สกว

รายงานวิจัยฉบับสมบูรณ์

โครงการ ระเบียบวิธีเชิงสถิติวิธีใหม่สำหรับการวิเคราะห์ข้อมูลและการประยุกต์

New Statistical Methods of data Analysis with Applications

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สัญญาเลขที่ BRG4680010

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว.ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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Contract Number BRG4680010

Final Report

Report Period 1 March 2003 – 28 February 2006

1. Project Title: New Statistical Methods of data Analysis with Applications

ระเบียบวิธีเชิงสถิติวิธีใหม่สำหรับการวิเคราะห์ข้อมูลและการประยุกต์

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4. Background and Rationale

Data analysis is an integral part of statistics, and statistical methods to analyze data arising from various sources abound in the literature. Nevertheless, the need for new methods of data analysis often arises because of emerging new and difficult data collection methods and innovative statistical problems. For example, in the context of environmental science, it is necessary to study the overall pollution level or state of the environment in a given region on the basis of several key factors such as the pollution level of air, land, water, and the like. It is thus necessary to suitably and meaningfully define an aggregate index which captures the essential features of the component indices and yet represents an overall picture of the environment. Data Envelopment Analysis (DEA) and its variations such as Multiple Criterion Decision Making (MCDM) are recent statistical techniques which are used to compute this aggregate index. In another context, Ranked Set Sampling (RSS) has been proved to be a very cost effective means of sampling from a population compared to the traditional simple random sampling. Many aspects of RSS have already been studied in the context of estimating population mean, variance, quantiles in a nonparametric setup as well as many other meaningful parametric functions while sampling from parametric models. As a third application, we mention that in the context of data analysis based on random or mixed effects models with the usual treatment and block effects arising in an additive form, a recent notion of generalized P-value has been very effective in some non-standard testing problems. These latter problems have the peculiarity that standard hypotheses testing methods do work. We have identified a few problems of this type, and we will apply generalized P-value to solve these problems. Quite often, there are multiple solutions to these testing problems, and it would then be natural to compare the competing test procedures in terms of both size and power. Measurement of agreements among different judges or individuals is another important example of a recent new statistical method which can be further developed and applied in some practical cases. Lastly, some statistical aspect of social network is yet another innovative application of statistical methods.

We believe that there are ample opportunities to carry out several statistical data analyses projects along the above lines which would be very helpful in the context of Thailand.

5. Project Objectives

The main objectives of this project are given as follows.

5.1 Data Envelopment Analysis (DEA) and Multiple Criteria Decision Making (MCDM)

5.1.1 Comparing and ranking estimators on the basis of MCDM in the following estimation problems:

(1) Two standard estimates of θ based on binomial distribution: B(n, θ)

(2) Three estimates of θ (1-θ) based on binomial distribution: B(n, θ)

(3) Four estimates of μ, including preliminary test estimators (PTEs) based on normal distribution : $N(\mu, \sigma^2)$

(4) Four estimates of a common mean μ based on two normal distributions $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$

5.1.2 Applying data integration methods to air pollution data from Bangkok, Thailand

5.2 Ranked Set Sampling (RSS)

5.2.1 Proposing quadratic nonnegative unbiased estimates of the population variance based on balanced RSS for some well known distributions such as uniform, exponential, and normal distributions

5.2.2 Applying RSS in a few situations and study the estimates of the population mean and variance.

5.2.3 Applying the RSS estimates of the population mean based on exponential distribution from Sinha et al.(1996) for proposing the RSS estimates of the reliability

5.2.4 Modifying the best linear unbiased estimate (BLUE) of the scale parameter using RSS for obtaining nonnegative unbiased estimates of the scale parameter based on Normal, Logistic, Double exponential, Two-parameter exponential, and Weibull distributions.

5.2.5 Applying the above nonnegative unbiased estimates of the scale parameter for estimating the population quantiles based on the same distributions

5.3 Analysis of Mixed/Random Effects Models

5.3.1 Estimating parameters of the first model (model I), i.e. a bivariate normal distribution with common mean by MLE and MM.

5.3.2 Estimating parameters of the second model (model II), i.e. a bivariate normal distribution with common variance by MLE and MM.

5.3.3 Studying the properties of the estimates, i.e. mean, variance.

5.3.4 Comparing the variance of the estimates between two methods for model I.

5.3.5 Applying model I to small sample data set.

5.3.6 Computing the power of the test of model I by simulation

5.4 Statistical Methods in Assessing Agreement

5.4.1 Part I Discrete variable

(1) Approximating Cohen's kappa statistic and compute its asymptotic variance.

(2) Proposing the modified Cohen's kappa estimates and computing their asymptotic variances.

(3) Determining the sample size for the desired power of the test based on modified Cohen's kappa statistics.

(4) Computing the power of the test for a given sample size based on the modified Cohen's kappa statistic.

(5) Proposing the standardized Cohen's kappa and find their properties, i.e. mean, variance

5.4.2 Part II Continuous variable

(1) Deriving the four test statistics for the hypothesis

$$H_0: \mu_x = \mu_y, \sigma_x = \sigma_y, \rho = \rho_0 \quad (*)$$

based on the bivariate normal population for the small sample size and the large sample size.

(2) Applying the four test procedures for testing hypothesis (*) to real data set on Diaspirin crosslinked hemoglobin (DCLHb) of 299 patients, measured by HemoCue method and its modification.

(3) Simulating the power of the tests for testing hypothesis (*).

5.5. Statistical Aspects of Social Network

5.5.1 Studying the social network, key parameters, descriptive features, and the basic concepts of dyadic models.

5.5.2 Applying the dyadic models to the Baghra Village Social Network data and describing the basic features of the data.

5.5.3 Analyzing the Baghra Village Social Network data by using dyadic models for V-Arrays, W-Arrays, and Y-Arrays and estimating maximum likelihood estimators for the parameters and associated standard errors.

5.5.4 Proposing the models of the dyadic relational network when the two classifications are crossed with each other and applying this to the Baghra Village Social Network data.

6. Research Activities

The activities in this project have been progressing very well, keeping up with the proposed plan. These activities can be shown as follows.

1) Data Envelopment Analysis (DEA) and Multiple Criterion Decision Making (MCDM)

Multiple Criteria Decision Making (MCDM) has recently been recognized as an efficient statistical method to combine component 'indices' arising from many 'sources' into a single overall meaningful index (Filar, 1999; Maitra, et.al., 2002) Such an index can be effectively used to compare relevant 'facilities'. The basic premise is a data matrix $X = (x_{ij})\ :\ K \times N$ where the rows represent facilities which need to be compared or ranked with respect to the element x_{ij}'s, the columns represent various sources of the elements x_{ij}'s and the x_{ij}'s themselves represent some quantitative information about the facilities. In the context of environmental science, the x_{ij}'s may represent levels of pollutants, facilities represent the sources of the pollutants (e.g., chemical or nuclear facilities) and the columns represent different types of pollution. In the context of an estimation problem, the x_{ij}'s may represent mean squared errors of different estimators, designated by rows, for different values of the unknown parameter, designated by columns. Since usually it is difficult to compare the facilities on a multiple scale, MCDM provides a statistical method to combine the elements in any row into a single value which can then be used to compare the rows on a linear scale.

The MCDM is a procedure to integrate multiple indicators into a single meaningful and overall index by combining $(x_{i1},\ldots,x_{iN})$ for row i across all indicators $j=1,2,\ldots,N$. We can define an Ideal Row as one with the smallest observed value for each column

$$IDR = (\min_i x_{i1}, \ldots, \min_i x_{iN}) = (u_1, \ldots, u_N) \qquad (1.1)$$

and a Negative-ideal Row (NIDR) as one with the largest observed value for each column

$$NIDR = (\max_i x_{i1}, \ldots, \max_i x_{iN}) = (v_1, \ldots, v_N). \qquad (1.2)$$

For any given row i, we now compute the distance of each row from Ideal row and from Negative Ideal row based on the L_2-norm by using the formulae :

$$L_2(i, IDR) = [\sum_{j=1}^{N} \frac{(x_{ij}-u_j)^2 w_j}{\sum_{i=1}^{K} x_{ij}^2}]^{1/2} \qquad (1.3)$$

$$L_2(i, NIDR) = [\sum_{j=1}^{N} \frac{(x_{ij}-v_j)^2 w_j}{\sum_{i=1}^{K} x_{ij}^2}]^{1/2} \qquad (1.4)$$

where $w_1, w_2, \ldots, w_N$ are suitably chosen nonnegative weights between 0 and 1. An objective way to select the weights (Maitra, et.al.,2002) is to use Shannon 's entropy (Shannon and Weaver, 1947) measure ϕ based on the proportion $p_{1j}, \ldots, p_{Kj}$ for the j th column where

$$p_{ij} = x_{ij} / \sum_{i=1}^{K} x_{ij} . \qquad (1.5)$$

For the j th column, ϕ_j is computed as

$$\phi_j = -\sum_{i=1}^{K} p_{ij} \log(p_{ij}) / [\log(K)]. \qquad (1.6)$$

Obviously, it is assumed here that x_{ij}'s are positive.

The quantity ϕ essentially provides a measure of closeness of the different proportions. The smaller the value of ϕ, the larger the variation among the proportions for classifying the rows. So we can select the weights as

$$w_j = (1-\phi_j)/[\sum_{j=1}^{N}(1-\phi_j)], \quad j=1,\ldots,N. \qquad (1.7)$$

In addition to Shannon's entropy measure, we can also use the sample variance (Maitra, et.al., 2002) of these proportions, given by

$$s^2_{j/prop} = \sum_{i=1}^{K} (p_{ij} - \bar{p}_j)^2 / (K-1). \quad (1.8)$$

If $\bar{x}_j$ and s_j^2 denote the mean and variance of x_{ij} in the jth column, $s^2_{j/prop}$ is directly proportional to $s_j^2/\bar{x}_j^2$, which is the square of the sample coefficient of variation cv_j for the jth column. Therefore we propose to use $w_j = cv_j$ for all j.

The various rows are now ranked based on an overall index I computed as

$$I_i = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, \quad i = 1, \dots, K. \quad (1.9)$$

In addition to L_2-norm we can also use the L_1-norm as a distance measure and rank the rows once again. The L_1-norm distance is defined as

$$L_1(i, IDR) = \sum_{j=1}^{N} \frac{|x_{ij} - u_j| w_j}{\sum_{i=1}^{K} x_{ij}} \quad (1.10)$$

$$L_1(i, NIDR) = \sum_{j=1}^{N} \frac{|x_{ij} - v_j| w_j}{\sum_{i=1}^{K} x_{ij}} \quad (1.11)$$

where w_j's are appropriate weights.

A '*continuous*' version of this setup would involve x_{ij}'s where the index j would vary '*continuously*'. In the context of the problem of comparing several estimators of a parameter, the x_{ij}'s are chosen to represent the mean squared errors of estimators which are functions of the parameter, denoted by θ. So the L_1-norm and L_2-norm would be redefined as

$$L_1(i, IDR) = \int_{\underline{\theta}}^{\bar{\theta}} [x_i(\theta) - u(\theta)] w(\theta) d\theta \quad (1.12)$$

$$L_1(i, NIDR) = \int_{\underline{\theta}}^{\bar{\theta}} [v(\theta) - x_i(\theta)] w(\theta) d\theta \quad (1.13)$$

$$L_2(i, IDR) = \sqrt{\int_{\underline{\theta}}^{\bar{\theta}} (x_i(\theta) - u(\theta))^2 w(\theta) d\theta} \quad (1.14)$$

$$L_2(i, NIDR) = \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} (x_i(\theta) - v(\theta))^2 w(\theta) d\theta} \qquad (1.15)$$

where $u(\theta) = \min_i \{x_i(\theta)\}$, $v(\theta) = \max_i \{x_i(\theta)\}$, and $\underline{\theta} \le \theta \le \overline{\theta}$.

We study and apply the theory of Multiple Criteria Decision Making (MCDM) to two kinds of problems. First we apply data integration methods to air pollution data from ten monitoring stations in Bangkok and meteorological data from five parts of Thailand. Second we compare and rank estimators on the basis of MCDM in the following estimation problems: two standard estimates of θ based on binomial distribution: $B(n,\theta)$, three estimates of $\theta(1-\theta)$ based on binomial distribution: $B(n,\theta)$, four estimates of μ, including Preliminary Test Estimators (PTEs), based on normal distribution: $N(\mu,\sigma^2)$, and six estimates of a common mean μ based on two normal distributions $N(\mu,\sigma_1^2)$ and $N(\mu,\sigma_2^2)$. (See papers in Appendices)

2) Ranked Set Sampling

Ranked set sampling is a procedure introduced by McIntyre (1952) which combines random sampling and the ability to rank the sampling units, with respect to the characteristic of interest, without making the actual measurements. The general protocol of an RSS (Patil, 2002) can be described as follows.

Step 1: Select k random samples of size k units from the population.

Step 2: Without yet knowing any values for the variable of interest, rank the units within each set based on a perception of relative values for this variable. This may be based on personal judgment or done with measurements of a covariate that is correlated with the variable of interest.

Step 3: Choose a sample for actual analysis by including the smallest ranked unit in the first set, then the second smallest ranked unit in the second set, continuing in this fashion until the largest ranked unit is selected in the last set.

Step 4: Repeat steps 1 through 3 for n cycles until the desired sample size, $N = kn$, is obtained for analysis.

In this study, we assume that the sample sizes are equal for both simple random sample (SRS) and ranked set sample (RSS); *i.e.* $N = k \times n$.

Let $X_1, X_2, \ldots, X_k$ be a simple random sample of size k on a random variable X with probability density function $f(x)$ and finite mean, μ and finite variance, σ^2. Let $X_{(i:k)} \equiv X_{(ii)}$ denote the *i*th order statistic from the *i*th sample of size *k* with mean $\mu_{(i:k)} \equiv \mu_{(i)}$ and variance $\sigma^2_{(i:k)} \equiv \sigma^2_{(i)}$. Let $X^{(j)}_{(i:k)} \equiv X^{(j)}_{(ii)}$ denote the *i*th order statistic from the *i*th sample of size *k* in the *j*th cycle ($j = 1, 2, \ldots, n$). A single cycle RSS with set size *k* may be displayed in a rectangular array such as the following:

$$\begin{array}{cccc} X_{(11)} & X_{(12)} & \cdots & X_{(1k)} \\ X_{(21)} & X_{(22)} & \cdots & X_{(2k)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{(k1)} & X_{(k2)} & \cdots & X_{(kk)} \end{array}$$

It is obvious that the new sample, $X_{(11)}, X_{(22)}, \ldots, X_{(kk)}$, known in the literature as a *Ranked Set Sample*, are independent but not identically distributed. Moreover, marginally, $X_{(ii)}$ is distributed as X_{ik}, the *i*th order statistic in a sample of size *k* from $F(x)$. Thus $E(X_{(ii)}) = \mu_{(i)}$ and $Var(X_{(ii)}) = \sigma^2_{(i)}$, for $i = 1, 2, \ldots, k$.

The relationships among $\mu, \sigma^2, \mu_{(i)}, \sigma^2_{(i)}$ for $i = 1, \ldots, k$ are

$$\mu = \sum_{i=1}^{k} \frac{\mu_{(i)}}{k} \qquad \text{and} \qquad \sigma^2 = \sum_{i=1}^{k} \frac{\sigma^2_{(i)}}{k} + \sum_{i=1}^{k} \frac{(\mu_{(i)} - \mu)^2}{k}.$$

McIntyre(1952) proposed

$$\hat{\mu}_{rss} = \bar{X}_{rss} = \sum_{i=1}^{k} X_{(ii)} / k \tag{2.1}$$

as a rival unbiased estimate of μ as opposed to $\hat{\mu}_{srs} = \bar{X}_{srs} = \sum_{i=1}^{k} X_i / k$.

Dell and Clutter(1972) provided the explicit expression for the variance of $\hat{\mu}_{rss}$ as

$$Var(\hat{\mu}_{rss}) = \frac{\sigma^2}{k} - \sum_{i=1}^{k} \frac{(\mu_{(i)} - \mu)^2}{k^2} = \sum_{i=1}^{k} \frac{\sigma^2_{(i)}}{k^2}. \tag{2.2}$$

Stokes(1980) suggested the following estimate of σ^2,

$$\hat{\sigma}^2_{Stokes} = \frac{\sum_{i=1}^{k}(X_{(ii)} - \overline{X}_{rss})^2}{k-1} \tag{2.3}$$

with
$$E(\hat{\sigma}^2_{Stokes}) = \sigma^2 + \sum_{i=1}^{k}\frac{(\mu_{(i)} - \mu)^2}{k(k-1)}$$

which shows that $\hat{\sigma}^2_{Stokes}$ is in general biased, and

$$Var(\hat{\sigma}^2_{Stokes}) = \frac{1}{(k-1)^2}\left[\left(\frac{k-1}{k}\right)^2\sum_{i=1}^{k}\mu_{4(i)} + 4\sum_{i=1}^{k}\tau^2_{(i)}\sigma^2_{(i)} + 4\left(\frac{k-1}{k}\right)\sum_{i=1}^{k}\tau_{(i)}\mu_{3(i)} + \frac{4}{k^2}\sum_{i<i'=1}^{k}\sigma^2_{(i)}\sigma^2_{(i')} - \frac{(k-1)^2}{k^2}\sum_{i=1}^{k}\sigma^4_{(i)}\right], \tag{2.4}$$

where $\mu_{l(i)} = E(X_{(ii)} - \mu_{(i)})^l$ for $l = 3,4$ and $\tau_{(i)} = \mu_{(i)} - \mu$.

To increase the efficiency of the RSS-based estimate of μ, McIntyre(1952) suggested replicating the entire RSS process several times. Quite generally, if $N = k \times n$ with $k \le n$, we can use an RSS procedure based on k units at a time, and repeat the process n times. A balanced RSS (BRSS) with set size k and the number of cycles n is displayed such as the following :

$$\begin{matrix} X^{(j)}_{(11)} & X^{(j)}_{(12)} & \cdots & X^{(j)}_{(1k)} \\ X^{(j)}_{(21)} & X^{(j)}_{(22)} & \cdots & X^{(j)}_{(2k)} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(j)}_{(k1)} & X^{(j)}_{(k2)} & \cdots & X^{(j)}_{(kk)} \end{matrix} \quad ; \text{for } j = 1, \ldots, n$$

where $X^{(j)}_{(11)}, \ldots, X^{(j)}_{(kk)}$, for all j and k, are independent.

Furthermore, $E(X^{(j)}_{(ii)}) = \mu_{(i)}$ and $Var(X^{(j)}_{(ii)}) = \sigma^2_{(i)}$; for all j.

The overall estimate of μ is then given by

$$\hat{\mu}_{brss}(N = k \times n) = \overline{X}_{rss} = \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\frac{X^{(j)}_{(ii)}}{k}\right]\Big/ n = \sum_{j=1}^{n}\frac{\overline{X}^{(j)}_{rss}}{n}$$

$$= \sum_{i=1}^{k}\left[\sum_{j=1}^{n}\frac{X^{(j)}_{(ii)}}{n}\right]\Big/ k = \sum_{i=1}^{k}\frac{\overline{X}_{(i)}}{k} . \tag{2.5}$$

The variance of $\hat{\mu}_{brss}(N = k \times n)$ is given by

$$Var(\hat{\mu}_{brss}(N = k \times n)) = \frac{1}{n}\left[\frac{\sigma^2}{k} - \sum_{i=1}^{k}\frac{(\mu_{(i)} - \mu)^2}{k^2}\right] = \sum_{i=1}^{k}\frac{\sigma_{(i)}^2}{k^2 n}. \quad (2.6)$$

Stokes (1980) suggested the following estimate of σ^2,

$$\hat{\sigma}_{Stokes}^2 = \sum_{j=1}^{n}\sum_{i=1}^{k}\frac{(X_{(ii)} - \overline{X}_{rss})^2}{kn-1} \quad (2.7)$$

where $\overline{X}_{rss} = \sum_{j=1}^{n}\sum_{i=1}^{k} X_{(ii)}^{(j)}/kn$. Again, $\hat{\sigma}_{Stokes}^2$ is biased with

$$E(\hat{\sigma}_{Stokes}^2) = \sigma^2 + \sum_{i=1}^{k}\frac{(\mu_{(i)} - \mu)^2}{k(kn-1)},$$

and $$Var(\hat{\sigma}_{Stokes}^2) = \frac{n}{(kn-1)^2}\left[\left(\frac{kn-1}{kn}\right)^2 \sum_{i=1}^{k}\mu_{4(i)} + 4\sum_{i=1}^{k}\tau_{(i)}^2\sigma_{(i)}^2 + 4\left(\frac{kn-1}{kn}\right)\sum_{i=1}^{k}\tau_{(i)}\mu_{3(i)}\right.$$

$$\left. + \frac{4}{k^2 n}\sum_{i<i'=1}^{k}\sigma_{(i)}^2\sigma_{(i')}^2 + \frac{2(n-1)-(kn-1)^2}{k^2 n^2}\sum_{i=1}^{k}\sigma_{(i)}^4\right]. \quad (2.8)$$

What we have described above can be called an equal allocation scheme (or balanced RSS) in the sense that each of the k order statistics is replicated an equal number of times, namely, n times.

It is quite possible to use unequal allocation (or unbalanced RSS, UBRSS) schemes as well. In this case, data are obtained by independently observing the ith order statistic n_i times, where $i = 1,\ldots,k$, resulting in a total of $N = n_1 + \cdots + n_k$ observations. Data can be denoted by $X_{(i)}^{(j)}, i = 1,\ldots,k, j = 1,\ldots,n_i$. It should be noted that each order statistics is obtained on the basis of a simple random sample of size k.

An unbiased estimate of μ is then constructed as

$$\hat{\mu}_{ubrss} = \frac{1}{k}\sum_{i=1}^{k}\overline{X}_{(i)} \quad (2.9)$$

where $\overline{X}_{(i)} = \sum_{j=1}^{n_i}\frac{X_{(i)}^{(j)}}{n_i}$, with its variance given by

$$Var(\hat{\mu}_{ubrss}) = \frac{1}{k^2}\sum_{i=1}^{k}\frac{\sigma_{(i)}^2}{n_i}. \quad (2.10)$$

The estimate of population variance σ^2 is

$$\hat{\sigma}_{ubrss}^2 = \sum_{i=1}^{k}\left(1+\frac{1}{k(n_i-1)}\right)\frac{S_{(i)}^2}{kn_i} + \sum_{i=1}^{k}\frac{(\overline{X}_{(i)} - \hat{\mu}_{ubrss})^2}{k} \quad (2.11)$$

where $S_{(i)}^2 = \sum_{j=1}^{n_i}(X_{(i)}^{(j)} - \overline{X}_{(i)})^2$ (See Perron and Sinha , 2004).

In most situations, the ranking may not be done perfectly. To tackle this problem, Stokes (1977) considered the case where the ranking is done on the basis of a concomitant variable X instead of judgment variable.

Following Stokes (1977), Yu and Lam(1997) assumed that the regression of Y on X is linear, i.e,

$$Y = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X) + \varepsilon$$

where X and ε are independent and ε has zero mean. It follows that ε has a variance given by $\sigma_Y^2(1-\rho^2)$.

Yu and Lam (1997) proposed an unbiased regression-type RSS estimate of the population mean, μ_Y, when the population mean, μ_X, of X is known. This is given by

$$\overline{Y}_{reg} = \overline{Y}_{rss} + \hat{B}(\mu_X - \overline{X}_{rss}), \quad (2.12)$$

where $\overline{Y}_{rss} = \frac{1}{kn}\sum_{j=1}^{n}\sum_{i=1}^{k}Y_{[ii]}^{(j)}$, $\overline{X}_{rss} = \frac{1}{kn}\sum_{j=1}^{n}\sum_{i=1}^{k}X_{(ii)}^{(j)}$ and

$$\hat{B} = \frac{\sum_{j=1}^{n}\sum_{i=1}^{k}(X_{(ii)}^{(j)} - \overline{X}_{rss})(Y_{[ii]}^{(j)} - \overline{Y}_{rss})}{\sum_{j=1}^{n}\sum_{i=1}^{k}(X_{(ii)}^{(j)} - \overline{X}_{rss})^2}$$

with $Var(\overline{Y}_{reg}) = \frac{\sigma_Y^2}{kn}(1-\rho^2)\left[1+E\left(\frac{\overline{Z}_{rss}^2}{S_z^2}\right)\right]$ (2.13)

where $Z_{(ii)}^{(j)} = \frac{X_{(ii)}^{(j)} - \mu_X}{\sigma_X}$, $\overline{Z}_{rss} = \frac{1}{kn}\sum_{j=1}^{n}\sum_{i=1}^{k}Z_{(ii)}^{(j)}$, $S_z^2 = \frac{1}{kn}\sum_{j=1}^{n}\sum_{i=1}^{k}(Z_{(ii)}^{(j)} - \overline{Z}_{rss})^2$.

We tackle several estimation problems and compare our proposed RSS estimates with the SRS naive estimates and apply this alternative strategy, RSS, in some real situations. Then methodologies of this study are given as followings:

1. Proposing quadratic nonnegative unbiased estimates of the population variance based on balanced RSS for some well-known distributions such as uniform, exponential, and normal distributions.
2. Applying RSS in a few situations and study the estimates of the population mean and variance.
3. Applying the RSS estimates of the population mean based on exponential distribution from Sinha *et al.* (1996) for proposing the RSS estimates of the reliability.
4. Modifying the best linear unbiased estimate (BLUE) of the scale parameter using RSS for obtaining nonnegative unbiased estimates of the scale parameter based on Normal, Logistic, Double exponential, Two-parameter exponential, and Weibull distributions.
5. Applying the above nonnegative unbiased estimates of the scale parameter for estimating the population quantiles based on the same distributions.

The results of this study will be seen in reprint of papers in the Appendices.

3) Analysis of Random/Mixed Linear Models

In this topic we address the statistical inference for the common mean of a bivariate normal population with unequal variances (Model I). We discuss estimation and tests for the common mean, the variances and the correlation coefficient. Both maximum likelihood and method of moments estimates are derived and some important features of both the methods are pointed out. Our application includes an Environmental Protection Agency (EPA) small data set on Reid Vapor Pressure (RVP) (Nussbaum and Sinha (1997); Yu, *et al.* (2002)).

Model specification and description of the data set

We assume that a random sample $\{(x_i, y_i), i = 1,2,\ldots,n\}$ is drawn from a bivariate normal population of (X,Y) with a common mean, with the parameters $(\mu, \mu, \sigma_1^2, \sigma_2^2, \rho)$. There are practical situations where the assumption of a common mean is valid. Our goal is to provide estimates and tests for the parameters under the above model. We consider both the maximum likelihood estimates as well as the moment estimates, and discuss their properties. We also provide relevant test statistics for testing hypothesis about the common mean, the variances, and the correlation coefficient.

Our data set illustrating a common mean scenario deals with the paired observations (x_i, y_i) representing field and lab data on RVP for 15 locations. This problem is motivated by the following practical issue in the context of the attempt by the EPA of the United States to evaluate the gasoline quality based on what is known as Reid Vapor Pressure (RVP). Occasionally, an EPA inspector would visit gas pumps in a city, take samples of gasoline of a particular brand, and measure RVP right at the spot which produces cheap and quick measurements. Once in a while, the inspector after measuring RVP at the spot will also ship a gasoline sample to a laboratory for a measurement of presumably higher precision at a higher cost, thus getting the pair (field, lab). Since usually laboratory measurements (Y) are much more expensive than field measurements (X) because of special packaging to be used to ship a gasoline sample from a field to a laboratory, not all the gasoline samples will be shipped to the laboratory and hence the resulting data would consist of many field measurements with occasional paired measurements obtained from both the field and laboratory. Our statistical analysis here is based on only the paired data reported below in Table 3.1 The scenario is such that the means are equal, but the variances are different.

Table 3.1 The field and lab data on RVP for new reformulated gasoline

X	Y	X	Y
8.03	8.28	8.60	8.52
8.64	8.63	7.83	7.92
9.14	9.28	7.88	7.89
7.86	7.85	8.56	8.48
8.70	8.62	7.83	7.95
9.28	9.14	7.99	8.32
7.86	7.86	7.56	7.60
7.83	7.90		

Statistical analysis of the model $N(\mu,\mu,\sigma_1,\sigma_2,\rho)$

Estimation of parameters

We discuss both the method of maximum likelihood (MLE) and the method of moments (MM) for estimating the four parameters: $\mu,\sigma_1^2,\sigma_2^2$ and ρ.

Method of maximum likelihood (MLE)

Since the joint p.d.f. of x and y is

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu)^2}{\sigma_1^2}+\frac{(y-\mu)^2}{\sigma_2^2}-\frac{2\rho}{\sigma_1\sigma_2}(x-\mu)(y-\mu)\right]},$$

for the given data set $\{x_i,y_i\}$, $i=1,2,\ldots,n$, the joint p.d.f. or the likelihood can be simplified to

$$L(\mu,\sigma_1,\sigma_2,\rho\mid \underline{x},\underline{y})=\frac{1}{\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)^n}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{S_x^2+n(\bar{x}-\mu)^2}{\sigma_1^2}+\frac{S_y^2+n(\bar{y}-\mu)^2}{\sigma_2^2}-\frac{2\rho}{\sigma_1\sigma_2}\left(S_{xy}+n(\bar{x}-\mu)(\bar{y}-\mu)\right)\right]}$$

where

$\bar{x}=\sum x_i/n$, $\bar{y}=\sum y_i/n$, $S_x^2=\sum(x_i-\bar{x})^2$, $S_y^2=\sum(y_i-\bar{y})^2$ and $S_{xy}=\sum(x_i-\bar{x})(y_i-\bar{y})$.

Obviously, the sufficient statistics are $\bar{x},\bar{y},S_x^2,S_y^2$ and S_{xy}, and these are not complete. Equating the first derivatives of $\ln L$ with respect to the parameters μ,σ_1,σ_2 and ρ to zero, and solving these equations, we get the MLEs as

$$\hat{\mu}_{MLE}=\frac{\bar{x}\left(S_y^2-S_{xy}\right)+\bar{y}\left(S_x^2-S_{xy}\right)}{S_x^2+S_y^2-2S_{xy}}, \tag{3.1}$$

$$\hat{\sigma}_{1,MLE}^2=\frac{1}{n}\left[S_x^2+\frac{n(\bar{x}-\bar{y})^2\left(S_x^2-S_{xy}\right)^2}{\left(S_x^2+S_y^2-2S_{xy}\right)^2}\right], \tag{3.2}$$

$$\hat{\sigma}_{2,MLE}^2=\frac{1}{n}\left[S_y^2+\frac{n(\bar{x}-\bar{y})^2\left(S_y^2-S_{xy}\right)^2}{\left(S_x^2+S_y^2-2S_{xy}\right)^2}\right], \tag{3.3}$$

$$\hat{\rho}_{MLE}=\frac{S_{xy}-\dfrac{n(\bar{x}-\bar{y})^2\left(S_x^2-S_{xy}\right)\left(S_y^2-S_{xy}\right)}{\left(S_x^2+S_y^2-2S_{xy}\right)^2}}{\sqrt{S_x^2+\dfrac{n(\bar{x}-\bar{y})^2\left(S_x^2-S_{xy}\right)^2}{\left(S_x^2+S_y^2-2S_{xy}\right)^2}}\cdot\sqrt{S_y^2+\dfrac{n(\bar{x}-\bar{y})^2\left(S_y^2-S_{xy}\right)^2}{\left(S_x^2+S_y^2-2S_{xy}\right)^2}}}. \tag{3.4}$$

Method of Moments (MM)

From $E(X)=\mu$, $E(Y)=\mu$, $E(X^2)=\mu^2+\sigma_1^2$, $E(Y^2)=\mu^2+\sigma_2^2$ and $E(XY)=Cov(X,Y)+\mu^2=\rho\sigma_1\sigma_2+\mu^2$, we can write:

$$\mu^2+\sigma_1^2=\frac{\sum_{i=1}^{n}x_i^2}{n},\quad \mu^2+\sigma_2^2=\frac{\sum_{i=1}^{n}y_i^2}{n}\text{ and }\mu^2+\rho\sigma_1\sigma_2=\frac{\sum_{i=1}^{n}x_iy_i}{n}.$$

Since an estimate of μ seems arbitrary and not unique, we choose to use the MLE of μ under the assumption that σ_1^2,σ_2^2 and ρ are known, and then replace σ_1^2,σ_2^2 and ρ by MM estimates.

Since $\hat{\mu}_{MLE}\left(\sigma_1^2,\sigma_2^2,\rho\right)=\dfrac{\bar{x}\sigma_2^2+\bar{y}\sigma_1^2-\rho\sigma_1\sigma_2(\bar{x}+\bar{y})}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}=\dfrac{\bar{x}\left(\sigma_2^2-\rho\sigma_1\sigma_2\right)+\bar{y}\left(\sigma_1^2-\rho\sigma_1\sigma_2\right)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$,

using $\sigma_1^2-\rho\sigma_1\sigma_2=\dfrac{\sum_{i=1}^{n}x_i^2}{n}-\dfrac{\sum_{i=1}^{n}x_iy_i}{n}$ and $\sigma_2^2-\rho\sigma_1\sigma_2=\dfrac{\sum_{i=1}^{n}y_i^2}{n}-\dfrac{\sum_{i=1}^{n}x_iy_i}{n}$,

we get

$$\hat{\mu}_{MM}=\frac{\bar{x}\left(\sum_{i=1}^{n}y_i^2-\sum_{i=1}^{n}x_iy_i\right)+\bar{y}\left(\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_iy_i\right)}{\sum_{i=1}^{n}(x_i-y_i)^2}. \tag{3.5}$$

Using (3.5), the MM estimates of σ_1^2,σ_2^2 and ρ are obtained as

$$\hat{\sigma}_{1,MM}^2=\frac{\sum_{i=1}^{n}x_i^2}{n}-\left[\frac{\bar{x}\left(\sum_{i=1}^{n}y_i^2-\sum_{i=1}^{n}x_iy_i\right)+\bar{y}\left(\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_iy_i\right)}{\sum_{i=1}^{n}(x_i-y_i)^2}\right]^2, \tag{3.6}$$

$$\hat{\sigma}_{2,MM}^2=\frac{\sum_{i=1}^{n}y_i^2}{n}-\left[\frac{\bar{x}\left(\sum_{i=1}^{n}y_i^2-\sum_{i=1}^{n}x_iy_i\right)+\bar{y}\left(\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_iy_i\right)}{\sum_{i=1}^{n}(x_i-y_i)^2}\right]^2, \tag{3.7}$$

$$\hat{\rho}_{MM}=\frac{\dfrac{\sum_{i=1}^{n}x_iy_i}{n}-\hat{\mu}_{MM}^2}{\sqrt{\hat{\sigma}_{1,MM}^2}\sqrt{\hat{\sigma}_{2,MM}^2}}. \tag{3.8}$$

Expressed in terms of $\bar{x}, \bar{y}, S_x^2, S_y^2$ and S_{xy}, the above estimates reduce to

$$\hat{\mu}_{MM} = \frac{\bar{x}\left(S_y^2 - S_{xy}\right) + \bar{y}\left(S_x^2 - S_{xy}\right)}{S_x^2 + S_y^2 - 2S_{xy} + n(\bar{x} - \bar{y})^2}, \tag{3.9}$$

$$\hat{\sigma}_{1,MM}^2 = \frac{S_x^2 + n\bar{x}^2}{n} - \left[\frac{\bar{x}\left(S_y^2 - S_{xy}\right) + \bar{y}\left(S_x^2 - S_{xy}\right)}{S_x^2 + S_y^2 - 2S_{xy} + n(\bar{x} - \bar{y})^2}\right]^2, \tag{3.10}$$

$$\hat{\sigma}_{2,MM}^2 = \frac{S_y^2 + n\bar{y}^2}{n} - \left[\frac{\bar{x}\left(S_y^2 - S_{xy}\right) + \bar{y}\left(S_x^2 - S_{xy}\right)}{S_x^2 + S_y^2 - 2S_{xy} + n(\bar{x} - \bar{y})^2}\right]^2, \tag{3.11}$$

$$\hat{\rho}_{MM} = \frac{\frac{S_{xy} + n\bar{x}\bar{y}}{n} - \left[\frac{\bar{x}\left(S_y^2 - S_{xy}\right) + \bar{y}\left(S_x^2 - S_{xy}\right)}{S_x^2 + S_y^2 - 2S_{xy} + n(\bar{x} - \bar{y})^2}\right]^2}{\sqrt{\frac{S_x^2 + n\bar{x}^2}{n} - \left[\frac{\bar{x}\left(S_y^2 - S_{xy}\right) + \bar{y}\left(S_x^2 - S_{xy}\right)}{S_x^2 + S_y^2 - 2S_{xy} + n(\bar{x} - \bar{y})^2}\right]^2} \cdot \sqrt{\frac{S_y^2 + n\bar{y}^2}{n} - \left[\frac{\bar{x}\left(S_y^2 - S_{xy}\right) + \bar{y}\left(S_x^2 - S_{xy}\right)}{S_x^2 + S_y^2 - 2S_{xy} + n(\bar{x} - \bar{y})^2}\right]^2}}. \tag{3.12}$$

Remark It is interesting to observe that unlike the MLEs of $\mu, \sigma_1^2, \sigma_2^2$ and ρ, the MM estimates given above do not satisfy the (full) equivariant conditions:

(i) $\hat{\mu}(a\overline{X} + b, a\overline{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) \neq a\hat{\mu}(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}) + b, \quad \forall a, b$ real.

(ii) $\hat{\sigma}_1^2(a\overline{X} + b, a\overline{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) \neq a^2\hat{\sigma}_1^2(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

(iii) $\hat{\sigma}_2^2(a\overline{X} + b, a\overline{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) \neq a^2\hat{\sigma}_2^2(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

(iv) $\hat{\rho}(a\overline{X} + b, a\overline{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) \neq \hat{\rho}(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

However, these estimates satisfy the less general equivariance property.

(i)' $\hat{\mu}(a\overline{X}, a\overline{Y}, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = a\hat{\mu}(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

(ii)' $\hat{\sigma}_1^2(a\overline{X}, a\overline{Y}, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = a^2\hat{\sigma}_1^2(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

(iii)' $\hat{\sigma}_2^2(a\overline{X}, a\overline{Y}, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = a^2\hat{\sigma}_2^2(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

(iv)' $\hat{\rho}(a\overline{X}, a\overline{Y}, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = \hat{\rho}(\overline{X}, \overline{Y}, S_x^2, S_y^2, S_{xy}), \quad \forall a, b$ real.

Properties of estimates

In this section we study the large sample properties of the MLE and MM estimates.The results given in the Appendix can be used to derive the expressions of large sample means and variances of the estimates of the parameters. This is done below by first expressing the ML and MM estimates in terms of

$$\underline{T} = (T_1, T_2, T_3, T_4, T_5) = \left(\bar{x}, \bar{y}, \frac{S_x^2}{n-1}, \frac{S_y^2}{n-1}, \frac{S_{xy}}{n-1} \right).$$

<u>MLEs</u>:

$$\hat{\mu}_{MLE} = \frac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{T_3 + T_4 - 2T_5},$$

$$\hat{\sigma}^2_{1,MLE} = T_3 + \frac{(T_1 - T_2)^2 (T_3 - T_5)^2}{(T_3 + T_4 - 2T_5)^2}, \ \hat{\sigma}^2_{2,MLE} = T_4 + \frac{(T_1 - T_2)^2 (T_4 - T_5)^2}{(T_3 + T_4 - 2T_5)^2},$$

$$\hat{\rho}_{MLE} = \frac{T_5 - \dfrac{(T_1 - T_2)^2 (T_3 - T_5)(T_4 - T_5)}{(T_3 + T_4 - 2T_5)^2}}{\sqrt{T_3 - \dfrac{(T_1 - T_2)^2 (T_3 - T_5)^2}{(T_3 + T_4 - 2T_5)^2}} \cdot \sqrt{T_4 - \dfrac{(T_1 - T_2)^2 (T_4 - T_5)^2}{(T_3 + T_4 - 2T_5)^2}}}.$$

<u>Moment Estimates:</u>

$$\hat{\mu}_{MM} = \frac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{T_3 + T_4 - 2T_5 + n(T_1 - T_2)^2},$$

$$\hat{\sigma}^2_{1,MM} = T_3 + T_1^2 - \left[\frac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{(T_3 + T_4 - 2T_5) + (T_1 - T_2)^2} \right]^2,$$

$$\hat{\sigma}^2_{2,MM} = T_4 + T_2^2 - \left[\frac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{(T_3 + T_4 - 2T_5) + (T_1 - T_2)^2} \right]^2,$$

$$\hat{\rho}_{MM} = \frac{T_5 + T_1 T_2 - \left[\dfrac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{(T_3 + T_4 - 2T_5) + (T_1 - T_2)^2} \right]^2}{\sqrt{T_3 + T_1^2 - \left[\dfrac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{(T_3 + T_4 - 2T_5) + (T_1 - T_2)^2} \right]^2} \cdot \sqrt{T_4 + T_2^2 - \left[\dfrac{T_1(T_4 - T_5) + T_2(T_3 - T_5)}{(T_3 + T_4 - 2T_5) + (T_1 - T_2)^2} \right]^2}}.$$

We now apply the general result for the mean and variance of $\phi(\underline{T})$ given in the Appendix and readily get the following results.

Method of MLE:

$$E(\hat{\mu}_{MLE}) = \mu\,,\ Var(\hat{\mu}_{MLE}) = \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n^2} + o\left(\frac{1}{n^2}\right),$$

$$E(\hat{\sigma}_{1,MLE}^2) = \sigma_1^2 + \frac{\sigma_1^2(\sigma_1-\rho\sigma_2)^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + o\left(\frac{1}{n}\right),\ Var(\hat{\sigma}_{1,MLE}^2) = \frac{2\sigma_1^4}{n} + O\left(\frac{1}{n^2}\right),$$

$$E(\hat{\sigma}_{2,MLE}^2) = \sigma_2^2 + \frac{\sigma_2^2(\sigma_1-\rho\sigma_2)^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + o\left(\frac{1}{n}\right),\ Var(\hat{\sigma}_{2,MLE}^2) = \frac{2\sigma_2^4}{n} + O\left(\frac{1}{n^2}\right),$$

$$E(\hat{\rho}_{MLE}) = \rho + \frac{\sigma_1\sigma_2(1-\rho^2)^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + o\left(\frac{1}{n}\right),\ Var(\hat{\rho}_{MLE}) = \frac{(1-\rho^2)^2}{n} + O\left(\frac{1}{n^2}\right).$$

Method of Moments:

$$E(\hat{\mu}_{MM}) = \mu - \frac{\mu}{n},\ Var(\hat{\mu}_{MM}) = \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + O\left(\frac{1}{n^2}\right),$$

$$E(\hat{\sigma}_{1,MM}^2) = \sigma_1^2 + O\left(\frac{1}{n}\right),$$

$$Var(\hat{\sigma}_{1,MM}^2) = \frac{2\sigma_1^2\left(\sigma_1^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2) + 2\mu^2(\sigma_1-\rho\sigma_2)^2\right)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + O\left(\frac{1}{n^2}\right),$$

$$E(\hat{\sigma}_{2,MM}^2) = \sigma_2^2 + O\left(\frac{1}{n}\right),$$

$$Var(\hat{\sigma}_{2,MM}^2) = \frac{2\sigma_2^2\left(\sigma_2^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2) + 2\mu^2(\sigma_2-\rho\sigma_1)^2\right)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} + O\left(\frac{1}{n^2}\right),$$

$$E(\hat{\rho}_{MM}) = \rho + O\left(\frac{1}{n}\right),$$

$$Var(\hat{\rho}_{MM})$$

$$= \frac{(1-\rho^2)(1+\rho^2)\left[\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2) + \mu^2(\sigma_1^2-\sigma_2^2)^2(\sigma_1^2+\sigma_2^2)^2\right]}{n\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)} + O\left(\frac{1}{n^2}\right)$$

$$\approx \frac{1-\rho^4}{n} \text{ if } \sigma_1^2 = \sigma_2^2.$$

Comparison of the estimates

In this section we compare the MLEs and MMs of $\mu, \sigma_1^2, \sigma_2^2$ and ρ based on their large sample properties. We first discuss on the basis of terms up to $O\left(\frac{1}{n}\right)$. Details are omitted.

(i) Estimation of μ

Since $Var(\hat{\mu}_{MLE}) = \dfrac{(1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n} = Var(\hat{\mu}_{MM})$, $\hat{\mu}_{MLE} \approx \hat{\mu}_{MM}$, up to $O\left(\frac{1}{n}\right)$.

(ii) Estimation of σ_1^2

The variance of $\hat{\sigma}_1^2$ for method of MLE and MM, up to $O\left(\frac{1}{n}\right)$, are

$$Var(\hat{\sigma}_{1,MLE}^2) = \frac{2\sigma_1^4}{n} \text{ and } Var(\hat{\sigma}_{1,MM}^2) = \frac{2\sigma_1^2\left(\sigma_1^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)+2\mu^2(\sigma_1-\rho\sigma_2)^2\right)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n}.$$

Obviously,

when $\mu = 0$, $Var(\hat{\sigma}_{1,MLE}^2) \approx Var(\hat{\sigma}_{1,MM}^2)$. So $\hat{\sigma}_{1,MLE}^2 \approx \hat{\sigma}_{1,MM}^2$, up to $O\left(\frac{1}{n}\right)$.

When $\mu \neq 0$, $Var(\hat{\sigma}_{1,MLE}^2) < Var(\hat{\sigma}_{1,MM}^2)$. So $\hat{\sigma}_{1,MLE}^2$ is better than $\hat{\sigma}_{1,MM}^2$, up to $O\left(\frac{1}{n}\right)$.

(iii) Estimation of σ_2^2

Simliarly,

$$Var(\hat{\sigma}_{2,MLE}^2) = \frac{2\sigma_2^4}{n} \text{ and } Var(\hat{\sigma}_{2,MM}^2) = \frac{2\sigma_2^2\left(\sigma_2^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)+2\mu^2(\sigma_2-\rho\sigma_1)^2\right)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)n}.$$

Again, when $\mu = 0$, $Var(\hat{\sigma}_{2,MLE}^2) \approx Var(\hat{\sigma}_{2,MM}^2)$. So $\hat{\sigma}_{2,MLE}^2 \approx \hat{\sigma}_{2,MM}^2$, up to $O\left(\frac{1}{n}\right)$.

When $\mu \neq 0$, $Var(\hat{\sigma}_{2,MLE}^2) < Var(\hat{\sigma}_{2,MM}^2)$. So $\hat{\sigma}_{2,MLE}^2$ is better than $\hat{\sigma}_{2,MM}^2$, up to $O\left(\frac{1}{n}\right)$.

(iv) Estimation of ρ

The variance of $\hat{\rho}$ for method of MLE and MM, up to $O\left(\frac{1}{n}\right)$, are the following

$Var(\hat{\rho}_{MLE}) = \dfrac{(1-\rho^2)^2}{n}$ and

$$Var(\hat{\rho}_{MM}) = \frac{(1-\rho^2)(1+\rho^2)\left[\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)+\mu^2(\sigma_1^2-\sigma_2^2)^2(\sigma_1^2+\sigma_2^2)^2\right]}{n\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}.$$

Obviously,

when $\mu = 0$, $Var(\hat{\rho}_{MLE}) \approx Var(\hat{\rho}_{MM})$. So $\hat{\rho}_{MLE} \approx \hat{\rho}_{MM}$, up to $O\left(\frac{1}{n}\right)$.

When $\mu \neq 0$, $Var(\hat{\rho}_{MLE}) < Var(\hat{\rho}_{MM})$. So $\hat{\rho}_{MLE}$ is better than $\hat{\rho}_{MM}$, up to $O\left(\frac{1}{n}\right)$.

We next discuss the comparison of the estimates based on terms up to $O\left(\frac{1}{n^2}\right)$. Our observations are the following. Details are omitted.

(i)' For $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MM}$,

when $\mu = 0$, $Var(\hat{\mu}_{MLE}) \approx Var(\hat{\mu}_{MM})$. So $\hat{\mu}_{MLE} \approx \hat{\mu}_{MM}$,

when $\mu \neq 0$, $Var(\hat{\mu}_{MLE}) < Var(\hat{\mu}_{MM})$. So $\hat{\mu}_{MLE}$ is better than $\hat{\mu}_{MM}$.

(ii)' For $\hat{\sigma}^2_{1,MLE}$ and $\hat{\sigma}^2_{1,MM}$,

when $\mu = 0$, $Var(\hat{\sigma}^2_{1,MLE}) < Var(\hat{\sigma}^2_{1,MM})$. So $\hat{\sigma}^2_{1,MLE}$ is better than $\hat{\sigma}^2_{1,MM}$,

when $\mu \neq 0$, the comparison is not straightforward.

(iii)' For $\hat{\sigma}^2_{2,MLE}$ and $\hat{\sigma}^2_{2,MM}$,

when $\mu = 0$, $Var(\hat{\sigma}^2_{2,MLE}) < Var(\hat{\sigma}^2_{2,MM})$. So $\hat{\sigma}^2_{2,MLE}$ is better than $\hat{\sigma}^2_{2,MM}$,

when $\mu \neq 0$, the comparison is not straightforward.

(iv)' For $\hat{\rho}_{MLE}$ and $\hat{\rho}_{MM}$,

when $\mu = 0$, $Var(\hat{\rho}_{MLE}) < Var(\hat{\rho}_{MM})$. So $\hat{\rho}_{MLE}$ is better than $\hat{\rho}_{MM}$,

when $\mu \neq 0$, the comparison is not straightforward.

Tests of hypotheses

We consider the problem of constructing large sample tests for suitable hypotheses of the basic parameters $\mu, \sigma_1^2, \sigma_2^2$ and ρ.

A. Test for $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$ where μ_0 is a given constant.

The test statistic is given by

Method of MLE: $T_{1\mu} = \dfrac{\hat{\mu}_{MLE} - \mu_0}{\sqrt{\hat{V}ar(\hat{\mu}_{MLE})}}$

Method of Moments: $T_{2\mu} = \dfrac{\hat{\mu}_{MM} - \mu_0}{\sqrt{\hat{V}ar(\hat{\mu}_{MM})}}$

where $\hat{V}ar(\hat{\mu}_{MLE})$ and $\hat{V}ar(\hat{\mu}_{MM})$ are the estimated variances up to order $O\left(\frac{1}{n}\right)$. We reject H_0 if $|T_{1\mu}| > Z_{\alpha/2}$ in case of $T_{1\mu}$, and if $|T_{2\mu}| > Z_{\alpha/2}$ in case of $T_{2\mu}$.

B. Test for $H_0 : \sigma_1^2 = \sigma_2^2$ against $H_1 : \sigma_1^2 \neq \sigma_2^2$.

The test statistic is given by

Method of MLE: $T_{1\sigma} = \dfrac{\hat{\sigma}_{1,MLE}^2 - \hat{\sigma}_{2,MLE}^2}{\sqrt{\hat{V}ar(\hat{\sigma}_{1,MLE}^2 - \hat{\sigma}_{2,MLE}^2)}}$

Method of Moments: $T_{2\sigma} = \dfrac{\hat{\sigma}_{1,MM}^2 - \hat{\sigma}_{2,MM}^2}{\sqrt{\hat{V}ar(\hat{\sigma}_{1,MM}^2 - \hat{\sigma}_{2,MM}^2)}}$

where $\hat{V}ar(\hat{\sigma}_{1,MLE}^2 - \hat{\sigma}_{2,MLE}^2)$ and $\hat{V}ar(\hat{\sigma}_{1,MM}^2 - \hat{\sigma}_{2,MM}^2)$ are the estimated variances up to order $O\left(\frac{1}{n}\right)$. We reject H_0 if $|T_{1\sigma}| > Z_{\alpha/2}$ in case of $T_{1\sigma}$, and if $|T_{2\sigma}| > Z_{\alpha/2}$ in case of $T_{2\sigma}$.

C. Test for $H_0 : \rho = \rho_0$ against $H_1 : \rho \neq \rho_0$ where ρ_0 is a given constant.

In case of $\hat{\rho}_{MLE}$, we test this hypothesis based on the variance stabilizing transformation h(ρ) = arctanh (ρ) since $E(\hat{\rho}_{MLE}) \approx \rho$ and $Var(\hat{\rho}_{MLE}) \approx \dfrac{(1-\rho^2)^2}{n}$.

