รายงานวิจัยฉบับสมบูรณ์

โครงการ

เสถียรภาพหลายรูปแบบและการทำให้มีเสถียรภาพสำหรับระบบควบคุมไม่เชิงเส้นที่มีตัวหน่วงและการประยุกต์

Various Stability and Stabilization for Nonlinear Control Systems with Delays and Applications

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

โดย

ผศ.ดร.คณิต มุกดาใส และคณะ

มิถุนายน 2557

สัญญาเลขที่ MRG5580006

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ผศ.ดร.คณิต มุกดาใส ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
(หัวหน้าโครงการวิจัยผู้รับทุน) มหาวิทยาลัยขอนแก่น

รศ.ดร.ปิยะพงศ์ เนียมทรัพย์ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์
(นักวิจัยที่ปรึกษา) มหาวิทยาลัยเชียงใหม่

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ชื่อนักวิจัย : ผศ.ดร. คณิต มุกดาใส
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น

นักวิจัยที่ปรึกษา: รศ.ดร. ปิยะพงศ์ เนียมทรัพย์
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

อีเมล์: kanit@kku.ac.th (คณิต มุกดาใส)
piyapong.n@cmu.ac.th (ปิยะพงศ์ เนียมทรัพย์)

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ในงานวิจัยนี้ เราได้สนใจปัญหาเสถียรภาพแบบเลขชี้กำลังทนทานสำหรับระบบไม่แน่นอนเวลาไม่ต่อเนื่องที่ขึ้นกับตัวแปรเชิงเส้นที่มีตัวหน่วงแปรผันตามเวลาเป็นช่วง อีกทั้งยังได้สนใจปัญหาเสถียรภาพแบบเลขชี้กำลังทนทาน สำหรับระบบไม่แน่นอนเวลาต่อเนื่องที่ขึ้นกับตัวแปรเชิงเส้นที่มีตัวหน่วงแปรผันตามเวลาหลายตัวและถูกรบกวนด้วยฟังก์ชันไม่เชิงเส้น ความไม่แน่นอนที่สนใจอยู่มีขอบเขต ฟังก์ชันไลปูนอฟ-คราซอฟสกีที่ขึ้นกับตัวแปร รูปแบบของไลบ์นิซ-นิวตันและอสมการเมตริกซ์เชิงเส้นถูกใช้เพื่อวิเคราะห์เสถียรภาพของระบบดังกล่าว รวมทั้งการประมาณค่าพื้นฐานและการใช้ประโยชน์จากอสมการศูนย์ เพื่อทำให้ได้มาซึ่งหลักเกณฑ์เสถียรภาพแบบเลขชี้กำลังทนทานรูปแบบใหม่ที่อยู่ในรูปอสมการเมตริกซ์เชิงเส้น ตัวอย่างเชิงตัวเลขที่ให้มาแสดงให้เห็นถึงศักยภาพของวิถีดังกล่าว

คำหลัก : เสถียรภาพแบบเลขชี้กำลังทนทาน, ระบบเวลาไม่ต่อเนื่องที่มีตัวหน่วง, ระบบเวลาต่อเนื่องที่มีตัวหน่วง, อสมการเมตริกซ์เชิงเส้น, วิธีไลปูนอฟ

Abstract

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Project Title: Various Stability and Stabilization for Nonlinear Control Systems with Delays and Applications

Investigator: Asst. Prof. Dr. Kanit Mukdasai
Department of Mathematics, Faculty of Science, Khon Kaen University

Mentor: Assoc. Prof. Dr. Piyapong Niamsup
Department of Mathematics, Faculty of Science, Chiang Mai University

E-mail Address: kanit@kku.ac.th (Kanit Mukdasai)
piyapong.n@cmu.ac.th (Piyapong Niamsup)

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This research work investigates the problems of robust exponential stability for uncertain linear parameter dependent discrete-time systems with interval time-varying delay and robust exponential stability for uncertain linear parameter dependent continuous-time systems with discrete and distributed time-varying delays and nonlinear perturbations. The uncertainty under consideration is norm-bounded uncertainty. Parameter dependent Lyapunov-Krasovskii functional, Leibniz-Newton formula and linear matrix inequality are proposed to analyze the stability. On the basis of the estimation and by utilizing free-weighting matrices, new robust delay-dependent exponential stability criteria are established in terms of linear matrix inequalities. Numerical examples are given to demonstrate the effectiveness and less conservativeness of the proposed methods.

Keywords: Robust exponential stability, Discrete-time delay system, Continuous-time delay system, Linear matrix inequality, Lyapunov method.

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา สำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยขอนแก่น ที่ได้ให้โอกาสผู้วิจัยได้รับทุนเพื่อเป็นการพัฒนาศักยภาพในการทำงานวิจัยอาจารย์รุ่นใหม่ในการทำงานวิจัยครั้งนี้

รองศาสตราจารย์ ดร.ปิยะพงศ์ เนียมทรัพย์ นักวิจัยที่ปรึกษาให้กับโครงการนี้ผู้ซึ่งอบรม สั่งสอนและถ่ายทอดความรู้ด้านต่างๆ จนผู้วิจัยสามารถทำงานวิจัยได้สำเร็จตามเป้าหมาย

คณะผู้ประเมินของวารสารวิชาการต่างๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปตีพิมพ์ในวารสารนั้นๆ

คณาจารย์ นักศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

ผศ.ดร.คณิต มุกดาใส
หัวหน้าโครงการวิจัย

Contents

Chapter 1

Executive Summary

Consider system described by the following state equation of the form

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) \\ \qquad +C(\alpha)\int_{t-g(t)}^{t} x(s)ds, \quad t > 0; \\ x(t) = \phi(t), \quad \dot{x}(t) = \psi(t), \quad t \in [-\bar{h}, 0], \quad \bar{h} = \max\{h, g\}, \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ is the state variable, $A(\alpha)$, $B(\alpha)$, $C(\alpha) \in R^{n\times n}$ are uncertain matrices belonging to the polytope

$$A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \quad B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i, \quad C(\alpha) = \sum_{i=1}^{N} \alpha_i C_i,$$

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i, B_i, C_i \in R^{n\times n}, \quad i = 1, ..., N.$$

$h(t)$ and $g(t)$ are discrete and distributed time-varying delays, respectively, satisfying

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq h_d, \quad 0 \leq g(t) \leq g,$$

where h, h_d, g are given positive real constants. Consider the initial functions $\phi(t), \psi(t) \in C([-\bar{h}, 0], R^n)$ ($C([-\bar{h}, 0], R^n)$ denotes the space of all continuous vector functions mapping $[-\bar{h}, 0]$ into R^n) with the norm $\|\phi\| = \sup_{t\in[-\bar{h},0]} \|\phi(t)\|$ and $\|\psi\| = \sup_{t\in[-\bar{h},0]} \|\psi(t)\|$. The uncertainties $f(.), g(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude of the form

$$f^T(t,x(t))f(t,x(t)) \leq \eta^2 x^T(t)x(t),$$
$$g^T(t,x(t-h(t)))g(t,x(t-h(t))) \leq \rho^2 x^T(t-h(t))x(t-h(t)),$$

where η, ρ are given real constants. Let us set $\|x\|$ denotes the Euclidean vector norm of $x \in R^n$; $x_t = \{x(t+s) : s \in [-h, 0]\}$, $\|x_t\| = \sup_{s\in[-h,0]} \|x(t+s)\|$.

Lemma 1.0.1 *Consider the system* (1.1)*. If there exist a Lyapunov-Krasovskii functional* $V(t, x_t)$ *and* $\lambda_1, \lambda_2, \lambda_3 > 0$ *such that for every solution* $x(t)$ *of the system, the following conditions hold,*

(i). $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii). $\dot{V}(t, x_t) \leq -\lambda_3 \|x(t)\|^2$,

then the solution of the system (1.1) *is robustly asymptotically stable.*

Definition 1.0.2 *The system* (1.1) *is robustly exponentially stable, if there exist positive real numbers* β *and* M *such that for each* $\phi(t), \psi(t) \in C([-\bar{h}, 0], R^n)$*, the solution* $x(t, \phi, \psi)$ *of the system* (1.1) *satisfies*

$$\|x(t, \phi, \psi)\| \leq M \max\{\|\phi\|, \|\psi\|\} e^{-\beta t}, \quad \forall t \in R^+.$$

Consider the following uncertain LPD discrete-time system with interval time-varying delay in the state

$$\begin{cases} x(k+1) = \Big[A(\alpha) + \Delta A(k)\Big] x(k) + \Big[B(\alpha) + \Delta B(k)\Big] x(k - h(k)); \\ x(s) = \phi(s), \quad \text{s} = -h_2, -h_2 + 1, \ldots, -1, 0, \end{cases} \tag{1.2}$$

where $k \in Z^+$, $x(k) \in R^n$ is the system state and $\phi(s)$ is a initial value at s. $A(\alpha)$, $B(\alpha) \in R^{n \times n}$ are uncertain matrices belonging to the polytope of the form

$$A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \quad B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i,$$

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i, B_i \in R^{n \times n}, \quad i = 1, ..., N.$$

$\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(k) = K(\alpha) \Delta(k) A_1(\alpha), \quad \Delta B(k) = K(\alpha) \Delta(k) B_1(\alpha),$$

$$A_1(\alpha) = \sum_{i=1}^{N} \alpha_i A_i^1, \quad B_1(\alpha) = \sum_{i=1}^{N} \alpha_i B_i^1, \quad K(\alpha) = \sum_{i=1}^{N} \alpha_i K_i,$$

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i^1, B_i^1 \in M^{n\times n}, \quad i = 1, ..., N.$$

The class of parametric uncertainties $\Delta(k)$ which satisfies

$$\Delta(k) = F(k)[I - JF(k)]^{-1},$$

is said to be admissible where J is a known matrix satisfying

$$I - JJ^T > 0,$$

and $F(k)$ is uncertain matrix satisfying

$$F(k)^T F(k) \leq I.$$

In addition, we assume that the time-varying delay $h(k)$ is upper and lower bounded. It satisfies the following assumption of the form

$$h_1 \leq h(k) \leq h_2,$$

where h_1 and h_2 are known positive integers.

Lemma 1.0.3 *The system* (1.2) *is said to be robustly asymptotically stable if there exists a positive definite function* $V : Z \times C([-h_2, 0], R^n) \rightarrow R^+$ *such that*

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) < 0,$$

along the solution of the system (1.2).

Definition 1.0.4 *The system* (1.2) *is said to be robustly exponentially stable if there exist constant scalars* $0 < a < 1$ *and* $b > 0$ *such that*

$$\|x(k)\|^2 \leq ba^k \sup_{-h_2 \leq l \leq 0} \|\phi(l)\|^2,$$

for all admissible uncertainties.

In this research, we shall investigate the problem of robust exponential stability for continuous-time linear parameter dependent (LPD) system with mixed time-varying delays and nonlinear perturbations. Based on combination of

Leibniz-Newton formula and linear matrix inequality, the use of suitable Lyapunov-Krasovskii functional, new delay-dependent exponential stability criteria will be obtained in terms of LMIs. Moreover, we investigate the problem of robust exponential stability for uncertain discrete-time LPD system with delay. The delay is of an interval type, which means that both lower and upper bounds for the time-varying delay are available. The uncertainty under consideration is norm-bounded uncertainty. Based on combination of the LMI technique and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability are obtained in terms of LMI. Finally, numerical examples will be given to show the effectiveness of the obtained results.

Chapter 2

Main Results

In this section, we present the robust exponential stability criteria for the continuous-time delay systems by using the combination of LMI technique and Lyapunov theory method. Moreover, we show our results on the robust exponential stability criteria for uncertain LPD discrete-time system with interval time-varying delays. Based on combination of the LMI technique and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability are obtained in terms of LMI.

2.1 Stability Continuous Time Delay System

We introduce the following notations for later use.

$$P_j(\alpha) = \sum_{i=1}^{N} \alpha_i P_i^j, \quad W_j(\alpha) = \sum_{i=1}^{N} \alpha_i W_i^j, \quad N_j(\alpha) = \sum_{i=1}^{N} \alpha_i N_i^j, \quad Q_j(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i^j,$$

$$M_l(\alpha) = \sum_{i=1}^{N} \alpha_i M_i^l, \quad R_s(\alpha) = \sum_{i=1}^{N} \alpha_i R_i^s, \quad \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0,$$

$$P_i^j, W_i^j, N_i^j, Q_i^j, M_i^l, R_i^s \in R^{n\times n}, \quad j = 1,2,...,6, \quad l = 1,2,...,5, \quad s = 1,2,3,$$

$$i = 1,2,...,N.$$

$$\prod\nolimits_{i,j,k} = \begin{bmatrix} \Sigma_{i,j,k}^{11} & \Sigma_{i,j,k}^{12} & \Sigma_{i,j}^{13} & \Sigma_{i,j}^{14} & \Sigma_{i,j}^{15} & \Sigma_{i}^{16} & \Sigma_{i,j,k}^{17} \\ * & \Sigma_{i,j,k}^{22} & \Sigma_{i,j}^{23} & \Sigma_{i,j}^{24} & \Sigma_{i,j}^{25} & \Sigma_{i}^{26} & \Sigma_{i,j,k}^{27} \\ * & * & \Sigma_{i}^{33} & \Sigma_{i}^{34} & \Sigma_{i}^{35} & -N_i^{3^T} & \Sigma_{i,j}^{37} \\ * & * & * & \Sigma_{i}^{44} & \Sigma_{i}^{45} & -N_i^{4^T} & \Sigma_{i,j}^{47} \\ * & * & * & * & \Sigma_{i}^{55} & \Sigma_{i}^{56} & \Sigma_{i,j}^{57} \\ * & * & * & * & * & \Sigma_{i}^{66} & 0 \\ * & * & * & * & * & * & \Sigma_{i,j,k}^{77} \end{bmatrix}, \tag{2.3}$$

