



# รายงานวิจัยฉบับสมบูรณ์

โครงการ “ปัญหาภาวะสมดุลและภาพฉายแบบนัยทั่วไปในปริภูมิบานาค”  
Equilibrium problems and generalized projections in Banach spaces  
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(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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**บทคัดย่อ:** ปัญหาภาวะสมดุลเป็นปัญหาหนึ่งในทางคณิตศาสตร์ที่มีความสำคัญและน่าสนใจ เนื่องจากปัญหาดังกล่าวสามารถใช้อธิบายปัญหาอื่นๆ เช่น อสมการผันแปร ปัญหาจุดตรึง และ ปัญหาต่างๆ ในการหาค่าเหมาะสมได้ ในงานวิจัยได้ศึกษาแบบแผนการทำซ้ำสำหรับปัญหาภาวะสมดุลแบบต่าง ๆ ผลลัพธ์ที่ได้รับเป็นการพัฒนาและครอบคลุมผลงานของนักคณิตศาสตร์จำนวนมากที่มีการศึกษามาก่อนหน้านี้

**คำหลัก:** ปัญหาภาวะสมดุล ภาพฉายแบบนัยทั่วไป ปัญหาจุดตรึง อสมการผันแปร

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**Abstract:** Equilibrium problem is one of important and interesting problems in mathematics. Because it includes many problems such as variational inequality, fixed point problem and many optimization type problems. We introduce and study many iteration schemes for various kinds of equilibrium problems in Banach spaces. Our results improve and unify the corresponding known results studied by many authors.

**Keywords:** equilibrium problem, generalized projection, fixed point problem, variational inequality

# กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และมหาวิทยาลัยขอนแก่น (มข.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนเพิ่มขีดความสามารถด้านการวิจัยของอาจารย์รุ่นกลางในสถาบันอุดมศึกษา ในการทำงานวิจัยครั้งนี้

ศาสตราจารย์ ดร. สมพงษ์ ธรรมพงษา ที่อบรมสั่งสอน ถ่ายทอด ความรู้ด้านต่างๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ที่ได้มีส่วนโครงการวิจัยในครั้งนี้

# Chapter 1

## Executive Summary

Let  $C$  be a closed and convex subset of a real Hilbert space  $\mathcal{H}$ . By an *equilibrium problem* we understand the problem of finding

$$\tilde{x} \in C \text{ such that } f(\tilde{x}, y) \geq 0 \quad \forall y \in C$$

where  $C$  is a given subset of a Banach space  $E$  and  $f : C \times C \rightarrow \mathbb{R}$  is a given function with  $f(x, x) = 0$  for all  $x \in C$ . We denote the set of all such  $\tilde{x}$  by  $\text{EP}(f)$ . Blum and Oettli [1] proved that under appropriate setting on the bifunction  $f$ , the solution set  $\text{EP}(f)$  coincides with the solution set of the following problems:

**Variational inequality:** For an operator  $A : C \rightarrow \mathcal{H}$ , the solution set for the variational inequality is given by

$$\text{VI}(C, A) := \{u \in C : \langle v - u, Au \rangle \geq 0 \forall v \in C\}.$$

If we set  $f(x, y) = \langle y - x, Ax \rangle$ , then it follows that  $\text{EP}(f) = \text{VI}(C, A)$ .

**Optimization:** For a convex, proper and lower semicontinuous function  $g : C \rightarrow (-\infty, \infty]$ , the set of minimizers of  $f$  is given by

$$\text{argmin}_C g := \{u \in C : g(u) = \min_{x \in C} g(x)\}.$$

If we set  $f(x, y) = g(y) - g(x)$ , then it follows that  $\text{EP}(f) = \text{argmin}_C g$ .

**Saddle point:** Suppose that  $\varphi : C_1 \times C_2 \rightarrow \mathbb{R}$  where  $C_1, C_2$  are closed convex subsets of  $\mathcal{H}$ . Recall that a point  $(\tilde{x}_1, \tilde{x}_2) \in C_1 \times C_2$  is a *saddle point* of  $\varphi$  if

$$\varphi(\tilde{x}_1, y_2) \leq \varphi(y_1, \tilde{x}_2) \quad \forall (y_1, y_2) \in C_1 \times C_2.$$

If we set  $C := C_1 \times C_2$  and define  $f : C \times C \rightarrow \mathbb{R}$  by

$$f((x_1, x_2), (y_1, y_2)) := \varphi(y_1, x_2) - \varphi(x_1, y_2),$$

then  $\text{EP}(f)$  coincides with the set of saddle points of  $\varphi$ .

**Fixed point problem:** For a given mapping  $T : C \rightarrow C$ , the fixed-point set of  $T$  is given by

$$\text{Fix}(T) := \{x \in C : x = Tx\}.$$

If we set  $f(x, y) := \langle x - Tx, y - x \rangle$ , then  $\text{EP}(f) = \text{Fix}(T)$ .

However, for each problem above, there are many independent corresponding schemes.

From now on, we assume that  $E$  is a Banach space with dual  $E^*$  and  $C$  is a closed convex subset of  $E$ . The value of  $x^*$  in the dual space  $E^*$  of a Banach space  $E$  at  $x \in E$  is denoted by  $\langle x, x^* \rangle$ . We assume from now on that a Banach space  $E$  is *smooth*, that is, the limit  $\lim_{t \rightarrow 0} \frac{1}{t}(\|x + ty\| - \|x\|)$  exists for all norm one elements  $x, y \in E$ . This implies that the *duality mapping*  $J$  from  $E$  to  $2^{E^*}$  defined by

$$x \mapsto Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

is single-valued and we do consider the singleton  $Jx$  as an element in  $E^*$ . If  $E$  is additional assumed to be *strictly convex*, that is, there are no distinct elements  $x, y \in E$  such that  $\|x\| = \|y\| = \frac{1}{2}\|x + y\| = 1$ , then  $J$  is one-to-one.

Let us note here that if  $E$  is a Hilbert space, then the duality mapping is just the identity mapping.

To deal with a solution of the equilibrium problem for a bifunction  $f : C \times C \rightarrow \mathbb{R}$ , the following assumptions are prerequisite:

(C1)  $f(x, x) = 0$  for all  $x \in C$ ;

(C2)  $f(x, y) + f(y, x) \leq 0$  for all  $x, y \in C$ ;

(C3)  $f(x, \cdot)$  is convex and lower semicontinuous for all  $x \in C$ ;

(C4)  $f$  is *maximal monotone*, that is, for each  $x \in C$  and  $x^* \in E^*$ ,

$$f(x, y) + \langle y - x, x^* \rangle \geq 0 \quad \text{for all } y \in C$$

whenever  $\langle z - x, x^* \rangle \geq f(z, x)$  for all  $z \in C$ .

It is noted (see [4]) that if  $f$  satisfies conditions (C1)–(C3) and the following condition:

(C4')  $\limsup_{t \downarrow 0} f((1-t)x + tz, y) \leq f(x, y)$  for all  $x, y, z \in C$ ,

then  $f$  satisfies condition (C4). The following are known: if  $E$  is a strictly convex, smooth and reflexive Banach space and  $C$  is a nonempty closed convex subset of  $E$ , then

- the function

$$x \mapsto \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \quad \forall y \in C \right\}$$

where  $x \in E$  and  $r > 0$  is well-defined and single-valued; so we write  $z = T_r^f(x)$ ;

- $\text{EP}(f) = \text{Fix}(T_r^f)$ .

The following three generalizations of a nearest point projection in a Hilbert space (see [2, 3]) are very interesting. For  $x \in E$  and  $x_0 \in C$

**Metric projection:**  $x_0 = P_C x$  if and only if  $\langle x_0 - y, J(x - x_0) \rangle \geq 0 \forall y \in C$ ;

**Alber's generalized projection:**  $x_0 = \Pi_C x$  if and only if

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \forall y \in C;$$

**Ibaraki–Takahashi's sunny generalized nonexpansive retraction:**

$$x_0 = R_C x \text{ if and only if } \langle x - x_0, Jx_0 - Jy \rangle \geq 0 \forall y \in C;$$

The study of equilibrium problems are very related to these three generalizations (see the paper in the appendix [A2]).

### Related References

1. Blum, Eugen; Oettli, Werner. *From optimization and variational inequalities to equilibrium problems*. Math. Student 63 (1994), no. 1-4, 123–145.
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# Chapter 2

## Main Results

Let us summarize our results obtained in this project. The full papers are appeared in the Appendix. The result in Section 2.x is described precisely in the paper [Ax].

### 2.1 Strong convergence via Halpern's method

We develop a new tool to conclude the strong convergence of the iterative scheme of Halpern type:

$$\begin{cases} x_1, u \in C \text{ are arbitrarily chosen;} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \quad (n \geq 1), \end{cases}$$

where  $C$  is a closed convex subset of a certain Banach space,  $T : C \rightarrow C$  is a strongly nonexpansive mapping with a fixed point, and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

When we apply our result to the problem of finding a zero of an accretive operator, our deduced result improves the recent result of Aoyama et al. [K.

Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.* 67 (2007) 2350–2360] by removing some assumptions on the parameters.

Finally we discuss the new sufficient condition studied by Song [Y. Song, A new sufficient condition for the strong convergence of Halpern type iterations. *Appl. Math. Comput.* 198 (2) (2008) 721–728; Y. Song, New strong convergence theorems for nonexpansive nonself-mappings without boundary conditions. *Comput. Math. Appl.* 56 (6) (2008) 1473–1478] and correct the main result of Song and Chai [Y. Song, X. Chai, Halpern iteration for firmly type nonexpansive mappings, *Nonlinear Anal.* 71 (10) (2009) 4500–4506].

## 2.2 Ray’s theorem for nonlinear problems

We prove the following result: Let  $C$  be a closed convex subset of a certain Banach space  $E$ . The following assertions are equivalent:

- $C$  is bounded;
- Every mapping  $T : C \rightarrow C$  satisfying

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \geq 0 \quad \forall x, y \in C,$$

has a fixed point;

- The equilibrium problem for a bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfying

$$(C1) \quad f(x, x) = 0 \text{ for all } x \in C;$$

$$(C2) \quad f(x, y) + f(y, x) \leq 0 \text{ for all } x, y \in C;$$

$$(C3) \quad f(x, \cdot) \text{ is convex and lower semicontinuous for all } x \in C;$$

(C4)  $f$  is *maximal monotone*, that is, for each  $x \in C$  and  $x^* \in E^*$ ,

$$f(x, y) + \langle y - x, x^* \rangle \geq 0 \quad \text{for all } y \in C$$

whenever  $\langle z - x, x^* \rangle \geq f(z, x)$  for all  $z \in C$ ;

has a solution;

- The variational inequality for an operator  $A : C \rightarrow E^*$  satisfying

(C5)  $\langle x - y, Ax - Ay \rangle \geq 0$  for all  $x, y \in C$ ;

(C6) for each  $x, y \in C$  the mapping  $t \mapsto A((1t)x + ty)$ , where  $t \in (0, 1)$  is continuous with respect to the weak\* topology of  $E^*$ .

has a solution.

We also discuss the dual equilibrium problems in the sense of Takahashi and Zembayashi [A strong convergence theorem for the equilibrium problem with a bifunction defined on the dual space of a Banach space. Fixed point theory and its applications, 197–209, Yokohama Publ., Yokohama, 2008].

## 2.3 Common fixed point theorem of Lipschitzian mappings

Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\{T_n\}$  be a family of  $L_n$ -Lipschitzian mappings of  $C$  into itself with a common fixed point and  $\lim_{n \rightarrow \infty} L_n = 1$ , that is,

- $\|T_n x - T_n y\| \leq L_n \|x - y\|$  for all  $x, y \in C$ ;
- there exists a point  $p \in C$  such that  $p = T_n p$  for all  $n \geq 1$ .

Suppose that  $\{\alpha_n\}$  is a sequence in  $[0, b] \subset [0, 1)$  and define an iterative sequence

$$\begin{cases} x_1 \in H \text{ is arbitrarily chosen;} \\ C_1 = C; \\ y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n; \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)(L_n^2 - 1)(\text{diam } C)^2\}; \\ x_{n+1} = P_{C_{n+1}} x_1. \end{cases}$$

Using the technique from the shrinking projection method we prove that the sequence  $\{x_n\}$  converges to a common fixed point of a countable family of Lipschitzian mappings if there exists a family  $\mathcal{T}$  of mappings of  $C$  into itself such that

- $\bigcap_{T \in \mathcal{T}} \text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$ ;
- $I - T$  is closed at zero for all  $T \in \mathcal{T}$ .

Our results include recent ones concerning asymptotically nonexpansive mappings due to Plubtieng and Ungchittrakool [Strong convergence of modified Ishikawa iteration for two asymptotically nonexpansive mappings and semigroups, *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 7, pp. 2306–2315, 2007] and Zegeye and Shahzad [Strong convergence theorems for a finite family of asymptotically nonexpansive mappings and semigroups, *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 12, pp. 4496–4503, 2008; and its corrigendum in *Nonlinear Analysis, Theory, Methods & Applications*, vol. 73, pp. 1905–1907, 2010] as special cases.

## 2.4 Zero of variational inequalities in Banach spaces

Let  $E$  be a 2-uniformly convex and smooth Banach space with the 2-uniform convexity constant  $c$  and  $C$  be a closed convex subset of  $E$ . Assume that an operator  $A : C \rightarrow E^*$  satisfies the following conditions:

(M1)  $A$  is  $\alpha$ -inverse strongly monotone where  $\alpha > 0$ ;

(M2')  $A^{-1}0 \neq \emptyset$ .

We prove the following two results:

1. Suppose that  $\{x_n\}$  is iteratively constructed by the following

$$\begin{cases} x_1 = x \in C \text{ arbitrarily chosen and} \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n) \quad (n \geq 1), \end{cases}$$

where  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some  $a, b$  such that  $0 < a < b < c^2\alpha/2$ . If the duality mapping  $J : E \rightarrow E^*$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to some element  $z \in A^{-1}0$ . Further  $z = \lim_{n \rightarrow \infty} \Pi_{A^{-1}0}(x_n)$ .

2. Suppose that  $\{y_n\}$  is iteratively constructed by the following

$$\begin{cases} y_1 = y \in C \text{ arbitrarily chosen and} \\ y_{n+1} = \Pi_C J^{-1}(\alpha_n Jy + (1 - \alpha_n)(Jy_n - \lambda_n Ay_n)) \quad (n \geq 1), \end{cases}$$

where  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some  $a, b$  with  $0 < a < b < c^2\alpha/2$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then the sequence  $\{y_n\}$  converges strongly to  $z = \Pi_{A^{-1}0}(y)$ .

This algorithm was studied by Iiduka and Takahashi [Weak convergence of a projection algorithm for variational inequalities in a Banach space, *J. Math. Anal. Appl.* 339 (2008) 668–679] for finding a solution of the variational inequality problem for an inverse strongly monotone operator in a Banach space. We first remark that, under the assumptions imposed on the operator in their paper, the iterative sequence converges weakly to a zero of the operator, not just a solution of the variational inequality problem. In our proof, slightly modified from the original, we do not assume the uniform smoothness of a space as was the case there. Finally, using Halpern’s type method, we modify this algorithm to obtain the strong convergence to a zero of an inverse strongly monotone operator which is nearest to the initial element of the algorithm in the sense of the Bergman distance associated with the function  $\frac{1}{2}\|\cdot\|^2$ . The technique of the proof is taken from [A1].

## 2.5 Strong convergence theorem for quasi-nonexpansive mappings

Based on the new tool proposed by Saejung (see [A1]), we improve the viscosity approximation process for approximation of a fixed point of a quasi-nonexpansive mapping in a Hilbert space proposed by Maingé [The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 74–79, 2010]. More precisely, we study an iterative sequence  $\{x_n\}$  by

$$\begin{cases} x_1 \in C \text{ is arbitrarily chosen;} \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n \quad (n \geq 1), \end{cases}$$

where  $C$  is a closed convex subset of a Hilbert space,  $T : C \rightarrow C$  is a strongly quasi-nonexpansive mapping,  $f : C \rightarrow C$  is a strict contraction, and  $\{\alpha_n\}$  is a

sequence in  $(0, 1)$  satisfying

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

An example beyond the scope of Maingé's result is given.

## 2.6 Convergence theorems for variational inequalities of accretive operators

Let  $C$  be a nonempty subset of a 2-uniformly smooth Banach space  $E$  and  $\alpha > 0$ . A mapping  $A : C \rightarrow E$  is said to be  $\alpha$ -inverse strongly accretive if

$$\langle Ax - Ay, J(x - y) \rangle \geq \alpha \|Ax - Ay\|^2$$

for all  $x, y \in C$ . The *variational inequality problem* for  $A : C \rightarrow E$  is the problem of finding an element  $u \in C$  such that

$$\langle Au, J(v - u) \rangle \geq 0 \quad \text{for all } v \in C.$$

The set of solutions of the problem above is denoted by  $S(C, A)$ .

In this section, we assume that  $Q : E \rightarrow C$  is a nonexpansive sunny retraction, that is,

- $Q(Qx + t(x - Qx)) = Qx$  for all  $t \geq 0$ ;
- $Q^2 = Q$ .

Suppose that  $\{x_n\}$  is iteratively defined by

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1) \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$ ,  $\{\lambda_n\} \subset (0, \frac{\alpha}{K^2}]$ , and  $K$  is the 2-uniform smoothness constant of  $E$ .

Under appropriate assumptions on the parameters  $\alpha_n$  and  $\lambda_n$ , we present two weak convergence theorems for inverse strongly accretive mappings in Banach spaces, which are supplements to the recent result of Aoyama et al. [Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl Art 2006, 13 (2006). ID 35390]. The first one is proved in the absence of the uniform convexity of a space while the second one is not. We also compare our results to the previous known ones.

## 2.7 Maingé's hierarchical optimization problems

Let  $H$  be a real Hilbert space. Recall that a mapping  $T : H \rightarrow H$  is called

- *$\mu$ -Lipschitzian* iff

$$\|Tx - Ty\| \leq \mu\|x - y\| \quad \text{for all } x, y \in H;$$

- *a contraction* iff it is  $\alpha$ -Lipschitzian with  $\alpha \in [0, 1[$ , and in this case, we also say that  $T$  is a contraction with the coefficient  $\alpha$ ;

- *$r$ -strongly monotone* iff

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2 \quad \text{for all } x, y \in H;$$

- *quasi-nonexpansive* iff  $\text{Fix}(T) \neq \emptyset$  and

$$\|Tx - q\| \leq \|x - q\| \quad \text{for all } (x, q) \in H \times \text{Fix}(T);$$

- $\beta$ -demicontractive iff  $\text{Fix}(T) \neq \emptyset$  and  $T$  satisfies

$$\langle x - Tx, x - q \rangle \geq \frac{1 - \beta}{2} \|x - Tx\|^2 \quad \text{for all } (x, q) \in H \times \text{Fix}(T);$$

(the inequality above is equivalent to

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \beta \|x - Tx\|^2)$$

- *strongly quasi-nonexpansive* iff  $T$  is quasi-nonexpansive and

$$x_n - Tx_n \rightarrow 0$$

whenever  $\{x_n\}$  is a bounded sequence in  $H$  and  $\|x_n - q\| - \|Tx_n - q\| \rightarrow 0$  for some  $q \in \text{Fix}(T)$ .

We are interested in the problem of finding an element

$$(x^*, y^*) \in \text{Fix}(S_1) \times \text{Fix}(S_2)$$

such that the following two inequalities are satisfied:

$$\begin{aligned} \langle x^* - f(y^*), x - x^* \rangle &\geq 0 \quad \forall x \in \text{Fix}(S_1), \\ \langle y^* - g(x^*), y - y^* \rangle &\geq 0 \quad \forall y \in \text{Fix}(S_2). \end{aligned}$$

where  $S_1, S_2 : H \rightarrow H$  are quasi-nonexpansive mappings and  $f, g : H \rightarrow H$  are contractions.