We use the test statistic:

$$T_{1\rho} = \sqrt{n}\big(h(r) - h(\rho_0)\big).$$

Also, based on $\hat{\rho}_{MM}$, we propose to use the test statistic

$$T_{2\rho} = \frac{\hat{\rho}_{MM} - \rho_0}{\sqrt{\hat{V}ar(\hat{\rho}_{MM})}}$$

where $\hat{V}ar(\hat{\rho}_{MM})$ is the estimated variance up to order $O\left(\frac{1}{n}\right)$, Under H_0, $T_{1\rho}$ and $T_{2\rho}$ follow asymptotic standard normal distributions.

An Application of Model I

We provide the numerical results of the analysis of the data set given in Table 3.1 The summary statistics are shown in Table 3.2

Table 3.2 Summary statistics for Data in Table 1

Data	Statistics
15	N
8.2393	$\bar{x}$
8.2827	$\bar{y}$
0.2848	s_x^2
0.2454	s_y^2
0.9716	R
3.9868	S_x^2
3.4355	S_y^2
3.5957	S_{xy}

Analysis of data set

We apply the basic bivariate model given earlier to analyze this data set. From the summary statistics in Table 2, we have T_1= 8.2393, T_2= 8.2827, T_3= 0.2848, T_4= 0.2454 and T_5= 0.2568. Based on these statistics, we can compute the estimated values of the parameters as shown in Table 3.3

Table 3.3 The estimated values of the MLE and MM estimates

MLE	estimated values	MM	estimated values
$\hat{\mu}_{MLE}$	8.3127	$\hat{\mu}_{MM}$	7.4631
$\hat{\sigma}^2_{1,MLE}$	0.2902	$\hat{\sigma}^2_{1,MM}$	12.4728
$\hat{\sigma}^2_{2,MLE}$	0.2463	$\hat{\sigma}^2_{2,MM}$	13.1494
$\hat{\rho}_{MLE}$	0.9690	$\hat{\rho}_{MM}$	0.9996

Since the sample size of the above data set is small (only 15), rather than using the large sample theory as developed in the previous section, we use the famous resampling technique to derived the cut-off points of the various test statistics mentioned in the previous section. Table 3.4 shows the 5% and 95% cut-off points of the three test statistics from 800 resample data sets.

Table 3.4 The 5% and 95% cut-off points for each test statistic

Hypothesis Testing	Method of MLE		Method of moments	
	5%	95%	5%	95%
$H_0:\mu=8$	0.7117	4.2795	-138.3927	3.6289
$H_0:\sigma_1^2=\sigma_2^2$	-0.4879	2.1540	-39.4765	0.5802
$H_0:\rho=0.9$	0.3964	4.5923	1.7341	143.1814

We now use the above cut-off points to carry out three hypotheses of interest.

A. Test for $H_0:\mu=8$ against $H_1:\mu\neq 8$.

The value of the test statistic is given by

Method of MLE: $$T_{1\mu}=\frac{\hat{\mu}_{MLE}-8}{\sqrt{\hat{V}ar(\hat{\mu}_{MLE})}}=2.4856$$

Method of Moments: $$T_{2\mu}=\frac{\hat{\mu}_{MM}-8}{\sqrt{\hat{V}ar(\hat{\mu}_{MM})}}=-12.7933$$

Since 0.7117 < $T_{1\mu}$ < 4.2795 and -138.3927 < $T_{2\mu}$ < 3.6289, so we accept H_0 by both the tests.

B. Test for $H_0:\sigma_1^2=\sigma_2^2$ against $H_1:\sigma_1^2\neq\sigma_2^2$.

The value of the test statistic is given by

Method of MLE: $$T_{1\sigma}=\frac{\hat{\sigma}^2_{1,MLE}-\hat{\sigma}^2_{2,MLE}}{\sqrt{\hat{V}ar(\hat{\sigma}^2_{1,MLE}-\hat{\sigma}^2_{2,MLE})}}=1.1642$$

Method of Moments: $$T_{2\sigma}=\frac{\hat{\sigma}^2_{1,MM}-\hat{\sigma}^2_{2,MM}}{\sqrt{\hat{V}ar(\hat{\sigma}^2_{1,MM}-\hat{\sigma}^2_{2,MM})}}=-3.9138$$

Since -0.4879 < $T_{1\sigma}$ < 2.1540 and -39.4765 < $T_{2\sigma}$ < 0.5802, so we accept H_0 by both the tests.

C. Test for $H_0:\rho=0.9$ against $H_1:\rho\neq 0.9$.

The value of the test statistic is given by

Method of MLE: $T_{1\rho} = \sqrt{n}\left(h(r) - h(\rho_0)\right) = 2.3371$

Method of Moments $T_{2\rho} = \dfrac{\hat{\rho}_{MM} - \rho_0}{\sqrt{\hat{V}ar(\hat{\rho}_{MM})}} = 19.0788$

Since 0.3964 < $T_{1\rho}$ < 4.5923, and 1.7341 < $T_{2\rho}$ < 143.1814, so we accept H_0 by both the tests.

Power of the proposed tests

In this section we provide some simulated results of power of the proposed tests in order to compare the tests derived by two methods: MLE and MM. We generate paired data form a bivariate normal distribution with n = 5, and 10. For testing $H_0 : \mu = 0$, we generate data under $\mu = 0$ (size) and different values of μ (power). For testing $H_0 : \sigma_1^2 = \sigma_2^2$ and $H_0 : \rho = 0.5$, we take $\mu = 0$ without loss of generality. We have computed $T_{1\mu}$, $T_{2\mu}$, $T_{1\sigma}$, $T_{2\sigma}$, $T_{1\rho}$ and $T_{2\rho}$ with 1000 runs. The results are shown in Tables 3.5 – 3.10.

Table 3.5 Simulated power of the test: $H_0 : \mu = 0$ when $\sigma_1^2 = 1$, $n = 5$

			μ					
σ_2^2	ρ	Test	-1.5	-1	-0.5	0.5	1	1.5
0.6	0.2	$T_{1\mu}$	0.051	0.051	0.051	0.051	0.051	0.051
		$T_{2\mu}$	0.168	0.116	0.070	0.066	0.099	0.150
	0.4	$T_{1\mu}$	0.051	0.051	0.051	0.051	0.051	0.051
		$T_{2\mu}$	0.192	0.117	0.071	0.063	0.106	0.164
	0.6	$T_{1\mu}$	0.051	0.051	0.051	0.051	0.051	0.051
		$T_{2\mu}$	0.221	0.142	0.080	0.070	0.122	0.194
1.5	0.2	$T_{1\mu}$	0.050	0.050	0.050	0.050	0.050	0.050
		$T_{2\mu}$	0.109	0.080	0.056	0.060	0.072	0.109
	0.4	$T_{1\mu}$	0.049	0.049	0.049	0.049	0.049	0.049
		$T_{2\mu}$	0.115	0.079	0.056	0.060	0.077	0.119
	0.6	$T_{1\mu}$	0.051	0.051	0.051	0.051	0.051	0.051
		$T_{2\mu}$	0.127	0.090	0.056	0.061	0.090	0.130

Table 3.6 Simulated power of the test: $H_0 : \mu = 0$ when $\sigma_1^2 = 1$, $n = 10$

σ_2^2	ρ	Test	μ					
			-1.5	-1	-0.5	0.5	1	1.5
0.6	0.2	$T_{1\mu}$	0.049	0.049	0.049	0.049	0.049	0.049
		$T_{2\mu}$	0.141	0.103	0.065	0.053	0.069	0.100
	0.4	$T_{1\mu}$	0.050	0.050	0.050	0.050	0.050	0.050
		$T_{2\mu}$	0.148	0.104	0.067	0.055	0.075	0.110
	0.6	$T_{1\mu}$	0.051	0.051	0.051	0.051	0.051	0.051
		$T_{2\mu}$	0.143	0.100	0.065	0.063	0.086	0.127
1.5	0.2	$T_{1\mu}$	0.050	0.050	0.050	0.050	0.050	0.050
		$T_{2\mu}$	0.103	0.078	0.056	0.054	0.068	0.090
	0.4	$T_{1\mu}$	0.050	0.050	0.050	0.050	0.050	0.050
		$T_{2\mu}$	0.112	0.085	0.060	0.054	0.070	0.095
	0.6	$T_{1\mu}$	0.050	0.050	0.050	0.050	0.050	0.050
		$T_{2\mu}$	0.118	0.091	0.062	0.051	0.073	0.102

Table 3.7 Simulated power of the test: $H_0 : \sigma_1^2 = \sigma_2^2$ when $\sigma_1^2 = 1$, $n = 5$

		σ_2^2			
ρ	Test	0.6	0.8	1.5	2
0.2	$T_{1\sigma}$	0.064	0.053	0.065	0.090
	$T_{2\sigma}$	0.061	0.050	0.059	0.095
0.4	$T_{1\sigma}$	0.071	0..059	0.059	0.078
	$T_{2\sigma}$	0.057	0.044	0.071	0.095
0.6	$T_{1\sigma}$	0.079	0.063	0.062	0.085
	$T_{2\sigma}$	0.045	0.041	0.058	0.071

Table 3.8 Simulated power of the test: $H_0 : \sigma_1^2 = \sigma_2^2$ when $\sigma_1^2 = 1$, $n = 10$

		σ_2^2			
ρ	Test	0.6	0.8	1.5	2
0.2	$T_{1\sigma}$	0.101	0.060	0.071	0.152
	$T_{2\sigma}$	0.144	0.076	0.077	0.142
0.4	$T_{1\sigma}$	0.128	0.063	0.083	0.171
	$T_{2\sigma}$	0.141	0.085	0.076	0.134
0.6	$T_{1\sigma}$	0.159	0.071	0.101	0.200
	$T_{2\sigma}$	0.151	0.071	0.077	0.148

Table 3.9 Simulated power of the test: $H_0 : \rho = 0.5$ when $\sigma_1^2 = 1$, $n = 5$

		ρ					
σ_2^2	Test	0.1	0.3	0.4	0.6	0.7	0.8
0.6	$T_{1\rho}$	0.045	0.044	0.045	0.050	0.046	0.041
	$T_{2\rho}$	0.083	0.062	0.055	0.048	0.046	0.039
0.8	$T_{1\rho}$	0.039	0.044	0.045	0.049	0.044	0.0.38
	$T_{2\rho}$	0.081	0.061	0.055	0.047	0.038	0.036
1.5	$T_{1\rho}$	0.053	0.048	0.048	0.046	0.045	0.043
	$T_{2\rho}$	0.085	0.068	0.056	0.048	0.036	0.041
2	$T_{1\rho}$	0.043	0.048	0.048	0.049	0.046	0.041
	$T_{2\rho}$	0.087	0.062	0.053	0.044	0.034	0.042

Table 3.10 Simulated power of the test: $H_0:\rho=0.5$ when $\sigma_1^2=1$, $n=10$

		ρ					
σ_2^2	Test	0.1	0.3	0.4	0.6	0.7	0.8
0.6	$T_{1\rho}$	0.046	0.044	0.047	0.046	0.049	0.051
	$T_{2\rho}$	0.085	0.065	0.058	0.047	0.051	0.048
0.8	$T_{1\rho}$	0.045	0.046	0.045	0.051	0.053	0.060
	$T_{2\rho}$	0.085	0.067	0.058	0.048	0.047	0.046
1.5	$T_{1\rho}$	0.047	0.045	0.047	0.048	0.055	0.063
	$T_{2\rho}$	0.085	0.066	0.055	0.052	0.048	0.042
2	$T_{1\rho}$	0.050	0.048	0.050	0.050	0.056	0.068
	$T_{2\rho}$	0.083	0.065	0.054	0.051	0.048	0.051

We can see from the tables that for testing $H_0:\mu=0$, the test based on $T_{2\mu}$ is better than the test based on $T_{1\mu}$. However, for testing $H_0:\sigma_1^2=\sigma_2^2$, it turns out that the test based on $T_{1\sigma}$ is better than the test based on $T_{2\sigma}$. For testing $H_0:\rho=0.5$, we see that the test based on $T_{2\rho}$ is better than the test based on $T_{1\rho}$ for $\rho<0.5$. The conclusion reverses for $\rho>0.5$

Next we consider the problem of drawing inferences on the common variance of a bivariate normal population with different mean (Model II). We discuss estimation and tests for the different mean, the variance and the correlation coefficient -both maximum likelihood and method of moments estimates- and point out some important features of both the methods. Our application includes data on Diaspirin crosslinked hemoglobin (DCLHb) of 299 patients, measured by HemoCue method and its modification (Hedayat *et al.*, 2002).

Model specification and description of the data set

We assume that a random sample $\{(x_i,y_i), i=1,2,...,n\}$ is drawn from a bivariate normal population of (X,Y) with a common variance, with the parameters $(\mu_1,\mu_2,\sigma^2,\rho)$. There are practical situations where the assumption of a common variance is valid. Our goal is to provide estimates and tests for the parameters under the above model. We consider both the maximum likelihood estimates as well as the moment estimates, and discuss their properties. We also provide relevant test statistics for testing hypothesis about the different mean, the common variance, and the correlation coefficient.

The data set deals with Diaspirin crosslinked hemoglobin (DCLHb) of 299 individuals, measured in two different ways, resulting in paired data $\{(x_i, y_i)\}$. DCLHb which is solution containing oxygen-carrying hemoglobin was created as a blood substitute to treat acute trauma patients and to replace blood loss during surgery. Measurements of DCLHb in patient's serum after infusion are routinely performed using a Sigma instrument. A method of measuring hemoglobin called the HemoCue photometer was modified to reproduce the Sigma instrument DCLHb results. To validate this modified method, serum samples from 299 patients over the analytical range of 50-2000 mg/dL were collected. DCLHb values of each sample were measured simultaneously with the HemoCue and Sigma methods. The problem here is to assess 'agreement' between the two methods in terms of their means, assuming that the variances are equal. Our statistical analysis here is based on only the paired data reported below in Table 3.11. The scenario is such that the variances are equal, but the means are different.

Statistical Analysis

Estimation of parameters

We discuss both the method of maximum likelihood (MLE) and the method of moments (MM) for estimating the four parameters: μ_1, μ_2, σ and ρ.

Method of MLE

Since the joint p.d.f. of x and y is

$$f(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma^2}\left[(x-\mu_1)^2+(y-\mu_2)^2-2\rho(x-\mu_1)(y-\mu_2)\right]},$$

for the given data set $\{x_i, y_i\}$, $i = 1,2,\ldots,n$, the joint p.d.f. or the likelihood can be simplified to

$$L(\mu_1,\mu_2,\sigma,\rho \mid \underline{x},\underline{y}) = \frac{1}{\left(2\pi\sigma^2\sqrt{1-\rho^2}\right)^n} e^{-\frac{1}{2(1-\rho^2)\sigma^2}\left[S_x^2+n(\bar{x}-\mu_1)^2+S_y^2+n(\bar{y}-\mu_2)^2-2\rho\left(S_{xy}+n(\bar{x}-\mu_1)(\bar{y}-\mu_1)\right)\right]}$$

where $\bar{x} = \sum x_i/n, \bar{y} = \sum y_i/n, S_x^2 = \sum (x_i - \bar{x})^2, S_y^2 = \sum (y_i - \bar{y})^2$ and

$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}).$$

Table 3.11 Data on DCLHb of 299 patients, measured by HemoCue method (X) and its modification (Y)

X	Y	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
1340	1330	100	100	180	180	50	50	50	50	320	310	270	270
100	100	1610	1600	50	50	60	60	1170	1180	1430	1430	80	80
80	80	150	140	70	70	1610	1600	330	330	90	90	70	70
1340	1340	1380	1380	1550	1560	200	200	60	60	210	220	210	200
530	550	520	510	80	80	90	90	90	90	1770	1770	50	50
240	240	1430	1420	90	90	160	160	310	310	600	600	1200	1220
60	60	330	330	70	70	50	50	90	90	360	350	1070	1080
70	80	90	90	1560	1560	1040	1040	360	360	70	70	400	400
50	50	90	90	710	720	1250	1250	1120	1120	130	130	120	120
1720	1770	1920	1950	190	190	150	150	1200	1210	680	680	80	90
460	460	70	60	100	100	1530	1530	210	210	110	110	790	790
130	140	880	990	720	720	440	440	60	60	360	360	680	680
50	50	1490	1490	710	710	1680	1740	1450	1450	70	70	80	80
60	60	330	320	80	80	500	500	100	100	70	70	850	850
270	270	1440	1420	670	670	100	100	360	360	60	60	340	340
80	80	1150	1130	270	280	80	70	950	940	130	130	450	450
910	910	810	820	80	70	50	60	80	80	420	420	130	130
650	700	400	400	1380	1380	1180	1230	1120	1170	340	340	850	850
90	80	110	120	670	670	80	70	450	460	230	220	240	240
900	890	1510	1520	440	430	1180	1160	150	150	400	400	430	430
130	140	1030	1010	140	140	460	460	740	750	140	130	50	50
80	80	370	370	90	80	190	190	880	880	390	380	50	50
1840	1830	150	150	1510	1380	1150	1150	540	540	50	70	360	360
430	430	800	800	860	870	530	520	230	230	570	560	400	400
80	90	580	580	90	90	450	450	1180	1180	530	540	110	110
90	100	220	240	160	160	200	200	1500	1510	410	420	980	980
160	160	90	90	230	240	1380	1300	760	760	60	60	200	200
500	500	530	530	520	510	320	320	130	130	460	460	50	50
60	60	470	470	1000	1010	170	180	1060	1080	50	50	480	480

Table 3.11 Data on DCLHb of 299 patients, measured by HemoCue method (X) and its modification (Y) (continued)

X	Y	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
200	200	200	210	450	450	1250	1250	770	770	330	340	100	100
130	130	100	100	300	270	580	580	360	380	690	690	160	160
110	110	1080	1080	150	150	430	430	450	460	120	120	80	80
90	90	50	50	280	280	180	180	50	50	90	90	760	760
910	910	570	570	80	80	1040	1040	230	230	790	790	590	590
770	770	360	370	820	790	420	420	460	460	570	520	700	700
800	810	240	240	510	520	870	870	360	370	130	140	130	130
50	50	50	60	130	130	1260	1230	640	640	1430	1400	230	230
620	610	1660	1740	1610	1610	840	840	110	110	420	420	50	50
210	210	50	60	70	70	320	320	1590	1590	670	680	500	500
190	190	1190	1150	50	50	70	70	500	510	80	80	500	500
90	90	70	80	1420	1440	70	60	810	810	300	300	60	60
140	140	1640	1660	330	330	1410	1440	80	90	210	220		
110	110	880	820	90	90	450	450	580	580	130	120		

Obviously, the sufficient statistics are $\bar{x}, \bar{y}, S_x^2, S_y^2$ and S_{xy}, and these are not complete. Equating the first derivatives of log L with respect to the parameters μ_1, μ_2, σ and ρ to zero, and solving these equations, we get the MLEs as

$$\hat{\mu}_{1,MLE} = \bar{x} \tag{3.13}$$

$$\hat{\mu}_{2,MLE} = \bar{y} \tag{3.14}$$

$$\hat{\sigma}^2_{MLE} = \frac{S_x^2 + S_y^2}{2n} \tag{3.15}$$

$$\hat{\rho}_{MLE} = \frac{2S_{xy}}{S_x^2 + S_y^2}. \tag{3.16}$$

Method of Moments

From $E(X) = \mu_1$, $E(Y) = \mu_2$, $E(X^2) = \mu_1^2 + \sigma^2$, $E(Y^2) = \mu_2^2 + \sigma^2$ and

$E(XY) = Cov(X,Y) + \mu_1\mu_2 = \rho\sigma^2 + \mu_1\mu_2$, we can write

$$\mu_1^2 + \sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n},\ \mu_2^2 + \sigma^2 = \frac{\sum_{i=1}^{n} y_i^2}{n}, \text{ and } \mu_1\mu_2 + \rho\sigma^2 = \frac{\sum_{i=1}^{n} x_i y_i}{n}.$$

Since $\mu_1 = \bar{x}$, $\mu_2 = \bar{y}$, so we get

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} = \frac{S_x^2}{n}$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} y_i^2}{n} - \bar{y}^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n} = \frac{S_y^2}{n}$$

$$\rho\sigma^2 = \frac{\sum_{i=1}^{n} x_i y_i}{n} - \overline{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{S_{xy}}{n}$$

Solving these equations, the MM estimates are obtained as

$$\hat{\mu}_{1,MM} = \bar{x} \tag{3.17}$$

$$\hat{\mu}_{2,MM} = \bar{y} \tag{3.18}$$

$$\hat{\sigma}_{MM}^2 = \frac{S_x^2 + S_y^2}{2n} \tag{3.19}$$

$$\hat{\rho}_{MM} = \frac{S_{xy}/n}{\hat{\sigma}^2} = \frac{2S_{xy}}{S_x^2 + S_y^2}. \tag{3.20}$$

Remark It is interesting to observe that the MM estimates are the same as the MLEs. Moreover, the MLEs and MM estimates satisfy the equivariant conditions:

(i) $\hat{\mu}(a\bar{X} + b, a\bar{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = a\hat{\mu}(\bar{X}, \bar{Y}, S_x^2, S_y^2, S_{xy}) + b,\quad \forall a,b$ real.

(ii) $\hat{\sigma}_1^2(a\bar{X} + b, a\bar{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = a^2\hat{\sigma}_1^2(\bar{X}, \bar{Y}, S_x^2, S_y^2, S_{xy}),\quad \forall a,b$ real.

(iii) $\hat{\sigma}_2^2(a\bar{X} + b, a\bar{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = a^2\hat{\sigma}_2^2(\bar{X}, \bar{Y}, S_x^2, S_y^2, S_{xy}),\quad \forall a,b$ real.

(iv) $\hat{\rho}(a\bar{X} + b, a\bar{Y} + b, a^2S_x^2, a^2S_y^2, a^2S_{xy}) = \hat{\rho}(\bar{X}, \bar{Y}, S_x^2, S_y^2, S_{xy}),\quad \forall a,b$ real.

Properties of estimates

We study the large sample properties of the MLE and MM estimates. This is done below by first expressing the ML and MM estimates in terms of

$$\underline{T} = (T_1, T_2, T_3, T_4, T_5) = \left(\bar{x}, \bar{y}, \frac{S_x^2}{n-1}, \frac{S_y^2}{n-1}, \frac{S_{xy}}{n-1} \right).$$

MLEs:

$$\hat{\mu}_{1,MLE} = T_1,\ \hat{\mu}_{2,MLE} = T_2,\ \hat{\sigma}^2_{MLE} = \frac{T_3 + T_4}{2}, \text{ and } \hat{\rho}_{MLE} = \frac{2T_5}{T_3 + T_4}.$$

Moment Estimates:

$$\hat{\mu}_{1,MM} = T_1,\ \hat{\mu}_{2,MM} = T_2,\ \hat{\sigma}^2_{MM} = \frac{T_3 + T_4}{2}, \text{ and } \hat{\rho}_{MM} = \frac{2T_5}{T_3 + T_4}.$$

We now apply the general result of the mean and variance of $\phi(\underline{T})$ and readily get the following results.

Method of MLE:

$$E(\hat{\mu}_{1,MLE}) = \mu_1,\ Var(\hat{\mu}_{1,MLE}) = \frac{\sigma^2}{n},\ E(\hat{\mu}_{2,MLE}) = \mu_2,\ Var(\hat{\mu}_{2,MLE}) = \frac{\sigma^2}{n},$$

$$E(\hat{\sigma}^2_{MLE}) = \sigma^2,\ Var(\hat{\sigma}^2_{MLE}) = \frac{\sigma^4(1+\rho^2)}{n},$$

$$E(\hat{\rho}_{MLE}) = \rho - \frac{\rho(1-\rho^2)}{n},\ Var(\hat{\rho}_{MLE}) = \frac{(1-\rho^2)^2}{n} + O\left(\frac{1}{n^2}\right).$$

Method of Moments:

$$E(\hat{\mu}_{1,MM}) = \mu_1,\ Var(\hat{\mu}_{1,MM}) = \frac{\sigma^2}{n},\ E(\hat{\mu}_{2,MM}) = \mu_2,\ Var(\hat{\mu}_{2,MM}) = \frac{\sigma^2}{n},$$

$$E(\hat{\sigma}^2_{MM}) = \sigma^2,\ Var(\hat{\sigma}^2_{MM}) = \frac{\sigma^4(1+\rho^2)}{n},$$

$$E(\hat{\rho}_{MM}) = \rho - \frac{\rho(1-\rho^2)}{n},\ Var(\hat{\rho}_{MM}) = \frac{(1-\rho^2)^2}{n} + O\left(\frac{1}{n^2}\right).$$

Tests of hypotheses

We consider the problem of constructing large sample test for suitable hypotheses of the basic parameters μ_1, μ_2, σ and ρ.

A. Test for $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$.

The test statistic is given by

Method of MLE: $$T_{1\mu} = \frac{\hat{\mu}_{1,MLE} - \hat{\mu}_{2,MLE}}{\sqrt{\hat{V}ar(\hat{\mu}_{1,MLE} - \hat{\mu}_{2,MLE})}}$$

Method of Moments: $$T_{2\mu} = \frac{\hat{\mu}_{1,MM} - \hat{\mu}_{2,MM}}{\sqrt{\hat{V}ar(\hat{\mu}_{1,MM} - \hat{\mu}_{2,MM})}}$$

where $\hat{V}ar(\hat{\mu}_{1,MLE} - \hat{\mu}_{2,MLE})$ and $\hat{V}ar(\hat{\mu}_{1,MM} - \hat{\mu}_{2,MM})$ are the estimated variances up to order of $O\left(\frac{1}{n}\right)$. We reject H_0 if $|T_{1\mu}| > Z_{\alpha/2}$ in case of $T_{1\mu}$, and if $|T_{2\mu}| > Z_{\alpha/2}$ in case of $T_{2\mu}$.

B. Test for $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$ where σ_0^2 is a given constant.

Test statistic is given by

Method of MLE: $$T_{1\sigma} = \frac{\hat{\sigma}^2_{MLE} - \sigma_0^2}{\sqrt{\hat{V}ar(\hat{\sigma}^2_{MLE})}}$$

Method of Moments: $$T_{2\sigma} = \frac{\hat{\sigma}^2_{MM} - \sigma_0^2}{\sqrt{\hat{V}ar(\hat{\sigma}^2_{MM})}}$$

where $\hat{V}ar(\hat{\sigma}^2_{MLE})$ and $\hat{V}ar(\hat{\sigma}^2_{MM})$ are the estimated variances up to order of $O\left(\frac{1}{n}\right)$. We reject H_0 if $|T_{1\sigma}| > Z_{\alpha/2}$ in case of $T_{1\sigma}$, and if $|T_{2\sigma}| > Z_{\alpha/2}$ in case of $T_{2\sigma}$.

C. Test for $H_0 : \rho = \rho_0$ against $H_1 : \rho \neq \rho_0$ where ρ_0 is a given constant.

In case of $\hat{\rho}_{MLE}$, we test this hypothesis based on the variance stabilizing transformation h(ρ) = arctanh (ρ) since $E(\hat{\rho}_{MLE}) \approx \rho$ and $Var(\hat{\rho}_{MLE}) \approx \frac{(1-\rho^2)^2}{n}$.

We use the test statistic:

$$T_{1\rho} = \sqrt{n}\left(h(r) - h(\rho_0)\right).$$

In case of $\hat{\rho}_{MM}$, we also test this hypothesis based on the variance stabilizing transformation by using the same test statistic as in case of $\hat{\rho}_{MLE}$.

Test statistic is given by

$$T_{2\rho} = \sqrt{n}\left(h(r) - h(\rho_0)\right).$$

Under H_0, $T_{1\rho}$ and $T_{2\rho}$ follow asymptotic standard normal distributions.

An Application of Model II

In this section we provide the numerical results of the analysis of the data set given in Table 1. The summary statistics are shown in Table 3.122.

Analysis of data set

We have applied all techniques described in above section to analyze this data set. Using the summary statistics in Table 3.12, we have T_1= 489.3311, T_2= 490.2676, T_3= 225060.6249, T_4= 225669.0557 and T_5 = 225203.5353. Based on these statistics, we can compute the estimated values as shown in Table 3.13.

Table3.12 Summary statistics for Data in Table1.

Data	Statistics
299	N
489.3311	$\bar{x}$
490.2676	$\bar{y}$
225060.6249	s_x^2
225669.0557	s_y^2
0.9993	r
67068066.22	S_x^2
67249378.6	S_y^2
67110653.51	S_{xy}

Table 3.13 The estimated values of the MLE and MM estimates.

The MLE estimate	The estimated value	The MM estimate	The estimated value
$\hat{\mu}_{1,MLE}$	489.3311	$\hat{\mu}_{1,MM}$	489.3311
$\hat{\mu}_{2,MLE}$	490.1676	$\hat{\mu}_{2,MM}$	490.1676
$\hat{\sigma}^2_{MLE}$	225364.8402	$\hat{\sigma}^2_{MM}$	225364.8402
$\hat{\rho}_{MLE}$	0.9993	$\hat{\rho}_{MM}$	0.9993

Since we obtained the same estimates for both methods, we now provide three hypotheses of interest based on only the method of MLE.

A. Test for $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$

The test statistic is given by

Method of MLE: $$T_{1\mu} = \frac{\hat{\mu}_{1,MLE} - \hat{\mu}_{2,MLE}}{\sqrt{\hat{V}ar(\hat{\mu}_{1,MLE} - \hat{\mu}_{2,MLE})}} = -0.9016$$

Since $-1.96 < T_{1\mu} < 1.96$, so we accept H_0.

B. Test for $H_0: \sigma^2 = 225364$ against $H_1: \sigma^2 \neq 225364$.

The test statistic is given by

Method of MLE: $$T_{1\sigma} = \frac{\hat{\sigma}^2_{MLE} - \sigma^2_0}{\sqrt{\hat{V}ar(\hat{\sigma}^2_{MLE})}} = 0.0000$$

Since $-1.96 < T_{1\sigma} < 1.96$, so we accept H_0.

C. Test for $H_0: \rho = 0.9$ against $H_1: \rho \neq 0.9$.

The test statistic is given by

Method of MLE: $$T_{1\rho} = \sqrt{n}\left(h(r) - h(\rho_0)\right) = 43.0959$$

Since $T_{1\rho} > 1.96$, so we reject H_0.

Power of the proposed tests

In this section we provide some simulated results of power of the proposed tests for method of MLE. We generate paired data form a bivariate normal distribution with n = 5 and 10. For testing $H_0: \sigma^2 = 1$, we generate data under $\sigma^2 = 1$ (size) and different values of σ^2 (power). For testing $H_0: \mu_1 = \mu_2$ and $H_0: \rho = 0.5$, we take $\mu_1 = 0$. We have computed $T_{1\mu}$, $T_{1\sigma}$ and $T_{1\rho}$ with 1000 runs. The results are shown in Tables 3.14 – 3.18.

We can see from the tables that for testing $H_0: \mu_1 = \mu_2$, when $\mu_2 < 0$, the power will increase if μ_2 decrease and ρ increase. On the other hand, when $\mu_2 > 0$, the power will increase if μ_2 and ρ increase. For testing $H_0: \sigma^2 = 1$, when σ^2 increase the power will decrease. For testing $H_0: \rho = 0.5$, it turns out that the power will increase as ρ increase.

Remark For testing $H_0 : \sigma^2 = 1$ and $H_0 : \rho = 0.5$, the powers do not depend on the values of μ_2, since the MLEs and MM estimates for σ^2 and ρ are not the function of μ_1 and μ_2.

Table 3.14 Simulated power of the test: $H_0 : \mu_1 = \mu_2$ when $\mu_1 = 0$, $\sigma^2 = 1$, $n = 5$

	μ_2					
ρ	-1.5	-1	-0.5	0.5	1	1.5
0.2	0.478	0.246	0.094	0.112	0.270	0.514
0.4	0.622	0.321	0.126	0.146	0.363	0.638
0.6	0.775	0.442	0.152	0.213	0.544	0.854
0.8	0.967	0.747	0.266	0.324	0.793	0.981

Table 3.15 Simulated power of the test: $H_0 : \mu_1 = \mu_2$ when $\mu_1 = 0$, $\sigma^2 = 1$, $n = 10$

	μ_2					
ρ	-1.5	-1	-0.5	0.5	1	1.5
0.2	0.932	0.645	0.207	0.181	0.578	0.902
0.4	0.977	0.770	0.268	0.236	0.709	0.964
0.6	0.998	0.899	0.353	0.350	0.871	0.993
0.8	1.000	0.997	0.657	0.629	0.989	1.000

Table 3.16 Simulated power of the test: $H_0:\sigma^2=1$ when $\mu_1=0$, n =5, 10

		n = 5				n = 10			
		σ^2				σ^2			
μ_2	ρ	0.6	0.8	1.5	2	0.6	0.8	1.5	2
-1.5	0.2	0.103	0.056	0.138	0.313	0.273	0.087	0.086	0.291
	0.4	0.114	0.052	0.114	0.257	0.266	0.096	0.194	0.440
	0.6	0.096	0.058	0.100	0.207	0.220	0.082	0.172	0.394
	0.8	0.082	0.059	0.114	0.209	0.159	0.061	0.154	0.380
-1	0.2	0.103	0.056	0.138	0.313	0.273	0.087	0.086	0.291
	0.4	0.114	0.052	0.114	0.257	0.266	0.096	0.194	0.440
	0.6	0.096	0.058	0.100	0.207	0.220	0.082	0.172	0.394
	0.8	0.082	0.059	0.114	0.209	0.159	0.061	0.154	0.380
-0.5	0.2	0.103	0.056	0.138	0.313	0.273	0.087	0.086	0.291
	0.4	0.114	0.052	0.114	0.257	0.266	0.096	0.194	0.440
	0.6	0.096	0.058	0.100	0.207	0.220	0.082	0.172	0.394
	0.8	0.082	0.059	0.114	0.209	0.159	0.061	0.154	0.380
0.5	0.2	0.103	0.056	0.138	0.313	0.273	0.087	0.086	0.291
	0.4	0.114	0.052	0.114	0.257	0.266	0.096	0.194	0.440
	0.6	0.096	0.058	0.100	0.207	0.220	0.082	0.172	0.394
	0.8	0.082	0.059	0.114	0.209	0.159	0.061	0.154	0.380
1	0.2	0.103	0.056	0.138	0.313	0.273	0.087	0.086	0.291
	0.4	0.114	0.052	0.114	0.257	0.266	0.096	0.194	0.440
	0.6	0.096	0.058	0.100	0.207	0.220	0.082	0.172	0.394
	0.8	0.082	0.059	0.114	0.209	0.159	0.061	0.154	0.380
1.5	0.2	0.103	0.056	0.138	0.313	0.273	0.087	0.086	0.291
	0.4	0.114	0.052	0.114	0.257	0.266	0.096	0.194	0.440
	0.6	0.096	0.058	0.100	0.207	0.220	0.082	0.172	0.394
	0.8	0.082	0.059	0.114	0.209	0.159	0.061	0.154	0.380

Table 3.17 Simulated power of the test: $H_0: \rho = 0.5$ when $\mu_1 = 0$, $\sigma^2 = 1$, $n = 5$

	ρ			
μ_2	0.2	0.4	0.6	0.8
-1.5	0.042	0.047	0.054	0.095
-1	0.042	0.047	0.054	0.095
-0.5	0.042	0.047	0.054	0.095
0.5	0.042	0.047	0.054	0.095
1	0.042	0.047	0.054	0.095
1.5	0.042	0.047	0.054	0.095

Table 3.18 Simulated power of the test: $H_0: \rho = 0.5$ when $\mu_1 = 0$, $\sigma^2 = 1$, $n = 10$

	ρ			
μ_2	0.2	0.4	0.6	0.8
-1.5	0.151	0.055	0.066	0.382
-1	0.151	0.055	0.066	0.382
-0.5	0.151	0.055	0.066	0.382
0.5	0.151	0.055	0.066	0.382
1	0.151	0.055	0.066	0.382
1.5	0.151	0.055	0.066	0.382

4) Statistical Methods in Assessing Agreement

In this study we address the statistical inference of assessing agreements in two parts according to the type of information that we collect: discrete and continuous.

4.1) Discrete Case

Measurements of agreement are of great importance for assessing the acceptability of new or generic process, methodology, and formulation in many fields of laboratory performance, instrument or assay validation, method comparisons, statistical process control, goodness of fit, and individual bioequivalence. There are numerous examples that illustrate these situations: the agreement of laboratory measurements collected in various laboratories, the agreement of a newly developed method with gold standard method, the agreement of manufacturing process measurements with specifications, the agreement of observed values with predicted values, and the agreement in bioavailability of a new or generic formulation with a commonly used formulation. By the way, measuring agreement has been used very often to designate the level of agreement between different data-generating sources referred to as raters. A rater could be a medical laboratory, a clinical chemist, a psychologist, a radiologist, a clinician, a nurse, a psychiatric classification system, or a general measurement instrument. One of the most popular indices of agreement was originally presented by Cohen (1960), namely Cohen's kappa statistic (κ_C), as a reliability index for measuring agreement between two raters employing nominal scales.

Basically, Cohen described that the kappa statistic ranges between -1 and 1. Later, Cohen (1968) extended the original kappa statistic by presenting the weighted kappa. It is useful when the data arise from an ordered scale but the original kappa statistic is useful if the data arise from a nominal scale. Formulas for estimators of standard errors for kappa can be found, for instance in Hildebrand *et al.* (1977) and in Liebetrau (1983). When the sample size is sufficiently large, Everitt (1968) and Fleiss *et al.* (1969) gave valid formulas for approximate large-sample mean and variance of the two statistics, kappa and weighted kappa.

Several authors have proposed guidelines for the interpretation of kappa statistic. Vide, for example, Landis and Koch (1977a), Kraemer (1979), Fleiss (1981), and Lantz and Nebenzahl (1996). Even a matter as simple as the range of kappa is not clear from the literature but many discussions of kappa state that it ranges from –1 to 1, with 0 indicating no agreement beyond that expected by chance and 1 indicating perfect agreement.

Extensions have also been made to allow for more than two raters (Light, 1971; Landis and Koch, 1977b; Conger, 1980; Gross, 1986; Berry and Mielke, 1988; Posner *et al.*, 1990; Lehmann *et al.*, 1995), more than two possible ratings (Flack *et al.*, 1988; Haley and Osberg, 1989; Donner and Eliasziw, 1992; Lau, 1993), ordinal data (Fleiss, 1971; Fleiss and Cohen, 1973; Cicchetti, 1972; Cicchetti, 1977; Gamsu, 1986) and continuous data (Conger, 1985; Rae, 1988).

In addition, many other applications of kappa statistic in a varitety of different contexts can be found in Spitzer *et al.* (1967), Fleiss *et al.* (1972), Kraemer and Bloch (1988), Boushka *et al.* (1990), Hutchinson (1993), Schouten (1993), Shoukri *et al.* (1995), Brenner and Kliebsch (1996), Cantor (1996), Donner and Klar (1996), Shoukri and Main (1996), Donner (1998), Robert and McNamee (1998), Pinfold *et al.* (2000), Barnhart and Williamson (2002), Blair (2002), Lin *et al.* (2002), Washington *et al.* (2003), Glenn *et al.* (2005), and Ruamviboonsuk *et al.* (2005, 2006).

A Brief Description of Cohen's Kappa and Its Features

Let us consider a reliability research where 2 raters, referred to as rater A and rater B, are required to classify subjects into one of 2 possible response categories. The 2 response categories, labeled as 1 and 2, are assumed to be disjoint. An interpretation of the probabilities π_{ij} for $i, j = 1, 2,$ in a cross-classification of the reliability research can be depicted as given in Table 4.1.

Table 4.1 Joint distribution of classification probabilities for 2 raters and 2 response categories

Rater A	Rater B		Total
	1	2	
1	π_{11}	π_{12}	$\pi_{1.}$
2	π_{21}	π_{22}	$\pi_{2.}$
Total	$\pi_{.1}$	$\pi_{.2}$	1

It follows from Table 2.1 that π_{ij} represents the chance that rater A classifies a subject into category i, while rater B classifies the *same* subject into category j, $1 \le i, j \le 2$.

In this set-up, Cohen's kappa for assessing agreement between the two raters is defined as

$$\kappa_C = \frac{\theta_o - \theta_e}{1 - \theta_e} \tag{4.1}$$

where $\theta_o = \pi_{11} + \pi_{22}$, $\theta_e = \pi_{1.}\pi_{.1} + \pi_{2.}\pi_{.2}$. (4.2)

In applications, if there are n subjects and n_{ij} represents the number of subjects classified in category i by rater A and in category j by rater B, the sample estimate of κ_C is given by

$$\hat{\kappa}_C = \frac{\hat{\theta}_o - \hat{\theta}_e}{1 - \hat{\theta}_e} \text{ (Cohen's kappa statistic)} \tag{4.3}$$

where $\hat{\pi}_{ij} = \frac{n_{ij}}{n}$, $\hat{\pi}_{i.} = \frac{n_{i.}}{n}$, $\hat{\pi}_{.j} = \frac{n_{.j}}{n}$, $\hat{\theta}_o = \frac{n_{11} + n_{22}}{n}$, $\hat{\theta}_e = \frac{n_{1.}n_{.1} + n_{2.}n_{.2}}{n^2}$. (4.4)

We now proceed to critically examine some features of κ_C.

The following properties of κ_C are well known.

(i) $\kappa_C = 1$ if and only if $\theta_o = 1$. This means that there are no controversial judgement by the two raters i.e., the disagreement cells [(1,2) and (2,1)] have zero probability each.

(ii) $\kappa_C = 0$ if and only if $\theta_o = \theta_e$. Technically, this holds if and only if

$$(\pi_{11} - \pi_{1.}\pi_{.1}) + (\pi_{22} - \pi_{2.}\pi_{.2}) = 0 \tag{4.5}$$

which, in its turn, implies that we necessarily have

$$\pi_{ij} = \pi_{i.}\pi_{.j}, \ 1 \le i, j \le 2. \tag{4.6}$$

That is, the two raters perform independently.

(iii) $\kappa_C = -1$ if and only if $\pi_{11} + \pi_{22} = 0$, $\pi_{12} = \pi_{21} = 0.5$. Technically, this means that both the agreement cells have zero probability each while the two disagreement cells are equally likely.

The case of "$\kappa_C = -1$" seems to impose too restrictive behavior on the part of the raters. When $\pi_{11} = \pi_{22} = 0$, there is already an indication of total disagreement between the two raters. Therefore, in such situations, irrespective of the values assumed by π_{12} and π_{21} $(0 < \pi_{12}, \pi_{21} < 1$, $\pi_{12} + \pi_{21} = 1)$, the kappa coefficient is desired to assume the value -1. With this in mind, let us set $\pi_{12} = \alpha$ and $\pi_{21} = 1 - \alpha$, $0 < \alpha < 1$ and analyze the situation with the purpose of modifying the definition of κ_C to deal with the full strength of disagreement between the two raters.

Modifications to Cohen's Kappa

Our first modification is aimed at the value $\kappa_C = -1$. We modify κ_C as

$$\kappa_{M1} = \frac{\theta_o - \theta_e}{A - \theta_e} \tag{4.7}$$

and suggest a value of A to take care of the situations:

$$\pi_{11} = \pi_{22} = 0, \ \pi_{12} = \alpha, \ \pi_{21} = 1 - \alpha, \ 0 < \alpha < 1 \text{ along with } \kappa_{M1} = -1. \tag{4.8}$$

Under (4.8), κ_{M1} reduces to

$$\kappa_{M1} = \frac{-2\alpha(1-\alpha)}{A - 2\alpha(1-\alpha)}. \quad (4.9)$$

and $\kappa_{M1} = -1$ yields

$$A = 4\alpha(1-\alpha). \quad (4.10)$$

In view of (4.8), we have $\pi_{1.} = \pi_{.2} = \alpha$ so that α has dual interpretation. Therefore, we may replace α by $\frac{\pi_{1.} + \pi_{.2}}{2}$ in (4.10). This yields

$$A = 4 \cdot \frac{\pi_{1.} + \pi_{.2}}{2} \cdot \frac{\pi_{.1} + \pi_{2.}}{2} = (\pi_{1.} + \pi_{.2})(\pi_{.1} + \pi_{2.}). \quad (4.11)$$

Next, substituting (4.11) in (4.7), we obtain

$$\kappa_{M1} = \frac{\theta_o - \theta_e}{(\pi_{1.} + \pi_{.2})(\pi_{.1} + \pi_{2.}) - \theta_e} \quad (4.12)$$

or equivalently,

$$\kappa_{M1} = \frac{\theta_o - \theta_e}{(\pi_{1.} + \pi_{.2})(\pi_{.1} + \pi_{2.}) - (\pi_{1.}\pi_{.1} + \pi_{.2}\pi_{2.})}. \quad (4.13)$$

Finally, upon simplification, we can rewrite the modified kappa κ_{M1} as

$$\kappa_{M1} = \frac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}}. \quad (4.14)$$

This modification is based on the analysis of situations leading to total disagreement between the two raters. We verify below that all the three essential features of κ_C are retained by κ_{M1}. Clearly, $\kappa_{M1} = 0$ if and only if $\theta_o = \theta_e$. The other two values (± 1) are examined below in Theorem 1.

Theorem 1 Let κ_{M1} be the modified kappa as defined in (4.14). Then

$$\kappa_{M1} = \begin{cases} 1 \text{ if and only if Table 2.1 is of the form} \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}, \text{ for } 0 < \alpha < 1 & (4.15\text{i}) \\ -1 \text{ if and only if Table 2.1 is of the form} \begin{pmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{pmatrix}, \text{ for } 0 < \alpha < 1. & (4.15\text{ii}) \end{cases}$$

Proof: We start with the proof of (4.15i).

Recall that $\kappa_{M1} = \frac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}}$.

Then $\frac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}} = 1$

$$\Leftrightarrow \pi_{11} + \pi_{22} - (\pi_{1.}\pi_{.1} + \pi_{2.}\pi_{.2}) - \pi_{1.}\pi_{2.} - \pi_{.1}\pi_{.2} = 0$$

$$\Leftrightarrow \pi_{11} + \pi_{22} = \pi_{1.}(\pi_{.1} + \pi_{2.}) + \pi_{.2}(\pi_{2.} + \pi_{.1})$$

$$\Leftrightarrow \pi_{11} + \pi_{22} = (\pi_{1.} + \pi_{.2})(\pi_{.1} + \pi_{2.})$$

$$\Leftrightarrow \pi_{11} + \pi_{22} = (1 + \pi_{12} - \pi_{21})(1 - \pi_{12} + \pi_{21})$$

$$\Leftrightarrow 1 - (\pi_{12} + \pi_{21}) = 1 - (\pi_{12} - \pi_{21})^2$$

$$\Leftrightarrow \pi_{12} + \pi_{21} = (\pi_{12} - \pi_{21})^2 .$$

Therefore, we obtain $\pi_{12} = \pi_{21} = 0$.

Next, we will prove (4.15ii).

Since $\kappa_{M1} = \frac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}}$, we then have

$$\frac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}} = -1$$

$$\Leftrightarrow \pi_{11} + \pi_{22} - (\pi_{1.}\pi_{.1} + \pi_{2.}\pi_{.2}) + \pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2} = 0$$

$$\Leftrightarrow \pi_{11} + \pi_{22} = \pi_{1.}(\pi_{.1} - \pi_{2.}) + \pi_{.2}(\pi_{2.} - \pi_{.1})$$

$$\Leftrightarrow \pi_{11} + \pi_{22} = (\pi_{1.} - \pi_{.2})(\pi_{.1} - \pi_{2.})$$

$$\Leftrightarrow \pi_{11} + \pi_{22} = (\pi_{11} - \pi_{22})^2 .$$

Hence, we establish that $\pi_{11} = \pi_{22} = 0$.

The proof of the theorem is now complete. □

In addition we illustrate some characteristics of the Cohen's kappa and modified kappa in the following theorems.

Theorem 2 Suppose $\pi_{12}, \pi_{21} \neq 0$, $\pi_{12}\pi_{21} > \pi_{11}\pi_{22}$ and $\pi_{12} + \pi_{21} > 0.5$, then $\kappa_C, \kappa_{M1} < 0$.

Proof: We first show that $\kappa_C < 0$. The Cohen's kappa is given by

$$\kappa_C = \frac{\theta_o - \theta_e}{1 - \theta_e}$$

$$= \frac{(\pi_{11} + \pi_{22}) - [(\pi_{11} + \pi_{12})(\pi_{11} + \pi_{21}) + (\pi_{21} + \pi_{22})(\pi_{12} + \pi_{22})]}{1 - [(\pi_{11} + \pi_{12})(\pi_{11} + \pi_{21}) + (\pi_{21} + \pi_{22})(\pi_{12} + \pi_{22})]}$$

$$= \frac{[1-(\pi_{12}+\pi_{21})]-[(\pi_{11}+\pi_{12})(\pi_{11}+\pi_{21})+(\pi_{21}+\pi_{22})(\pi_{12}+\pi_{22})]}{1-[(\pi_{11}+\pi_{12})(\pi_{11}+\pi_{21})+(\pi_{21}+\pi_{22})(\pi_{12}+\pi_{22})]}. \quad (4.16)$$

Replacing 1 by $(\pi_{11}+\pi_{12}+\pi_{21}+\pi_{22})(\pi_{11}+\pi_{12}+\pi_{21}+\pi_{22})$ in (3.10), we have

$$\kappa_C = \frac{[(\pi_{11}(\pi_{12}+\pi_{21})+\pi_{22}(\pi_{12}+\pi_{21}))+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2]-(\pi_{12}+\pi_{21})}{[(\pi_{11}(\pi_{12}+\pi_{21})+\pi_{22}(\pi_{12}+\pi_{21}))+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2]}$$

$$= \frac{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2-(\pi_{12}+\pi_{21})}{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2}$$

$$= \frac{(\pi_{11}+\pi_{22}-1)(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2}{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2}. \quad (4.17)$$

Again, we substitute $1=(\pi_{11}+\pi_{12}+\pi_{21}+\pi_{22})(\pi_{11}+\pi_{12}+\pi_{21}+\pi_{22})$ in (3.11) and re-write it as

$$\kappa_C = \frac{2(\pi_{11}\pi_{22}-\pi_{12}\pi_{21})}{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2}. \quad (4.18)$$

We now consider the right hand side of (4.18). The denominator is always positive, that is $(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2>0$. For the numerator, we recall $\pi_{12}\pi_{21}>\pi_{11}\pi_{22}$, so we can obviously see that $\kappa_C<0$.

Next, we prove that $\kappa_{M1}<0$. Recall

$$\kappa_{M1} = \frac{\theta_o-\theta_e}{\pi_{1.}\pi_{2.}+\pi_{.1}\pi_{.2}}$$

$$= \frac{(\pi_{11}+\pi_{22})-[(\pi_{11}+\pi_{12})(\pi_{11}+\pi_{21})+(\pi_{21}+\pi_{22})(\pi_{12}+\pi_{22})]}{[(\pi_{11}+\pi_{12})(\pi_{21}+\pi_{22})+(\pi_{11}+\pi_{21})(\pi_{12}+\pi_{22})]}. \quad (4.19)$$

Now $(\pi_{11}+\pi_{22})$ is replaced by

$(\pi_{11}+\pi_{12}+\pi_{21}+\pi_{22})(\pi_{11}+\pi_{12}+\pi_{21}+\pi_{22})-(\pi_{12}+\pi_{21})$ and, therefore, (4.19) can be written in the form

$$\kappa_{M1} = \frac{[(\pi_{11}(\pi_{12}+\pi_{21})+\pi_{22}(\pi_{12}+\pi_{21}))+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2]-(\pi_{12}+\pi_{21})}{[(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+2\pi_{12}\pi_{21}]}$$

$$= \frac{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2-(\pi_{12}+\pi_{21})}{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+2\pi_{12}\pi_{21}}$$

$$= \frac{(\pi_{11}+\pi_{22}-1)(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2}{(\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21})+2\pi_{11}\pi_{22}+\pi_{12}^2+\pi_{21}^2}. \quad (4.20)$$

Next, we make the substitution using the relation

$$1 = (\pi_{11} + \pi_{12} + \pi_{21} + \pi_{22})(\pi_{11} + \pi_{12} + \pi_{21} + \pi_{22})$$

and (4.20) thus becomes

$$\kappa_{M1} = \frac{2(\pi_{11}\pi_{22} - \pi_{12}\pi_{21})}{(\pi_{11} + \pi_{22})(\pi_{12} + \pi_{21}) + 2\pi_{11}\pi_{22} + 2\pi_{12}\pi_{21}} \tag{4.21}$$

which is negative whenever $\pi_{12}\pi_{21} > \pi_{11}\pi_{22}$.

This completes the proof. □

Corollary 1 Suppose $\pi_{11}, \pi_{22} \neq 0$, $\pi_{12}\pi_{21} < \pi_{11}\pi_{22}$ and $\pi_{11} + \pi_{22} > 0.5$, then $\kappa_C, \kappa_{M1} > 0$.