where

$$
\begin{aligned}
\Sigma_{i,j,k}^{11} &= 2\beta P_i^1 + P_i^1 A_j + A_i^T P_j^1 + P_i^2 + h^2 A_i^T P_j^5 A_k - e^{-2\beta h} P_i^5 + Q_i^1 + Q_i^{1^T} \\
&\quad + Q_i^{4^T} A_j + A_i^T Q_j^4 + N_i^{1^T} + N_i^1 + W_i^{1^T} A_j + A_i^T W_j^1 \\
&\quad + hM_i^{1^T} + hM_i^1 + h^2 M_i^3 + \epsilon_1 \eta^2 I + g^2 P_i^6, \\
\Sigma_{i,j,k}^{12} &= P_i^1 B_j + Q_i^2 - Q_i^{1^T} + A_i^T Q_j^5 + Q_i^{4^T} B_j + h^2 A_i^T P_j^5 B_k + e^{-2\beta h} P_i^5 \\
&\quad + hR_i^{2^T} - hM_i^{1^T} - N_i^{1^T} + N_i^2 + W_i^{1^T} B_j + W_i^{2^T} A_j + hM_i^2 + h^2 M_i^4, \\
\Sigma_{i,j}^{13} &= N_i^3 + W_i^{1^T} + A_i^T W_j^3 + P_i^1 + Q_i^{4^T} + h^2 A_i^T P_j^5, \\
\Sigma_{i,j}^{14} &= N_i^4 + W_i^{1^T} + A_i^T W_j^4 + P_i^1 + Q_i^{4^T} + h^2 A_i^T P_j^5, \\
\Sigma_{i,j}^{15} &= N_i^5 - W_i^{1^T} + A_i^T W_j^5 + Q_i^3 - Q_i^{4^T} + A_i^T Q_j^6, \\
\Sigma_{i}^{16} &= -Q_i^{1^T} - N_i^{1^T} + N_i^6, \\
\Sigma_{i,j,k}^{17} &= h^2 A_i^T P_j^5 C_k + W_i^{1^T} C_j + A_i^T W_j^6 + P_i^1 C_j + Q_i^{4^T} C_j, \\
\Sigma_{i,j,k}^{22} &= -Q_i^{2^T} - Q_i^2 + Q_i^{5^T} B_j + B_i^T Q_j^5 - e^{-2\beta h} P_i^2 + h_d P_i^2 + h^2 B_i^T P_j^5 B_k \\
&\quad - e^{-2\beta h} P_i^5 + h^2 R_i^1 - N_i^{2^T} - N_i^2 + W_i^{2^T} B_j + B_i^T W_j^2 \\
&\quad - hR_i^{2^T} - hR_i^2 - hM_i^{2^T} - hM_i^2 + h^2 M_i^5 + \epsilon_2 \rho^2 I, \\
\Sigma_{i,j}^{23} &= W_i^{2^T} - N_i^3 + B_i^T W_j^3 + Q_i^{5^T} + h^2 B_i^T P_j^5, \\
\Sigma_{i,j}^{24} &= W_i^{2^T} - N_i^4 + B_i^T W_j^4 + Q_i^{5^T} + h^2 B_i^T P_j^5, \\
\Sigma_{i,j}^{25} &= -W_i^{2^T} - N_i^5 + B_i^T W_j^5 + B_i^T Q_j^6 - Q_i^3 - Q_i^{5^T}, \\
\Sigma_{i}^{26} &= -Q_i^{2^T} - N_i^{2^T} - N_i^6, \\
\Sigma_{i,j,k}^{27} &= h^2 B_i^T P_j^5 C_k + W_i^{2^T} C_j + B_i^T W_j^6 + Q_i^{5^T} C_j, \\
\Sigma_{i}^{33} &= W_i^{3^T} + W_i^3 + h^2 P_i^5 - \epsilon_1 I, \\
\Sigma_{i}^{34} &= W_i^{3^T} + W_i^4 + h^2 P_i^5, \\
\Sigma_{i}^{35} &= -W_i^{3^T} + W_i^5 + Q_i^6, \\
\Sigma_{i,j}^{37} &= h^2 P_i^5 C_j + W_i^{3^T} C_j + W_i^6, \\
\Sigma_{i}^{44} &= W_i^{4^T} + W_i^4 + h^2 P_i^5 - \epsilon_2 I, \\
\Sigma_{i}^{45} &= -W_i^{4^T} + W_i^5 + Q_i^6, \\
\Sigma_{i,j}^{47} &= h^2 P_i^5 C_j + W_i^{4^T} C_j + W_i^6,
\end{aligned}
$$

$$\begin{aligned}
\Sigma_i^{55} &= -W_i^{5^T} - W_i^5 - Q_i^{6^T} - Q_i^6 + h^2 P_i^3 + h^2 P_i^4, \\
\Sigma_i^{56} &= -Q_i^{3^T} - N_i^{5^T}, \\
\Sigma_{i,j}^{57} &= W_i^{5^T} C_j - W_i^6 + Q_i^{6^T} C_j, \\
\Sigma_i^{66} &= -N_i^{6^T} - N_i^6 - e^{-2\beta h} P_i^4, \\
\Sigma_{i,j,k}^{77} &= -e^{2\beta g} P_i^6 + h^2 C_i^T P_j^5 C_k + C_i^T W_j^6 + W_i^{6^T} C_j.
\end{aligned}$$

Theorem 2.1.1 *For given positive real constants h, h_d, g, η and ρ, system (1.1) is robustly exponentially stable with a decay rate β, if there exist positive definite symmetric matrices P_i^s, any appropriate dimensional matrices W_i^s, Q_i^s, N_i^s, M_i^r, R_i^t, $s = 1, 2, \ldots, 6$, $r = 1, 2, \ldots, 5$, $t = 1, 2, 3$, $i = 1, 2, ..., N$ and positive real constants ϵ_1 and ϵ_2 satisfying the following LMIs:*

$$\begin{bmatrix} R_i^1 & R_i^2 \\ * & R_i^3 \end{bmatrix} > 0, \quad i = 1, 2, ..., N, \tag{2.4}$$

$$\begin{bmatrix} e^{-2\beta h} P_i^3 - R_i^3 & M_i^1 & M_i^2 \\ * & M_i^3 & M_i^4 \\ * & * & M_i^5 \end{bmatrix} \geq 0, \quad i = 1, 2, ..., N, \tag{2.5}$$

$$\prod\nolimits_{i,i,i} < -I, \quad i = 1, 2, ..., N, \tag{2.6}$$

$$\prod\nolimits_{i,i,j} + \prod\nolimits_{i,j,i} + \prod\nolimits_{j,i,i} < \frac{1}{(N-1)^2} I,$$
$$i = 1, 2, ..., N, i \neq j, j = 1, 2, ..., N, \tag{2.7}$$

$$\prod\nolimits_{i,j,k} + \prod\nolimits_{i,k,j} + \prod\nolimits_{j,i,k} + \prod\nolimits_{j,k,i} + \prod\nolimits_{k,i,j} + \prod\nolimits_{k,j,i} < \frac{6}{(N-1)^2} I,$$
$$i = 1, 2, ..., N-2, j = i+1, ..., N-1, k = j+1, ..., N. \tag{2.8}$$

Moreover, the solution $x(t, \phi, \psi)$ satisfies the inequality

$$\|x(t, \phi, \psi)\| \leq \sqrt{\frac{N}{\lambda_{\min}(P_1(\alpha))}} \max\left[\|\phi\|, \|\psi\|\right] e^{-\beta t}, \quad \forall t \in R^+. \tag{2.9}$$

where $N = \lambda_{\max}(P_1(\alpha)) + h\lambda_{\max}(P_2(\alpha)) + h^3\lambda_{\max}(P_3(\alpha)) + h^3\lambda_{\max}(P_4(\alpha)) + h^3\lambda_{\max}(P_5(\alpha)) + h^3\lambda_{\max}\Big(\begin{bmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha) \end{bmatrix} \Big).$

If $A(\alpha) = A$, $B(\alpha) = B$ and $C(\alpha) = 0$ when A and B are appropriate dimensional constant matrices, then system (1.1) reduces to the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - h(t)) + f(t, x(t)) + g(t, x(t - h(t))), \quad t > 0; \\ x(t) = \phi(t), \quad \dot{x}(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \tag{2.10}$$

Corollary 2.1.2 *For given positive real constants* h, h_d, η *and* ρ*, system* (2.10) *is exponentially stable with a decay rate* β*, if there exist positive definite symmetric matrices* P_i*,* $i = 1, 2, \ldots, 5$*, any appropriate dimensional matrices* $Q_i, N_i, i = 1, 2, \ldots, 6$*,* $W_i, M_i, i = 1, 2, \ldots, 5$*,* $R_i, i = 1, 2, 3$ *and positive real constants* ϵ_1 *and* ϵ_2 *satisfying the following LMIs:*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0, \tag{2.11}$$

$$\begin{bmatrix} e^{-2\beta h} P_3 - R_3 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{2.12}$$

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} \\ * & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & -N_3^T \\ * & * & * & \Sigma_{44} & \Sigma_{45} & -N_4^T \\ * & * & * & * & \Sigma_{55} & \Sigma_{56} \\ * & * & * & * & * & \Sigma_{66} \end{bmatrix} < 0, \tag{2.13}$$

where

$$
\begin{aligned}
\Sigma_{11} &= 2\beta P_1 + P_1 A + A^T P_1 + P_2 + h^2 A^T P_5 A - e^{-2\beta h} P_5 + Q_1 + Q_1^T + Q_4^T A \\
&\quad + A^T Q_4 + N_1^T + N_1 + W_1^T A + A^T W_1 + h M_1^T + h M_1 + h^2 M_3 + \epsilon_1 \eta^2 I, \\
\Sigma_{12} &= P_1 B + Q_2 - Q_1^T + A^T Q_5 + Q_4^T B + h^2 A^T P_5 B + e^{-2\beta h} P_5 + h R_2^T \\
&\quad - h M_1^T - N_1^T + N_2 + W_1^T B + W_2^T A + h M_2 + h^2 M_4, \\
\Sigma_{13} &= N_3 + W_1^T + A^T W_3 + P_1 + Q_4^T + h^2 A^T P_5, \\
\Sigma_{14} &= N_4 + W_1^T + A^T W_4 + P_1 + Q_4^T + h^2 A^T P_5, \\
\Sigma_{15} &= N_5 - W_1^T + A^T W_5 + Q_3 - Q_4^T + A^T Q_6, \\
\Sigma_{16} &= -Q_1^T - N_1^T + N_6, \\
\Sigma_{22} &= -Q_2^T - Q_2 + Q_5^T B + B^T Q_5 - e^{-2\beta h} P_2 + h_d P_2 + h^2 B^T P_5 B - e^{-2\beta h} P_5 \\
&\quad + h^2 R_1 - N_2^T - N_2 + W_2^T B + B^T W_2 - h R_2^T - h R_2 - h M_2^T \\
&\quad - h M_2 + h^2 M_5 + \epsilon_2 \rho^2 I, \\
\Sigma_{23} &= W_2^T - N_3 + B^T W_3 + Q_5^T + h^2 B^T P_5, \\
\Sigma_{24} &= W_2^T - N_4 + B^T W_4 + Q_5^T + h^2 B^T P_5, \\
\Sigma_{25} &= -W_2^T - N_5 + B^T W_5 + B^T Q_6 - Q_3 - Q_5^T, \\
\Sigma_{26} &= -Q_2^T - N_2^T - N_6, \\
\Sigma_{33} &= W_3^T + W_3 + h^2 P_5 - \epsilon_1 I, \\
\Sigma_{34} &= W_3^T + W_4 + h^2 P_5, \\
\Sigma_{35} &= -W_3^T + W_5 + Q_6, \\
\Sigma_{44} &= W_4^T + W_4 + h^2 P_5 - \epsilon_2 I, \\
\Sigma_{45} &= -W_4^T + W_5 + Q_6, \\
\Sigma_{55} &= -W_5^T - W_5 - Q_6^T - Q_6 + h^2 P_3 + h^2 P_4, \\
\Sigma_{56} &= -Q_3^T - N_5^T, \\
\Sigma_{66} &= -N_6^T - N_6 - e^{-2\beta h} P_4.
\end{aligned}
$$

Moreover, the solution $x(t,\phi,\psi)$ satisfies the inequality

$$
\|x(t,\phi,\psi)\| \le \sqrt{\frac{N}{\lambda_{\min}(P_1)}} \max\left[\|\phi\|, \|\psi\|\right] e^{-\beta t}, \quad \forall t \in R^+,
$$

where $N = \lambda_{\max}(P_1) + h\lambda_{\max}(P_2) + h^3\lambda_{\max}(P_3) + h^3\lambda_{\max}(P_4) + h^3\lambda_{\max}(P_5) + h^3\lambda_{\max}\Big(\begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}\Big)$.

2.2 Stability Discrete Time Delay System

We introduce the following notation for later use,

$$\widehat{A}(\alpha) = A(\alpha) + \Delta A(k), \quad \widehat{B}(\alpha) = B(\alpha) + \Delta B(k), \quad \hat{h} = h_2 - h_1 + 1. \quad (2.14)$$

Lemma 2.2.1 *For any* $\widehat{A}(\alpha), \widehat{B}(\alpha), \hat{h}$ *in* (2.14)*,* $P(\alpha)$ *and* $Q(\alpha)$ *given by*

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \quad Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i, \quad \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, \quad i = 1, ..., N,$$

are parameter dependent positive definite Lyapunov matrices such that

$$\begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) - P(\alpha) + \hat{h}Q(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) - Q(\alpha) \end{bmatrix} < 0, \quad (2.15)$$

if and only if,

$$\begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 & A(\alpha)^T P(\alpha) & \epsilon^{-1} A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^T P(\alpha) & \epsilon^{-1} B_1(\alpha)^T & 0 \\ * & * & -P(\alpha) & 0 & \epsilon P(\alpha) K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < 0. \quad (2.16)$$

Lemma 2.2.2 *If there exist positive definite symmetric matrices* P_i*,* Q_i*,* $i = 1, 2, ..., N$*,* *and positive real numbers* ϵ, ζ *such that*

$$\begin{bmatrix} -P_i + \hat{h}Q_i & 0 & A_i^T P_i & \epsilon^{-1} A_i^{1T} & 0 \\ * & -Q_i & B_i^T P_i & \epsilon^{-1} B_i^{1T} & 0 \\ * & * & -P_i & 0 & \epsilon P_i K_i \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < -\zeta I,$$

$$i = 1, 2, ...N, \tag{2.17}$$

$$\begin{bmatrix} -P_i + \hat{h}Q_i & 0 & A_i^T P_j & \epsilon^{-1} A_i^{1T} & 0 \\ * & -Q_i & B_i^T P_j & \epsilon^{-1} B_i^{1T} & 0 \\ * & * & -P_i & 0 & \epsilon P_i K_j \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix}$$

$$+ \begin{bmatrix} -P_j + \hat{h}Q_j & 0 & A_j^T P_i & \epsilon^{-1} A_j^{1T} & 0 \\ * & -Q_j & B_j^T P_i & \epsilon^{-1} B_j^{1T} & 0 \\ * & * & -P_j & 0 & \epsilon P_j K_i \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < \frac{2\zeta I}{N-1},$$

$$i = 1, ..., N-1, j = i+1, ..., N. \tag{2.18}$$

Then, for any $A(\alpha), A_1(\alpha), B(\alpha), B_1(\alpha), K(\alpha), \hat{h}$ *in* (2.14), $P(\alpha)$ *and* $Q(\alpha)$ *are parameter dependent positive definite Lyapunov matrices in Lemma 2.2.1 such that* (2.16) *holds.*

Theorem 2.2.3 *The system* (1.2) *is robustly exponentially stable if the LMI conditions* (2.17)-(2.18) *are feasible.*

Output

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 Kanit Mukdasai, Akkharaphong Wongphat and Piyapong Niamsup, Robust Exponential Stability Criteria of LPD Systems with Mixed Time-Varying Delays and Nonlinear Perturbations, Abstract and Applied Analysis, Volume 2012, Article ID 348418, 20 pages.