The aims of this research is to investigate an iterative method for solving a system of variational inequalities with fixed-point set constraints. Our scheme can be regarded as a more general variant of the algorithm proposed by Maingé [New approach to solving a system of variational inequalities and hierarchical problems. J. Optim. Theory Appl. 138 (2008) 459–477]. Using the technique proposed in [A1], we propose an alternative analysis that allows us to relax some assumption imposed in his paper for convergence of the

considered method. As a complementary result, we show how to adapt these processes to the case when the constraints involve operators belonging to the class of hemi-contractive mappings; this goes beyond the scope of Maingé's result. Recall that  $T : H \rightarrow H$  is *hemi-contractive* if  $\text{Fix}(T) \neq \emptyset$  and  $T$  satisfies

$$\langle x - Tx, x - q \rangle \geq 0 \quad \text{for all } (x, q) \in H \times \text{Fix}(T).$$

## 2.8 A system of variational inequalities for accretive operators

Katchang and Kumam [An iterative algorithm for finding a common solution of fixed points and a general system of variational inequalities for two inverse strongly accretive operators. *Positivity* 15, 281–295 (2011)] studied the following problem:

Find  $(\hat{x}, \hat{y}) \in C \times C$  such that, for each  $x \in C$ , the following inequalities

$$\begin{cases} \langle \lambda A\hat{y} + \hat{x} - \hat{y}, j(x - \hat{x}) \rangle \geq 0; \\ \langle \mu B\hat{x} + \hat{y} - \hat{x}, j(x - \hat{y}) \rangle \geq 0 \end{cases}$$

holds for some  $j(x - \hat{x}) \in J(x - \hat{x})$  and  $j(x - \hat{y}) \in J(x - \hat{y})$ ,

where  $C$  is a closed convex subset of a Banach space  $E$ ,  $A, B : C \rightarrow E$  are possibly nonlinear operators, and  $\lambda, \mu$  are two positive real numbers.

We give a very short and simple proof of their result. In our proof, we do not assume the uniform convexity of a space. Our result is a *strict* generalization of their result because there exists a strictly convex and 2-uniformly smooth space which is not uniformly convex.

# Appendix

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# Halpern's iteration in Banach spaces<sup>☆</sup>

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## ABSTRACT

We prove strong convergence theorems for a sequence which is generated by Halpern's iteration. We also apply our result for finding zeros of an accretive operator. Our result improves the recent result of Aoyama et al. [K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.* 67 (2007) 2350–2360] by removing some assumptions on the parameters. Finally we discuss the new sufficient condition studied by Song [Y. Song, A new sufficient condition for the strong convergence of Halpern type iterations. *Appl. Math. Comput.* 198 (2) (2008) 721–728; Y. Song, New strong convergence theorems for nonexpansive nonself-mappings without boundary conditions. *Comput. Math. Appl.* 56 (6) (2008) 1473–1478] and correct the main result of Song and Chai [Y. Song, X. Chai, Halpern iteration for firmly type nonexpansive mappings, *Nonlinear Anal.* 71 (10) (2009) 4500–4506].

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## 1. Introduction

Let  $C$  be a subset of a Banach space  $E$ . A mapping  $T : C \rightarrow E$  is called *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . A point  $x \in C$  such that  $x = Tx$  is called a *fixed point* of  $T$ . The problem of finding fixed points of nonexpansive mappings has been widely investigated by many authors (see [1–17] and the references therein). In 1967, Halpern proved that the sequence  $\{x_n\}$ , iteratively generated by

$$x_1, u \in C; \quad x_{n+1} = n^{-\theta}u + (1 - n^{-\theta})Tx_n \quad (n \geq 1)$$

where  $C$  is a closed convex subset of a Hilbert space and  $\theta \in (0, 1)$ , converges strongly to a fixed point of  $T$ .

Many mathematicians [5,18–20,8] have investigated and extended Halpern's iteration to certain Banach spaces and replaced the sequence  $\{n^{-\theta}\}$  with the sequence  $\{\alpha_n\} \subset (0, 1)$  which satisfies the following conditions:

(C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;

(C2)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;

(C3)  $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$  or  $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ .

Recently, Suzuki [13] discussed several sufficient conditions on the sequence  $\{\alpha_n\} \subset (0, 1)$  that guarantee the convergence to a fixed point of nonexpansive mappings.

On the other hand, Chidume and Chidume [3] and Suzuki [12] independently discovered that together just conditions (C1) and (C2) are sufficient for the strong convergence of the following iterative sequence:

$$x_1, u \in C; \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n)$$

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to a fixed point of  $T$ .

The purpose of this paper is to provide a nice subclass of nonexpansive mappings that together with conditions (C1) and (C2) are sufficient for the strong convergence of Halpern's iteration to a fixed point of  $T$ . Even in Hilbert spaces, our result is new. We apply our result for finding zeros of an accretive operator. Finally, we discuss the new sufficient condition studied by Song [9,10] and we correct the main results of Song and Chai [11].

## 2. Preliminaries

Since geometry of Banach spaces plays an important role in the theory of nonexpansive mappings, we recall some definitions and facts that are needed in our study.

Let  $E$  be a real Banach space and  $E^*$  its dual. We denote the value of  $f \in E^*$  at  $x \in E$  by  $\langle x, f \rangle$ . The *normalized duality mapping*  $J : E \rightarrow 2^{E^*}$  is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for each } x \in E.$$

The following inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j \rangle$$

holds for all  $x, y \in E$  and  $j \in J(x + y)$ .

A Banach space  $E$  is said to be

- *strictly convex* if  $\frac{1}{2}\|x + y\| < 1$  for all norm one elements  $x, y \in E$  and  $x \neq y$ ;
- *uniformly convex* if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\frac{1}{2}\|x + y\| \leq 1 - \delta$  for all norm one elements  $x, y \in E$  and  $\|x - y\| \geq \varepsilon$ ;
- *smooth* if the limit  $\lim_{t \rightarrow 0} \frac{1}{t}(\|x + ty\| - \|x\|)$  exists for all norm one elements  $x, y \in E$ .

It is known that every uniformly convex Banach space is reflexive and strictly convex. If  $E$  is smooth, then the normalized duality mapping  $J$  is single valued and in this case the image of  $J$  is treated as a member of  $E^*$ .

We say that a norm of Banach space  $E$  is *uniformly Gâteaux differentiable* if the limit  $\lim_{t \rightarrow 0} \frac{1}{t}(\|x + ty\| - \|x\|)$  exists for all norm one elements  $x, y \in E$  and its value is attained uniformly for all norm one elements  $x \in E$ . In this case,  $J : E \rightarrow E^*$  is uniformly norm-to-weak\* continuous on each bounded subset of  $E$ .

Let  $C$  be a subset of a Banach space  $E$  and let  $D$  be a subset of  $C$ . A mapping  $Q : C \rightarrow D$  is *sunny* if

$$Q(Qx + t(x - Qx)) = Qx$$

for all  $x \in C$  and  $t \geq 0$  such that  $Qx + t(x - Qx) \in C$ . We say that  $Q$  is a *retraction* if  $Q^2 = Q$ . In a smooth Banach space, a retraction  $Q : C \rightarrow D$  is sunny and nonexpansive, where  $C$  is convex, if and only if  $\langle x - Qx, J(z - Qx) \rangle \leq 0$  for all  $x \in C$  and  $z \in D$  (see [21]).

The following lemma is observed by Maingé.

**Lemma 1** ([6, Lemma 2.1]). *Let  $\{\gamma_n\}$  be a sequence of real numbers such that there exists a subsequence  $\{\gamma_{n_j}\}$  of  $\{\gamma_n\}$  such that  $\gamma_{n_j} < \gamma_{n_j+1}$  for all  $j \in \mathbb{N}$ . Then there exists a nondecreasing sequence  $\{m_k\}$  of  $\mathbb{N}$  such that  $\lim_{k \rightarrow \infty} m_k = \infty$  and the following properties are satisfied by all (sufficiently large) numbers  $k \in \mathbb{N}$ :*

$$\gamma_{m_k} \leq \gamma_{m_k+1} \quad \text{and} \quad \gamma_k \leq \gamma_{m_k+1}.$$

In fact,  $m_k$  is the largest number  $n$  in the set  $\{1, \dots, k\}$  such that the condition  $\gamma_n < \gamma_{n+1}$  holds.

We will need the following lemma.

**Lemma 2** ([1, Lemma 2.3]). *Let  $\{s_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  a sequence of real numbers in  $[0, 1]$  with  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\{u_n\}$  a sequence of nonnegative real numbers with  $\sum_{n=1}^{\infty} u_n < \infty$ , and  $\{t_n\}$  a sequence of real numbers with  $\limsup_{n \rightarrow \infty} t_n \leq 0$ . Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + u_n \quad \text{for all } n \in \mathbb{N}.$$

Then  $\lim_{n \rightarrow \infty} s_n = 0$ .

### 3. Main result

#### 3.1. Convergence theorem for a single mapping

The following lemma is extracted from [1]. The proof is given here for sake of completeness.

**Lemma 3.** *Let  $E$  be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Let  $C$  be a closed and convex subset of  $E$  and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$  and suppose that one of the following holds:*

- every bounded, closed and convex subset  $D$  of  $C$  such that  $T(D) \subset D$  contains a fixed point of  $T$ ;
- $E$  is strictly convex.

Suppose that  $\{x_n\}$  is a bounded sequence in  $C$  such that  $x_n - Tx_n \rightarrow 0$  and  $u \in C$ . Then

$$\limsup_{n \rightarrow \infty} \langle u - p, J(x_n - p) \rangle \leq 0,$$

where  $p \in F(T)$  and satisfies the following inequality

$$\langle u - p, J(q - p) \rangle \leq 0 \quad \text{for all } q \in F(T).$$

**Proof.** For each  $t \in (0, 1)$ , there exists a unique element  $z_t \in C$  such that

$$z_t = tu + (1 - t)Tz_t.$$

In fact, it follows since  $x \mapsto tu + (1 - t)Tx$  is a contraction. It is known that  $\{z_t\}$  converges strongly to  $p \in F(T)$  as  $t \downarrow 0$ . Moreover,  $\langle u - p, J(q - p) \rangle \leq 0$  for all  $q \in F(T)$ .

We next consider the following

$$\begin{aligned} \|x_n - z_t\|^2 &= \|t(x_n - u) + (1 - t)(x_n - Tz_t)\|^2 \\ &\leq (1 - t)^2 \|x_n - Tz_t\|^2 + 2t \langle x_n - u, J(x_n - z_t) \rangle \\ &= (1 - t)^2 \|x_n - Tz_t\|^2 + 2t \|x_n - z_t\|^2 + 2t \langle z_t - u, J(x_n - z_t) \rangle \\ &\leq (1 - t)^2 (\|x_n - Tx_n\| + \|Tx_n - Tz_t\|)^2 + 2t \|x_n - z_t\|^2 + 2t \langle z_t - u, J(x_n - z_t) \rangle \\ &\leq (1 - t)^2 (\|x_n - Tx_n\| + \|x_n - z_t\|)^2 + 2t \|x_n - z_t\|^2 + 2t \langle z_t - u, J(x_n - z_t) \rangle. \end{aligned}$$

Then

$$\limsup_{n \rightarrow \infty} \|x_n - z_t\|^2 \leq ((1 - t)^2 + 2t) \limsup_{n \rightarrow \infty} \|x_n - z_t\|^2 + 2t \limsup_{n \rightarrow \infty} \langle z_t - u, J(x_n - z_t) \rangle.$$

Consequently,

$$\limsup_{n \rightarrow \infty} \langle u - z_t, J(x_n - z_t) \rangle \leq \frac{t}{2} \limsup_{n \rightarrow \infty} \|x_n - z_t\|^2.$$

For a given number  $\varepsilon > 0$ , by the uniform norm-to-weak\* continuity of the duality mapping  $J$  on bounded sets and the (strong) convergence of  $\{z_t\}$ , there exists a number  $\delta > 0$  such that

$$\langle u - p, J(x_n - p) - J(x_n - z_t) \rangle < \varepsilon \quad \text{and} \quad \|z_t - p\| \|x_n - z_t\| < \varepsilon$$

for all  $0 < t < \delta$  and for all  $n \in \mathbb{N}$ . This implies that

$$\begin{aligned} \langle u - p, J(x_n - p) \rangle &= \langle u - p, J(x_n - p) - J(x_n - z_t) \rangle + \langle u - p - (u - z_t), J(x_n - z_t) \rangle + \langle u - z_t, J(x_n - z_t) \rangle \\ &\leq \langle u - p, J(x_n - p) - J(x_n - z_t) \rangle + \|z_t - p\| \|x_n - z_t\| + \langle u - z_t, J(x_n - z_t) \rangle \\ &< 2\varepsilon + \langle u - z_t, J(x_n - z_t) \rangle \end{aligned}$$

for all  $0 < t < \delta$  and for all  $n \in \mathbb{N}$ . Consequently,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - p, J(x_n - p) \rangle &\leq 2\varepsilon + \limsup_{n \rightarrow \infty} \langle u - z_t, J(x_n - z_t) \rangle \\ &\leq 2\varepsilon + \frac{t}{2} \limsup_{n \rightarrow \infty} \|x_n - z_t\|^2. \end{aligned}$$

Letting  $t \downarrow 0$ , we have

$$\limsup_{n \rightarrow \infty} \langle u - p, J(x_n - p) \rangle \leq 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, this gives the desired result.  $\square$

A mapping  $T : C \rightarrow C$  is said to be *strongly nonexpansive* [22] if  $T$  is nonexpansive and

$$x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$$

whenever  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0$ .

**Theorem 4.** Let  $E$  be a Banach space such that one of the following conditions is satisfied:

- (1)  $E$  is uniformly smooth;
- (2)  $E$  is reflexive, strictly convex and its norm is uniformly Gâteaux differentiable.

Let  $T$  be a strongly nonexpansive mapping from a closed and convex subset  $C$  of  $E$  into itself. Suppose that  $u \in C$  and  $\{x_n\}$  is a sequence generated iteratively by  $x_1 \in C$  and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to a fixed point  $Qu$  of  $T$ , where  $Q$  is a sunny nonexpansive retraction from  $C$  onto  $F(T)$ .

**Proof.** Let  $p = Qu \in F(T)$ . Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(u - p) + (1 - \alpha_n)(Tx_n - p)\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|Tx_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \max\{\|u - p\|, \|x_n - p\|\}. \end{aligned}$$

In particular,  $\{x_n\}$  is bounded.

Notice that if there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\lim_{k \rightarrow \infty} (\|x_{n_{k+1}} - p\| - \|x_{n_k} - p\|) = 0,$$

then it follows from  $\lim_{k \rightarrow \infty} \alpha_{n_k} = 0$  that

$$\lim_{k \rightarrow \infty} (\|Tx_{n_k} - p\| - \|x_{n_k} - p\|) = 0.$$

Consequently, since  $T$  is strongly nonexpansive,  $x_{n_k} - Tx_{n_k} \rightarrow 0$ . It follows from Lemma 3 that

$$\limsup_{k \rightarrow \infty} \langle u - p, J(x_{n_k} - p) \rangle \leq 0.$$

Let us consider the following two cases.

Case 1.  $\|x_{n+1} - p\| \leq \|x_n - p\|$  for all sufficiently large  $n$ . In this case,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists and hence  $\lim_{n \rightarrow \infty} (\|x_{n+1} - p\| - \|x_n - p\|) = 0$ . This implies that

$$\limsup_{n \rightarrow \infty} \langle u - p, J(x_n - p) \rangle \leq 0.$$

Finally, by the subdifferential inequality and the nonexpansiveness of  $T$ , we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(u - p) + (1 - \alpha_n)(Tx_n - p)\|^2 \\ &\leq (1 - \alpha_n)^2 \|Tx_n - p\|^2 + 2\alpha_n \langle u - p, J(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n) \|x_n - p\|^2 + 2\alpha_n \langle u - p, J(x_{n+1} - p) \rangle. \end{aligned}$$

It follows immediately from Lemma 2 that  $\lim_{n \rightarrow \infty} \|x_n - p\|^2 = 0$ .

Case 2. there exists a subsequence  $\{\|x_{n_j} - p\|\}$  of  $\{\|x_n - p\|\}$  such that  $\|x_{n_j} - p\| < \|x_{n_{j+1}} - p\|$  for all  $j \in \mathbb{N}$ . By Lemma 1, there exists a strictly increasing sequence  $\{m_k\}$  of positive integers such that the following properties are satisfied by all numbers  $k \in \mathbb{N}$ :

$$\|x_{m_k} - p\| \leq \|x_{m_{k+1}} - p\| \quad \text{and} \quad \|x_k - p\| \leq \|x_{m_{k+1}} - p\|.$$

Consequently,

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} (\|x_{m_{k+1}} - p\| - \|x_{m_k} - p\|) \\ &\leq \limsup_{n \rightarrow \infty} (\|x_{n+1} - p\| - \|x_n - p\|) \\ &\leq \limsup_{n \rightarrow \infty} \alpha_n (\|u - p\| - \|x_n - p\|) = 0. \end{aligned}$$

It follows from the remark above that

$$\limsup_{k \rightarrow \infty} \langle u - p, J(x_{m_k} - p) \rangle \leq 0.$$

Moreover, since

$$\|x_{m_{k+1}} - p\|^2 \leq (1 - \alpha_{m_k}) \|x_{m_k} - p\|^2 + 2\alpha_{m_k} \langle u - p, J(x_{m_{k+1}} - p) \rangle$$

we have

$$\begin{aligned} \alpha_{m_k} \|x_{m_k} - p\|^2 &\leq \|x_{m_k} - p\|^2 - \|x_{m_{k+1}} - p\|^2 + 2\alpha_{m_k} \langle u - p, J(x_{m_{k+1}} - p) \rangle \\ &\leq 2\alpha_{m_k} \langle u - p, J(x_{m_{k+1}} - p) \rangle. \end{aligned}$$

In particular,

$$\lim_{k \rightarrow \infty} \|x_{m_k} - p\|^2 \leq 2 \limsup_{k \rightarrow \infty} \langle u - p, J(x_{m_{k+1}} - p) \rangle \leq 0.$$

This and  $\lim_{k \rightarrow \infty} (\|x_{m_{k+1}} - p\| - \|x_{m_k} - p\|) = 0$  implies that

$$\limsup_{k \rightarrow \infty} \|x_k - p\|^2 \leq \lim_{k \rightarrow \infty} \|x_{m_{k+1}} - p\|^2 = 0.$$

The proof is finished.  $\square$

Now we prove that **Theorem 4** remains true if the space admits weakly sequentially continuous duality mapping. A continuous and strictly increasing function  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a *gauge function* if  $\varphi(0) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . The mapping  $J_\varphi : E \rightarrow 2^{E^*}$  is defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \varphi(\|x\|)\}$$

and is called the *duality mapping with the gauge function*  $\varphi$ . We say that a Banach space  $E$  has a *weakly sequentially continuous duality mapping* if there exists a gauge function  $\varphi$  such that the duality mapping with the gauge function  $\varphi$  is single-valued and weak-to-weak\* sequentially continuous. Note that every  $\ell^p$  space where  $1 < p < \infty$  has a weakly sequentially continuous duality mapping  $J_\varphi$  with  $\varphi(t) = t^{p-1}$ . It is clear that the duality mapping  $J_\varphi$  with the gauge function  $\varphi(t) \equiv t$  is the normalized duality mapping. For more details, we refer to [23].

**Lemma 5 (Subdifferential Inequality).** *Let  $E$  be a Banach space and  $\varphi$  be a gauge function. Define  $\Phi(t) = \int_0^t \varphi(s) ds$  for all  $t \in [0, \infty)$ . Then the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j \rangle$$

for all  $x, y \in E$  and  $j \in J_\varphi(x + y)$ . In particular,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j \rangle$$

for all  $x, y \in E$  and  $j \in J(x + y)$ .

Inspired by [16, Theorems 3.1 and 3.2], we prove the following result.