Theorem 3 $\kappa_C = \kappa_{M1}$ if and only if $\pi_{1.} = \pi_{.1}$ or equivalently, $\pi_{2.} = \pi_{.2}$.

Proof: Our proof of this claim begins with

$$\kappa_C = \kappa_{M1} \Rightarrow \pi_{1.} = \pi_{.1} \text{ or } \pi_{2.} = \pi_{.2}.$$

Suppose $\kappa_C = \kappa_{M1}$, or $\dfrac{\theta_o - \theta_e}{1 - \theta_e} = \dfrac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}}$,

or equivalently, $1 - \pi_{1.}\pi_{.1} - \pi_{2.}\pi_{.2} = \pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}$.

We then substitute $1 = (\pi_{1.} + \pi_{2.})(\pi_{.1} + \pi_{.2})$ in the above expression. So it takes the form

$$\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2} - \pi_{1.}\pi_{.2} - \pi_{2.}\pi_{.1} = 0,$$

or $$\pi_{1.}(\pi_{2.} - \pi_{.2}) + \pi_{.1}(\pi_{.2} - \pi_{2.}) = 0,$$

or equivalently, $$(\pi_{1.} - \pi_{.1})(\pi_{.2} - \pi_{2.}) = 0.$$

That is $\pi_{1.} = \pi_{.1}$ or $\pi_{2.} = \pi_{.2}$.

To prove the converse of this claim

$$\pi_{1.} = \pi_{.1} \text{ or } \pi_{2.} = \pi_{.2} \Rightarrow \kappa_C = \kappa_{M1},$$

we assume that $\pi_{1.} = \pi_{.1}$ which is equivalent to $\pi_{2.} = \pi_{.2}$.

Therefore, $$\kappa_C = \frac{\theta_o - \theta_e}{1 - \theta_e}$$

$$= \frac{\pi_{11} + \pi_{22} - \pi_{1.}\pi_{.1} - \pi_{2.}\pi_{.2}}{1 - \pi_{1.}\pi_{.1} - \pi_{2.}\pi_{.2}}$$

$$= \frac{\pi_{11} + \pi_{22} - \pi_{1.}^2 - \pi_{2.}^2}{(\pi_{1.} + \pi_{2.})(\pi_{.1} + \pi_{.2}) - \pi_{1.}^2 - \pi_{2.}^2}$$

$$= \frac{\pi_{11} + \pi_{22} - \pi_{1.}^2 - \pi_{2.}^2}{(\pi_{1.} + \pi_{2.})^2 - \pi_{1.}^2 - \pi_{2.}^2}$$

$$= \frac{\pi_{11} + \pi_{22} - \pi_{1.}^2 - \pi_{2.}^2}{2\pi_{1.}\pi_{2.}}$$

$$= \frac{\pi_{11} + \pi_{22} - \pi_{1.}\pi_{.1} - \pi_{2.}\pi_{.2}}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}}$$

$$= \frac{\theta_o - \theta_e}{\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2}}$$

$$= \kappa_{M1}.$$

Thus, we can conclude that $\kappa_C = \kappa_{M1} \Leftrightarrow \pi_{1.} = \pi_{.1}$ or $\pi_{2.} = \pi_{.2}$. □

It has been noted by Cantor (1996) that all of the values $\pi_{2.}, \pi_{.2}, \theta_e, \theta_o, \pi_{22}, \pi_{11}, \pi_{12}$, and π_{21} can be determined by $\pi_{1.}, \pi_{.1}$ and κ_C. That is,

$$\pi_{2.} = 1 - \pi_{1.},\ \pi_{.2} = 1 - \pi_{.1},\ \theta_e = \pi_{1.}\pi_{.1} + \pi_{2.}\pi_{.2},\ \theta_o = \kappa_C(1 - \theta_e) + \theta_e,$$

$$\pi_{22} = (\theta_o - \pi_{1.} + \pi_{.2})/2,\ \pi_{11} = \theta_o - \pi_{22},\ \pi_{12} = \pi_{1.} - \pi_{11}, \text{ and } \pi_{21} = \pi_{.1} - \pi_{11}. \qquad (4.22)$$

Since $\theta_o - \theta_e = \kappa_{M1}(\pi_{1.}\pi_{2.} + \pi_{.1}\pi_{.2})$, it turns out that an analogous result also holds in terms of $\pi_{1.}, \pi_{.1}$ and κ_{M1}.

Remark 1 In the above, we have dealt with the situations: Kappa = 0, ± 1. It is only natural to expect that Kappa$\rightarrow +1(-1)$ whenever $\pi_{11} + \pi_{22} \rightarrow 1(0)$ and $\pi_{12} + \pi_{21} \rightarrow 0(1)$. We will consider two scenarios and examine such situations below.

Table 4.2 Scenario I

Rater A	Rater B		Total
	1	2	
1	0.40	0.03	0.43
2	0.03	0.54	0.57
Total	0.43	0.57	1.00

Table 4.3 Scenario II

Rater A	Rater B		Total
	1	2	
1	0.93	0.02	0.95
2	0.04	0.01	0.05
Total	0.97	0.03	1.00

For scenario I : $\kappa_C = 0.8776$ and $\kappa_{M1} = 0.8776$.

For scenario II : $\kappa_C = 0.2208$ and $\kappa_{M1} = 0.2219$.

The Cohen's kappa κ_C and modified kappa κ_{M1} surprisingly indicate a low level of agreement between raters A and B following the second scenario. In fact, it is difficult to explain why the raters would have a high level of agreement in scenario I and a low level of agreement in scenario II. Let us examine two more scenarios.

Table 4.4 Scenario III

Rater A	Rater B		Total
	1	2	
1	0.03	0.40	0.43
2	0.54	0.03	0.57
Total	0.57	0.43	1.00

Table 4.5 Scenario IV

Rater A	Rater B		Total
	1	2	
1	0.02	0.93	0.95
2	0.01	0.04	0.05
Total	0.03	0.97	1.00

For scenario III : $\kappa_C = -0.8439$ and $\kappa_{M1} = -0.8776$.

For scenario IV: $\kappa_C = -0.0184$ and $\kappa_{M1} = -0.2219$.

Once more, it is hard to explain why the raters would have a much higher level of disagreement in scenario III and the level of disagreement for scenario IV is not at all high. It turns out that neither Cohen's kappa κ_C nor the modified kappa κ_{M1} may serve the purpose of assessing agreement/disagreement in all situations. We attempt one more modification below.

In order to obtain a good expression for assessing agreement or disagreement, it seems necessary to modify Cohen's kappa once more. Towards this, we start with a study of the maximum and minimum of Cohen's kappa for given value of θ_o as presented below.

Theorem 4 For given θ_o, $0 < \theta_o < 1$,

$$\theta_{e,\max} = \theta_o + \frac{(1-\theta_o)^2}{2}, \tag{4.23}$$

$$\theta_{e,\min} = \theta_o\left(1 - \frac{\theta_o}{2}\right). \tag{4.24}$$

Further,

$$\kappa_{C,\max} = \frac{\theta_o^2}{1+(1-\theta_o)^2}, \tag{4.25}$$

$$\kappa_{C,\min} = -\left(\frac{1-\theta_o}{1+\theta_o}\right). \tag{4.26}$$

Proof: Recall that $\theta_e = \pi_{1.}\pi_{.1} + \pi_{2.}\pi_{.2}$

$$= (\pi_{11}+\pi_{12})(\pi_{11}+\pi_{21}) + (\pi_{21}+\pi_{22})(\pi_{12}+\pi_{22})$$

$$= \pi_{11}^2 + \pi_{22}^2 + (\pi_{11}+\pi_{22})(\pi_{12}+\pi_{21}) + 2\pi_{12}\pi_{21}.$$

Since $\theta_o = \pi_{11} + \pi_{22}$, then $\theta_e = \pi_{11}^2 + \pi_{22}^2 + \theta_o(1-\theta_0) + 2\pi_{12}\pi_{21}$.

We first maximize θ_e by maximizing $(\pi_{12}\pi_{21})$ and $(\pi_{11}^2 + \pi_{22}^2)$ simultaneously.

Clearly, $\max(\pi_{12}\pi_{21})$ subject to $\pi_{12} + \pi_{21} = 1 - \theta_o$ occurs when $\pi_{12} = \pi_{21} = \frac{1-\theta_o}{2}$. Further, $\max(\pi_{11}^2 + \pi_{22}^2)$ subject to $\pi_{11} + \pi_{22} = \theta_o$ occurs when $\pi_{11} = \theta_o, \pi_{22} = 0$ or $\pi_{11} = 0, \pi_{22} = \theta_o$. Both the solutions are simultaneously feasible.

Therefore, we have $\theta_{e,\max} = \theta_o^2 + \theta_o(1-\theta_o) + 2\left(\frac{1-\theta_o}{2}\right)^2 = \theta_o + \frac{(1-\theta_o)^2}{2}$.

Hence $\theta_{e,\max} = \theta_o + \dfrac{(1-\theta_o)^2}{2}$ if and only if Table 2.1 reads as

$$\begin{pmatrix} \theta_o & \dfrac{1-\theta_o}{2} \\ \dfrac{1-\theta_o}{2} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \dfrac{1-\theta_o}{2} \\ \dfrac{1-\theta_o}{2} & \theta_o \end{pmatrix}.$$

Since κ_C is decreasing function in θ_e, we also deduce that

$$\kappa_{C,\min} = \frac{\theta_o - \theta_{e,\max}}{1-\theta_{e,\max}} = -\left(\frac{1-\theta_o}{1+\theta_o}\right), \text{ upon simplification.}$$

Next, to minimize θ_e, we need to minimize $(\pi_{12}\pi_{21})$ and $(\pi_{11}^2 + \pi_{22}^2)$ simultaneously.

Again, $\min(\pi_{12}\pi_{21})$ subject to $\pi_{12} + \pi_{21} = 1 - \theta_o$ occurs when $\pi_{12} = 1-\theta_o, \pi_{21} = 0$ or $\pi_{12} = 0, \pi_{21} = 1-\theta_o$. Further, $\min(\pi_{11}^2 + \pi_{22}^2)$ subject to $\pi_{11} + \pi_{22} = \theta_o$ occurs when $\pi_{11} = \pi_{22} = \dfrac{\theta_o}{2}$. Both the solutions are simultaneously feasible.

Then, we have $\theta_{e,\min} = 2\left(\dfrac{\theta_o}{2}\right)^2 + \theta_o(1-\theta_o) = \theta_o\left(1 - \dfrac{\theta_o}{2}\right)$.

Hence $\theta_{e,\min} = \theta_o\left(1 - \dfrac{\theta_o}{2}\right)$ if and only if Table 2.1 reads as

$$\begin{pmatrix} \dfrac{\theta_o}{2} & 1-\theta_o \\ 0 & \dfrac{\theta_o}{2} \end{pmatrix} \text{ or } \begin{pmatrix} \dfrac{\theta_o}{2} & 0 \\ 1-\theta_o & \dfrac{\theta_o}{2} \end{pmatrix}.$$

Since κ_C is decreasing function in θ_e, we also derive

$$\kappa_{C,\max} = \frac{\theta_o - \theta_{e,\min}}{1-\theta_{e,\min}} = \frac{\theta_o^2}{1+(1-\theta_o)^2}, \text{ upon simplification.}$$

The proof of this theorem is now complete. □

In the next theorem, we give analogous results when θ_e is specified.

Theorem 5 For given θ_e, $0 < \theta_e < 1$,

(i) $\theta_e = \dfrac{1}{2}$

$$\theta_{o,\max} = 1,\ \theta_{o,\min} = 0,\ \kappa_{C,\max} = 1,\ \kappa_{C,\min} = -1. \tag{4.27}$$

(ii) $\theta_e > \frac{1}{2}$

$$\theta_{o,\max} = 1,\ \theta_{o,\min} = \sqrt{2\theta_e - 1},\ \kappa_{C,\max} = 1,\ \kappa_{C,\min} = \frac{\sqrt{2\theta_e - 1} - \theta_e}{1-\theta_e}. \tag{4.28}$$

(iii) $\theta_e < \frac{1}{2}$

$$\theta_{o,\max} = 1 - \sqrt{1-2\theta_e},\ \theta_{o,\min} = 0,\ \kappa_{C,\max} = \frac{1-\sqrt{1-2\theta_e} - \theta_e}{1-\theta_e},\ \kappa_{C,\min} = -\left(\frac{\theta_e}{1-\theta_e}\right). \tag{4.29}$$

Proof:

(i) $\theta_e = \frac{1}{2}$

Obviously, $\theta_{o,\max} = 1$ if and only if Table 2.1 reads as $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $\theta_{o,\min} = 0$ if and only if Table 2.1 reads as $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

Hence $\kappa_{C,\max} = 1$ and $\kappa_{C,\min} = -1$.

(ii) $\theta_e > \frac{1}{2}$

Recall that $\theta_e = \pi_{11}^2 + \pi_{22}^2 + (\pi_{11} + \pi_{22})(\pi_{12} + \pi_{21}) + 2\pi_{12}\pi_{21}$.

Since $\theta_o = \pi_{11} + \pi_{22}$, then $\theta_e = \pi_{11}^2 + \pi_{22}^2 + \theta_o(1-\theta_0) + 2\pi_{12}\pi_{21}$.

We first minimize θ_o by maximizing $(\pi_{12}\pi_{21})$ and $(\pi_{11}^2 + \pi_{22}^2)$ for given θ_e.

Note that $\max(\pi_{12}\pi_{21})$ subject to $\pi_{12} + \pi_{21} = 1 - \theta_o$ occurs when $\pi_{12} = \pi_{21} = \frac{1-\theta_o}{2}$. Further, $\max(\pi_{11}^2 + \pi_{22}^2)$ subject to $\pi_{11} + \pi_{22} = \theta_o$ occurs when $\pi_{11} = \theta_o, \pi_{22} = 0$ or $\pi_{11} = 0, \pi_{22} = \theta_o$. Both the solutions are simultaneously feasible.

Then, we have $\theta_e = \theta_o^2 + \theta_o(1-\theta_o) + 2\left(\frac{1-\theta_o}{2}\right)^2 = \theta_o + \frac{(1-\theta_o)^2}{2}$,

or equivalently, $\theta_o^2 = 2\theta_e - 1$ which yields $\theta_o = \sqrt{2\theta_e - 1}$.

Hence $\theta_{o,\min} = \sqrt{2\theta_e - 1}$ if and only if Table 2.1 reads as

$$\begin{pmatrix} \sqrt{2\theta_e-1} & \frac{1-\sqrt{2\theta_e-1}}{2} \\ \frac{1-\sqrt{2\theta_e-1}}{2} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \frac{1-\sqrt{2\theta_e-1}}{2} \\ \frac{1-\sqrt{2\theta_e-1}}{2} & \sqrt{2\theta_e-1} \end{pmatrix}.$$

Consequently, $\kappa_{C,\min} = \dfrac{\theta_{o,\min}-\theta_e}{1-\theta_e} = \dfrac{\sqrt{2\theta_e-1}-\theta_e}{1-\theta_e}$.

Next, to maximize θ_o, we set $\pi_{12}=\pi_{21}=0$ and $\pi_{11}+\pi_{22}=1$. For given θ_e, the solutions to π_{11} and π_{22} are $\dfrac{1\pm\sqrt{2\theta_e-1}}{2}$. Accordingly, $\theta_{o,\max}=1$ and, hence, $\kappa_{C,\max}=1$. Further, Table 2.1 reads as $\begin{pmatrix} \frac{1\pm\sqrt{2\theta_e-1}}{2} & 0 \\ 0 & \frac{1\mp\sqrt{2\theta_e-1}}{2} \end{pmatrix}$.

(iii) $\theta_e < \dfrac{1}{2}$

Recall that $\theta_e = \pi_{11}^2+\pi_{22}^2+\theta_o(1-\theta_0)+2\pi_{12}\pi_{21}$,

or equivalently, $\theta_o(1-\theta_0)=\theta_e-\pi_{11}^2-\pi_{22}^2-2\pi_{12}\pi_{21}$.

For $\theta_e<\dfrac{1}{2}$, $\pi_{11}+\pi_{22}=1$ $(\pi_{12}=\pi_{21}=0)$ is not feasible because $\pi_{11}+\pi_{22}=1$ and $\pi_{11}^2+\pi_{22}^2=\theta_e<\dfrac{1}{2}$ are contradictory.

Set, therefore, $\pi_{11}+\pi_{22}=1-\delta$, $\pi_{12}+\pi_{21}=\delta$, so that $\pi_{11}^2+\pi_{22}^2+(1-\delta)\delta+2\pi_{12}\pi_{21}=\theta_e$

or equivalently, $(1-\delta)\delta=\theta_e-(\pi_{11}^2+\pi_{22}^2)-2\pi_{12}\pi_{21}$.

Since $\pi_{11}^2+\pi_{22}^2 \ge \dfrac{(1-\delta)^2}{2}$ and $\pi_{12}\pi_{21}\ge 0$, we deduce $(1-\delta)\delta \le \theta_e - \dfrac{(1-\delta)^2}{2}$,

or equivalently, $\delta \ge \sqrt{1-2\theta_e}$. That is, $\pi_{11}+\pi_{22}=1-\delta\le 1-\sqrt{1-2\theta_e}$.

Therefore, $\theta_{o,\max}=1-\sqrt{1-2\theta_e}$ if and only if Table 2.1 reads as

$$\begin{pmatrix} \frac{1-\sqrt{1-2\theta_e}}{2} & \sqrt{1-2\theta_e} \\ 0 & \frac{1-\sqrt{1-2\theta_e}}{2} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1-\sqrt{1-2\theta_e}}{2} & 0 \\ \sqrt{1-2\theta_e} & \frac{1-\sqrt{1-2\theta_e}}{2} \end{pmatrix}.$$

Consequently, we have $\kappa_{C,\max} = \dfrac{1-\sqrt{1-2\theta_e}-\theta_e}{1-\theta_e} = 1 - \dfrac{\sqrt{1-2\theta_e}}{1-\theta_e}$.

Next, to minimize θ_o, we re-write $\theta_o(1-\theta_0) = \theta_e - \pi_{11}^2 - \pi_{22}^2 - 2\pi_{12}\pi_{21}$ as

$$\begin{aligned}\theta_o(1-\theta_0) &= \theta_e - (\pi_{11}+\pi_{22})^2 + 2(\pi_{11}\pi_{22} - \pi_{12}\pi_{21}) \\ &= \theta_e - \theta_o^2 + 2(\pi_{11}\pi_{22} - \pi_{12}\pi_{21})\end{aligned}$$

i.e., $\theta_o = \theta_e + 2(\pi_{11}\pi_{22} - \pi_{12}\pi_{21})$. Therefore, we minimize $(\pi_{11}\pi_{22})$ and maximize $(\pi_{12}\pi_{21})$ simultaneously.

Set $\pi_{11} = \pi_{22} = 0$, then we obtain $\theta_e = 2\pi_{12}\pi_{21}$ and $\pi_{12} + \pi_{21} = 1$.

Since $\pi_{12} - \pi_{21} = \pm\sqrt{1-2\theta_e}$, we get $\pi_{12} = \dfrac{1\pm\sqrt{1-2\theta_e}}{2}$ and $\pi_{21} = \dfrac{1\mp\sqrt{1-2\theta_e}}{2}$.

Thus, $\theta_{o,\min} = 0$ if and only if Table 2.1 reads as $\begin{pmatrix} 0 & \dfrac{1\pm\sqrt{1-2\theta_e}}{2} \\ \dfrac{1\mp\sqrt{1-2\theta_e}}{2} & 0 \end{pmatrix}$.

And as a result, we have $\kappa_{C,\min} = -\left(\dfrac{\theta_e}{1-\theta_e}\right)$.

This completes the proof. □

Remark 2 We may now suggest the following modification to κ_C based on its standardization.

$$\kappa_{M2} = \frac{\kappa_{C,\max} - \kappa_C}{\kappa_{C,\max} - \kappa_{C,\min}}$$

Thus, in effect, we will have

(i) $\kappa_{M2}(\theta_o) = \dfrac{\kappa_{C,\max}(\theta_o) - \kappa_C}{\kappa_{C,\max}(\theta_o) - \kappa_{C,\min}(\theta_o)} = \dfrac{\dfrac{\theta_o^2}{1+(1-\theta_o)^2} - \kappa_C}{\dfrac{\theta_o^2}{1+(1-\theta_o)^2} + \dfrac{1-\theta_o}{1+\theta_o}}$ when θ_o is specified.

(ii) $\kappa_{M2}(\theta_e) = \dfrac{\kappa_{C,\max}(\theta_e) - \kappa_C}{\kappa_{C,\max}(\theta_e) - \kappa_{C,\min}(\theta_e)}$ when θ_e is specified.

$$= \begin{cases} \dfrac{1-\kappa_C}{2}, & \text{if } \theta_e = \dfrac{1}{2} \\[2ex] \dfrac{1-\kappa_C}{1-\dfrac{\sqrt{2\theta_e - 1} - \theta_e}{1-\theta_e}}, & \text{if } \theta_e > \dfrac{1}{2} \\[4ex] \dfrac{\dfrac{1-\sqrt{1-2\theta_e}-\theta_e}{1-\theta_e} - \kappa_C}{\dfrac{1-\sqrt{1-2\theta_e}}{1-\theta_e}}, & \text{if } \theta_e < \dfrac{1}{2}. \end{cases}$$

Remark 3 In view of the expressions for $\kappa_{C,\max}$ and $\kappa_{C,\min}$ for given θ_o and the underlying structures for the cell probabilities, we can explain the results in Tables 3.1-3.4. In Tables 3.1 and 3.2, $\theta_o = 0.94$. Therefore,

$$\kappa_{C,\max} = \frac{\theta_o^2}{1+(1-\theta_o)^2} = 0.8804 \text{ corresponding to}$$

$$\begin{pmatrix} \dfrac{\theta_o}{2} & 1-\theta_o \\ 0 & \dfrac{\theta_o}{2} \end{pmatrix} = \begin{pmatrix} 0.47 & 0.06 \\ 0 & 0.47 \end{pmatrix}, \text{ or } \begin{pmatrix} 0.47 & 0 \\ 0.06 & 0.47 \end{pmatrix}$$

and

$$\kappa_{C,\min} = -\left(\frac{1-\theta_o}{1+\theta_o}\right) = -0.0309 \text{ corresponding to}$$

$$\begin{pmatrix} \theta_o & \dfrac{1-\theta_o}{2} \\ \dfrac{1-\theta_o}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0.94 & 0.03 \\ 0.03 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 0.03 \\ 0.03 & 0.94 \end{pmatrix}.$$

The value of κ_C in scenario I is given by $\kappa_C = 0.8776$ which is very close to $\kappa_{C,\max}$. This is because the structure in Table 4.2 [scenario I] closely resembles the structure corresponding to $\kappa_{C,\max}$ given above. Similar explanation holds for the value of κ_C in Table 4.3 [scenario II]. Also, the results in Tables 4.4 and 4.5 can be justified along similar lines of argument.

4.2) Continuous Case

Making inference about the common mean is one of the oldest problems in statistical inference. Several authors have made notable contributions to this topic. Beginning with Cohen and Sackrowitz (1984) considered several tests, including a normal approximate test for testing about the common mean. Furthermore, Cohen and Sackrowitz (1977) deal with the hypothesis testing for the common mean and for balanced incomplete blocks designs (BIBD) and an application of this shown in Cohen and Sackrowitz (1989). Recently, Krishnamoorthy and Lu (2003) presented procedures for hypothesis testing and interval estimation of the common mean of several normal populations based on the concepts of generalized *p*-value and generalized confidence limit. Later, Krishnamoorthy and Lu (2004) compared the five tests for the common mean of several multivariate normal populations. In addition, test for common variance and test for the correlation coefficient also are the interesting problems. Mathew, Sinha and Zhou (1993) addressed the testing hypothesis concerning a common mean vector of two independent linear models having different variances and the testing hypothesis concerning a common variance component in linear models involving two variance components. Inferences on the correlation coefficient in bivariate normal populations from ranked set samples for means and variances known case mentioned in Stokes (1980).

In this topic we are interested in combining the problems testing for common mean, common variance, and common correlation coefficient into one overall problem testing by using four test procedures for the small sample size and the large sample size. These test procedures based on a bivariate normal population.

Derivation of Test Procedures for Small Sample Size

Assuming that X and Y have a bivariate normal distribution with means μ_x and μ_y, variances σ_x^2 and σ_y^2, correlation coefficient ρ and covariance of X and Y is $\rho\sigma_x\sigma_y$. We denote this by

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}\right], \qquad (2.1)$$

where $-\infty < \mu_x, \mu_y < \infty$, $\sigma_x, \sigma_y > 0$, $-1 < \rho < 1$.

We wish to combine the problems testing for common mean, common variance, and common correlation coefficient into one overall problem testing

$$H_0 : \mu_x = \mu_y, \sigma_x = \sigma_y, \rho = \rho_0 \quad \text{versus} \quad H_1 : H_0 \text{ is not true,} \qquad (2.2)$$

where ρ_0 is a given value.

In addition, we use 4 test procedures for testing to our problem based upon the small sample size and the large sample size. We begin with the testing problem (2.2) for n is small sample size.

First test procedure, we use the statistic λ_1^* from the Likelihood Ratio Test (LRT) as derived in the following.

Suppose $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $i = 1, 2, \ldots, n$ follow n paired data of bivariate normal distribution.

So the joint probability density function (pdf) of x_i and y_i is given by

$$f(x_i, y_i) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_i-\mu_x)^2}{\sigma_x^2} + \frac{(y_i-\mu_y)^2}{\sigma_y^2} - 2\rho\frac{(x_i-\mu_x)(y_i-\mu_y)}{\sigma_x\sigma_y}\right]}, \qquad (2.3)$$

then the likelihood is

$$L(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho | \text{ data}) = \frac{1}{\left[2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}\right]^n} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{\sum_{i=1}^{n}(x_i-\mu_x)^2}{\sigma_x^2} + \frac{\sum_{i=1}^{n}(y_i-\mu_y)^2}{\sigma_y^2} - 2\rho\frac{\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y)}{\sigma_x\sigma_y}\right]}. \qquad (2.4)$$

Define $\bar{x} = \sum x_i/n$, $\bar{y} = \sum y_i/n$, $S_x^2 = \sum (x_i - \bar{x})^2$, $S_y^2 = \sum (y_i - \bar{y})^2$

and $S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y})$. (2.5)

It is obvious that we have

$$\sum_{i=1}^{n}(x_i - \mu_x)^2 = S_x^2 + n(\bar{x} - \mu_x)^2,$$

$$\sum_{i=1}^{n}(y_i - \mu_y)^2 = S_y^2 + n(\bar{y} - \mu_y)^2, \qquad (2.6)$$

and $$\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y) = S_{xy} + n(\bar{x} - \mu_x)(\bar{y} - \mu_y).$$

After substituting (2.6) into (2.4) and rearranging the terms considerably, we then obtain the likelihood function

$$L=\frac{1}{\left[2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}\right]^n}e^{-\frac{n}{2(1-\rho^2)}\left[\frac{(\bar{x}-\mu_x)^2}{\sigma_x^2}+\frac{(\bar{y}-\mu_y)^2}{\sigma_y^2}-2\rho\left(\frac{\bar{x}-\mu_x}{\sigma_x}\right)\left(\frac{\bar{y}-\mu_y}{\sigma_y}\right)\right]}\cdot e^{-\frac{1}{2(1-\rho^2)}\left[\frac{S_x^2}{\sigma_x^2}+\frac{S_y^2}{\sigma_y^2}-2\rho\frac{S_{xy}}{\sigma_x\sigma_y}\right]}. \quad (2.7)$$

Since the likelihood ratio test statistic λ is defined as the ratio of the maximum value of the likelihood when the parameters are restricted based on the null hypothesis to the maximum value of the data without the additional assumption.

For our problem, the likelihood ratio test statistic is given by

$$\lambda=\frac{\underset{H_0:\mu_x=\mu_y,\sigma_x=\sigma_y,\rho=\rho_0}{\text{Max}} L(\mu_x,\mu_x,\sigma_x^2,\sigma_x^2,\rho_0|\text{data})}{\underset{\text{Unrestricted}}{\text{Max}} L(\mu_x,\mu_y,\sigma_x^2,\sigma_y^2,\rho|\text{data})}. \quad (2.8)$$

We first evaluate $\underset{H_0:\mu_x=\mu_y,\sigma_x=\sigma_y,\rho=\rho_0}{\text{Max}} L(\mu_x,\mu_x,\sigma_x^2,\sigma_x^2,\rho_0|\text{data})$.

Under the null hypothesis, the likelihood function in equation (1.7) becomes

$$L=\frac{1}{\left[2\pi\sigma_x^2\sqrt{1-\rho_0^2}\right]^n}e^{-\frac{n}{2(1-\rho_0^2)\sigma_x^2}\left[(\bar{x}-\mu_x)^2+(\bar{y}-\mu_x)^2-2\rho_0(\bar{x}-\mu_x)(\bar{y}-\mu_x)\right]}\cdot e^{-\frac{1}{2(1-\rho_0^2)\sigma_x^2}\left[S_x^2+S_y^2-2\rho_0S_{xy}\right]}, \quad (2.9)$$

and so the log-likelihood function is

$$\log \underset{H_0}{L}=-n\log 2\pi-\frac{n}{2}\log\left(1-\rho_0^2\right)-2n\log\sigma_x$$

$$-\frac{n}{2\left(1-\rho_0^2\right)\sigma_x^2}\left[(\bar{x}-\mu_x)^2+(\bar{y}-\mu_x)^2-2\rho_0(\bar{x}-\mu_x)(\bar{y}-\mu_x)\right]$$

$$-\frac{1}{2\left(1-\rho_0^2\right)\sigma_x^2}\left[S_x^2+S_y^2-2\rho_0S_{xy}\right]. \quad (2.10)$$

After differentiating (2.10) with respect to μ_x and σ_x respectively, then the results are

$$\frac{\partial\log \underset{H_0}{L}}{\partial\mu_x}=-\frac{n}{2\left(1-\rho_0^2\right)\sigma_x^2}\left[-2(\bar{x}-\mu_x)-2(\bar{y}-\mu_x)-2\rho_0\left[-(\bar{x}+\bar{y})+2\mu_x\right]\right] \quad (2.11)$$

and

$$\frac{\partial\log \underset{H_0}{L}}{\partial\sigma_x}=\frac{-2n}{\sigma_x}+\frac{n}{\left(1-\rho_0^2\right)\sigma_x^3}\left[(\bar{x}-\mu_x)^2+(\bar{y}-\mu_x)^2-2\rho_0(\bar{x}-\mu_x)(\bar{y}-\mu_x)\right]$$

$$+\frac{1}{\left(1-\rho_0^2\right)\sigma_x^3}\left[S_x^2+S_y^2-2\rho_0S_{xy}\right]. \quad (2.12)$$

The estimators of μ_x and σ_x^2 are obtained by solving the equations

$$(1-\rho_0)\bar{x}+(1-\rho_0)\bar{y}-2(1-\rho_0)\mu_x=0 \quad (2.13)$$

and

$$-2n(1-\rho_0^2)\sigma_x^2+n\left[(\bar{x}-\mu_x)^2+(\bar{y}-\mu_x)^2-2\rho_0(\bar{x}-\mu_x)(\bar{y}-\mu_x)\right]+S_x^2+S_y^2-2\rho_0 S_{xy}=0 \quad (2.14)$$

respectively.

Hence the estimators are given by

$$\hat{\hat{\mu}}_x=\frac{\bar{x}+\bar{y}}{2} \quad (2.15)$$

and

$$\hat{\hat{\sigma}}_x^2=\frac{n\left[\left(\bar{x}-\hat{\hat{\mu}}_x\right)^2+\left(\bar{y}-\hat{\hat{\mu}}_x\right)^2-2\rho_0\left(\bar{x}-\hat{\hat{\mu}}_x\right)\left(\bar{y}-\hat{\hat{\mu}}_x\right)\right]+S_x^2+S_y^2-2\rho_0 S_{xy}}{2n\left(1-\rho_0^2\right)}. \quad (2.16)$$

By substituting (2.15) into (2.16), we have

$$\hat{\hat{\sigma}}_x^2=\frac{1}{2n\left(1-\rho_0^2\right)}\left\{\frac{n}{2}(1+\rho_0)(\bar{x}-\bar{y})^2+S_x^2+S_y^2-2\rho_0 S_{xy}\right\}. \quad (2.17)$$

We can substitute (2.15) and (2.17) in (2.9) and then writing the result as

$$\underset{H_0:\mu_x=\mu_y,\sigma_x=\sigma_y,\rho=\rho_0}{\text{Max}} L(\mu_x,\mu_x,\sigma_x^2,\sigma_x^2,\rho_0|\text{data})=\left(\frac{n\sqrt{1-\rho_0^2}}{\pi\left[\frac{n}{2}(1+\rho_0)(\bar{x}-\bar{y})^2+S_x^2+S_y^2-2\rho_0 S_{xy}\right]}\right)^n e^{-n}. \quad (2.18)$$

Next, we find $\underset{\text{Unrestricted}}{\text{Max}} L(\mu_x,\mu_y,\sigma_x^2,\sigma_y^2,\rho|\text{data})$.

The likelihood function under unrestricted assumption as given in equation (2.7), and therefore the log-likelihood function is

$$\begin{aligned}\log L=&-n\log 2\pi-n\log\sigma_x-n\log\sigma_y-\frac{n}{2}\log(1-\rho)^2\\&-\frac{n}{2(1-\rho^2)}\left[\frac{(\bar{x}-\mu_x)^2}{\sigma_x^2}+\frac{(\bar{y}-\mu_y)^2}{\sigma_y^2}-2\rho\left(\frac{\bar{x}-\mu_x}{\sigma_x}\right)\left(\frac{\bar{y}-\mu_y}{\sigma_y}\right)\right]\\&-\frac{1}{2(1-\rho^2)}\left[\frac{S_x^2}{\sigma_x^2}+\frac{S_y^2}{\sigma_y^2}-2\rho\frac{S_{xy}}{\sigma_x\sigma_y}\right].\end{aligned} \quad (2.19)$$

After differentiating (2.19) with respect to μ_x and μ_y respectively, then the results are the following equations:

$$\frac{\partial \log L}{\partial \mu_x} = \frac{-n}{2(1-\rho^2)}\left[\frac{-2(\bar{x}-\mu_x)}{\sigma_x^2} + \frac{2\rho(\bar{y}-\mu_y)}{\sigma_x \sigma_y}\right] \tag{2.20}$$

and

$$\frac{\partial \log L}{\partial \mu_y} = \frac{-n}{2(1-\rho^2)}\left[\frac{-2(\bar{y}-\mu_y)}{\sigma_y^2} + \frac{2\rho(\bar{x}-\mu_x)}{\sigma_x \sigma_y}\right]. \tag{2.21}$$

By equating (2.20) and (2.21) to zero, we have

$$\mu_x - \rho\frac{\sigma_x}{\sigma_y}\mu_y = \bar{x} - \rho\frac{\sigma_x}{\sigma_y}\bar{y} \tag{2.22}$$

and

$$\rho\frac{\sigma_y}{\sigma_x}\mu_x - \mu_y = \rho\frac{\sigma_y}{\sigma_x}\bar{x} - \bar{y}. \tag{2.23}$$

The estimators of μ_x and μ_y are obtained by solving the equations (2.22) and (2.23) thus the estimators are given by

$$\hat{\mu}_x = \bar{x} \text{ and } \hat{\mu}_y = \bar{y}. \tag{2.24}$$

In the same way, the estimators of σ_x^2, σ_y^2 and ρ are easily obtained by equating the first derivatives of (2.19) with respect to the parameters σ_x, σ_y and ρ to zero, and solving these equations, leading to

$$\hat{\sigma}_x^2 = \frac{S_x^2}{n},\ \hat{\sigma}_y^2 = \frac{S_y^2}{n}, \text{ and } \hat{\rho} = \frac{S_{xy}}{S_x S_y} \text{ or } \hat{\rho}\hat{\sigma}_x\hat{\sigma}_y = \hat{\sigma}_{xy} = \frac{S_{xy}}{n}. \tag{2.25}$$

By replacing (2.24) and (2.25) into the equation (2.7), we get

$$\underset{\text{Unrestricted}}{\text{Max}}\ L(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho|\text{data}) = \left[\frac{n}{2\pi\sqrt{S_x^2 S_y^2 - S_{xy}^2}}\right]^n e^{-n}. \tag{2.26}$$

After substituting (2.18) and (2.26) into the equation (2.8), this leads to

$$\lambda = \left(\frac{4\sqrt{(1-\rho_0^2)(S_x^2 S_y^2 - S_{xy}^2)}}{n(1+\rho_0)(\bar{x}-\bar{y})^2 + 2\{S_x^2 + S_y^2 - 2\rho_0 S_{xy}\}}\right)^n. \tag{2.27}$$

Since any constant can be ignored, thus we decide as follows:

$$\text{Reject } H_0 \text{ if } \lambda_1^* = \frac{\sqrt{S_x^2 S_y^2 - S_{xy}^2}}{n(1+\rho_0)(\bar{x}-\bar{y})^2 + 2\{S_x^2 + S_y^2 - 2\rho_0 S_{xy}\}} < d_1, \tag{2.28}$$

where d_1 is a generic constant.

Second test procedure, following Lin *et al.* (2002), the sample counterpart of concordance correlation coefficient is given as

$$r_c = \frac{2rs_x s_y}{s_x^2 + s_y^2 + (\bar{y}-\bar{x})^2}, \tag{2.29}$$

where $\bar{x}, \bar{y}, s_x^2, s_y^2$, and s_{xy} represent the usual sample based on n paired observations on (x, y) and each with divisor n.

This means that s_x^2 and s_y^2 can be put in the form

$$s_x^2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{S_x^2}{n}$$

$$\text{and} \quad s_y^2 = \frac{\sum(y_i - \bar{y})^2}{n} = \frac{S_y^2}{n}. \tag{2.30}$$

Under H_0, we then substitute (2.30) in (2.29), so the expression takes the form

$$\lambda_2^* = \frac{2\rho_0 S_x S_y}{S_x^2 + S_y^2 + n(\bar{y}-\bar{x})^2}. \tag{2.31}$$

That is,

$$\text{Reject } H_0 \text{ if } \lambda_2^* = \frac{2\rho_0 S_x S_y}{S_x^2 + S_y^2 + n(\bar{y}-\bar{x})^2} < d_2, \tag{2.32}$$

where d_2 is a generic constant.

Third test procedure, Lin, L. (2002) showed that

$$W = \ln e^2 \sim N(\omega = \ln \varepsilon^2, \sigma_W^2) \tag{2.33}$$

$$\text{where} \quad \varepsilon^2 = (\mu_y - \mu_x)^2 + \sigma_y^2 + \sigma_x^2 - 2\sigma_{yx}, \tag{2.34}$$

$$e^2 = (\bar{y}-\bar{x})^2 + s_y^2 + s_x^2 - 2s_{yx}, \tag{2.35}$$

$$\text{and} \quad \sigma_W^2 = \frac{2[1 - (\mu_y - \mu_x)^4 / e^4]}{n-2}. \tag{2.36}$$

Under H_0, it follows that

$$\frac{\sqrt{n-2}}{\sqrt{2}}\left\{\ln\left[(\bar{y}-\bar{x})^2+s_y^2+s_x^2-2s_{xy}\right]-\ln\left[2\sigma_x^2(1-\rho)\right]\right\}\sim N(0,1). \qquad (2.37)$$

Once again, when (2.37) are substituted by (2.30) and any constant can be ignored, so we decide as follows:

$$\text{Reject } H_0 \text{ if } \lambda_3^* = \frac{n(\bar{y}-\bar{x})^2 - 2\rho_0 S_x S_y}{S_x^2+S_y^2} > d_3, \qquad (2.38)$$

where d_3 is a generic constant.

Fourth test procedure, we use the statistic

$$-2\ln P_1 - 2\ln P_2 - 2\ln P_3 \qquad (2.39)$$

where

$$d_i = x_i - y_i;\ i=1,\ldots,n, \qquad (2.40)$$

$$s_d^2 = \frac{\sum(d_i-\bar{d})^2}{(n-1)}, \qquad (2.41)$$

$$\bar{d} = \frac{\sum d_i}{n}, \qquad (2.42)$$

$$u_i = x_i + y_i;\ i=1,\ldots,n, \qquad (2.43)$$

$$v_i = x_i - y_i;\ i=1,\ldots,n, \qquad (2.44)$$

$$r_{uv} = \frac{\sum(u_i-\bar{u})(v_i-\bar{v})}{\sqrt{\sum(u_i-\bar{u})^2\sum(v_i-\bar{v})^2}}, \qquad (2.45)$$

$$r = r_{xy} = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2\sum(y_i-\bar{y})^2}}, \qquad (2.46)$$

$$P_1 = \Pr\left[|t_{n-1}| > \frac{\sqrt{n}|\bar{d}|}{s_d}\right], \qquad (2.47)$$

$$P_2 = \Pr\left[|t_{n-2}| > \frac{\sqrt{n-2}|r_{uv}|}{\sqrt{1-r_{uv}^2}}\right], \qquad (2.48)$$

and $$P_3 = \Pr\left[|N(0,1)| > \sqrt{n-3}\left|\left\{\frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) - \frac{1}{2}\ln\left(\frac{1+\rho_0}{1-\rho_0}\right)\right\}\right|\right]. \tag{2.49}$$

Thus the implement able form of the level α test will look tike this:

$$\text{Reject } H_0 \text{ if } \lambda_4^* = -2\ln P_1 - 2\ln P_2 - 2\ln P_3 > \chi^2_{\alpha;6}. \tag{2.50}$$

We may summarize the test procedures for our testing problem (2.2) as follows:

Test Procedure 1: Reject H_0 if $\lambda_1^* = \dfrac{\sqrt{S_x^2 S_y^2 - S_{xy}^2}}{n(1+\rho_0)(\bar{x}-\bar{y})^2 + 2\{S_x^2 + S_y^2 - 2\rho_0 S_{xy}\}} < d_1$.

Test Procedure 2: Reject H_0 if $\lambda_2^* = \dfrac{2\rho_0 S_x S_y}{S_x^2 + S_y^2 + n(\bar{y}-\bar{x})^2} < d_2$.

Test Procedure 3: Reject H_0 if $\lambda_3^* = \dfrac{n(\bar{y}-\bar{x})^2 - 2\rho_0 S_x S_y}{S_x^2 + S_y^2} > d_3$.

Test Procedure 4: Reject H_0 if $\lambda_4^* = -2\ln P_1 - 2\ln P_2 - 2\ln P_3 > \chi^2_{\alpha;6}$.

Computation of the Cut-off Points

In this section we compute d_1, d_2, and d_3 for test procedure 1, 2, and 3 respectively. This is done below by first expressing λ_1^*, λ_2^*, and λ_3^* in term of V_1, V_2, and V_3.

Define $S_x^2 \sim \sigma_x^2 V_1$, $S_y^2 \sim \sigma_x^2 V_2$, $S_{xy} \sim \rho_0 \sigma_x^2 V_3$, and $\dfrac{n(\bar{x}-\bar{y})^2}{2\sigma_x^2(1-\rho_0)} \sim \chi^2_{1d.f.}$ (3.1)

Recall that

$$\lambda_1^* = \frac{\sqrt{S_x^2 S_y^2 - S_{xy}^2}}{n(1+\rho_0)(\bar{x}-\bar{y})^2 + 2\{S_x^2 + S_y^2 - 2\rho_0 S_{xy}\}}, \tag{3.2}$$

$$\lambda_2^* = \frac{2rs_x s_y}{s_x^2 + s_y^2 + (\bar{y}-\bar{x})^2}, \tag{3.3}$$

$$\lambda_3^* = \frac{(\bar{y}-\bar{x})^2 - 2s_{xy}}{s_x^2 + s_y^2}, \tag{3.4}$$

and $$\lambda_4^* = -2\ln P_1 - 2\ln P_2 - 2\ln P_3. \tag{3.5}$$

We then substitute (3.1) in (3.2), (3.3), and (3.4), so it take the form

$$\lambda_1^* = \frac{\sqrt{V_1V_2 - \rho_0^2 V_3^2}}{2\{(1-\rho_0^2)\chi_1^2 + \{V_1 + V_2 - 2\rho_0^2 V_3\}\}}, \tag{3.6}$$

$$\lambda_2^* = \frac{2\rho_0\sqrt{V_1V_2}}{V_1 + V_2 + 2(1-\rho_0)\chi_1^2}, \tag{3.7}$$

and $$\lambda_3^* = \frac{2\{(1-\rho_0)\chi_1^2 - \rho_0\sqrt{V_1V_2}\}}{V_1 + V_2}. \tag{3.8}$$

We now use the following steps to compute d_1, d_2, and d_3.

Step 1: Generate the standard normal distribution $N(0,1)$ for computing the chi-square distribution with 1 degree of freedom $\chi_1^2 = [N(0,1)]^2$.

Step 2: Simulate $n-1$ independent paired samples of bivariate normal distribution subject to

$$\begin{pmatrix} u_i \\ w_i \end{pmatrix} \sim N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}\right]; \text{ for } i = 1,\ldots,n-1.$$

Step 3: Evaluate V_1, V_2, and V_3 by using the formulae:

$$V_1 = u_1^2 + \cdots + u_{n-1}^2,$$

$$V_2 = w_1^2 + \cdots + w_{n-1}^2, \tag{3.9}$$

and $$V_3 = \sum_{i=1}^{n-1} u_i w_i.$$

Step 4: Calculate λ_1^*, λ_2^*, and λ_3^* from equations (3.6), (3.7), and (3.8) respectively.

Step 5: Repeat step 1 through 4 for 1000 times by using $\rho_0 = 0.7, 0.8, 0.9$ and $n = 5, 10, 15, 20$.

Step 6: Find each value of d_1, d_2, and d_3 for the level $\alpha = 1\%, 5\%$.

All cut-off points are shown in Table 3.1.

Table 3.1 Cut-off points for $n = 5, 10, 15, 20$, $\rho_0 = 0.7, 0.8, 0.9$, and $\alpha = 1\%, 5\%$.

α	ρ_0	n	d_1	d_2	d_3
0.01	0.7	5	0.08357	0.24412	0.32555
		10	0.19761	0.47031	-0.34353
		15	0.31165	0.69650	-0.70036
		20	0.41133	0.72618	-0.83907
	0.8	5	0.09788	0.33229	0.02564
		10	0.23447	0.61392	-0.53683
		15	0.37106	0.79665	-0.80141
		20	0.49238	0.82992	-0.84482
	0.9	5	0.13314	0.49368	-0.33277
		10	0.32284	0.77020	-0.75790
		15	0.51254	0.89550	-0.91159
		20	0.68581	0.93365	-0.95043
0.05	0.7	5	0.14472	0.38018	-0.10107
		10	0.23358	0.55080	-0.49281
		15	0.32245	0.71140	-0.73922
		20	0.42569	0.74171	-0.88563
	0.8	5	0.16970	0.50348	-0.33914
		10	0.27726	0.67172	-0.64366
		15	0.38482	0.81303	-0.84584
		20	0.50765	0.84767	-0.89166
	0.9	5	0.23413	0.67620	-0.62293
		10	0.38469	0.82106	-0.81179
		15	0.53525	0.91466	-0.95157
		20	0.70224	0.95363	-0.99211

Power Computation

In this section we also address the computation of the power which corresponding to our 4 test procedures. We can brief the power of test as follows:

First Test: Power $= \Pr[T_1 < d_1]$ (4.1)

$$\text{where } T_1 = \frac{\sqrt{S_x^2 S_y^2 - S_{xy}^2}}{n(1+\rho_0)(\bar{x}-\bar{y})^2 + 2\{S_x^2 + S_y^2 - 2\rho_0 S_{xy}\}}. \quad (4.2)$$

Second Test: Power $= \Pr[T_2 < d_2]$ (4.3)

$$\text{where } T_2 = \frac{2\rho_0 S_x S_y}{S_x^2 + S_y^2 + n(\bar{y}-\bar{x})^2}. \quad (4.4)$$

Third Test: Power $= \Pr[T_3 > d_3]$ (4.5)

$$\text{where } T_3 = \frac{n(\bar{y}-\bar{x})^2 - 2\rho_0 S_x S_y}{S_x^2 + S_y^2}. \tag{4.6}$$

Fourth Test: Power $= \Pr[T_4 > \chi^2_{\alpha;6}]$ (4.7)

$$\text{where } T_4 = -2\ln P_1 - 2\ln P_2 - 2\ln P_3. \tag{4.8}$$

To compute power of the test for test procedure 1-4 we use the following steps.

Step 1: Simulate n independent paired samples of bivariate normal distribution subject to $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim N_2\left[\begin{pmatrix} 0 \\ \delta_1 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1\sqrt{\delta_2} \\ \rho_1\sqrt{\delta_2} & \delta_2 \end{pmatrix}\right]$; for $i = 1,\ldots,n$, under the alternative hypothesis H_1.

Step 2: Calculate $\bar{x}, \bar{y}, S_x^2, S_y^2$ and S_{xy} by using equation (2.5) for test procedure 1-3. Compute d_i, s_d^2, $\bar{d}$, u_i, v_i, r_{uv}, r, P_1, P_2, and P_3 for test procedure 4 from equations (2.40)-(2.49) respectively.

Step 3: Evaluate the values of T_1, T_2, T_3 and T_4 by applying equations (4.2), (4.4), (4.6) and (4.8) respectively.

Step 4: Repeat step 1 through 3 for 1000 times by using $\delta_1 = 0.5, 1, 1.5, 2$; δ_2 (>1 say) $= 1.5, 2, 2.5$ and δ_2 (<1 say) $= 0.5, 0.7, 0.9$; ρ_1 (for $\rho_0 = 0.7$) $= 0.6, 0.65, 0.75, 0.8$, ρ_1 (for $\rho_0 = 0.8$) $= 0.6, 0.7, 0.85, 0.9$ and ρ_1 (for $\rho_0 = 0.9$) $= 0.7, 0.8, 0.95$.

Step 6: Compute the power of T_1, T_2, T_3, and T_4 as follows.

$$\text{Power of } T_1 = \frac{\text{The number of data with value of } T_1 \text{ less than } d_1}{1000}. \tag{4.9}$$

$$\text{Power of } T_2 = \frac{\text{The number of data with value of } T_2 \text{ less than } d_2}{1000}. \tag{4.10}$$

$$\text{Power of } T_3 = \frac{\text{The number of data with value of } T_3 \text{ more than } d_3}{1000}. \tag{4.11}$$

$$\text{Power of } T_4 = \frac{\text{The number of data with value of } T_4 \text{ more than } \chi^2_{\alpha;6}}{1000}. \tag{4.11}$$

Derivation of Test Procedures for Large Sample Size

In this section we discuss the testing problem (2.2) for large values of n. We are now in a position to derive the expressions of large sample means and variances of the values λ_1^*, λ_2^* and λ_3^* by applying large sample theory. This is done below by first expressing these estimates in term of $\underline{T} = (T_1, T_2, T_3, T_4, T_5)$.