1.2 Kanit Mukdasai, Robust Exponential Stability for LPD Discrete-Time System with Interval Time-Varying Delay, Journal of Applied Mathematics, Volume 2012, Article ID 237430, 13 pages.

2. การนำผลงานวิจัยไปใช้ประโยชน์

ผลงานวิจัยที่ได้มามีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการ และเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอนและมีเครือข่ายความร่วมมือสร้างกระแสความสนใจในวงกว้าง

3. อื่น ๆ: การเสนอผลงานในที่ประชุมวิชาการ

3.1 วันที่ 30 มิ.ย. - 4 ก.ค. 2556

หัวข้อ: Delay-dependent β-stability criteria for neutral system with mixed time-varying delays and nonlinear perturbations

ชื่อการประชุม : The Asian Mathematical Conference 2013, Busan, Korea on June 30, Sunday through July 4, Thursday, 2013.

Appendix

A1 **Kanit Mukdasai**, Akkharaphong Wongphat and Piyapong Niamsup, Robust Exponential Stability Criteria of LPD Systems with Mixed Time-Varying Delays and Nonlinear Perturbations, Abstract and Applied Analysis, Volume 2012, Article ID 348418, 20 pages.

A2 **Kanit Mukdasai**, Robust Exponential Stability for LPD Discrete-Time System with Interval Time-Varying Delay, Journal of Applied Mathematics, Volume 2012, Article ID 237430, 13 pages.

A1. **Kanit Mukdasai**, Akkharaphong Wongphat and Piyapong Niamsup, Robust Exponential Stability Criteria of LPD Systems with Mixed Time-Varying Delays and Nonlinear Perturbations, Abstract and Applied Analysis, Volume 2012, Article ID 348418, 20 pages.

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Research Article

Robust Exponential Stability Criteria of LPD Systems with Mixed Time-Varying Delays and Nonlinear Perturbations

Kanit Mukdasai,[1,2] Akkharaphong Wongphat,[1] and Piyapong Niamsup[2,3]

[1] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*
[3] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

Correspondence should be addressed to Piyapong Niamsup, piyapong.n@cmu.ac.th

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This paper investigates the problem of robust exponential stability for linear parameter-dependent (LPD) systems with discrete and distributed time-varying delays and nonlinear perturbations. Parameter dependent Lyapunov-Krasovskii functional, Leibniz-Newton formula, and linear matrix inequality are proposed to analyze the stability. On the basis of the estimation and by utilizing free-weighting matrices, new delay-dependent exponential stability criteria are established in terms of linear matrix inequalities (LMIs). Numerical examples are given to demonstrate the effectiveness and less conservativeness of the proposed methods.

1. Introduction

Over the past decades, dynamical systems with state delays have attracted much interest in the literature over the half century, especially in the last decade. Since time delay is frequently a source of instability or poor performances in various systems such as electric, chemical processes, and long transmission line in pneumatic systems [1]. The problems of stability and stabilization for dynamical systems with or without state delays have been intensively studied in the past years by many researchers in mathematics and control communities [2, 3]. Stability criteria for dynamical systems with time delay is generally divided into two classes: delay-independent one and delay-dependent one. Delay-independent stability criteria tends to be more conservative, especially for small size delay, such criteria do not give any information on the size of the delay. On the other hand, delay-dependent stability criteria concerned with the size of the delay and usually provide a maximal delay size. Various stability of linear continuous-time and discrete-time systems subject to time-invariant parametric uncertainty have received considerable attention. An important class of linear time-invariant parametric uncertain system is linear parameter-dependent (LPD) system in which the uncertain state matrices are in the polytope consisting of all convex combination of known matrices. Most of sufficient (or necessary and sufficient) conditions have been obtained via Lyapunov-Krasovskii theory approaches in which parameter-dependent Lyapunov-Krasovskii functional has been employed. These conditions are always expressed in terms of linear matrix inequalities (LMIs). The results have been obtained for robust stability for LPD systems in which time delay occurs in state variable such as [4–6] which present sufficient conditions for robust stability of LPD continuous-time system with delays.

Recently, many researchers have studied the problem of stability for time-delay systems with nonlinear perturbations such as [7] which considers the robust stability for a class of linear systems with interval time-varying delay and nonlinear perturbations. In [8], exponential stability of time-delay systems with nonlinear uncertainties is studied. Based on the Lyapunov theory approach and the approaches of decomposing the matrix, a new exponential stability criterion is derived in terms of LMI. In [9], they propose a new delay-dependent stability criterion in terms of linear matrix inequality for dynamic systems with time-varying delays and nonlinear perturbations by using Lyapunov theory. However, many researchers have studied the problem of stability for systems with discrete and distributed delays such as [10] which presented some stability conditions for uncertain neutral systems with discrete and distributed delays. The robust stability of uncertain linear neutral systems with discrete and distributed delays has been studied in [11]. In [12, 13], they studied the problem of stability for linear switching system with discrete and distributed delays. Moreover, a descriptor model transformation and a corresponding Lyapunov-Krasovskii functionals have been introduced for stability analysis of systems with delays in [14, 15].

In this paper, we will investigate the problems of robust exponential stability for LPD system with mixed time-varying delays and nonlinear perturbations. Based on the combination of Leibniz-Newton formula and linear matrix inequality, the use of suitable Lyapunov-Krasovskii functional, new delay-dependent exponential stability criteria will be obtained in terms of LMIs. Finally, numerical examples will be given to show the effectiveness of the obtained results.

2. Problem Formulation and Preliminaries

We introduce some notations, definition, and lemmas that will be used throughout the paper. R^+ denotes the set of all real nonnegative numbers; R^n denotes the n-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in R^n$; $R^{n\times r}$ denotes the set $n\times r$ real matrices; A^T denotes the transpose of the matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\text{Re } \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{\text{Re } \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\max}(A(\alpha)) = \max\{\lambda_{\max}(A_i) : i = 1,2,\ldots,N\}$; $\lambda_{\min}(A(\alpha)) = \min\{\lambda_{\min}(A_i) : i = 1,2,\ldots,N\}$; matrix A is called a semipositive definite ($A \geq 0$) if $x^T Ax \geq 0$, for all $x \in R^n$; A is a positive definite ($A > 0$) if $x^T Ax > 0$ for all $x \neq 0$; matrix B is called a seminegative definite ($B \leq 0$) if $x^T Bx \leq 0$, for all $x \in R^n$; B is a negative definite ($B < 0$) if $x^T Bx < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $C([-\overline{h}, 0], R^n)$ denotes the space of all continuous vector functions mapping $[-\overline{h}, 0]$ into R^n

where $\overline{h} = \max\{h, g\}$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the system described by the following state equation of the form

$$\begin{aligned}\dot{x}(t) &= A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) \\ &\quad + C(\alpha)\int_{t-g(t)}^{t} x(s)ds, \quad t > 0; \\ x(t) &= \phi(t), \quad \dot{x}(t) = \varphi(t), \quad t \in \left[-\overline{h}, 0\right],\end{aligned} \tag{2.1}$$

where $x(t) \in R^n$ is the state variable, $A(\alpha), B(\alpha), C(\alpha) \in R^{n\times n}$ are uncertain matrices belonging to the polytope

$$\begin{aligned}&A(\alpha) = \sum_{i=1}^{N}\alpha_i A_i, \quad B(\alpha) = \sum_{i=1}^{N}\alpha_i B_i, \quad C(\alpha) = \sum_{i=1}^{N}\alpha_i C_i, \\ &\sum_{i=1}^{N}\alpha_i = 1, \quad \alpha_i \geq 0, A_i,\ B_i, C_i \in R^{n\times n},\ i = 1,\dots,N.\end{aligned} \tag{2.2}$$

$h(t)$ and $g(t)$ are discrete and distributed time-varying delays, respectively, satisfying

$$0 \leq h(t) \leq h, \qquad \dot{h}(t) \leq h_d, \qquad 0 \leq g(t) \leq g, \tag{2.3}$$

where h, h_d, g are given positive real constants. Consider the initial functions $\phi(t)$, $\varphi(t) \in C([-\overline{h}, 0], R^n)$ with the norm $\|\phi\| = \sup_{t\in[-h,0]}\|\phi(t)\|$ and $\|\varphi\| = \sup_{t\in[-h,0]}\|\varphi(t)\|$. The uncertainties $f(\cdot), g(\cdot)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude of the form

$$\begin{aligned}f^T(t,x(t))f(t,x(t)) &\leq \eta^2 x^T(t)x(t), \\ g^T(t,x(t-h(t)))g(t,x(t-h(t))) &\leq \rho^2 x^T(t-h(t))x(t-h(t)),\end{aligned} \tag{2.4}$$

where η, ρ are given real constants.

Definition 2.1. The system (2.1) is robustly exponentially stable, if there exist positive real numbers β and M such that for each $\phi(t), \varphi(t) \in C([-\overline{h}, 0], R^n)$, the solution $x(t, \phi, \varphi)$ of the system (2.1) satisfies

$$\|x(t,\phi,\varphi)\| \leq M\max\{\|\phi\|, \|\varphi\|\}e^{-\beta t}, \quad \forall t \in R^+. \tag{2.5}$$

Lemma 2.2 (Schur complement lemma, see [9]). *Given constant symmetric matrices X, Y, Z where $Y > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0. \tag{2.6}$$

Lemma 2.3 (Jensen's inequality, see [1]). *For any constant matrix $Q \in R^{n\times n}$, $Q = Q^T > 0$, scalar $h > 0$, vector function $\dot{x} : [0, h] \to R^n$ such that the integrations concerned are well defined, then*

$$-h\int_{-h}^{0} \dot{x}^T(s+t)Q\dot{x}(s+t)ds \le -\left(\int_{-h}^{0} \dot{x}(s+t)ds\right)^T Q\left(\int_{-h}^{0} \dot{x}(s+t)ds\right). \tag{2.7}$$

Rearranging the term $\int_{-h}^{0} \dot{x}(s+t)ds$ with $x(t) - x(t-h)$, one can yield the following inequality:

$$-h\int_{-h}^{0} \dot{x}^T(s+t)Q\dot{x}(s+t)ds \le \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ Q & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}. \tag{2.8}$$

Lemma 2.4 (see [16]). *Let $x(t) \in R^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $X, M_i \in R^{n\times n}$, $i = 1, 2, \ldots, 5$ and a scalar function $h := h(t) \ge 0$:*

$$\begin{aligned} -\int_{t-h}^{t} \dot{x}^T(s)X\dot{x}(s)ds \le{}& \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ -M_1 + M_2^T & -M_2^T - M_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ &+ h\begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 \\ M_4^T & M_5 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}, \end{aligned} \tag{2.9}$$

where

$$\begin{bmatrix} X & M_1 & M_2 \\ M_1^T & M_3 & M_4 \\ M_2^T & M_4^T & M_5 \end{bmatrix} \ge 0. \tag{2.10}$$

3. Main Results

In this section, we first study the robust exponential stability criteria for the system (2.1) by using the combination of linear matrix inequality (LMI) technique and Lyapunov theory method. We introduce the following notations for later use:

$$P_j(\alpha) = \sum_{i=1}^{N}\alpha_i P_i^j, \quad W_j(\alpha) = \sum_{i=1}^{N}\alpha_i W_i^j, \quad N_j(\alpha) = \sum_{i=1}^{N}\alpha_i N_i^j, \quad Q_j(\alpha) = \sum_{i=1}^{N}\alpha_i Q_i^j,$$

$$M_l(\alpha) = \sum_{i=1}^{N}\alpha_i M_i^l, \quad R_s(\alpha) = \sum_{i=1}^{N}\alpha_i R_i^s, \quad \sum_{i=1}^{N}\alpha_i = 1, \quad \alpha_i \geq 0, \tag{3.1}$$

$$P_i^j, W_i^j, N_i^j, Q_i^j, M_i^l, R_i^s \in R^{n\times n}, \quad j = 1,2,\ldots,6, \ l = 1,2,\ldots,5, \ s = 1,2,3, \ i = 1,2,\ldots,N;$$

$$\prod_{i,j,k} = \begin{bmatrix} \Sigma_{i,j,k}^{11} & \Sigma_{i,j,k}^{12} & \Sigma_{i,j}^{13} & \Sigma_{i,j}^{14} & \Sigma_{i,j}^{15} & \Sigma_{i}^{16} & \Sigma_{i,j,k}^{17} \\ * & \Sigma_{i,j,k}^{22} & \Sigma_{i,j}^{23} & \Sigma_{i,j}^{24} & \Sigma_{i,j}^{25} & \Sigma_{i}^{26} & \Sigma_{i,j,k}^{27} \\ * & * & \Sigma_{i}^{33} & \Sigma_{i}^{34} & \Sigma_{i}^{35} & -N_i^{3^T} & \Sigma_{i,j}^{37} \\ * & * & * & \Sigma_{i}^{44} & \Sigma_{i}^{45} & -N_i^{4^T} & \Sigma_{i,j}^{47} \\ * & * & * & * & \Sigma_{i}^{55} & \Sigma_{i}^{56} & \Sigma_{i,j}^{57} \\ * & * & * & * & * & \Sigma_{i}^{66} & 0 \\ * & * & * & * & * & * & \Sigma_{i,j,k}^{77} \end{bmatrix}, \tag{3.2}$$

where

$$\begin{aligned}
\Sigma_{i,j,k}^{11} &= 2\beta P_i^1 + P_i^1 A_j + A_i^T P_j^1 + P_i^2 + h^2 A_i^T P_j^5 A_k - e^{-2\beta h} P_i^5 + Q_i^1 + Q_i^{1^T} + Q_i^{4^T} A_j + A_i^T Q_j^4 \\
&\quad + N_i^{1^T} + N_i^1 + W_i^{1^T} A_j + A_i^T W_j^1 + hM_i^{1^T} + hM_i^1 + h^2 M_i^3 + \epsilon_1 \eta^2 I + g^2 P_i^6, \\
\Sigma_{i,j,k}^{12} &= P_i^1 B_j + Q_i^2 - Q_i^{1^T} + A_i^T Q_j^5 + Q_i^{4^T} B_j + h^2 A_i^T P_j^5 B_k + e^{-2\beta h} P_i^5 + hR_i^{2^T} - hM_i^{1^T} \\
&\quad - N_i^{1^T} + N_i^2 + W_i^{1^T} B_j + W_i^{2^T} A_j + hM_i^2 + h^2 M_i^4, \\
\Sigma_{i,j}^{13} &= N_i^3 + W_i^{1^T} + A_i^T W_j^3 + P_i^1 + Q_i^{4^T} + h^2 A_i^T P_j^5, \\
\Sigma_{i,j}^{14} &= N_i^4 + W_i^{1^T} + A_i^T W_j^4 + P_i^1 + Q_i^{4^T} + h^2 A_i^T P_j^5, \\
\Sigma_{i,j}^{15} &= N_i^5 - W_i^{1^T} + A_i^T W_j^5 + Q_i^3 - Q_i^{4^T} + A_i^T Q_j^6,
\end{aligned}$$