**Lemma 6.** *Let  $E$  be a reflexive Banach space having a weakly sequentially continuous duality mapping  $J_\varphi$ . Let  $C$  be a closed and convex subset of  $E$  and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Suppose that  $\{x_n\}$  is a bounded sequence in  $C$  such that  $x_n - Tx_n \rightarrow 0$  and  $u \in C$ . Then*

$$\limsup_{n \rightarrow \infty} \langle u - p, J_\varphi(x_n - p) \rangle \leq 0,$$

where  $p \in F(T)$  and satisfies the following inequality

$$\langle u - p, J_\varphi(q - p) \rangle \leq 0 \quad \text{for all } q \in F(T).$$

**Proof.** It is proved in [16, Theorems 3.1 and 3.2] that such an element  $p \in F(T)$  does exist. To see that  $\limsup_{n \rightarrow \infty} \langle u - p, J_\varphi(x_n - p) \rangle \leq 0$ , we extract a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightharpoonup q$  and

$$\lim_{k \rightarrow \infty} \langle u - p, J_\varphi(x_{n_k} - p) \rangle = \limsup_{n \rightarrow \infty} \langle u - p, J_\varphi(x_n - p) \rangle.$$

Notice that, since  $J_\varphi$  is weakly sequentially continuous, we have (see [24])

$$\limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - Tq\|) = \limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - q\|) + \Phi(\|q - Tq\|).$$

But we know that

$$\limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - Tq\|) = \limsup_{k \rightarrow \infty} \Phi(\|Tx_{n_k} - Tq\|) \leq \limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - q\|).$$

This implies that  $\Phi(\|q - Tq\|) = 0$  and hence  $q \in F(T)$ . We now conclude that

$$\lim_{k \rightarrow \infty} \langle u - p, J_\varphi(x_{n_k} - p) \rangle = \langle u - p, J_\varphi(q - p) \rangle \leq 0.$$

The proof is finished.  $\square$

Using Lemmas 5, 6 and modifying the proof of Theorem 4, we obtain the following result, the proof of which we omit.

**Theorem 7.** Let  $E$  be a reflexive Banach space having a weakly sequentially continuous duality mapping. Let  $T$  be a strongly nonexpansive mapping from a closed and convex subset  $C$  of  $E$  into itself. Suppose that  $u \in C$  and  $\{x_n\}$  is a sequence generated iteratively by  $x_1 \in C$  and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to a fixed point  $Qu$  of  $T$ , where  $Q$  is a sunny nonexpansive retraction from  $C$  onto  $F(T)$ .

In Hilbert space setting, we have the following result.

**Corollary 8.** Let  $C$  be a closed and convex subset of a Hilbert space  $H$  and let  $T : C \rightarrow C$  be a strongly quasi-nonexpansive mapping, that is,  $T$  has the following properties:

- $F(T) \neq \emptyset$ ;
- $\|Tx - p\| \leq \|x - p\|$  for all  $x \in C$  and  $p \in F(T)$ ;
- $x_n - Tx_n \rightarrow 0$  whenever  $\{x_n\}$  is a bounded sequence in  $C$  such that  $\|x_n - p\| - \|Tx_n - p\| \rightarrow 0$  for some  $p \in F(T)$ .

Suppose in addition that  $I - T$  is demi-closed at zero. Suppose that  $u \in C$  and  $\{x_n\}$  is a sequence generated iteratively by  $x_1 \in C$  and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to a fixed point  $Pu$  of  $T$ , where  $P$  is a metric projection from  $C$  onto  $F(T)$ .

**Proof.** It is easy to see that  $\langle u - Pu, q - Pu \rangle \leq 0$  for all  $q \in F(T)$ . Moreover, if  $\{z_n\}$  is a sequence in  $C$  such that  $z_n - Tz_n \rightarrow 0$  and  $z_n \rightarrow q$ , then  $q \in F(T)$  and hence  $\limsup_{n \rightarrow \infty} \langle u - Pu, z_n - Pu \rangle \leq 0$ . We now follow the proof of Theorem 4 and obtain the result.  $\square$

**Remark 9.** The iteration studied by Maingé [6, Theorem 3.1], when a contraction in the scheme is a constant mapping, is a special case of our result. In fact, the mapping  $T_w = (1 - w)I + wT$ , where  $w \in (0, 1)$  and  $T$  is quasi-nonexpansive, is a strongly quasi-nonexpansive mapping.

### 3.2. Convergence theorem for a family of mappings

Employing the idea in [1], we obtain the following theorem, whose proof is left for the reader to verify. Recall that a sequence of nonexpansive mappings  $\{T_n\}$  from  $C$  into itself is a *strongly nonexpansive sequence* if

$$x_n - y_n - (T_n x_n - T_n y_n) \rightarrow 0$$

whenever  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \rightarrow 0$ .

**Theorem 10.** Let  $C$  be a closed and convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $\{T_n\}_{n=1}^{\infty} : C \rightarrow C$  be a strongly nonexpansive sequence. Suppose that the following conditions are satisfied:

(M1) For each bounded subset  $B$  of  $C$ ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty;$$

(M2) Let  $T : C \rightarrow C$  be defined by  $Tz = \lim_{n \rightarrow \infty} T_n z$  for all  $z \in C$  and suppose that

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

Suppose that  $u \in C$  and  $\{x_n\}$  is a sequence generated iteratively by  $x_1 \in C$  and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_n x_n,$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to  $Qu$ , where  $Q$  is a sunny nonexpansive retraction from  $C$  onto  $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ .

We will apply this theorem for finding zeros of an accretive operator in the next section.

#### 4. Zeros of an accretive operator

Let  $E$  be a Banach space. An operator  $A \subset E \times E$  with the domain  $D(A) = \{x \in E : Ax \neq \emptyset\}$  and range  $R(A) = \bigcup \{Ax : x \in D(A)\}$  is called accretive if for any  $(x_1, y_1), (x_2, y_2) \in A$  there exists  $j \in J(x_1 - x_2)$  such that  $\langle y_1 - y_2, j \rangle \geq 0$ . It is known that  $A$  is accretive if and only if  $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$  for all  $(x_1, y_1), (x_2, y_2) \in A$  and  $r > 0$ . We say that an accretive operator  $A$  satisfies the range condition if the closure of  $D(A)$  is contained in the range of the operator  $I + rA$  for all  $r > 0$ , where  $I$  is the identity operator, that is,  $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$ . Denote by  $J_r$  the resolvent of an accretive operator  $A$ , where  $r > 0$ , which defined as follows:

$$J_r x = x_r \Leftrightarrow x \in x_r + rAx_r.$$

In other words,  $J_r = (I + rA)^{-1}$ . For more details, we refer to [25]. It is known that  $J_r : R(I + rA) \rightarrow D(A)$  is nonexpansive and  $F(J_r) = A^{-1}0 := \{x \in D(A) : 0 \in Ax\}$ .

We prove a strong convergence theorem for the problem of finding zeros of an accretive operator which is an improvement of the recent results in [1,26].

The following observation follows from the fact that firmly nonexpansive mapping is strongly nonexpansive in the presence of uniform convexity (see [22]). Recall that a mapping  $T : C \rightarrow E$ , where  $C$  is a subset of a Banach space  $E$ , is firmly nonexpansive if  $\|Tx - Ty\| \leq \|t(x - y) + (1 - t)(Tx - Ty)\|$  for all  $x, y \in C$  and  $t > 0$ . Note that in the definition above we can consider only  $0 < t \leq 1$ .

**Lemma 11.** Let  $E$  be a uniformly convex Banach space and  $A \subset E \times E$  an accretive operator such that  $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$  for some closed and convex subset  $C$  of  $E$ .

- (1) For each  $r > 0$ , the resolvent  $J_r$  is firmly nonexpansive and hence strongly nonexpansive.
- (2) If  $\{r_n\}$  is a sequence in  $(0, \infty)$ , then  $\{J_{r_n}\}$  is a strongly nonexpansive sequence.
- (3) If  $\{r_n\}$  is a sequence in  $(0, \infty)$  such that  $\inf\{r_n : n \in \mathbb{N}\} > 0$  and  $\sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$ , then the sequence  $\{J_{r_n}\}$  satisfies conditions (M1) and (M2).

**Proof.** (1) Let  $x, y \in C$ . Then  $(J_r x, \frac{1}{r}(x - J_r x)), (J_r y, \frac{1}{r}(y - J_r y)) \in A$ . Since  $A$  is accretive,

$$\begin{aligned} \|J_r x - J_r y\| &\leq \left\| J_r x - J_r y + \frac{s}{r}(x - J_r x - (y - J_r y)) \right\| \\ &= \left\| \frac{s}{r}(x - y) + \left(1 - \frac{s}{r}\right)(J_r x - J_r y) \right\| \end{aligned}$$

for all  $0 < s \leq r$ . This implies that  $J_r$  is firmly nonexpansive.

(2) Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|J_{r_n} x_n - J_{r_n} y_n\| \rightarrow 0$ . Suppose that  $\{n_k\}$  is a subsequence of  $\{n\}$  such that  $\lim_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| = a$ . We show that  $x_{n_k} - y_{n_k} - (J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}) \rightarrow 0$ . We may assume that  $a > 0$ . Otherwise, we are done. Since  $A$  is accretive,

$$\begin{aligned} \|J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}\|^2 &\leq \left\| J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k} + \frac{r_{n_k}}{2} \left( \frac{1}{r_{n_k}}(x_{n_k} - J_{r_{n_k}} x_{n_k}) - \frac{1}{r_{n_k}}(y_{n_k} - J_{r_{n_k}} y_{n_k}) \right) \right\|^2 \\ &= \left\| \frac{1}{2}(J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}) + \frac{1}{2}(x_{n_k} - y_{n_k}) \right\|^2 \\ &\leq \frac{1}{2} \|J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}\|^2 + \frac{1}{2} \|x_{n_k} - y_{n_k}\|^2. \end{aligned}$$

Note that

$$\lim_{k \rightarrow \infty} \|J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}\| = a,$$

and

$$\lim_{k \rightarrow \infty} \left\| \frac{1}{2}(J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}) + \frac{1}{2}(x_{n_k} - y_{n_k}) \right\| = a.$$

It follows by the uniform convexity of  $E$  that

$$x_{n_k} - y_{n_k} - (J_{r_{n_k}} x_{n_k} - J_{r_{n_k}} y_{n_k}) \rightarrow 0.$$

Therefore  $x_n - y_n - (J_{r_n} x_n - J_{r_n} y_n) \rightarrow 0$ . This means that  $\{J_{r_n}\}$  is a strongly nonexpansive sequence.

(3) is proved in [1].  $\square$

As a consequence of the preceding result and [Theorem 10](#), we have the following theorem which is an improvement of [1, [Theorem 4.3](#)].

**Theorem 12.** *Let  $E$  be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let  $C$  be a nonempty, closed convex subset of  $E$ . Let  $A \subset E \times E$  be an accretive operator such that  $A^{-1}0 \neq \emptyset$  and  $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ . Let  $\{\alpha_n\}$  be a sequence of  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Let  $\{r_n\}$  be a sequence of positive real numbers such that  $\inf\{r_n : n \in \mathbb{N}\} > 0$  and  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ . Let  $u \in C$  and  $\{x_n\}$  be a sequence defined by  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n \quad \text{for all } n \in \mathbb{N}.$$

Then  $\{x_n\}$  converges strongly to  $Qu$ , where  $Q$  is the sunny nonexpansive retraction of  $C$  onto  $A^{-1}0$ .

**Remark 13.** The conclusion of [Theorem 12](#) above is proved in [1, [Theorem 4.3](#)] under the superfluous condition  $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$  or  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ .

## 5. Remarks on Song's sufficient condition

Recently, Song [[9,10](#)] gave a new sufficient condition for the strong convergence of sequences generated by Halpern's iteration. We quote his result here.

**Theorem 14** ([\[9, Theorem 3.1\]](#)). *Let  $C$  be a closed and convex subset of a Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Assume that  $\{x_n\}$  is a sequence in  $C$  generated by  $u, x_1 \in C$  and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n \quad \text{for all } n \in \mathbb{N},$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . For a Banach limit  $\mu$ , let

$$K_\mu = \left\{ z \in C : \mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2 \right\}.$$

Suppose that  $K_\mu \cap F(T) \neq \emptyset$  for all Banach limits  $\mu$  and  $x_{n+1} - x_n \rightarrow 0$ . Then  $\{x_n\}$  converges strongly to some fixed point of  $T$ .

Recall that a continuous linear functional  $\mu$  on  $\ell_\infty$  (the Banach space of bounded real sequences) is called a *Banach limit* if  $\|\mu\| = \mu(1, 1, \dots) = 1$  and  $\mu_n(a_n) = \mu_n(a_{n+1})$  for all  $\{a_n\} \in \ell_\infty$ .

The key point of the proof of Song's result is to show that  $\limsup_{n \rightarrow \infty} \langle u - p, J(x_n - p) \rangle \leq 0$  by making use of the following lemma.

**Lemma 15** ([\[8, Proposition 2\]](#)). *Let  $\{a_1, a_2, \dots\} \in \ell^\infty$  be such that  $\mu_n(a_n) \leq 0$  for all Banach limits  $\mu$  and  $\limsup_{n \rightarrow \infty} (a_{n+1} - a_n) \leq 0$ . Then  $\limsup_{n \rightarrow \infty} a_n \leq 0$ .*

Unfortunately, to apply this lemma when  $a_n := \langle u - p, J(x_n - p) \rangle$ , the condition that  $\mu_n \langle u - p, J(x_n - p) \rangle \leq 0$  for all Banach limits  $\mu$  seems to be overlooked to verify.

Later, Song and Chai [[11](#)] used the result above to prove a strong convergence theorem for firmly type nonexpansive mappings. Hence the proofs of their main results ([\[11, Theorems 3.1 and 3.3\]](#)) are also questionable. Recall that for a subset  $C$  of a Banach space  $E$ , a mapping  $T : C \rightarrow C$  is *firmly type nonexpansive* if there exists a positive real number  $k$  such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k\|x - Tx - (y - Ty)\|^2 \quad \text{for all } x, y \in C.$$

It is clear that every firmly type nonexpansive mapping is strongly nonexpansive. Hence it follows from our [Theorem 4](#) that their results remain true.

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## Research Article

# Ray's Theorem for Firmly Nonexpansive-Like Mappings and Equilibrium Problems in Banach Spaces

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We prove that every firmly nonexpansive-like mapping from a closed convex subset  $C$  of a smooth, strictly convex and reflexive Banach space into itself has a fixed point if and only if  $C$  is bounded. We obtain a necessary and sufficient condition for the existence of solutions of an equilibrium problem and of a variational inequality problem defined in a Banach space.

## 1. Introduction

Let  $C$  be a subset of a Banach space  $E$ . A mapping  $T : C \rightarrow E$  is *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . In 1965, it was proved independently by Browder [1], Göhde [2], and Kirk [3] that if  $C$  is a bounded closed convex subset of a Hilbert space and  $T : C \rightarrow C$  is nonexpansive, then  $T$  has a fixed point. Combining the results above, Ray [4] obtained the following interesting result (see [5] for a simpler proof).

**Theorem 1.1.** *Let  $C$  be a closed and convex subset of a Hilbert space. Then the following statements are equivalent:*

- (i)  $\text{Fix}(T) := \{x \in C : x = Tx\} \neq \emptyset$  for every nonexpansive mapping  $T : C \rightarrow C$ ;
- (ii)  $C$  is bounded.

It is well known that, for each subset  $C$  of a Hilbert space  $H$ , a mapping  $T : C \rightarrow H$  is nonexpansive if and only if  $S := (1/2)(I + T)$  is *firmly nonexpansive*, that is, the following

inequality is satisfied by all  $x, y \in C$ :

$$\langle Sx - Sy, (x - Sx) - (y - Sy) \rangle \geq 0. \quad (1.1)$$

In this case,  $\text{Fix}(T) = \text{Fix}(S)$ . We can restate Ray's theorem in the following form.

**Theorem 1.2.** *Let  $C$  be a closed and convex subset of a Hilbert space. Then the following statements are equivalent:*

- (i)  $\text{Fix}(S) \neq \emptyset$  for every firmly nonexpansive mapping  $S : C \rightarrow C$ ;
- (ii)  $C$  is bounded.

To extend this result to the framework of Banach spaces, let us recall some definitions and related known facts. The value of  $x^*$  in the dual space  $E^*$  of a Banach space  $E$  at  $x \in E$  is denoted by  $\langle x, x^* \rangle$ . We assume from now on that a Banach space  $E$  is *smooth*, that is, the limit  $\lim_{t \rightarrow 0} (1/t)(\|x + ty\| - \|x\|)$  exists for all norm one elements  $x, y \in E$ . This implies that the duality mapping  $J$  from  $E$  to  $2^{E^*}$  defined by

$$x \mapsto Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (1.2)$$

is single-valued and we do consider the singleton  $Jx$  as an element in  $E^*$ . If  $E$  is additionally assumed to be *strictly convex*, that is, there are no distinct elements  $x, y \in E$  such that  $\|x\| = \|y\| = (1/2)\|x + y\| = 1$ , then  $J$  is one-to-one. Let us note here that if  $E$  is a Hilbert space, then the duality mapping is just the identity mapping.

The following three generalizations of firmly nonexpansive mappings in Hilbert spaces were introduced by Aoyama et al. [6]. For a subset  $C$  of a (smooth) Banach space  $E$ , a mapping  $T : C \rightarrow E$  is of

- (i) *type (P)* (or *firmly nonexpansive-like*) if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \geq 0 \quad \forall x, y \in C, \quad (1.3)$$

- (ii) *type (Q)* (or *firmly nonexpansive type*) if

$$\langle Tx - Ty, (Jx - JT_x) - (Jy - JT_y) \rangle \geq 0 \quad \forall x, y \in C, \quad (1.4)$$

- (iii) *type (R)* (or *firmly generalized nonexpansive*) if

$$\langle (x - Tx) - (y - Ty), JT_x - JT_y \rangle \geq 0 \quad \forall x, y \in C. \quad (1.5)$$

Recently, Takahashi et al. [7] successfully proved the following theorem.

**Theorem 1.3.** *Let  $C$  be a closed and convex subset of a smooth, strictly convex and reflexive Banach space. Then the following statements are equivalent:*

- (i)  $\text{Fix}(T) \neq \emptyset$  for every mapping  $T : C \rightarrow C$  which is of type (Q);
- (ii)  $C$  is bounded.

As a direct consequence of the duality theorem [8], we obtain the following result (see also [9]).

**Theorem 1.4.** *Let  $C$  be a closed subset of a smooth, strictly convex and reflexive Banach space such that  $J C$  is closed and convex. Then the following statements are equivalent:*

- (i)  $\text{Fix}(T) \neq \emptyset$  for every mapping  $T : C \rightarrow C$  which is of type (R);
- (ii)  $C$  is bounded.

The purpose of this short paper is to prove the analogue of these results for mappings of type (P). Let us note that our result is different from the existence theorems obtained recently by Aoyama and Kohsaka [10]. We also obtain a necessary and sufficient condition for the existence of solutions of certain equilibrium problems and of variational inequality problems in Banach spaces.

## 2. Ray's Theorem for Mappings of Type (P) and Equilibrium Problems

The following result was proved by Aoyama et al. [6].

**Theorem 2.1.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space, and let  $C$  be a bounded, closed and convex subset of  $E$ . If a mapping  $T : C \rightarrow C$  is of type (P), then  $T$  has a fixed point.*

Let  $C$  be a closed and convex subset of a Banach space  $E$ . An *equilibrium problem* for a bifunction  $f : C \times C \rightarrow \mathbb{R}$  is the problem of finding an element  $\hat{x} \in C$  such that

$$f(\hat{x}, y) \geq 0 \quad \forall y \in C. \quad (2.1)$$

We denote the set of solutions of the equilibrium problem for  $f$  by  $\text{EP}(f)$ . We assume that a bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfies the following conditions (see [11]):

- (C1)  $f(x, x) = 0$  for all  $x \in C$ ;
- (C2)  $f(x, y) + f(y, x) \leq 0$  for all  $x, y \in C$ ;
- (C3)  $f(x, \cdot)$  is convex and lower semicontinuous for all  $x \in C$ ;
- (C4)  $f$  is *maximal monotone*, that is, for each  $x \in C$  and  $x^* \in E^*$ ,

$$f(x, y) + \langle y - x, x^* \rangle \geq 0 \quad \forall y \in C \quad (2.2)$$

whenever  $\langle z - x, x^* \rangle \geq f(z, x)$  for all  $z \in C$ .