Define $T_1 = \bar{x}$, $T_2 = \bar{y}$, $T_3 = \dfrac{S_x^2}{n-1} = \dfrac{\sum(x_i - \bar{x})}{n-1}$, $T_4 = \dfrac{S_y^2}{n-1} = \dfrac{\sum(y_i - \bar{y})}{n-1}$,

and $T_5 = \dfrac{S_{xy}}{n-1} = \dfrac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n-1}$. (5.1)

We then substitute (5.1) in (3.2), (3.3), and (3.4), and simplify the terms considerably for large n, so it take the form

$$\lambda_1^* \approx \frac{\sqrt{T_3T_4 - T_5^2}}{(1+\rho_0)(T_1 - T_2)^2 + 2\{T_3 + T_4 - 2\rho_0 T_5\}}, \tag{5.2}$$

$$\lambda_2^* \approx \frac{2\rho_0\sqrt{T_3T_4}}{T_3 + T_4 + (T_2 - T_1)^2}, \tag{5.3}$$

and
$$\lambda_3^* \approx \frac{(T_2 - T_1)^2 - 2\rho_0\sqrt{T_3T_4}}{T_3 + T_4}. \tag{5.4}$$

By using the general result for the mean and variance, see Appendix II, then the mean and variance of the estimates λ_1^*, λ_2^* and λ_3^* are as follows.

$$E(\lambda_1^*) \approx \frac{1}{4\sqrt{1-\rho_0^2}} - \frac{3}{8n\sqrt{1-\rho_0^2}} \approx \frac{1}{4\sqrt{1-\rho_0^2}} + O\left(\frac{1}{n}\right), \tag{5.5}$$

$$V(\lambda_1^*) \approx \frac{3}{32n^2(1-\rho_0^2)} \approx O\left(\frac{1}{n^2}\right). \tag{5.6}$$

$$E(\lambda_2^*) \approx \rho_0 + \frac{\rho_0(\rho_0^2 + 2\rho_0 - 3)}{2n} \approx \rho_0 + O\left(\frac{1}{n}\right), \tag{5.7}$$

$$V(\lambda_2^*) \approx \frac{\rho_0^2(\rho_0^4 + 2\rho_0^2 - 8\rho_0 + 5)}{2n^2} \approx O\left(\frac{1}{n^2}\right). \tag{5.8}$$

$$E(\lambda_3^*) \approx -\rho_0 + \frac{2 - \rho_0 - \rho_0^3}{2n} \approx -\rho_0 + O\left(\frac{1}{n}\right), \tag{5.9}$$

$$V(\lambda_3^*) \approx \frac{\rho_0^6 - 2\rho_0^4 + 5\rho_0^2 - 8\rho_0 + 4}{2n^2} \approx O\left(\frac{1}{n^2}\right). \tag{5.10}$$

When n is large, we can summarize the test procedures for our testing problem (2.2) as follows:

Test Procedure 1: Reject H_0 if $\lambda_1^* < d_1 \Leftrightarrow \dfrac{\lambda_1^* - E(\lambda_1^*)}{\sqrt{\hat{Var}(\lambda_1^*)}} < \dfrac{d_1 - E(\lambda_1^*)}{\sqrt{\hat{Var}(\lambda_1^*)}}$. (5.11)

Test Procedure 2: Reject H_0 if $\lambda_2^* < d_2 \Leftrightarrow \dfrac{\lambda_2^* - E(\lambda_2^*)}{\sqrt{\hat{Var}(\lambda_2^*)}} < \dfrac{d_2 - E(\lambda_2^*)}{\sqrt{\hat{Var}(\lambda_2^*)}}$. (5.12)

Test Procedure 3: Reject H_0 if $\lambda_3^* > d_3 \Leftrightarrow \dfrac{\lambda_3^* - E(\lambda_3^*)}{\sqrt{\hat{Var}(\lambda_3^*)}} > \dfrac{d_3 - E(\lambda_3^*)}{\sqrt{\hat{Var}(\lambda_3^*)}}$. (5.13)

Test Procedure 4: Reject H_0 if $\lambda_4^* = -2\ln P_1 - 2\ln P_2 - 2\ln P_3 > \chi^2_{\alpha;6}$ (5.14)

where $d_i = x_i - y_i; i = 1,\ldots,n,$ (5.15)

$$s_d^2 = \frac{\sum (d_i - \bar{d})^2}{(n-1)}, \tag{5.16}$$

$$\bar{d} = \frac{\sum d_i}{n}, \tag{5.17}$$

$$u_i = x_i + y_i;\ i = 1,\ldots,n, \tag{5.18}$$

$$v_i = x_i - y_i;\ i = 1,\ldots,n, \tag{5.19}$$

$$r_{uv} = \frac{\sum (u_i - \bar{u})(v_i - \bar{v})}{\sqrt{\sum (u_i - \bar{u})^2 \sum (v_i - \bar{v})^2}}, \tag{5.20}$$

$$r = r_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}, \tag{5.21}$$

$$P_1 = \Pr\left[|N(0,1)| > \frac{\sqrt{n}|\bar{d}|}{s_d}\right], \tag{5.22}$$

$$P_2 = \Pr\left[|N(0,1)| > \frac{\sqrt{n-2}|r_{uv}|}{\sqrt{1 - r_{uv}^2}}\right], \tag{5.23}$$

and $$P_3 = \Pr\left[|N(0,1)| > \sqrt{n-3}\left|\left\{\frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) - \frac{1}{2}\ln\left(\frac{1+\rho_0}{1-\rho_0}\right)\right\}\right|\right]. \quad (5.24)$$

Numerical results

Here we provide the numerical results of the analysis for our testing problem (2.2) by using two data sets as follows: 1) data set for small sample size 2) data set for large sample size.

Analysis of data set for small sample size

First data set, refers to Yu *et al.* (2002), is the paired observations (x_i, y_i) representing field and lab data on RVP for 15 locations as given in Table 6.1.1. This problem is motivated by the following practical issue in the context of the attempt by the Environmental Protection Agency (EPA) of the United States to evaluate the gasoline quality based on what is known as Reid Vapor Pressure (RVP). Occasionally, an EPA inspector would visit gas pumps in a city, take samples of gasoline of a particular brand, and measure RVP right at the spot which produces cheap and quick measurements. Once in a while, the inspector after measuring RVP at the spot will also ship a gasoline sample to a laboratory for a measurement of presumably higher precision at a higher cost, thus getting the pair (field, lab). Since usually laboratory measurements (Y) are much more expensive than field measurements (X) because of special packaging to be used to ship a gasoline sample from a field to a laboratory, not all the gasoline samples will be shipped to the laboratory and hence the resulting data would consist of many field measurements with occasional paired measurements obtained from both the field and laboratory. Our statistical analysis here is based on only the paired data. The scenario is such that the means are equal, but the variances are different. In addition, the summary statistics for first data set are shown in Table 6.1.2.

Table 6.1.1 Field and lab data on RVP for new reformulated gasoline

X	8.03	8.64	9.14	7.86	8.70	9.28	7.86	7.83	8.60	7.83	7.88	8.56	7.83	7.99	7.56
Y	8 28	8.63	9.28	7.85	8.62	9.14	7.86	7.90	8.52	7.92	7.89	8.48	7.95	8.32	7.60

Table 6.1.2 Summary statistics for first data set

n	$\bar{x}$	$\bar{y}$	s_x^2	s_y^2	S_x^2	S_y^2	S_{xy}	r	$\bar{d}$	s_d^2	$\bar{u}$	r_{xy}
15	8.2393	8.2827	0.2848	0.2454	3.9868	3.4355	3.5957	0.9716	-0.0433	0.0165	16.5220	0.3002

For small values of n, we apply our test procedures to analyze this data set for testing (2.2). By replacing the statistics in Table 6.1.2 into equations (3.2)-(3.4) when $\rho_0 = 0.7, 0.8, 0.9,$ we then obtain the values of λ_1^*, λ_2^* and λ_3^* respectively. Once again, we substitute the statistics in Table 6.1.2 into equations (3.47)-(3.49), we then get the values of P_1, P_2, and P_3 for $\rho_0 = 0.7, 0.8, 0.9$. Next, we substitute these values in (3.5) to obtain λ_4^* as shown in Table 6.1.3.

Table 6.1.3 Values of the estimates $\lambda_1^*, \lambda_2^*, \lambda_3^*$ and λ_4^* for $\rho_0 = 0.7, 0.8, 0.9$

ρ_0	λ_1^*	λ_2^*	λ_3^*	λ_4^*
0.7	0.18159	0.69542	-0.69426	27.96632
0.8	0.25850	0.79476	-0.79398	21.28919
0.9	0.44843	0.89411	-0.89371	13.04986

It follows from Table 2.1 that $\lambda_1^* < d_1$, $\lambda_2^* < d_2$, and $\lambda_3^* < d_3$ for all $\rho_0 = 0.7, 0.8, 0.9$.

Since $\chi^2_{0.01,6} = 16.81187$ and $\chi^2_{0.05,6} = 12.59158$, we then have $\lambda_4^* < \chi^2_{\alpha;6}$; $\alpha = 0.01, 0.05$ for all $\rho_0 = 0.7, 0.8, 0.9$.

Hence, based on $\rho_0 = 0.7, 0.8, 0.9$, we reject H_0 for all test procedures 1-4 at significance level $\alpha = 0.01, 0.05$, that is, $\mu_x \neq \mu_y$ or $\sigma_x \neq \sigma_y$ or $\rho \neq \rho_0$.

Analysis of data set for large sample size

Following Lin *et al.* (2002), second data set deals with Diaspirin crosslinked hemoglobin (DCLHb) of 299 individuals, measured in two different ways, resulting in paired data $\{(x_i, y_i)\}$ as displayed in Table 6.2.1. DCLHb which is solution containing oxygen-carrying hemoglobin, was created as a blood substitute to treat acute trauma patients and to replace blood loss during surgery. Measurements of DCLHb in patient's serum after infusion are routinely performed using a Sigma instrument. A method of measuring hemoglobin called the HemoCue photometer was modified to reproduce the Sigma instrument DCLHb results. To validate this modified method, serum samples from 299 patients over the analytical range of 50-2000 mg/dL were collected. DCLHb values of each sample were measured simultaneously with the HemoCue and Sigma methods. The problem here is to assess 'agreement' between the two methods in terms of their means, assuming that the variances are equal. Here we apply the results of Model II along with the use of large sample theory. Furthermore, the summary statistics for second data set are shown in Table 6.2.2.

For large values of n, we also apply our test procedures to analyze the second data set for testing (2.2). By replacing $\rho_0 = 0.7, 0.8, 0.9$ into equations (5.5)-(5.10), we then get that

$$\begin{aligned} &\text{for } \rho_0 = 0.7: E(\lambda_1^*) \to 0.3501,\ E(\lambda_2^*) \to 0.7,\ E(\lambda_3^*) \to -0.7, \\ &\text{for } \rho_0 = 0.8: E(\lambda_1^*) \to 0.4167,\ E(\lambda_2^*) \to 0.8,\ E(\lambda_3^*) \to -0.8, \\ &\text{for } \rho_0 = 0.9: E(\lambda_1^*) \to 0.5735,\ E(\lambda_2^*) \to 0.9,\ E(\lambda_3^*) \to -0.9, \end{aligned} \qquad (6.2.1)$$

and the variances of λ_1^*, λ_2^* and λ_3^* tend to zero for all $\rho_0 = 0.7, 0.8, 0.9$.

The values of T_1, T_2, T_3, T_4, and T_5 are estimated by substituting the statistics in Table 6.2.2 into (5.1). Next we replace these values into equations (5.2)-(5.4) when $\rho_0 = 0.7, 0.8, 0.9$, then the values of the estimates λ_1^*, λ_2^* and λ_3^* are obtained as shown in Table 6.2.3.

Table 6.2.1 Data on DCLHb of 299 patients, measured by HemoCue method (X) and its modification (Y)

X	Y	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
1340	1330	100	100	180	180	50	50	50	50	320	310	270	270
100	100	1610	1600	50	50	60	60	1170	1180	1430	1430	80	80
80	80	150	140	70	70	1610	1600	330	330	90	90	70	70
1340	1340	1380	1380	1550	1560	200	200	60	60	210	220	210	200
530	550	520	510	80	80	90	90	90	90	1770	1770	50	50
240	240	1430	1420	90	90	160	160	310	310	600	600	1200	1220
60	60	330	330	70	70	50	50	90	90	360	350	1070	1080
70	80	90	90	1560	1560	1040	1040	360	360	70	70	400	400
50	50	90	90	710	720	1250	1250	1120	1120	130	130	120	120
1720	1770	1920	1950	190	190	150	150	1200	1210	680	680	80	90
460	460	70	60	100	100	1530	1530	210	210	110	110	790	790
130	140	880	990	720	720	440	440	60	60	360	360	680	680
50	50	1490	1490	710	710	1680	1740	1450	1450	70	70	80	80
60	60	330	320	80	80	500	500	100	100	70	70	850	850
270	270	1440	1420	670	670	100	100	360	360	60	60	340	340
80	80	1150	1130	270	280	80	70	950	940	130	130	450	450
910	910	810	820	80	70	50	60	80	80	420	420	130	130
650	700	400	400	1380	1380	1180	1230	1120	1170	340	340	850	850
90	80	110	120	670	670	80	70	450	460	230	220	240	240
900	890	1510	1520	440	430	1180	1160	150	150	400	400	430	430
130	140	1030	1010	140	140	460	460	740	750	140	130	50	50
80	80	370	370	90	80	190	190	880	880	390	380	50	50
1840	1830	150	150	1510	1380	1150	1150	540	540	50	70	360	360
430	430	800	800	860	870	530	520	230	230	570	560	400	400
80	90	580	580	90	90	450	450	1180	1180	530	540	110	110
90	100	220	240	160	160	200	200	1500	1510	410	420	980	980
160	160	90	90	230	240	1380	1300	760	760	60	60	200	200
500	500	530	530	520	510	320	320	130	130	460	460	50	50
60	60	470	470	1000	1010	170	180	1060	1080	50	50	480	480
200	200	200	210	450	450	1250	1250	770	770	330	340	100	100
130	130	100	100	300	270	580	580	360	380	690	690	160	160
110	110	1080	1080	150	150	430	430	450	460	120	120	80	80
90	90	50	50	280	280	180	180	50	50	90	90	760	760
910	910	570	570	80	80	1040	1040	230	230	790	790	590	590
770	770	360	370	820	790	420	420	460	460	570	520	700	700
800	810	240	240	510	520	870	870	360	370	130	140	130	130
50	50	50	60	130	130	1260	1230	640	640	1430	1400	230	230
620	610	1660	1740	1610	1610	840	840	110	110	420	420	50	50
210	210	50	60	70	70	320	320	1590	1590	670	680	500	500
190	190	1190	1150	50	50	70	70	500	510	80	80	500	500
90	90	70	80	1420	1440	70	60	810	810	300	300	60	60
140	140	1640	1660	330	330	1410	1440	80	90	210	220		
110	110	880	820	90	90	450	450	580	580	130	120		

Table 6.2.2 Summary statistics for second data set

n	$\bar{x}$	$\bar{y}$	s_x^2	s_y^2	S_x^2	S_y^2	S_{xy}
299	489.3311	490.2676	225060.6249	225669.0557	67068066.22	67249378.6	67110653.51

r	$\bar{d}$	s_d^2	$\bar{u}$	r_{uv}
0.9993	-0.9365	322.6100	979.5987	-0.0357

Table 6.2.3 Values of the estimates λ_1^*, λ_2^*, and λ_3^* for $\rho_0 = 0.7, 0.8, 0.9$

ρ_0	λ_1^*	λ_2^*	λ_3^*
0.7	0.0315	0.6999	-0.6999
0.8	0.0471	0.7999	-0.7999
0.9	0.0939	0.8999	-0.8999

Again, for test procedure 4 we substitute the statistics in Table 6.2.2 into equations (5.22)-(5.24), we then get the values of P_1, P_2, and P_3 for $\rho_0 = 0.7, 0.8, 0.9$. Next, we substitute these values in $-2\ln P_1 - 2\ln P_2 - 2\ln P_3$, this lead to $\lambda_4^* \to \infty$ for all $\rho_0 = 0.7, 0.8, 0.9$.

By replacing the statistics in Table 6.2.3, (6.2.1) and $\lambda_4^* \to \infty$ into equations (5.11)-(5.14), then based on $\rho_0 = 0.7, 0.8, 0.9$, we reject H_0 for all test procedures 1-4 at significance level $\alpha = 0.01, 0.05$, that is, $\mu_x \neq \mu_y$ or $\sigma_x \neq \sigma_y$ or $\rho \neq \rho_0$.

It should be noted that the conclusion for four test procedures are the same when we apply with real data for small sample size and large sample size based on a bivariate normal population. This can also concentrate on the other meaningful bivariate distributions such as bivariate exponential, bivariate lognormal, bivariate gamma, etc. We propose to undertake this study in the future.

5) Statistical Aspects of Social Network

The analytical and statistical study of social networks is relatively new in the area of social sciences. In the context of social sciences, a network is referred to as a social network. The social network perspective focuses on social relationships or ties (links) among *actors* in the society. The actors or units of a social network can be different according to the relationship under consideration such as individuals, families, households, villages or towns, regions, rural or urban areas. Social relationships can be involuntary like ties of kinship or voluntarily established in courses of interaction, mutual consent etc. Illustrative examples include friendship providing help and support at a time of crisis, flow of goods, traffic etc, choice of individuals to spend leisure time together, preference in marriage and the like. Incidentally a relationship can be negative also.

In a social network we are interested in a dyadic relation, which involves a pair of actors and the (possible) tie(s) between them. Dyadic analyses focus on the properties of pair-wise relationships, such as whether a specific dyadic relationship exists between the members of any pair of them or not, or whether it flows in both directions or only in one direction because social relationships are not necessarily symmetric. Asymmetric relations are also not uncommon in societal contexts. Social network analyses call for special statistical techniques as the former deals with dyads and sometimes, dyadic relations may exist only through a large number of intermediaries.

Some aspects of social networks including models for social behavior have been studied by several authors such as Goodman (1979), Wasserman (1980), Holland and Leinhardt (1981), Wasserman *et al.* (1985), Rao and Bandyopadhyay (1987), Iacobucci and Wasserman (1988), Iacobucci (1989), Freeman (1992), and Sinha (1997).

The main objectives of this study are to provide a description of dyadic models and take up their applications in the context of a real survey data. We also propose models of the dyadic relational network for two-way crossed classifications and apply these models to the same survey data.

Theoretical Background of Dyadic Models

As usual, we start with N individuals forming a network based on a certain meaningful "criterion" exhibiting some form of social choice between any pair of individuals. The possibilities of choice involving the individuals (i) and (j) are:

$$(I)\ (0,0);\quad (II)\ (1,0);\quad (III)\ (0,1);\quad (IV)\ (1,1)$$

with obvious interpretations in terms of "go" or "no go". The body of data arising from a network in the form of the incidence matrix of order $N \times N$ is referred to as X-data. In general, X-data is not symmetric. For modelling purpose, we convert X-data into what is called the "Y-array". It is a symmetric matrix of order $2N \times 2N$ and it is based on classification of each individual's choice of 0 and 1 with reference to every other individual. Thus, for example, whereas X_{ij} assumes the values 0 or 1, we convert it into $Y_{(r,s);(i,j)}$ which is a 2×2 matrix with the two rows designated as $r = 0,1$ and also the two columns designated as $s = 0,1$ and the entries of this matrix are composed of three 0's and only one 1, depending on the nature of X_{ij} and X_{ji}. Thus, for example, if $X_{ij} = 0 = X_{ji}$, then there is no arc in between i and j in any direction and so consequently, $r = s = 0$. Therefore, the "cell" $(0,0)$ in the expression for $Y_{(r,s);(i,j)}$ will receive the value 1 and the rest will assume value 0.

The contemplated model for Y-array dwells on specific expressions for the multinomial cell probabilities corresponding to the four cells designated by (r,s) for every pair of individuals (i,j). Denoting by $P[(r,s);(i,j)]$ the probability that the cell (r,s) receives the value 1 and the rest assume the value 0 each, the model, simply stated in terms of $LogP$ is explicitly given by:

$$LogP[(0,0);(i,j)] = \lambda_{ij}; \quad (1)$$

$$LogP[(1,0);(i,j)] = \lambda_{ij} + \theta + \alpha_i + \beta_j; \quad (2)$$

$$LogP[(0,1);(i,j)] = \lambda_{ij} + \theta + \alpha_j + \beta_i; \quad (3)$$

$$LogP[(1,1);(i,j)] = \lambda_{ij} + 2\theta + \alpha_i + \beta_j + \alpha_j + \beta_i + (\alpha\beta). \quad (4)$$

In the above expressions, for every pair $(i,j), \lambda_{ij}$ is a normalizing constant since the four probability expressions add up to unity. This gives

$$\lambda_{ij} = -ln[1 + e^{\theta+\alpha_i+\beta_j} + e^{\theta+\alpha_j+\beta_i} + e^{2\theta+(\alpha_i+\beta_j)+(\alpha_j+\beta_i)+(\alpha\beta)}]. \quad (5)$$

The other parameters are interpreted as follows :

(a) θ represents the "movement" parameter and for each move [by either i or j or by both], its presence in the model expression for probability is thus incorporated;

(b) α [respectively, β] represents the "outdegree" aspect [respectively, the "indegree" aspect] of an individual and it is individual-specific in general terms;

(c) $(\alpha\beta)$ represents the "reciprocity" aspect of the pair of individuals concerned and it is regarded as a global phenomenon [so that its dependence on specific individuals' features is ruled out].

More general models are discussed in Wasserman and Faust (1999).

Note that while the indegree and/or outdegree aspects are assumed to be individual-specific, the reciprocity aspect is taken to be a "global" phenomenon and not pair-wise individuals-specific. The same is true of θ as a parameter indicating the "global" aspect of movement of individuals in the network. It receives weight 2 for the last case when both the individuals move to each other. We must also note that there are altogether $N(N-1)$ ordered pairs of individuals indexed by (i,j) in terms of X-data and in terms of Y-array, we have unordered pairs of the type (i,j). Further to this, in the model described above, there are altogether $1+N+N+1=2(N+1)$ parameters. As a matter of convention, it is assumed that $\sum\alpha_i=\sum\beta_i=0$ so that, in effect, there are $2N$ parameters to be estimated. For a given pair of individuals, the likelihood function is readily expressible in terms of the multinomial [4-cell] probabilities, using the indicator functions. Hence the joint log-likelihood function for all such unordered pairs of individuals can be written down in a routine manner. The central problem towards finding maximum likelihood estimates of the parameters lies in the dependence of the λ_{ij}'s on the others viz., $\theta,\alpha_i,\alpha_j,\beta_i,\beta_j,(\alpha\beta)$. Therefore, the likelihood equations are not analytically tractable and one has to take recourse to statistical computing.

Having described a general model as above, we can now restrict to some simplified models which are motivated by considerations of possible "grouping" among the individuals. In a sociological context, these groupings may be prompted by, for example, family size, occupation, caste, kinship and the like. If we consider one such "external factor", then the individuals may be classified into several disjoint categories and the category-specific out-degree and in-degree parameters may be more appropriate to use. That means we can dispense with individual-specific α_i and β_j parameters and replace them by the associated group representatives. Thus, for two groups or categories [like small/large family size], we may use α_I and α_{II} along with the induced constraint $N_I\alpha_I+N_{II}\alpha_{II}=0$. The same is true of the β's, the in-degree parameters. Of course, θ and $(\alpha\beta)$ remain unaltered. This leads to what has been referred to as W-array. It may be noted that under the W-array, the number of parameters reduces to $1+(C-1)+(C-1)+1=2C$ where C is the number of categories induced by the external factor.

For the W-array involving two groups, the model description is given by

$$LogP[(0,0);(i,j)]=\lambda_{I,I}; \quad (6)$$

$$LogP[(1,0);(i,j)]=\lambda_{I,I}+\theta+\alpha_I+\beta_I; \quad (7)$$

$$LogP[(0,1);(i,j)]=\lambda_{I,I}+\theta+\alpha_I+\beta_I; \quad (8)$$

$$LogP[(1,1);(i,j)] = \lambda_{I,I} + 2\theta + 2\alpha_I + 2\beta_I + (\alpha\beta). \qquad (9)$$

when the two individuals are *both* in Category I. Similar descriptions apply for the cases: Both are in Category II or one is in Category I while the other is in Category II.

$$LogP[(0,0);(i,j)] = \lambda_{I,II}; \qquad (10)$$

$$LogP[(1,0);(i,j)] = \lambda_{I,II} + \theta + \alpha_I + \beta_{II}; \qquad (11)$$

$$LogP[(0,1);(i,j)] = \lambda_{I,II} + \theta + \alpha_{II} + \beta_I; \qquad (12)$$

$$LogP[(1,1);(i,j)] = \lambda_{I,II} + 2\theta + \alpha_I + \beta_{II} + \alpha_{II} + \beta_I + (\alpha\beta), \qquad (13)$$

when " i " is in Category I while " j " is in Category II.

$$LogP[(0,0);(i,j)] = \lambda_{II,I}; \qquad (14)$$

$$LogP[(1,0);(i,j)] = \lambda_{II,I} + \theta + \alpha_{II} + \beta_I; \qquad (15)$$

$$LogP[(0,1);(i,j)] = \lambda_{II,I} + \theta + \alpha_I + \beta_{II}; \qquad (16)$$

$$LogP[(1,1);(i,j)] = \lambda_{II,I} + 2\theta + \alpha_{II} + \beta_I + \alpha_I + \beta_{II} + (\alpha\beta), \qquad (17)$$

when " i " is in Category II while " j " is in Category I.

$$LogP[(0,0);(i,j)] = \lambda_{II,II}; \qquad (18)$$

$$LogP[(1,0);(i,j)] = \lambda_{II,II} + \theta + \alpha_{II} + \beta_{II}; \qquad (19)$$

$$LogP[(0,1);(i,j)] = \lambda_{II,II} + \theta + \alpha_{II} + \beta_{II}; \qquad (20)$$

$$LogP[(1,1);(i,j)] = \lambda_{II,II} + 2\theta + 2\alpha_{II} + 2\beta_{II} + (\alpha\beta), \qquad (21)$$

when the two individuals are *both* in Category II.

Further, in the first and the last situations, the model description remains the same no matter which individual belongs to which Category (go or no go) vide (7)-(8); (19)-(20). Moreover, for this model involving two groups, there are altogether 6 parameters viz.,

$\theta, \alpha_I, \alpha_{II}, \beta_I, \beta_{II}, (\alpha\beta)$ and there are two defining relations viz., $N_I\alpha_I + N_{II}\alpha_{II} = 0 = N_I\beta_I + N_{II}\beta_{II}$. So effectively, we have $\theta, (\alpha\beta)$, $\alpha_1 (= \alpha_I)$, $\beta_1 (= \beta_I)$ only as the free parameters to be estimated.

There is yet another simplified version of the Y-array as well. For example, in situations where individual α's and β's are insignificant or sufficiently small compared to the other parameters, we can ignore them in the model description. This is referred to as V-array and the model with only two parameters reduces to

$$LogP[(0,0);(i,j)] = \lambda; \qquad (22)$$

$$LogP[(1,0);(i,j)] = \lambda + \theta; \qquad (23)$$

$$LogP[(0,1);(i,j)] = \lambda + \theta; \qquad (24)$$

$$LogP[(1,1);(i,j)] = \lambda + 2\theta + (\alpha\beta). \qquad (25)$$

For a mature reader we can provide further analytical results as follows. These are needed for the purpose of estimation of the model parameters. Normally, we use the maximum likelihood method of estimation of the parameters and hence we need to explicitly write down the joint likelihood function under different models. In this context it is important to know how the likelihood depends on the data, or rather, what are the *summary statistics* that are involved in the likelihood function. These summary statistics are referred to as *sufficient statistics*. For the Y-array, the summary statistics are the individual out-degrees and in-degrees i.e., d_i, e_i-values apart from s, the total number of reciprocal pairs. The likelihood involves $\sum d_i$ as well which is the same as $\sum e_i$. For the W-array, the sufficient statistics are $\sum_I d_i, \sum_{II} d_i, \sum_I e_i, \sum_{II} e_i$ and s. Finally, for the V-array, all we need are $\sum d_i$ [which is the same as $\sum e_i$] and s.

Rao and Bandyopadhyay (1987) discuss a very special case of this model.

Main Results

We will now elaborate on the nature of the solutions under the V-and W-arrays. These are relatively much easier to tackle unlike the Y-array which is computation-intensive. For the V-array, the log-likelihood is given by

$$LogL = \lambda.{}^{N}C_2 + \theta(\sum d_i) + (\alpha\beta)s \tag{26}$$

where λ is given by

$$\lambda = -ln[1 + 2e^{\theta} + e^{2\theta+(\alpha\beta)}]. \tag{27}$$

First note that

$$d\lambda/d\theta = -2e^{\theta}[1 + e^{\theta+(\alpha\beta)}]/[1 + 2e^{\theta} + e^{2\theta+(\alpha\beta)}] \tag{28}$$

and that

$$d\lambda/d(\alpha\beta) = -e^{2\theta+(\alpha\beta)}/[1 + 2e^{\theta} + e^{2\theta+(\alpha\beta)}]. \tag{29}$$

Next note that

$$dLogL/d\theta = {}^{N}C_2.d\lambda/d\theta + D; \tag{30}$$

$$dLogL/d(\alpha\beta) = {}^{N}C_2.d\lambda/d(\alpha\beta) + s. \tag{31}$$

Here $D = \sum d_i$. From the above, we can easily write down the two equations. Set $D* = [\frac{D}{2s} - 1]^{-1}$. It follows that

$$e^{\theta+(\alpha\beta)} = D*. \tag{32}$$

We now use this defining relation involving θ and $\alpha\beta$ in the first equation (30) and derive, upon simplification,

$$e^{\theta} = \frac{D-2s}{2\left[\binom{N}{2} - (D-s)\right]}, \tag{33}$$

whence we can solve for $\hat{\theta}$. Subsequently, we solve for $(\alpha\beta)$ from the defining relation given above.

$$e^{(\alpha\beta)} = \frac{D^{*}}{e^{\theta}} = \frac{2D^{*}\left[\binom{N}{2} - (D-s)\right]}{D-2s} \tag{34}$$

For the W-array involving two categories I and II with sizes N_I and N_{II} respectively, the dyadic model is given above in (6)-(9) when both the individuals belong to the first category. Note that in this case the $\lambda_{I,I}$ parameter is given by the relation

$$\lambda_{I,I} = -ln[1 + 2e^{\theta+\alpha_I+\beta_I} + e^{2\theta+2(\alpha_I+\beta_I)+(\alpha\beta)}]. \quad (35)$$

Similar expressions are available for $\lambda_{I,II}$ and $\lambda_{II,II}$. Then we can write

$$LogL = \lambda_{I,I}N_I(N_I-1)/2 + N_IN_{II}\lambda_{I,II} + \lambda_{II,II}N_{II}(N_{II}-1)/2 +$$
$$\theta[(b+2c)+(e+f+2g)+(q+2r)]+$$
$$\alpha[(b+2c+f+g)-N_I/N_{II}(e+g+q+2r)]+$$
$$\beta[(b+2c+e+g)-N_I/N_{II}(f+g+q+2r)]+(\alpha\beta)[c+g+r] \quad (36)$$

where we have used the symbols $a,b,c;d,e,f,g;p,q,r$ to denote the frequencies corresponding to the cells: $[(0,0),(0,1),(1,1);(0,0),(0,1),(1,0),(1,1);(0,0),(0,1),(1,1)]$
the first set of three correspond to Category I, the second set of four correspond to the Crossed Categories $I \times II$ and the last set of three correspond to the Category II.
Moreover, α denotes α_I and β stands for β_I. We have naturally expressed α_{II} and β_{II} in terms of α and β respectively. It facilitates to draw Table 1 showing the frequency counts in different cells.

Table 1 Frequencies of W-array

		G1		G2	
		0	1	0	1
G1	0	a	b	d	e
	1	b	c	f	g
G2	0	d	f	p	q
	1	e	g	q	r

Note that

(i) $a+b+c=\binom{N_I}{2}$

(ii) $d+e+f+g=N_IN_{II}$ (37)

(iii) $p+q+r=\binom{N_{II}}{2}$

As we have mentioned before, the difficulty in obtaining explicit solutions to the likelihood equations lies in the complex nature of dependence of the λ's on other parameters. These parameters have *exclusive* and *common* roles to play. Thus, for example, θ and $(\alpha\beta)$ are common whereas the others are exclusive to the Cross-Categories. Fortunately, we have only four independent parameters to estimate from the data. Moreover, the likelihood equations can be written down in a routine manner. It would help to "initiate" the solution by writing down some initial conditions.

We suggest :

$$\alpha^{(0)} = (b+f+c+g)/N_I(N-1); \quad (38)$$

$$\beta^{(0)} = (b+e+c+g)/N_I(N-1); (\alpha\beta)^{(0)} = (c+g+r)/{}^{N}C_2 \quad (39)$$

and then we start solving for $\theta^{(1)}$. The cycle goes like: $\theta, (\alpha\beta), \alpha, \beta$.

Numerical Example

There was an extensive survey in the village of Baghra near Giridih, Bihar, India in the year 2002. It was a multi-purpose survey covering, among other items of enquiry, information on "financial help" received from HHs in the village during the reference period of 12-month duration in 2001-2002 till the date of the survey. It resulted in a social network covering a total of 104 HHs. A substantial number of these HHs were "isolates" scattered over four [4] main communities: Muslim, Koiri, Gowala and Turi and these communities collectively covered more than 90 percent of the HHs. The network is exhibited in Appendix. We took up the study of modelling of the dyadic relations among the HHs in the communities separately. For this, again, we deleted the isolates in each community. Thus, effectively, we are left with 14 Muslim HHs, 36 Koiri HHs, 12 Gowala HHs and 20 Turi HHs.

For each of the communities listed above, we examined the dyadic relational models based on W-arrays. For this, we needed to adhere to a "grouping" mechanism and we used two criteria: HHs [Small Family / Large Family] and Occupation [Primarily Agriculture / Non-Agriculture]. Thus for each community, we have fitted dyadic interaction models for data in the form of W-arrays resulting from both the grouping criteria. The results are given in Table 2 below.

Table 2 Parameter Estimates and their SE of Fitted Model for W-Array Data

	$\hat{\theta}$ (SE)	$\widehat{(\alpha\beta)}$ (SE)	$\hat{\alpha}$ (SE)	$\hat{\beta}$ (SE)
Muslim : HHs Size[1]	-1.9623 (0.26)	1.0907 (0.69)	0.0561 (0.21)	-0.1376 (0.21)
Muslim : Occupation[2]	-1.9501 (0.26)	1.0687 (0.69)	-0.0221 (0.16)	-0.0221 (0.16)
Koiri : HHs Size[3]	-3.6596 (0.20)	2.8806 (0.57)	-0.0065 (0.19)	-0.1733 (0.19)
Koiri : Occupation[4]	-3.7442 (0.22)	2.8782 (0.59)	-0.1740 (0.27)	0.6205 (0.25)
	-	-	-0.1050 (0.23)	-0.5187 (0.25)
Gowala : HHs Size[5]	-2.2654 (0.38)	0.4889 (1.23)	-0.5634 (0.38)	-0.1199 (0.35)
Gowala : Occupation[6]	-2.2256 (0.37)	0.6472 (1.21)	0.1878 (0.22)	-0.0854 (0.18)
Turi : HHs Size[7]	-2.5866 (0.28)	1.6199 (0.79)	0.0169 (0.23)	-0.2242 (0.23)
Turi : Occupation[8]	-2.6977 (0.29)	1.8608 (0.80)	-0.1531 (0.21)	0.2014 (0.22)

1: Small Family size ≤ 5 vs. Large Family Size ≥ 6;

2: Carrying Coal vs. Misc. others;

3: Small Family size ≤ 6 vs. Large Family Size ≥ 7;

4: 3 Categories of Occupation : Agri. Labourer / Daily Labourer vs. Factory Labourer / Cloth Mill Worker - cum - Labourer/ Water Carrier / Coal Carrier / Mason / Helper vs. All others combined;

5: Small Family Size ≤ 6 vs. Large Family Size ≥ 7;

6: Agri. Labourer vs. Rest;

7: Small Family Size ≤ 4 vs. Large Family Size ≥ 5;

8: Rickshaw Puller vs. Rest

In the same spirit, we also examined the dyadic relational models based on V-arrays. For this, there is no need for grouping. The results are given in Table 3 below.

Table 3 Parameter Estimates and their SE of Fitted Model for V-Array Data

	$\hat{\theta}$ (SE)	$\widehat{(\alpha\beta)}$ (SE)
Muslim	1.9683 (0.26)	1.1034 (0.69)
Koiri	-3.6829 (0.21)	2.949 (0.57)
Gowala	-2.1751 (0.35)	0.5656 (1.19)
Turi	-2.689 (0.28)	1.843 (0.79)

For the V-array, we have also tested the hypotheses about common θ and $(\alpha\beta)$. First, we test the common θ parameter

$$H_0 : \theta_M = \theta_K = \theta_G = \theta_T .$$

The Chi-square statistic has the value 31.018 with 3 degree of freedom. We reject H_{04} since $31.018 > \chi^2_{0.05}(3) = 7.815$. So there is no significant common θ parameter.

Next, we test the common $(\alpha\beta)$ parameter

$$H_0 : (\alpha\beta)_M = (\alpha\beta)_K = (\alpha\beta)_G = (\alpha\beta)_T .$$

The Chi-square statistic has the value 5.9569 with 3 degree of freedom. We accept H_{05} since $5.9569 < \chi^2_{0.05}(3) = 7.815$ and the estimated common $(\alpha\beta)$ parameter is 1.9689.

So the movement parameters of all 4 communities have a difference, but there is a common reciprocal parameter of all 4 communities.

For the W-array, we have also tested the hypotheses of significance of the α and β parameters – those whose estimates are around 0.10 or smaller in absolute value. The tests were based on the computation of the maximum likelihood under the original model and then under the restricted model, assuming the insignificance of the parameter(s) being tested. Then the difference of the two log-likelihoods is asymptotically Chi-square with 1 df for every single parameter being tested.

The results indicate

(a) For Muslim Community, $\alpha = 0$ for both HHs and Occupation and, further, $\beta = 0$ for Occupation only; Chi-square values are 0.135, 0.019 and 0.019, respectively

(b) For Koiri Community, $\alpha = 0$ for HHs; Chi-square value is 0.1751

(c) For Gowala Community, $\beta = 0$ for both HHs and Occupation; Chi-square values are 0.1178 and 0.2183, respectively

(d) For Turi Community, $\alpha = 0$ for HHs; Chi-square value is 0.001

After deleting the insignificant model parameters, all others have been estimated using method of maximum likelihood once more. The results are given in Table 4 below.

Table 4 Parameter Estimates and their SE of Fitted Model for W-Array Data after deletion of insignificant parameters

	$\hat{\theta}$ (SE)	$\widehat{(\alpha\beta)}$ (SE)	$\hat{\alpha}$ (SE)	$\hat{\beta}$ (SE)
Muslim: HHs Size	-1.9568 (0.26)	1.0733 (0.69)	-	-0.1316 (0.21)
Muslim: Occupation	-1.9683 (0.26)	1.1034 (0.69)	-	-
Koiri: HHs Size	-3.6606 (0.20)	2.8844 (0.57)	-	-0.1733 (0.19)
Koiri: Occupation	-3.7442 (0.22)	2.8782 (0.59)	-0.1740 (0.27)	0.6205 (0.25)
	-	-	-0.1050 (0.23)	-0.5187 (0.25)
Gowala: HHs Size	-2.2713 (0.38)	0.5833 (1.19)	-0.5567 (0.38)	-
Gowala: Occupation	-2.2087 (0.36)	0.5694 (1.19)	0.1898 (0.22)	-
Turi: HHs Size	-2.6857 (0.29)	1.8016 (0.79)	-	-0.2213 (0.24)
Turi: Occupation	-2.6977 (0.29)	1.8608 (0.80)	-0.1531 (0.21)	0.2014 (0.22)

Remark 1 It is evident from the computations in Tables 2 and 3 that we can also take up the problem of simultaneous testing of significance of α and β with respect to each of the communities. It is a routine task to compute the log-likelihood values under the usual [full]

model and under the sub-model when the null hypothesis [of insignificance of the parameters] is assumed to be true.

Next, we take up the problem of modelling the dyadic relational network when the two classifications are crossed with each other, thus resulting into 4 Crossed Categories with respect to HHs Size and Occupation [with 2 options for each] in the next section.

Crossed Classification of Dyadic Models

We propose to model the dyadic relational network when the two classifications are crossed with each other, thus resulting into 4 Crossed Categories with respect to HH Size and Occupation [with 2 options for each]. It can be drawn as Table 5 showing the frequency counts of each of the 4 Crossed Categories, denoted as C1, C2, C3, and C4.

Table 5 The frequency counts of each of the 4 Crossed Categories

FS \ O		*I* A	*II* NA
I	SF	$NC1$	$NC2$
II	LF	$NC3$	$NC4$

O = Occupation : A = Agriculture, NA = Non-Agriculture

FS = Family size : SF = Small Family, LF = Large Family

Under this set up the dyadic model for a pair of HHs in W-array will depend on the categories (C1-C4) to which the HHs belong. We may refer to { (Ci, Ci), i = 1, 2, 3, 4} as pure categories while { (Ci, Cj), $1 \leq i \neq j \leq 4$} as mixed categories. Since in a W-array, the individual HHs do not have specific roles, (Ci, Cj) and (Cj,Ci) will have identical model descriptions. Thus, in effect there are 10 Crossed Categories viz., (C1, C1), (C2, C2), (C3, C3), (C4, C4), (C1, C2), (C1,C3), (C1, C4), (C2, C3), (C2, C4) and (C3, C4).

For this model, there are altogether 10 parameters viz., $\theta, (\alpha\beta), \alpha_{I,I}, \alpha_{I,II}, \alpha_{II,I}, \alpha_{II,II}$, $\beta_{I,I}, \beta_{I,II}, \beta_{II,I}, \beta_{II,II}$ and the defining relations involving α 's and β 's are given by

$$NC1\alpha_{I,I} + NC2\alpha_{I,II} + NC3\alpha_{II,I} + NC4\alpha_{II,II}$$
$$= 0 = NC1\beta_{I,I} + NC2\beta_{I,II} + NC3\beta_{II,I} + NC4\beta_{II,II}.$$

Sufficient statistics are $d_{I,I}$, $d_{I,II}$, $d_{II,I}$, $d_{II,II}$, $e_{I,I}$, $e_{I,II}$, $e_{II,I}$, $e_{II,II}$ and s, where $d_{I,I}$ = sum of out-degrees of all HHs in (I,I) Category,

$e_{I,I}$ = sum of in-degrees of all HHs in (I,I) Category, and similarly for other d's and e's; s = total number of reciprocal pairs.

For the W-array, the log-likelihood function of each case is given as

$$\log L_1 = \lambda_1 \binom{NC1}{2} + \theta \sum_{i\in(I,I)} d_{i;I,I} + \alpha_{I,I} \sum_{i\in(I,I)} d_{i;I,I} + \beta_{I,I} \sum_{i\in(I,I)} e_{i;I,I} + (\alpha\beta) s_{(I,I),(I,I)}$$

$$\log L_2 = \lambda_2 (NC1 \times NC2) + \theta \sum_{i\in(I,I)} d_{i;I,II} + \theta \sum_{i\in(I,II)} d_{i;I,I} + (\alpha_{I,I} + \beta_{I,II}) \sum_{i\in(I,I)} d_{i;I,II}$$
$$+ (\alpha_{I,II} + \beta_{I,I}) \sum_{i\in(I,II)} d_{i;I,I} + (\alpha\beta) s_{(I,I),(I,II)}$$

$$\log L_3 = \lambda_3 (NC1 \times NC3) + \theta \sum_{i\in(I,I)} d_{i;II,I} + \theta \sum_{i\in(II,I)} d_{i;I,I} + (\alpha_{I,I} + \beta_{II,I}) \sum_{i\in(I,I)} d_{i;II,I}$$
$$+ (\alpha_{II,I} + \beta_{I,I}) \sum_{i\in(II,I)} d_{i;I,I} + (\alpha\beta) s_{(I,I),(II,I)}$$

$$\log L_4 = \lambda_4 (NC1 \times NC4) + m \sum_{i\in(I,I)} d_{i;II,II} + \theta \sum_{i\in(II,II)} d_{i;I,I} + (\alpha_{I,I} + \beta_{II,II}) \sum_{i\in(I,I)} d_{i;II,II}$$
$$+ (\alpha_{II,II} + \beta_{I,I}) \sum_{i\in(II,II)} d_{i;I,I} + (\alpha\beta) s_{(I,I),(II,II)}$$

$$\log L_5 = \lambda_5 \binom{NC2}{2} + \theta \sum_{i\in(I,II)} d_{i;I,II} + \alpha_{I,II} \sum_{i\in(I,II)} d_{i;I,II} + \beta_{I,II} \sum_{i\in(I,II)} e_{i;I,II} + (\alpha\beta) s_{(I,II),(I,II)}$$

$$\log L_6 = \lambda_6 (NC2 \times NC3) + \theta \sum_{i\in(I,II)} d_{i;II,I} + m \sum_{i\in(II,I)} d_{i;I,II} + (\alpha_{I,II} + \beta_{II,I}) \sum_{i\in(I,II)} d_{i;II,I}$$
$$+ (\alpha_{II,I} + \beta_{I,II}) \sum_{i\in(II,I)} d_{i;I,II} + (\alpha\beta) s_{(I,II),(II,I)}$$

$$\log L_7 = \lambda_7 (NC2 \times NC4) + \theta \sum_{i\in(I,II)} d_{i,II,II} + m \sum_{i\in(II,II)} d_{i;I,II} + (\alpha_{I,II} + \beta_{II,II}) \sum_{i\in(I,II)} d_{i;II,II}$$
$$+ (\alpha_{II,II} + \beta_{I,II}) \sum_{i\in(II,II)} d_{i;I,II} + (\alpha\beta) s_{(I,II),(II,II)}$$

$$\log L_8 = \lambda_8 \binom{NC3}{2} + \theta \sum_{i \in (II,I)} d_{i;II,I} + \alpha_{II,I} \sum_{i \in (II,I)} d_{i;II,I} + \beta_{II,I} \sum_{i \in (II,I)} e_{i;II,I} + (\alpha\beta) s_{(II,I),(II,I)}$$

$$\log L_9 = \lambda_9 (NC3 \times NC4) + \theta \sum_{i \in (II,I)} d_{i;II,II} + \theta \sum_{i \in (II,II)} d_{i;II,I} + (\alpha_{II,I} + \beta_{II,II}) \sum_{i \in (II,I)} d_{i;II,II}$$
$$+ (\alpha_{II,II} + \beta_{II,I}) \sum_{i \in (II,II)} d_{i;II,I} + (\alpha\beta) s_{(II,I),(II,II)}$$

$$\log L_{10} = \lambda_{10} \binom{NC4}{2} + \theta \sum_{i \in (II,II)} d_{i;II,II} + \alpha_{II,II} \sum_{i \in (II,II)} d_{i;II,II} + \beta_{II,II} \sum_{i \in (II,II)} e_{i;II,II} + (\alpha\beta) s_{(II,II),(II,II)}$$

where the λ_i, $(i = 1,2,\ldots,10)$ parameters are given by

$$\lambda_1 = -\ln\left[1 + 2e^{\theta+\alpha_{I,I}+\beta_{I,I}} + e^{2\theta+2(\alpha_{I,I}+\beta_{I,I})+(\alpha\beta)}\right]$$

$$\lambda_2 = -\ln\left[1 + e^{\theta+\alpha_{I,I}+\beta_{I,II}} + e^{\theta+\alpha_{I,II}+\beta_{I,I}} + e^{2\theta+\alpha_{I,I}+\beta_{I,II}+\alpha_{I,II}+\beta_{I,I}+(\alpha\beta)}\right]$$

$$\lambda_3 = -\ln\left[1 + e^{\theta+\alpha_{I,I}+\beta_{II,I}} + e^{\theta+\alpha_{II,I}+\beta_{I,I}} + e^{2\theta+\alpha_{I,I}+\beta_{II,I}+\alpha_{II,I}+\beta_{I,I}+(\alpha\beta)}\right]$$

$$\lambda_4 = -\ln\left[1 + e^{\theta+\alpha_{I,I}+\beta_{II,II}} + e^{\theta+\alpha_{II,II}+\beta_{I,I}} + e^{2\theta+\alpha_{I,I}+\beta_{II,II}+\alpha_{II,II}+\beta_{I,I}+(\alpha\beta)}\right]$$

$$\lambda_5 = -\ln\left[1 + 2e^{\theta+\alpha_{I,II}+\beta_{I,II}} + e^{2\theta+2(\alpha_{I,II}+\beta_{I,II})+(\alpha\beta)}\right]$$

$$\lambda_6 = -\ln\left[1 + e^{\theta+\alpha_{I,II}+\beta_{II,I}} + e^{\theta+\alpha_{II,I}+\beta_{I,II}} + e^{2\theta+\alpha_{I,II}+\beta_{II,I}+\alpha_{II,I}+\beta_{I,II}+(\alpha\beta)}\right]$$

$$\lambda_7 = -\ln\left[1 + e^{\theta+\alpha_{I,II}+\beta_{II,II}} + e^{\theta+\alpha_{II,II}+\beta_{I,II}} + e^{2\theta+\alpha_{I,II}+\beta_{II,II}+\alpha_{II,II}+\beta_{I,II}+(\alpha\beta)}\right]$$

$$\lambda_8 = -\ln\left[1 + 2e^{\theta+\alpha_{II,I}+\beta_{II,I}} + e^{2m+2(\alpha_{II,I}+\beta_{II,I})+(\alpha\beta)}\right]$$

$$\lambda_9 = -\ln\left[1 + e^{\theta+\alpha_{II,I}+\beta_{II,II}} + e^{\theta+\alpha_{II,II}+\beta_{II,I}} + e^{2\theta+\alpha_{II,I}+\beta_{II,II}+\alpha_{II,II}+\beta_{II,I}+(\alpha\beta)}\right]$$

$$\lambda_{10} = -\ln\left[1 + 2e^{\theta+\alpha_{II,II}+\beta_{II,II}} + e^{2\theta+2(\alpha_{II,II}+\beta_{II,II})+(\alpha\beta)}\right]$$

The log-likelihood is given as

$$\log L = \sum_{i=1}^{10} \log L_i$$

$$= \sum \lambda_i M_i + \theta \sum d_i + \alpha_{I,I}(d_{I,I}) + \alpha_{I,II}(d_{I,II}) + \alpha_{II,I}(d_{II,I}) + \alpha_{II,II}(d_{II,II})$$

$$+\beta_{I,I}\left(e_{I,I}\right)+\beta_{I,II}\left(e_{I,II}\right)+\beta_{II,I}\left(e_{II,I}\right)+\beta_{II,II}\left(e_{II,II}\right)+\left(\alpha\beta\right)s,$$

where M_i = Coefficients of λ_i, $1 \le i \le 10$.

The maximum likelihood method is needed to estimate the parameters of the above model. The initial solutions are also helpful. We have the marginal with respect to family size $\hat{\alpha}_{I.}$, $\hat{\beta}_{I.}$ and the marginal with respect to occupation $\hat{\alpha}_{.I}$, $\hat{\beta}_{.I}$.

(a) $\hat{\left(\alpha\beta\right)}^{(0)}$ as an average of the two.

$$N_{I.} = N_{I,I} + N_{I,II} \quad , \quad N_{II.} = N_{II,I} + N_{II,II}$$

$$\frac{N_{I,I}\alpha_{I,I} + N_{I,II}\alpha_{I,II}}{N_{I,I} + N_{I,II}} = \alpha_{I.}$$

$$\frac{N_{II,I}\alpha_{II,I} + N_{II,II}\alpha_{II,II}}{N_{II,I} + N_{II,II}} = \alpha_{II.}$$

$$N_{I.}\alpha_{I.} + N_{II.}\alpha_{II.} = 0$$

(b) $N_{I,I}\hat{\alpha}_{I,I}^{(0)} = \hat{\alpha}_{I.}\left(N_{I,I} + N_{I,II}\right)$

So $\hat{\alpha}_{I,I}^{(0)} = \dfrac{\hat{\alpha}_{I.}\left(N_{I,I} + N_{I,II}\right)}{N_{I,I}}$.

(c) $N_{II,I}\hat{\alpha}_{II,I}^{(0)} = \hat{\alpha}_{II.}\left(N_{II,I} + N_{II,II}\right) + \hat{\alpha}_{.I}\left(N_{I,I} + N_{II,I}\right)$

So $\hat{\alpha}_{II,I}^{(0)} = \dfrac{\hat{\alpha}_{II.}\left(N_{II,I} + N_{II,II}\right) + \hat{\alpha}_{.I}\left(N_{I,I} + N_{II,I}\right)}{N_{II,I}}$.

(d) $N_{II,II}\hat{\alpha}_{II,II}^{(0)} = \hat{\alpha}_{.II}\left(N_{I,II} + N_{II,II}\right)$

So $\hat{\alpha}_{II,II}^{(0)} = \dfrac{\hat{\alpha}_{.II}\left(N_{I,II} + N_{II,II}\right)}{N_{II,II}}$.

We also replace $\alpha_{I,II}$ by $-\dfrac{NC1}{NC2}\alpha_{I,I} - \dfrac{NC3}{NC2}\alpha_{II,I} - \dfrac{NC4}{NC2}\alpha_{II,II}$, and

$\beta_{I,II}$ by $-\frac{NC1}{NC2}\beta_{I,I}-\frac{NC3}{NC2}\beta_{II,I}-\frac{NC4}{NC2}\beta_{II,II}$ in deriving the maximum likelihood estimates.

Then we start solving for first iterative parameters. The same cycles go on.

Numerical Example (Continued)

From the previous example of Baghra village data, we used two classifications: HHs [Small Family / Large Family] and Occupation [Primarily Agriculture / Non-Agriculture]. Thus for each community, we have fitted the dyadic relational network model when the two classifications are crossed with each other. The results are given in Table 6 below.