$$\Sigma_i^{16} = -Q_i^{1^T} - N_i^{1^T} + N_i^6,$$

$$\Sigma_{i,j,k}^{17} = h^2 A_i^T P_j^5 C_k + W_i^{1^T} C_j + A_i^T W_j^6 + P_i^1 C_j + Q_i^{4^T} C_j,$$

$$\Sigma_{i,j,k}^{22} = -Q_i^{2^T} - Q_i^2 + Q_i^{5^T} B_j + B_i^T Q_j^5 - e^{-2\beta h} P_i^2 + h_d P_i^2 + h^2 B_i^T P_j^5 B_k - e^{-2\beta h} P_i^5 + h^2 R_i^1$$
$$- N_i^{2^T} - N_i^2 + W_i^{2^T} B_j + B_i^T W_j^2 - hR_i^{2^T} - hR_i^2 - hM_i^{2^T} - hM_i^2 + h^2 M_i^5 + \epsilon_2 \rho^2 I,$$

$$\Sigma_{i,j}^{23} = W_i^{2^T} - N_i^3 + B_i^T W_j^3 + Q_i^{5^T} + h^2 B_i^T P_j^5,$$

$$\Sigma_{i,j}^{24} = W_i^{2^T} - N_i^4 + B_i^T W_j^4 + Q_i^{5^T} + h^2 B_i^T P_j^5,$$

$$\Sigma_{i,j}^{25} = -W_i^{2^T} - N_i^5 + B_i^T W_j^5 + B_i^T Q_j^6 - Q_i^3 - Q_i^{5^T},$$

$$\Sigma_i^{26} = -Q_i^{2^T} - N_i^{2^T} - N_i^6,$$

$$\Sigma_{i,j,k}^{27} = h^2 B_i^T P_j^5 C_k + W_i^{2^T} C_j + B_i^T W_j^6 + Q_i^{5^T} C_j,$$

$$\Sigma_i^{33} = W_i^{3^T} + W_i^3 + h^2 P_i^5 - \epsilon_1 I,$$

$$\Sigma_i^{34} = W_i^{3^T} + W_i^4 + h^2 P_i^5,$$

$$\Sigma_i^{35} = -W_i^{3^T} + W_i^5 + Q_i^6,$$

$$\Sigma_{i,j}^{37} = h^2 P_i^5 C_j + W_i^{3^T} C_j + W_i^6,$$

$$\Sigma_i^{44} = W_i^{4^T} + W_i^4 + h^2 P_i^5 - \epsilon_2 I,$$

$$\Sigma_i^{45} = -W_i^{4^T} + W_i^5 + Q_i^6,$$

$$\Sigma_{i,j}^{47} = h^2 P_i^5 C_j + W_i^{4^T} C_j + W_i^6,$$

$$\Sigma_i^{55} = -W_i^{5^T} - W_i^5 - Q_i^{6^T} - Q_i^6 + h^2 P_i^3 + h^2 P_i^4,$$

$$\Sigma_i^{56} = -Q_i^{3^T} - N_i^{5^T},$$

$$\Sigma_{i,j}^{57} = W_i^{5^T} C_j - W_i^6 + Q_i^{6^T} C_j,$$

$$\Sigma_i^{66} = -N_i^{6^T} - N_i^6 - e^{-2\beta h} P_i^4,$$

$$\Sigma_{i,j,k}^{77} = -e^{2\beta g} P_i^6 + h^2 C_i^T P_j^5 C_k + C_i^T W_j^6 + W_i^{6^T} C_j. \tag{3.3}$$

Theorem 3.1. *For given positive real constants h, h_d, g, η and ρ, system* (2.1) *is robustly exponentially stable with a decay rate β, if there exist positive definite symmetric matrices P_i^s, any* *appropriate dimensional matrices* W_i^s, Q_i^s, N_i^s, M_i^r, R_i^t, $s = 1,2,\ldots,6$, $r = 1,2,\ldots,5$, $t = 1, 2, 3$, $i = 1, 2, \ldots, N$ *and positive real constants* ϵ_1 *and* ϵ_2 *satisfying the following LMIs:*

$$\begin{bmatrix} R_i^1 & R_i^2 \\ * & R_i^3 \end{bmatrix} > 0, \quad i = 1,2,\ldots,N, \tag{3.4}$$

$$\begin{bmatrix} e^{-2\beta h}P_i^3 - R_i^3 & M_i^1 & M_i^2 \\ * & M_i^3 & M_i^4 \\ * & * & M_i^5 \end{bmatrix} \geq 0, \quad i = 1,2,\ldots,N, \tag{3.5}$$

$$\prod_{i,i,i} < -I, \quad i = 1,2,\ldots,N, \tag{3.6}$$

$$\prod_{i,i,j} + \prod_{i,j,i} + \prod_{j,i,i} < \frac{1}{(N-1)^2}I, \quad i = 1,2,\ldots,N,\ i \neq j,\ j = 1,2,\ldots,N, \tag{3.7}$$

$$\begin{gathered} \prod_{i,j,k} + \prod_{i,k,j} + \prod_{j,i,k} + \prod_{j,k,i} + \prod_{k,i,j} + \prod_{k,j,i} < \frac{6}{(N-1)^2}I, \\ i = 1,2,\ldots,\ N-2, \quad j = i+1,\ldots,\ N-1, k = j+1,\ldots,N. \end{gathered} \tag{3.8}$$

Moreover, the solution $x(t,\phi,\psi)$ *satisfies the inequality*

$$\|x(t,\phi,\psi)\| \leq \sqrt{\frac{N}{\lambda_{\min}(P_1(\alpha))}} \max[\|\phi\|, \|\psi\|] e^{-\beta t}, \quad \forall t \in R^+, \tag{3.9}$$

where $N = \lambda_{\max}(P_1(\alpha)) + h\lambda_{\max}(P_2(\alpha)) + h^3\lambda_{\max}(P_3(\alpha)) + h^3\lambda_{\max}(P_4(\alpha)) + h^3\lambda_{\max}(P_5(\alpha)) + h^3\lambda_{\max}(\left[\begin{smallmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha) \end{smallmatrix}\right])$.

Proof. Choose a parameter-dependent Lyapunov-Krasovskii functional candidate for the system (2.1) of the form

$$V(t) = \sum_{i=1}^{7} V_i(t), \tag{3.10}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)P_1(\alpha)x(t) \\ &= \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1(\alpha) & 0 & 0 \\ Q_1(\alpha) & Q_2(\alpha) & Q_3(\alpha) \\ Q_4(\alpha) & Q_5(\alpha) & Q_6(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
V_2(t) &= \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) P_2(\alpha) x(s) ds, \\
V_3(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s) P_3(\alpha) \dot{x}(s) ds\, d\theta, \\
V_4(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s) P_4(\alpha) \dot{x}(s) ds\, d\theta, \\
V_5(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s) P_5(\alpha) \dot{x}(s) ds\, d\theta, \\
V_6(t) &= h \int_{-h}^{t} \int_{\theta-h(\theta)}^{\theta} e^{2\beta(\theta-t)} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha) \end{bmatrix} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix} ds\, d\theta, \\
V_7(t) &= g \int_{-g}^{0} \int_{t+\theta}^{t} e^{2\beta(s-t)} x^T(s) P_6(\alpha) x(s) ds d\theta.
\end{aligned}
\tag{3.11}
$$

Calculating the time derivatives of $V_i(t)$, $i = 1, 2, 3, \ldots, 6$, along the trajectory of (2.1) yields

$$
\begin{aligned}
\dot{V}_1(t) &= 2 \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} P_1(\alpha) & Q_1^T(\alpha) & Q_4^T(\alpha) \\ 0 & Q_2^T(\alpha) & Q_5^T(\alpha) \\ 0 & Q_3^T(\alpha) & Q_6^T(\alpha) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ 0 \\ 0 \end{bmatrix} \\
&= 2 \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} P_1(\alpha) & Q_1^T(\alpha) & Q_4^T(\alpha) \\ 0 & Q_2^T(\alpha) & Q_5^T(\alpha) \\ 0 & Q_3^T(\alpha) & Q_6^T(\alpha) \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \omega_{31} \end{bmatrix},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\omega_{11} &= A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t, x(t)) + g(t, x(t-h(t))) + C(\alpha) \int_{t-g(t)}^{t} x(s) ds, \\
\omega_{21} &= x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \dot{x}(s) ds, \\
\omega_{31} &= A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t, x(t)) + g(t, x(t-h(t))) + C(\alpha) \int_{t-g(t)}^{t} x(s) ds - \dot{x}(t).
\end{aligned}
\tag{3.13}
$$

Taking the time-derivative of $V_2(t)$ leads to

$$\begin{aligned}\dot{V}_2(t) &= x^T(t)P_2(\alpha)x(t) - \left(1-\dot{h}(t)\right)e^{-2\beta h(t)}x^T(t-h(t))P_2(\alpha)x(t-h(t)) - 2\beta V_2(t)\\ &\leq x^T(t)P_2(\alpha)x(t) - e^{-2\beta h}x^T(t-h(t))P_2(\alpha)x(t-h(t)) + h_d x^T(t-h(t))P_2(\alpha)x(t-h(t))\\ &\quad - 2\beta V_2(t).\end{aligned} \tag{3.14}$$

Obviously, for any scalar $s \in [t-h, t]$, we get $e^{-2\beta h} \leq e^{-2\beta(s-t)} \leq 1$. Together with Lemma 2.3 (Jensen's inequality), we obtain

$$\begin{aligned}\dot{V}_3(t) &= h^2\dot{x}^T(t)P_3(\alpha)\dot{x}(t) - h\int_{-h}^{0} e^{2\beta s}\dot{x}^T(t+s)P_3(\alpha)\dot{x}(t+s)ds - 2\beta V_3(t)\\ &\leq h^2\dot{x}^T(t)P_3(\alpha)\dot{x}(t) - h\int_{t-h}^{t} e^{2\beta(s-t)}\dot{x}^T(s)P_3(\alpha)\dot{x}(s)ds - 2\beta V_3(t)\\ &\leq h^2\dot{x}^T(t)P_3(\alpha)\dot{x}(t) - he^{-2\beta h}\int_{t-h}^{t}\dot{x}^T(s)P_3(\alpha)\dot{x}(s)ds - 2\beta V_3(t).\end{aligned} \tag{3.15}$$

Following the estimation of $\dot{V}_3(t)$, we have

$$\begin{aligned}\dot{V}_4(t) &\leq h^2\dot{x}^T(t)P_4(\alpha)\dot{x}(t) - he^{-2\beta h}\int_{t-h}^{t}\dot{x}^T(s)P_4(\alpha)\dot{x}(s)ds - 2\beta V_4(t)\\ &\leq h^2\dot{x}^T(t)P_4(\alpha)\dot{x}(t) - e^{-2\beta h}\int_{t-h}^{t}\dot{x}^T(s)ds P_4(\alpha)\int_{t-h}^{t}\dot{x}(s)ds - 2\beta V_4(t)\\ &\leq h^2\dot{x}^T(t)P_4(\alpha)\dot{x}(t) - e^{-2\beta h}\int_{t-h(t)}^{t}\dot{x}^T(s)ds P_4(\alpha)\int_{t-h(t)}^{t}\dot{x}(s)ds - 2\beta V_4(t).\end{aligned} \tag{3.16}$$

From (3.16), it follows that

$$\begin{aligned}\dot{V}_5(t) &\leq h^2\dot{x}^T(t)P_5(\alpha)\dot{x}(t) - e^{-2\beta h}\int_{t-h(t)}^{t}\dot{x}^T(s)ds P_5(\alpha)\int_{t-h(t)}^{t}\dot{x}(s)ds - 2\beta V_5(t)\\ &= h^2\dot{x}^T(t)P_5(\alpha)\dot{x}(t) - e^{-2\beta h}\left[x^T(t) - x^T(t-h(t))\right]P_5(\alpha)[x(t) - x(t-h(t))] - 2\beta V_5(t)\\ &= h^2\left[A(\alpha)x(t) + B(\alpha)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))) + C(\alpha)\int_{t-g(t)}^{t}x(s)ds\right]^T\end{aligned}$$

$$\times P_5(\alpha)\left[A(\alpha)x(t)+B(\alpha)x(t-h(t))+f(t,x(t))+g(t,x(t-h(t)))\right.$$
$$\left.+C(\alpha)\int_{t-g(t)}^{t}x(s)ds\right]-e^{-2\beta h}\left[x^T(t)-x^T(t-h(t))\right]P_5(\alpha)[x(t)-x(t-h(t))]$$
$$-2\beta V_5(t). \tag{3.17}$$

Taking the time derivative of $V_6(t)$ and $V_7(t)$, we obtain

$$\begin{aligned}\dot{V}_6(t) &= hh(t)x^T(t-h(t))R_1(\alpha)x(t-h(t))+2hx^T(t-h(t))R_2(\alpha)x(t)\\&\quad -2hx^T(t-h(t))R_2(\alpha)x(t-h(t))+h\int_{t-h}^{t}\dot{x}^T(s)R_3(\alpha)\dot{x}(s)ds-2\beta V_6(t)\\&\le h^2x^T(t-h(t))R_1(\alpha)x(t-h(t))+2hx^T(t-h(t))R_2(\alpha)x(t)\\&\quad -2hx^T(t-h(t))R_2(\alpha)x(t-h(t))+h\int_{t-h}^{t}\dot{x}^T(s)R_3(\alpha)\dot{x}(s)ds-2\beta V_6(t);\\\dot{V}_7(t) &\le g^2x^T(t)P_6(\alpha)x(t)-e^{-2\beta g}\int_{t-g(t)}^{t}x^T(s)dsP_6(\alpha)\int_{t-g(t)}^{t}x(s)ds-2\beta V_7(t).\end{aligned} \tag{3.18}$$

From the Leibinz-Newton formula, the following equation is true for any real matrices $N_i(\alpha)$, $i=1,2,\ldots,6$ with appropriate dimensions

$$\begin{aligned}2&\left[x^T(t)N_1^T(\alpha)+x^T(t-h(t))N_2^T(\alpha)+f^T(t,x(t))N_3^T(\alpha)+g^T(t,x(t-h(t)))N_4^T(\alpha)\right.\\&\left.+\dot{x}^T(t)N_5^T(\alpha)+\int_{t-h(t)}^{t}\dot{x}^T(s)dsN_6^T(\alpha)\right]\times\left[x(t)-x(t-h(t))-\int_{t-h(t)}^{t}\dot{x}(s)ds\right]=0.\end{aligned} \tag{3.19}$$