*Remark 2.2.* It is noted (see [12]) that if  $f$  satisfies conditions (C1)–(C3) and the following condition:

$$(C4') \limsup_{t \downarrow 0} f((1-t)x + tz, y) \leq f(x, y) \text{ for all } x, y, z \in C,$$

then  $f$  satisfies condition (C4).

**Lemma 2.3** (see [12]). *Let  $C$  be a closed and convex subset of a smooth, strictly convex and reflexive Banach space  $E$  and  $f : C \times C \rightarrow \mathbb{R}$  satisfy conditions (C1)–(C4). Then for each  $x \in E$ , there exists a unique element  $z \in C$  such that*

$$f(z, y) + \langle y - z, J(z - x) \rangle \geq 0 \quad \forall y \in C. \quad (2.3)$$

Employing the methods in [5, 7], we obtain the following result.

**Theorem 2.4.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space and  $C$  a closed and convex subset of  $E$ . The following statements are equivalent.*

- (a)  $\text{Fix}(T) \neq \emptyset$  for every mapping  $T : C \rightarrow C$  which is of type (P);
- (b)  $\text{EP}(f) \neq \emptyset$  for every bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfying conditions (C1)–(C4);
- (c)  $\text{EP}(f) \neq \emptyset$  for every bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfying conditions (C1)–(C3) and (C4');
- (d)  $C$  is bounded.

*Proof.* (a) $\Rightarrow$ (b) Assume that a bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfies conditions (C1)–(C4). We define  $T : E \rightarrow C$  by  $x \mapsto Tx = z \in C$  where  $z$  is given by Lemma 2.3. The mapping  $T$  is of type (P). In fact, for  $x, x' \in E$ , we have  $Tx, Tx' \in C$  and hence

$$\begin{aligned} f(Tx, Tx') + \langle Tx' - Tx, J(Tx - x) \rangle &\geq 0, \\ f(Tx', Tx) + \langle Tx - Tx', J(Tx' - x') \rangle &\geq 0. \end{aligned} \quad (2.4)$$

By the condition (C2),

$$\langle Tx - Tx', J(x - Tx) - J(x' - Tx') \rangle \geq 0. \quad (2.5)$$

In particular, the restriction of  $T$  to the closed and convex subset  $C$  is of type (P). It then follows from (a) that  $\text{EP}(f) = \text{Fix}(T) \neq \emptyset$ .

(b) $\Rightarrow$ (c) It follows directly from Remark 2.2.

(c) $\Rightarrow$ (d) We suppose that  $C$  is not bounded. By the uniform boundedness theorem, there exists an element  $x^* \in E^*$  such that  $\inf\{\langle x, x^* \rangle : x \in C\} = -\infty$ . We define  $f : C \times C \rightarrow \mathbb{R}$  by

$$f(x, y) = \langle y - x, x^* \rangle \quad \forall x, y \in C. \quad (2.6)$$

It is clear that  $f$  satisfies conditions (C1)–(C3) and (C4'). Moreover,  $EP(f) = \emptyset$  since

$$\begin{aligned} p \in EP(f) &\iff \langle y - p, x^* \rangle \geq 0 \quad \forall y \in C \\ &\iff -\infty = \inf\{\langle y, x^* \rangle : y \in C\} \geq \langle p, x^* \rangle. \end{aligned} \quad (2.7)$$

(d) $\Rightarrow$ (a) This is Theorem 2.1.  $\square$

Let  $C$  be a subset of a Banach space  $E$ . We now discuss a *variational inequality problem* for a mapping  $A : C \rightarrow E^*$ , that is, the problem of finding an element  $\hat{x} \in C$  such that  $\langle y - \hat{x}, A\hat{x} \rangle \geq 0$  for all  $y \in C$  and the set of solutions of this problem is denoted by  $VI(C, A)$ . Recall that a mapping  $A : C \rightarrow E^*$  is said to be

- (i) *monotone* if  $\langle x - y, Ax - Ay \rangle \geq 0$  for all  $x, y \in C$ ;
- (ii) *hemicontinuous* if for each  $x, y \in C$  the mapping  $t \mapsto A((1-t)x + ty)$ , where  $t \in [0, 1]$ , is continuous with respect to the weak\* topology of  $E^*$ ;
- (iii) *demicontinuous* if  $\{Ax_n\}$  converges to  $Ax$  with respect to the weak\* topology of  $E^*$  whenever  $\{x_n\}$  is a sequence in  $C$  such that it converges strongly to  $x \in C$ .

It is known that if  $C$  is a nonempty weakly compact and convex subset of a reflexive Banach space  $E$  and  $A : C \rightarrow E^*$  is monotone and hemicontinuous, then  $VI(C, A) \neq \emptyset$  (see e.g., [13]).

As a consequence of Theorem 2.4, we obtain a necessary and sufficient condition for the existence of solutions of a variational inequality problem.

**Corollary 2.5.** *Let  $E$  be a reflexive Banach space and  $C$  a nonempty, closed and convex subset of  $E$ . Then the following statements are equivalent:*

- (a)  $VI(C, A) \neq \emptyset$  for every monotone and hemicontinuous mapping  $A : C \rightarrow E^*$ ;
- (b)  $VI(C, A) \neq \emptyset$  for every monotone and demicontinuous mapping  $A : C \rightarrow E^*$ ;
- (c)  $C$  is bounded.

*Proof.* (a) $\Rightarrow$ (b) It is clear since demicontinuity implies hemicontinuity.

(b) $\Rightarrow$ (c) To see this, let us note that there is an equivalent norm on  $E$  such that the underlying space equipped with this new norm is smooth and strictly convex (see [14, 15]). Moreover, the monotonicity and demicontinuity of any mapping  $A : C \rightarrow E^*$  remain unaltered with respect to this renorming. We now assume in addition that  $E$  is smooth and strictly convex. Suppose that  $C$  is not bounded. By Theorem 2.4, there exists a fixed point-free mapping  $T : C \rightarrow C$  such that it is of type (P). We define  $A : C \rightarrow E^*$  by

$$Ax = J(x - Tx) \quad \forall x \in C. \quad (2.8)$$

For each  $x, y \in C$ , we have  $\langle x - y, Ax - Ay \rangle = \langle x - y, J(x - Tx) - J(y - Ty) \rangle \geq 0$ , that is,  $A$  is monotone. Moreover, it is proved in [6, Theorem 7.3] that  $A$  is demicontinuous. Therefore,  $VI(C, A) = \text{Fix}(T) = \emptyset$ .

(c) $\Rightarrow$ (a) It is a corollary of [13, Theorem 7.1.8.]  $\square$

We finally discuss an equilibrium problem defined in the dual space of a Banach space. This problem was initiated by Takahashi and Zembayashi [16]. Let  $C$  be a closed subset of a

smooth, strictly convex and reflexive Banach space  $E$  such that  $JC$  is closed and convex. We assume that a bifunction  $f^* : JC \times JC \rightarrow \mathbb{R}$  satisfies the following conditions:

- (D1)  $f^*(Jx, Jx) = 0$  for all  $x \in C$ ;
- (D2)  $f^*(Jx, Jy) + f^*(Jy, Jx) \leq 0$  for all  $x, y \in C$ ;
- (D3)  $f^*(Jx, \cdot)$  is convex and lower semicontinuous for all  $x \in C$ ;
- (D4)  $f^*$  is *maximal monotone* (with respect to  $JC$ ), that is, for each  $x \in C$  and  $u \in E$ ,

$$f^*(Jx, Jy) + \langle u, Jy - Jx \rangle \geq 0 \quad \forall y \in C \quad (2.9)$$

whenever  $\langle u, Jz - Jx \rangle \geq f^*(Jz, Jx)$  for all  $z \in C$ .

In [16], a bifunction is assumed to satisfy conditions (D1)–(D3) and

$$(D4') \limsup_{t \downarrow 0} f^*((1-t)Jx + tJz, Jy) \leq f^*(Jx, Jy) \text{ for all } x, y, z \in C.$$

We are interested in the problem of finding an element  $\hat{x} \in C$  such that

$$f^*(J\hat{x}, Jy) \geq 0 \quad \forall y \in C \quad (2.10)$$

and the set of solutions of this problem is denoted by  $EP^*(f^*)$ .

The following lemma was proved by Takahashi and Zembayashi ([16], Lemma 2.10) where the bifunction satisfies conditions (D1)–(D3) and (D4'). However, it can be proved that the conclusion remains true under the conditions (D1)–(D4). We also note that the uniform smoothness assumption on a space in [16, Lemma 2.10] is a misprint.

**Lemma 2.6.** *Let  $C$  be a closed subset of a smooth, strictly convex and reflexive Banach space  $E$  such that  $JC$  is closed and convex. Suppose that a bifunction  $f^* : JC \times JC \rightarrow \mathbb{R}$  satisfies conditions (D1)–(D4). Then for each  $x \in E$  there exists a unique element  $z \in C$  such that*

$$f^*(Jz, Jy) + \langle z - x, Jy - Jz \rangle \geq 0 \quad \forall y \in C. \quad (2.11)$$

Moreover, if  $T : E \rightarrow C$  is defined by  $x \mapsto Tx = z$  where  $z$  is given above, then  $T$  is of type (R).

Based on the preceding lemma and Theorem 2.4, we obtain the result whose proof is omitted.

**Theorem 2.7.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space, and let  $C$  be a closed subset of  $E$  such that  $JC$  is closed and convex. The following statements are equivalent:*

- (i)  $\text{Fix}(T) \neq \emptyset$  for every mapping  $T : C \rightarrow C$  which is of type (R);
- (ii)  $EP^*(f^*) \neq \emptyset$  for every bifunction  $f^* : JC \times JC \rightarrow \mathbb{R}$  satisfying conditions (D1)–(D4);
- (iii)  $EP^*(f^*) \neq \emptyset$  for every bifunction  $f^* : JC \times JC \rightarrow \mathbb{R}$  satisfying conditions (D1)–(D3) and (D4');
- (iv)  $C$  is bounded.

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## Research Article

# Strong Convergence Theorems for a Countable Family of Lipschitzian Mappings

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We modify the iterative method introduced by Kim and Xu (2006) for a countable family of Lipschitzian mappings by the hybrid method of Takahashi et al. (2008). Our results include recent ones concerning asymptotically nonexpansive mappings due to Plubtieng and Ungchittrakool (2007) and Zegeye and Shahzad (2008, 2010) as special cases.

## 1. Introduction

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . A mapping  $T : C \rightarrow C$  is said to be *Lipschitzian* if there exists a positive constant  $L$  such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in C. \quad (1.1)$$

In this case,  $T$  is also said to be  $L$ -Lipschitzian. Clearly, if  $T$  is  $L_1$ -Lipschitzian and  $L_1 < L_2$ , then  $T$  is  $L_2$ -Lipschitzian. Throughout the paper, we assume that every Lipschitzian mapping is  $L$ -Lipschitzian with  $L \geq 1$ . If  $L = 1$ , then  $T$  is known as a nonexpansive mapping. We denote by  $F(T)$  the set of fixed points of  $T$ . If  $C$  is nonempty bounded closed convex and  $T$  is a nonexpansive of  $C$  into itself, then  $F(T) \neq \emptyset$  (see [1]). There are many methods for approximating fixed points of a nonexpansive mapping. In 1953, Mann [2] introduced the iteration as follows: a sequence  $\{x_n\}$  defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad (1.2)$$

where the initial guess element  $x_1 \in C$  is arbitrary and  $\{\alpha_n\}$  is a real sequence in  $[0, 1]$ . Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [3]. In an infinite-dimensional Hilbert space, Mann iteration can conclude only weak convergence [4]. Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [5] proposed the following modification of Mann iteration method (1.2):

$$\begin{aligned} x_1 &= x \in C \text{ is arbitrary,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{1.3}$$

where  $P_K$  denotes the metric projection from  $H$  onto a closed convex subset  $K$  of  $H$ . They prove that if the sequence  $\{\alpha_n\}$  bounded above from one, then  $\{x_n\}$  defined by (1.3) converges strongly to  $P_{F(T)}x$ . Takahashi et al. [6] modified (1.3) so-called the *shrinking projection method* for a countable family of nonexpansive mappings  $\{T_n\}_{n=1}^\infty$  as follows:

$$\begin{aligned} x_1 &= x \in H, \\ C_1 &= C, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{1.4}$$

and prove that if the sequence  $\{\alpha_n\}$  bounded above from one, then  $\{x_n\}$  defined by (1.4) converges strongly to  $P_{\bigcap_{n=1}^\infty F(T_n)}x$ .

Recently, the present authors [7] extended (1.3) to obtain a strong convergence theorem for common fixed points of a countable family of  $L_n$ -Lipschitzian mappings  $\{T_n\}_{n=1}^\infty$  by

$$\begin{aligned} x_1 &= x \in C \text{ is arbitrary,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_n &= \left\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\right\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{1.5}$$

where  $\theta_n = (1 - \alpha_n)(L_n^2 - 1)(\text{diam } C)^2 \rightarrow 0$  as  $n \rightarrow \infty$  and prove that  $\{x_n\}$  defined by (1.5) converges strongly to  $P_{\bigcap_{n=1}^\infty F(T_n)}x$ .

In this paper, we establish strong convergence theorems for finding common fixed points of a countable family of Lipschitzian mappings in a real Hilbert space. Moreover, we also apply our results for asymptotically nonexpansive mappings.

## 2. Preliminaries

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Then,

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{2.2}$$

for all  $x, y \in H$  and  $\lambda \in [0, 1]$ . For any  $n$  points  $x_1, x_2, \dots, x_n$  in  $H$ , the following generalized identity holds:

$$\left\| \sum_{i=1}^n \lambda_i x_i \right\|^2 = \sum_{i=1}^n \lambda_i \|x_i\|^2 - \sum_{i < j} \lambda_i \lambda_j \|x_i - x_j\|^2, \tag{2.3}$$

where  $\lambda_i \in [0, 1]$  and  $\sum_{i=1}^n \lambda_i = 1$ .

We write  $x_n \rightarrow x$  ( $x_n \rightharpoonup x$ , resp.) if  $\{x_n\}$  converges strongly (weakly, resp.) to  $x$ . It is also known that  $H$  satisfies:

- (1) the *Opial's condition* [8] that is, for any sequence  $\{x_n\}$  with  $x_n \rightharpoonup x$ ,

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \tag{2.4}$$

holds for every  $y \in H$  with  $y \neq x$

- (2) the *Kadec-Klee property* [9, 10]; that is, for any sequence  $\{x_n\}$  with  $x_n \rightharpoonup x$  and  $\|x_n\| \rightarrow \|x\|$  together imply  $x_n \rightarrow x$ .

Let  $C$  be a nonempty closed convex subset of  $H$ . Then, for any  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C. \tag{2.5}$$

Such a mapping  $P_C$  is called the *metric projection* of  $H$  onto  $C$ . We know that  $P_C$  is nonexpansive. Furthermore, for  $x \in H$  and  $z \in C$ ,

$$z = P_C x \quad \text{iff} \quad \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.6}$$

To deal with a family of mappings, the following conditions are introduced: let  $C$  be a subset of a Banach space, let  $\{T_n\}$  and  $\mathcal{T}$  be families of mappings of  $C$  with  $\bigcap_{n=1}^\infty F(T_n) = F(\mathcal{T}) \neq \emptyset$ , where  $F(\mathcal{T})$  is the set of all common fixed points of all mappings in  $\mathcal{T}$ .  $\{T_n\}$  is said to satisfy

(a) the *AKTT-condition* [11] if for each bounded subset  $B$  of  $C$ ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty, \quad (2.7)$$

(b) the *NST-condition (I) with  $\mathcal{T}$*  [12] if for each bounded sequence  $\{z_n\}$  in  $C$ ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \text{ implies } \lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \quad \forall T \in \mathcal{T}, \quad (2.8)$$

(c) the *NST-condition (II)* [12] if for each bounded sequence  $\{z_n\}$  in  $C$ ,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0 \text{ implies } \lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0 \quad \forall m \in \mathbb{N}, \quad (2.9)$$

(d) *NST\*-condition with  $\mathcal{T}$*  [13] if for each bounded sequence  $\{z_n\}$  in  $C$ ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|z_n - z_{n+1}\| = 0, \quad (2.10)$$

imply  $\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0$  for all  $T \in \mathcal{T}$ .

In particular, if  $\mathcal{T} = \{T\}$ , then we simply say that  $\{T_n\}$  satisfies the *NST-condition (I) with  $T$*  (*NST\*-condition with  $T$ , resp.*) rather than *NST-condition (I) with  $\{T\}$*  (*NST\*-condition with  $\{T\}$ , resp.*).

*Remark 2.1.* It follows directly from the definitions above that

- (i) if  $\{T_n\}$  satisfies the *NST-condition (I) with  $\mathcal{T}$* , then  $\{T_n\}$  satisfies the *NST\*-condition with  $\mathcal{T}$*
- (ii) if  $\{T_n\}$  satisfies the *NST-condition (II)*, then  $\{T_n\}$  satisfies the *NST\*-condition with  $\{T_n\}$* .

**Lemma 2.2** (see [11, Lemma 3.2]). *Let  $C$  be a nonempty closed subset of a Banach space, and let  $\{T_n\}$  be a family of mappings of  $C$  into itself which satisfies the *AKTT-condition*, then the mapping  $T : C \rightarrow C$  defined by*

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in C \quad (2.11)$$

*satisfies*

$$\lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0, \quad (2.12)$$

*for each bounded subset  $B$  of  $C$ .*

From now on, we will write  $(\{T_n\}, T)$  satisfies *AKTT-condition* if  $\{T_n\}$  satisfies *AKTT-condition* and  $T$  is defined by (2.11).

**Lemma 2.3** (see [13, Lemma 2.6]). *Let  $C$  be a nonempty closed subset of a Banach space. Suppose that  $(\{T_n\}, T)$  satisfies AKTT-condition and  $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ . Then,  $\{T_n\}$  satisfies the NST-condition (I) with  $T$ . Consequently,  $\{T_n\}$  satisfies the NST\*-condition with  $T$ .*

*Remark 2.4.* There are families of mappings  $\{T_n\}$  and  $\mathcal{T}$  such that

- (1)  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ , and
- (2)  $\{T_n\}$  fails the NST-condition (I) with  $\mathcal{T}$  and the NST-condition (II).

The following example shows that the NST\*-condition with  $\mathcal{T}$  is strictly weaker than NST-condition (I) with  $\mathcal{T}$  and the NST-condition (II).

*Example 2.5* (see [13, Example 2.9]). Let  $H := \mathbb{R}^2$  and  $C := [0, 1] \times [0, 1]$ . Define  $T_1, T_2 : C \rightarrow C$  as follows:

$$T_1(x, y) = (x, 1 - y), \quad T_2(x, y) = (1 - x, y), \tag{2.13}$$

for all  $(x, y) \in C$ . Hence,  $T_1$  and  $T_2$  are nonexpansive mappings with

$$F(T_1) \cap F(T_2) = \left( [0, 1] \times \left\{ \frac{1}{2} \right\} \right) \cap \left( \left\{ \frac{1}{2} \right\} \times [0, 1] \right) = \left\{ \left( \frac{1}{2}, \frac{1}{2} \right) \right\} \neq \emptyset. \tag{2.14}$$

Let  $T_n = T_{n-1(\bmod 2)+1}$ . Then,  $\{T_n\}$  satisfies NST\*-condition but it fails NST-condition (I) with  $\mathcal{T}$  and the NST-condition (II).

**Lemma 2.6.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\{T_n\}$  and  $\{S_n\}$  be two families of  $t_n$ -Lipschitzian and  $s_n$ -Lipschitzian mappings of  $C$  into itself, respectively. Let  $\{U_n\}$  be a family of mappings of  $C$  into itself defined by*

$$U_n = T_n(\beta_n I + (1 - \beta_n)S_n) \quad \forall n \in \mathbb{N}, \tag{2.15}$$

where  $\{\beta_n\}$  is a sequence in  $[a, b]$  for some  $a, b \in (0, 1)$  and  $I$  is an identity mapping. Assume that  $\{t_n\}$  and  $\{s_n\}$  are two sequences such that  $t_n \rightarrow 1$  and  $s_n \rightarrow 1$ . Then, the following statements hold.