Table 6 Parameter Estimates and their SE of Fitted Model for Crossed Model

	$\hat{\theta}$ (SE)	$\widehat{(\alpha\beta)}$ (SE)	$\hat{\alpha}_{I,I}$ (SE)	$\hat{\alpha}_{II,I}$ (SE)	$\hat{\alpha}_{II,II}$ (SE)	$\hat{\beta}_{I,I}$ (SE)	$\hat{\beta}_{II,I}$ (SE)	$\hat{\beta}_{II,II}$ (SE)
Muslim	-2.0006	1.1532	-0.1303	0.0525	-0.7810	-0.0441	0.0453	0.3872
	(0.27)	(0.70)	(0.42)	(0.25)	(1.00)	(0.41)	(0.24)	(0.69)
Koiri	-3.8005	3.0266	-0.0413	-0.2396	0.1863	-0.7322	0.8135	0.0352
	(0.23)	(0.60)	(0.60)	(0.35)	(0.76)	(0.79)	(0.29)	(0.26)
Gowala	-2.6750	0.7690	0.0461	0.8298	0.3699	-0.2689	0.0348	0.4756
	(0.92)	(1.25)	(0.95)	(0.93)	(1.08)	(0.45)	(0.44)	(0.61)
Turi	-2.7594	1.9527	-0.1300	-0.1121	-0.0284	-0.0467	0.1865	0.0553
	(0.30)	(0.81)	(0.32)	(0.52)	(0.37)	(0.32)	(0.48)	(0.36)

The movement parameters (θ) of all 4 communities are highly negatively large. Therefore, we expect a large number of (0, 0) cells in each community which is otherwise evident from the survey data. The excess effects of α and β are not necessarily pronounced.

For the crossed model, we have tested the hypotheses of equality of each of the α and β parameters and the $(\alpha\beta)$ parameter across all the 4 communities. The test is asymptotically Chi-square with 3 degree of freedom for every single parameter being tested. The results are as follows:

(a) For $\alpha_{I,I}$ parameter, $(\alpha_{I,I})_M=(\alpha_{I,I})_K=(\alpha_{I,I})_G=(\alpha_{I,I})_T$ for crossed classification with Chi-square value computed as 0.048 and the expected common $\alpha_{I,J}$ parameter is -0.1067.

(b) For $\alpha_{II,I}$ parameter, $(\alpha_{II,I})_M = (\alpha_{II,I})_K = (\alpha_{II,I})_G = (\alpha_{II,I})_T$ for crossed classification 1.339 Chi-square value computed as 1.339 and the expected common $\alpha_{II,I}$ parameter is -0.0209.

(c) For $\alpha_{II,II}$ parameter, $(\alpha_{II,II})_M = (\alpha_{II,II})_K = (\alpha_{II,II})_G = (\alpha_{II,II})_T$ for crossed classification with Chi-square value computed as 0.783 and the expected common $\alpha_{II,II}$ parameter is -0.0324.

(d) For $\beta_{I,I}$ parameter, $(\beta_{I,I})_M = (\beta_{I,I})_K = (\beta_{I,I})_G = (\beta_{I,I})_T$ for crossed classification with Chi-square value computed as 0.779 and the expected common $\beta_{I,I}$ parameter is -0.1451.

(e) For $\beta_{II,I}$ parameter, $(\beta_{II,I})_M = (\beta_{II,I})_K = (\beta_{II,I})_G = (\beta_{II,I})_T$ for crossed classification with Chi-square value computed as 4.606 and the expected common $\beta_{II,I}$ parameter is 0.2962.

(f) For $\beta_{II,II}$ parameter, $(\beta_{II,II})_M = (\beta_{II,II})_K = (\beta_{II,II})_G = (\beta_{II,II})_T$ for crossed classification with Chi-square value computed as 0.623 and the expected common $\beta_{II,II}$ parameter is 0.1159.

(g) For $(\alpha\beta)$ parameter, $(\alpha\beta)_M = (\alpha\beta)_K = (\alpha\beta)_G = (\alpha\beta)_T$ for crossed classification with Chi-square value computed as 5.384 and the expected common $(\alpha\beta)$ parameter is 2.0377.

For the crossed model, we have also tested for each community the hypotheses of significance of the α and β parameters whose estimates are around 0.10 or smaller in absolute value. The tests were based on the computation of the maximum likelihood under the original model and then under the restricted model, assuming the insignificance of the parameters being tested. The difference of the two log-likelihoods is asymptotically Chi-square with 1 degree of freedom for every single parameter being tested.

The results show that

(a) For Muslim community, $\alpha_{I,I} = 0$, $\alpha_{II,I} = 0$, $\beta_{I,I} = 0$, $\beta_{II,I} = 0$ for crossed classification with Chi-square values computed as 0.132, 0.0602, 0.0158, and 0.0392, respectively.

(b) For Koiri community, $\alpha_{I,I} = 0$, $\alpha_{II,II} = 0$, and $\beta_{II,II} = 0$ for crossed classification with Chi-square values computed as 0.03, 0.426, and 0.008 respectively.

(c) For Gowala community, $\alpha_{I,I} = 0$ and $\beta_{II,I} = 0$ for crossed classification with Chi-square values computed as 0.0564 and 0.0486 respectively.

(d) For Turi community, $\alpha_{I,I} = 0$, $\alpha_{II,I} = 0$, $\alpha_{II,II} = 0$, $\beta_{I,I} = 0$, $\beta_{II,I} = 0$, $\beta_{II,II} = 0$ for crossed classification with Chi-square values computed as 0.2928, 0.1748, 0.1946, 0.1936, 0.14, and 0.196 respectively.

After deleting the insignificant model parameters, all others have been estimated using method of maximum likelihood once more. The results are given in Table 7 below.

Table 7 Parameter estimates and their SE of fitted model for W-array data using crossed classification after deletion of insignificant parameters.

	$\hat{\theta}$ (SE)	$\widehat{(\alpha\beta)}$ (SE)	$\hat{\alpha}_{I,I}$ (SE)	$\hat{\alpha}_{II,I}$ (SE)	$\hat{\alpha}_{II,II}$ (SE)	$\hat{\beta}_{I,I}$ (SE)	$\hat{\beta}_{II,I}$ (SE)	$\hat{\beta}_{II,II}$ (SE)
Muslim	-1.9783	1.1184	-	-	-0.7904	-	-	0.4017
	(0.26)	(0.70)			(0.87)			(0.65)
Koiri	-3.7353	2.9330	-	-0.1352	-	-0.6228	0.7710	-
	(0.22)	(0.19)		(0.30)		(0.68)	(0.28)	
Gowala	-2.5828	0.7790	-	0.7512	0.2925	-0.2286	-	0.4649
	(0.53)	(1.20)		(0.62)	(0.84)	(0.34)		(0.58)
Turi	-2.6660	1.7986	-	-	-	-	-	-
	(0.28)	(0.79)						

Conclusion

For V-array the parameter estimates showed that all 4 communities have very little tendency towards "dyadic movement" between HHs within community, but when the movement take place, there are some tendency for these dyadic relations to be reciprocated. All 4 communities can be concluded in the same way i.e., the two-way ties is very rare event. Mostly the HHs are self-sufficient and one-way ties are also rare. The 4 communities have a comm reciprocal parameter, but there are different movement parameters.

We have already discussed the salient feature of model fitting with respect classification of Occupation vs. HH Size. Below we discuss some feat classification and HH Size classification, taken one at a time.

(d) For Turi community, $\alpha_{I,I}=0$, $\alpha_{II,I}=0$, $\alpha_{II,II}=0$, $\beta_{I,I}=0$, $\beta_{II,I}=0$, $\beta_{II,II}=0$ for crossed classification with Chi-square values computed as 0.2928, 0.1748, 0.1946, 0.1936, 0.14, and 0.196 respectively.

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	$\hat{\theta}$ (SE)	$\widehat{(\alpha\beta)}$ (SE)	$\hat{\alpha}_{I,I}$ (SE)	$\hat{\alpha}_{II,I}$ (SE)	$\hat{\alpha}_{II,II}$ (SE)	$\hat{\beta}_{I,I}$ (SE)	$\hat{\beta}_{II,I}$ (SE)	$\hat{\beta}_{II,II}$ (SE)
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	(0.26)	(0.70)			(0.87)			(0.65)
Koiri	-3.7353	2.9330	-	-0.1352	-	-0.6228	0.7710	-
	(0.22)	(0.19)		(0.30)		(0.68)	(0.28)	
Gowala	-2.5828	0.7790	-	0.7512	0.2925	-0.2286	-	0.4649
	(0.53)	(1.20)		(0.62)	(0.84)	(0.34)		(0.58)
Turi	-2.6660	1.7986	-	-	-	-	-	-
	(0.28)	(0.79)						

Conclusion

For V-array the parameter estimates showed that all 4 communities have very little tendency towards "dyadic movement" between HHs within community, but when the movement take place, there are some tendency for these dyadic relations to be reciprocated. All 4 communities can be concluded in the same way i.e., the two-way ties is very rare event. Mostly the HHs are self-sufficient and one-way ties are also rare. The 4 communities have a common reciprocal parameter, but there are different movement parameters.

We have already discussed the salient feature of model fitting with respect to the crossed classification of Occupation vs. HH Size. Below we discuss some feature of Occupation classification and HH Size classification, taken one at a time.

a) For each of these classifications, it turns out that the movement parameters (θ) are negatively large, thereby indicating prevalence of (0,0) cells. This is true for all the communities.

b) The out-degree and in-degree parameters are not significant for either classification. This is also true for all the communities.

In fine, therefore, there is predominantly a scenario of "rare" movement among HHs of all the communities. Only when the movements are visible, one-way ties are less frequent than the two-way ties [($\alpha\beta$) is significant compared to α or β].

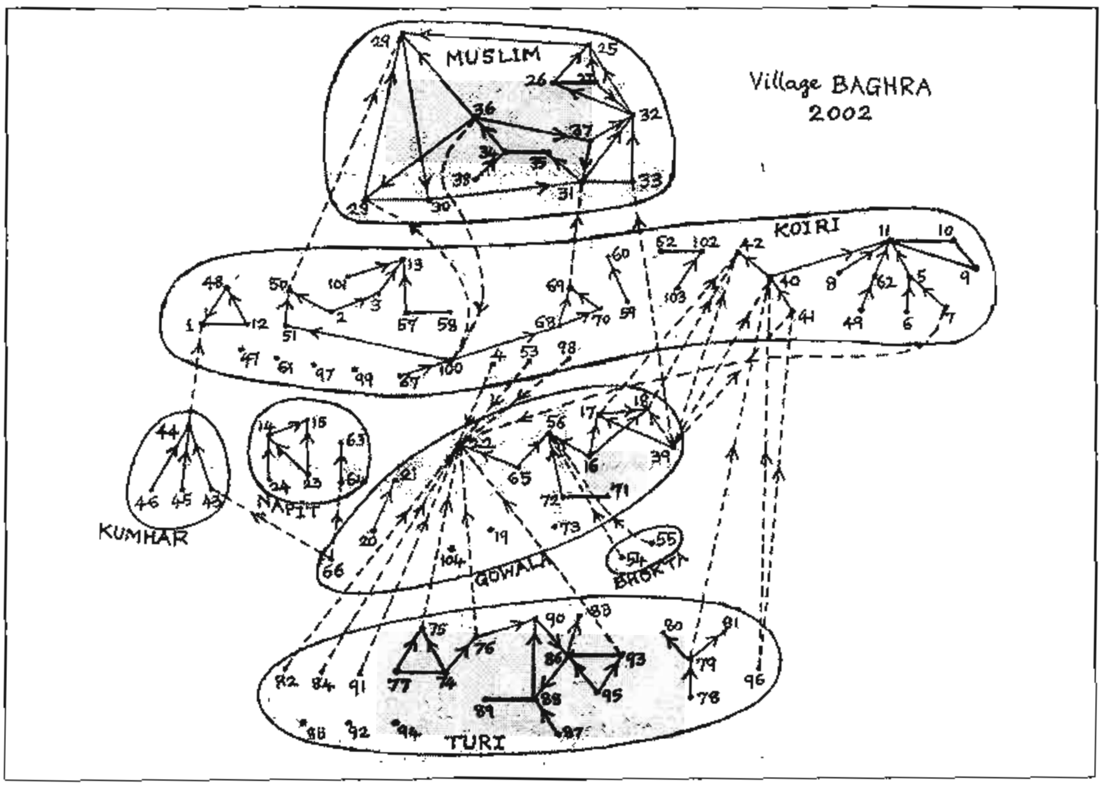

Figure 1 Network of "mutual help relation" among HHs in Baghra near the town of Giridih in the state of Bihar, India (2002).

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Table 1: Proposed and Actual Activity

Activity : Proposed (<.......>) Actual (<——>)	Year 1 : Month		Year 2 : Month		Year 3 : Month	
	1-6	7-12	13-18	19-24	25-30	31-36
1. Search literature	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>
2. Develop new statistical methods	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>
3. Data collection	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>
4. Data analysis with applications	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>
5. Write papers and submit to journals for publication	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>
6. Progress report	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	<.......> <——>	
7. Final report						<.......> <——>

7. Overall Output

7.1 Summary Table

Table 2: Outputs

Outputs	**Proposed No. / Actual No.**
1. Papers submitted in international journals	7 / 12
2. Papers submitted in international published book	0 / 1
2. Papers accepted for publication*	7 / 7
3. Papers appeared in international journals*	7 / 6
4. Papers presented in international conferences	3 / 10

(*See Appendix)

7.2 Papers submitted and /or appeared in international journals

7.2.1 **Tiensuwan, M.** and Sarikavanij, S. (2003). On estimation of population variance based on a ranked set sample, *Journal of Applied Statistical Science*, Vol.12, No.4, pp.283-295. (**Indexed/Abstracted** in MathSciNet, Statistical Theory and Method Abstracts)

7.2.2. Sarikavanij, S. and **Tiensuwan, M.**(2003). On Two Novel Applications of Ranked Set Sampling. *Calcutta Statistical Association Bulletin*, Vol. 54, Nos. 213-214, pp. 105-114. (**Indexed/Abstracted** in MathSciNet, Statistical Theory and Method Abstracts)

7.2.3 Sarikavanij, S.; **Tiensuwan, M.**; and Sinha, B.K.(2004). Estimation of Reliability Based on Exponential Distribution and Ranked Set Sample. *Pakistan Journal of Statistics*, Vol. 20(1), pp. 31-36.
(**Indexed/Abstracted** in MathSciNet, Statistical Theory and Method Abstracts)

7.2.4 Lertprapai, S.; **Tiensuwan, M**; Sinha, B.K.(2004). A statistical approach to combining environmental indices with an application to air pollution data from Bangkok, Thailand. *Pakistan Journal of Statistics*, Vol.20(2), pp. 245-261.(Impact factor: -)
(**Indexed/Abstracted** in MathSciNet, Statistical Theory and Method Abstracts)

7.2.5 Lertprapai, S.; **Tiensuwan, M.**; and Sinha, B.K (2004) On a comparison of two standard estimates of a binomial proportion based on multiple criteria decision making method. *Journal of Statistical Theory and Aplications*, Vol.3, no.2, pp.141-149. (Impact factor: -) (**Indexed/Abstracted** in MathSciNet)

7.2.6 Lertprapai, S. ;**Tiensuwan, M.** ; and Sinha, B.(2004). On a comparison of three estimators of binomial variance by multiple criteria decision making method. *International Journal of Statistical Sciences*, Vol.3, pp.105-117. (Impact factor: -)
(**Indexed/Abstracted** in Statistical Theory and Method Abstracts)

7.2.7 **Tiensuwan, M.**; Lertprapai, S. and Sinha, B.(2005). On a comparison of several competing estimates of a univariate normal mean by multiple criteria decision making method
(**To appear** in *Communications in Statistics-Simulation and Computation*, Vol. 35, No.4, 2006)
(Impact factor: 0.220; Journal Citation Reports, 2003)

7.2.8 **Tiensuwan, M.;** Lertprapai, S. and Sinha, B.(2005). On a comparison of four estimates of a common mean by multiple criteria decision making method.
(**Submitted to** *Journal of Statistical Research*)(Impact factor: -)
(**Indexed/Abstracted** in Statistical Theory and Method Abstracts)

7.2.9 Tiensuwan M., **Sarikavanij S.**, and Sinha, B.K. (2005). Nonnegative unbiased estimation of scale parameters and associates quantiles based on a ranked set sample.
(**Submitted to** *Communications in Statistics-Simulation and Computation*)
(**Impact factor** = 0.220; Source: Journal Citation Reports, 2003)

7.2.10 Sinha, B. K., Yimprayoon, P. and **Tiensuwan, M.(2006).** Cohen's Kappa: A Critical Review and Some Modifications.
(**Submitted to** *Calcutta Statistical Association Bulletin*)
(**Indexed/Abstracted** in MathSciNet, Statistical Theory and Method Abstracts)

7.2.11 Sunthornworasiri, N., **Tiensuwan, M.,** and Sinha, B.K. (2005). Statistical Inference for a Bivariate Normal Population with a Common Mean.
(**Submitted to** *International Journal of Statistical Sciences*)
(**Indexed/Abstracted** in Statistical Theory and Method Abstracts)

7.2.12 Em-ot, P., **Tiensuwan, M.,** and Sinha, B.K. (2006). Some Aspects of Stochastic Modeling of Dyadic Relations in Social Networks :Theory and Applications.
(**Submitted to** *Journal of Statistical Theory and Aplications*)
(**Indexed/Abstracted** in MathSciNet)

7.3 Paper submitted to an international published book

7.3.1 Yimprayoon, P. ,**Tiensuwan, M.,** and Sinha, B.K. (2006). Some Statistical Aspects of Assessing Agreements: Theory and Applications.
(**Submitted to** Festschrift Tarmo Pukkila on his 60th birthday; The Festschrift will be a published book of scholarly papers. It will be given to Dr.Pukkila in a specific Session of the Fifteenth International Workshop on Matrices and Statistics, to be held in Uppsala during June 13-17, 2006)

7.4 Papers presented in International Conferences

7.4.1 Lertprapai, S.; **Tiensuwan, M.**; and Sinha, B.(2003). On a comparison of two standard estimates of a binomial proportion based on multiple criteria decision making method. *The Fifth International Triennial Calcutta Symposium on Probability and Statistics*, December 28-31, 2003, Department of Statistics and Calcutta Statistical Association, Department of Statistics Calcutta University, Kolkata, India.

7.4.2 Sarikavanij, S. and **Tiensuwan, M.** (2004). On Two Novel Applications of Ranked Set Sampling. *The XXII International Biometric Conference in parallel with the Australia Statistical Conference* 11-16 July 2004 Cairns Conventional Centre, Cairns, Queensland, Australia.

7.4.3 Lertprapai, S. and **Tiensuwan, M.**(2004). On Some Data Integration Methods with an Application to Meteorological Data in Thailand. *The XXII International Biometric Conference in parallel with the Australia Statistical Conference* 11-16 July 2004 Cairns Conventional Centre, Cairns, Queensland, Australia.

7.4.4 **Tiensuwan, M.,** Lertprapai, S., and Sinha, BK. (2005). Application of Multiple Criteria Decision Making to Air Pollution Data from Bangkok, Thailand. *The 4th Annual Hawaii International Conference Statistics, Mathematics and Related Field,* January 9-11, 2005, Sheraton Waikiki, Honolulu, Hawaii, USA.

7.4.5 **Tiensuwan, M,**; Lertprapai, S; Sinha, BK. (2005). On a Comparison of two Standard Estimators of a Binomial Proportion by Multiple Criteria Decision Making Method. *The 2nd International Symposium on Mathematical, Statistical, and Computer Sciences 2005 (MSCS 2005), KMITL Sci.J.* Vol.5 No.1, pp.420-435. **(Keynote speaker)**

7.4.6 Sarikavanij, S.; **Tiensuwan, M.**; Sinha, BK. (2005). Nonnegative Unbiased Estimation of Scale Parameters Based on a Ranked Set Sample. *International Conference in Mathematics and Applications(ICMA-MU2005),* Department of Mathematics, Faculty of Science, Mahidol University, December 15-17, 2005, Chaophya Park Hotel, Bangkok, Thailand.

7.4.7 Lertprapai, S.; **Tiensuwan, M.**; Sinha, BK. (2005). On a Comparison of Six Estimates of a Common Mean by Multiple Criteria Decision Making Method. *International Conference in Mathematics and Applications(ICMA-MU2005),* Department of Mathematics, Faculty of Science, Mahidol University, December 15-17, 2005, Chaophya Park Hotel, Bangkok, Thailand.

7.4.8 Sunthornworasiri, N.; **Tiensuwan, M.**; Sinha, BK. (2005). Statistical Inference for a Bivariate Normal Population with a Common Mean. *International Conference in Mathematics and Applications(ICMA-MU2005),* Department of Mathematics, Faculty of Science, Mahidol University, December 15-17, 2005, Chaophya Park Hotel, Bangkok, Thailand.

7.4.9 Yimprayoon, P; **Tiensuwan, M.**; Sinha, BK.(2005). Some aspects of assessing agreements based on Cohen's kappa. *International Conference in Mathematics and Applications (ICMA-MU2005),* Department of Mathematics, Faculty of Science, Mahidol University, December 15-17, 2005, Chaophya Park Hotel, Bangkok, Thailand.

7.4.10 Em-ot, P.; **Tiensuwan, M.**; Sinha, BK. (2005). Some aspects of stochastic modeling of dyadic relations in social networks: theory and applications. *International Conference in Mathematics and Applications(ICMA-MU2005),* Department of Mathematics, Faculty of Science, Mahidol University, December 15-17, 2005, Chaophya Park Hotel, Bangkok, Thailand.

7.5 Other research papers appeared/accepted in International journals

7.5.1 Ruamviboonsuk, P.;**Tiensuwan, M.** ; Kunawut, C.; and Massayaanon, P. (2003). Repeatability of an Automated Landolt C Test, Compared to the ETDRS Testing. *American Journal of Ophthalmology*, Vol.136, Issue 4, pp. 662-669. (Impact factor: 1.978)(**Journal Citation Reports,** 2002-Science Edition)

7.5.2 Kengluecha A, Singhasivanon P, **Tiensuwan M**, Jones JW, Sithiprasasna R. Water Quality and Breeding Habitats of Anopheline Mosquito in North-Western Thailand. *SEA J of Trop Med and Public Health.* Vol.36, No.1. pp. 46-53.
(**Indexed and Abstracted** in Index Medicus/MEDLINE)

7.5.3 Ruamviboonsuk, P.; Wongcumchang, N.; Surawongsin, P; Panyawatananukul, E.; **Tiensuwan, M.**(2005) The Screening for Diabetic Retinopathy in Rural Area Using Single-Field, 2.3 Million-Pixel, Digital Fundus Images. *J Med Assoc Thai*, 88; pp.176-180.
(**Indexed and Abstracted** in Index Medicus/MEDLINE)

7.5.4 **Tiensuwan, M.**; Yimprayoon, P.; Lenbury, Y. (2005). Application of Log-linear Models to Cancer Patients: A Case Study of Data from the National Cancer Institute of Thailand. *SEA J of Trop Med and Public Health.* Vol 36 No. 5 September, pp. 1283-1291.
(**Indexed/Abstracted** in MEDLINE)

7.5.5 Ruamviboonsuk, P.; Teerasuwanajak, K.; **Tiensuwan, M.**; Yuttitham, K. (2006). Interobserver agreement ophthalmic care personnel in the interpretation of single-field digital fundus images for diabetic retinopathy screening. (To appear in *Ophthalmology*) (**Impact factor** = 3.20; Source: Journal Citation Reports, 2004)

7.6 Graduated Students

7.6.1 Ph.D. Students

1. Miss Satinee Lertprapai
2. Miss Sukuman Sarikavanij

7.6.2 M.Sc. Students

1. Miss Sugunya Rattanapornpong
2. Miss Pharita Am-ot

8. Appendices

8.1 Manuscripts submitted and /or appeared in international journals Pages 106-190

8.1.1 **Tiensuwan, M.** and Sarikavanij, S. (2003). On estimation of population variance based on a ranked set sample, *Journal of Applied Statistical Science*, Vol.12, No.4, pp.283-295.

8.1.2 Sarikavanij, S. and **Tiensuwan, M.**(2003). On Two Novel Applications of Ranked Set Sampling. *Calcutta Statistical Association Bulletin*, Vol. 54, Nos. 213-214, pp. 105-114.

8.1.3 Sarikavanij, S.; **Tiensuwan, M.**; and Sinha, B.K.(2004). Estimation of Reliability Based on Exponential Distribution and Ranked Set Sample. *Pakistan Journal of Statistics*, Vol. 20(1), pp. 31-36.

8.1.4 Lertprapai, S.; **Tiensuwan, M**; Sinha, B.K.(2004). A statistical approach to combining environmental indices with an application to air pollution data from Bangkok, Thailand. *Pakistan Journal of Statistics*, Vol.20(2), pp. 245-261.

8.1.5 Lertprapai, S.; **Tiensuwan, M.**; and Sinha, B.K (2004) On a comparison of two standard estimates of a binomial proportion based on multiple criteria decision making method. *Journal of Statistical Theory and Aplications*, Vol.3, no.2, pp.141-149.

8.1.6 Lertprapai, S. ;**Tiensuwan, M.** ; and Sinha, B.(2004). On a comparison of three estimators of binomial variance by multiple criteria decision making method. *International Journal of Statistical Sciences*, Vol.3, pp.105-117.

8.1.7 **Tiensuwan, M.**; Lertprapai, S. and Sinha, B.(2005). On a comparison of several competing estimates of a univariate normal mean by multiple criteria decision making method. (To appear in *Communications in Statistics-Simulation and Computation*, Vol. 35, No.4, 2006).

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Date....Feb. 28, 2006......

Journal of Applied Statistical Science
Volume 12, Number 4, pp. 283-295
ISSN 1067-5817

ON ESTIMATION OF POPULATION VARIANCE BASED ON A RANKED SET SAMPLE

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SUMMARY

Ranked set sampling (RSS) is by now a well known sampling strategy which provides more efficient estimates of a population mean compared to the traditional simple random sampling (SRS). Although a lot of work on RSS has been done related to estimation of the mean, very little is known about estimation of variance. In this paper the problem of variance estimation based on RSS is studied in detail. It is shown that, based on a single cycle RSS, there does not exist an unbiased estimate of population variance. For multiple cycles in the balanced case, nonnegative quadratic unbiased estimates of population variance are derived. We compare our estimates with an unbiased estimate based on SRS for uniform, exponential and normal distributions. The results show that our proposed estimates are more efficient than the one based on SRS.

Keywords: Ranked set sampling, Simple random sampling, Variance.
AMS 1991 Subject Classification: 62H12

1. INTRODUCTION

In sampling situations when the variable of interest from the experimental units can be more easily ranked than quantified, it turns out that, for estimation of the population mean, an old concept of McIntyre (1952), namely, "*Ranked Set Sampling*" (RSS) is highly beneficial and much superior to the standard simple random sampling (SRS). Fortunately, in many agricultural and environmental studies, it is indeed possible to rank the experimental or sampling units without actually measuring them. See Hall and Dell (1966).

Ranked set sampling (RSS) is a procedure introduced by McIntyre (1952) which combines random sampling and the ability to rank the sampling units, with respect to the characteristic of interest, without making the actual measurements. Early work on RSS concentrated on estimation of the population mean. Several authors have proposed the use of RSS in the collection of environmental data, discussed the efficiency of McIntyre's original sampling scheme, and investigated the efficiency of RSS. In 1966, Halls and Dell applied McIntyre's method, coining it "ranked set sampling" for estimating the weights of browse and herbage in a pine-hardwood forest of east Texas. For theoretical justification of the use of RSS, we refer to Takahasi and Wakimoto (1968).

Stokes (1980) extended the technique of RSS to propose an estimator of variance, which is asymptotically unbiased regardless of the presence of errors in ranking. Furthermore, the asymptotic efficiency of this estimator, relative to the one based on the same number of quantified observations from a simple random sample, is greater than unity for any underlying distribution, even if ranking errors occur.

In this paper we address unbiased estimation of the population variance based on an RSS.

2. Theoretical Background

2.1 Simple Random Sampling (SRS)

Let $X_1, X_2, \ldots, X_N$ denote a sample of independent identically distributed (i.i.d.) random variables, each X_i having mean μ and variance σ^2. Then a standard unbiased estimator of μ is $\hat{\mu}_{SRS} = \sum_{i=1}^{N} X_i \Big/ N$ with $Var(\hat{\mu}_{SRS}) = \sigma^2/N$.

Analogously, a standard unbiased estimator of σ^2 is

$$\hat{\sigma}^2_{SRS} = \frac{\sum_{i=1}^{N}(X_i - \overline{X}_N)^2}{N-1} \tag{2.1}$$

$$\text{with Var}(\hat{\sigma}^2_{SRS}) = \frac{\mu_4}{N} - \frac{(N-3)\sigma^4}{N(N-1)} \tag{2.2}$$

where $\mu_4 = E(X-\mu)^4$.

2.2 Ranked Set Sampling (RSS)

Let $X_{(i:k)} \equiv X_{(ii)}$ denote the ith order statistic from the ith set of k observations with mean $\mu_{(i:k)} \equiv \mu_{(i)}$ and variance $\sigma^2_{(i:k)} \equiv \sigma^2_{(i)}$. Let $X^{(j)}_{(i:k)} \equiv X^{(j)}_{(ii)}$ denote the ith order statistic from the ith sample of size k in the jth cycle (j = 1, 2, ... , n). A single cycle of RSS with set size N may be displayed in a rectangular array such as the following:

$$\begin{matrix} X_{(11)} & X_{(12)} & \cdots & X_{(1N)} \\ X_{(21)} & X_{(22)} & \cdots & X_{(2N)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{(N1)} & X_{(N2)} & \cdots & X_{(NN)} \end{matrix}$$

The important point to emphasize is that although RSS requires identification of as many as N^2 experimental or sampling units, only N of them, namely, $\{X_{(11)}, \ldots, X_{(NN)}\}$, are actually measured, thus making a comparison of this sampling strategy with SRS of the same size N meaningful.It is obvious that the new sample $X_{(11)}, X_{(22)}, \ldots, X_{(NN)}$, known in the literature as a *Ranked Set Sample*, are independent but not identically distributed. Moreover, marginally, $X_{(ii)}$ is distributed as $X_{(1i)}$, for all $i = 1, \ldots, N$, so that

$$E(X_{(ii)}) = \mu_{(i)} \text{ and } Var(X_{(ii)}) = \sigma^2_{(i)}$$

for $i = 1, 2, \ldots, N$.

McIntyre (1952) proposed

$$\hat{\mu}_{RSS} = \overline{X}_{RSS} = \sum_{i=1}^{N} X_{(ii)} \Big/ N \tag{2.3}$$

as a rival unbiased estimator of μ as opposed to $\overline{X}_N$.

Dell and Clutter (1972) provided the following explicit expression for the variance of $\hat{\mu}_{RSS}$ where $\mu_{(i)}$ is the mean of $X_{i:N}$.

$$Var(\hat{\mu}_{RSS}) = \frac{\sigma^2}{N} - \sum_{i=1}^{N} \frac{(\mu_{(i)} - \mu)^2}{N^2}. \tag{2.4}$$

For increased efficiency, we can use multiple cycle RSS. Quite generally, if $N = k \times n$ with $k \le n$, we can use an RSS procedure based on k units at a time, and repeat the process n times. A balanced RSS (BRSS) with set size k and the number of cycles n is displayed such as the following:

$$\begin{matrix} X^{(j)}_{(11)} & X^{(j)}_{(12)} & \cdots & X^{(j)}_{(1k)} \\ X^{(j)}_{(21)} & X^{(j)}_{(22)} & \cdots & X^{(j)}_{(2k)} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(j)}_{(k1)} & X^{(j)}_{(k2)} & \cdots & X^{(j)}_{(kk)} \end{matrix} \quad ; \text{for } j = 1, \ldots, n$$

where $X^{(j)}_{(11)}, \ldots, X^{(j)}_{(kk)}$, for all j, are independent. Furthermore,

$$E(X^{(j)}_{(ii)}) = \mu_{(i)} \text{ and } Var(X^{(j)}_{(ii)}) = \sigma^2_{(i)}, \text{ for all } j.$$

The overall estimator of μ is then given by

$$\hat{\mu}_{BRSS}(N = k \times n) = \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\frac{X_{(ii)}^{(j)}}{k}\right]\Big/ n = \sum_{j=1}^{n}\frac{\overline{X}_{RSS}^{(j)}}{n} \tag{2.5}$$

$$= \sum_{i=1}^{k}\left[\sum_{j=1}^{n}\frac{X_{(ii)}^{(j)}}{n}\right]\Big/ k = \sum_{i=1}^{k}\frac{\overline{X}_{(ii)}}{k}.$$

The variance of $\hat{\mu}_{BRSS}(N = k \times n)$ is given by

$$Var(\hat{\mu}_{BRSS}(N = k \times n)) = \left[\frac{\sigma^2}{k} - \sum_{i=1}^{k}\frac{(\mu_{(i:k)} - \mu)^2}{k^2}\right]\Big/ n \tag{2.6}$$

thus indicating its superiority over $\overline{X}_N$ based on $N = k \times n$ units.

The relationships between $\mu, \sigma^2, \mu_{(1)}, \ldots, \mu_{(k)}, \sigma^2_{(1)}, \ldots, \sigma^2_{(k)}$ are

$$\mu = \sum_{i=1}^{k}\frac{\mu_{(i)}}{k} \text{ and } \sigma^2 = \sum_{i=1}^{k}\frac{\sigma^2_{(i)}}{k} + \sum_{i=1}^{k}\frac{(\mu_{(i)} - \mu)^2}{k}.$$

What we have described above can be called an equal allocation scheme (or balanced RSS) in the sense that each of the k order statistics is replicated an equal number of times, namely, n times.

Now let us briefly explain how the population variance, σ^2 is estimated on the basis of a RSS. For a single cycle RSS, Stokes (1980) suggested the following estimate of σ^2:

$$\hat{\sigma}^2_{RSS/Stokes} = \frac{\sum_{i=1}^{k}(X_{(ii)} - \overline{X}_{RSS})^2}{k-1} \tag{2.7}$$

where $\overline{X}_{RSS} = \sum_{i=1}^{k} X_{(ii)}/k$, with

$$E(\hat{\sigma}^2_{RSS/Stokes}) = \sigma^2 + \sum_{i=1}^{k}\frac{(\mu_{(i:k)} - \mu)^2}{k(k-1)} \tag{2.8}$$

which shows that $\hat{\sigma}^2_{RSS/Stokes}$ is in general biased, and

$$Var(\hat{\sigma}^2_{RSS/Stokes}) = \frac{1}{(k-1)^2}\left[\left(\frac{k-1}{k}\right)^2\sum_{i=1}^{k}\mu_{4(i)} + 4\sum_{i=1}^{k}\tau^2_{(i)}\sigma^2_{(i)} + 4\left(\frac{k-1}{k}\right)\sum_{i=1}^{k}\tau_{(i)}\mu_{3(i)}\right.$$

$$+\frac{4}{k^2}\sum_{i<i'=1}^{k}\sigma_{(i)}^2\sigma_{(i')}^2-\frac{(k-1)^2}{k^2}\sum_{i=1}^{k}\sigma_{(i)}^4\Bigg] \tag{2.9}$$

where $\mu_{l(i)} = E(X_{(i:k)} - \mu_{(i:k)})^l$ for $l = 3,4$, $\sigma_{(i)}^2 = \sigma_{(i:k)}^2$ and $\tau_{(i)} = \mu_{(i:k)} - \mu$.

For a balanced RSS (BRSS) with $N = kn$ where k is the set size and n is the number of cycles, Stokes (1980) suggested

$$\hat{\sigma}_{BRSS/Stokes}^2 = \frac{\sum_{j=1}^{n}\sum_{i=1}^{k}(X_{(ii)}^{(j)}-\overline{X})^2}{kn-1} \tag{2.10}$$

where $\overline{X} = \sum_{j=1}^{n}\sum_{i=1}^{k}X_{(ii)}^{(j)}\big/kn$. Again, $\hat{\sigma}_{BRSS/Stokes}^2$ is biased for σ^2 with

$$E(\hat{\sigma}_{BRSS/Stokes}^2) = \sigma^2 + \sum_{i=1}^{k}\frac{(\mu_{(i:k)}-\mu)^2}{k(kn-1)} \tag{2.11}$$

and

$$Var(\hat{\sigma}_{BRSS/Stokes}^2) = \frac{n}{(kn-1)^2}\left[\left(\frac{kn-1}{kn}\right)^2\sum_{i=1}^{k}\mu_{4(i)} + 4\sum_{i=1}^{k}\tau_{(i)}^2\sigma_{(i)}^2 + 4\left(\frac{kn-1}{kn}\right)\sum_{i=1}^{k}\tau_{(i)}\mu_{3(i)}\right.$$

$$\left.+\frac{4}{k^2 n}\sum_{i<i'=1}^{k}\sigma_{(i)}^2\sigma_{(i')}^2 + \frac{2(n-1)-(kn-1)^2}{k^2n^2}\sum_{i=1}^{k}\sigma_{(i)}^4\right]. \tag{2.12}$$

Our objective in this paper is to propose two unbiased estimates of σ^2 based on a balanced RSS. We will derive the variances of our proposed estimates of σ^2 and demonstrate that $Var(\hat{\sigma}_{BRSS/propose}^2) < Var(\hat{\sigma}_{SRS}^2)$ for three well known distributions: Uniform, Exponential and Normal. It may be noted that our proposed estimates of σ^2 are always nonnegative and unbiased, regardless of the underlying parent population.

3. MAIN RESULTS

Ranked set sampling is considered as two cases, single cycle $n = 1$ and multiple cycles $n > 1$. Based on a ranked set sample of a single cycle, we show that an unbiased estimate of σ^2 does not exist. For multiple cycles in the balanced case, we derive two nonnegative quadratic unbiased estimators of σ^2.

3.1 Nonexistence of an Unbiased Variance Estimator Based on a Single Cycle RSS

Let $\mathbf{X} = [X_{(11)}, X_{(22)}, \ldots, X_{(kk)}]'$, $\mathbf{1}_k = [1, 1, \ldots, 1]'_{k\times 1}$,

$E(\mathbf{X}) = [\mu_{(1)}, \mu_{(2)}, \ldots, \mu_{(k)}]' = \mu_{(\cdot)}$,

and

$$Var(\mathbf{X}) = \begin{bmatrix} \sigma^2_{(1)} & & 0 \\ & \ddots & \\ 0 & & \sigma^2_{(k)} \end{bmatrix} = \mathrm{diag}[\sigma^2_{(1)}, \sigma^2_{(2)}, \ldots, \sigma^2_{(k)}] = \sum.$$

Suppose $\hat{\sigma}^2 = \mathbf{X}'A\mathbf{X}$ is an unbiased estimator of σ^2 where A is a matrix $k \times k$ so that $E(\hat{\sigma}^2) = E(\mathbf{X}'A\mathbf{X}) = \sigma^2$. Since $E(\mathbf{X}'A\mathbf{X}) = \mu'_{(\cdot)}A\mu_{(\cdot)} + \sum_{i=1}^{k} a_{ii}\sigma^2_{(i)}$ and from a basic identity, $\sigma^2 = \sum_{i=1}^{k} \frac{\sigma^2_{(i)}}{k} + \sum_{i=1}^{k} \frac{(\mu_{(i)} - \mu)^2}{k}$, for unbiasedness to hold, we must have

$$\mu'_{(\cdot)}A\mu_{(\cdot)} = \sum_{i=1}^{k} \frac{(\mu_{(i)} - \mu)^2}{k} \tag{3.1}$$

and

$$\sum_{i=1}^{k} a_{ii}\sigma^2_{(i)} = \sum_{i=1}^{k} \frac{\sigma^2_{(i)}}{k}. \tag{3.2}$$

From (3.1), we get $A = \frac{1}{k}\left[I_k - \frac{\mathbf{1}_k\mathbf{1}'_k}{k}\right]$, so that $a_{ii} = \frac{1}{k}\left[1 - \frac{1}{k}\right] = \frac{k-1}{k^2}$. From (3.2), we get $a_{ii} = \frac{1}{k}$, which is a contradiction. Hence the result.

3.2 Proposed Variance Estimators Based on a BRSS

Consider the basic identity $\sigma^2 = \sum_{i=1}^{k} \frac{\sigma^2_{(i)}}{k} + \sum_{i=1}^{k} \frac{(\mu_{(i)} - \mu)^2}{k}$, rewritten as

$$\sigma^2 = \frac{1}{k}\left[\sum_{i=1}^{k} \sigma^2_{(i)} + \sum_{i=1}^{k} \mu^2_{(i)}\right] - \mu^2. \tag{3.3}$$

From this basic identity (3.3), it is clear that an unbiased estimate of σ^2 can be obtained by plugging in unbiased estimates of $(\sigma_{(i)}^2 + \mu_{(i)}^2)$ and μ^2. Since $\sum_{j=1}^{n}(X_{(ii)}^{(j)})^2/n$ is an unbiased estimate of the former term and $\overline{X}_{RSS}^{(j)}\overline{X}_{RSS}^{(j')}$ for $j \neq j'$ is an unbiased estimate of μ^2, it follows easily that an unbiased estimate of σ^2 is given by

$$\hat{\sigma}_{1BRSS/propose}^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{(X_{(ii)}^{(j)})^2}{kn} - \sum_{j\neq j'=1}^{n}\frac{\overline{X}_{RSS}^{(j)}\overline{X}_{RSS}^{(j')}}{n(n-1)}. \tag{3.4}$$

The above estimate can be readily simplified in terms of B and W which represent, respectively, the between and within sum of squares of the entire balanced data, defined as

$$B = k\sum_{j=1}^{n}(\overline{X}_{RSS}^{(j)} - \overline{X})^2 \ , W = \sum_{j=1}^{n}\sum_{i=1}^{k}(X_{(ii)}^{(j)} - \overline{X}_{RSS}^{(j)})^2 \text{ with } \overline{X} = \sum_{j=1}^{n}\overline{X}_{RSS}^{(j)}/n .$$

Then, $\hat{\sigma}_{1BRSS/propose}^2$ becomes

$$\hat{\sigma}_{1BRSS/propose}^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{(X_{(ii)}^{(j)})^2}{kn} - \sum_{j\neq j'=1}^{n}\frac{\overline{X}_{RSS}^{(j)}\overline{X}_{RSS}^{(j')}}{n(n-1)} = \frac{W}{kn} + \frac{B}{k(n-1)}, \tag{3.5}$$

which is a nonnegative unbiased estimator.

Following the same idea as above, we can suggest another nonnegative unbiased estimator of σ^2. It is easy to verify that

$$E(W) = n(k-1)\left[\sigma^2 + \sum_{i=1}^{k}\tau_{(i)}^2 / k(k-1)\right], \tag{3.6}$$

$$E(W^*) = (n-1)\sum_{i=1}^{k}\sigma_{(i)}^2, \tag{3.7}$$

$$E(B) = \frac{(n-1)}{k}\sum_{i=1}^{k}\sigma_{(i)}^2, \tag{3.8}$$

$$E(B^*) = \frac{k-1}{k}\sum_{i=1}^{k}\sigma_{(i)}^2 + n\sum_{i=1}^{k}\tau_{(i)}^2, \tag{3.9}$$

where

$$B^* = n\sum_{i=1}^{k}(\overline{X}_{(i)} - \overline{\overline{X}})^2 \tag{3.10}$$

$$W^{*} = \sum_{i=1}^{k}\sum_{j=1}^{n}(X_{i:i}^{(j)} - \bar{X}_i)^2. \tag{3.11}$$

Our proposed second estimator of σ^2 is given by.

$$\hat{\sigma}^2_{2BRSS/propose} = \frac{kn-k+1}{k^2 n(n-1)}W^{*} + \frac{B^{*}}{kn}. \tag{3.12}$$

3.3 Derivation of Variances of Our Proposed Estimators

In this section, we compute the variances of our proposed estimators. First, we will state the variances of $W, B, B^{*}, W^{*}, Cov(W,B)$, and $Cov(W^{*},B^{*})$ as follows.

Define $T_1 = \sum_{i=1}^{k}\mu_{4(i)}$, $T_2 = \sum_{i=1}^{k}\tau_{(i)}^2\sigma_{(i)}^2$, $T_3 = \sum_{i=1}^{k}\tau_{(i)}\mu_{3(i)}$,

$$T_4 = \sum_{i<i'=1}^{k}\sigma_{(i)}^2\sigma_{(i')}^2, \text{ and } T_5 = \sum_{i=1}^{k}\sigma_{(i)}^4. \tag{3.13}$$

Lemma 3.1 $Var(W) = n\left[\left(\frac{k-1}{k}\right)^2 T_1 + 4T_2 + 4\left(\frac{k-1}{k}\right)T_3 + \frac{4}{k^2}T_4 - \frac{(k-1)^2}{k^2}T_5\right].$

Lemma 3.2 $Var(B) = \frac{(n-1)}{k^2 n}\left[(n-1)\{T_1 + 6T_4\} - (n-3)\left(\sum_{i=1}^{k}\sigma_{(i)}^2\right)^2\right].$

Lemma 3.3 $Var(B^{*}) = \left(1-\frac{1}{k}\right)^2\frac{T_1}{n} + 4nT_2 + \left(1-\frac{1}{k}\right)4T_3 + \frac{4T_4}{k^2} + \left(\frac{2n-3}{n}\right)\left(1-\frac{1}{k}\right)^2 T_5.$

Lemma 3.4 $Var(W^{*}) = \frac{(n-1)^2}{n}T_1 - \frac{(n-3)(n-1)}{n}T_5.$

Lemma 3.5 $Cov(W,B) = \frac{1}{2}\left\{\frac{2(k-1)(n-1)}{k^2}T_1 + \frac{4(n-1)}{k}T_3\right.$

$$\left. - \frac{1}{k^2}\left(4(n-1)\left(\sum_{i=1}^{k}\sigma_{(i)}^2\right)^2 + (n-1)(2k-6)T_5\right)\right\}.$$

Lemma 3.6 $Cov(W^{*},B^{*}) = \frac{(k-1)(n-1)}{kn}T_1 + 2(n-1)T_3 - \frac{3(k-1)(n-1)}{kn}T_5$.

Now, we can use the previous lemmas to compute the variances of our proposed estimates of σ^2 as in the following theorems. Details are omitted.

Theorem 3.1 $Var(\hat{\sigma}^2_{1BRSS/propose}) = \frac{T_1+4T_2+4T_3}{k^2n} - \frac{1}{k^2n}T_5 + \frac{2}{k^4n(n-1)}\left(\sum_{i=1}^{k}\sigma^2_{(i)}\right)^2$.

Theorem 3.2 $Var(\hat{\sigma}^2_{2BRSS/propose}) =$

$$\frac{T_1+4T_2+4T_3}{k^2n} - \frac{(k^2n^2-k^2n-2)}{k^4n^2(n-1)}T_5 + \frac{2}{k^4n^2}\left(\sum_{i=1}^{k}\sigma^2_{(i)}\right)^2.$$

3.4 Computation of Variances of Our Proposed Estimators for Uniform Exponential and Normal Distributions

Proposition 3.1 For the uniform (0,1) distribution, the variances of our proposed estimators are given as

$$Var(\hat{\sigma}^2_{1BRSS/propose}) = \frac{1}{30(k+1)(k+2)n} + \frac{1}{18(k+1)^2k^2n(n-1)}.$$

$$Var(\hat{\sigma}^2_{2BRSS/propose}) = \frac{k^2+2k+2}{15k^3(k+1)^3)k+2)(n-1)} + \frac{k^5+2k^4+k^3-2k^2-4k-4}{30k^3n(k+1)^3(k+2)}$$

$$+\frac{5k^3+9k^2-2k-12}{90k^3n^2(k+1)^3(k+2)}.$$

Proposition 3.2 For a standard exponential distribution, the variance of our proposed estimator is given as

$$Var(\hat{\sigma}^2_{1BRSS/propose})$$

$$= \left[\frac{2(10k^2n-10k^2-8kn+8k+2n-1)}{k^4n(n-1)}\right]\sum_{i=1}^{k}\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)^4$$

$$+\left[\frac{8(2k-1)(k-1)}{k^4n}\right]\sum_{i=1}^{k}\sum_{l\neq l'=1}^{i}\left(\frac{1}{k-l+1}\right)^3\left(\frac{1}{k-l'+1}\right)$$

$$+\left[\frac{2(3k^2n-3k^2-4kn+4k+2n-1)}{k^4n(n-1)}\right]\sum_{i=1}^{k}\sum_{l\neq l'=1}^{i}\left(\frac{1}{k-l+1}\right)^2\left(\frac{1}{k-l'+1}\right)^2$$

$$+\left[\frac{4(k-1)^2}{k^4n}\right]\sum_{i=1}^{k}\sum_{l\neq l'\neq l''=1}^{i}\left(\frac{1}{k-l+1}\right)^2\left(\frac{1}{k-l'+1}\right)\left(\frac{1}{k-l''+1}\right)$$

$$-\frac{16}{k^3n}\sum_{i\neq i'=1}^{k}\left\{\left[\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)\right]\left[\sum_{l=1}^{i'}\left(\frac{1}{k-l+1}\right)^3\right]\right\}$$

$$+\frac{2(2n-1)}{k^4n(n-1)}\sum_{i\neq i'=1}^{k}\left\{\left[\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)^2\right]\left[\sum_{l=1}^{i'}\left(\frac{1}{k-l+1}\right)^2\right]\right\}$$

$$+\frac{4}{k^4n}\sum_{i\neq i'=1}^{k}\left\{\left[\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)^2\right]\left[\sum_{l\neq l'=1}^{i'}\left(\frac{1}{k-l+1}\right)\left(\frac{1}{k-l'+1}\right)\right]\right\}$$

$$-\frac{8}{k^3n}\sum_{i\neq i'=1}^{k}\left\{\left[\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)\right]\left[\sum_{l\neq l'=1}^{i'}\left(\frac{1}{k-l+1}\right)^2\left(\frac{1}{k-l'+1}\right)\right]\right\}$$

$$+\frac{4}{k^4n}\left[\sum_{i=1}^{k}\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)^2\right]\left[\sum_{i\neq i'=1}^{k}\left\{\left[\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)\right]\left[\sum_{l=1}^{i'}\left(\frac{1}{k-l+1}\right)\right]\right\}\right].$$

The variance of the other estimator has a similar expression. We omit details.

Proposition 3.3 For a standard normal distribution, the variance of our proposed estimator is given as

$$Var(\hat{\sigma}^2_{IBRSS/propose})$$

$$=\frac{2kn-2k+2}{k^2n(n-1)}+\frac{7kn-7k+12}{6(2\pi)^{\frac{3}{2}}k^3n(n-1)}\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_3(x)e^{-\frac{3}{2}x^2}dx\right)$$

$$+\frac{1}{24(2\pi)^{\frac{5}{2}}k^2n}\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_5(x)e^{-\frac{5}{2}x^2}dx\right)-\frac{4}{(2\pi)^2k^3n(n-1)}\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)^2$$

$$-\frac{2}{(2\pi)^2k^3n}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)\right]^2$$

$$+\frac{2}{(2\pi)^{\frac{7}{2}}k^3n}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)\right]\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)\left(\int_{-\infty}^{+\infty}P_3(x)e^{-\frac{3}{2}x^2}dx\right)\right]$$

$$+\frac{2}{(2\pi)^{\frac{7}{2}}k^4n}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)\right]^2\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_3(x)e^{-\frac{3}{2}x^2}dx\right)\right]$$

$$-\frac{4}{(2\pi)^4k^4n}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)\right]^2\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)^2\right]$$

$$-\frac{2}{3(2\pi)^3k^3n}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)\right]\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_4(x)e^{-2x^2}dx\right)\right]$$

$$+\frac{1}{2(2\pi)^3k^4n(n-1)}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_3(x)e^{-\frac{3}{2}x^2}dx\right)\right]^2$$

$$+\frac{2}{(2\pi)^4k^4n(n-1)}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)^2\right]^2$$

$$-\frac{2}{(2\pi)^{\frac{7}{2}}k^4n(n-1)}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_2(x)e^{-x^2}dx\right)^2\right]\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_3(x)e^{-\frac{3}{2}x^2}dx\right)\right]$$

$$-\frac{1}{4(2\pi)^3k^2n}\sum_{i=1}^{k}\left(\int_{-\infty}^{+\infty}P_3(x)e^{-\frac{3}{2}x^2}dx\right)^2.$$

In the above, $P_2(x)$-$P_5(x)$ are suitable polynomials defined in Bose and Gupta(1959).

The variance of the other estimator again has a similar expression. We omit details.