From the utilization of zero equation, the following equation is true for any real matrices W_i, $i=1,2,\ldots,5$ with appropriate dimensions

$$\begin{aligned}2&\left[x^T(t)W_1^T(\alpha)+x^T(t-h(t))W_2^T(\alpha)+f^T(t,x(t))W_3^T(\alpha)+g^T(t,x(t-h(t)))W_4^T(\alpha)\right.\\&\left.+\dot{x}^T(t)W_5^T(\alpha)+\int_{t-g(t)}^{t}x^T(s)dsW_6^T(\alpha)\right]\\&\times\left[A(\alpha)x(t)+B(\alpha)x(t-h(t))+f(t,x(t))+g(t,x(t-h(t)))+C(\alpha)\int_{t-g(t)}^{t}x(s)ds-\dot{x}(t)\right]=0.\end{aligned} \tag{3.20}$$

From (2.4), we obtain for any positive real constants ϵ_1 and ϵ_2,

$$\begin{aligned} 0 &\le \epsilon_1 \eta^2 x^T(t)x(t) - \epsilon_1 f^T(t,x(t)) f(t,x(t)), \\ 0 &\le \epsilon_2 \rho^2 x^T(t-h(t))x(t-h(t)) - \epsilon_2 g^T(t,x(t-h(t)))g(t,x(t-h(t))). \end{aligned} \tag{3.21}$$

By (3.5), Lemma 2.4 and the integral term of the right-hand side of $\dot{V}_3(t)$ and $\dot{V}_6(t)$, we obtain

$$\begin{aligned} &-h\int_{t-h}^{t} \dot{x}^T(s)\left[e^{-2\beta h}P_3(\alpha) - R_3(\alpha)\right]\dot{x}(s)ds \\ &\quad \le h\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} M_1^T(\alpha)+M_1(\alpha) & -M_1^T(\alpha)+M_2(\alpha) \\ -M_1(\alpha)+M_2^T(\alpha) & -M_2^T(\alpha)-M_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ &\qquad + h^2 \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} M_3(\alpha) & M_4(\alpha) \\ M_4^T(\alpha) & M_5(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}. \end{aligned} \tag{3.22}$$

According to (3.12)–(3.22), it is straightforward to see that

$$\dot{V}(t) \le \zeta^T(t)\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_i\alpha_j\alpha_k\prod_{i,j,k}\zeta(t) - 2\beta V(t), \tag{3.23}$$

where $\zeta^T(t) = [x^T(t),\ x^T(t-h(t)),\ f^T(t,x(t)),\ g^T(t,\ x(t-h(t))),\ \dot{x}^T(t),\ \int_{t-h(t)}^{t}\dot{x}^T(s)ds,$ $\int_{t-g(t)}^{t} x^T(s)ds]$ and $\prod_{i,j,k}$ is defined in (3.2). The facts that $\sum_{i=1}^{N}\alpha_i = 1$, we obtain the following identities:

$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_i\alpha_j\alpha_k\prod_{i,j,k} &= \sum_{i=1}^{N}\alpha_i^3\prod_{i,i,i} + \sum_{i=1}^{N}\sum_{i\ne j,j=1}^{N}\alpha_i^2\alpha_j\left[\prod_{i,i,j}+\prod_{i,j,i}+\prod_{j,i,i}\right] \\ &\quad + \sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\alpha_i\alpha_j\alpha_l\left[\prod_{i,j,k}+\prod_{i,k,j}+\prod_{j,i,k}+\prod_{j,k,i}+\prod_{k,i,j}+\prod_{k,j,i}\right]. \end{aligned} \tag{3.24}$$

We define Φ and Λ as

$$\begin{aligned} \Phi &\equiv \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i(\alpha_i-\alpha_j)^2 = (N-1)\sum_{i=1}^{N}\alpha_i^3 - \sum_{i=1}^{N}\sum_{j\ne i;j=1}^{N}\alpha_i^2\alpha_j \ge 0, \\ \Lambda &\equiv \sum_{i=1}^{N}\sum_{j\ne i;j=1}^{N-1}\sum_{k\ne i;k=2}^{N}\alpha_i\left[\alpha_j-\alpha_k\right]^2 = (N-2)\sum_{i=1}^{N}\sum_{j\ne i;j=1}^{N-1}\alpha_i^2\alpha_j - 6\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\alpha_i\alpha_j\alpha_l \ge 0. \end{aligned} \tag{3.25}$$

From $(N-1)\Phi + \Lambda \geq 0$, we obtain

$$\sum_{i=1}^{N}\alpha_i^3 - \frac{1}{(N-1)^2}\sum_{i=1}^{N}\sum_{i\neq j,j=1}^{N}\alpha_i^2\alpha_j - \frac{6}{(N-1)^2}\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N}\alpha_i\alpha_j\alpha_l \geq 0. \tag{3.26}$$

By (3.23)–(3.26), if the conditions (3.6)–(3.8) are true, then

$$\dot{V}(t) + 2\beta V(t) \leq 0, \quad \forall t \in R^+, \tag{3.27}$$

which gives

$$V(t) \leq V(0)e^{-2\beta t}, \quad \forall t \in R^+. \tag{3.28}$$

From (3.28), it is easy to see that

$$\begin{gathered}\lambda_{\min}(P_1(\alpha))\|x(t)\|^2 \leq V(t) \leq V(0)e^{-2\beta t},\\ V(0) = \sum_{i=1}^{6}V_i(0),\end{gathered} \tag{3.29}$$

where

$$\begin{gathered}V_1(0) = x^T(0)P_1(\alpha)x(0),\\ V_2(0) = \int_{-h(0)}^{t} e^{2\beta s}x^T(s)P_2(\alpha)x(s)ds,\\ V_3(0) = h\int_{-h}^{0}\int_{\theta}^{0} e^{2\beta s}\dot{x}^T(s)P_3(\alpha)\dot{x}(s)ds\,d\theta,\\ V_4(0) = h\int_{-h}^{0}\int_{\theta}^{0} e^{2\beta s}\dot{x}^T(s)P_4(\alpha)\dot{x}(s)ds\,d\theta,\\ V_5(0) = h\int_{-h}^{0}\int_{\theta}^{0} e^{2\beta s}\dot{x}^T(s)P_5(\alpha)\dot{x}(s)ds\,d\theta,\\ V_6(0) = h\int_{-h}^{0}\int_{\theta-h(\theta)}^{0} e^{2\beta\theta}\begin{bmatrix}x(\theta-h(\theta))\\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix}R_1(\alpha) & R_2(\alpha)\\ R_2^T(\alpha) & R_3(\alpha)\end{bmatrix}\begin{bmatrix}x(\theta-h(\theta))\\ \dot{x}(s)\end{bmatrix}ds\,d\theta,\\ V_7(0) = g\int_{-g}^{0}\int_{\theta}^{0} e^{2\beta s}x^T(s)P_6(\alpha)x(s)ds\,d\theta.\end{gathered} \tag{3.30}$$

Therefore, we get

$$\lambda_{\min}(P_1(\alpha))\|x(t)\|^2 \leq V(0)e^{-2\beta t} \leq N\max\left[\|\phi\|, \|\varphi\|\right]^2 e^{-2\beta t}, \tag{3.31}$$

where $N = \lambda_{\max}(P_1(\alpha)) + h\lambda_{\max}(P_2(\alpha)) + h^3\lambda_{\max}(P_3(\alpha)) + h^3\lambda_{\max}(P_4(\alpha)) + h^3\lambda_{\max}(P_5(\alpha)) + h^3\lambda_{\max}(\left[\begin{smallmatrix} R_1(\alpha) & R_2(\alpha) \\ R_2^T(\alpha) & R_3(\alpha)\end{smallmatrix}\right])$. From (3.31), we get

$$\|x(t,\phi,\varphi)\| \le \sqrt{\frac{N}{\lambda_{\min}(P_1(\alpha))}} \max[\|\phi\|, \|\varphi\|] e^{-\beta t}, \quad \forall t \in R^+. \tag{3.32}$$

This means that system (2.1) is robustly exponentially stable. The proof of the theorem is complete. □

If $A(\alpha) = A$, $B(\alpha) = B$ and $C(\alpha) = 0$ when A and B are appropriate dimensional constant matrices, then system (2.1) reduces to the following system:

$$\begin{gathered} \dot{x}(t) = Ax(t) + Bx(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))), \quad t > 0; \\ x(t) = \phi(t), \quad \dot{x}(t) = \varphi(t), \quad t \in [-h, 0]. \end{gathered} \tag{3.33}$$

Take the Lyapunov-Krasovskii functional as

$$V(t) = \sum_{i=1}^{6} V_i(t), \tag{3.34}$$

where

$$\begin{gathered} V_1(t) = x^T(t)P_1x(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 & 0 \\ Q_1 & Q_2 & Q_3 \\ Q_4 & Q_5 & Q_6 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}, \\ V_2(t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s)P_2x(s)ds, \\ V_3(t) = h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s)P_3\dot{x}(s)ds\,d\theta, \\ V_4(t) = h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s)P_4\dot{x}(s)ds\,d\theta, \\ V_5(t) = h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\beta(s-t)} \dot{x}^T(s)P_5\dot{x}(s)ds\,d\theta, \\ V_6(t) = h\int_{-h}^{t}\int_{\theta-h(\theta)}^{\theta} e^{2\beta(\theta-t)} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} \begin{bmatrix} x(\theta-h(\theta)) \\ \dot{x}(s) \end{bmatrix} ds\,d\theta. \end{gathered} \tag{3.35}$$

According to Theorem 3.1, we have the following Corollary 3.2 for the delay-dependent exponential stability criteria of system (3.33).

Corollary 3.2. *For given positive real constants h, h_d, η and ρ, system* (3.33) *is exponentially stable with a decay rate β, if there exist positive definite symmetric matrices P_i, $i = 1, 2, \ldots, 5$, any appropriate* *dimensional matrices $Q_i, N_i, i = 1, 2, \ldots, 6$, $W_i, M_i, i = 1, 2, \ldots, 5$, $R_i, i = 1, 2, 3$ and positive real constants ϵ_1 and ϵ_2 satisfying the following LMIs:*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0,$$
$$\begin{bmatrix} e^{-2\beta h}P_3 - R_3 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0,$$
$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} \\ * & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & -N_3^T \\ * & * & * & \Sigma_{44} & \Sigma_{45} & -N_4^T \\ * & * & * & * & \Sigma_{55} & \Sigma_{56} \\ * & * & * & * & * & \Sigma_{66} \end{bmatrix} < 0, \tag{3.36}$$

where

$$\begin{aligned}
\Sigma_{11} &= 2\beta P_1 + P_1 A + A^T P_1 + P_2 + h^2 A^T P_5 A - e^{-2\beta h} P_5 + Q_1 + Q_1^T + Q_4^T A + A^T Q_4 \\
&\quad + N_1^T + N_1 + W_1^T A + A^T W_1 + hM_1^T + hM_1 + h^2 M_3 + \epsilon_1 \eta^2 I, \\
\Sigma_{12} &= P_1 B + Q_2 - Q_1^T + A^T Q_5 + Q_4^T B + h^2 A^T P_5 B + e^{-2\beta h} P_5 + hR_2^T - hM_1^T \\
&\quad - N_1^T + N_2 + W_1^T B + W_2^T A + hM_2 + h^2 M_4, \\
\Sigma_{13} &= N_3 + W_1^T + A^T W_3 + P_1 + Q_4^T + h^2 A^T P_5, \\
\Sigma_{14} &= N_4 + W_1^T + A^T W_4 + P_1 + Q_4^T + h^2 A^T P_5, \\
\Sigma_{15} &= N_5 - W_1^T + A^T W_5 + Q_3 - Q_4^T + A^T Q_6, \\
\Sigma_{16} &= -Q_1^T - N_1^T + N_6, \\
\Sigma_{22} &= Q_2^T - Q_2 + Q_5^T B + B^T Q_5 - e^{-2\beta h} P_2 + h_d P_2 + h^2 B^T P_5 B - e^{-2\beta h} P_5 + h^2 R_1 \\
&\quad - N_2^T - N_2 + W_2^T B + B^T W_2 - hR_2^T - hR_2 - hM_2^T - hM_2 + h^2 M_5 + \epsilon_2 \rho^2 I, \\
\Sigma_{23} &= W_2^T - N_3 + B^T W_3 + Q_5^T + h^2 B^T P_5, \\
\Sigma_{24} &= W_2^T - N_4 + B^T W_4 + Q_5^T + h^2 B^T P_5, \\
\Sigma_{25} &= -W_2^T - N_5 + B^T W_5 + B^T Q_6 - Q_3 - Q_5^T, \\
\Sigma_{26} &= -Q_2^T - N_2^T - N_6, \\
\Sigma_{33} &= W_3^T + W_3 + h^2 P_5 - \epsilon_1 I,
\end{aligned}$$

$$\Sigma_{34} = W_3^T + W_4 + h^2 P_5,$$
$$\Sigma_{35} = -W_3^T + W_5 + Q_6,$$
$$\Sigma_{44} = W_4^T + W_4 + h^2 P_5 - \epsilon_2 I,$$
$$\Sigma_{45} = -W_4^T + W_5 + Q_6,$$
$$\Sigma_{55} = -W_5^T - W_5 - Q_6^T - Q_6 + h^2 P_3 + h^2 P_4,$$
$$\Sigma_{56} = -Q_3^T - N_5^T,$$
$$\Sigma_{66} = -N_6^T - N_6 - e^{-2\beta h} P_4. \tag{3.37}$$

Moreover, the solution $x(t, \phi, \psi)$ *satisfies the inequality*

$$\|x(t, \phi, \psi)\| \leq \sqrt{\frac{N}{\lambda_{\min}(P_1)}} \max[\|\phi\|, \|\psi\|] e^{-\beta t}, \quad \forall t \in R^+, \tag{3.38}$$

where $N = \lambda_{\max}(P_1) + h\lambda_{\max}(P_2) + h^3\lambda_{\max}(P_3) + h^3\lambda_{\max}(P_4) + h^3\lambda_{\max}(P_5) + h^3\lambda_{\max}\left(\begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}\right)$.

4. Numerical Examples

In order to show the effectiveness of the approaches presented in Section 3, four numerical examples are provided.