- (i)  $\{U_n\}$  is a family of  $L_n$ -Lipschitzian mappings of  $C$  into itself, where  $L_n = (\beta_n t_n^2 + (1 - \beta_n) t_n^2 s_n^2)^{1/2}$  and  $L_n \rightarrow 1$ .
- (ii) Suppose that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are families of mappings of  $C$  into itself such that  $F(\mathcal{T}_1) = \bigcap_{n=1}^{\infty} F(T_n)$ ,  $F(\mathcal{T}_2) = \bigcap_{n=1}^{\infty} F(S_n)$  and  $F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \neq \emptyset$ . If  $\{T_n\}$  and  $\{S_n\}$  satisfy the NST\*-condition with  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , respectively, then  $\{U_n\}$  satisfies the NST\*-condition with  $\mathcal{T}_1 \cup \mathcal{T}_2$  and

$$\bigcap_{n=1}^{\infty} F(U_n) = F(\mathcal{T}_1) \cap F(\mathcal{T}_2). \tag{2.16}$$

*Proof.* (i) We first observe that

$$\begin{aligned}
\|U_n x - U_n y\|^2 &\leq t_n^2 \|\beta_n(x - y) + (1 - \beta_n)(S_n x - S_n y)\|^2 \\
&\leq t_n^2 \left( \beta_n \|x - y\|^2 + (1 - \beta_n) \|S_n x - S_n y\|^2 \right) \\
&\leq \beta_n t_n^2 \|x - y\|^2 + (1 - \beta_n) t_n^2 s_n^2 \|x - y\|^2 \\
&= L_n^2 \|x - y\|^2,
\end{aligned} \tag{2.17}$$

for all  $x, y \in C$ . That is,  $U_n$  is  $L_n$ -Lipschitzian. Since  $t_n \rightarrow 1$  and  $s_n \rightarrow 1$ , it follows that  $L_n \rightarrow 1$ .

(ii) Let  $\{z_n\}$  be a bounded sequence in  $C$  such that  $\lim_{n \rightarrow \infty} \|z_n - U_n z_n\| = \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0$ . Let  $p \in F(\mathcal{T}_1) \cap F(\mathcal{T}_2)$ , and let  $M = \sup\{\|z_n - U_n z_n\|, \|z_n - p\| : n \in \mathbb{N}\}$ . Then

$$\begin{aligned}
\|z_n - p\|^2 &\leq (\|z_n - U_n z_n\| + \|U_n z_n - p\|)^2 \\
&= \|z_n - U_n z_n\|^2 + 2\|z_n - U_n z_n\| \|U_n z_n - p\| + \|U_n z_n - p\|^2 \\
&\leq 3M\|z_n - U_n z_n\| + \|T_n(\beta_n z_n + (1 - \beta_n)S_n z_n) - T_n p\|^2 \\
&\leq 3M\|z_n - U_n z_n\| + t_n^2 \|\beta_n(z_n - p) + (1 - \beta_n)(S_n z_n - p)\|^2 \\
&= 3M\|z_n - U_n z_n\| + \beta_n t_n^2 \|z_n - p\|^2 + (1 - \beta_n) t_n^2 \|S_n z_n - p\|^2 \\
&\quad - \beta_n(1 - \beta_n) t_n^2 \|z_n - S_n z_n\|^2 \\
&\leq 3M\|z_n - U_n z_n\| + \beta_n t_n^2 \|z_n - p\|^2 + (1 - \beta_n) t_n^2 s_n^2 \|z_n - p\|^2 \\
&\quad - a(1 - b) \|z_n - S_n z_n\|^2 \\
&= 3M\|z_n - U_n z_n\| + L_n^2 \|z_n - p\|^2 - a(1 - b) \|z_n - S_n z_n\|^2,
\end{aligned} \tag{2.18}$$

for all  $n \in \mathbb{N}$ . In particular,

$$a(1 - b) \|z_n - S_n z_n\|^2 \leq 3M\|z_n - U_n z_n\| + (L_n^2 - 1) \|z_n - p\|^2. \tag{2.19}$$

So, we get

$$\lim_{n \rightarrow \infty} \|z_n - S_n z_n\| = 0. \tag{2.20}$$

Since  $\{S_n\}$  satisfies the NST\*-condition with  $\mathcal{T}_2$ , we have

$$\lim_{n \rightarrow \infty} \|z_n - S_n z_n\| = 0 \quad \forall S \in \mathcal{T}_2. \tag{2.21}$$

Since

$$\begin{aligned} \|z_n - T_n z_n\| &\leq \|z_n - U_n z_n\| + \|U_n z_n - T_n z_n\| \\ &\leq \|z_n - U_n z_n\| + (1 - \beta_n) t_n \|z_n - S_n z_n\|, \end{aligned} \tag{2.22}$$

it follows that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0. \tag{2.23}$$

Since  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}_1$ , we have

$$\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \quad \forall T \in \mathcal{T}_1. \tag{2.24}$$

It is easy to see that  $F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \subset \bigcap_{n=1}^{\infty} F(U_n)$ . To see the reverse inclusion, let  $z \in \bigcap_{n=1}^{\infty} F(U_n)$  follow the first part of the proof above but now let  $z_n \equiv z$ . Then,  $z \in F(\mathcal{T}_1) \cap F(\mathcal{T}_2) = F(\mathcal{T}_1 \cup \mathcal{T}_2)$ . Hence,  $\{U_n\}$  satisfies the NST\*-condition with  $\mathcal{T}_1 \cup \mathcal{T}_2$ .  $\square$

**Lemma 2.7.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\{T_{ni}\}_{n=1}^{\infty}$  be families of  $L_{ni}$ -Lipschitzian mappings of  $C$  into itself for  $i = 1, 2, \dots, r$ , respectively. Let  $\{T_n\}$  be a family of mappings of  $C$  into itself defined by*

$$T_n = \sum_{i=1}^r \beta_{ni} T_{ni} \quad \forall n \in \mathbb{N}, \tag{2.25}$$

where  $\{\beta_{ni}\}_{n=1}^{\infty}$  are sequences in  $[a, b]$  for some  $a, b \in (0, 1)$  satisfying  $\sum_{i=1}^r \beta_{ni} = 1$  for all  $n \in \mathbb{N}$ . Assume that  $\{L_{ni}\}_{n=1}^{\infty}$  are sequences such that  $L_{ni} \rightarrow 1$  as  $n \rightarrow \infty$  for all  $i = 1, 2, \dots, r$ . Then, the following statements hold.

- (i)  $\{T_n\}$  is a family of  $L_n$ -Lipschitzian mappings of  $C$  into itself, where  $L_n = (\sum_{i=1}^r \beta_{ni} L_{ni}^2)^{1/2}$  and  $L_n \rightarrow 1$ .
- (ii) Suppose that  $\mathcal{T}_i$  are families of mappings of  $C$  into itself such that  $F(\mathcal{T}_i) = \bigcap_{n=1}^{\infty} F(T_{ni})$  for  $i = 1, 2, \dots, r$  and  $\bigcap_{i=1}^r F(\mathcal{T}_i) \neq \emptyset$ . If  $\{T_{ni}\}$  satisfies the NST\*-condition with  $\mathcal{T}_i$  for all  $i = 1, 2, \dots, r$ , then  $\{T_n\}$  satisfies the NST\*-condition with  $\bigcup_{i=1}^r \mathcal{T}_i$  and

$$\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{i=1}^r F(\mathcal{T}_i). \tag{2.26}$$

*Proof.* (i) From (2.3), we have

$$\begin{aligned}
 \|T_n x - T_n y\|^2 &= \left\| \sum_{i=1}^r \beta_{ni} (T_{ni} x - T_{ni} y) \right\|^2 \\
 &\leq \sum_{i=1}^r \beta_{ni} \|T_{ni} x - T_{ni} y\|^2 \\
 &\leq \left( \sum_{i=1}^r \beta_{ni} L_{ni}^2 \right) \|x - y\|^2 \\
 &= L_n^2 \|x - y\|^2,
 \end{aligned} \tag{2.27}$$

for all  $x, y \in C$ . That is,  $T_n$  is  $L_n$ -Lipschitzian. Since  $L_{ni} \rightarrow 1$  for  $i = 1, 2, \dots, r$  and  $\sum_{i=1}^r \beta_{ni} = 1$ , it follows that  $L_n \rightarrow 1$ .

(ii) Let  $\{z_n\}$  be a bounded sequence in  $C$  such that  $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0$ . Let  $p \in \bigcap_{i=1}^r F(\mathcal{T}_i)$ , and let  $M = \sup\{\|z_n - T_n z_n\|, \|z_n - p\| : n \in \mathbb{N}\}$ ; it follows from (2.3) that

$$\begin{aligned}
 \|z_n - p\|^2 &\leq (\|z_n - T_n z_n\| + \|T_n z_n - p\|)^2 \\
 &= \|z_n - T_n z_n\|^2 + 2\|z_n - T_n z_n\| \|T_n z_n - p\| + \|T_n z_n - p\|^2 \\
 &\leq 3M\|z_n - T_n z_n\| + \left\| \sum_{i=1}^r \beta_{ni} (T_{ni} z_n - p) \right\|^2 \\
 &= 3M\|z_n - T_n z_n\| + \sum_{i=1}^r \beta_{ni} \|T_{ni} z_n - p\|^2 - \sum_{i < j} \beta_{ni} \beta_{nj} \|T_{ni} z_n - T_{nj} z_n\|^2 \\
 &\leq 3M\|z_n - T_n z_n\| + \sum_{i=1}^r \beta_{ni} L_{ni}^2 \|z_n - p\|^2 - a^2 \sum_{i < j} \|T_{ni} z_n - T_{nj} z_n\|^2 \\
 &= 3M\|z_n - T_n z_n\| + L_n^2 \|z_n - p\|^2 - a^2 \sum_{i < j} \|T_{ni} z_n - T_{nj} z_n\|^2.
 \end{aligned} \tag{2.28}$$

So, by (i), we get

$$\lim_{n \rightarrow \infty} \|T_{ni} z_n - T_{nj} z_n\| = 0 \quad \forall i, j \in \{1, 2, \dots, r\}. \tag{2.29}$$

For each  $k = 1, 2, \dots, r$ , we have

$$\begin{aligned}
 \|z_n - T_{nk}z_n\| &\leq \|z_n - T_n z_n\| + \|T_n z_n - T_{nk}z_n\| \\
 &= \|z_n - T_n z_n\| + \left\| \sum_{i=1}^r \beta_{ni} (T_{ni}z_n - T_{nk}z_n) \right\| \\
 &\leq \|z_n - T_n z_n\| + \sum_{i=1}^r \beta_{ni} \|T_{ni}z_n - T_{nk}z_n\| \longrightarrow 0.
 \end{aligned} \tag{2.30}$$

Since each family  $\{T_{nk}\}_{n=1}^\infty$  satisfies the NST\*-condition with  $\mathcal{T}_k$ ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \quad \forall T \in \bigcup_{i=1}^r \mathcal{T}_i. \tag{2.31}$$

It is easy to see that  $\bigcap_{i=1}^r F(\mathcal{T}_i) \subset \bigcap_{n=1}^\infty F(T_n)$ . To see the reverse inclusion, let  $z \in \bigcap_{n=1}^\infty F(T_n)$ . Follow the first part of the proof above but now let  $z_n \equiv z$ . Then,  $z \in \bigcap_{i=1}^r F(\mathcal{T}_i) = F(\bigcup_{i=1}^r \mathcal{T}_i)$ . Hence,  $\{T_n\}$  satisfies the NST\*-condition with  $\bigcup_{i=1}^r \mathcal{T}_i$ .  $\square$

**Lemma 2.8.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ , and let  $\{T_n\}$  be a family of  $L_n$ -Lipschitzian mappings of  $C$  into itself with  $L_n \rightarrow 1$  and  $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ . If  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ , where  $\mathcal{T}$  is a family of mappings of  $C$  into itself such that  $F(\mathcal{T}) = \bigcap_{n=1}^\infty F(T_n)$ , then  $F(\mathcal{T})$  is closed and convex.*

*Proof.* It follows from the continuity of  $T_n$  that  $F(T_n)$  is closed and so is  $F(\mathcal{T}) = \bigcap_{n=1}^\infty F(T_n)$ . Now, we prove that  $F(\mathcal{T})$  is convex. To this end, let  $x, y \in F(\mathcal{T})$ . Put  $z = tx + (1-t)y$ , where  $t \in (0, 1)$ . From (2.2), we have

$$\begin{aligned}
 \|z - T_n z\|^2 &= t\|x - T_n z\|^2 + (1-t)\|y - T_n z\|^2 - t(1-t)\|x - y\|^2 \\
 &\leq tL_n^2\|x - z\|^2 + (1-t)L_n^2\|y - z\|^2 - t(1-t)\|x - y\|^2 \\
 &= t(1-t)(L_n^2 - 1)\|x - y\|^2.
 \end{aligned} \tag{2.32}$$

So, we get

$$\lim_{n \rightarrow \infty} \|z - T_n z\| = 0. \tag{2.33}$$

Since  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ , we have  $\|z - Tz\| = 0$  for all  $T \in \mathcal{T}$ . Then,  $z \in F(\mathcal{T})$  and so  $F(\mathcal{T})$  is convex.  $\square$

*Remark 2.9.* The conclusions of Lemmas 2.6, 2.7, and 2.8 remain true if we replace a Hilbert space with a uniformly convex Banach space. Recall a Banach space  $X$  is *uniformly convex* if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \varepsilon$  imply  $\|(x + y)/2\| \leq 1 - \delta$ .

### 3. Main Results

In this section, using the method introduced by Takahashi et al. [6], we obtain a strong convergence theorem for a countable family of Lipschitzian mappings.

Recall that a mapping  $T : C \rightarrow C$  is *closed* (*demiclosed*, resp.) at  $y$  if whenever  $\{x_n\}$  is a sequence in  $C$  satisfying  $x_n \rightarrow x$  ( $x_n \rightharpoonup x$ , resp.) and  $Tx_n \rightarrow y$ , then  $x \in C$  and  $Tx = y$ .

**Theorem 3.1.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\{T_n\}$  be a family of  $L_n$ -Lipschitzian mappings of  $C$  into itself with a common fixed point. Assume that  $\{\alpha_n\}$  is a sequence in  $[0, b]$  for some  $b \in (0, 1)$ . For  $x_1 = x \in H$  and  $C_1 = C$ , one defines a sequence  $\{x_n\}$  of  $C$  as follows:*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} &= \left\{ z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \\ x_{n+1} &= P_{C_{n+1}} x, \quad n = 1, 2, 3, \dots, \end{aligned} \quad (3.1)$$

where

$$\theta_n = (1 - \alpha_n) (L_n^2 - 1) (\text{diam } C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (3.2)$$

Suppose that  $\mathcal{T}$  is a family of mappings of  $C$  into itself such that  $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$  and  $I - T$  is closed at 0 for all  $T \in \mathcal{T}$ . If  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ , then  $\{x_n\}$  converges strongly to  $P_{F(\mathcal{T})} x$ .

*Proof.* By Lemma 2.8, we have  $F(\mathcal{T})$  is closed and convex. We now prove that  $C_n$  is closed and convex for each  $n \in \mathbb{N}$  by induction. It is obvious that  $C_1 = C$  is closed and convex. Assume that  $C_k$  is closed and convex for some  $k \in \mathbb{N}$ . For  $z \in C_k$ , we know that

$$\|y_k - z\|^2 \leq \|x_k - z\|^2 + \theta_k \quad (3.3)$$

is equivalent to

$$2\langle x_k - y_k, z \rangle \leq \|x_k\|^2 - \|y_k\|^2 + \theta_k. \quad (3.4)$$

It follows that  $C_{k+1}$  is closed and convex. Next, we show that

$$F(\mathcal{T}) \subset C_n \quad \forall n \in \mathbb{N}. \quad (3.5)$$

It is clear that  $F(\mathcal{T}) \subset C_1 = C$ . Suppose that  $F(\mathcal{T}) \subset C_k$  for some  $k \in \mathbb{N}$ . Then, for  $p \in F(\mathcal{T})$ ,

$$\begin{aligned} \|y_k - p\|^2 &= \|\alpha_k x_k + (1 - \alpha_k) T_k x_k - p\|^2 \\ &\leq \alpha_k \|x_k - p\|^2 + (1 - \alpha_k) \|T_k x_k - p\|^2 \\ &\leq \alpha_k \|x_k - p\|^2 + (1 - \alpha_k) L_k^2 \|x_k - p\|^2 \\ &= \|x_k - p\|^2 + (1 - \alpha_k) (L_k^2 - 1) \|x_k - p\|^2 \\ &\leq \|x_k - p\|^2 + \theta_k, \end{aligned} \tag{3.6}$$

we have  $p \in C_{k+1}$ . Therefore, we obtain (3.5). Now, the sequence  $\{x_n\}$  is well defined. As  $x_n = P_{C_n} x$ ,

$$\|x_n - x\| \leq \|z - x\| \quad \forall z \in C_n, \forall n \in \mathbb{N}. \tag{3.7}$$

In particular, since  $F(\mathcal{T}) \subset C_n$ ,

$$\|x_n - x\| \leq \|p - x\| \quad \forall p \in F(\mathcal{T}), \forall n \in \mathbb{N}. \tag{3.8}$$

On the other hand, from  $x_n = P_{C_n} x$  and  $x_{n+1} \in C_{n+1} \subset C_n$ , we have

$$\|x_n - x\| \leq \|x_{n+1} - x\| \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Therefore,  $\{\|x_n - x\|\}$  is nondecreasing and bounded. So,

$$\lim_{n \rightarrow \infty} \|x_n - x\| \text{ exists.} \tag{3.10}$$

Noticing again that  $x_n = P_{C_n} x$  and for any positive integer  $k$ ,  $x_{n+k} \in C_{n+k-1} \subset C_n$ , we have

$$\langle x_n - x_{n+k}, x - x_n \rangle \geq 0. \tag{3.11}$$

It follows from (2.1) that

$$\begin{aligned} \|x_{n+k} - x_n\|^2 &= \|(x_{n+k} - x) - (x_n - x)\|^2 \\ &= \|x_{n+k} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+k} - x, x_n - x \rangle \\ &\leq \|x_{n+k} - x\|^2 - \|x_n - x\|^2. \end{aligned} \tag{3.12}$$

It then follows from the existence of  $\lim_{n \rightarrow \infty} \|x_n - x\|^2$  that  $\{x_n\}$  is a Cauchy sequence. Moreover,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

We now assume that  $x_n \rightarrow w$  for some  $w \in C$ . Now, since  $x_{n+1} \in C_{n+1}$  and  $C_{n+1} \subset C_n$ ,  $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n$  which implies that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\theta_n} \rightarrow 0. \quad (3.14)$$

From  $\alpha_n \leq b < 1$ , we get

$$\begin{aligned} \|x_n - T_n x_n\| &= \frac{1}{1 - \alpha_n} \|y_n - x_n\| \\ &\leq \frac{1}{1 - b} (\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\|) \rightarrow 0. \end{aligned} \quad (3.15)$$

Since  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ , we have

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0 \quad \forall T \in \mathcal{T}. \quad (3.16)$$

Since  $I - T$  is closed at 0 for all  $T \in \mathcal{T}$ , we have  $(I - T)w = 0$ . This implies that  $w \in F(\mathcal{T})$ . Furthermore, by (3.8),

$$\|w - x\| = \lim_{n \rightarrow \infty} \|x_n - x\| \leq \|p - x\| \quad \forall p \in F(\mathcal{T}). \quad (3.17)$$

Hence,  $w = P_{F(\mathcal{T})}x$ . This completes the proof.  $\square$

**Lemma 3.2** (see [9, Theorem 10.4]). *Let  $C$  be a nonempty closed convex subset of a real Hilbert space, and let  $T : C \rightarrow C$  be a nonexpansive mapping. Then,  $I - T$  is demiclosed at 0.*

It is not difficult to see from the proof of Theorem 3.1 that the boundedness of  $C$  can be discarded if  $\{T_n\}$  is a family of nonexpansive mappings. So, we immediately obtain the following theorem.