3.5 Computation of Variance of Unbiased Estimator of Population Variance Based on SRS for Uniform, Exponential and Normal Distributions

In section 2, the variance of $\hat{\sigma}^2_{SRS}$, in (2.2), is given as

$$Var(\hat{\sigma}^2_{SRS}) = \frac{\mu_4}{N} - \frac{(N-3)\sigma^4}{N(N-1)}.$$

Next, let us compute $Var(\hat{\sigma}^2_{SRS})$ for uniform, exponential and normal distributions.

For Uniform (0,1) distribution, we get $Var(\hat{\sigma}^2_{SRS}) = \frac{4N+6}{720N(N-1)}$.

For a standard exponential distribution, we get $Var(\hat{\sigma}^2_{SRS}) = \frac{8N-6}{N(N-1)}$. For a standard normal distribution, we get $Var(\hat{\sigma}^2_{SRS}) = \frac{2}{N-1}$.

3.6 Comparison of the Estimators

Now we compare $\hat{\sigma}^2_{SRS}$ and $\hat{\sigma}^2_{1BRSS/propose}$, $\hat{\sigma}^2_{2BRSS/propose}$, for uniform, exponential and normal distributions by computing the relative efficiency (R.E.) of $\hat{\sigma}^2_{BRSS/propose}$ relative to $\hat{\sigma}^2_{SRS}$ defined as follows.

$$\text{R.E.} = \left(\frac{Var(\hat{\sigma}^2_{SRS})}{Var(\hat{\sigma}^2_{BRSS/propose})} - 1 \right) \times 100\%.$$

Table 3.1 The relative efficiency (R.E.) of $\hat{\sigma}^2_{BRSS/propose}$ relative to $\hat{\sigma}^2_{SRS}$ for the uniform (0,1) distribution where k = 2, 3, 4, 5 and n = 10, 15.

	(n, k)							
$\hat{\sigma}^2_{BRSS/propose}$	(10, 2)	(10, 3)	(10, 4)	(10, 5)	(15, 2)	(15, 3)	(15, 4)	(15, 5)
$\hat{\sigma}^2_{1BRSS/propose}$	6.58	17.66	31.19	45.88	4.47	15.51	29.14	43.93
$\hat{\sigma}^2_{2BRSS/propose}$	6.89	17.86	31.32	45.98	4.61	15.60	29.20	43.97

Table 3.2 The relative efficiency (R.E.) of $\hat{\sigma}^2_{BRSS/propose}$ relative to $\hat{\sigma}^2_{SRS}$ for a standard exponential distribution where $k = 2, 3, 4, 5$ and $n = 10, 15$ $\hat{\sigma}^2_{BRSS/propose}$.

	(n, k)							
$\hat{\sigma}^2_{BRSS/propose}$	(10, 2)	(10, 3)	(10, 4)	(10, 5)	(15, 2)	(15, 3)	(15, 4)	(15, 5)
$\hat{\sigma}^2_{1BRSS/propose}$	3.74	8.08	12.53	16.94	3.58	7.90	12.38	16.79
$\hat{\sigma}^2_{2BRSS/propose}$	3.77	8.09	12.55	16.95	3.59	7.91	12.38	16.80

Table 3.3 The relative efficiency (R.E.) of $\hat{\sigma}^2_{BRSS/propose}$ relative to $\hat{\sigma}^2_{SRS}$ for a standard normal distribution where $k = 2, 3, 4, 5$ and $n = 10, 15$.

	(n, k)							
$\hat{\sigma}^2_{BRSS/propose}$	(10, 2)	(10, 3)	(10, 4)	(10, 5)	(15, 2)	(15, 3)	(15, 4)	(15, 5)
$\hat{\sigma}^2_{1BRSS/propose}$	2.61	10.74	20.23	30.04	1.76	9.91	19.46	29.33
$\hat{\sigma}^2_{2BRSS/propose}$	2.74	10.82	20.28	30.08	1.81	9.94	19.49	29.35

ACKNOWLEDGEMENTS

Our sincere thanks are due to Professor Bimal K. Sinha of the University Maryland Baltimore County (UMBC) for his critical comments and thoughtful suggestions which led to much improvement in the final version. We would also like to thank the Institute for the Promotion of Teaching Science and Technology (IPST) for granting a Development and Promotion of Science and Technology Talents Project (DPST) for the second author. Further, the first author would like to thank the Thailand Research Fund (TRF) for research grant.

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Calcutta Statistical Association Bulletin
Vol. 54, March & June 2003, Nos. 213-214

ON TWO NOVEL APPLICATIONS OF RANKED SET SAMPLING

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ABSTRACT: In this paper we discuss two case studies which clearly indicate the advantages of using a ranked set sample (RSS) over those of a simple random sample (SRS). The applications of RSS considered here cover single family homes sales data, and tree data. It is demonstrated that in each case RSS is much more efficient than SRS for estimation of population mean.

Keywords and phrases: Ranked set sample. Simple random sample, Balanced ranked set sample, Unbalanced ranked set sample.

AMS(2000) Subject Classification: Primary 62D05; Secondary 62F10

1. INTRODUCTION

In this paper we discuss two case studies which clearly indicate the advantages of using a ranked set sample (RSS) over those of a simple random sample (SRS). The applications of RSS considered here cover single family homes sales data (Bowerman and O'Connell, 1993), and tree data (Platt et al., 1988). The objective in these studies is not so much in a direct application of RSS to efficiently solve an estimation problem but rather to strongly support the fact that use of RSS over SRS does provide substantial advantages for estimation of a population mean. Thus in these studies samples from the entire population are made available so that a comparison

[1]Supported by a fellowship from the Institute for the Promotion of Teaching Science and Technology (IPST)
[2]Supported by a fellowship from Thailand Research Fund (TRF)

can be made among the population parameters and their estimates which are obtained via SRS and RSS. Moreover, in case of unbalanced RSS, since the population values are all available, it is indeed possible then to use the optimum choice of the *replications* of different order statistics.

For a description and some basic results on RSS, we refer to Stokes (1980), Takahasi and Wakimoto (1968), and Sinha et al. (1995). For other applications of RSS, we refer to Cobby et al. (1985), Halls and Dell (1966), and Sarikavanij and Tiensuwan (2003).

1. Applications of RSS

In this section we describe two applications of RSS. All relevant and required formulae on RSS are given in Appendix B.

2.1. Single family homes sales price data. In this application of RSS, we consider 63 single family homes sales price data given in Appendix A and compare the effectiveness of SRS and RSS based on a sample of size 15. The data are obtained from Table 4.2, pages 138-139, of Bowerman and O'Connell (1993). We consider these 63 values as comprising the population so that the population mean is 78.8 and the population variance is 1206.94. An SRS of size 15 is drawn from the population of 63 values, and the sample mean and sample variance are obtained as $\bar{x} = 70.5$ and $s^2 = 1306.5$, resulting in the estimate of the variance of the sample mean as 87. To draw an RSS of size 15, we first draw an SRS of size 45 out of 63 values, then divide the sample data at random into 5 sets of 9 each, then divide the 9 values within each set at random into a 3×3 square, and finally we draw an RSS of size 3 from each set. These values $\{X_{(r)i}\}$ are given below in Table 2.1.1.

Table 2.1.1. Ranked set sample of single family homes sales price

	i				
r	1	2	3	4	5
1	53.4	44.5	65.0	68.0	34.0
2	60.0	100.0	101.0	52.0	73.0
3	115.0	139.0	69.0	149.0	94.0

Several statistics can now be routinely computed based on the above values and various formulae available in Stokes (1980) and Perron and Sinha (2003). See Appendix B. These are given below in Table 2.1.2. It is clear that RSS is doing much better than SRS for estimation of the population mean.

Table 2.1.2. Estimates of population mean and variance of single family homes sales price based on RSS

$\hat{\mu}_{RSS}$	$\hat{var}(\hat{\mu}_{RSS})$	$\hat{\sigma}^2_{Stokes}$	$\hat{\sigma}^2_1$	$\hat{\sigma}^2_2$	$\hat{\sigma}^2_3$	$\hat{\sigma}^2_4$
81.13	39.5	1163.70	1110.32	1125.62	1186.80	926.77

We also generated 20 sets of RSS from the above population values and computed various summary statistics. These are given below in Table 2.1.3.

Table 2.1.3. Estimates of population mean and variance of single family homes sales price for 20 RSS samples

sample	$\hat{\mu}_{RSS}$	$\hat{var}(\hat{\mu}_{RSS})$	$\hat{\sigma}^2_{Stokes}$	$\hat{\sigma}^2_1$	$\hat{\sigma}^2_2$	$\hat{\sigma}^2_3$	$\hat{\sigma}^2_4$
1	81.13	39.50	1163.70	1110.32	1125.62	1186.80	926.77
2	83.16	42.45	1508.82	1436.24	1450.68	1508.43	1263.01
3	85.66	36.32	1417.18	1387.01	1359.02	1247.05	1722.92
4	76.27	30.92	713.75	712.85	607.08	634.03	902.00
5	78.33	19.72	575.90	558.75	557.22	551.12	577.06
6	80.53	55.19	950.89	936.61	942.68	966.97	863.77
7	72.85	28.11	551.74	545.45	543.06	533.52	574.07
8	77.81	53.95	1060.38	1051.76	1052.03	1053.12	1048.51
9	78.69	21.89	1138.41	1079.90	1084.41	1102.42	1025.85
10	76.55	47.60	747.75	746.28	745.50	742.30	755.69
11	88.92	78.02	2346.64	2278.20	2268.21	2228.27	2398.03
12	75.89	25.87	766.12	760.90	740.91	660.97	1000.74
13	85.97	60.33	2194.37	2128.62	2108.40	2027.54	2371.22
14	73.96	29.24	800.31	786.70	776.20	734.21	912.66
15	76	33.62	1059.61	1022.21	1022.59	1024.08	1017.74
16	82.79	74.33	2005.68	1978.58	1946.29	1817.14	2366.02
17	78.68	51.86	1088.85	1063.77	1068.12	1035.52	1011.58
18	72.7	27.12	693.41	665.79	674.30	708.35	563.65
19	86.04	53.62	2353.34	2262.49	2250.07	2200.37	2411.59
20	76.57	96.97	2000.21	2037.16	2047.83	2090.53	1909.08

From the above table, it is evident that even allowing for sampling fluctuations, RSS offers much improvement compared to SRS. Some salient features of the sampling distributions of the above statistics are summarized in the following Table 2.1.4.

Table 2.1.4. Summary statistics of estimates of population mean and variance for 20 RSS samples

variable	Mean	S.D.	Median
$\hat{\mu}_{RSS}$	79.43	4.70	78.51
$\hat{var}(\hat{\mu}_{RSS})$	45.33	20.56	40.97
$\hat{\sigma}^2_{Stokes}$	1261.80	611.86	1079.12
$\hat{\sigma}^2_1$	1227.48	590.82	1057.77
$\hat{\sigma}^2_2$	1223.01	587.10	1060.08
$\hat{\sigma}^2_3$	1205.14	575.26	1069.32
$\hat{\sigma}^2_4$	1281.10	656.40	1014.66

2.2. Tree data. In this application of RSS, we refer to an important data set from Platt et al. (1988) related to 399 conifer (pinus palustris) trees. The original data were collected on seven variables of which we have used only two: X, the diameter in centimeters at breast height, and Y, the entire height in feet. To make an easy application of RSS, we deleted the last 3 observations. This truncated data set is reported in Sarikavanij and Tiensuwan (2003).

Treating this as the population, we computed the population means $\mu_x = 20.9641$, $\mu_y = 52.6768$, the population variances $\sigma_x^2 = 309.545$, $\sigma_y^2 = 3253.44$. Considering its random decomposition into 44 squares each of dimension 3×3, we computed the variances of the three order statistics $\sigma^2_{y(1)} = 156.37$, $\sigma^2_{y(2)} = 1140.40$ and $\sigma^2_{y(3)} = 4174.35$ only for Y, which can be used when we consider an unbalanced RSS.

To compare SRS and RSS, we drew an SRS of size $n = 30$ and computed the sample mean and the sample variance for Y. For RSS, we drew a total of 90 simple random sample of trees from the above population, divided these observations into 10 sets of 9 each, then divided each set at random into a 3×3 square, and finally performed RSS based on X-values, which eventually led to RSS Y-values. For UBRSS, we used the replications $n_1 = 2$, $n_2 = 6$ and $n_3 = 22$ for the three order statistics in the usual way, which are proportional to 1 : 3.8 : 4.8. All these sample values are reported in Table 2.2.1.

Based on the sample values given in Table 2.2.1, we can easily compute estimates of the population mean and variance of Y as well as the estimates of the variance of the estimated mean of Y by each of the three methods: SRS, BRSS, UBRSS. Again, the formulae to compute these elements appear in Stokes (1980) and Perron and Sinha (2003). See Appendix B. These are reported below in Table 2.2.2. It is obvious from this table that both the BRSS and the UBRSS are doing much better than SRS.

Table 2.2.1. SRS, BRSS and UBRSS samples

No.	SRS sample	No.	BRSS sample	No.	UBRSS sample
1	219	1	8	1	22
2	16	2	38	1	66
3	11	3	103	2	16
4	94	1	5	2	33
5	33	2	40	2	36
6	28	3	223	2	38
7	203	1	6	2	26
8	180	2	70	2	22
9	16	3	78	3	38
10	154	1	4	3	37
11	5	2	27	3	140
12	6	3	232	3	208
13	11	1	21	3	105
14	32	2	16	3	107
15	19	3	96	3	21
16	43	1	26	3	92
17	8	2	82	3	104
18	85	3	106	3	92
19	25	1	11	3	222
20	35	2	12	3	109
21	119	3	91	3	38
22	17	1	20	3	120
23	68	2	40	3	87
24	152	3	176	3	244
25	19	1	6	3	84
26	23	2	9	3	33
27	222	3	38	3	105
28	223	1	6	3	111
29	9	2	11	3	113
30	3	3	154	3	192

Table 2.2.2. Estimates of population mean and variance of the height of tree

Method	Mean(y)	Varinace(y)	Variance of mean(y)
Population	52.68	3253.44	8.32
SRS	69.27	5773.03	192.43
BRSS	58.5	$\hat{\sigma}^2_{Stokes}$=4203.36	53.29
		$\hat{\sigma}^2_1$=4116.50	
		$\hat{\sigma}^2_2$=4116.54	
		$\hat{\sigma}^2_3$=4116.88	
		$\hat{\sigma}^2_4$=4115.50	
UBRSS	60.56	2008.90	74.23

To study the behavior of RSS from a frequentist point of view, we also generated 20 sets of RSS from the above population and computed various summary statistics. These are given below in Table 2.2.3 (a).(b). It is thus clear that even allowing for sampling fluctuations, RSS does much better than SRS.

Table 2.2.3(a). Estimates of population mean and variance of the height of tree for 20 RSS samples

sample	$\hat{\mu}_{RSS}$	$\hat{v}ar(\hat{\mu}_{RSS})$	$\hat{\sigma}^2_{Stokes}$	$\hat{\sigma}^2_1$	$\hat{\sigma}^2_2$	$\hat{\sigma}^2_3$	$\hat{\sigma}^2_4$
1	50.33	78.60	3019.30	2951.76	2997.15	3405.68	1726.18
2	60.17	116.54	5409.11	5311.50	5345.35	5649.92	4397.78
3	55.37	62.55	3250.52	3195.03	3204.71	3291.90	2933.45
4	50.67	71.16	3744.02	3706.69	3690.30	3543.61	4147.02
5	50.97	54.89	2413.27	2403.63	2387.72	2244.61	2832.96
6	52.47	76.41	3279.98	3235.54	3247.06	3350.77	2924.41
7	55.87	83.11	3751.91	3700.50	3700.96	3795.10	3445.08
8	40.53	52.67	2170.88	2149.55	2151.18	2165.87	2105.49
9	48.70	81.93	4016.70	3976.14	3964.74	3862.14	4283.96
10	58.87	106.86	3983.77	3990.92	3938.51	3466.84	5405.93
11	64.80	102.96	4142.30	4117.18	4107.19	4017.21	4387.12
12	60.63	66.88	4742.03	4636.67	4650.84	4778.45	4253.85
13	52.53	49.50	2902.40	2839.61	2855.15	2995.07	2419.87
14	52.53	60.76	2698.46	2679.74	2669.28	2675.18	2962.04
15	49.23	83.24	3321.43	3274.17	3293.95	3471.96	2740.16
16	42.83	41.33	1943.94	1934.88	1920.47	1790.78	2323.94
17	56.67	107.75	3705.20	3691.55	3689.44	3670.45	3748.53
18	42.13	68.84	3375.02	3331.02	3331.36	3334.24	3322.38
19	54.4	81.51	4378.52	4309.94	4314.09	435[illegible].39	419[illegible].04
20	48.17	53.26	2471.87	2430.33	2442.74	2549.86	2109.47

Table 2.2.3(b). Summary statistics of estimates of population mean and variance based on 20 RSS samples

variable	Mean	S.D	Median
$\hat{\mu}_{RSS}$	52.84	5.71	52.5
$\hat{v}ar(\hat{\mu}_{RSS})$	75.04	21.11	73.79
$\hat{\sigma}^2_{Stokes}$	3435.03	881.07	3348.23
$\hat{\sigma}^2_1$	3393.34	865.74	3302.60
$\hat{\sigma}^2_2$	3395.36	869.08	3312.66
$\hat{\sigma}^2_3$	3415.55	910.46	3436.26
$\hat{\sigma}^2_4$	3333.38	983.92	3142.21

To use the RSS regression estimate of the mean of Y, we generated afresh 25 SRS on Y and an equal number of BRSS samples on both X and Y, using $k = n = 5$. This is done in the usual way by first selecting RSS X-values and then the corresponding Y-values (judgment ordered), resulting in $(x_{(r)i}, y_{[r]i})$. These values are reported below in Table 2.2.4. We have used the results from Yu and Lam (1997) to compute the regression-based RSS estimate of μ_y and also its estimated variance. See Appendix B. Other results are also obtained in a similar way. These are all reported in Table 2.2.5. As expected, we find the performance of the regression-based RSS estimate of μ_y to be very impressive compared to those of SRS- and BRSS-based estimates.

Table 2.2.4. SRS and BRSS samples

No.	SRS sample	No.	BRSS sample	No.	SRS sample	No.	BRSS sample
1	12	1	(13.2,38)	14	34	4	(18.82)
2	77	2	(3.2,2)	15	140	5	(57.8,188)
3	41	3	(18.1,21)	16	25	1	(2.5,3)
4	7	4	(47.9,137)	17	26	2	(3.7,6)
5	82	5	(45.9,202)	18	70	3	(12.7,38)
6	9	1	(2.6,4)	19	85	4	(36.4,103)
7	7	2	(3.1,4)	20	10	5	(39.8,196)
8	5	3	(18.4,22)	21	239	1	(3,4)
9	176	4	(34,99)	22	5	2	(14.1,40)
10	34	5	(41.1,105)	23	13	3	(19.9,55)
11	105	1	(12.5,34)	24	34	4	(41.8,92)
12	26	2	(13.4,21)	25	105	5	(36.7,77)
13	3	3	(21.3,40)				

Table 2.2.5. Estimates of population mean and variance of the height of tree

Method	Mean(y)	Varinace(y)	Variance of mean(y)
Population	52.68	3253.44	8.22
SRS	54.8	3634.42	145.38
BRSS	64.52	$\hat{\sigma}^2_{Stokes}$=3897.76	36.76
		$\hat{\sigma}^2_1$=3780.18	
		$\hat{\sigma}^2_2$=3778.61	
		$\hat{\sigma}^2_3$=3772.31	
		$\hat{\sigma}^2_4$=3811.69	
REG	59.50	$\hat{\sigma}^2_{Stokes}$=3897.76	(based on $\hat{\sigma}^2_{Stokes}$) 26.69
		$\hat{\sigma}^2_1$=3780.18	(based on $\hat{\sigma}^2_1$) 25.89
		$\hat{\sigma}^2_2$=3778.61	(based on $\hat{\sigma}^2_2$) 25.88
		$\hat{\sigma}^2_3$=3772.31	(based on $\hat{\sigma}^2_3$) 25.83
		$\hat{\sigma}^2_4$=3811.69	(based on $\hat{\sigma}^2_4$) 26.10

Appendix A: Sales price of sixty-three single-family residences

Residence, i	Sales Price, y ($\times$\$1000)	Residence, i	Sales Price, y ($\times$\$1000)	Residence, i	Sales Price, y ($\times$\$1000)
1	53.5	22	87.9	43	90.0
2	49.0	23	80.0	44	83.0
3	50.5	24	94.0	45	115.0
4	49.9	25	74.0	46	50.0
5	52.0	26	69.0	47	55.2
6	55.0	27	63.0	48	61.0
7	80.5	28	67.5	49	147.0
8	86.0	29	35.0	50	210.0
9	69.0	30	142.5	51	60.0
10	149.0	31	92.2	53	100.0
11	46.0	32	56.0	53	44.5
12	38.0	33	63.0	54	55.0
13	49.5	34	60.0	55	53.4
14	105.0	35	34.0	56	65.0
15	152.5	36	52.0	57	73.0
16	85.0	37	75.0	58	40.0
17	60.0	38	93.0	59	141.0
18	58.5	39	60.0	60	68.0
19	101.0	40	73.0	61	139.0
20	79.4	41	71.0	62	140.0
21	125.0	42	83.0	63	55.0

Appendix B: Selected Formulae

Balanced data: $\{x_{(r)i}\}$, $r = 1, \ldots, k$ (set size); $i = 1, \ldots, n$ (number of cycles).

1. $\hat{\mu}_{brss} = \bar{x} = \sum_{i=1}^{n} \sum_{r=1}^{k} x_{(r)i}/kn$
2. $\hat{\sigma}^2_{Stokes} = \sum_{i=1}^{n} \sum_{r=1}^{k} (x_{(r)i} - \bar{x})^2/(kn - 1)$
3. $\hat{\sigma}_1^2 = W/kn + B/k(n-1)$
4. $\hat{\sigma}_2^2 = [(kn - k + 1)/(k^2 n(n-1))]W^* + B^*/kn$
5. $\hat{\sigma}_3^2 = W/kn + W^*/(k^2(n-1))$
6. $\hat{\sigma}_4^2 = [(kn - k + 1)/(kn(n-1))]B + B^*/kn$
7. $W = \sum_{i=1}^{n} \sum_{r=1}^{k} (x_{(r)i} - \bar{x}_{rss(i)})^2$
8. $W^* = \sum_{r=1}^{k} \sum_{i=1}^{n} (x_{(r)i} - \bar{x}_{(r)})^2$
9. $B = k \sum_{i=1}^{n} (\bar{x}_{rss(i)} - \bar{x}_{brss})^2$

10. $B^* = n\sum_{r=1}^{k}(\bar{x}_{(r)} - \bar{x}_{brss})^2$

11. $\bar{x}_{rss(i)} = \sum_{r=1}^{k} x_{(r)i}/k$

12. $\bar{x}_{(r)} = \sum_{i=1}^{n} x_{(r)i}/n$

13. $\bar{x}_{brss} = \bar{x} = \sum_{i=1}^{n} \bar{x}_{rss(i)}/n = \sum_{i=1}^{n}[\sum_{r=1}^{k} x_{(r)i}/k]/n$

14. $\hat{v}ar(\hat{\mu}_{brss}) = \hat{v}ar(\bar{x}_{brss}) = \sum_{r=1}^{k} \hat{\sigma}_{(r)}^2/k^2 n$

15. $\hat{\sigma}_{(r)}^2 = \sum_{i=1}^{n}(x_{(r)i} - \bar{x}_{(r)})^2/(n-1)$

16. $\hat{\rho}_{brss} = \frac{\sum_{i=1}^{n}\sum_{r=1}^{k}(x_{(r)i}-\bar{x}_{brss})(y_{[r]i}-\bar{y}_{brss})}{\sqrt{\sum_{i=1}^{n}\sum_{r=1}^{k}(x_{(r)i}-\bar{x}_{brss})^2\sum_{i=1}^{n}\sum_{r=1}^{k}(y_{[r]i}-\bar{y}_{brss})^2}}$

$y_{[r]i}$: judgment ordered value.

Unblanced data: $\{x_{(r)i}\}$, $i = 1, ..., n_r$; $r = 1, ..., k$.

17. $\hat{\mu}_{ubrss} = \sum_{r=1}^{k}[\sum_{i=1}^{n_r} x_{(r)i}/n_r]/k$

18. $\hat{v}ar(\hat{\mu}_{ubrss}) = \sum_{r=1}^{k} \hat{\sigma}_{(r)}^2/k^2 n_r$

19. $\hat{\sigma}_{(r)}^2 = \sum_{i=1}^{n_r}(x_{(r)i} - \bar{x}_{(r)})^2/(n_r - 1)$

20. $\bar{x}_{(r)} = \sum_{i=1}^{n_r} x_{(r)i}/n_r$

21. $\hat{\sigma}_{ubrss}^2 = \sum_{r=1}^{k}[1 + 1/k(n_r - 1)]s_{(r)}^2/kn_r + \sum_{r=1}^{k}(\bar{x}_{(r)} - \hat{\mu}_{ubrss})^2/k$

22. $s_{(r)}^2 = \sum_{i=1}^{n_r}(x_{(r)i} - \bar{x}_{(r)})^2$.

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BRG4680010 Pak J. Statist.
2004 Vol. 20(1) pp 31 – 36 Report 36 months

ESTIMATION OF RELIABILITY BASED ON EXPONENTIAL DISTRIBUTION AND RANKED SET SAMPLE

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ABSTRACT

Assume X (strength) $\sim(1/\theta_1)e^{-x/\theta_1}$, $x>0$, $\theta_1>0$, independent of Y (stress) $\sim(1/\theta_2)e^{-y/\theta_2}$, $y>0$, $\theta_2>0$. In this paper we consider the problem of estimation of the reliability $R(\theta_1,\theta_2)=P(X>Y)$. We consider both simple random sample (SRS) and ranked set sample (RSS), and provide several estimates of R along with their comparisons.

1. INTRODUCTION
2.

In this paper we consider the problem of estimation of the reliability $R(\theta_1,\theta_2)=P(X>Y)$, based on $X_1,\dots,X_N \sim iid \sim X$ where X is the strength with pdf, $f(x)=(1/\theta_1)e^{-x/\theta_1}$, and $Y_1,\dots,Y_M \sim iid \sim Y$ where Y is the stress with pdf, $f(y)=(1/\theta_2)e^{-y/\theta_2}$, and X and Y are independent. We consider both simple random sample (SRS) and ranked set sample (RSS), and provide several estimates of R. Under RSS, we have used three estimates of R. The comparisons of the estimates of R are conducted for large sample sizes as well as small sample sizes.

For details about RSS, we refer to Stokes (1980), McIntyre (1952), Takahasi and Wakimoto (1968), Dell and Clutter (1972) and Sinha, Sinha and Purkayastha (1995).

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Since $X \sim Exp(\theta_1)$ and $Y \sim Exp(\theta_2)$ $R(\theta_1,\theta_2)=\theta_1/(\theta_1+\theta_2)$. For estimation of $R(\theta_1,\theta_2)$ based on SRS, let $X_1,\ldots,X_N \sim iid \sim Exp(\theta_1)$, $Y_1,\ldots,Y_M \sim iid \sim Exp(\theta_2)$. Obviously standard estimates of θ_1 and θ_2 are $\overline{X}$ and $\overline{Y}$, respectively. So we use

$$\hat{R}_{SRS}(\theta_1,\theta_2)=\frac{\overline{X}}{\overline{X}+\overline{Y}}. \tag{1}$$

By the Central Limit Theorem, for large N and M, $\overline{X} \sim N(\theta_1,\frac{\theta_1^2}{N})$, $\overline{Y} \sim N(\theta_2,\frac{\theta_2^2}{M})$.
Therefore, by using standard Taylor expansion, we get for large N and M:

$$\hat{R}_{SRS}(\theta_1,\theta_2) \sim N\left[R(\theta_1,\theta_2),\frac{\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}\left(\frac{1}{N}+\frac{1}{M}\right)\right]. \tag{2}$$

For using RSS, we write $N=kn$ and $M=sm$, and draw RSS from the X-population as $\{X_{(ii)}^{(j)}\}$, $i=1,\ldots,k$; $j=1,\ldots,n$ and from the Y-population as $\{Y_{(ii)}^{(j)}\}$, $i=1,\ldots,s$; $j=1,\ldots,m$, (see McIntyre (1952)). From McIntyre (1952) and Sinha, Sinha and Purkayastha (1995), the estimates of θ_1 and θ_2 based on RSS are obtained as

$$\hat{\theta}_{1Mc}=\sum_{j=1}^{n}\sum_{i=1}^{k}\frac{X_{(ii)}^{(j)}}{kn}, \qquad \hat{\theta}_{2Mc}=\sum_{j=1}^{m}\sum_{i=1}^{s}\frac{Y_{(ii)}^{(j)}}{ms} \tag{3}$$

$$\hat{\theta}_{1Blue}=\left[\sum_{j=1}^{n}\sum_{i=1}^{k}\frac{X_{(ii)}^{(j)}}{c_{i,k}a_{i,k}}\right]\Big/\left[\sum_{j=1}^{n}\sum_{i=1}^{k}\frac{1}{a_{i,k}}\right], \quad \hat{\theta}_{2Blue}=\left[\sum_{j=1}^{m}\sum_{i=1}^{s}\frac{Y_{(ii)}^{(j)}}{c_{i,s}a_{i,s}}\right]\Big/\left[\sum_{j=1}^{m}\sum_{i=1}^{s}\frac{1}{a_{i,s}}\right] \tag{4}$$

where $a_{i:k}=\frac{d_{i:k}}{c_{i:k}^2}$, $d_{i,k}=\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)^2$, $c_{i:k}=\sum_{l=1}^{i}\left(\frac{1}{k-l+1}\right)$, $a_{i:s}=\frac{d_{i:s}}{c_{i:s}^2}$,
$d_{i:s}=\sum_{l=1}^{i}\left(\frac{1}{s-l+1}\right)^2$, $c_{i:s}=\sum_{l=1}^{i}\left(\frac{1}{s-l+1}\right)$. Here $\hat{\theta}_{Blue}$ is the best linear unbiased estimate of θ based on RSS-data. For $\hat{\theta}_{Opt}$, our strategy is a variation of the usual RSS sample, which is based on always drawing the r^{th} order statistic from each row of all the cycles, r depending on the set size, resulting in $X_{(ir_k)}^{(j)}$, $i=1,\ldots,k$; $j=1,\ldots,n$ and $Y_{(ir_s)}^{(j)}$, $i=1,\ldots,s$; $j=1,\ldots,m$. Following Sinha et al. (1995), we use

$$\hat{\theta}_{1Opt}=\left[\sum_{j=1}^{n}\sum_{i=1}^{k}\frac{X_{(ir_k)}^{(j)}}{c_{r,k}}\right]\Big/kn, \qquad \hat{\theta}_{2Opt}=\left[\sum_{j=1}^{m}\sum_{i=1}^{s}\frac{X_{(ir_s)}^{(j)}}{c_{r,s}}\right]\Big/sm. \tag{5}$$

Here r_k is such that $a_{r:k}$ is the smallest among $a_{1:k},\dots,a_{k:k}$ and r_s is such that $a_{r:s}$ is the smallest among $a_{1:s},\dots,a_{s:s}$.

Once θ_1 and θ_2 are estimated as above, an estimate of $R(\theta_1,\theta_2)$ is obtained by $\hat{R}_{RSS}(\theta_1,\theta_2)=\dfrac{\hat{\theta}_{1RSS}}{\hat{\theta}_{1RSS}+\hat{\theta}_{2RSS}}$. To study the large sample properties of $\hat{R}_{RSS}(\theta_1,\theta_2)$, we first state the following theorem where proof follows from the CLT.

Theorem 2.1 For large *n* and *m*, the distributions of the estimates of θ_1 and θ_2 based on RSS are given by

$$\text{a) } \hat{\theta}_{1Mc}\sim N\left[\theta_1,\frac{\theta_1^2}{k^2n}\sum_{i=1}^{k}d_{i:k}\right],\qquad \hat{\theta}_{2Mc}\sim N\left[\theta_2,\frac{\theta_2^2}{s^2m}\sum_{i=1}^{s}d_{i:s}\right] \tag{6}$$

$$\text{b) } \hat{\theta}_{1Blue}\sim N\left[\theta_1,\frac{\theta_1^2}{n}\left(\sum_{i=1}^{k}\frac{1}{a_{i:k}}\right)^{-1}\right],\qquad \hat{\theta}_{2Blue}\sim N\left[\theta_2,\frac{\theta_2^2}{m}\left(\sum_{i=1}^{s}\frac{1}{a_{i:s}}\right)^{-1}\right] \tag{7}$$

$$\text{c) } \hat{\theta}_{1Opt}\sim N\left[\theta_1,\frac{\theta_1^2a_{r:k}}{kn}\right],\qquad \hat{\theta}_{2Opt}\sim N\left[\theta_2,\frac{\theta_2^2a_{r:s}}{sm}\right] \tag{8}$$

The large sample distributions of $\hat{R}_{RSS}(\theta_1,\theta_2)$ are stated below.

Theorem 2.2 For large *n* and *m*, the distributions of the estimates of $R(\theta_1,\theta_2)$ based on RSS are given by the following:

$$\text{a) } \hat{R}_{Mc}(\theta_1,\theta_2)\sim N\left[R(\theta_1,\theta_2),\frac{\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}\left(\sum_{i=1}^{k}d_{i:k}/k^2n+\sum_{i=1}^{s}d_{i:s}/s^2m\right)\right] \tag{9}$$

$$\text{b) } \hat{R}_{Blue}(\theta_1,\theta_2)\sim N\left[R(\theta_1,\theta_2),\frac{\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}\left(\frac{1}{n}\left(\sum_{i=1}^{k}\frac{1}{a_{i:k}}\right)^{-1}+\frac{1}{m}\left(\sum_{i=1}^{s}\frac{1}{a_{i:s}}\right)^{-1}\right)\right] \tag{10}$$

$$\text{c) } \hat{R}_{Opt}(\theta_1,\theta_2)\sim N\left[R(\theta_1,\theta_2),\frac{\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4}\left(\frac{a_{r:k}}{kn}+\frac{a_{r:s}}{sm}\right)\right] \tag{11}$$

Proof. Follows from Theorem 2:1 and Taylor expansion.

3. COMPARISON OF ESTIMATES

In this section we provide a comparison of the above estimates of $R(\theta_1,\theta_2)$. We first mention about the large sample result.

Theorem 3.1 For large n and m, $Var(\hat{R}_{Opt}) < Var(\hat{R}_{Blue}) < Var(\hat{R}_{Mc}) < Var(\hat{R}_{SRS})$.

Proof. 1. To compare $Var(\hat{R}_{SRS})$ with $Var(\hat{R}_{Mc})$ is equivalent to comparing $1/N$ with $\sum_{i=1}^{k} d_{i:k}/k^2 n$ and $1/M$ with $\sum_{i=1}^{s} d_{i:s}/s^2 m$.

Since $N=kn$ and $\sum_{i=1}^{k} d_{i:k} \le k$, so $\sum_{i=1}^{k} d_{i:k}/k^2 n = \sum_{i=1}^{k} d_{i:k}/Nk \le 1/N$. Similarly we get $\sum_{i=1}^{s} d_{i:s}/sM \le 1/M$. So $Var(\hat{R}_{Mc}) < Var(\hat{R}_{SRS})$.

2. To compare $Var(\hat{R}_{Mc})$ with $Var(\hat{R}_{Blue})$ is equivalent to comparing $\sum_{i=1}^{k} d_{i:k}/k^2 n$ with $\frac{1}{n}\left(\sum_{i=1}^{k}\frac{1}{a_{i:k}}\right)^{-1}$ and $\sum_{i=1}^{s} d_{i:s}/s^2 m$ with $\frac{1}{m}\left(\sum_{i=1}^{s}\frac{1}{a_{i:s}}\right)^{-1}$.

From the numerical computations, we verified that $\left(\sum_{i=1}^{k}\frac{1}{a_{i:k}}\right)^{-1} < \sum_{i=1}^{k} d_{i:k}/k^2$ for all k. Of course the same is true for comparing $\sum_{i=1}^{s} d_{i:s}/s^2$ with $\left(\sum_{i=1}^{s}\frac{1}{a_{i:s}}\right)^{-1}$.

So $Var(\hat{R}_{Blue}) < Var(\hat{R}_{Mc})$.

3. To compare $Var(\hat{R}_{Blue})$ with $Var(\hat{R}_{Opt})$ is equivalent to comparing $\frac{1}{n}\left(\sum_{i=1}^{k}\frac{1}{a_{i:k}}\right)^{-1}$ with $\frac{a_{r:k}}{kn}$ and $\frac{1}{m}\left(\sum_{i=1}^{s}\frac{1}{a_{i:s}}\right)^{-1}$ with $\frac{a_{r:s}}{sm}$.

Since $a_{r:k} \le a_{i:k}, \forall i$, then $\frac{1}{a_{r:k}} \ge \frac{1}{a_{i:k}}$ and $\frac{k}{a_{r:k}} \ge \sum_{i=1}^{k}\left(\frac{1}{a_{i:k}}\right)$, so $\frac{a_{r:k}}{k} \le \left(\sum_{i=1}^{k}\frac{1}{a_{i:k}}\right)^{-1}$.

Similarly, we get $\frac{a_{r:s}}{s} \le \left(\sum_{i=1}^{s}\frac{1}{a_{i:s}}\right)^{-1}$. So $Var(\hat{R}_{Opt}) < Var(\hat{R}_{Blue})$

This completes the proof.

We conclude this paper with a small sample comparison of the above estimates of $R(\theta_1,\theta_2)$ based on 1000 simulations using SAS. We have taken $N=M=10$, and $n=m=5$, $k=s=2$. The table below shows the bias and the variance of the proposed estimates of $R(\theta_1,\theta_2)$.

Table 3.1 Comparison of estimates of $R(\theta_1,\theta_2)$ in small samples

	$\theta_1=1,\ \theta_2=1$ R=0.5		$\theta_1=1,\ \theta_2=2$ R=0.33		$\theta_1=1,\ \theta_2=3$ R=0.25		$\theta_1=1,\ \theta_2=4$ R=0.2	
	bias	var	bias	var	bias	var	bias	var
SRS	0.00179	0.01041	0.00781	0.00842	0.00933	0.00625	0.00943	0.00472
McIntyre	0.00618	0.00797	0.01031	0.00648	0.01082	0.00477	0.01036	0.00357
Blue	-0.00358	0.00884	0.00205	0.00705	0.00399	0.00515	0.00459	0.00383
Optimum	-0.00292	0.00807	0.00744	0.00656	0.00841	0.00484	0.00831	0.00362

Table 3.1 (continued)

	$\theta_1=2,\ \theta_2=1$ R=0.67		$\theta_1=3,\ \theta_2=1$ R=0.75		$\theta_1=4,\ \theta_2=1$ R=0.8	
	bias	var	bias	var	bias	var
SRS	-0.00469	0.00863	-0.00677	0.00648	-0.00729	0.00492
McIntyre	0.00073	0.00645	-0.00146	0.00475	-0.00234	0.00355
Blue	-0.00854	0.00732	-0.00961	0.00546	-0.00949	0.00413
Optimum	-0.00221	0.00651	-0.00396	0.00478	-0.00448	0.00357

It follows from the above table that, even in small samples, the estimates of $R(\theta_1,\theta_2)$ based on RSS have both smaller bias and smaller variance compared to the SRS-based estimate. It also happens that the estimate of $R(\theta_1,\theta_2)$ based on McIntyre procedure is marginally better than the two other RSS-based estimates.

ACKNOWLEDGEMENT

* Supported by a fellowship from the Institute for the Promotion of Teaching Science and Technology (IPST).
** Supported by a fellowship from Thailand Research Fund (TRF).
*** Supported by UMBC's Presidential Research Professorship grant.

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Pak J. Statist.
2004 Vol. 20(2) pp 245 – 261

A STATISTICAL APPROACH TO COMBINING ENVIRONMENTAL INDICES WITH AN APPLICATION TO AIR POLLUTION DATA FROM BANGKOK, THAILAND

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ABSTRACT

In this paper we concentrate on the air pollution data measured as carbon monoxide, nitrogen dioxide, sulfur dioxide and ozone from ten monitoring stations in Bangkok, Thailand and apply Multiple Criteria Decision Making (MCDM) method to compute an overall air pollution index for these stations and compare them. We also study robustness of these overall indices.

1. INTRODUCTION

Multiple Criteria Decision Making (MCDM) has recently been recognized as an efficient statistical method to combine component 'indices' arising from many 'sources' into a single overall meaningful index. Such an index can be effectively used to compare relevant 'facilities'. The basic premise is a data matrix $X = (x_{ij}) : K \times N$ where the rows represent facilities which need to be compared or ranked with respect to the element x_{ij}'s, the columns represent various sources of the elements x_{ij}'s and x_{ij}'s themselves represent some quantitative information about the facilities. In the context of environmental science, the x_{ij}'s may represent levels of pollutants, facilities represent the sources of the pollutants (e.g., chemical or nuclear facilities) and the columns represent different types of pollution. Since usually it is difficult to compare the facilities on a multiple scale, MCDM provides a statistical method to combine the elements in any row into a single value which can then be used to compare the rows on a linear scale.

In this paper we briefly review MCDM in Section 2 and apply this technique and some of its variations to the air pollution data from Bangkok, Thailand in Section 3. Some conclusions are drawn in Section 4.

2. MCDM AND ITS MODIFICATIONS

In this section we briefly describe the Multiple Criteria Decision Making procedure and some of its variations.

2.1 A Brief Description

MCDM is a procedure to integrate multiple indicators into a single meaningful and overall index by combining $(x_{i1},\ldots,x_{iN})$ for row i across all indicators $j=1,2,\ldots,N$. We can define an Ideal Row as one with the smallest observed value for each column

$$\text{IDR} = (\min_i x_{i1},\ldots,\min_i x_{iN}) = (u_1,\ldots,u_N)$$

and a Negative-ideal Row (NIDR) as one with the largest observed value for each column

$$\text{NIDR} = (\max_i x_{i1},\ldots,\max_i x_{iN}) = (v_1,\ldots,v_N).$$

For any given row i, we now compute the distance of each row from Ideal row and from Negative Ideal row based on the L_2-norm by using the formulae :

$$L_2(i,\text{IDR}) = [\sum_{j=1}^{N}\frac{(x_{ij}-u_j)^2 w_j}{\sum_{i=1}^{K} x_{ij}^2}]^{1/2}$$

$$L_2(i,\text{NIDR}) = [\sum_{j=1}^{N}\frac{(x_{ij}-v_j)^2 w_j}{\sum_{i=1}^{K} x_{ij}^2}]^{1/2}$$

where $w_1,w_2,\ldots,w_N$ are suitably chosen nonnegative weights between 0 and 1. The denominator above plays the role of a 'norming' factor. An objective way to select the weights is to use Shannon's [4] entropy measure φ based on the proportion $p_{1j},\ldots,p_{Kj}$ for the j th column where

$$p_{ij} = x_{ij} / \sum_{i=1}^{K} x_{ij}\ .$$

For the j th column, ϕ_j is computed as

$$\phi_j = \sum_{i=1}^{K} p_{ij}\ln(p_{ij})/[\ln(K)].$$

The quantity φ essentially provides a measure of closeness of the different proportions. The smaller the value of φ, the larger the variation among the proportions for classifying the rows. So we can select the weights as

$$w_j = (1-\phi_j)/[\sum_{j=1}^{N}(1-\phi_j)] \quad , \quad j = 1, \ldots, N .$$

In addition to Shannon's entropy measure, we can also use the sample variance of these proportions, given by

$$s^2_{j,\,prop} = \sum_{j=1}^{N}(p_{ij} - \bar{p}_j)^2 / (K-1) .$$

If $\bar{x}_j$ and s_j^2 denote the mean and variance of x_{ij} in the jth column, $s^2_{j,\,prop}$ is directly proportional to $s_j^2/\bar{x}_j^2$, which is the square of the sample coefficient of variation cv_j. Therefore we propose to use $w_j = cv_j$.

The various rows are now ranked based on an overall index I computed as

$$I_i = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, \quad i = 1, \ldots, K .$$

In addition to L_2-norm we can also use the L_1-norm as a distance measure and rank the rows once again. L_1-norm distance is defined below and the denominator again is used as a 'norming' factor.

$$L_1(i, IDR) = \sum_{j=1}^{N}\frac{|x_{ij} - u_j| w_j}{\sum_{i=1}^{K} x_{ij}} .$$

$$L_1(i, NIDR) = \sum_{j=1}^{N}\frac{|x_{ij} - v_j| w_j}{\sum_{i=1}^{K} x_{ij}} .$$

2.2 Modifications of MCDM

Here we describe two modifications of MCDM (Sinha and Shah, 2002). Let $d_i = [d_{i1}, d_{i2}, \ldots, d_{iN}]$ which represents the row-vector of d_{ij}'s, distance of x_{ij} from $\min_i x_{ij}$, $1 \le i \le K, 1 \le j \le N$, for i-th row involving N columns, and

$d_i^- = [d_{i1}^-, d_{i2}^-, \ldots, d_{iN}^-]$ which represents the row-vector of d_{ij}^-'s, distance of x_{ij} from $\max_i x_{ij}$, $1 \le i \le K, 1 \le j \le N$, for i-th row involving N columns.

Modification I:

$$L_i(d, d^-) = \left[\sum_j w_j (d_{ij}^k / d_{ij}^{-k}) / \left[\sum_i x_{ij}^2\right]^{k/2}\right]^{1/2} + \left[\sum_j{}' w_j R_j^k / \left[\sum_i x_{ij}^2\right]^{k/2}\right]^{1/2}$$

where Σ refers to all j for which $d_{ij}^- > 0$ while Σ' refers to all j for which $d_{ij}^- = 0$ and R_j is a finite quantity of our choice subject to $R_j \ge \max [d_{ij} / d_{ij}^-]$ taken over all i for which $d_{ij}^- > 0$.

Modification II:

$$L_i(d, d^-) = \left[\sum_j (w_j d_{ij}^k) / \left[\sum_i x_{ij}^2\right]^{k/2}\right]^{1/2} + \left[\sum_j{}' (w_j / d_j^{-k}) / \left[\sum_i x_{ij}^2\right]^{k/2}\right]^{1/2} + \left[\sum_j{}'' (w_j / r_j^2) / \left[\sum_i x_{ij}^2\right]^{k/2}\right]^{1/2}$$

where $r_j \ge \min\{d_{ij}^-\}$ being taken over all $d_{ij}^- > 0$ and Σ' refers to all j for which $d_{ij}^- > 0$ while Σ'' refers to all j for which $d_{ij}^- = 0$. In the above k is a positive number.

To check the robustness of various sets of ranks produced by different methods, we will compute Spearman's rank correlation (SRC) coefficient :

$$r = 1 - \frac{1}{K(K^2 - 1)} \sum_{i=1}^{K} \Delta_i^2$$

where Δ_i = difference between ranks. It is obvious that a large value of r signifies a good agreement.

2.3 Electre Method

Electre Method (Sinha and Shah, 2002) is used for comparing the status of two locations rather than ranking all of them together. We begin with the $K \times N$ data matrix X of observations and proceed as follows:

Step 1:

Transform $X = [X_1, X_2, \ldots, X_N]$ to $R = [R_1, R_2, \ldots, R_N]$ where $R_i = \dfrac{X_i}{\|X_i\|^2}$.

Step 2:

Transform R to $V = RW$ where $W = \text{diag}[w_1, w_2, \ldots, w_N]$.

Step 3:

Construct two matrices **C** and **D**

where $c_{ij} = \sum_{k: v_{ik} \le v_{jk}} w_k$ and $d_{ij} = \dfrac{\max_{k: v_{ik} < v_{jk}} |v_{ik} - v_{jk}|}{\max_k |v_{ik} - v_{jk}|}$.

Compute $\bar{c} = \dfrac{\sum \sum_{i \ne j} c_{ij}}{K(K-1)}$ and $\bar{d} = \dfrac{\sum \sum_{i \ne j} d_{ij}}{K(K-1)}$.

Step 4:

Construct matrices F and G such that

$$f_{ij} = \begin{cases} 1 \ ; c_{ij} \le \bar{c} \\ 0 \ ; \text{otherwise} \end{cases} \quad \text{and} \quad g_{ij} = \begin{cases} 1 \ ; d_{ij} \le \bar{d} \\ 0 \ ; \text{otherwise} \end{cases}.$$

Step 5:

Define matrix E where $e_{ij} = f_{ij} \cdot g_{ij}$.

It should be noted that the weights w_i's are obtained as discussed before, and that $e_{ij} = 0$ means that row i is better than row j.

3. AN APPLICATION

In this section we apply the previously described MCDM method and its modifications to the air pollution data from Bangkok, Thailand. The main air pollutants in Bangkok are carbon monoxide (CO_2), nitrogen dioxide (NO_2) and sulfur dioxide (SO_2) which are released directly from motor vehicles. The photochemical reaction on the oxide of nitrogen is ozone (O_3) which is a secondary pollutant.

The data sets were provided by the Pollution Control Department of Thailand and were recorded by 10 monitoring stations in Bangkok during 1998 – 2001. The monitoring stations are as follows:

1. Ramkhamheang University
2. National Housing Authority
3. Huai Khwang
4. Nonsee Vitaya School
5. Singharatpitayakom School
6. Thonburi
7. Chokchai 4
8. Dindaeng
9. Meteorological Department
10. Ratburana.

The locations of 10 monitoring stations in Bangkok are shown in Figure 1.

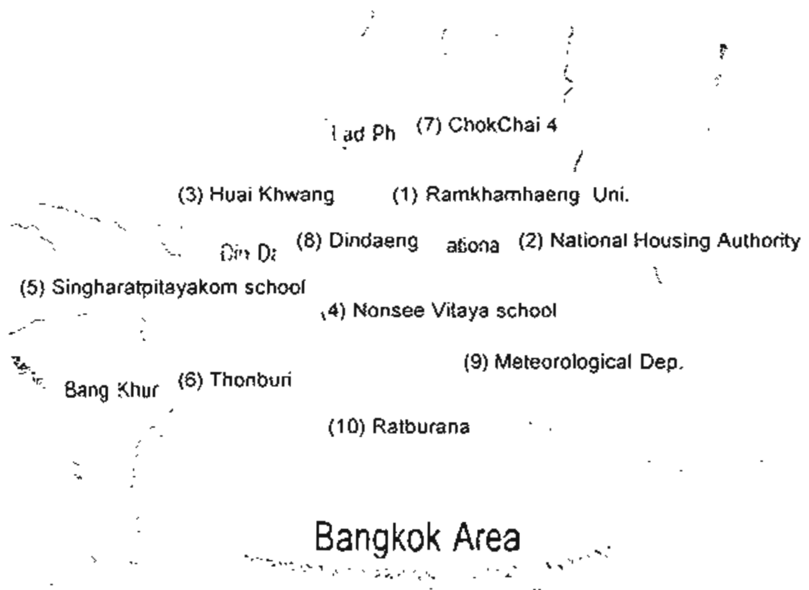

Figure 1 : Location of 10 monitoring stations in Bangkok area

At each station, the signals from the instruments were sampled every five seconds and hourly average values were calculated and stored. For our analysis, we have used the annual averages of each pollutant. The entire data set appears in a Technical Report (Lertprapai et al., 2003).

To apply the MCDM method, we use both the distance measures L_1 and L_2 as well as the two choices of weights based on phi and coefficient of variation (cv). We show below the results in four sets of the values of combined indices for each year. The final ranks of the rows are then based on the average index. We also compute the standard deviation to show the closeness of the four indices in a row. Tables 1 – 4 present all the results for years 1998 - 2001.

Table 1: Results of MCDM Method on Air Pollution Data in 1998.

Monitoring Station	L1		L2		Mean	SD	Rank
	W1	W2	W1	W2			
1) Ramkhamheang University	0.3574	0.3610	0.3891	0.3922	0.3749	0.0183	6
2) National Housing Authority	0.2983	0.3023	0.3327	0.3359	0.3173	0.0197	4
3) Huai Khwang	0.4423	0.4461	0.4390	0.4425	0.4425	0.0029	9
4) Nonsee Vitaya school	0.2934	0.2896	0.3271	0.3235	0.3084	0.0196	3
5) Singharatpitayakom school	0.3707	0.3685	0.3932	0.3937	0.3815	0.0138	7
6) Thonburi	0.4255	0.4293	0.4281	0.4316	0.4286	0.0025	8
7) Chokchai 4	0.3685	0.3665	0.3716	0.3698	0.3691	0.0021	5
8) Dindaeng	0.8054	0.7983	0.6909	0.6855	0.7450	0.0657	10
9) Meteorological Department	0.0387	0.0402	0.0492	0.0503	0.0446	0.0060	1
10) Ratburana	0.1217	0.1236	0.1828	0.1858	0.1535	0.0356	2

Table 2: Results of MCDM Method on Air Pollution Data in 1999.