Example 4.1. Consider the LPD time-delay system (2.1) with the following parameters ($N = 3$):

$$A_1 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & 0.5 \end{bmatrix},$$
$$f(t, x(t)) = \begin{bmatrix} 0.2 \sin t x_1(t) \\ 0.2 \cos t x_2(t) \end{bmatrix}, \quad g(t, x(t - h(t))) = \begin{bmatrix} 0.3 \sin t x_1(t - h(t)) \\ 0.3 \cos t x_2(t - h(t)) \end{bmatrix},$$
$$h(t) = 0.2134 \sin^2\left(\frac{0.3t}{0.4268}\right), \quad g(t) = 0.4 \cos^2(t), \quad \phi(t) = \begin{bmatrix} -7 \\ 5 \end{bmatrix}, \quad t \in [-0.4, 0]. \tag{4.1}$$

It is easy to see that $h_d = 0.3$, $\eta = 0.2$, $\rho = 0.3$, and $g = 0.4$. Find the discrete delay time h to guarantee system (2.1) with the above parameters to be robustly exponentially stable with a decay rate $\beta = 0.15$.

Solution 1. By using the LMI Toolbox in Matlab (with accuracy 0.01) and conditions (3.4)–(3.8) of Theorem 3.1, this system is robustly exponentially stable for discrete delay satisfying $h = 0.2134$ and

$$P_1^1 = \begin{bmatrix} 225.4987 & -143.6565 \\ -143.6565 & 300.6876 \end{bmatrix}, \quad P_2^1 = \begin{bmatrix} 316.6633 & 0.3122 \\ 0.3122 & 426.8816 \end{bmatrix}, \quad P_3^1 = \begin{bmatrix} 361.5534 & -0.4974 \\ -0.4974 & 306.5360 \end{bmatrix},$$

$$P_1^2 = \begin{bmatrix} 153.3987 & -105.3621 \\ -105.3621 & 242.2669 \end{bmatrix}, \quad P_2^2 = \begin{bmatrix} 284.5598 & -17.5849 \\ -17.5849 & 306.8985 \end{bmatrix}, \quad P_3^2 = \begin{bmatrix} 283.8792 & -0.3896 \\ -0.3896 & 254.6003 \end{bmatrix},$$

$$P_1^3 = \begin{bmatrix} 484.7945 & 31.1598 \\ 31.1598 & 398.0539 \end{bmatrix}, \quad P_2^3 = \begin{bmatrix} 380.0951 & 9.9856 \\ 9.9856 & 387.3373 \end{bmatrix}, \quad P_3^3 = \begin{bmatrix} 391.5353 & -0.0741 \\ -0.0741 & 392.5712 \end{bmatrix},$$

$$P_1^4 = \begin{bmatrix} 285.1892 & -5.6549 \\ -5.6549 & 230.3520 \end{bmatrix}, \quad P_2^4 = \begin{bmatrix} 229.7296 & -0.3010 \\ -0.3010 & 239.7532 \end{bmatrix}, \quad P_3^4 = \begin{bmatrix} 239.9236 & 0.0430 \\ 0.0430 & 251.9889 \end{bmatrix},$$

$$P_1^5 = \begin{bmatrix} 285.3879 & -8.7124 \\ -8.7124 & 227.7653 \end{bmatrix}, \quad P_2^5 = \begin{bmatrix} 231.9963 & -0.7154 \\ -0.7154 & 239.6643 \end{bmatrix}, \quad P_3^5 = \begin{bmatrix} 239.9146 & -0.0007 \\ -0.0007 & 251.9987 \end{bmatrix},$$

$$P_1^6 = \begin{bmatrix} 195.2662 & -48.4444 \\ -48.4444 & 261.3741 \end{bmatrix}, \quad P_2^6 = \begin{bmatrix} 285.1103 & -2.8941 \\ -2.8941 & 286.4945 \end{bmatrix}, \quad P_3^6 = \begin{bmatrix} 291.3897 & 2.4418 \\ 2.4418 & 303.1851 \end{bmatrix},$$

$$Q_1^1 = \begin{bmatrix} 153.8322 & -6.2058 \\ -6.2058 & 511.1849 \end{bmatrix}, \quad Q_2^1 = \begin{bmatrix} 1,057.7 & 119.5 \\ 119.5 & 1,113.8 \end{bmatrix}, \quad Q_3^1 = \begin{bmatrix} -442.4 & 2.6 \\ 2.6 & -1,942.6 \end{bmatrix},$$

$$Q_1^2 = \begin{bmatrix} -1,438.7 & 472.6 \\ 472.6 & 503.4 \end{bmatrix}, \quad Q_2^2 = \begin{bmatrix} -3,162.4 & -435.4 \\ -435.4 & -107.8 \end{bmatrix}, \quad Q_3^2 = \begin{bmatrix} 1,383.2 & -0.1 \\ -0.1 & 1,874.5 \end{bmatrix},$$

$$Q_1^3 = \begin{bmatrix} -1,217.0 & -623.5 \\ -623.5 & -135.3 \end{bmatrix}, \quad Q_2^3 = \begin{bmatrix} -9.6 & -1,232.0 \\ -1,232.0 & 91.0 \end{bmatrix}, \quad Q_3^3 = \begin{bmatrix} 564.8910 & -1.6044 \\ -1.6044 & -90.0137 \end{bmatrix},$$

$$Q_1^4 = \begin{bmatrix} -61.6 & -1,470.3 \\ -1,470.3 & -4,166.8 \end{bmatrix}, \quad Q_2^4 = \begin{bmatrix} -4,191.2 & 13.6 \\ 13.6 & -4,402.9 \end{bmatrix}, \quad Q_3^4 = \begin{bmatrix} 3,768.2 & -0.5 \\ -0.5 & -89.7 \end{bmatrix},$$

$$Q_1^5 = \begin{bmatrix} 214.0 & -15,185.0 \\ -15,185.0 & 15,214.0 \end{bmatrix}, \quad Q_2^5 = \begin{bmatrix} 7,593.6 & 7,529.0 \\ 7,529.0 & 1,325.3 \end{bmatrix}, \quad Q_3^5 = \begin{bmatrix} 3,542.1 & 0.0 \\ 0.0 & -3,023.0 \end{bmatrix},$$

$$Q_1^6 = \begin{bmatrix} -40.4375 & 9.9389 \\ 9.9389 & 28.8514 \end{bmatrix}, \quad Q_2^6 = \begin{bmatrix} 54.7453 & -6.5632 \\ -6.5632 & 45.3682 \end{bmatrix}, \quad Q_3^6 = \begin{bmatrix} 53.4114 & -0.2701 \\ -0.2701 & 42.9778 \end{bmatrix},$$

$$N_1^1 = \begin{bmatrix} -123.7682 & 9.3117 \\ 9.3117 & -461.4212 \end{bmatrix}, \quad N_2^1 = \begin{bmatrix} -1,007.0 & -119.6 \\ -119.6 & -1,067.4 \end{bmatrix}, \quad N_3^1 = \begin{bmatrix} 483.1 & -2.9 \\ -2.9 & 1,962.9 \end{bmatrix},$$

$$N_1^2 = \begin{bmatrix} 1,415.9 & -464.4 \\ -464.4 & -534.9 \end{bmatrix}, \quad N_2^2 = \begin{bmatrix} 3,122.6 & 434.1 \\ 434.1 & 64.8 \end{bmatrix}, \quad N_3^2 = \begin{bmatrix} -1,429.4 & 0.1 \\ 0.1 & -1,922.0 \end{bmatrix},$$

$$N_1^3 = \begin{bmatrix} 38.2260 & -6.0579 \\ -6.0579 & -13.3891 \end{bmatrix}, \quad N_2^3 = \begin{bmatrix} -13.4412 & 6.7579 \\ 6.7579 & 9.3172 \end{bmatrix}, \quad N_3^3 = \begin{bmatrix} -9.1322 & -0.0352 \\ -0.0352 & -24.2347 \end{bmatrix},$$

$$N_1^4 = \begin{bmatrix} 31.0135 & -6.3425 \\ -6.3425 & -14.6572 \end{bmatrix}, \quad N_2^4 = \begin{bmatrix} -14.1576 & 6.0573 \\ 6.0573 & 7.3776 \end{bmatrix}, \quad N_3^4 = \begin{bmatrix} -9.6531 & 0.0282 \\ 0.0282 & -23.7517 \end{bmatrix},$$

$$N_1^5 = \begin{bmatrix} 1,234.7 & 657.0 \\ 657.0 & 110.7 \end{bmatrix}, \quad N_2^5 = \begin{bmatrix} -18.2 & 1,241.1 \\ 1,241.1 & -134.2 \end{bmatrix}, \quad N_3^5 = \begin{bmatrix} -604.6748 & 1.6013 \\ 1.6013 & 49.2281 \end{bmatrix},$$

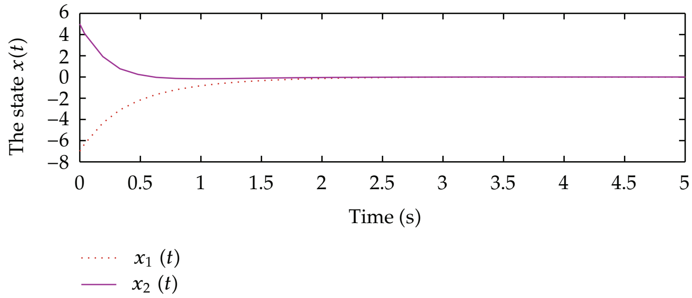

Figure 1: State trajectories $x_1(t)$ and $x_2(t)$ of LPD time-delay system (2.1) with (4.1), $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ by using program dde45lin with Matlab.

$$N_1^6 = \begin{bmatrix} 14.9282 & -0.7772 \\ -0.7772 & 33.2457 \end{bmatrix}, \quad N_2^6 = \begin{bmatrix} 36.9436 & 0.4195 \\ 0.4195 & 33.4742 \end{bmatrix}, \quad N_3^6 = \begin{bmatrix} 33.2443 & -0.0821 \\ -0.0821 & 25.6059 \end{bmatrix},$$
$$M_1^1 = \begin{bmatrix} 222.2533 & 104.5561 \\ 104.5561 & 54.1789 \end{bmatrix}, \quad M_2^1 = \begin{bmatrix} 2.1314 & 0.0476 \\ 0.0476 & 3.1486 \end{bmatrix}, \quad M_3^1 = \begin{bmatrix} 2.0463 & -0.0108 \\ -0.0108 & 1.4850 \end{bmatrix},$$
$$M_1^2 = \begin{bmatrix} -11.9251 & -13.1913 \\ -13.1913 & -5.6759 \end{bmatrix}, \quad M_2^2 = \begin{bmatrix} 1.4344 & -2.6047 \\ -2.6047 & 3.3445 \end{bmatrix}, \quad M_3^2 = \begin{bmatrix} 1.0670 & -0.0153 \\ -0.0153 & 0.0826 \end{bmatrix},$$
$$M_1^3 = \begin{bmatrix} 393.2821 & 56.3348 \\ 56.3348 & 295.3780 \end{bmatrix}, \quad M_2^3 = \begin{bmatrix} 269.8280 & -0.1703 \\ -0.1703 & 270.2562 \end{bmatrix}, \quad M_3^3 = \begin{bmatrix} 270.0284 & -0.0192 \\ -0.0192 & 268.5208 \end{bmatrix},$$
$$M_1^4 = \begin{bmatrix} -63.2823 & -34.4679 \\ -34.4679 & -20.2534 \end{bmatrix}, \quad M_2^4 = \begin{bmatrix} -0.8017 & -0.1312 \\ -0.1312 & -0.8706 \end{bmatrix}, \quad M_3^4 = \begin{bmatrix} -0.6584 & -0.0064 \\ -0.0064 & -1.2408 \end{bmatrix},$$
$$M_1^5 = \begin{bmatrix} 208.1212 & -29.9392 \\ -29.9392 & 252.5413 \end{bmatrix}, \quad M_2^5 = \begin{bmatrix} 268.6888 & 0.6640 \\ 0.6640 & 268.0517 \end{bmatrix}, \quad M_3^5 = \begin{bmatrix} 268.9342 & -0.0012 \\ -0.0012 & 268.6650 \end{bmatrix},$$
$$R_1^1 = \begin{bmatrix} 196.3135 & -36.7812 \\ -36.7812 & 248.7288 \end{bmatrix}, \quad R_2^1 = \begin{bmatrix} 268.6863 & 0.6631 \\ 0.6631 & 268.0488 \end{bmatrix}, \quad R_3^1 = \begin{bmatrix} 268.9326 & -0.0013 \\ -0.0013 & 268.6593 \end{bmatrix},$$
$$R_1^2 = \begin{bmatrix} 28.5196 & 7.4817 \\ 7.4817 & 4.9413 \end{bmatrix}, \quad R_2^2 = \begin{bmatrix} 1.4407 & -2.6031 \\ -2.6031 & 3.3546 \end{bmatrix}, \quad R_3^2 = \begin{bmatrix} 1.0720 & -0.0153 \\ -0.0153 & 0.0894 \end{bmatrix},$$
$$R_1^3 = \begin{bmatrix} 158.8158 & -17.7934 \\ -17.7934 & 171.3160 \end{bmatrix}, \quad R_2^3 = \begin{bmatrix} 178.2539 & 4.6824 \\ 4.6824 & 181.6405 \end{bmatrix}, \quad R_3^3 = \begin{bmatrix} 183.6198 & -0.0347 \\ -0.0347 & 184.1092 \end{bmatrix}, \tag{4.2}$$

and $\epsilon_1 = 414.9151$ and $\epsilon_2 = 381.9944$. It is known that the maximum value of h for the stability of this system is $h = 0.6246$. The stability is also assured for $h < 0.6246$. The numerical solution $x_1(t)$ and $x_2(t)$ of (2.1) with (4.1) are plotted in Figure 1.

Example 4.2. Consider the following linear systems, which are considered in [17]:

Table 1: Upper bounds of time delays in Example 4.2 for various conditions.

	$\eta = 0, \rho = 0.1$ $h_d = 0.5$	$\eta = 0, \rho = 0.1$ $h_d \geq 1$	$\eta = 0.1, \rho = 0.1$ $h_d = 0.5$	$\eta = 0.1, \rho = 0.1$ $h_d \geq 1$
Cao and Lam [18] (2000)	0.5467	—	0.4950	—
Zuo and Wang [3] (2006)	1.1424	—	1.0097	—
Chen et al. [17] (2008)	1.1425	0.7355	1.0097	0.7147
Corollary 3.2	2.0925	0.8412	1.8235	0.8406

Table 2: Upper bounds of time delays in Example 4.3 for various conditions.

	$\eta = 0.05, \rho = 0.1$			
	$\beta = 0$	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$
Kwon and Park [19] (2008)	3.40	1.36	0.76	0.55
Corollary 3.2	2.87	1.53	0.97	0.75

Table 3: Comparison of convergence rate obtained for Corollary 3.2 and from [8, 20] in Example 4.4.

Method	Year	Convergence rate β
Mondié and Kharitonov [20]	2005	0.470
Nam [8]	2009	1.153
Corollary 3.2	2012	1.410

$$\dot{x}(t) = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix} x(t - h(t)) + f(t, x(t)) + g(t, x(t - h(t))), \tag{4.3}$$

where $\|f(t, x(t))\| \leq \eta \|x(t)\|$, $\|g(t, x(t - h(t)))\| \leq \rho \|x(t - h(t))\|$.