**Theorem 3.3.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\{T_n\}$  and  $\mathcal{T}$  be two families of nonexpansive mappings of  $C$  into itself such that  $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$  and suppose that  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ . Assume that  $\{\alpha_n\}$  is a sequence in  $[0, b]$  for some  $b \in (0, 1)$ . Then, the sequence  $\{x_n\}$  in  $C$  defined by (1.4) converges strongly to  $P_{F(\mathcal{T})}x$ .*

*Remark 3.4.* Theorem 3.3 includes [6, Theorem 3.3] as a special case since the NST-condition (I) with  $\mathcal{T}$  implies the NST\*-condition with  $\mathcal{T}$ .

**Theorem 3.5.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\{T_n\}$  be a family of  $L_n$ -Lipschitzian mappings of  $C$  into itself with a common fixed point. Suppose that  $\mathcal{T}$  is a family of mappings from  $C$  into itself such that  $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$  and  $I - T$  is demiclosed at 0 for all  $T \in \mathcal{T}$ . Assume that  $\{\alpha_n\}$  is a sequence in  $[0, b]$  for some  $b \in (0, 1)$ . If  $\{T_n\}$  satisfies the NST\*-condition with  $\mathcal{T}$ , then the sequence  $\{x_n\}$  in  $C$  defined by (1.5) converges strongly to  $P_{F(\mathcal{T})}x$ .*

*Proof.* The proof is analogous to the proof of [7, Theorem 10] and Theorem 3.1, so it is omitted.  $\square$

#### 4. Deduced Results

Let  $C$  be a subset of a real Hilbert space  $H$ . A mapping  $T : C \rightarrow C$  is said to be an *asymptotically nonexpansive* if there exists a sequence  $\{k_n\}$  of real numbers such that  $k_n \in [1, \infty)$ ,  $k_n \rightarrow 1$ , and

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \forall x, y \in C, n \in \mathbb{N}. \quad (4.1)$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] as an important generalization of the class of nonexpansive mappings. They proved that if  $C$  is nonempty bounded closed convex and  $T$  is an asymptotically nonexpansive self-mapping of  $C$ , then  $T$  has a fixed point.

In this section, we use the NST\*-condition to obtain recent results proved by Kim and Xu [15], Plubtieng and Ungchittrakool [16], and Zegeye and Shahzad [17, 18]. We start with the following auxiliary result.

**Lemma 4.1.** *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$ , and let  $T$  be an asymptotically nonexpansive mappings of  $C$  into itself with a sequence  $\{k_n\}$  in  $[1, \infty)$  satisfying  $k_n \rightarrow 1$  and  $F(T) \neq \emptyset$ . Then,  $\{T^n\}$  is a family of  $k_n$ -Lipschitzian mappings of  $C$  into itself and satisfies the NST\*-condition with  $T$ .*

*Proof.* We note that  $\{T^n\}$  is a family of  $k_n$ -Lipschitzian mappings of  $C$  into itself. Let  $\{z_n\}$  be a bounded sequence in  $C$  such that

$$\lim_{n \rightarrow \infty} \|z_n - T^n z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0. \quad (4.2)$$

Since

$$\begin{aligned} \|z_{n+1} - T z_{n+1}\| &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + \|T^{n+1} z_{n+1} - T z_{n+1}\| \\ &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + k_1 \|T^n z_{n+1} - z_{n+1}\| \\ &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + k_1 (\|T^n z_{n+1} - T^n z_n\| + \|T^n z_n - z_n\| + \|z_{n+1} - z_n\|) \\ &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + k_1 (k_n + 1) \|z_{n+1} - z_n\| + k_1 \|T^n z_n - z_n\|, \end{aligned} \quad (4.3)$$

it follows that

$$\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0. \quad (4.4)$$

It is easy to see that  $F(T) \subset \bigcap_{n=1}^{\infty} F(T^n)$ . To see the reverse inclusion, let  $z \in \bigcap_{n=1}^{\infty} F(T^n)$  following from the first part of the proof above, but now let  $z_n \equiv z$ . Then,  $z \in \bigcap_{n=1}^{\infty} F(T^n)$ , and hence  $\bigcap_{n=1}^{\infty} F(T^n) \subset F(T)$ . This implies that  $\{T^n\}$  satisfies the NST\*-condition with  $T$ .  $\square$

**Lemma 4.2** (see [19]). *Let  $C$  be a nonempty bounded closed convex subset of a Hilbert space  $H$ , and let  $T$  be an asymptotically nonexpansive mappings of  $C$  into itself. Then,  $I - T$  is demiclosed at 0.*

Using Theorem 3.1 and Lemmas 2.6 and 4.1, we have the following result.

**Theorem 4.3.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ , and let  $S, T$  be two asymptotically nonexpansive mappings of  $C$  into itself with sequences  $\{s_n\}$  and  $\{t_n\}$ , respectively, and  $F(S) \cap F(T) \neq \emptyset$ . Assume that  $\{\alpha_n\}$  is a sequence in  $[0, b]$  and  $\{\beta_n\}$  is a sequence in  $[a, b]$  for some  $a, b \in (0, 1)$ . For  $x_1 = x \in H$  and  $C_1 = C$ , one defines a sequence  $\{x_n\}$  of  $C$  as follows:*

$$\begin{aligned} z_n &= \beta_n x_n + (1 - \beta_n) S^n x_n, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\ C_{n+1} &= \left\{ z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \\ x_{n+1} &= P_{C_{n+1}} x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{4.5}$$

where

$$\theta_n = (1 - \alpha_n) \left( (t_n^2 - 1) + (1 - \beta_n) t_n^2 (s_n^2 - 1) \right) (\text{diam } C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.6}$$

Then,  $\{x_n\}$  converges strongly to  $P_{F(S) \cap F(T)} x$ .

Using Theorem 3.5 and Lemmas 2.6 and 4.1, we have the following result.

**Theorem 4.4** (see [16, Theorem 3.1]). *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ , and let  $S, T$  be two asymptotically nonexpansive mappings of  $C$  into itself with sequences  $\{s_n\}$  and  $\{t_n\}$ , respectively, and  $F(S) \cap F(T) \neq \emptyset$ . Assume that  $\{\alpha_n\}$  is a sequence in  $[0, b]$  and  $\{\beta_n\}$  is a sequence in  $[a, b]$  for some  $a, b \in (0, 1)$ . For  $x_1 = x \in C$ , one defines a sequence  $\{x_n\}$  of  $C$  as follows:*

$$\begin{aligned} z_n &= \beta_n x_n + (1 - \beta_n) S^n x_n, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\ C_n &= \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \\ Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{4.7}$$

where

$$\theta_n = (1 - \alpha_n) \left( (t_n^2 - 1) + (1 - \beta_n) t_n^2 (s_n^2 - 1) \right) (\text{diam } C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.8}$$

Then,  $\{x_n\}$  converges strongly to  $P_{F(S) \cap F(T)} x$ .

Using Theorem 3.1 and Lemmas 2.7 and 4.1, we have the following result.

**Theorem 4.5.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\{T_i\}_{i=1}^r$  be a finite family of asymptotically nonexpansive mappings of  $C$  into itself with sequences  $\{k_{ni}\}$  for  $i = 1, 2, \dots, r$ , respectively, and suppose that  $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ . Assume that  $\{\alpha_{ni}\}_{n=1}^\infty$  are sequences in  $[0, 1)$  such that  $\alpha_{n0} \leq b < 1$ ,  $\alpha_{ni} \geq a > 0$  for some  $a, b \in (0, 1)$  and  $\sum_{i=0}^r \alpha_{ni} = 1$  for all  $n \in \mathbb{N}$ . For  $x_1 = x \in H$  and  $C_1 = C$ , one defines a sequence  $\{x_n\}$  of  $C$  as follows:*

$$\begin{aligned} y_n &= \alpha_{n0}x_n + \alpha_{n1}T_1^n x_n + \alpha_{n2}T_2^n x_n + \dots + \alpha_{nr}T_r^n x_n, \\ C_{n+1} &= \left\{ z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \\ x_{n+1} &= P_{C_{n+1}}x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{4.9}$$

where

$$\theta_n = \left( \alpha_{n1}(k_{n1}^2 - 1) + \alpha_{n2}(k_{n2}^2 - 1) + \dots + \alpha_{nr}(k_{nr}^2 - 1) \right) (\text{diam } C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.10}$$

Then,  $\{x_n\}$  converges strongly to  $P_{\bigcap_{i=1}^r F(T_i)}x$ .

Using Theorem 3.5 and Lemmas 2.7 and 4.1, we have the following two results which were proved by Zegeye and Shahzad [17, 18].

**Theorem 4.6.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\{T_i\}_{i=1}^r$  be a finite family of asymptotically nonexpansive mappings of  $C$  with sequences  $\{k_{ni}\}$  for  $i = 1, 2, \dots, r$ , respectively, and suppose that  $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ . Assume that  $\{\alpha_{ni}\}_{n=1}^\infty$  are sequences in  $[0, 1)$  such that  $\alpha_{n0} \leq b < 1$ ,  $\alpha_{ni} \geq a > 0$  for some  $a, b \in (0, 1)$  and  $\sum_{i=0}^r \alpha_{ni} = 1$  for all  $n \in \mathbb{N}$ . For  $x_1 = x \in C$ , one defines a sequence  $\{x_n\}$  of  $C$  as follows:*

$$\begin{aligned} y_n &= \alpha_{n0}x_n + \alpha_{n1}T_1^n x_n + \alpha_{n2}T_2^n x_n + \dots + \alpha_{nr}T_r^n x_n, \\ C_n &= \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \\ Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\ x_{n+1} &= P_{C_n \cap Q_n}x, \quad n = 1, 2, 3, \dots, \end{aligned} \tag{4.11}$$

where

$$\theta_n = \left( \alpha_{n1}(k_{n1}^2 - 1) + \alpha_{n2}(k_{n2}^2 - 1) + \dots + \alpha_{nr}(k_{nr}^2 - 1) \right) (\text{diam } C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.12}$$

Then,  $\{x_n\}$  converges strongly to  $P_{\bigcap_{i=1}^r F(T_i)}x$ .

**Theorem 4.7.** Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\mathfrak{T}_i = \{T_i(t) : t \in \mathbb{R}^+, i = 1, 2, \dots, r\}$  be a finite family of asymptotically nonexpansive semigroups such that  $F = \bigcap_{i=1}^r F(\mathfrak{T}_i) \neq \emptyset$ . Assume that  $\{\alpha_{ni}\}_{n=1}^{\infty}$  are sequences in  $[0, 1)$  such that

$$\alpha_{n0} \leq b < 1, \quad \alpha_{ni} \geq a > 0, \quad (4.13)$$

for some  $a, b \in (0, 1)$  and  $\sum_{i=0}^r \alpha_{ni} = 1$  for all  $n \in \mathbb{N}$ . Let  $\{t_{ni}, i = 1, 2, \dots, r\}$  be finite positive and divergent real sequences. For  $x_1 = x \in C$ , one defines a sequence  $\{x_n\}$  of  $C$  as follows:

$$\begin{aligned} y_n &= \alpha_{n0}x_n + \alpha_{n1} \frac{1}{t_{n1}} \int_0^{t_{n1}} T_1(u)x_n du + \alpha_{n2} \frac{1}{t_{n2}} \int_0^{t_{n2}} T_2(u)x_n du \\ &\quad + \dots + \alpha_{nr} \frac{1}{t_{nr}} \int_0^{t_{nr}} T_r(u)x_n du, \end{aligned} \quad (4.14)$$

$$C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\},$$

$$Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\},$$

$$x_{n+1} = P_{C_n \cap Q_n} x, \quad n = 1, 2, 3, \dots,$$

where

$$\bar{\theta}_n = \left( \alpha_{n1} (L_{u1}^2 - 1) + \alpha_{n2} (L_{u2}^2 - 1) + \dots + \alpha_{nr} (L_{ur}^2 - 1) \right) (\text{diam } C)^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.15)$$

with  $L_{ui} = (1/t_{ni}) \int_0^{t_{ni}} L_i^{T_i} du$ . Then  $\{x_n\}$  converges strongly to  $P_F x$ .

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# Approximation of zeros of inverse strongly monotone operators in Banach spaces

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## ABSTRACT

In this paper, we consider the projection algorithm studied by Iiduka and Takahashi (2008) [10] for finding a solution of the variational inequality problem for an inverse strongly monotone operator in a Banach space. We first remark that, under the assumptions imposed on the operator in their paper, the iterative sequence converges weakly to a zero of the operator, not just a solution of the variational inequality problem. In our proof, slightly modified from the original, we do not assume the uniform smoothness of a space as was the case there. Finally, using Halpern's type method, we modify this algorithm to obtain the strong convergence to a zero of an inverse strongly monotone operator which is nearest to the initial element of the algorithm in the sense of the Bergman distance associated with the function  $\frac{1}{2} \|\cdot\|^2$ .

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## 1. Introduction

Let  $E$  be a real Banach space with the dual space  $E^*$ . The value of a functional  $x^* \in E^*$  at  $x \in E$  is denoted by  $\langle x, x^* \rangle$ . For a subset  $C$  of  $E$ , the *variational inequality problem* [1,2] of an operator  $A : C \rightarrow E^*$  is the problem of finding an element  $u \in C$  such that

$$\langle v - u, Au \rangle \geq 0 \quad \text{for all } v \in C.$$

The set of solutions of the problem above is denoted by  $VI(C, A)$ , that is,  $VI(C, A) = \{u \in C : \langle v - u, Au \rangle \geq 0 \text{ for all } v \in C\}$ . This problem is interesting and has been extensively studied by many authors because it includes various problems in many branches in mathematics and sciences, for example, convex optimization problems, complementarity problems, etc. Relations between convex functions and variational inequalities were considered by Rockafellar [3] and Moreau [4]. The fact that complementarity problems can be deduced from variational inequalities was given by Karamardian [5].

If  $E$  is a Hilbert space, one of several methods for finding an element in  $VI(C, A)$  is the *projection algorithm*, which is defined as follows:

$$\begin{cases} x_1 = x \in C \text{ arbitrarily chosen and} \\ x_{n+1} = P_C(x_n - \lambda_n Ax_n) \quad (n \geq 1), \end{cases} \quad (1)$$

where  $P_C$  is the metric projection of a Hilbert space  $E$  onto  $C$  and  $\{\lambda_n\}$  is a certain sequence of real numbers. The convergence result of the sequence  $\{x_n\}$  generated by the algorithm (1) was studied by many authors (e.g., see [6–8]).

If  $E$  is a Banach space, Alber [9] discussed the projection algorithm above with the metric projection  $P_C$  replaced by the generalized projection  $\Pi_C$ . He also proved a convergence theorem for uniformly monotone operators with a certain growth condition. Inspired by Alber's work, Iiduka and Takahashi [10] proved the following result for an inverse strongly monotone operator. The related definitions and notion will be given in the next section.

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**Theorem 1.1.** *Let  $E$  be a 2-uniformly convex and uniformly smooth Banach space with the 2-uniform convexity constant  $c$  and let  $C$  be a closed convex subset of  $E$ . Assume that an operator  $A : C \rightarrow E^*$  satisfies the following conditions:*

- (M1)  $A$  is  $\alpha$ -inverse strongly monotone where  $\alpha > 0$ ;
- (M2)  $VI(C, A) \neq \emptyset$ ;
- (M3)  $\|Ay\| \leq \|Ay - Au\|$  for all  $y \in C$  and  $u \in VI(C, A)$ .

Suppose that  $\{x_n\}$  is iteratively constructed by the following

$$\begin{cases} x_1 = x \in C \text{ arbitrarily chosen and} \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n) \quad (n \geq 1), \end{cases} \tag{2}$$

where  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some real numbers  $a, b$  such that  $0 < a < b < c^2\alpha/2$ . If the duality mapping  $J : E \rightarrow E^*$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to some element  $z \in VI(C, A)$ . Further  $z = \lim_{n \rightarrow \infty} \Pi_{VI(C, A)}(x_n)$ .

The purpose of this paper is to give a remark on the condition (M3) imposed on the operator  $A$  in Theorem 1.1. This condition implies that the solution set  $VI(C, A)$  is identical to the set of zeros of the operator  $A$ , that is,  $VI(C, A) = \{u \in C : Au = 0\}$ . Moreover, we slightly modify their original proof and obtain the result without assuming the uniform smoothness condition. Finally, using Halpern’s type method, we modify this algorithm to obtain the strong convergence to a zero of the operator  $A$  which is nearest to the initial element of the modified algorithm in the sense of the Bergman distance associated with the function  $\frac{1}{2} \|\cdot\|^2$ .

## 2. Preliminaries

Let  $E$  be a real Banach space with the dual space  $E^*$ . When  $\{x_n\}$  is a sequence in  $E$ , we denote the strong convergence and weak convergence of  $\{x_n\}$  to  $x \in E$  by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$ , respectively. The set of all weakly cluster points of  $\{x_n\}$  is denoted by  $\omega_w\{x_n\}$ , that is,  $\omega_w\{x_n\} = \{x \in E : \text{there exists } \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\}$ . The duality mapping  $J : E \rightarrow 2^{E^*}$  is the point-to-set mapping defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (\text{for all } x \in E).$$

Let us recall some geometric properties which are used in this paper. A Banach space  $E$  is said to be

- *strictly convex* if for any  $\|x\| = \|y\| = 1$  with  $\frac{1}{2}\|x + y\| = 1$ , we have  $x = y$ ;
- *smooth* if  $\lim_{t \rightarrow 0} \frac{1}{t}(\|x + ty\| - \|x\|)$  exists for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$ .

The uniform character of two properties above are related to the following moduli:

$$\begin{aligned} \delta_E(\varepsilon) &:= \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\} \quad (\varepsilon \in [0, 2]) \\ \varrho_E(\tau) &:= \sup \left\{ \frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : \|x\| = \|y\| = 1 \right\} \quad (\tau \geq 0). \end{aligned}$$

We say that  $E$  is

- *uniformly convex* if  $\delta_E(\varepsilon) > 0$  for all  $\varepsilon \in (0, 2)$ ;
- *uniformly smooth* if  $\lim_{\tau \rightarrow 0} \frac{\varrho_E(\tau)}{\tau} = 0$ .

It is known that  $E$  is uniformly convex (uniformly smooth, resp.) if and only if  $E^*$  is uniformly smooth (uniformly convex, resp.). To deal with some estimates, we also need the stronger property than uniform convexity. A space  $E$  is *2-uniformly convex* if there exists a constant  $k > 0$  such that  $\delta_E(\varepsilon) \geq k\varepsilon^2$  for all  $\varepsilon \in (0, 2)$ . It is known that  $E$  is 2-uniformly convex if and only if there exists a constant  $c > 0$  such that

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, j_x \rangle + \frac{c^2}{2}\|y\|^2$$

for all  $x, y \in E$  and  $j_x \in Jx$  (see [11–13]). The largest constant  $c$  such that the inequality above holds is called the *2-uniform convexity constant* of  $E$ . In particular, if  $E$  is 2-uniformly convex with the 2-uniform convexity constant  $c$ , then

$$\|x - y\| \leq \frac{2}{c^2} \|j_x - j_y\| \tag{3}$$

for all  $x, y \in E$  and  $j_x \in Jx, j_y \in Jy$ .

Let us recall some known fact concerning the geometric properties and the properties of the duality mapping (for more information, we refer to Cioranescu’s book [14] and its review by Reich [15]):

- If  $E$  is smooth, then  $Jx$  is a singleton for all  $x \in E$ . In this case, we treat each singleton  $Jx$  as the element in  $Jx$  and we consider  $J$  as a single-valued mapping  $J : E \rightarrow E^*$ .
- If  $E$  is smooth, strictly convex and reflexive, then the duality mapping  $J : E \rightarrow E^*$  is one-to-one and onto. Moreover,  $J$  is strictly monotone, that is,  $\langle x - y, Jx - Jy \rangle > 0$  for all  $x, y \in E$  with  $x \neq y$ .
- If  $E$  is smooth, strictly convex and reflexive, then  $J^{-1}$  is single-valued, bijective and it is the duality mapping from  $E^*$  into  $E$ .
- If  $E$  is uniformly smooth, then  $J : E \rightarrow E^*$  is norm-to-norm uniformly continuous on bounded sets.
- We say that the duality mapping  $J : E \rightarrow E^*$  is weakly sequentially continuous if  $x_n \rightarrow x$  implies  $Jx_n \xrightarrow{*} Jx$ . Here  $\xrightarrow{*}$  denotes the weak\* convergence the dual space  $E^*$ .

