause when the process is in-control, and $E(FA)$ denote the expected number of false alarms per cycle, which is given by:

$$E(FA) = B'(I-Q)^{-1} f, \tag{3.58}$$

where $f'=(0,0,1,0,0)$ (Chen, 2009). Then the expected length of searching period due to false alarm (T3) can be expressed by $t_0 E(FA)$

The time to identify and correct the assignable cause following an action signal (T4) is a constant t_1.

Aggregating the foregoing four time intervals, the expected length of a production cycle would be expressed by:

$$E(T) = M + t_0 E(FA) + t_1 \tag{3.59}$$

Let V_0 is the hourly profit earned when the process is operating in control state;

V_1 is the hourly profit earned when the process is operating in out-of-control state;

C_0 is the average search cost if the given signal is false;

C_1 is the average cost to discover the assignable cause and adjust the process to in-control state;

s is the cost for each inspected item;

then the expected net profit during the a production cycle is given by:

$$E(C) = V_0\left(\frac{1}{\lambda}\right) + V_1\left(M - \frac{1}{\lambda}\right) - C_0 E(FA) - C_1 - sE(N) \tag{3.60}$$

where $E(N)$ is the average numbers of inspected items during production cycle, and it is given by:

$$E(N) = B'(I - Q)^{-1}\eta, \tag{3.61}$$

where $\eta' = (n_1, n_2, n_2, n_1, n_2)$ is the vector of sample sizes corresponding to the five transient taken for next sampling.
Finally, the expected loss per hour $E(L)$ is given by

$$E(L) = V_0 - \frac{E(C)}{E(T)}. \tag{3.62}$$

3.3 The Economic Model of VP MEWMA Control Chart and VPSC MEWMA Control Chart by Genetic Algorithm

The solution procedure is carried out using Genetic Algorithms (GA) with MATLAB 7.6.0 (R2008a) software (Appendix B.1) to obtain the optimal values of n_1, n_2, h_1, h_2 that minimize $(E(L))$ VP MEWMA control chart and n_2, h_1, h_2 that minimize $(E(L))$ VPSC MEWMA control chart.

The GA, based on the concept of natural genetics, is directed toward a random optimization search technique. The GA solves problems using the approach inspired by the process of Darwinian evolution. The current GA in science and engineering refers to the models introduced and investigated by Holland (1992). In the GA, the solution of a problem is called a ''chromosome''. A chromosome is composed of genes (i.e., features or characters). Although there are several kinds of numerical optimization methods, such as neural network, gradient-based search, GA, etc., the GA has advantages in the following aspects:

1. The operation of GA uses the fitness function values and the stochastic way (not deterministic rule) to guide the search direction of finding the optimal solution. Therefore the GA can be applied for many kinds of optimization problems.
2. The GA can lead to a global optimum by mutation and crossover technique to avoid trapping in the local optimum.
3. The GA is able to search for many possible solutions (or chromosomes) at the same time. Hence, it can obtain the global optimal solution efficiently. Based on these points, GA is considered as an appropriate technique for solving the problems of combinatorial optimization and has been successfully applied in many areas to solve optimization problems (e.g., Jensen, 2003; Chou, et al. , 2006) ; Chou and Chen, 2006). The solution procedure for our example using the GA by MATLAB is briefly described as follows:

Step1. Initialization:

One hundred initial solutions that satisfy the constraint condition of each test variable are randomly produced. Meanwhile, the constraint condition for each test variable is set as follows: $0 < r \le 1,\ \ h_2 < h_1\ \ and\ \ h_1, h_2 > 0,\ \ 2 \le n_1 < n_2 \le 100$

Step2. Evaluation:

The fitness of each solution (Appendix B) is evaluated by calculating the value of fitness function. The fitness function for our example is the cost function shown in Equation (3.62).

Step3. Selection:

The survivors (i.e., 30 solutions) are selected for the next generation according to the better fitness of chromosomes. (In the first generation, the chromosome with the highest cost is replaced by the chromosome with the lowest cost.)

Step4. Crossover:

A pairs of survivors (from the 30 solutions) are selected randomly as the parents used for crossover operations to produce new chromosomes (or children) for the next generation. In this example, we apply the arithmetical crossover method with crossover rate 0.8 as follows:

$$D_1 = 0.8R + 0.2M$$
$$D_2 = 0.2R + 0.8M$$

where D_1 is the first new chromosome, D_2 is the second new chromosome, and R and M are the parents chromosomes. If 30 parents are randomly selected, then there are 60 children that will be produced. Thus, the population size increases to 90 (i.e., 30 parents + 60 children) in this step.

Step5. Mutation:

Suppose that the mutation rate is 0.1. In this example, we use non-uniform method to carry out the mutation operation. Since we have 90 solutions, we can randomly select nine chromosomes (i.e., $90 \times 0.1 = 9$) to mutate some parameters (or genes).

Step6.

Repeat Step 2 to Step 5 until the stopping criteria is found. In this example, we use until cannot be changed value as our stopping criteria.

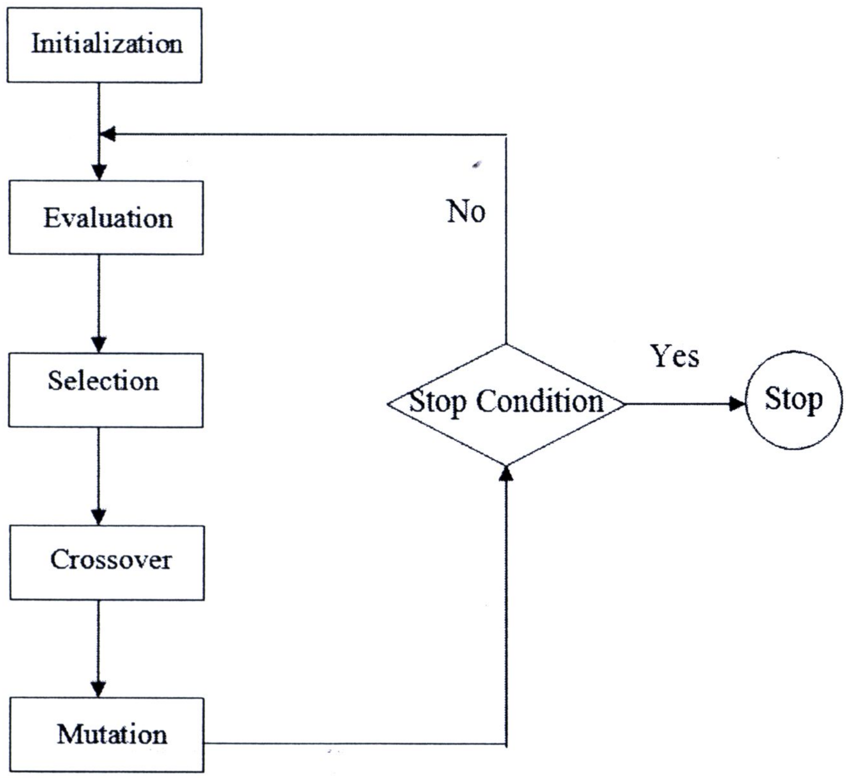

Figure 3.3 The solution procedure using genetic algorithm