By Corollary 3.2 to the system (4.3), we can obtain the maximum upper bounds of the time delay under different values of η, ρ, and h_d as shown in Table 1. From Table 1, we see that Corollary 3.2 gives larger delay bounds than some of the recent results in literatures.

Example 4.3. Consider the following linear systems, which are considered in [19]:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h) + f(t, x(t)) + g(t, x(t - h)), \tag{4.4}$$

where $\|f(t, x(t))\| \leq \eta \|x(t)\|$, $\|g(t, x(t - h(t)))\| \leq \rho \|x(t - h(t))\|$. By using Corollary 3.2 to the system (4.4), we obtain the maximum upper bounds of the time delay for different values of η, ρ, and h_d as shown in Table 2. From Table 2, it can be seen that Corollary 3.2 gives larger delay bounds than the recent results in [19].

Example 4.4. Consider the following linear systems, which is considered in [8]:

$$\dot{x}(t) = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0 \\ 4 & 0.1 \end{bmatrix} x(t - h) + f(t, x(t)) + g(t, x(t - h)), \tag{4.5}$$

where $\|f(t, x(t))\| \leq 0.2\|x(t)\|$, $\|g(t, x(t-h))\| \leq 0.2\|x(t-h)\|$. The maximum value of convergence rate is 1.410 by using Corollary 3.2 for system (4.5). From Table 3, we can see that Corollary 3.2 gives larger convergence rate than the results in [8, 20].

5. Conclusions

The problem of robust exponential stability for LPD systems with time-varying delays and nonlinear perturbations was studied. Based on the combination of Leibniz-Newton formula and linear matrix inequality, the use of suitable Lyapunov-Krasovskii functional, new delay-dependent exponential stability criteria are formulated in terms of LMIs. Numerical examples have shown significant improvements over some existing results.

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Research Article

Robust Exponential Stability for LPD Discrete-Time System with Interval Time-Varying Delay

Kanit Mukdasai[1, 2]

[1] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Si Ayuthaya Road, Bangkok 10400, Thailand*

Correspondence should be addressed to Kanit Mukdasai, kanit@kku.ac.th

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This paper investigates the problem of robust exponential stability for uncertain linear-parameter dependent (LPD) discrete-time system with delay. The delay is of an interval type, which means that both lower and upper bounds for the time-varying delay are available. The uncertainty under consideration is norm-bounded uncertainty. Based on combination of the linear matrix inequality (LMI) technique and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability are obtained in terms of LMI. Numerical examples are given to demonstrate the effectiveness and less conservativeness of the proposed methods.

1. Introduction

Over the past decades, the problem of stability analysis of delay discrete-time systems has been widely investigated by many researchers. Because the existence of time delay is frequent, a source of oscillation instability performances degradation of systems. Stability criteria for discrete-time systems with time delay is generally divided into two classes: delay-independent ones and delay-dependent ones. Delay-independent stability criteria tend to be more conservative, especially for small-size delay; such criteria do not give any information on the size of the delay. On the other hand, delay-dependent stability criteria is concerned with the size of the delay and usually provide a maximal delay size. Moreover, robust stability of linear continuous-time and discrete-time systems subject to time-invariant parametric uncertainty has received considerable attention. An important class of linear time-invariant parametric uncertain system is linear parameter-dependent (LPD) system in which the uncertain state matrices are in the polytope consisting of all convex combination of known matrices. To address this problem, several results have been obtained in terms of sufficient (or necessary and sufficient) conditions; see [1–15] and references cited therein. Most of these conditions have been obtained via the Lyapunov theory approaches in which parameter dependent Lyapunov functions have been employed. These conditions are always expressed in terms of linear matrix inequalities (LMIs) which can be solved numerically by using available tools such as LMI toolbox in MATLAB. The results have been obtained for robust stability for LPD systems in which time delay occur in state variable such as [6, 11, 14] present sufficient conditions for robust stability of LPD continuous-time system with delays. However, a few results have been obtained for robust stability for LPD discrete-time systems with delay.

In this paper, we deal with the problem of robust exponential stability for uncertain LPD discrete-time system with interval time-varying delay. Combined with the linear matrix inequality technique and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability are obtained in terms of LMI. Finally, numerical examples have demonstrated the effectiveness of the criteria.

2. Problem Formulation and Preliminaries

We introduce some notations and definitions that will be used throughout the paper. $\mathbb{Z}^+$ denotes the set of non negative integer numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in \mathbb{R}^n$; that is, $\|x\|^2 = x^T x$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimensions; A^T denotes transpose of the Matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda : \lambda \in \lambda(A)\}$; Matrix A is called semi positive definite $(A \geq 0)$ if $x^T A x \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite $(A > 0)$ if $x^T A x > 0$ for all $x \neq 0$; Matrix B is called semi-negative definite $(B \leq 0)$ if $x^T B x \leq 0$, for all $x \in \mathbb{R}^n$; B is negative definite $(B < 0)$ if $x^T B x < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the following uncertain LPD discrete-time system with interval time-varying delay in the state

$$\begin{gathered} x(k+1) = [A(\alpha) + \Delta A(k)]x(k) + [B(\alpha) + \Delta B(k)]x(k-h(k)), \\ x(s) = \phi(s), \quad s = -h_2, \ldots, -1, 0, \end{gathered} \tag{2.1}$$

where $k \in \mathbb{Z}^+$, $x(k) \in \mathbb{R}^n$ is the system state and $\phi(s)$ is a initial value at s. $A(\alpha), B(\alpha) \in M^{n\times n}$ are uncertain matrices belonging to the polytope of the form

$$\begin{gathered} A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \qquad B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i, \\ \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i, B_i \in M^{n\times n}, \quad i = 1, \ldots, N. \end{gathered} \tag{2.2}$$

$\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we assumed to be of the form

$$\Delta A(k) = K(\alpha)\Delta(k)A_1(\alpha), \qquad \Delta B(k) = K(\alpha)\Delta(k)B_1(\alpha),$$
$$A_1(\alpha) = \sum_{i=1}^{N}\alpha_i A_i^1, \quad B_1(\alpha) = \sum_{i=1}^{N}\alpha_i B_i^1, \quad K(\alpha) = \sum_{i=1}^{N}\alpha_i K_i,$$
$$\sum_{i=1}^{N}\alpha_i = 1, \quad \alpha_i \geq 0, \quad A_i^1, B_i^1 \in M^{n\times n}, \quad i = 1,\ldots,N. \tag{2.3}$$

The class of parametric uncertainties $\Delta(k)$, which satisfies

$$\Delta(k) = F(k)[I - JF(k)]^{-1}, \tag{2.4}$$

is said to be admissible where J is a known matrix satisfying

$$I - JJ^T > 0, \tag{2.5}$$

and $F(k)$ is uncertain matrix satisfying

$$F(k)^T F(k) \leq I. \tag{2.6}$$

In addition, we assume that the time-varying delay $h(k)$ is upper and lower bounded. It satisfies the following assumption of the form

$$h_1 \leq h(k) \leq h_2, \tag{2.7}$$

where h_1 and h_2 are known positive integers.

Definition 2.1. The uncertain LPD discrete-time-delayed system in (2.1) is said to be robustly exponentially stable if there exist constant scalars $0 < a < 1$ and $b > 0$ such that

$$\|x(k)\|^2 \leq ba^k \sup_{-h_2\leq l\leq 0} \|\phi(l)\|^2, \tag{2.8}$$

for all admissible uncertainties.

Lemma 2.2 (see [5] (Schur complement lemma)). *Given constant matrices X, Y, Z of appropriate dimensions with $Y > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0. \tag{2.9}$$

Lemma 2.3 (see [2]). *Given constant matrices M_1, M_2, and M_3 of appropriate dimensions with $M_1 = M_1^T$. Then,*

$$M_1 + M_2\Delta(k)M_3 + M_3^T\Delta(k)^T M_2^T < 0, \tag{2.10}$$

where $\Delta(k) = F(k)[I - JF(k)]^{-1}, F(k)^T F(k) \le I$, for all $k \in \mathbb{Z}^+$ if and only if

$$M_1 + \left[\epsilon^{-1}M_3^T \;\; \epsilon M_2\right]\begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1}\left[\epsilon^{-1}M_3^T \;\; \epsilon M_2\right]^T < 0, \tag{2.11}$$

for some scalar $\epsilon > 0$.

3. Main Results

In this section, we present our main results on the robust exponential stability criteria for uncertain LPD discrete-time system with interval time-varying delays. We introduce the following notation for later use:

$$\widehat{A}(\alpha) = A(\alpha) + \Delta A(k), \qquad \widehat{B}(\alpha) = B(\alpha) + \Delta B(k), \qquad \widehat{h} = h_2 - h_1 + 1. \tag{3.1}$$

Lemma 3.1. *For any $\widehat{A}(\alpha), \widehat{B}(\alpha), \widehat{h}$ in* (3.1), *$P(\alpha)$ and $Q(\alpha)$ given by*

$$P(\alpha) = \sum_{i=1}^{N}\alpha_i P_i, \qquad Q(\alpha) = \sum_{i=1}^{N}\alpha_i Q_i, \quad \sum_{i=1}^{N}\alpha_i = 1,\ \alpha_i \ge 0,\ i = 1,\dots,N, \tag{3.2}$$

are parameter-dependent positive definite Lyapunov matrices such that

$$\begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) - P(\alpha) + \widehat{h}Q(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) - Q(\alpha) \end{bmatrix} < 0, \tag{3.3}$$

if and only if

$$\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T P(\alpha) & \epsilon^{-1}A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^T P(\alpha) & \epsilon^{-1}B_1(\alpha)^T & 0 \\ * & * & -P(\alpha) & 0 & \epsilon P(\alpha)K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < 0. \tag{3.4}$$

Proof. Consider

$$
\begin{aligned}
&\begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) - P(\alpha) + \widehat{h}Q(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) - Q(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \widehat{A}^T(\alpha) \\ \widehat{B}^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} \widehat{A}(\alpha) & \widehat{B}(\alpha) \end{bmatrix}.
\end{aligned} \tag{3.5}
$$

We assume that

$$
\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \overline{A}^T(\alpha) \\ \overline{B}^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} \overline{A}(\alpha) & \overline{B}(\alpha) \end{bmatrix} < 0. \tag{3.6}
$$

Using Lemma 2.2, we obtain

$$
\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T + [K(\alpha)\Delta(k)A_1(\alpha)]^T \\ * & -Q(\alpha) & B(\alpha)^T + [K(\alpha)\Delta(k)B_1(\alpha)]^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} < 0. \tag{3.7}
$$

We rewrite the latter inequality as

$$
\begin{aligned}
&\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T \\ * & -Q(\alpha) & B(\alpha)^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K(\alpha) \end{bmatrix} \Delta(k) \begin{bmatrix} A_1(\alpha) & B_1(\alpha) & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} A_1(\alpha) & B_1(\alpha) & 0 \end{bmatrix}^T \Delta(k)^T \begin{bmatrix} 0 \\ 0 \\ K(\alpha) \end{bmatrix}^T < 0.
\end{aligned} \tag{3.8}
$$

Using Lemma 2.3, inequality (3.8) holds if and only if there exists $\epsilon > 0$ such that

$$
\begin{aligned}
&\begin{bmatrix} -P(\alpha) + \widehat{h}Q(\alpha) & 0 & A(\alpha)^T \\ * & -Q(\alpha) & B(\alpha)^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} \epsilon^{-1}A_1(\alpha)^T & 0 \\ \epsilon^{-1}B_1(\alpha)^T & 0 \\ 0 & \epsilon K(\alpha) \end{bmatrix} \begin{bmatrix} I & -J \\ -J & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}A_1(\alpha)^T & 0 \\ \epsilon^{-1}B_1(\alpha)^T & 0 \\ 0 & \epsilon K(\alpha) \end{bmatrix}^T < 0.
\end{aligned} \tag{3.9}
$$

If we apply to (3.9), then we obtain

$$\begin{bmatrix} -P(\alpha)+\hat{h}Q(\alpha) & 0 & A(\alpha)^T & \epsilon^{-1}A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^T & \epsilon^{-1}B_1(\alpha)^T & 0 \\ * & * & -P(\alpha)^{-1} & 0 & \epsilon K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < 0. \tag{3.10}$$

Premultiplying (3.10) by $\operatorname{diag}\{I, I, P(\alpha), I, I\}$ and postmultiplying by $\operatorname{diag}\{I, I, P(\alpha), I, I\}$, we get that (3.4) and the lemma is proved. □

Lemma 3.2. *If there exist positive definite symmetric matrices P_i, Q_i, $i = 1, 2, \ldots, N$, and positive real numbers ϵ, ζ such that*

$$\begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^TP_i & \epsilon^{-1}A_i^{1^T} & 0 \\ * & -Q_i & B_i^TP_i & \epsilon^{-1}B_i^{1^T} & 0 \\ * & * & -P_i & 0 & \epsilon P_iK_i \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < -\zeta I, \quad i = 1, 2, \ldots, N,$$

$$\begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^TP_j & \epsilon^{-1}A_i^{1^T} & 0 \\ * & -Q_i & B_i^TP_j & \epsilon^{-1}B_i^{1^T} & 0 \\ * & * & -P_i & 0 & \epsilon P_iK_j \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} + \begin{bmatrix} -P_j+\hat{h}Q_j & 0 & A_j^TP_i & \epsilon^{-1}A_j^{1^T} & 0 \\ * & -Q_j & B_j^TP_i & \epsilon^{-1}B_j^{1^T} & 0 \\ * & * & -P_j & 0 & \epsilon P_jK_i \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix} < \frac{2\zeta I}{N-1},$$
$$i = 1, \ldots, N-1, \quad j = i+1, \ldots, N, \tag{3.11}$$

then, for any $A(\alpha), A_1(\alpha), B(\alpha), B_1(\alpha), K(\alpha), \hat{h}$ in (3.1)*, $P(\alpha)$ and $Q(\alpha)$ are parameter-dependent positive definite Lyapunov matrices in Lemma 3.1 such that* (3.4) *holds.*

Proof. Consider

$$\begin{bmatrix} -P(\alpha)+\hat{h}Q(\alpha) & 0 & A(\alpha)^TP(\alpha) & \epsilon^{-1}A_1(\alpha)^T & 0 \\ * & -Q(\alpha) & B(\alpha)^TP(\alpha) & \epsilon^{-1}B_1(\alpha)^T & 0 \\ * & * & -P(\alpha) & 0 & \epsilon P(\alpha)K(\alpha) \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix}$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j \begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^TP_j & \epsilon^{-1}A_i^{1^T} & 0 \\ * & -Q_i & B_i^TP_j & \epsilon^{-1}B_i^{1^T} & 0 \\ * & * & -P_i & 0 & \epsilon P_iK_j \\ * & * & * & -I & J \\ * & * & * & * & -I \end{bmatrix}. \tag{3.12}$$