Let  $E$  be a smooth Banach space. Let  $\varphi : E \times E \rightarrow [0, \infty)$  be defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad (x, y \in E).$$

This function  $\varphi$  is very similar to the distance function but it is not, for example, it is not symmetry and fails the triangle inequality in general. In Hilbert space setting,  $\varphi(x, y) = \|x - y\|^2$  because  $Jy = y$ . It should be noted that  $\frac{1}{2}\varphi$  is the Bregman distance associated with the function  $\frac{1}{2}\|\cdot\|^2$  (for more detail, we refer to [16]). Alber [9] introduced the following type of projection: let  $C$  be a closed and convex subset of a reflexive, smooth and strictly convex Banach space  $E$ , for each  $x \in E$ , there corresponds a unique element  $x_0 \in C$ , denoted by  $\Pi_C x$ , such that

$$\varphi(x_0, x) = \min\{\varphi(y, x) : y \in C\}.$$

The mapping  $\Pi_C$  is called the *generalized projection* from  $E$  onto  $C$ . If  $E$  is a Hilbert space, then  $\Pi_C$  is just the metric projection from  $E$  onto  $C$ . Moreover, it is known that the following two inequalities hold for all  $x \in E$  and for all  $y \in C$ :

- $\langle y - \Pi_C x, J\Pi_C x - Jx \rangle \geq 0$ ;
- $\varphi(y, \Pi_C x) \leq \varphi(y, x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$ .

The following results are essentially needed for our main result.

**Lemma 2.1** ([9,17]). *Let  $E$  be a smooth and uniformly convex Banach space. Suppose that  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $E$  such that one of them is bounded. If  $\lim_{n \rightarrow \infty} \varphi(x_n, y_n) = 0$ , then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

**Lemma 2.2** ([10]). *Let  $C$  be a closed convex subset of a smooth and uniformly convex Banach space  $E$ . Let  $\{x_n\}$  be a sequence in  $E$  such that*

$$\varphi(y, x_{n+1}) \leq \varphi(y, x_n) \quad \text{for all } y \in C \text{ and } n \geq 1.$$

*Then the sequence  $\{\Pi_C x_n\}$  converges strongly to some element  $z \in C$ .*

We make use of the following functional  $V : E \times E^* \rightarrow [0, \infty)$  studied in [9]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2$$

for all  $x \in E$  and  $x^* \in E^*$ . In other words,  $V(x, x^*) = \varphi(x, J^{-1}(x^*))$  for all  $x \in E$  and  $x^* \in E^*$ . The following two inequalities are consequences of the subdifferential inequality and the convexity of the function  $\|\cdot\|^2$ , respectively.

**Lemma 2.3** ([9]). *Let  $E$  be a reflexive, smooth and strictly convex Banach space. Then the following statements hold*

- $V(x, x^*) \leq V(x, x^* + y^*) + 2\langle x - J^{-1}(x^*), y^* \rangle$ ;
- $V(x, \alpha x^* + (1 - \alpha)y^*) \leq \alpha V(x, x^*) + (1 - \alpha)V(x, y^*)$ ;

*for all  $x \in E, x^*, y^* \in E^*$  and  $\alpha \in (0, 1)$ .*

The following two lemmas are our main tool for proving the strong convergence of the modification of the projection algorithm.

**Lemma 2.4** ([18]). *Let  $\{\gamma_n\}$  be a sequence of real numbers such that there exists a subsequence  $\{\gamma_{n_j}\}$  of  $\{\gamma_n\}$  such that  $\gamma_{n_j} < \gamma_{n_{j+1}}$  for all  $j \geq 1$ . Then there exists a nondecreasing sequence  $\{m_k\}$  of positive integers such that the following two inequalities:*

$$\gamma_{m_k} \leq \gamma_{m_{k+1}} \quad \text{and} \quad \gamma_k \leq \gamma_{m_{k+1}}$$

*hold for all (sufficiently large) numbers  $k$ . In fact,  $m_k$  is the largest number  $n$  in the set  $\{1, 2, \dots, k\}$  such that the condition  $\gamma_n < \gamma_{n+1}$  holds.*

**Lemma 2.5** ([19–21]). *Let  $\{s_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  be a sequence in  $(0, 1)$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $\{t_n\}$  be a sequence of real numbers with  $\limsup_{n \rightarrow \infty} t_n \leq 0$ . Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n \quad \text{for all } n \geq 1.$$

*Then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

In virtue of Lemma 2.4, the condition  $\limsup_{n \rightarrow \infty} t_n \leq 0$  in Lemma 2.5 can be slightly weakened as follows. In fact, the proof of the following lemma is extracted from the recent work of the first author [22].

**Lemma 2.6.** *Let  $\{s_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  be a sequence in  $(0, 1)$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $\{t_n\}$  be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n \quad \text{for all } n \geq 1.$$

*If  $\limsup_{k \rightarrow \infty} t_{n_k} \leq 0$  for every subsequence  $\{s_{n_k}\}$  of  $\{s_n\}$  satisfying  $\liminf_{k \rightarrow \infty} (s_{n_{k+1}} - s_{n_k}) \geq 0$ , then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

**Proof.** The proof is split into two cases.

Case 1: There exists an  $n_0 \in \mathbb{N}$  such that  $s_{n+1} \leq s_n$  for all  $n \geq n_0$ . It follows then that  $\liminf_{n \rightarrow \infty} (s_{n+1} - s_n) = 0$ . Hence  $\limsup_{n \rightarrow \infty} t_n \leq 0$ . The conclusion follows from Lemma 2.5.

Case 2: There exists a subsequence  $\{s_{m_j}\}$  of  $\{s_n\}$  such that  $s_{m_j} < s_{m_{j+1}}$  for all  $j \in \mathbb{N}$ . In this case, we can apply Lemma 2.4 to find a nondecreasing sequence  $\{n_k\}$  of  $\{n\}$  such that  $n_k \rightarrow \infty$  and the following two inequalities:

$$s_{n_k} \leq s_{n_{k+1}} \quad \text{and} \quad s_k \leq s_{n_{k+1}}$$

hold for all (sufficiently large) numbers  $k$ . It follows from the first inequality that  $\liminf_{k \rightarrow \infty} (s_{n_{k+1}} - s_{n_k}) \geq 0$ . This implies that  $\limsup_{k \rightarrow \infty} t_{n_k} \leq 0$ . Moreover, by the first inequality again, we have

$$s_{n_{k+1}} \leq (1 - \alpha_{n_k})s_{n_k} + \alpha_{n_k} t_{n_k} \leq (1 - \alpha_{n_k})s_{n_{k+1}} + \alpha_{n_k} t_{n_k}.$$

In particular, since each  $\alpha_{n_k} > 0$ , we have  $s_{n_{k+1}} \leq t_{n_k}$ . Finally, it follows from the second inequality that

$$\limsup_{k \rightarrow \infty} s_k \leq \limsup_{k \rightarrow \infty} s_{n_{k+1}} \leq \limsup_{k \rightarrow \infty} t_{n_k} \leq 0.$$

Hence  $\lim_{k \rightarrow \infty} s_k = 0$ . This completes the proof.  $\square$

### 3. Results

#### 3.1. A remark on the condition (M3) imposed on the operator

**Proposition 3.1.** *Suppose that  $C$  is a subset of a Banach space  $E$  and  $A : C \rightarrow E^*$  is an operator satisfying condition (M3). Then the solution set  $VI(C, A)$  is identical to the set of zeros of  $A$ , that is,  $VI(C, A) = \{u \in C : Au = 0\}$ .*

**Proof.** For convenience, we write  $A^{-1}0 = \{u \in C : Au = 0\}$ . It is clear that  $A^{-1}0 \subset VI(C, A)$ . To see the reverse inclusion, if  $VI(C, A) = \emptyset$ , then  $VI(C, A) \subset A^{-1}0$  holds trivially. Next, we assume that  $VI(C, A) \neq \emptyset$  and let  $p \in VI(C, A)$ . It follows from condition (M3) that  $\|Ap\| \leq \|Ap - Ap\| = 0$ , that is,  $Ap = 0$  and hence  $p \in A^{-1}0$ . This also implies  $VI(C, A) \subset A^{-1}0$ . Hence  $VI(C, A) = A^{-1}0$ .  $\square$

**Remark 3.2.** As a consequence of Proposition 3.1, Theorem 3.1 of [10] (see Theorem 1.1) provides an algorithm for finding an element in  $A^{-1}0$ . This refines their result.

From now on, we will consider the problem of finding a zero of an operator instead of finding a solution of a variational inequality problem.

#### 3.2. A recapitulation of Iiduka and Takahashi's result

Recall that an operator  $A : C \rightarrow E^*$  is  $\alpha$ -inverse strongly monotone, where  $\alpha > 0$ , if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all  $x, y \in C$ .

The following easy observation also plays an important role in our main result.

**Proposition 3.3.** *Let  $C$  be a closed convex subset of a Banach space  $E$ . Assume that  $A : C \rightarrow E^*$  is  $\alpha$ -inverse strongly monotone where  $\alpha > 0$ . Suppose that  $\{x_n\}$  is a sequence of  $C$  such that  $Ax_n \rightarrow 0$ . Then  $\omega_w\{x_n\} \subset A^{-1}0$ .*

**Proof.** Let  $y \in \omega_w\{x_n\}$ . Then there is a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow y$ . It is clear that  $y \in C$ . Since  $A$  is  $\alpha$ -inverse strongly monotone, we have

$$\alpha \|Ay - Ax_{n_k}\|^2 \leq \langle y - x_{n_k}, Ay - Ax_{n_k} \rangle \rightarrow 0.$$

In particular,  $Ax_{n_k} \rightarrow Ay$ . Because  $Ax_n \rightarrow 0$ , so  $Ay = 0$ . Hence  $\omega_w\{x_n\} \subset A^{-1}0$ .  $\square$

We recapitulate the result given by Iiduka and Takahashi without assuming the uniform smoothness condition.

**Theorem 3.4.** Let  $E$  be a 2-uniformly convex and smooth Banach space with the 2-uniform convexity constant  $c$  and let  $C$  be a closed convex subset of  $E$ . Assume that an operator  $A : C \rightarrow E^*$  satisfies the following conditions:

- (M1)  $A$  is  $\alpha$ -inverse strongly monotone where  $\alpha > 0$ ;  
 (M2')  $A^{-1}0 \neq \emptyset$ .

Suppose that  $\{x_n\}$  is iteratively constructed by the following

$$\begin{cases} x_1 = x \in C \text{ arbitrarily chosen and} \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n) \quad (n \geq 1), \end{cases} \quad (4)$$

where  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some real numbers  $a, b$  such that  $0 < a < b < c^2\alpha/2$ . If the duality mapping  $J : E \rightarrow E^*$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to some element  $z \in A^{-1}0$ . Further  $z = \lim_{n \rightarrow \infty} \Pi_{A^{-1}0}(x_n)$ .

To prove Theorem 3.4, we first prove the following result which is interesting in its own right.

**Proposition 3.5.** Let  $E$  be a 2-uniformly convex and smooth Banach space with the 2-uniform convexity constant  $c$  and let  $C$  be a closed convex subset of  $E$ . Suppose that  $A : C \rightarrow E^*$  is an  $\alpha$ -inverse strongly monotone operator where  $\alpha > 0$  and  $A^{-1}0 \neq \emptyset$ . Then the following inequality

$$\varphi(u, J^{-1}(Jx - \lambda Ax)) \leq \varphi(u, x) + 2\lambda \left( \frac{2\lambda}{c^2} - \alpha \right) \|Ax\|^2$$

holds for all  $x \in C$ ,  $u \in A^{-1}0$ , and  $\lambda \in \mathbb{R}$ . In particular, if  $\lambda \in (0, c^2\alpha/2)$ , then

- $\varphi(u, J^{-1}(Jx - \lambda Ax)) \leq \varphi(u, x)$  all  $x \in C$  and  $u \in A^{-1}0$ ;
- $A^{-1}0$  is closed convex and hence  $\Pi_{A^{-1}0}$  is well defined (see [23, Lemma 2.5], and [24, Proposition 2.4]).

**Proof.** Let  $u \in A^{-1}0$ ,  $x \in C$  and  $\lambda \in \mathbb{R}$ . Note that  $\langle x - u, Ax \rangle = \langle x - u, Ax - Au \rangle \geq \alpha \|Ax - Au\|^2 = \alpha \|Ax\|^2$ . It then follows from the inequality (3) that

$$\begin{aligned} \langle u - J^{-1}(Jx - \lambda Ax), Ax \rangle &= \langle u - x, Ax \rangle + \langle x - J^{-1}(Jx - \lambda Ax), Ax \rangle \\ &\leq -\alpha \|Ax\|^2 + \|x - J^{-1}(Jx - \lambda Ax)\| \|Ax\| \\ &\leq -\alpha \|Ax\|^2 + \frac{2}{c^2} \|Jx - (Jx - \lambda Ax)\| \|Ax\| \\ &= \left( \frac{2\lambda}{c^2} - \alpha \right) \|Ax\|^2. \end{aligned}$$

By Lemma 2.3 and the inequality above, we have

$$\begin{aligned} \varphi(u, J^{-1}(Jx - \lambda Ax)) &= V(u, Jx - \lambda Ax) \\ &\leq V(u, Jx - \lambda Ax + \lambda Ax) + 2\langle u - J^{-1}(Jx - \lambda Ax), \lambda Ax \rangle \\ &= V(u, Jx) + 2\lambda \langle u - J^{-1}(Jx - \lambda Ax), Ax \rangle \\ &\leq \varphi(u, x) + 2\lambda \left( \frac{2\lambda}{c^2} - \alpha \right) \|Ax\|^2. \end{aligned}$$

This completes the proof.  $\square$

We are now ready to prove Theorem 3.4.

**Proof of Theorem 3.4.** We first show that  $Ax_n \rightarrow 0$ . Let  $u \in A^{-1}0$  and  $n \geq 1$ . By Proposition 3.5 and the property of  $\Pi_C$ , we have

$$\begin{aligned} \varphi(u, x_{n+1}) &= \varphi(u, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &\leq \varphi(u, J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &\leq \varphi(u, x_n) + 2\lambda_n \left( \frac{2\lambda_n}{c^2} - \alpha \right) \|Ax_n\|^2. \end{aligned}$$

In particular, by the assumption on  $\{\lambda_n\}$ , we get that  $\varphi(u, x_{n+1}) \leq \varphi(u, x_n)$ . Then  $\lim_{n \rightarrow \infty} \varphi(u, x_n)$  exists, and so  $\varphi(u, x_n) - \varphi(u, x_{n+1}) \rightarrow 0$ . It follows from  $\lambda_n \in [a, b] \subset (0, c^2\alpha/2)$  that

$$0 \leq 2a \left( \alpha - \frac{2b}{c^2} \right) \|Ax_n\|^2 \leq 2\lambda_n \left( \alpha - \frac{2\lambda_n}{c^2} \right) \|Ax_n\|^2 \leq \varphi(u, x_n) - \varphi(u, x_{n+1}) \rightarrow 0.$$

Therefore,  $Ax_n \rightarrow 0$ . Notice that, by Proposition 3.3, we have  $\omega_w \{x_n\} \subset A^{-1}0$ . Put  $u_n = \Pi_{A^{-1}0}(x_n)$ . Applying Lemma 2.2, it follows from  $\varphi(u, x_{n+1}) \leq \varphi(u, x_n)$  that there exists  $z \in A^{-1}0$  such that  $u_n \rightarrow z$ . Finally, we prove that  $x_n \rightarrow z$ . Let  $\{x_{n_k}\}$  be a subsequence of  $\{x_n\}$  such that  $x_{n_k} \rightarrow \tilde{z} \in \omega_w \{x_n\} \subset A^{-1}0$ . It follows from the property of  $\Pi_{A^{-1}0}$  that the following inequality

$$\langle \tilde{z} - u_{n_k}, Ju_{n_k} - Jx_{n_k} \rangle \geq 0$$

holds for all  $k \in \mathbb{N}$ . Since  $u_n \rightarrow z$ ,  $x_{n_k} \rightarrow \tilde{z}$  and  $J$  is weakly sequentially continuous, we have

$$\langle \tilde{z} - z, Jz - J\tilde{z} \rangle \geq 0.$$

By the strict monotonicity of  $J$ , we get  $z = \tilde{z}$ . Since every subsequence of  $\{x_n\}$  has a further subsequence which converges weakly to  $z$ , we can conclude that  $x_n \rightarrow z = \lim_{n \rightarrow \infty} \Pi_{A^{-1}0}(x_n)$  and the proof is finished.  $\square$

**Remark 3.6.** The proof of Theorem 3.4 is different from the original one given by Iiduka and Takahashi (see the proof of Theorem 3.1 of [10]). In fact, we do not use the intrinsic property of a maximal monotone operator (the maximality of the operator  $A$  joining with its corresponding normal cone, see e.g., [25]) and so we do not have to use the uniform continuity (on bounded sets) of the duality mapping as was the case in Iiduka and Takahashi’s proof. Let us note that the latter type of continuity follows from the uniform smoothness property of a space.

**Remark 3.7.** There exists a 2-uniformly convex and smooth space which is not uniformly smooth. The following example is suggested by Professor Simeon Reich and Professor Yoav Benyamini. For each  $n \in \mathbb{N}$ , suppose that  $E_n := (\mathbb{R}^2, \|\cdot\|_n)$  is a two-dimensional space with its modulus of convexity of power type 2 and its modulus of smoothness of power type  $q_n$  where  $q_n \downarrow 1$ . (Attaching to each side of the square  $[-1, 1]^2$  a part of a disk of radius  $R > 1$ , we obtain a convex set  $K$  which is 2-uniformly convex, but not smooth. We then smooth it by replacing each corner with a part of a small disk of radius  $r < 1$ , which is tangent to the boundary of  $K$ . The resulting set is still 2-uniformly convex (with estimates that do not depend on  $r$ ) and uniformly smooth, but its modulus of smoothness deteriorates as  $r \downarrow 0$ .) Next, we consider  $E := \ell^2(E_n) = \{\{x_n\} : x_n \in E_n \text{ for all } n \in \mathbb{N}, \sum_{n=1}^{\infty} \|x_n\|_n^2 < \infty\}$  and  $\|\{x_n\}\| := (\sum_{n=1}^{\infty} \|x_n\|_n^2)^{1/2}$  for all  $\{x_n\} \in E$ . Since the family  $\{E_n\}$  is not “uniformly” uniformly smooth, the space  $E$  is smooth but not uniformly smooth (for example see [26]). We may assume that  $c$  is the 2-uniform convexity of all  $E_n$ ’s. To see that  $E$  is 2-uniformly convex, we show that the following inequality holds for all  $\mathbf{x}, \mathbf{y} \in E$  and  $\mathbf{j} \in J\mathbf{x}$ :

$$\|\mathbf{x} + \mathbf{y}\|^2 \geq \|\mathbf{x}\|^2 + 2\langle \mathbf{y}, \mathbf{j} \rangle + \frac{c^2}{2} \|\mathbf{y}\|^2. \tag{5}$$

For convenience, we write  $\mathbf{x} = \{x_n\}$  and  $\mathbf{y} = \{y_n\}$  where  $x_n, y_n \in E_n$  for all  $n \in \mathbb{N}$ . Furthermore, we can write  $\mathbf{j} = \{x_n^*\}$  where  $x_n^* \in E_n^*$  and  $x_n^* \in Jx_n$  for all  $n \in \mathbb{N}$ . It follows from the 2-uniform convexity of each  $E_n$  that

$$\|x_n + y_n\|_n^2 \geq \|x_n\|_n^2 + 2\langle y_n, x_n^* \rangle + \frac{c^2}{2} \|y_n\|_n^2.$$

Consequently, the inequality (5) holds immediately.