Monitoring Station	L1		L2		Mean	SD	Rank
	W1	W2	W1	W2			
1) Ramkhamheang University	0.4414	0.4374	0.4683	0.4667	0.4534	0.0163	9
2) National Housing Authority	0.3281	0.3154	0.3709	0.3634	0.3445	0.0269	3
3) Huai Khwang	0.4271	0.4172	0.4287	0.4190	0.4230	0.0058	7
4) Nonsee Vitaya school	0.3519	0.3367	0.3745	0.3621	0.3563	0.0160	4
5) Singharatpitayakom school	0.4193	0.4004	0.4328	0.4169	0.4174	0.0133	6
6) Thonburi	0.4499	0.4451	0.4464	0.4411	0.4456	0.0036	8
7) Chokchai 4	0.3654	0.3609	0.3665	0.3615	0.3636	0.0028	5
8) Dindaeng	0.7386	0.7341	0.6180	0.6146	0.6763	0.0694	10
9) Meteorological Department	0.1890	0.1984	0.2448	0.2518	0.2210	0.0319	2
10) Ratburana	0.1382	0.1404	0.1831	0.1851	0.1617	0.0259	1

Table 3: Results of MCDM Method on Air Pollution Data in 2000.

Monitoring Station	L1		L2		Mean	SD	Rank
	W1	W2	W1	W2			
1) Ramkhamheang University	0.4031	0.4041	0.4141	0.4150	0.4091	0.0064	9
2) National Housing Authority	0.2499	0.2470	0.2951	0.2930	0.2713	0.0264	4
3) Huai Khwang	0.3090	0.3190	0.3038	0.3126	0.3111	0.0064	7
4) Nonsee Vitaya school	0.2067	0.2305	0.2485	0.2660	0.2379	0.0254	2
5) Singharatpitayakom school	0.3392	0.3315	0.3600	0.3558	0.3466	0.0135	8
6) Thonburi	0.2907	0.2958	0.2909	0.2948	0.2930	0.0026	5
7) Chokchai 4	0.3010	0.3143	0.2929	0.3045	0.3032	0.0089	6
8) Dindaeng	0.7350	0.7544	0.6410	0.6489	0.6948	0.0582	10
9) Meteorological Department	0.1420	0.1320	0.1599	0.1548	0.1472	0.0126	1
10) Ratburana	0.2370	0.2243	0.3049	0.2983	0.2661	0.0414	3

Table 4: Results of MCDM Method on Air Pollution Data in 2001.

Monitoring Station	L1		L2		Mean	SD	Rank
	W1	W2	W1	W2			
1) Ramkhamheang University	0.4144	0.3669	0.4595	0.4356	0.4191	0.0394	8
2) National Housing Authority	0.3074	0.3006	0.3512	0.3411	0.3250	0.0248	7
3) Huai Khwang	0.2739	0.2891	0.2730	0.2850	0.2803	0.0081	6
4) Nonsee Vitaya school	0.1967	0.2082	0.2155	0.2226	0.2107	0.0111	3
5) Singharatpitayakom school	0.4501	0.4371	0.4554	0.4414	0.4460	0.0083	9
6) Thonburi	0.1853	0.2000	0.1865	0.1991	0.1927	0.0079	2
7) Chokchai 4	0.2237	0.2342	0.2238	0.2331	0.2287	0.0057	4
8) Dindaeng	0.5948	0.6508	0.5424	0.5680	0.5890	0.0464	10
9) Meteorological Department	0.1684	0.1544	0.1954	0.1852	0.1759	0.0181	1
10) Ratburana	0.2131	0.1946	0.2675	0.2539	0.2323	0.0341	5

From Tables 1-4, we observe that most often (1998, 2000, 2001) first rank is Meteorological Department station which means this station is expected to be good in terms of air pollution. On the other hand, Dindaeng station performed poorly. We selected these two stations to represent their performances graphically in Figures 2 – 3. These figures also depict their ranks for each season separately, rainy, summer and winter, along with the overall ranks. Details of seasonal analyses appear in the Technical Report. Returning to the air pollution data sets, we have applied Modifications I and II using various value of k to see ranks afresh. These are reported in Tables 5 – 12. The values of Spearman's rank correlation of two sets of ranks between MCDM method and Modifications I and II are shown in Table 13. Tables 14-15 show these values for Modifications I and II within themselves for different values of k. The robustness of the ranks is obvious in view of the large values of Spearman's rank correlation uniformly in all cases.

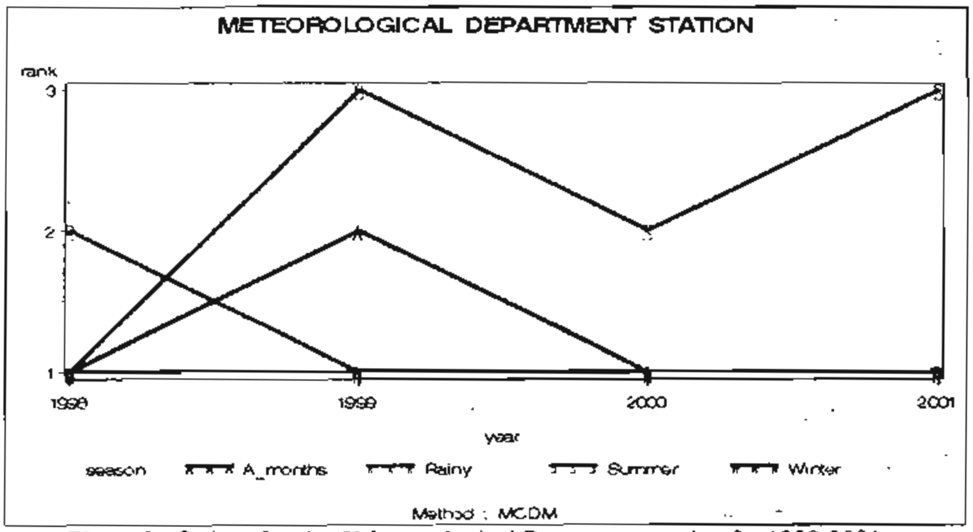

Figure 2 : Order of rank of Meteorological Department station for 1998-2001.

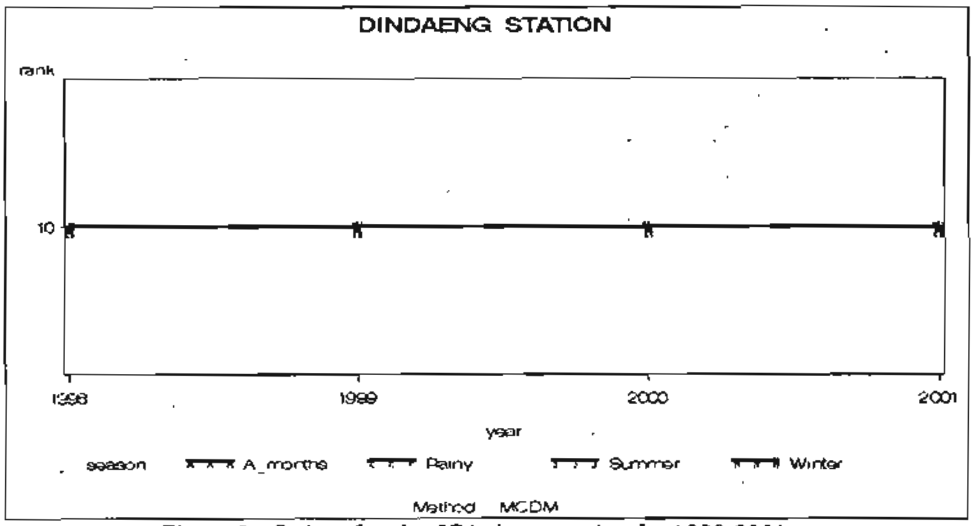

Figure 3 : Order of rank of Dindaeng station for 1998-2001.

Table 5: Results of Modification I for 1998.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamhcang University	0.4108	10	0.2828	10	0.2065	10	0.1558	10	0.1198	10
2) National Housing Authority	0.1671	3	0.0805	3	0.0405	4	0.0208	4	0.0108	4
3) Huai Khwang	0.2304	6	0.1188	5	0.0627	5	0.0335	5	0.0180	5
4) Nonsee Vitaya School	0.1730	4	0.0819	4	0.0404	3	0.0205	3	0.0105	3
5) Singharatpitayakom School	0.3344	9	0.2449	9	0.1882	9	0.1472	9	0.1158	9
6) Thonburi	0.2561	7	0.1523	7	0.0944	7	0.0594	7	0.0376	7
7) Chokchai 4	0.2274	5	0.1261	6	0.0726	6	0.0425	6	0.0251	6
8) Dindaeng	0.2742	8	0.1642	8	0.1005	8	0.0623	8	0.0389	8
9) Meteorological Department	0.0345	1	0.0073	1	0.0016	1	0.0003	1	0.0001	1
10) Ratburana	0.1039	2	0.0389	2	0.0146	2	0.0055	2	0.0021	2

Table 6: Results of Modification II for 1998.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.6604	8	0.4202	7	0.2824	8	0.197	9	0.1413	9
2) National Housing Authority	0.5505	3	0.3467	4	0.226	4	0.1505	4	0.1017	5
3) Huai Khwang	0.6569	7	0.4246	8	0.2788	7	0.185	6	0.1239	6
4) Nonsee Vitaya School	0.5537	4	0.3426	3	0.2212	3	0.1471	3	0.1	3
5) Singharatpitayakom School	0.6155	5	0.3996	6	0.2703	6	0.1879	7	0.1334	8
6) Thonburi	0.6673	9	0.4347	9	0.2858	9	0.1887	8	0.125	7
7) Chokchai 4	0.6257	6	0.3934	5	0.2495	5	0.1589	5	0.1015	4
8) Dindaeng	0.8476	10	0.6385	10	0.4953	10	0.3902	10	0.3108	10
9) Meteorological Department	0.3119	1	0.1526	1	0.0773	1	0.04	1	0.0211	1
10) Ratburana	0.4187	2	0.2455	2	0.148	2	0.0905	2	0.0558	2

Table 7: Results of Modification I for 1999.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.2396	10	0.1050	8	0.0466	5	0.0208	4	0.0094	4
2) National Housing Authority	0.1482	2	0.0649	2	0.0291	2	0.0132	2	0.0060	2
3) Huai Khwang	0.2067	7	0.1002	7	0.0495	7	0.0247	7	0.0124	6
4) Nonsee Vitaya School	0.1520	3	0.0688	3	0.0321	3	0.0152	3	0.0073	3
5) Singharatpitayakom School	0.1927	6	0.0919	4	0.0453	4	0.0228	5	0.0116	5
6) Thonburi	0.2276	8	0.1179	9	0.0628	9	0.0340	9	0.0186	9
7) Chokchai 4	0.1921	5	0.0931	5	0.0468	6	0.0241	6	0.0125	7
8) Dindaeng	0.2342	9	0.1261	10	0.0688	10	0.0379	10	0.0210	10
9) Meteorological Department	0.1757	4	0.0966	6	0.0544	8	0.0310	8	0.0177	8
10) Ratburana	0.0855	1	0.0275	1	0.0091	1	0.0030	1	0.0010	1

Table 8: Results of Modification II for 1999.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.6537	9	0.4397	9	0.3188	9	0.2423	9	0.1898	9
2) National Housing Authority	0.553	3	0.3578	3	0.2403	5	0.1653	5	0.1157	5
3) Huai Khwang	0.6285	7	0.4071	7	0.2701	7	0.182	6	0.1241	6
4) Nonsee Vitaya School	0.5676	4	0.3629	4	0.2373	3	0.1571	4	0.1051	4
5) Singharatpitayakom School	0.6138	6	0.4001	6	0.269	6	0.1842	7	0.1278	7
6) Thonburi	0.652	8	0.4273	8	0.285	8	0.1921	8	0.1305	8
7) Chokchai 4	0.5992	5	0.3752	5	0.2383	4	0.1524	3	0.0981	3
8) Dindaeng	0.7974	10	0.5967	10	0.4617	10	0.3632	10	0.2888	10
9) Meteorological Department	0.4888	2	0.3073	2	0.1947	2	0.1236	2	0.0786	2
10) Ratburana	0.4172	1	0.2363	1	0.1401	1	0.0849	1	0.0521	1

Table 9: Results of Modification I for 2000.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.3624	10	0.2161	10	0.1348	10	0.0865	10	0.0566	10
2) National Housing Authority	0.1502	2	0.0638	2	0.0278	2	0.0123	2	0.0055	2
3) Huai Khwang	0.2027	6	0.1032	6	0.0542	6	0.0289	6	0.0155	6
4) Nonsee Vitaya School	0.1989	4	0.1320	8	0.0908	8	0.0631	8	0.0439	8
5) Singharatpitayakom School	0.2100	8	0.1025	5	0.0508	5	0.0255	5	0.0129	5
6) Thonburi	0.2076	7	0.1095	7	0.0594	7	0.0326	7	0.0180	7
7) Chokchai 4	0.1998	5	0.0981	4	0.0491	4	0.0248	4	0.0126	4
8) Dindaeng	0.2719	9	0.1697	9	0.1094	9	0.0722	9	0.0485	9
9) Meteorological Department	0.1257	1	0.0512	1	0.0214	1	0.0090	1	0.0038	1
10) Ratburana	0.1550	3	0.0731	3	0.0357	3	0.0178	3	0.0089	3

Table 10: Results of Modification II for 2000.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.6788	9	0.4461	9	0.3146	9	0.2317	9	0.1754	9
2) National Housing Authority	0.5353	4	0.3331	3	0.2171	6	0.1455	6	0.0994	6
3) Huai Khwang	0.5949	7	0.3576	7	0.2165	5	0.1315	5	0.0801	5
4) Nonsee Vitaya School	0.5111	2	0.3072	2	0.1892	2	0.1189	2	0.0761	2
5) Singharatpitayakom School	0.6024	8	0.385	8	0.2568	8	0.1766	8	0.1244	8
6) Thonburi	0.5856	5	0.3508	6	0.2121	4	0.1289	4	0.0788	4
7) Chokchai 4	0.5863	6	0.3486	5	0.209	3	0.126	3	0.0762	3
8) Dindaeng	0.8251	10	0.6191	10	0.486	10	0.3925	10	0.3234	10
9) Meteorological Department	0.4527	1	0.2481	1	0.1374	1	0.0765	1	0.0427	1
10) Ratburana	0.5246	3	0.3367	4	0.2272	7	0.1578	7	0.1117	7

Table 11: Results of Modification I for 2001.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.1658	6	0.0724	5	0.0318	5	0.0140	4	0.0062	3
2) National Housing Authority	0.1616	5	0.0789	6	0.0416	7	0.0232	7	0.0133	7
3) Huai Khwang	0.2043	8	0.1088	8	0.0595	8	0.0329	8	0.0182	8
4) Nonsee Vitaya School	0.1218	3	0.0519	3	0.0250	3	0.0129	3	0.0069	5
5) Singharatpitayakom School	0.2624	10	0.1467	10	0.0837	10	0.0484	10	0.0282	9
6) Thonburi	0.1500	4	0.0674	4	0.0311	4	0.0145	5	0.0068	4
7) Chokchai 4	0.1676	7	0.0802	7	0.0397	6	0.0199	6	0.0100	6
8) Dindaeng	0.2331	9	0.1368	9	0.0808	9	0.0479	9	0.0284	10
9) Meteorological Department	0.1083	2	0.0390	2	0.0143	2	0.0053	2	0.0020	2
10) Ratburana	0.0946	1	0.0330	1	0.0119	1	0.0044	1	0.0016	1

Table 12 : Results of Modification II for 2001.

Monitoring Station	k = 1	Rank	k = 1.5	Rank	k = 2	Rank	k = 2.5	Rank	k = 3	Rank
1) Ramkhamheang University	0.6731	8	0.4809	9	0.3613	9	0.279	9	0.2194	9
2) National Housing Authority	0.5902	6	0.384	7	0.2571	7	0.1751	7	0.1206	7
3) Huai Khwang	0.5999	7	0.3667	6	0.2259	6	0.1398	6	0.0869	6
4) Nonsee Vitaya School	0.5169	2	0.3014	3	0.1784	3	0.1064	3	0.0636	3
5) Singharatpitayakom School	0.7084	9	0.4761	8	0.3255	8	0.2251	8	0.1571	8
6) Thonburi	0.5207	3	0.2983	2	0.173	2	0.1012	1	0.0597	1
7) Chokchai 4	0.5541	5	0.3263	5	0.194	4	0.116	4	0.0697	4
8) Dindaeng	0.7948	10	0.5889	10	0.4497	10	0.349	10	0.274	10
9) Meteorological Department	0.4939	1	0.2872	1	0.1706	1	0.1024	2	0.0619	2
10) Ratburana	0.521	4	0.3244	4	0.2062	5	0.1324	5	0.0855	5

Table 13: Spearman's Rank Correlations Between MCDM and Modifications I and II.

Year	Modification I					Modification II				
	k=1	k=1.5	k=2	k=2.5	k=3	k=1	k=1.5	k=2	k=2.5	k=3
1998	0.7818	0.7333	0.7455	0.7455	0.7455	0.9030	0.9758	0.9394	0.8909	0.8667
1999	0.9515	0.8545	0.6364	0.6000	0.5758	1.0000	1.0000	0.9636	0.9394	0.9394
2000	0.9030	0.6364	0.6364	0.6364	0.6364	1.0000	0.9758	0.7939	0.7939	0.7939
2001	0.7333	0.7212	0.7576	0.6848	0.6182	0.9636	0.9758	0.9879	0.9758	0.9758

Table 14: Spearman's Rank Correlations for Modification I for Different Values of k.

Year	k=1, k=1.5	k=1, k=2	k=1, k=2.5	k=1, k=3	k=1.5, k=2	k=1.5, k=2.5	k=1.5, k=3	k=2, k=2.5	k=2, k=3	k=2.5, k=3
1998	0.9879	0.9758	0.9758	0.9758	0.9879	0.9879	0.9879	1.0000	1.0000	1.0000
1999	0.9152	0.7091	0.6606	0.6364	0.9152	0.8667	0.8424	0.9879	0.9758	0.9879
2000	0.8424	0.8424	0.8424	0.8424	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2001	0.9879	0.9636	0.9394	0.8788	0.9879	0.9758	0.9212	0.9879	0.9394	0.9515

Table 15: Spearman's Rank Correlations for Modification II for Different Values of k.

Year	k=1, k=1.5	k=1, k=2	k=1, k=2.5	k=1, k=3	k=1.5, k=2	k=1.5, k=2.5	k=1.5, k=3	k=2, k=2.5	k=2, k=3	k=2.5, k=3
1998	0.9636	0.9758	0.9394	0.8606	0.9879	0.9394	0.8909	0.9758	0.9273	0.9758
1999	1.0000	0.9636	0.9394	0.9394	0.9636	0.9394	0.9394	0.9758	0.9758	1.0000
2000	0.9758	0.7939	0.7939	0.7939	0.8182	0.8182	0.8182	1.0000	1.0000	1.0000
2001	0.9636	0.9515	0.9273	0.9273	0.9879	0.9758	0.9758	0.9879	0.9879	1.0000

The rest of this section is devoted to a discussion of the Electre Method described in Section 2.3. For every year (1998 – 2001), we begin with the data matrix X, and follow steps 1-5 to eventually obtain the matrix **E**. The four E-matrices are shown below in Table 16.

Table 16: E-matrices.

Station	1	2	3	4	5	6	7	8	9	10
1	0	1	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	0	0	1	1
3	1	1	0	1	1	0	0	0	1	1
4	1	1	0	0	0	0	0	0	1	1
5	1	1	0	1	0	0	0	0	1	1
6	1	1	1	1	1	0	1	0	1	1
7	1	1	1	1	1	0	0	0	1	1
8	1	1	1	1	1	1	1	0	1	1
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	1	0

Year 1998

Station	1	2	3	4	5	6	7	8	9	10
1	0	1	0	1	0	0	0	0	1	1
2	0	0	0	1	0	0	0	0	1	1
3	1	1	0	1	1	1	0	0	1	1
4	0	0	0	0	0	0	1	0	0	1
5	1	1	0	1	0	0	0	0	1	1
6	1	1	0	1	1	0	1	0	1	1
7	1	1	1	0	1	0	0	0	1	1
8	1	1	1	1	1	1	1	0	1	1
9	0	0	0	1	0	0	0	0	0	1
10	0	0	0	0	0	0	0	0	0	0

Year 1999

Station	1	2	3	4	5	6	7	8	9	10
1	0	1	0	1	0	0	0	0	1	1
2	0	0	0	1	0	0	0	0	1	1
3	1	1	0	1	1	1	1	0	1	1
4	0	0	0	0	0	0	0	0	0	0
5	1	1	0	1	0	0	0	0	1	1
6	1	1	0	1	1	0	1	0	1	1
7	1	1	0	1	1	0	0	0	1	1
8	1	1	1	1	1	1	1	0	1	1
9	0	0	0	1	0	0	0	0	0	1
10	0	0	0	1	0	0	0	0	0	0

Year 2000

Station	1	2	3	4	5	6	7	8	9	10
1	0	1	0	1	0	0	0	0	1	1
2	0	0	0	1	0	0	0	0	1	1
3	1	1	0	1	0	1	1	0	1	1
4	0	0	0	0	0	0	0	0	0	1
5	1	1	1	1	0	1	1	0	1	1
6	1	1	0	1	0	0	0	0	1	1
7	1	1	0	1	0	1	0	0	1	1
8	1	1	1	1	1	1	1	0	1	1
9	0	0	0	1	0	0	0	0	0	1
10	0	0	0	0	0	0	0	0	0	0

Year 2001

From the above table, we can conclude that in 1998, Meteorological Department station is the best and Ratburana station is the second. For 1999, the best station is Ratburana and the second best stations are Meteorological Department and Nonsee Vitaya school. In 2000, Nonsee Vitaya school and Ratburana are the best station and the second best station, respectively. Finally, in 2001, the best station is Ratburana and the second best station is Nonsee Vitaya School. In addition, the worst station is Dindeang for every year.

4. CONCLUSION

This paper presents a statistical study of the four air pollutants in the four-year period (1998-2001) from ten monitoring stations in Bangkok, Thailand using MCDM method. MCDM method is used to integrate the various columns of a data matrix so that each row is endowed with a single overall index, summarizing the different component indices over columns, thus making a ranking of the rows and hence their comparison feasible. Some modifications of MCDM are also used to rank the stations.

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Journal of Statistical Theory and Applications
Volume 3, Number 2, 2004, pp. 141-149
ISSN 1538-7887

On a comparison of two standard estimates of a binomial proportion based on Multiple Criteria Decision Making method

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Abstract

In this paper we consider the problem of estimation of a binomial proportion θ based on $X \sim B(n,\theta)$, n known, $0<\theta<1$. We compare two standard estimates $T_1 = X/n$ and $T_2 = [X+\sqrt{n}/2]/[n+\sqrt{n}]$ on the basis of Multiple Criteria Decision Making (MCDM) procedure. Our recommendation is that we should use T_2 rather than T_1 for small values of n.

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* Research supported by a scholarship under the Ministry Staff Development Project of Ministry of University Affairs, Thailand.

** Research supported by a grant from Thailand Research Fund (TRF).

*** Research supported by UMBC's Presidential Research Professorship grant.

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1. INTRODUCTION

consider the problem of estimation of a binomial proportion θ based on $X \sim B(n,\theta)$, n known, $0 < \theta < 1$. In this paper we compare $T_1 = X/n$, and $T_2 = [X + \sqrt{n}/2]/[n + \sqrt{n}]$ on the basis of Multiple Criteria Decision Making (MCDM) method. This method is briefly decribed in Section 2 and Section 3 contains the main results for our problem. Our recommendation is to use T_2 rather than T_1 for most reasonable values of n.

2. A brief description of MCDM procedure

In the context of a '*discrete*' data matrix $X = (x_{ij}): K \times N$ where x_{ij}'s represent '*risk*' of ith '*source*' for jth '*category*', and we need to compare the K rows simultaneously with respect to all the N columns, MCDM is a novel statistical procedure to integrate the multiple indicators $(x_{i1},\ldots,x_{iN})$ for row i across all indicators into a single meaningful and overall index. This is done by defining an Ideal Row with the smallest observed value for each column as

$$IDR = (min_i\, x_{i1},\ldots,min_i\, x_{iN}) = (u_1,\ldots,u_N)$$

and a Negative-ideal Row (NIDR) with the largest observed value for each column as

$$NIDR = (max_i\, x_{i1},\ldots,max_i\, x_{iN}) = (v_1,\ldots,v_N).$$

For any given row i, we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under L_1-norm, we compute

$$L_1(i,IDR) = \sum_{j=1}^{N}\left[x_{ij} - u_j\right]w_j$$

$$L_1(i,NIDR) = \sum_{j=1}^{N}\left[v_j - x_{ij}\right]w_j$$

where w_j's are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) = \frac{L_1(i,IDR)}{L_1(i,IDR) + L_1(i,NIDR)}, \quad i = 1,\ldots,K. \qquad (2.1)$$

Similarly, under L_2-norm, we compute

Comparison of Estimates

$$L_2(i,IDR) = [\sum_{j=1}^{N}(x_{ij}-u_j)^2 w_j]^{1/2}$$

$$L_2(i,NIDR) = [\sum_{j=1}^{N}(x_{ij}-v_j)^2 w_j]^{1/2}$$

and compare the rows based on

$$L_2(Index_i) = \frac{L_2(i,IDR)}{L_2(i,IDR)+L_2(i,NIDR)}, \quad i = 1,\dots,K. \tag{2.2}$$

A '*continuous*' version of this setup would involve $x_{ij}'s$ where the index j would vary '*continuously*'. In the context of the problem of comparing several estimates for estimation of θ, $x_{ij}'s$ are chosen to represent the mean squared errors of the estimates for various values of θ, and L_1-norm and L_2-norm would be redefined as

$$L_1(i,IDR) = \int_{\underline{\theta}}^{\bar{\theta}}[x_i(\theta)-u(\theta)]w(\theta)d\theta \tag{2.3}$$

$$L_1(i,NIDR) = \int_{\underline{\theta}}^{\bar{\theta}}[v(\theta)-x_i(\theta)]w(\theta)d\theta \tag{2.4}$$

$$L_2(i,IDR) = \sqrt{\int_{\underline{\theta}}^{\bar{\theta}}(x_i(\theta)-u(\theta))^2 w(\theta)d\theta} \tag{2.5}$$

$$L_2(i,NIDR) = \sqrt{\int_{\underline{\theta}}^{\bar{\theta}}(x_i(\theta)-v(\theta))^2 w(\theta)d\theta}\,. \tag{2.6}$$

Comparison of estimates is then based on the overall index defined as

$$L_1(Index_i) = \frac{\int_{\underline{\theta}}^{\bar{\theta}}[x_i(\theta)-u(\theta)]w(\theta)d\theta}{\int_{\underline{\theta}}^{\bar{\theta}}[x_i(\theta)-u(\theta)]w(\theta)d\theta + \int_{\underline{\theta}}^{\bar{\theta}}[v(\theta)-x_i(\theta)]w(\theta)d\theta},$$

$$L_2(Index_i) = \frac{\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} (x_i(\theta) - u(\theta))^2 w(\theta)\, d\theta}}{\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} (x_i(\theta) - u(\theta))^2 w(\theta)\, d\theta} + \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} (x_i(\theta) - v(\theta))^2 w(\theta)\, d\theta}},$$

$i = 1, \dots, K$.

3. Main Result

In this section we consider two standard loss functions, namely, absolute error loss (L_1-norm) and squared error loss (L_2-norm).

3.1. L_1-norm. We first prove a general result in the case of L_1-norm, showing that the MCDM approach in this case is equivalent to a standard Bayesian approach. Suppose $T_1, \dots, T_K$ are estimates of θ to be compared with respect to their mean squared errors (MSE) $\text{MSE}(T_i) = x_i(\theta),\ i = 1, \dots, K,$ where $\underline{\theta} \le \theta \le \overline{\theta}$.

Theorem 3.1 : Under L_1-norm, T_i is better than T_j if

$$\int_{\underline{\theta}}^{\overline{\theta}} x_i(\theta)\, w(\theta)\, d\theta < \int_{\underline{\theta}}^{\overline{\theta}} x_j(\theta)\, w(\theta)\, d\theta.$$

(3.1)

Proof : T_i is better than T_j if

$$\frac{\int_{\underline{\theta}}^{\overline{\theta}} [x_i(\theta) - u(\theta)]\, w(\theta)\, d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} [v(\theta) - x_i(\theta)]\, w(\theta)\, d\theta} < \frac{\int_{\underline{\theta}}^{\overline{\theta}} [x_j(\theta) - u(\theta)]\, w(\theta)\, d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} [v(\theta) - x_j(\theta)]\, w(\theta)\, d\theta}$$

$\Leftrightarrow$

$$\left[\int_{\underline{\theta}}^{\overline{\theta}} x_i(\theta)\, w(\theta)\, d\theta - A\right]\left[B - \int_{\underline{\theta}}^{\overline{\theta}} x_j(\theta)\, w(\theta)\, d\theta\right] < \left[\int_{\underline{\theta}}^{\overline{\theta}} x_j(\theta)\, w(\theta)\, d\theta - A\right]\left[B - \int_{\underline{\theta}}^{\overline{\theta}} x_i(\theta)\, w(\theta)\, d\theta\right]$$

where $A = \int_{\underline{\theta}}^{\overline{\theta}} u(\theta)\, w(\theta)\, d\theta$ and $B = \int_{\underline{\theta}}^{\overline{\theta}} v(\theta)\, w(\theta)\, d\theta$. Since $[B - A] > 0$, T_i is better than T_j if

$$\int_{\underline{\theta}}^{\bar{\theta}} x_i(\theta)\, w(\theta)\, d\theta \quad < \quad \int_{\underline{\theta}}^{\bar{\theta}} x_j(\theta)\, w(\theta)\, d\theta .$$

This completes the proof.

Corollary 3.2 : Let θ be a binomial proportion, $0<\theta<1$. If the weight function is defined by $w_1(\theta) = \theta^{\alpha-1}(1-\theta)^{\beta-1}$ with $\alpha=\beta=\sqrt{n}/2$, $T_2(x)=[x+\sqrt{n}/2]/[n+\sqrt{n}]$ is the best estimate of θ under the MCDM approach.

The robustness of $T_2(x)$ for some other choices of α and β can be seen from the following cases where we mention values of n for which $T_2(x)$ is better than $T_1(x)$.

Case 1 : $\alpha=\beta=1 : n\in[1,19]$. **Case 2** : $\alpha=\beta=1.5 : n\in[1,41]$.

Case 3 : $\alpha=\beta=2 : n\in[1,71]$. **Case 4** : $\alpha=\beta=3 : n\in[1,155]$.

Following Filar et al. (1999), we now consider two additional choices of $w(\theta)$. The first one, denoted by $w_2(\theta)$, is based on the notion of entropy between MSE(T_1) and MSE(T_2) for various values of θ, and the second one, denoted by $w_3(\theta)$, is based on the coefficient of variation of MSE(T_1) and MSE(T_2) for various values of θ.

In the context of binomial parameter estimation problem, recall that $X\sim B(n,\theta)$, $T_1(x)=x/n$, $T_2(x)=[x+\sqrt{n}/2]/[n+\sqrt{n}]$, MSE$(T_1)=\dfrac{\theta(1-\theta)}{n}$ and MSE(T_2) = $\dfrac{n}{4(n+\sqrt{n})^2}$. It then readily turns out that

$$w_2(\theta) = \frac{1-\varphi(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} [1-\varphi(\theta)]\, d\theta} \tag{3.2}$$

where

$$\varphi(\theta) = -\frac{1}{\ln 2}\left\{\left(\frac{\frac{\theta(1-\theta)}{n}}{\frac{\theta(1-\theta)}{n}+\frac{n}{4(n+\sqrt{n})^2}}\right)\cdot \log\left(\frac{\frac{\theta(1-\theta)}{n}}{\frac{\theta(1-\theta)}{n}+\frac{n}{4(n+\sqrt{n})^2}}\right)+\right.$$

$$\left(\frac{\frac{n}{4(n+\sqrt{n})^2}}{\frac{\theta(1-\theta)}{n}+\frac{n}{4(n+\sqrt{n})^2}} \right) \cdot \log \left(\frac{\frac{n}{4(n+\sqrt{n})^2}}{\frac{\theta(1-\theta)}{n}+\frac{n}{4(n+\sqrt{n})^2}} \right) \Bigg\}$$

and

$$w_3(\theta) = \frac{\left| \frac{\theta(1-\theta)}{n} - \frac{n}{4(n+\sqrt{n})^2} \right|}{\frac{\theta(1-\theta)}{n} + \frac{n}{4(n+\sqrt{n})^2}}. \tag{3.3}$$

Verification of (3.1) for these two weight functions has been carried out for various values of n, using MATHEMATICA. The results are stated below.

Corollary 3.3 : Let θ be a binomial proportion, $0 < \theta < 1$. Under $w_2(\theta)$, $T_2(x)$ is better than $T_1(x)$ for all $n \geq 1$.

Corollary 3.4 : Let θ be a binomial proportion, $0 < \theta < 1$. Under $w_3(\theta)$, $T_2(x)$ is better than $T_1(x)$ for all $n \in [1,19]$.

3.2. L₂-norm. In the case of L₂-norm with a general weight function $w(\theta)$, proceeding as before, it is easily seen that T_i is better than T_j if

$$\frac{\int_{\underline{\theta}}^{\bar{\theta}} [x_i(\theta) - u(\theta)]^2 w(\theta) d\theta}{\int_{\underline{\theta}}^{\bar{\theta}} [x_i(\theta) - v(\theta)]^2 w(\theta) d\theta} < \frac{\int_{\underline{\theta}}^{\bar{\theta}} [x_j(\theta) - u(\theta)]^2 w(\theta) d\theta}{\int_{\underline{\theta}}^{\bar{\theta}} [x_j(\theta) - v(\theta)]^2 w(\theta) d\theta}. \tag{3.4}$$

We now simplify (3.4) in the case of our problem of comparison of T_1 and T_2 for estimation of the binomial proportion θ. Obviously $\text{MSE}(T_2) \leq \text{MSE}(T_1)$ holds whenever $c_1(n) < \theta < c_2(n)$, where

$$c_1(n) = \frac{1 - \sqrt{1 - \left(\frac{n}{n+\sqrt{n}} \right)^2}}{2} \tag{3.5}$$

and

$$c_2(n)=\frac{1+\sqrt{1-\left(\frac{n}{n+\sqrt{n}}\right)^2}}{2}. \tag{3.6}$$

Moreover, the Ideal row and Negative-ideal row are easily obtained as

$u(\theta): IDR$ =

$$\left\{\frac{\theta(1-\theta)}{n}: \theta < c_1(n),\ \frac{n}{4(n+\sqrt{n})^2}: c_1(n)<\theta<c_2(n),\ \frac{\theta(1-\theta)}{n}: \theta > c_2(n)\right\}$$

$v(\theta): NIDR =$

$$\left\{\frac{n}{4(n+\sqrt{n})^2}: \theta < c_1(n),\ \frac{\theta(1-\theta)}{n}: c_1(n)<\theta<c_2(n),\ \frac{n}{4(n+\sqrt{n})^2}: \theta > c_2(n)\right\}$$

.

(3.4) is now simplified as

$$\int_{\theta<c_1(n)\cup\theta>c_2(n)}\left(\frac{n}{4(n+\sqrt{n})^2}-\frac{\theta(1-\theta)}{n}\right)^2 w(\theta)\,d\theta < \int_{c_1(n)}^{c_2(n)}\left(\frac{\theta(1-\theta)}{n}-\frac{n}{4(n+\sqrt{n})^2}\right)^2 w(\theta)d\theta$$

$\Leftrightarrow$

$$\int_0^1\left(\frac{n}{4(n+\sqrt{n})^2}-\frac{\theta(1-\theta)}{n}\right)^2 w(\theta)\,d\theta < 2\int_{c_1(n)}^{c_2(n)}\left(\frac{n}{4(n+\sqrt{n})^2}-\frac{\theta(1-\theta)}{n}\right)^2 w(\theta)\,d\theta$$

. (3.7)

Theorem 3.5 : Let θ be a binomial proportion, $0<\theta<1$. If the weight function is defined by $w(\theta)=\theta^{\alpha-1}(1-\theta)^{\beta-1}$ for some $\alpha,\beta>0$, then $T_2(x)$ is better than $T_1(x)$ based on L_2-norm if n satisfies (3.8).

Proof: Since $w(\theta)=\theta^{\alpha-1}(1-\theta)^{\beta-1}$, (3.7) reduces to

$$\int_0^1\left(\frac{n^2}{16(n+\sqrt{n})^4}-\frac{\theta(1-\theta)}{2(n+\sqrt{n})^2}+\frac{\theta^2(1-\theta)^2}{n^2}\right)\theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta$$

$$< \int_{c_1(n)}^{c_2(n)} \left(\frac{n^2}{8(n+\sqrt{n})^4} - \frac{\theta(1-\theta)}{(n+\sqrt{n})^2} + \frac{2\theta^2(1-\theta)^2}{n^2} \right) \theta^{\alpha-1}(1-\theta)^{\beta-1}\, d\theta$$

$\Leftrightarrow$

$$\frac{n^2}{16(n+\sqrt{n})^4} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} - \frac{1}{2(n+\sqrt{n})^2} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} + \frac{1}{n^2} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+4)}$$

$$< \int_{c_1(n)}^{c_2(n)} \left(\frac{n^2\theta^{\alpha-1}(1-\theta)^{\beta-1}}{8(n+\sqrt{n})^4} - \frac{\theta^{\alpha}(1-\theta)^{\beta}}{(n+\sqrt{n})^2} + \frac{2\theta^{\alpha+1}(1-\theta)^{\beta+1}}{n^2} \right) d\theta . \qquad (3.8)$$

We now consider some special cases of α and β and indicate values of n for which (3.8) holds.

Case 1 : $\alpha = \beta = 1 : n \in [1,14]$. **Case 2 :** $\alpha = \beta = 1.5 : n \in [1,26]$.

Case 3 : $\alpha = \beta = 2 : n \in [1,41]$. **Case 4 :** $\alpha = \beta = 3 : n \in [1,82]$.

Case 5 : $\alpha = \beta = \sqrt{n}/2 : n \in I^+$.

It is interesting to observe that in this case also, T_2 outperforms T_1 for all values of n under the minimax prior distribution.

As in the case of L_1-norm, here also we considered the other two weight functions $w_2(\theta)$ and $w_3(\theta)$, and verified (3.7) for various values of n, using MATHEMATICA. The results are stated below.

Corollary 3.6 : Let θ be a binomial proportion, $0 < \theta < 1$. If the weight function is defined by (3.2) then T_2 is better than T_1 based on L_2-norm for all $n \geq 1$.

Corollary 3.7 : Let θ be a binomial proportion, $0 < \theta < 1$. If the weight function is defined by (3.3) then T_2 is better than T_1 based on L_2-norm if $n \in [1,14]$.

4. Conclusion

Based on the above analysis under L_1- and L_2- norms, our recommendation is to use T_2 rather than T_1 for small values of n.

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International Journal of Statistical Sciences *ISSN 1683-5603*
Vol. 3 (Special), 2004, pp 105-117

On a comparison of three estimators of binomial variance by multiple criteria decision making method

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[Received March 1, 2004; Accepted July 2, 2004]

Abstract

In this paper we consider the problem of estimation of $\theta(1-\theta)$ based on $X \sim B(n,\theta)$, n being known and $0<\theta<1$, θ being unknown. We compare three standard estimators $T_1 = \frac{X}{n}\left(1-\frac{X}{n}\right)$, $T_2 = \frac{X(n-X)}{n(n-1)}$, and $T_3 = \frac{X(n-X)+n\sqrt{n}/2+n/4}{(n+\sqrt{n})^2}$ on the basis of Multiple Criteria Decision Making (MCDM) procedure. MCDM is a novel statistical procedure to compare several competing estimators of a parameter. It turns out that our preference is mostly for T_1.

Keywords and Phrases: Binomial distribution, variance, minimax, multiple criteria decision making.

AMS Classification: Primary 62P12; Secondary 62H30.

*Research supported by a scholarship under the Staff Development Project of the Commission on Higher Education, Thailand.
†Research supported by a grant from The Thailand Research Fund (TRF).
‡Research supported by UMBC's Presidential Research Professorship grant.

1 Introduction

We consider the problem of estimation of $\theta(1-\theta)$ based on $X \sim B(n,\theta)$. Here n is known and $0 < \theta < 1$, θ being unknown. It is well known that there are three standard estimators of $\theta(1-\theta)$, namely, $T_1 = \frac{X}{n}\left(1-\frac{X}{n}\right)$, the maximum likelihood estimate [1], $T_2 = \frac{X(n-X)}{n(n-1)}$, the minimum variance unbiased estimate, and $T_3 = \frac{X(n-X)+n\sqrt{n}/2+n/4}{(n+\sqrt{n})^2}$, based on the minimax estimate of θ. In this paper we compare T_1, T_2 and T_3 on the basis of Multiple Criteria Decision Making (MCDM) method. This method is briefly described in Section 2 and Section 3 contains the main results of this paper. It turns out that most often T_1 is the preferred choice. For detailed discussions on MCDM, we refer to Zeleny [5].

2 A brief description of MCDM procedure

In the context of a '*discrete*' data matrix $X = (x_{ij}) : K \times N$ where x_{ij}'s represent '*risk*' of ith '*source*' for jth '*category*', and we need to compare the K rows simultaneously with respect to all the N columns, MCDM is a novel statistical procedure to integrate the multiple indicators $(x_{i1}, \ldots, x_{iN})$ for row i across all indicators into a single meaningful and overall index. This is done by defining an Ideal Row (IDR) with the smallest observed value for each column as

$$IDR = (\min_i x_{i1}, \ldots, \min_i x_{iN}) = (u_1, \ldots, u_N), \text{ say}$$

and a Negative-ideal Row (NIDR) with the largest observed value for each column as

$$NIDR = (\max_i x_{i1}, \ldots, \max_i x_{iN}) = (v_1, \ldots, v_N), \text{ say.}$$

For any given row i, we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under L_1-norm, we compute

$$L_1(i, IDR) = \sum_{j=1}^{N} |x_{ij} - u_j|\, w_j = \sum_{j=1}^{N} [x_{ij} - u_j]\; w_j$$

$$L_1(i, NIDR) = \sum_{j=1}^{N} |x_{ij} - v_j|\, w_j = \sum_{j=1}^{N} [v_j - x_{ij}]\; w_j$$

where w_j's are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1, \ldots, K. \qquad (1)$$

Similarly, under L_2-norm, we compute

$$L_2(i, IDR) = [\sum_{j=1}^{N} (x_{ij} - u_j)^2 w_j]^{1/2}$$

$$L_2(i, NIDR) = [\sum_{j=1}^{N} (x_{ij} - v_j)^2 w_j]^{1/2}$$

and compare the rows based on

$$L_2(Index_i) = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, \quad i = 1, \ldots, K. \tag{2}$$

A '*continuous*' version of this setup would involve x_{ij}*'s* where the index j would vary '*continuously*'. In the context of our problem of comparing T_1, T_2 and T_3 for estimation of $\theta(1-\theta)$, obviously $K = 3$, x_{ij}*'s* are chosen to represent the mean squared errors of T_1, T_2 and T_3 for various values of θ, and L_1-norm and L_2-norm would be redefined as

$$L_1(i, IDR) = \int_0^1 [x_i(\theta) - u(\theta)]\, w(\theta)\, d\theta \tag{3}$$

$$L_1(i, NIDR) = \int_0^1 [v(\theta) - x_i(\theta)]\, w(\theta)\, d\theta \tag{4}$$

$$L_2(i, IDR) = \sqrt{\int_0^1 (x_i(\theta) - u(\theta))^2 w(\theta)\, d\theta} \tag{5}$$

$$L_2(i, NIDR) = \sqrt{\int_0^1 (x_i(\theta) - v(\theta))^2 w(\theta)\, d\theta} \tag{6}$$

where $u(\theta) = \min_i \{x_i(\theta)\}$ and $v(\theta) = \max_i \{x_i(\theta)\}$.

3 Main Results

We first start with the mean squared errors of T_1, $T_2$and T_3, given below. For details of derivation, we refer to Technical Report [3].

$$\text{MSE}(T_1) = B_1(n)\,\theta + C_1(n)\,\theta^2 + D_1(n)\,\theta^3 + E_1(n)\,\theta^4 \tag{7}$$

where $B_1(n) = \frac{n\,(n-1)^2}{n^4}$, $C_1(n) = \frac{(5n-7)\,(n-n^2)+n^2}{n^4}$,

$D_1(n) = \frac{(2n-3)\,(4n^2-4n)-2n^2}{n^4}$ and $E_1(n) = \frac{(2n-3)\,(2n-2n^2)+n^2}{n^4}$.

$$\text{MSE}\,(T_2) = B_2(n)\,\theta + C_2(n)\,\theta^2 + D_2(n)\,\theta^3 + E_2(n)\,\theta^4 \quad (8)$$

where $B_2(n) = \frac{1}{n}$, $C_2(n) = \frac{(7-5n)}{n\,(n-1)}$, $D_2(n) = \frac{4(2n-3)}{n\,(n-1)}$ and $E_2(n) = \frac{-2\,(2n-3)}{n\,(n-1)}$.

$$\text{MSE}\,(T_3) = A_3(n) + B_3(n)\,\theta + C_3(n)\,\theta^2 + D_3(n)\,\theta^3 + E_3(n)\,\theta^4 \quad (9)$$

where $A_3(n) = \frac{(n\sqrt{n}/2+n/4)^2}{(n+\sqrt{n})^4}$,

$B_3(n) = \frac{-2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4)+n\,(n-1)^2}{(n+\sqrt{n})^4}$,

$C_3(n) = \frac{2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4)+(2n+2n\sqrt{n})^2-n\,(5n-7)(n-1)}{(n+\sqrt{n})^4}$,

$D_3(n) = \frac{-2\,(2n+2n\sqrt{n})^2+4n(2n-3)(n-1)}{(n+\sqrt{n})^4}$

and $E_3(n) = \frac{(2n+2n\sqrt{n})^2-2n\,(2n-3)(n-1)}{(n+\sqrt{n})^4}$.

Writing $x_1(\theta) = \text{MSE}(T_1)$, $x_2(\theta) = \text{MSE}(T_2)$ and $x_3(\theta) = \text{MSE}(T_3)$, we present in Figure 1 their graphical patterns for $n = 5,10,15,20$. It is interesting to note the bimodal nature of $x_1(\theta)$and $x_2(\theta)$, and convex nature of $x_3(\theta)$.

Since $0 < \theta < 1$, the intersection of three graphs can separate the interval of θ into seven intervals $(0 < c_1(n) < c_2(n) < c_3(n) < c_4(n) < c_5(n) < c_6(n) < 1)$. Obviously, MSE$(T_1)$= MSE$(T_2)$ holds whenever $\theta = c_3(n),\ c_4(n)$ where

$$c_3(n) = \frac{6-17n+9n^2-\sqrt{12-64n+109n^2-62n^3+9n^4}}{2\,(6-17n+9n^2)}$$

and

$$c_4(n) = \frac{6-17n+9n^2+\sqrt{12-64n+109n^2-62n^3+9n^4}}{2\,(6-17n+9n^2)}.$$

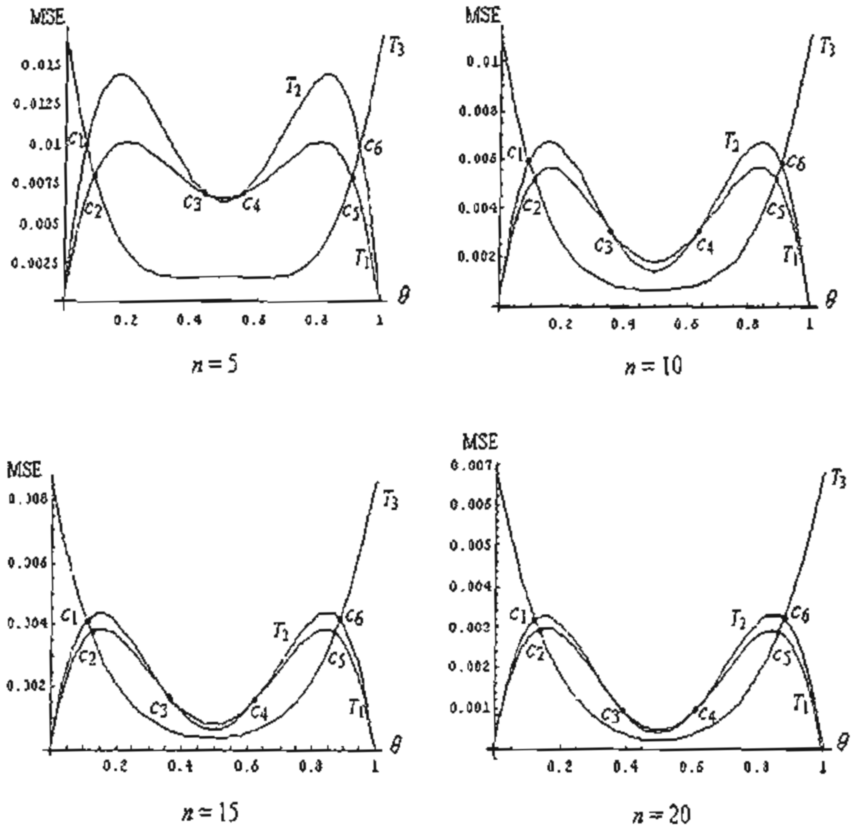

Figure 1: Graphical illustration of mean squared errors for n = 5,10,15, 20.

Likewise, MSE(T_1)= MSE(T_3) holds whenever $\theta = c_2(n),\ c_5(n)$ where

$$\begin{aligned} c_2(n) \ = \ & 24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}- \\ & \Big((-24-48\sqrt{n}-4n+88n^{3/2}+72n^2-32n^{5/2}-44n^3-8n^{7/2})^2- \\ & 4(24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}) \\ & (2+4\sqrt{n}-8n^{3/2}-6n^2+4n^{5/2}+6n^3+2n^{7/2}- \\ & \Big(4+16\sqrt{n}+16n-32n^{3/2}-88n^2-32n^{5/2}+114n^3+ \\ & 128n^{7/2}-37n^4-124n^{9/2}-26n^5+52n^{11/2}+25n^6)^{\ 1/2})\Big)^{1/2} / \\ & (2\,(24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2})) \end{aligned}$$

and

$$\begin{aligned} c_5(n) &= 24 + 48\sqrt{n} + 4n - 88n^{3/2} - 72n^2 + 32n^{5/2} + 44n^3 + 8n^{7/2} + \\ &\Big((-24 - 48\sqrt{n} - 4n + 88n^{3/2} + 72n^2 - 32n^{5/2} - 44n^3 - 8n^{7/2})^2 - \\ &4(24 + 48\sqrt{n} + 4n - 88n^{3/2} - 72n^2 + 32n^{5/2} + 44n^3 + 8n^{7/2}) \\ &(2 + 4\sqrt{n} - 8n^{3/2} - 6n^2 + 4n^{5/2} + 6n^3 + 2n^{7/2} - \\ &\Big(4 + 16\sqrt{n} + 16n - 32n^{3/2} - 88n^2 - 32n^{5/2} + 114n^3 + \\ &128n^{7/2} - 37n^4 - 124n^{9/2} - 26n^5 + 52n^{11/2} + 25n^6\Big)^{1/2})\Big)^{1/2} / \\ &(2(24 + 48\sqrt{n} + 4n - 88n^{3/2} - 72n^2 + 32n^{5/2} + 44n^3 + 8n^{7/2})). \end{aligned}$$

Lastly, MSE(T_2)= MSE(T_3) holds whenever $\theta = c_1(n),\ c_6(n)$ where

$$\begin{aligned} c_1(n) &= -48 - 56\sqrt{n} + 24n + 40n^{3/2} + 8n^2 - ((48 + 56\sqrt{n} - 24n - 40n^{3/2} - 8n^2)^2 \\ &-4(-48 - 56\sqrt{n} + 24n + 40n^{3/2} + 8n^2)(-4 - 5\sqrt{n} + 2n + 5n^{3/2} + 2n^2 \\ &-\sqrt{16 + 34\sqrt{n} - 16n - 78n^{3/2} - 24n^2 + 44n^{5/2} + 24n^3}\ ))^{1/2} / \\ &(2(-48 - 56\sqrt{n} + 24n + 40n^{3/2} + 8n^2)) \end{aligned}$$

and

$$\begin{aligned} c_6(n) &= -48 - 56\sqrt{n} + 24n + 40n^{3/2} + 8n^2 + ((48 + 56\sqrt{n} - 24n - 40n^{3/2} - 8n^2)^2 \\ &-4(-48 - 56\sqrt{n} + 24n + 40n^{3/2} + 8n^2)(-4 - 5\sqrt{n} + 2n + 5n^{3/2} + 2n^2 \\ &-\sqrt{16 + 34\sqrt{n} - 16n - 78n^{3/2} - 24n^2 + 44n^{5/2} + 24n^3}\ ))^{1/2} / \\ &(2(-48 - 56\sqrt{n} + 24n + 40n^{3/2} + 8n^2)). \end{aligned}$$

Moreover, the Ideal row and Negative-ideal row are as follows :

$$IDR : u(\theta) = \{x_1(\theta) : \theta < c_2(n),\ x_3(\theta) : c_2(n) < \theta < c_5(n),\ x_1(\theta) : \theta > c_5(n)\}. \quad (10)$$

$$\begin{aligned} NIDR : v(\theta) &= \{x_3(\theta) : \theta < c_1(n),\ x_2(\theta) : c_1(n) < \theta < c_3(n), \\ &x_1(\theta) : c_3(n) < \theta < c_4(n), x_2(\theta) : c_4(n) < \theta < c_6(n), \\ &x_3(\theta) : \theta > c_6(n)\}. \end{aligned} \quad (11)$$

Since we are dealing with a continuous parameter θ, $0 < \theta < 1$, a proper formulation of the MCDM procedure as described earlier in (3)-(6) can be given as follows.