Using the fact that $\sum_{i=1}^{N}\alpha_i = 1$, we obtain the following identities:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j A_i B_j = \sum_{i=1}^{N}\alpha^2 A_i B_i + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\left[A_i B_j + A_j B_i\right],$$
$$(N-1)\sum_{i=1}^{N}\alpha_i^2\zeta - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\zeta = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\alpha_i - \alpha_j\right]^2\zeta \geq 0. \tag{3.13}$$

Then, it follows from (3.11), (3.12), and (3.13) that (3.4) holds. The proof of the lemma is complete. □

Theorem 3.3. *The system* (2.1) *is robustly exponentially stable if the LMI conditions* (3.11) *are feasible.*

Proof. Consider the following Lyapunov-Krasovskii function for system (2.1) of the form

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k)), \tag{3.14}$$

where

$$V_1(x(k)) = x^T(k)P(\alpha)x(k), \qquad V_2(x(k)) = \sum_{i=k-h(k)}^{k-1} x^T(i)Q(\alpha)x(i),$$
$$V_3(x(k)) = \sum_{j=-h_2+2}^{-h_1+1}\sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l). \tag{3.15}$$

A Lyapunov-Krasovskii difference for the system (2.1) is defined as

$$\Delta V(x(k)) = \Delta V_1(x(k)) + \Delta V_2(x(k)) + \Delta V_3(x(k)). \tag{3.16}$$

Taking the difference of $V_1(x(k))$ and $V_2(x(k))$, the increments of $V_1(x(k))$ and $V_2(x(k))$ are

$$\begin{aligned}\Delta V_1(x(k)) &= V_1(x(k+1)) - V_1(x(k))\\ &= x^T(k+1)P(\alpha)x(k+1) - x^T(k)P(\alpha)x(k)\\ &= x^T(k)\widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha)x(k) + x^T(k-h(k))\widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha)x(k)\\ &\quad + x^T(k-h(k))\widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha)x(k-h(k))\\ &\quad + x^T(k)\widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha)x(k-h(k)) - x^T(k)P(\alpha)x(k),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\Delta V_2(x(k)) &= V_2(x(k+1)) - V_2(x(k))\\ &= \sum_{i=k+1-h(k+1)}^{k} x^T(i)Q(\alpha)x(i) - \sum_{i=k-h(k)}^{k-1} x^T(i)Q(\alpha)x(i)\\ &= x^T(k)Q(\alpha)x(k) - x^T(k-h(k))Q(\alpha)x(k-h(k))\\ &\quad + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Q(\alpha)x(i)\\ &\quad + \sum_{i=k+1-h_1}^{k-1} x^T(i)Q(\alpha)x(i).\end{aligned} \tag{3.18}$$

Form $h(k) \geq h_1$, the two last terms of the right-hand side of the latter equality yield

$$\sum_{i=k+1-h_1}^{k-1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Q(\alpha)x(i) \leq 0. \tag{3.19}$$

Thus, we obtain

$$\begin{aligned}\Delta V_2(x(k)) \leq\ & x^T(k)Q(\alpha)x(k) - x^T(k-h(k))Q(\alpha)x(k-h(k))\\ &+ \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i).\end{aligned} \tag{3.20}$$

The increment of $V_3(x(k))$ is easily computed as

$$\begin{aligned}\Delta V_3(x(k)) &= V_3(x(k+1)) - V_3(x(k))\\ &= \sum_{j=-h_2+2}^{-h_1+1}\left[x^T(k)Q(\alpha)x(k) + \sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l) - \sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l)\right]\\ &= (h_2-h_1)x^T(k)Q(\alpha)x(k) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i).\end{aligned} \tag{3.21}$$

It is easy to see that

$$\begin{aligned}\Delta V_2(x(k)) + \Delta V_3(x(k)) \leq\ & (h_2-h_1+1)x^T(k)Q(\alpha)x(k) - x_{k-h}^T Q(\alpha)x_{k-h}\\ &+ \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i),\end{aligned} \tag{3.22}$$

for simplicity, we let $x(k-h(k)) = x_{k-h}$. Since, $h(k) \le h_2$, we obtain that

$$\sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i) \le 0. \tag{3.23}$$

Therefore, we conclude that

$$\begin{aligned}\Delta V(x(k)) \le\ & x^T(k)\widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha)x(k) + x_{k-h}^T\widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha)x(k)\\ & + x^T(k)\widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha)x_{k-h} + x_{k-h}^T\widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha)x_{k-h}\\ & - x^T(k)P(\alpha)x(k) + (h_2 - h_1 + 1)x^T(k)Q(\alpha)x(k)\\ & - x_{k-h}^TQ(\alpha)x_{k-h}.\end{aligned} \tag{3.24}$$

It follows form (3.24) that

$$\Delta V(x(k)) \le Y^T\begin{bmatrix}\Delta_{11}(\alpha) & \widehat{A}^T(\alpha)P(\alpha)\widehat{B}(\alpha)\\ \widehat{B}^T(\alpha)P(\alpha)\widehat{A}(\alpha) & \widehat{B}^T(\alpha)P(\alpha)\widehat{B}(\alpha) - Q(\alpha)\end{bmatrix}Y, \tag{3.25}$$

where $\Delta_{11}(\alpha) = \widehat{A}^T(\alpha)P(\alpha)\widehat{A}(\alpha) - P(\alpha) + \widehat{h}Q(\alpha)$ and $Y^T = [x(k)^T\, x(k-h(k))^T]$. By (3.11), (3.25), and Lemma 3.1, and 3.2, we obtain

$$\Delta V(x(k)) < -\omega\|x\|^2, \tag{3.26}$$

where $\omega > 0$. By (3.14), it is easy to see that

$$V(x(k)) \le \beta_1\|x\|^2 + \beta_1\widehat{h}\sum_{i=k-h_2}^{k-1}\|x(i)\|^2, \tag{3.27}$$

where

$$\beta_1 = \max\{\lambda_{\max}(P_i), \lambda_{\max}(Q_i);\ i = 1, 2, \ldots, N\}. \tag{3.28}$$

It can be shown that there always exists a scalar $\theta > 1$ satisfying

$$(\theta - 1)\beta_1 - \lambda\theta + h_2\theta^{h_2}(\theta - 1)\beta_1\widehat{h} = 0. \tag{3.29}$$

For any scalar $\theta > 1$, it follows from (3.26) and (3.27) that

$$\begin{aligned}
\theta^{k+1}V(x(k+1)) - \theta^{k+1}V(x(k)) &= \theta^{k+1}(V(x(k+1)) - V(x(k))) + \theta^{k}(\theta-1)V(x(k)) \\
&< \alpha_1(\theta)\theta^{k}\|x(k)\|^2 + \alpha_2(\theta)\theta^{k}\sum_{i=k-h_2}^{k-1}\|x(i)\|^2,
\end{aligned} \tag{3.30}$$

where

$$\alpha_1(\theta) = (\theta-1)\beta_1 - \lambda\theta, \qquad \alpha_2(\theta) = (\theta-1)\beta_1\widehat{h}. \tag{3.31}$$

Therefore, for any integer $T \geq h_2 + 1$, summing up both sides of (3.30) from 0 to $T-1$ gives

$$\theta^{T}V(x(T)) - V(x(0)) \leq \alpha_1(\theta)\sum_{i=0}^{T-1}\theta^{i}\|x(i)\|^2 + \alpha_2(\theta)\sum_{i=0}^{T-1}\sum_{l=i-h_2}^{i-1}\theta^{i}\|x(l)\|^2. \tag{3.32}$$

For $h_2 \geq 1$,

$$\begin{aligned}
\sum_{i=0}^{T-1}\sum_{l=i-h_2}^{i-1}\theta^{i}\|x(l)\|^2 &\leq \sum_{l=-h_2}^{-1}\sum_{i=0}^{l+h_2}\theta^{i}\|x(l)\|^2 + \sum_{l=0}^{T-1-h_2}\sum_{i=l+1}^{l+h_2}\theta^{i}\|x(l)\|^2 \\
&\quad + \sum_{l=T-h_2}^{T-1}\sum_{i=l+1}^{T-1}\theta^{i}\|x(l)\|^2 \\
&\leq h_2\sum_{l=-h_2}^{-1}\theta^{l+h_2}\|x(l)\|^2 + h_2\sum_{l=0}^{T-1-h_2}\theta^{l+h_2}\|x(l)\|^2 \\
&\quad + h_2\sum_{l=T-h_2}^{T-1}\theta^{l+h_2}\|x(l)\|^2 \\
&\leq h_2(h_2+1)\theta^{h_2}\sup_{-h_2\leq l\leq 0}\left\|\phi(l)\right\|^2 + h_2\theta^{h_2}\sum_{l=1}^{T-1}\theta^{l}\|x(l)\|^2.
\end{aligned} \tag{3.33}$$

From (3.32) and (3.33), we obtain

$$\begin{aligned}
\theta^{T}V(x(T)) &\leq V(x(0)) + h_2(h_2+1)\theta^{h_2}\alpha_2(\theta)\sup_{-h_2\leq l\leq 0}\left\|\phi(l)\right\|^2 \\
&\quad + \left[\alpha_1(\theta) + \alpha_2(\theta)h_2\theta^{h_2}\right]\sum_{l=0}^{T-1}\theta^{l}\|x(l)\|^2.
\end{aligned} \tag{3.34}$$

Observe

$$V(x(T)) \geq \gamma \|x(T)\|^2, \qquad V(x(0)) \leq \left(\beta_1 + \beta_1 \widehat{h} h_2\right) \sup_{-h_2 \leq l \leq 0} \|\phi(l)\|^2,$$
$$\gamma = \min\{\lambda_{\min}(P_i);\ i = 1, 2, \ldots, N\}. \tag{3.35}$$

Then, it follows from (3.29), (3.33), and (3.35) that

$$\|x(T)\|^2 \leq \frac{h_2\left[(h_2+1)\theta^{h_2}\alpha_2(\theta) + \beta_1 + \beta_1 \widehat{h} h_2\right]}{\gamma} \left(\frac{1}{\theta}\right)^T \sup_{-h_2 \leq l \leq 0} \|\phi(l)\|^2. \tag{3.36}$$

By Definition 2.1, this means that the system (2.1) is robustly exponentially stable. The proof of the theorem is complete. □

4. Numerical Example

Example 4.1. Consider the following uncertain LPD discrete-time system with time-varying delays (2.1) where $h(k) = 2 + \cos(k\pi/2)$, that is,. $h_1 = 1, h_2 = 3$ and

$$A_1 = \begin{bmatrix} -0.6 & 0.02 \\ 0.02 & -0.6 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.7 & 0.03 \\ 0.03 & -0.7 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -0.6 & 0.02 \\ 0.02 & -0.08 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -0.8 & 0.03 \\ 0.03 & -0.09 \end{bmatrix}, \qquad A_1^1 = \begin{bmatrix} 0.005 & 0.0001 \\ 0.0001 & 0.005 \end{bmatrix}, \qquad A_2^1 = \begin{bmatrix} 0.006 & 0.0002 \\ 0.0002 & 0.006 \end{bmatrix},$$
$$B_1^1 = \begin{bmatrix} -0.007 & 0.0005 \\ 0.0005 & -0.007 \end{bmatrix}, \qquad B_2^1 = \begin{bmatrix} -0.004 & 0.0002 \\ 0.0002 & -0.004 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.01 & 0.003 \\ 0.003 & 0.01 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0.02 & 0.001 \\ 0.001 & 0.02 \end{bmatrix}, \tag{4.1}$$

and $J = \left[\begin{smallmatrix} 0.001 & 0 \\ 0 & 0.001 \end{smallmatrix}\right]$. By using LMI Toolbox in MATLAB, we use condition (3.11) in Theorem 3.3 for this example. The solutions of LMI verify as follows of the form $\epsilon = 1$, $P_1 = \left[\begin{smallmatrix} 31.3635 & 1.2365 \\ 1.2365 & 29.4763 \end{smallmatrix}\right]$, $P_2 = \left[\begin{smallmatrix} 37.6354 & 0.2543 \\ 0.2543 & 41.3745 \end{smallmatrix}\right]$, $Q_1 = \left[\begin{smallmatrix} 9.4325 & 0.5587 \\ 0.5587 & 11.4534 \end{smallmatrix}\right]$, and $Q_2 = \left[\begin{smallmatrix} 10.8564 & 1.3856 \\ 1.3856 & 11.9781 \end{smallmatrix}\right]$ (see Figure 1).

Example 4.2. Consider the following the LPD discrete-time system with time-varying delays (2.1) where, $\Delta A(k) = \Delta B(k) = 0$ with

$$A_1 = \begin{bmatrix} 0.60 & 0 \\ 0.01 & 0.60 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.80 & 0 \\ 0.05 & 0.70 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.10 & 0 \\ 0.20 & 0.10 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.10 & 0 \\ -0.20 & -0.10 \end{bmatrix}. \tag{4.2}$$

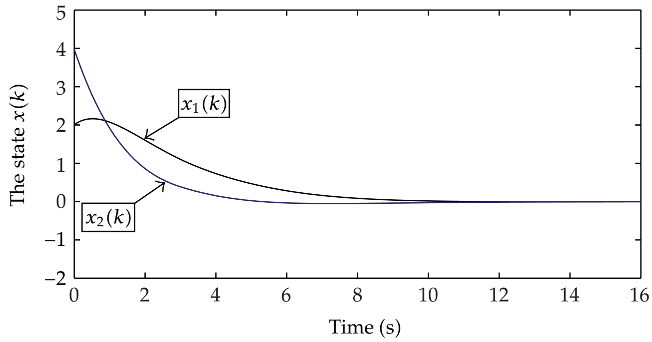

Figure 1: The simulation solution of the states $x_1(k)$ and $x_2(k)$ in Example 4.1 for uncertain LPD discrete-time delayed system with initial conditions $x_1(k) = 2$ and $x_2(k) = 4$, $k = -3, -2, -1, 0$, and $\alpha_1 = \alpha_2 = 1/2$ by using the method of Runge-Kutta order $4(h = 0.01)$ with Matlab.

Table 1: Comparison of the maximum allowed time delay h_2.

Methods	h_2 ($h_1 = 2$)	h_2 ($h_1 = 4$)	h_2 ($h_1 = 5$)	h_2 ($h_1 = 7$)
Liuet al. [7] 2006	2	4	5	7
Our results	4	6	7	9

Table 1 lists the comparison of the upper-bound delay for asymptotic stability of system (2.1) where $\Delta A(k) = \Delta B(k) = 0$ by different method. We apply Theorem 3.3 and see from Table 1 that our result is superior to those in [7, Theorem 3.2].

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