3.1 Analysis based on L_1-norm

For $i = 1$, applying equations (3) and (4), we get

$$L_1(1, IDR) = \int_{c_2(n)}^{c_5(n)} (x_1(\theta) - x_3(\theta))\, w(\theta)\, d\theta,$$

$$L_1(1, NIDR) = \int_{\theta < c_1(n) \cup \theta > c_6(n)} (x_3(\theta) - x_1(\theta))\, w(\theta)\, d\theta + \int_{c_1(n)}^{c_{[illegible]}(n)} (x_2(\theta) - x_1(\theta))\, w(\theta)\, d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_1(\theta))\, w(\theta)\, d\theta.$$

For $i = 2$, applying equations (3) and (4), we obtain

$$L_1(2, IDR) = \int_{\theta < c_2(n) \cup \theta > c_5(n)} (x_2(\theta) - x_1(\theta))\, w(\theta)\, d\theta + \int_{c_2(n)}^{c_5(n)} (x_2(\theta) - x_3(\theta))\, w(\theta)\, d\theta,$$

$$L_1(2, NIDR) = \int_{\theta < c_1(n) \cup \theta > c_6(n)} (x_3(\theta) - x_2(\theta))\, w(\theta)\, d\theta + \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_2(\theta))\, w(\theta)\, d\theta.$$

For $i = 3$, applying equations (3) and (4), we obtain

$$L_1(3, IDR) = \int_{\theta < c_2(n) \cup \theta > c_5(n)} (x_3(\theta) - x_1(\theta))\, w(\theta)\, d\theta,$$

$$L_1(3, NIDR) = \int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_3(\theta))\, w(\theta)\, d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_3(\theta))\, w(\theta)\, d\theta + \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_3(\theta))\, w(\theta)\, d\theta.$$

The overall index can then be computed from equation (1). It is clear that for the purpose of comparison of the three estimates, we can work with

$$L_1(Index_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1, 2, 3.$$

3.2 Analysis based on L_2-norm

For $i = 1$, applying equations (5) and (6), we get

$$L_2(1, IDR) = \sqrt{\int_{c_2(n)}^{c_5(n)} (x_1(\theta) - x_3(\theta))^2 \, w(\theta) \, d\theta} \, ,$$

$$L_2(1, NIDR) = \sqrt{\int_{\theta < c_1(n) \cup \theta > c_6(n)} (x_3(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta + \int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta}$$

For $i = 2$, applying equations (5) and (6), we obtain

$$L_2(2, IDR) = \sqrt{\int_{\theta < c_2(n) \cup \theta > c_5(n)} (x_2(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta + \int_{c_3(n)}^{c_5(n)} (x_2(\theta) - x_3(\theta))^2 w(\theta) \, d\theta} \, .$$

$$L_2(2, NIDR) = \sqrt{\int_{\theta < c_1(n) \cup \theta > c_6(n)} (x_3(\theta) - x_2(\theta))^2 \, w(\theta) \, d\theta + \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_2(\theta))^2 \, w(\theta) \, d\theta} \, .$$

For $i = 3$, applying equations (5) and (6), we obtain

$$L_2(3, IDR) = \sqrt{\int_{\theta < c_2(n) \cup \theta > c_5(n)} (x_3(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta} \, ,$$

$$L_2(3, NIDR) = \sqrt{\int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_3(\theta))^2 \, w(\theta) \, d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_3(\theta))^2 \, w(\theta) \, d\theta + \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_3(\theta))^2 \, w(\theta) \, d\theta}$$

Under L_2-norm also, the overall index can be computed from equation (2) for each value of n.

3.3 Choice of weight functions

Our first weight function $w_1(\theta)$ is defined by $w_1(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$ for some $\alpha, \beta > 0$, which is a conjugate prior for the binomial parameter θ. Following Filar et al. [2], we also consider two additional choices of $w(\theta)$. The first one, denoted by $w_2(\theta)$, is based on the notion of entropy among $x_1(\theta)$, $x_2(\theta)$ and $x_3(\theta)$ for various values of θ, and the second one, denoted by $w_3(\theta)$, is based on the coefficient of variation of $x_1(\theta)$, $x_2(\theta)$ and $x_3(\theta)$ for various values of θ (Vide [4]). It turns out that

$$w_2(\theta) = \frac{1-\phi(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} [1-\phi(\theta)]\, d\theta} \tag{12}$$

where $\phi(\theta) = -\frac{1}{\log 3} \sum_{i=1}^{3} \left\{ \frac{x_i(\theta)}{\sum_{i=1}^{3} x_i(\theta)} \cdot \log \left[\frac{x_i(\theta)}{\sum_{i=1}^{3} x_i(\theta)} \right] \right\}$,

and

$$w_3(\theta) = \frac{\sqrt{2\left(x_1^2(\theta) + x_2^2(\theta) + x_3^2(\theta) - x_1(\theta)x_2(\theta) - x_1(\theta)x_3(\theta) - x_2(\theta)x_3(\theta)\right)}}{x_1(\theta) + x_2(\theta) + x_3(\theta)}. \tag{13}$$

For details of above derivation, we refer to Technical Report [3]. These expressions can be readily computed using the functions $x_1(\theta)$, $x_2(\theta)$ and $x_3(\theta)$ given in (3.1)-(3.3).

3.4 Comparison of estimators

We report in Table 1 the ranks of the three estimators when compared on the basis of the weight function $w_1(\theta)$. In Table 2, we provide the ranks for the two other weight functions $w_2(\theta)$ and $w_3(\theta)$.

Table 1: Rank of three estimators using weight $w_1(\theta)$*

		L_1		L_2	
		rank ($\alpha = \beta = 1$)	rank ($\alpha = \beta = \sqrt{n}/2$)	rank ($\alpha = \beta = 1$)	rank ($\alpha = \beta = \sqrt{n}/2$)
n =5	T_1	2	2	2	2
	T_2	3	3	3	3
	T_3	1	1	1	1
n=10	T_1	2	2	1	2
	T_2	3	3	2	3
	T_3	1	1	3	1
n=15	T_1	1	2	1	2
	T_2	3	3	3	3
	T_3	2	1	2	1
n=20	T_1	1	2	1	2
	T_2	2	3	2	3
	T_3	3	1	3	1

* Rank 1 = best, Rank 3 = worst

Table 2: Rank of three estimators using weights $w_2(\theta)$and $w_3(\theta)$*

		L_1		L_2	
		$w_2(\theta)$	$w_3(\theta)$	$w_2(\theta)$	$w_3(\theta)$
n=5	T_1	1	2	1	2
	T_2	2	3	2	3
	T_3	3	1	3	1
n=10	T_1	1	1	1	1
	T_2	2	3	2	2
	T_3	3	2	3	3
n=15	T_1	1	1	1	1
	T_2	2	2	2	2
	T_3	3	3	3	3
n=20	T_1	1	1	1	1
	T_2	2	2	2	2
	T_3	3	3	3	3

* Rank 1 = best, Rank 3 = worst

4 Conclusion

Based on the above analysis under L_1- and L_2- norms, we conclude that, for small values of n, our preference is uniformly for T_1. Under the weight function $w_1(\theta)$, T_3 also has some advantages. Of the three estimators studied in this paper, it turns out that T_2 is improper since $T_2(x) > \frac{1}{4}$ whenever $\frac{n-\sqrt{n}}{2} < x < \frac{n+\sqrt{n}}{2}$. On the other hand, both T_1 and T_3 are seen to be proper estimators. Therefore, one should use the truncated version T_2^* of T_2 and compute its mse and then compare it with the other two. This will improve the performance of T_2 and possibly make it preferable over T_1 and T_3. We propose to undertake this study in future.

Acknowledgement

We thank an anonymous referee for suggestions leading to an improvement in the presentation of our ideas.

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Appendix

Derivation of equations (3.1)-(3.3).
Recall that

$$T_1 = \frac{X}{n}\left(1 - \frac{X}{n}\right), \; T_2 = \frac{X(n-X)}{n(n-1)}, \; T_3 = \frac{X(n-X) + n\sqrt{n}/2 + n/4}{(n+\sqrt{n})^2}$$

Let $\delta_\theta = \theta(1-\theta)$. It is easy to verify that $E(X) = n\theta$, $E(X^2) = n\theta(1+n\theta-\theta)$
$E(X^3) = n\theta(1+3n\theta+n^2\theta^2-3n\theta^2-3\theta+2\theta^2)$, and
$E(X^4) = n\theta(1-7\theta+7n\theta+12\theta^2-18n\theta^2+6n^2\theta^2-6\theta^3+11n\theta^3-6n^2\theta^3+n^3\theta^3)$.
Moreover, $V(X) = n\theta(1-\theta)$, and one can easily show
$V(X^2) = n\theta(1-7\theta+6n\theta+12\theta^2-16n\theta^2+4n^2\theta^2-6\theta^3+10n\theta^3-4n^2\theta^3)$, and
$Cov(X,X^2) = n\theta\,(1+2n\theta-2n\theta^2+2\theta^2-3\theta)$.
Note that $E(T_1) = \dfrac{\theta(1-\theta)(n-1)}{n}$, $E(T_2) = \theta(1-\theta)$, and
$E(T_3) = \left\{\theta(1-\theta)(n-1)+\dfrac{\sqrt{n}}{2}+\dfrac{1}{4}\right\}\cdot\dfrac{n}{(n+\sqrt{n})^2}$.
Hence, $(E(T_1)-\delta_\theta)^2 = \dfrac{1}{n^4}\,(n^2\theta^4-2n^2\theta^3+n^2\theta^2)$, $(E(T_2)-\delta_\theta)^2 = 0$, and
$(E(T_3)-\delta_\theta)^2 = \dfrac{1}{(n+\sqrt{n})^4}\left(\left(\dfrac{n\sqrt{n}}{2}+\dfrac{n}{4}\right)-2\,\theta\,(1-\theta)\,n\,(1+\sqrt{n})\right)^2$.
Furthermore,

$$\begin{aligned}
V(T_1) &= V\left(\frac{X(n-X)}{n^2}\right) = \frac{1}{n^4}\left(n^2\,V(X)+V(X^2)-2\,n\,Cov(X,X^2)\right)\\
&= \frac{1}{n^4}\left(n\,(n-1)^2\theta+(5n-7)(n-n^2)\,\theta^2+(2n-3)(4n^2-4n)\,\theta^3\right)+\\
&\quad \frac{1}{n^4}\left((2n-3)(2n-2n^2)\,\theta^4\right),
\end{aligned}$$

$$\begin{aligned}
V(T_2) &= V\left(\frac{X(n-X)}{n\,(n-1)}\right) = \frac{1}{n^2(n-1)^2}\left(n^2\,V(X)+V(X^2)-2\,n\,Cov(X,X^2)\right)\\
&= \frac{1}{n^2(n-1)^2}\left(n(n-1)^2\theta+(5n-7)(n-n^2)\theta^2+(2n-3)(4n^2-4n)\theta^3\right)+\\
&\quad \frac{1}{n^2(n-1)^2}\left((2n-3)(2n-2n^2)\theta^4\right),
\end{aligned}$$

$$\begin{aligned}
V(T_3) &= V\left(\frac{X(n-X)+n\sqrt{n}/2+n/4}{(n+\sqrt{n})^2}\right)\\
&= \frac{1}{(n+\sqrt{n})^4}\left(n^2V(X)+V(X^2)-2nCov(X,X^2)\right)\\
&= \frac{1}{(n+\sqrt{n})^4}\left(n(n-1)^2\theta+(5n-7)(n-n^2)\theta^2+(2n-3)(4n^2-4n)\theta^3\right)+\\
&\quad \frac{1}{(n+\sqrt{n})^4}\left((2n-3)(2n-2n^2)\theta^4\right).
\end{aligned}$$

We are now in a position to compute the mean squared errors of the three estimators.

1. $\mathrm{MSE}(T_1) = E(T_1-\delta_\theta)^2 = V(T_1)+(E(T_1)-\delta_\theta)^2$

$$= \frac{1}{n^4}\left\{n(n-1)^2\theta + \left((5n-7)(n-n^2)+n^2\right)\theta^2 + \left((2n-3)(4n^2-4n)-2n^2\right)\theta^3 + \left((2n-3)(2n-2n^2)+n^2\right)\theta^4\right\}.$$

2. $\mathrm{MSE}(T_2) = E(T_2-\delta_\theta)^2 = V(T_2)$

$$= \frac{1}{n^2(n-1)^2}\{n(n-1)^2\theta - n(5n-7)(n-1)\theta^2 + 4n(2n-3)(n-1)\theta^3 - 2n(2n-3)(n-1)\theta^4\}.$$

3. $\mathrm{MSE}(T_3) = E(T_3-\delta_\theta)^2 = V(T_3) + (E(T_3)-\delta_\theta)^2$

$$\begin{aligned} = \frac{1}{(n+\sqrt{n})^4}\{&[(2n+2n\sqrt{n})^2-2n(2n-3)(n-1)]\theta^4 \\ &+[-2(2n+2n\sqrt{n})^2+4n(2n-3)(n-1)]\theta^3 \\ &+[2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4)+(2n+2n\sqrt{n})^2-n(5n-7)(n-1)]\theta^2 \\ &+[-2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4)+n(n-1)^2]\theta + (n\sqrt{n}/2+n/4)^2\}.\end{aligned}$$

Finally, a justification of (13) follows from the following observation. Referring to [2] and [4], note that

$$w_3(\theta) = \frac{s}{\bar{x}} = \frac{\sqrt{\frac{1}{3}\sum_{i=1}^{3}(x_i(\theta)-\bar{x}(\theta))^2}}{\frac{1}{3}\sum_{i=1}^{3}x_i(\theta)}.$$

Since

$$\sum_{i=1}^{3}(x_i(\theta)-\bar{x}(\theta))^2 = \frac{2}{3}\left[x_1^2(\theta)+x_2^2(\theta)+x_3^2(\theta)-x_1(\theta)x_2(\theta)-x_1(\theta)x_3(\theta)-x_2(\theta)x_3(\theta)\right],$$

(13) follows.

On a Comparison of Several Competing Estimates of a Univariate Normal Mean by Multiple Criteria Decision Making Method

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ABSTRACT

In this paper we consider the problem of estimation of the mean of a univariate normal population with an unknown variance when uncertain non-sample prior information about the mean is available. We compare four estimators of the mean, including pre-test and shrinkage estimators. The performances of the estimators are compared based on the Multiple Criteria Decision Making (MCDM) procedure in order to find the best estimator.

Key Words: Multiple criteria decision making; pre-test estimate; shrinkage estimate; univariate normal mean. **AMS 2000 Subject Classification:** 62C25; 62G05

1. Introduction

We consider the problem of estimation of the mean μ of a normal population $N(\mu,\sigma^2)$ based on a random sample $X_1, X_2, \ldots, X_n$ of size *n*. Assume that uncertain non-sample prior information on the value of μ is available in the form of a null hypothesis $H_0 : \mu = \mu_0$. In this context, Khan and Saleh (2001) discussed three biased estimators of μ given by:

1) $\hat{\mu}_n(d) = d\tilde{\mu}_n + (1-d)\mu_0$, $0 \le d \le 1$, the restricted estimator (RE) with a coefficient of distrust *d*, where d is the degree of distrust in the null hypothesis, $H_0 : \mu = \mu_0$, and $\tilde{\mu}_n = \bar{x}$, the maximum likelihood estimator of μ.

2) $\hat{\mu}_n^{PTE}(d) = \hat{\mu}_n(d) I(|t_v| < t_{\alpha/2}) + \tilde{\mu}_n I(|t_v| \ge t_{\alpha/2})$, $|t_v| = \dfrac{|\sqrt{n}(\tilde{\mu}_n - \mu_0)|}{S_n}$

$= \tilde{\mu}_n - (1-d)(\tilde{\mu}_n - \mu_0)\, I(|t_v| < t_{\alpha/2})$, the preliminary test estimator (PTE) as a linear combination of the maximum likelihood estimator and the RE, where *I*(*A*) is an indicator function of the set A and $t_{\alpha/2}$ is the critical value chosen for the two-side $\alpha-$ level test based on the Student-t distribution with $\nu = n-1$ degree of freedom.

3) $\hat{\mu}_n^s = \mu_0 + \left\{1 - \frac{cS_n}{\left|\sqrt{n}(\tilde{\mu}_n - \mu_0)\right|}\right\}(\tilde{\mu}_n - \mu_0)$, the shrinkage estimator (SE) by using the preliminary test approach, where c is a constant function of ν.

In this paper we compare the unbiased estimator $\bar{x}$ (Casella and Berger, 1990) and the three biased estimators as described above on the basis of mean squared errors (MSE) (Casella and Berger, 1990) by using Multiple Criteria Decision Making (MCDM) method for searching the best estimator. This method is briefly described in Section 2, and Section 3 contains the mean squared errors of each estimator. Section 4 describes the main results. Our recommendation is to use $\bar{x}$ from the MCDM point of view. Among the biased estimators, the shrinkage estimator $\hat{\mu}_n^s$ performs the best.

2. A Brief Description of MCDM Procedure

In the context of a '*discrete*' risk matrix $X = (x_{ij}) : K \times N$ where x_{ij}'s represent '*risk*' of i th '*estimator*' for j th '*parameter point*', and we need to compare K estimators simultaneously with respect to all the N parameter points, MCDM is a novel statistical procedure to integrate the multiple risks $(x_{i1}, \ldots, x_{iN})$ for the i th estimator into a single meaningful and overall risk factor (see Filar et al., 1999 and Maitra et al. 2002). The K estimators are then compared on the basis of these integrated risk factors. Integration of risks is done by defining an Ideal Row with the smallest observed value for each column as

$$IDR = (\min_i x_{i1}, \ldots, \min_i x_{iN}) = (u_1, \ldots, u_N)$$

and a Negative-ideal Row (NIDR) with the largest observed value for each column as

$$NIDR = (\max_i x_{i1}, \ldots, \max_i x_{iN}) = (v_1, \ldots, v_N).$$

For any given row i, we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under L_1-norm (Zeleny, 1982), we compute

$$L_1(i, IDR) = \sum_{j=1}^{N} (x_{ij} - u_j)\, w_j$$

$$L_1(i,NIDR) = \sum_{j=1}^{N}(v_j - x_{ij})\, w_j$$

where $w_j's$ are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) = \frac{L_1(i,IDR)}{L_1(i,IDR) + L_1(i,NIDR)}, \quad i = 1, \ldots, K. \tag{2.1}$$

Similarly, under L_2-norm (Zeleny, 1982), we compute

$$L_2(i,IDR) = [\sum_{j=1}^{N}(x_{ij} - u_j)^2 w_j]^{1/2},$$

$$L_2(i,NIDR) = [\sum_{j=1}^{N}(x_{ij} - v_j)^2 w_j]^{1/2},$$

and compare the rows based on

$$L_2(Index_i) = \frac{L_2(i,IDR)}{L_2(i,IDR) + L_2(i,NIDR)}, \quad i = 1, \ldots, K. \tag{2.2}$$

A '*continuous*' version of this setup which is relevant for our problem would involve $x_{ij}'s$ representing risks or MSEs where the index j would vary '*continuously*'. In the context of our problem of comparing $\tilde{\mu}_n$, $\hat{\mu}_n(d)$, $\hat{\mu}_n^{PTE}(d)$ and $\hat{\mu}_n^s$ for estimation of μ, obviously $K = 4$, $x_{ij}'s$ represent the mean squared errors of $\tilde{\mu}_n$, $\hat{\mu}_n(d)$, $\hat{\mu}_n^{PTE}(d)$ and $\hat{\mu}_n^s$ which are functions of a real-valued parameter θ (see below), and L_1-norm and L_2-norm would be redefined as

$$L_1(i,IDR) = \int_{\underline{\theta}}^{\bar{\theta}} (x_i(\theta) - u(\theta))\, w(\theta)\, d\theta \tag{2.3}$$

$$L_1(i,NIDR) = \int_{\underline{\theta}}^{\bar{\theta}} (v(\theta) - x_i(\theta))\, w(\theta)\, d\theta \tag{2.4}$$

$$L_2(i,IDR) = \sqrt{\int_{\underline{\theta}}^{\bar{\theta}} (x_i(\theta) - u(\theta))^2\, w(\theta)\, d\theta} \tag{2.5}$$

$$L_2(i,NIDR) = \sqrt{\int_{\underline{\theta}}^{\bar{\theta}} (x_i(\theta) - v(\theta))^2\, w(\theta)\, d\theta} \tag{2.6}$$

where $u(\theta) = \min_i \{x_i(\theta)\}$ and $v(\theta) = \max_i \{x_i(\theta)\}$, $i = 1, 2, 3, 4$.

3. Mean Squared Errors

We first start with the mean squared errors of $\tilde{\mu}_n$, $\hat{\mu}_n(d)$, $\hat{\mu}_n^{PTE}(d)$ and $\hat{\mu}_n^s$. We refer to Khan and Saleh (2001) for details.

$$MSE(\tilde{\mu}_n) = \frac{\sigma^2}{n}.$$

$$MSE(\hat{\mu}_n(d)) = \frac{\sigma^2}{n}\left(d^2 + (1-d)^2\Delta^2\right), \quad \Delta = \frac{\sqrt{n}(\mu-\mu_0)}{\sigma}.$$

$$MSE(\hat{\mu}_n^{PTE}(d)) = \frac{\sigma^2}{n}\left(1-(1-d^2)G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)+(1-d)\Delta^2\left\{2G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)-(1+d)G_{5,\nu}\left(\frac{1}{5}F_\alpha;\Delta^2\right)\right\}\right).$$

where $G_{p,q}(\cdot;\Delta^2)$ is the c.d.f. of a non-central F-distribution with (p,q) degrees of freedom and non-centrality parameter Δ^2, and Δ^2 is the departure constant from the null hypothesis.

$$MSE(\hat{\mu}_n^s) = \frac{\sigma^2}{n}\left(1-\frac{2}{\pi}K_n^2[2e^{-\frac{\Delta^2}{2}}-1]\right), \quad K_n = \sqrt{\frac{2}{n-1}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Since σ^2/n is a common term so we ignore it. Writing $\tilde{\mu}_n$, $\hat{\mu}_n(d)$, $\hat{\mu}_n^{PTE}(d)$ and $\hat{\mu}_n^s$ as T_1, T_2, T_3 and T_4, respectively, we get

$$MSE(T_1) = 1,$$

$$MSE(T_2) = d^2 + (1-d)^2\Delta^2,$$

$$MSE(T_3) = 1-(1-d^2)G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)+(1-d)\Delta^2\left\{2G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)-(1+d)G_{5,\nu}\left(\frac{1}{5}F_\alpha;\Delta^2\right)\right\}$$

$$MSE(T_4) = 1-\frac{2}{\pi}K_n^2[2e^{-\frac{\Delta^2}{2}}-1], \quad K_n = \sqrt{\frac{2}{n-1}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

In order to apply the MCDM procedure, we need to evaluate the above MSEs for various values of Δ^2. First, we consider

$$MSE(T_3) = 1-(1-d^2)\ G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)+(1-d)\Delta^2\left\{2G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)-(1+d)G_{5,\nu}\left(\frac{1}{5}F_\alpha;\Delta^2\right)\right\}.$$

Let $F_{1,\nu;\alpha} = c$. Note that

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$$G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right) = \Pr\left[\chi_3^2(\Delta^2) \le \frac{c}{\nu}\ \chi_\nu^2\right]$$

$$= \sum_{j=0}^{\infty} e^{-\frac{\Delta^2}{2}} \frac{\left(\frac{\Delta^2}{2}\right)^j}{j!} \Pr\left[\frac{\chi_{3+2j}^2}{\chi_\nu^2} \le \frac{c}{\nu}\right],$$

and in the same manner

$$G_{5,\nu}\left(\frac{1}{5}F_\alpha;\Delta^2\right) = \sum_{j=0}^{\infty} e^{-\frac{\Delta^2}{2}} \frac{\left(\frac{\Delta^2}{2}\right)^j}{j!} \Pr\left[\frac{\chi_{5+2j}^2}{\chi_\nu^2} \le \frac{c}{\nu}\right].$$

The above probabilities are computed by simulation (1000 replications) for $\alpha = 0.05$, 0.25 and 0.5 and appear in Tables 1, 2 and 3 respectively.

In view of entries in Tables 1, 2 and 3, we can approximate $G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)$ and $G_{5,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right)$ as

$$G_{3,\nu}\left(\frac{1}{3}F_\alpha;\Delta^2\right) \approx \sum_{j=0}^{20} e^{-\frac{\Delta^2}{2}} \frac{\left(\frac{\Delta^2}{2}\right)^j}{j!} \underbrace{\Pr\left[\chi_{3+2j}^2 \le \frac{c}{\nu}\chi_\nu^2\right]}_{\text{use from Table 1, 2 and 3}}$$

$$G_{5,\nu}\left(\frac{1}{5}F_\alpha;\Delta^2\right) \approx \sum_{j=0}^{20} e^{-\frac{\Delta^2}{2}} \frac{\left(\frac{\Delta^2}{2}\right)^j}{j!} \underbrace{\Pr\left[\chi_{5+2j}^2 \le \frac{c}{\nu}\chi_\nu^2\right]}_{\text{use from Table 1, 2 and 3}}$$

Table 1 The simulated probabilities for n = 5, 10, 15; $\alpha = 0.05$

j	Simulated Probability					
	$n = 5$		$n = 10$		$n = 15$	
	$\frac{\chi^2_{3+2j}}{\chi^2_4} \le \frac{F_{1,4;.05}}{4}$	$\frac{\chi^2_{5+2j}}{\chi^2_4} \le \frac{F_{1,4;.05}}{4}$	$\frac{\chi^2_{3+2j}}{\chi^2_9} \le \frac{F_{1,9;.05}}{9}$	$\frac{\chi^2_{5+2j}}{\chi^2_9} \le \frac{F_{1,9;.05}}{9}$	$\frac{\chi^2_{3+2j}}{\chi^2_{14}} \le \frac{F_{1,14;.05}}{14}$	$\frac{\chi^2_{5+2j}}{\chi^2_{14}} \le \frac{F_{1,14;.05}}{14}$
0	0.805	0.655	0.43	0.225	0.26	0.098
1	0.655	0.522	0.225	0.089	0.098	0.027
2	0.522	0.396	0.089	0.055	0.027	0.015
3	0.396	0.317	0.055	0.02	0.015	0.004
4	0.317	0.221	0.02	0.013	0.004	0.005
5	0.221	0.165	0.013	0.002	0.005	0
6	0.165	0.122	0.002	0.003	0	0
7	0.122	0.088	0.003	0.001	0	0
8	0.088	0.073	0.001	0.001	0	0
9	0.073	0.041	0.001	0	0	0
10	0.041	0.03	0	0	0	0
11	0.03	0.02	0	0	0	0
12	0.02	0.015	0	0	0	0
13	0.015	0.01	0	0	0	0
14	0.01	0.004	0	0	0	0
15	0.004	0.006	0	0	0	0
16	0.006	0.003	0	0	0	0
17	0.003	0.001	0	0	0	0
18	0.001	0	0	0	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0

$F_{1,4;.05} = 7.7086$, $F_{1,9;.05} = 5.1173$, $F_{1,14;.05} = 4.6001$.

Table 2 The simulated probabilities for n = 5, 10, 15; $\alpha = 0.25$

j	Simulated Probability					
	$n=5$		$n=10$		$n=15$	
	$\frac{\chi^2_{3+2j}}{\chi^2_4} \le \frac{F_{1,4,.05}}{4}$	$\frac{\chi^2_{5+2j}}{\chi^2_4} \le \frac{F_{1,4;.05}}{4}$	$\frac{\chi^2_{3+2j}}{\chi^2_9} \le \frac{F_{1,9;.05}}{9}$	$\frac{\chi^2_{5+2j}}{\chi^2_9} \le \frac{F_{1,9;.05}}{9}$	$\frac{\chi^2_{3+2j}}{\chi^2_{14}} \le \frac{F_{1,14;.05}}{14}$	$\frac{\chi^2_{5+2j}}{\chi^2_{14}} \le \frac{F_{1,14;.05}}{14}$
0	0.355	0.162	0.13	0.031	0.079	0.009
1	0.162	0.057	0.031	0.005	0.009	0.001
2	0.057	0.028	0.005	0.002	0.001	0.001
3	0.028	0.009	0.002	0	0.001	0
4	0.009	0.008	0	0	0	0
5	0.008	0.001	0	0	0	0
6	0.001	0.001	0	0	0	0
7	0.001	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0

$F_{1,4;.25}$ = 1.8074, $F_{1,9;.25}$ = 1.5121, $F_{1,14;.25}$ = 1.4403.

Table 3 The simulated probabilities for $n = 5, 10, 15$; $\alpha = 0.5$

j	Simulated Probability					
	$n = 5$		$n = 10$		$n = 15$	
	$\frac{\chi^2_{3+2j}}{\chi^2_4} \le \frac{F_{1,4;.05}}{4}$	$\frac{\chi^2_{5+2j}}{\chi^2_4} \le \frac{F_{1,4;.05}}{4}$	$\frac{\chi^2_{3+2j}}{\chi^2_9} \le \frac{F_{1,9;.05}}{9}$	$\frac{\chi^2_{5+2j}}{\chi^2_9} \le \frac{F_{1,9;.05}}{9}$	$\frac{\chi^2_{3+2j}}{\chi^2_{14}} \le \frac{F_{1,14;.05}}{14}$	$\frac{\chi^2_{5+2j}}{\chi^2_{14}} \le \frac{F_{1,14;.05}}{14}$
0	0.107	0.02	0.031	0.002	0.013	0
1	0.02	0.001	0.002	0.001	0	0
2	0.001	0.001	0.001	0	0	0
3	0.001	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0

$F_{1,4;.5} = 0.5486$, $F_{1,9;.5} = 0.4938$, $F_{1,14;.5} = 0.4794$.

4. Main Results

In this section we compute the mean squared errors of each estimator and compare them based on MCDM. We consider three values of d ($d = 0, 0.25, 0.5$) and three values of n ($n = 5, 10, 15$) for each α ($\alpha = 0.05, 0.25, 0.5$) and denote Δ^2 by θ. It may be mentioned that $d = 0$ corresponds to the usual preliminary test estimator. We have considered the range $0 < \theta < 10$ and show only the result in two cases, i.e. Case I: $d = 0.25$, $n = 5$, $\alpha = 0.05$ and Case II: $d = 0.5$, $n = 10$, $\alpha = 0.05$. The others can be obtained in the same manner.

Case I: $d = 0.25$, $n = 5$, $\alpha = 0.05$

$$MSE(T_1) = 1,$$

$$MSE(T_2) = (0.25)^2 + (1-0.25)^2\theta = 0.0625 + 0.5625\theta,$$

$$MSE(T_3) = 1-(1-d^2)G_{3,4}\left(\frac{1}{3}F_\alpha;\theta\right)+(1-d)\theta\left\{2G_{3,4}\left(\frac{1}{3}F_\alpha;\theta\right)-(1+d)G_{5,4}\left(\frac{1}{5}F_\alpha;\theta\right)\right\}$$

$$= e^{-\theta/2}(-0.754688+e^{\theta/2}+0.286406\theta+0.185391\theta^2+0.04373344\theta^3+0.00540967\theta^4$$
$$+0.000644\theta^5+0.00004268\theta^6+2.7117\times10^{-6}\theta^7+1.47792\times10^{-7}\theta^8+5.78966\times10^{-9}\theta^9$$
$$+3.72135\times10^{-10}\theta^{10}+8.63766\times10^{-12}\theta^{11}+3.11546\times10^{-13}\theta^{12}+7.84746\times10^{-15}\theta^{13}$$
$$+2.5073\times10^{-16}\theta^{14}+7.78882\times10^{-18}\theta^{15}+4.64923\times10^{-21}\theta^{16}+4.45216\times10^{-21}\theta^{17}$$
$$+7.5856\times10^{-23}\theta^{18}+8.93738\times10^{-25}\theta^{19})$$

$$MSE(T_4) = 1-\frac{2}{\pi}K_5^2[2e^{-\frac{\theta}{2}}-1]\ ,\ K_5=\sqrt{\frac{2}{5-1}}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5-1}{2}\right)} = \sqrt{\frac{2}{4}}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(2)} = \frac{3}{4}\sqrt{\frac{\pi}{2}}$$

$$= 1.5625-1.125e^{-\theta/2}.$$

We present their graphical patterns for $d = 0.25$, $n = 5$ and $\alpha = 0.05$ in Figure 1.

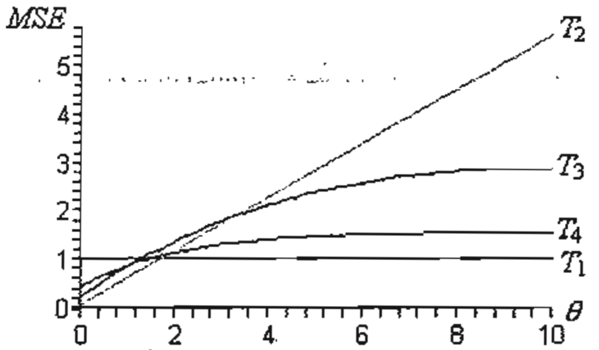

Figure 1 Graphical illustration of mean squared errors for $d = 0.25$, $n = 5$ and $\alpha = 0.05$.

Case II: $d = 0.5, n = 10, \alpha = 0.05$

$MSE(T_1) = 1,$

$MSE(T_2) = (0.5)^2+(1-0.5)^2\theta = 0.25+0.25\theta,$

$$MSE(T_3) = 1-(1-d^2)G_{3,9}\left(\frac{1}{3}F_\alpha;\theta\right)+(1-d)\theta\left\{2G_{3,9}\left(\frac{1}{3}F_\alpha;\theta\right)-(1+d)G_{5,9}\left(\frac{1}{5}F_\alpha;\theta\right)\right\}$$

$$= 1-0.3225e^{-\theta/2}+0.176875e^{-\theta/2}\theta+0.0707813e^{-\theta/2}\theta^2+0.005109e^{-\theta/2}\theta^3$$
$$+0.000794271e^{-\theta/2}\theta^4+0.0000241536e^{-\theta/2}\theta^5+2.96224\times10^{-6}e^{-\theta/2}\theta^6$$
$$-8.91307\times10^{-9}e^{-\theta/2}\theta^7+3.41506\times10^{-9}e^{-\theta/2}\theta^8+2.01836\times10^{-11}e^{-\theta/2}\theta^9$$
$$+5.38229\times10^{-12}e^{-\theta/2}\theta^{10}$$

$$MSE(T_4) = 1-\frac{2}{\pi}K_{10}^2[2e^{-\frac{\theta}{2}}-1]\ ,\ K_{10}=\sqrt{\frac{2}{10-1}}\frac{\Gamma\left(\frac{10}{2}\right)}{\Gamma\left(\frac{10-1}{2}\right)} = \sqrt{\frac{2}{9}}\frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)} = \frac{128}{105}\sqrt{\frac{2}{\pi}}$$

$$= 1-0.6018(-1+2e^{-\theta/2}).$$

We present their graphical patterns for $d = 0.5$, $n = 10$ and $\alpha = 0.05$ in Figure 2.

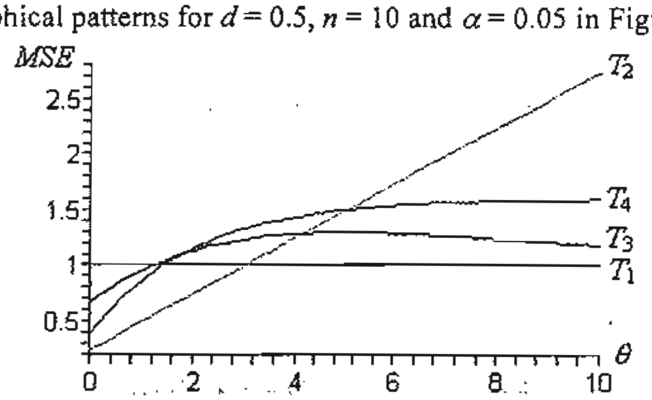

Figure 2 Graphical illustration of mean squared errors for $d = 0.5$, $n = 10$ and $\alpha = 0.05$.

Since $0 < \theta < 10$, the intersection of four graphs can separate the interval of θ into seven intervals $(0 < c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < 10)$. Moreover, the Ideal row and Negative-ideal row are as follows:

$$IDR: u(\theta) = \{u_1(\theta):\theta < c_1,\ u_2(\theta):c_1 < \theta < c_2, u_3(\theta):c_2 < \theta < c_3, u_4(\theta):c_3 < \theta < c_4,$$
$$u_5(\theta):c_4 < \theta < c_5, u_6(\theta):c_5 < \theta < c_6, u_7(\theta):\theta > c_6\}$$

$$NIDR: v(\theta) = \{v_1(\theta):\theta < c_1,\ v_2(\theta):c_1 < \theta < c_2, v_3(\theta):c_2 < \theta < c_3, v_4(\theta):c_3 < \theta < c_4,$$
$$v_5(\theta):c_4 < \theta < c_5, v_6(\theta):c_5 < \theta < c_6, v_7(\theta):\theta > c_6\}$$

The Ideal row and Negative-ideal row for each interval are shown in Tables 4, 5 and 6 for α = 0.05, 0.25 and 0.5 respectively.

Table 4 The *IDR* and *NIDR* for each interval at $\alpha = 0.05$

d	n	$\theta < c_1$		$c_1<\theta<c_2$		$c_2<\theta<c_3$		$c_3<\theta<c_4$		$c_4<\theta<c_5$		$c_5<\theta<c_6$		$c_6<\theta$	
		u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6	u_7	v_7
0.0	5	T_2	T_1	T_2	T_1	T_4	T_1	T_4	T_3	T_4	T_3	T_1	T_3	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
0.25	5	T_2	T_1	T_2	T_1	T_2	T_3	T_2	T_3	T_1	T_3	T_1	T_3	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_2	T_3	T_1	T_3	T_1	T_3	T_1	T_2	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_2	T_3	T_1	T_3	T_1	T_3	T_1	T_2	T_1	T_2
0.5	5	T_2	T_1	T_2	T_4	T_2	T_4	T_1	T_4	T_1	T_3	T_1	T_3	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2

Table 5 The *IDR* and *NIDR* for each interval at $\alpha = 0.25$

d	n	$\theta < c_1$		$c_1<\theta<c_2$		$c_2<\theta<c_3$		$c_3<\theta<c_4$		$c_4<\theta<c_5$		$c_5<\theta<c_6$		$c_6<\theta$	
		u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6	u_7	v_7
0.0	5	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
0.25	5	T_2	T_1	T_2	T_3	T_2	T_3	T_1	T_3	T_1	T_3	T_1	T_2	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_2	T_3	T_1	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
0.5	5	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2

Table 6 The *IDR* and *NIDR* for each interval at $\alpha = 0.5$

d	n	$\theta < c_1$		$c_1<\theta<c_2$		$c_2<\theta<c_3$		$c_3<\theta<c_4$		$c_4<\theta<c_5$		$c_5<\theta<c_6$		$c_6<\theta$	
		u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6	u_7	v_7
0.0	5	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_4	T_3	T_4	T_3	T_4	T_2	T_1	T_2	T_1	T_2
0.25	5	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
0.5	5	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	10	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2
	15	T_2	T_1	T_2	T_3	T_2	T_3	T_2	T_4	T_1	T_4	T_1	T_4	T_1	T_2

4.1. Analysis based on L_1-norm

For $i = 1, 2, 3, 4$, applying equations (2.3) and (2.4), we get

$$
\begin{aligned}
L_1(i, IDR) = \; & \int_0^{c_1} (x_i(\theta) - u_1(\theta)) w(\theta)\, d\theta + \int_{c_1}^{c_2} (x_i(\theta) - u_2(\theta)) w(\theta)\, d\theta + \int_{c_2}^{c_3} (x_i(\theta) - u_3(\theta)) w(\theta)\, d\theta + \\
& \int_{c_3}^{c_4} (x_i(\theta) - u_4(\theta)) w(\theta)\, d\theta + \int_{c_4}^{c_5} (x_i(\theta) - u_5(\theta)) w(\theta)\, d\theta + \int_{c_5}^{c_6} (x_i(\theta) - u_6(\theta)) w(\theta)\, d\theta + \\
& \int_{c_6}^{10} (x_i(\theta) - u_7(\theta)) w(\theta)\, d\theta,
\end{aligned}
$$

$$
\begin{aligned}
L_1(i, NIDR) = & \int_0^{c_1}(v_1(\theta)-x_i(\theta))w(\theta)d\theta + \int_{c_1}^{c_2}(v_2(\theta)-x_i(\theta))w(\theta)d\theta + \int_{c_2}^{c_3}(v_3(\theta)-x_i(\theta))w(\theta)d\theta + \\
& \int_{c_3}^{c_4}(v_4(\theta)-x_i(\theta))w(\theta)d\theta + \int_{c_4}^{c_5}(v_5(\theta)-x_i(\theta))w(\theta)d\theta + \int_{c_5}^{c_6}(v_6(\theta)-x_i(\theta))w(\theta)d\theta + \\
& \int_{c_6}^{10}(v_7(\theta)-x_i(\theta))w(\theta)d\theta.
\end{aligned}
$$

The overall index can then be computed from equation (2.1).

4.2. Analysis based on L_2-norm

For i = 1,2,3,4, applying equations (2.5) and (2.6), we get

$$
L_2(i, IDR) = \sqrt{\begin{aligned}
& \int_0^{c_1}(x_i(\theta)-u_1(\theta))^2 w(\theta)d\theta + \int_{c_1}^{c_2}(x_i(\theta)-u_2(\theta))^2 w(\theta)d\theta + \int_{c_2}^{c_3}(x_i(\theta)-u_3(\theta))^2 w(\theta)d\theta + \\
& \int_{c_3}^{c_4}(x_i(\theta)-u_4(\theta))^2 w(\theta)d\theta + \int_{c_4}^{c_5}(x_i(\theta)-u_5(\theta))^2 w(\theta)d\theta + \int_{c_5}^{c_6}(x_i(\theta)-u_6(\theta))^2 w(\theta)d\theta + \\
& \int_{c_6}^{10}(x_i(\theta)-u_7(\theta))^2 w(\theta)d\theta
\end{aligned}}
$$

$$
L_2(i, NIDR) = \sqrt{\begin{aligned}
& \int_0^{c_1}(v_1(\theta)-x_i(\theta))^2 w(\theta)d\theta + \int_{c_1}^{c_2}(v_2(\theta)-x_i(\theta))^2 w(\theta)d\theta + \int_{c_2}^{c_3}(v_3(\theta)-x_i(\theta))^2 w(\theta)d\theta + \\
& \int_{c_3}^{c_4}(v_4(\theta)-x_i(\theta))^2 w(\theta)d\theta + \int_{c_4}^{c_5}(v_5(\theta)-x_i(\theta))^2 w(\theta)d\theta + \int_{c_5}^{c_6}(v_6(\theta)-x_i(\theta))^2 w(\theta)d\theta + \\
& \int_{c_6}^{10}(v_7(\theta)-x_i(\theta))^2 w(\theta)d\theta
\end{aligned}}
$$

Under L_2-norm also, the overall index can be computed from equation (2.2).

4.3. Choice of weight functions

Our first weight function $w_1(\theta)$ is defined by $w_1(\theta) = 1$. Following Filar et al. (1999), we also consider two additional choices of $w(\theta)$. The first one, denoted by $w_2(\theta)$, is based on the notion of entropy among $x_1(\theta)$, $x_2(\theta)$, $x_3(\theta)$ and $x_4(\theta)$ for various values of θ, and the second one, denoted by $w_3(\theta)$, is based on the coefficient of variation of $x_1(\theta)$, $x_2(\theta)$, $x_3(\theta)$ and $x_4(\theta)$ for various values of θ. It turns out that

$$w_2(\theta) = \frac{1-\phi(\theta)}{\int_0^{10}[\ 1-\phi(\theta)\ ]\ d\theta}$$

where $$\phi(\theta) = -\frac{1}{\log 4}\sum_{i=1}^{4}\left\{\frac{x_i(\theta)}{\sum_{i=1}^{4}x_i(\theta)}.\log\left[\frac{x_i(\theta)}{\sum_{i=1}^{4}x_i(\theta)}\right]\right\},$$

and

$$w_3(\theta) = \frac{\sqrt{\begin{array}{l}3x_1^2(\theta)+3x_2^2(\theta)+3x_3^2(\theta)+3x_4^2(\theta)-2x_3(\theta)x_4(\theta)-2x_2(\theta)(x_3(\theta)+x_4(\theta)\ )\\ -2x_1(\theta)(x_2(\theta)+x_3(\theta)+x_4(\theta))\end{array}}}{\sum_{i=1}^{4}x_i(\theta\ )}.$$

4.4. Comparison of the estimators

We report summary of the ranks of the four estimators when compared using L_1-norm and L_2-norm on the basis of the weight functions $w_1(\theta)$, $w_2(\theta)$ and $w_3(\theta)$ for $\alpha = 0.05$, 0.25 and 0.50 in Tables 7, 8 and 9, respectively.

Table 7 Rank of four estimates using weights $w_1(\theta)$, $w_2(\theta)$ and $w_3(\theta)$ at $\alpha = 0.05^*$

d	n	T	L_1-norm			L_2-norm		
			w_1	w_2	w_3	w_1	w_2	w_3
0.0	5	T_1	1	2	1	1	2	1
		T_2	4	4	4	4	4	4
		T_3	3	3	3	3	3	3
		T_4	2	1	2	2	1	2
	10	T_1	1	1	1	1	1	1
		T_2	4	4	4	4	4	4
		T_3	3	3	3	3	3	3
		T_4	2	2	2	2	2	2
	15	T_1	1	1	1	1	1	1
		T_2	3	2	4	4	3	4
		T_3	2	4	3	3	4	3
		T_4	2	3	2	2	2	2

Table 7 (Continued)

d	n	T	L_1-norm			L_2-norm		
			w_1	w_2	w_3	w_1	w_2	w_3
0.25	5	T_1	1	1	1	1	1	1
		T_2	4	4	4	4	4	4
		T_3	3	3	3	3	3	3
		T_4	2	2	2	2	2	2
	10	T_1	1	1	1	1	1	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	3	2
		T_4	3	2	3	3	2	3
	15	T_1	1	1	1	1	1	1
		T_2	4	2	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	4	3	3	3	3
0.5	5	T_1	1	2	1	1	2	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	3	2
		T_4	3	1	3	3	1	3
	10	T_1	1	1	1	1	1	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	3	2
		T_4	3	2	3	3	2	3
	15	T_1	1	1	1	1	1	1
		T_2	4	2	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	4	3	3	3	3

*Rank 1 = best, Rank 3 = worst

Table 8 Rank of four estimates using weights $w_1(\theta)$, $w_2(\theta)$ and $w_3(\theta)$ at $\alpha = 0.25^*$

d	n	T	L_1-norm			L_2-norm		
			w_1	w_2	w_3	w_1	w_2	w_3
0.0	5	T_1	1	1	1	1	2	1
		T_2	4	4	4	4	4	4
		T_3	3	3	2	3	3	2
		T_4	2	2	3	2	1	3
	10	T_1	1	1	1	1	1	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	3	2
		T_4	3	2	3	3	2	3
	15	T_1	1	1	1	1	1	1
		T_2	4	2	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	4	3	3	3	3

Table 8 (Continued)

d	n	T	L_1-norm			L_2-norm		
			w_1	w_2	w_3	w_1	w_2	w_3
0.0	5	T_1	1	1	1	1	2	1
		T_2	4	4	4	4	4	4
		T_3	3	3	2	3	3	2
		T_4	2	2	3	2	1	3
	10	T_1	1	1	1	1	1	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	3	2
		T_4	3	2	3	3	2	3
	15	T_1	1	1	1	1	1	1
		T_2	4	2	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	4	3	3	3	3
0.25	5	T_1	1	2	1	1	3	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	1	3	3	1	3
	10	T_1	1	1	1	1	3	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	2	3	3	1	3
	15	T_1	1	2	1	1	1	1
		T_2	4	1	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	4	3	3	3	3
0.5	5	T_1	1	2	1	1	3	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	1	3	3	1	3
	10	T_1	1	1	1	1	3	2
		T_2	4	4	4	4	4	4
		T_3	2	2	2	2	2	1
		T_4	3	3	3	3	1	3
	15	T_1	1	2	1	1	1	2
		T_2	4	1	4	4	4	4
		T_3	2	3	2	2	2	1
		T_4	3	4	3	3	3	3

*Rank 1 = best, Rank 3 = worst

Table 9 Rank of four estimates using weights $w_1(\theta)$, $w_2(\theta)$ and $w_3(\theta)$ at $\alpha = 0.5^*$

d	n	T	L_1-norm			L_2-norm		
			w_1	w_2	w_3	w_1	w_2	w_3
0.0	5	T_1	1	2	1	1	3	1
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	2	2
		T_4	3	1	3	3	1	3
	10	T_1	1	1	1	1	3	2
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	2	1
		T_4	3	2	3	3	1	3
	15	T_1	1	2	1	1	1	2
		T_2	4	1	4	4	4	4
		T_3	2	3	2	2	2	1
		T_4	3	4	3	3	3	3
0.25	5	T_1	1	2	1	1	3	2
		T_2	4	4	4	4	4	4
		T_3	2	3	2	2	2	1
		T_4	3	1	3	3	1	3
	10	T_1	1	1	1	1	3	2
		T_2	4	4	4	4	4	4
		T_3	2	2	2	2	2	1
		T_4	3	3	3	3	1	3
	15	T_1	1	2	1	1	1	2
		T_2	4	1	4	4	4	4
		T_3	2	3	2	2	2	1
		T_4	3	4	3	3	3	3
0.5	5	T_1	1	2	1	2	3	2
		T_2	4	4	4	4	4	4
		T_3	2	3	2	1	2	1
		T_4	3	1	3	3	1	3
	10	T_1	1	1	1	2	3	2
		T_2	4	4	4	4	4	4
		T_3	2	2	2	1	2	1
		T_4	3	3	3	3	1	3
	15	T_1	1	2	1	1	1	2
		T_2	4	1	4	4	4	4
		T_3	2	3	2	2	2	1
		T_4	3	4	3	3	3	3

*Rank 1 = best, Rank 3 = worst

5. Conclusion

It is obvious from Figures 1 and 2 that there is no one estimator which dominates the other estimators in terms of mean squared errors. Based on L_1- and L_2- norms, we conclude that our preference is uniformly for T_1 under three weights $w_1(\theta)$, $w_2(\theta)$ and $w_3(\theta)$ for all values of α. Moreover, among the biased estimators T_2, T_3 and T_4, it turns out that T_3 and T_4 have some edge over T_2 for $w_1(\theta)$, $w_2(\theta)$ and $w_3(\theta)$ for all values of α.

6. Acknowledgements

We are extremely thankful to a referee for some helpful comments which improved the presentation. We would also like to thank the Thailand Research Fund and the Commission on Higher Education for the financial support.

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