### 3.3. Strong convergence theorem

In this subsection, we use the idea of Halpern’s type iteration [27–29] and modify the projection algorithm in the preceding subsection to obtain strong convergence. The key point of the proof is based on our Proposition 3.5 and the useful Lemma 2.6 inspired by Maingé’s result [18] and the first author’s result [22].

**Theorem 3.8.** Let  $E$  be a 2-uniformly convex and smooth Banach space with the 2-uniform convexity constant  $c$  and let  $C$  be a closed convex subset of  $E$ . Assume that an operator  $A : C \rightarrow E^*$  satisfies the following conditions:

- (M1)  $A$  is  $\alpha$ -inverse strongly monotone where  $\alpha > 0$ ;
- (M2')  $A^{-1}0 \neq \emptyset$ .

Suppose that  $\{x_n\}$  is iteratively constructed by the following

$$\begin{cases} x_1 = x \in C \text{ arbitrarily chosen and} \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n x + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)) \quad (n \geq 1), \end{cases} \tag{6}$$

where  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some real numbers  $a, b$  with  $0 < a < b < c^2\alpha/2$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then the sequence  $\{x_n\}$  converges strongly to  $z = \Pi_{A^{-1}0}(x)$ .

Note that  $A^{-1}0$  is closed convex and hence  $\Pi_{A^{-1}0}$  is well defined. Put  $z := \Pi_{A^{-1}0}x$ . To prove Theorem 3.8, we need the following lemmas.

**Lemma 3.9.** *The following inequality holds:*

$$\varphi(z, x_{n+1}) \leq \alpha_n \varphi(z, x) + (1 - \alpha_n) \varphi(z, J^{-1}(Jx_n - \lambda_n Ax_n))$$

for all  $n \in \mathbb{N}$ . In particular,

$$\varphi(z, x_{n+1}) \leq \alpha_n \varphi(z, x) + (1 - \alpha_n) \varphi(z, x_n)$$

and hence the sequence  $\{x_n\}$  is bounded.

**Proof.** It follows from the properties of  $\Pi_C$  and  $V$  that

$$\begin{aligned} \varphi(z, x_{n+1}) &= \varphi(z, \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n))) \\ &\leq \varphi(z, J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n))) \\ &= V(z, \alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)) \\ &\leq \alpha_n V(z, Jx) + (1 - \alpha_n) V(z, Jx_n - \lambda_n Ax_n) \\ &= \alpha_n \varphi(z, x) + (1 - \alpha_n) \varphi(z, J^{-1}(Jx_n - \lambda_n Ax_n)). \end{aligned}$$

This together with Proposition 3.5 implies that

$$\varphi(z, x_{n+1}) \leq \alpha_n \varphi(z, x) + (1 - \alpha_n) \varphi(z, x_n) \leq \max\{\varphi(z, x), \varphi(z, x_n)\}.$$

By induction, we get  $\varphi(z, x_n) \leq \varphi(z, x)$  for all  $n \geq 1$ . In particular,  $\{x_n\}$  is bounded.  $\square$

**Lemma 3.10.** *The following inequality holds:*

$$\varphi(z, x_{n+1}) \leq (1 - \alpha_n) \varphi(z, x_n) + 2\alpha_n \langle z - J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)), Jz - Jx \rangle$$

for all  $n \in \mathbb{N}$ .

**Proof.** It follows from the property of  $\Pi_C$ , Lemma 2.3, and Proposition 3.5 that

$$\begin{aligned} \varphi(z, x_{n+1}) &= \varphi(z, \Pi_C J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n))) \\ &\leq \varphi(z, J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n))) \\ &= V(z, \alpha_n Jz + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n) - \alpha_n (Jz - Jx)) \\ &\leq V(z, \alpha_n Jz + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)) + 2\langle z - J^{-1}(\alpha_n Jx + (1 - \alpha_n) \\ &\quad \times (Jx_n - \lambda_n Ax_n)), \alpha_n (Jz - Jx) \rangle \\ &\leq \alpha_n V(z, Jz) + (1 - \alpha_n) V(z, Jx_n - \lambda_n Ax_n) + 2\langle z - J^{-1}(\alpha_n Jx + (1 - \alpha_n) \\ &\quad \times (Jx_n - \lambda_n Ax_n)), \alpha_n (Jz - Jx) \rangle \\ &= (1 - \alpha_n) \varphi(z, J^{-1}(Jx_n - \lambda_n Ax_n)) + 2\alpha_n \langle z - J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)), Jz - Jx \rangle \\ &\leq (1 - \alpha_n) \varphi(z, x_n) + 2\alpha_n \langle z - J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)), Jz - Jx \rangle. \quad \square \end{aligned}$$

We are now ready to prove Theorem 3.8.

**Proof of Theorem 3.8.** Set  $s_n = \varphi(z, x_n)$  and  $t_n = 2\langle z - J^{-1}(\alpha_n Jx + (1 - \alpha_n)(Jx_n - \lambda_n Ax_n)), Jz - Jx \rangle$ . It follows from Lemma 3.10 that

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n t_n.$$

To apply Lemma 2.6, it suffices to show that  $\limsup_{k \rightarrow \infty} t_{n_k} \leq 0$  for every subsequence  $\{s_{n_k}\}$  of  $\{s_n\}$  satisfying  $\liminf_{k \rightarrow \infty} (s_{n_{k+1}} - s_{n_k}) \geq 0$ . Suppose that

$$\liminf_{k \rightarrow \infty} (s_{n_{k+1}} - s_{n_k}) = \liminf_{k \rightarrow \infty} (\varphi(z, x_{n_{k+1}}) - \varphi(z, x_{n_k})) \geq 0.$$

We first show that  $Ax_{n_k} \rightarrow 0$ . It follows from  $\alpha_n \rightarrow 0$ , the boundedness of  $\{x_n\}$  and Lemma 3.9 that

$$\begin{aligned} 0 &\leq \liminf_{k \rightarrow \infty} (\varphi(z, x_{n_{k+1}}) - \varphi(z, x_{n_k})) \\ &\leq \limsup_{k \rightarrow \infty} (\varphi(z, x_{n_{k+1}}) - \varphi(z, x_{n_k})) \\ &\leq \limsup_{n \rightarrow \infty} (\varphi(z, x_{n+1}) - \varphi(z, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n \varphi(z, x) + (1 - \alpha_n) \varphi(z, x_n) - \varphi(z, x_n)) \\ &= \limsup_{n \rightarrow \infty} \alpha_n (\varphi(z, x) - \varphi(z, x_n)) = 0. \end{aligned}$$

Consequently,

$$\lim_{k \rightarrow \infty} (\varphi(z, x_{n_{k+1}}) - \varphi(z, x_{n_k})) = 0. \tag{7}$$

By Proposition 3.5 and Lemma 3.9, we have

$$\begin{aligned} \varphi(z, x_{n_{k+1}}) &\leq \alpha_{n_k} \varphi(z, x) + (1 - \alpha_{n_k}) \varphi(z, J^{-1}(Jx_{n_k} - \lambda_{n_k} Ax_{n_k})) \\ &\leq \alpha_{n_k} \varphi(z, x) + (1 - \alpha_{n_k}) \left( \varphi(z, x_{n_k}) + 2\lambda_{n_k} \left( \frac{2\lambda_{n_k}}{c^2} - \alpha \right) \|Ax_{n_k}\|^2 \right). \end{aligned}$$

This together with  $\lambda_n \in [a, b] \subset (0, c^2\alpha/2)$  implies that

$$0 \leq 2(1 - \alpha_{n_k})a \left( \alpha - \frac{2b}{c^2} \right) \|Ax_{n_k}\|^2 \leq \alpha_{n_k} \varphi(z, x) + (1 - \alpha_{n_k}) \varphi(z, x_{n_k}) - \varphi(z, x_{n_{k+1}}).$$

It follows from  $\alpha_n \rightarrow 0$  and (7) that  $Ax_{n_k} \rightarrow 0$ .

Next, we show that  $\limsup_{k \rightarrow \infty} \langle z - x_{n_k}, Jz - Jx \rangle \leq 0$ . Choose a subsequence  $\{x_{n_{k_l}}\}$  of  $\{x_{n_k}\}$  such that

$$\limsup_{k \rightarrow \infty} \langle z - x_{n_k}, Jz - Jx \rangle = \lim_{l \rightarrow \infty} \langle z - x_{n_{k_l}}, Jz - Jx \rangle.$$

Notice that, by Proposition 3.3 and  $Ax_{n_k} \rightarrow 0$ , we have  $\omega_w\{x_{n_k}\} \subset A^{-1}0$ . Since  $\{x_{n_k}\}$  is bounded, we may assume without loss of generality that  $x_{n_{k_l}} \rightarrow y \in A^{-1}0$ . Using the property of  $\Pi_{A^{-1}0}$  gives

$$\lim_{l \rightarrow \infty} \langle z - x_{n_{k_l}}, Jz - Jx \rangle = \langle z - y, Jz - Jx \rangle \leq 0.$$

Since 2-uniform convexity implies uniform convexity, the duality mapping  $J^{-1} : E^* \rightarrow E$  is norm-to-norm uniformly continuous on bounded sets. It follows then from  $Ax_{n_k} \rightarrow 0$  and  $\alpha_{n_k} \rightarrow 0$  that

$$J^{-1}(\alpha_{n_k} Jx + (1 - \alpha_{n_k})(Jx_{n_k} - \lambda_{n_k} Ax_{n_k})) - x_{n_k} \rightarrow 0.$$

Hence

$$\begin{aligned} \limsup_{k \rightarrow \infty} t_{n_k} &= 2 \limsup_{k \rightarrow \infty} \langle z - J^{-1}(\alpha_{n_k} Jx + (1 - \alpha_{n_k})(Jx_{n_k} - \lambda_{n_k} Ax_{n_k})), Jz - Jx \rangle \\ &= 2 \limsup_{k \rightarrow \infty} \langle z - x_{n_k}, Jz - Jx \rangle \\ &= 2 \langle z - y, Jz - Jx \rangle \leq 0. \end{aligned}$$

Therefore, by Lemma 2.6, we have  $\lim_{n \rightarrow \infty} \varphi(z, x_n) = 0$ . It follows then from Lemma 2.1 that the sequence  $\{x_n\}$  converges strongly to  $z$ , as desired. The proof is now finished.  $\square$

**Remark 3.11.** Theorem 3.8 is, in some aspect, an improvement of Theorem 1.1.

- Since we do not need the weak sequential continuity of the duality mapping in Theorem 3.8 as was the case in Theorem 1.1, we can apply our Theorem 3.8 in the Lebesgue function space  $L^p$  where  $1 < p < \infty$  and  $p \neq 2$  while this space is not applicable for Theorem 1.1 (see [30]).
- In most cases, strong convergence is more desirable than weak convergence.

#### 4. Applications

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space. To prove this, we need the following lemma proved by Baillon and Haddad [31]:

**Lemma 4.1** ([31]). *Let  $E$  be a Banach space. Suppose that  $f$  is a continuously Fréchet differentiable and convex function on  $E$ . If the gradient  $\nabla f$  of  $f$  is  $1/\alpha$ -Lipschitz continuous, then  $\nabla f$  is  $\alpha$ -inverse strongly monotone.*

**Theorem 4.2.** *Let  $E$  be a 2-uniformly convex and smooth Banach space with the 2-uniform convexity constant  $c$ . Assume that a function  $f : E \rightarrow \mathbb{R}$  satisfies the following conditions:*

- $f$  is continuously Fréchet differentiable and convex on  $E$  and  $\nabla f$  is  $1/\alpha$ -Lipschitz continuous;
- $S := \arg \min_{y \in E} f(y) = \{z \in E : f(z) = \min_{y \in E} f(y)\} \neq \emptyset$ .

Let  $\{\lambda_n\}$  be a sequence in  $[a, b]$  for some real numbers  $a, b$  with  $0 < a < b < c^2\alpha/2$ .

(a) Suppose that  $\{x_n\}$  is iteratively constructed by the following

$$\begin{cases} x_1 = x \in E \text{ arbitrarily chosen and} \\ x_{n+1} = J^{-1}(Jx_n - \lambda_n \nabla f(x_n)) \quad (n \geq 1). \end{cases} \quad (8)$$

If the duality mapping  $J : E \rightarrow E^*$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to some element  $z \in S$ . Further  $z = \lim_{n \rightarrow \infty} \Pi_S(x_n)$ .

(b) If  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , then the sequence  $\{y_n\}$  iteratively constructed by the following

$$\begin{cases} y_1 = y \in E \text{ arbitrarily chosen and} \\ y_{n+1} = J^{-1}(\alpha_n Jy + (1 - \alpha_n)(Jy_n - \lambda_n \nabla f(y_n))) \quad (n \geq 1) \end{cases} \quad (9)$$

converges strongly to  $z = \Pi_S(y)$ .

**Proof.** It follows from Lemma 4.1 that  $\nabla f$  is  $\alpha$ -inverse strongly monotone. Notice that  $(\nabla f)^{-1} = \arg \min_{y \in E} f(y)$ . Then, by Theorems 3.4 and 3.8, we obtain the results.  $\square$

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## Research Article

# On the Strong Convergence of Viscosity Approximation Process for Quasinonexpansive Mappings in Hilbert Spaces

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We improve the viscosity approximation process for approximation of a fixed point of a quasinonexpansive mapping in a Hilbert space proposed by Maingé (2010). An example beyond the scope of the previously known result is given.

## 1. Introduction

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\|\cdot\|$ . In this paper, we denote the strong and weak convergence by  $\rightarrow$  and  $\rightharpoonup$ , respectively. For a subset  $C$  of  $H$ , a mapping  $T : C \rightarrow C$  is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ ; and it is *quasinonexpansive* if its *fixed-point set*  $\text{Fix}(T) := \{x \in C : x = Tx\}$  is nonempty and  $\|Tx - p\| \leq \|x - p\|$  for all  $x \in C$  and  $p \in \text{Fix}(T)$ . It is clear that every nonexpansive mapping with a nonempty fixed-point set is quasinonexpansive, but the converse is not true. The process for approximation of a fixed point of a nonexpansive or quasinonexpansive mapping is one of interesting problems in mathematics and it has been investigated by many researchers. One of the effective processes for this problem is given by Moudafi [1]. Let  $C$  be a closed convex subset of  $H$ , and  $T : C \rightarrow C$  is a nonexpansive mapping with a nonempty fixed-point set  $\text{Fix}(T)$ . Moudafi proposed the following scheme which is known as Moudafi's viscosity approximation process:

$$\begin{aligned} x_1 &= x \in C \quad \text{arbitrarily chosen,} \\ x_{n+1} &= \frac{\varepsilon_n}{1 + \varepsilon_n} f(x_n) + \frac{1}{1 + \varepsilon_n} Tx_n, \end{aligned} \tag{1.1}$$

where  $f : C \rightarrow C$  is a *contraction*; that is, there exists an  $\alpha \in [0, 1)$  such that  $\|f(x) - f(y)\| \leq \alpha\|x - y\|$  for all  $x, y \in C$  and  $\{\varepsilon_n\}$  is a sequence in  $(0, 1)$  satisfying

- (M1)  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ ,
- (M2)  $\sum_{n=1}^{\infty} \varepsilon_n = \infty$ ,
- (M3)  $\lim_{n \rightarrow \infty} (1/\varepsilon_n - 1/\varepsilon_{n+1}) = 0$ .

It was also proved that  $\{x_n\}$  converges to an element  $z \in \text{Fix}(T)$  satisfying the following inequality:

$$\langle f(z) - z, q - z \rangle \leq 0 \quad (1.2)$$

for all  $q \in \text{Fix}(T)$ .

In the literature, Moudafi's scheme has been widely studied and extended (see [2–5] and references therein). For example, Xu [6] improved this result to a Banach space. The interesting improvement of this result given by Maingé [7] is our starting point. His result is given below.

**Theorem 1.1.** *Let  $C$  be a closed convex subset of a Hilbert space  $H$ , and  $T : C \rightarrow C$  is a quasicontractive mapping such that  $I - T$  is demiclosed at zero, that is,  $z \in \text{Fix}(T)$  whenever  $\{z_n\}$  is a sequence in  $C$  such that  $z_n \rightarrow z$  and  $z_n - Tz_n \rightarrow 0$ . Suppose that  $f : C \rightarrow C$  is a contraction. Let  $\{x_n\}$  be a sequence in  $C$  defined by*

$$\begin{aligned} x_1 &= x \in C \quad \text{arbitrarily chosen,} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)((1 - \omega)I + \omega T)x_n, \end{aligned} \quad (1.3)$$

where  $\omega \in (0, 1)$ ,  $I$  is an identity mapping, and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,
- (C2)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then the sequence  $\{x_n\}$  converges to an element  $z \in \text{Fix}(T)$  and the following inequality holds

$$\langle f(z) - z, q - z \rangle \leq 0 \quad (1.4)$$

for all  $q \in \text{Fix}(T)$ .

It should be noted that Maingé's result is more widely applicable than Moudafi's. However, after a careful reading, we find that there is a small mistake in Maingé's proof. The following fact (see [7, Remark 2.1(i3)]) is used: if  $T : C \rightarrow C$  is quasicontractive and  $T_\omega := (1 - \omega)I + \omega T$  where  $\omega \in (0, 1]$ , then

$$\langle x - T_\omega x, x - q \rangle \geq \omega \|x - Tx\|^2 \quad (1.5)$$

for all  $x \in C$  and  $q \in \text{Fix}(T)$ . Note that the inequality above is equivalent to

$$\langle x - Tx, x - q \rangle \geq \|x - Tx\|^2. \quad (1.6)$$

But this fails; for example, let us consider the nonexpansive mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $Tx = -x$  for all  $x \in \mathbb{R}$ . It is clear that  $\text{Fix}(T) = \{0\}$  and  $\langle x - Tx, x - q \rangle = 2x^2 \not\geq 4x^2 = \|x - Tx\|^2$ .

Recall the following identities in a Hilbert space  $H$ : for  $x, y \in H, \omega \in [0, 1]$

$$(i) \|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2;$$

$$(ii) \|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - (1 - \omega)\omega\|x - y\|^2.$$

The correction of Maingé's result is as follows.

**Proposition 1.2.** *Let  $C$  be a subset of a Hilbert space and  $T : C \rightarrow C$  be a mapping with a nonempty fixed-point set  $\text{Fix}(T)$ . Suppose that  $T_\omega := (1 - \omega)I + \omega T$  where  $\omega \in (0, 1]$ . Then  $T$  is quasicontractive if and only if*

$$\langle x - T_\omega x, x - q \rangle \geq \frac{\omega}{2}\|x - Tx\|^2 \quad (1.7)$$

for all  $x \in C$  and  $q \in \text{Fix}(T)$ .

*Proof.* Notice that  $x - T_\omega x = \omega(x - Tx)$  and

$$\begin{aligned} \|T_\omega x - q\|^2 &= \|(T_\omega x - x) + (x - q)\|^2 \\ &= \|T_\omega x - x\|^2 + 2\langle T_\omega x - x, x - q \rangle + \|x - q\|^2 \\ &= \omega^2\|x - Tx\|^2 + 2\langle T_\omega x - x, x - q \rangle + \|x - q\|^2. \end{aligned} \quad (1.8)$$

On the other hand,

$$\begin{aligned} \|T_\omega x - q\|^2 &= \|(1 - \omega)(x - q) + \omega(Tx - q)\|^2 \\ &= (1 - \omega)\|x - q\|^2 + \omega\|Tx - q\|^2 - (1 - \omega)\omega\|x - Tx\|^2. \end{aligned} \quad (1.9)$$

Hence

$$2\langle x - T_\omega x, x - q \rangle = \omega\left(\|x - q\|^2 - \|Tx - q\|^2\right) + \omega\|x - Tx\|^2. \quad (1.10)$$

□

*Remark 1.3.* Unfortunately, this effects the main result (see [7, Theorem 3.1]) in Maingé's paper. More precisely, inequality (32) of its proof (page 78, line 22) should read

$$\begin{aligned} &\frac{1}{2}\|x_{n+1} - z\|^2 - \frac{1}{2}\|x_n - z\|^2 + \left(\frac{1}{2} - \omega\right)\omega(1 - \alpha_n)\|x_n - Tx_n\|^2 \\ &\leq \alpha_n\left(\alpha_n\|f(x_n) - x_n\|^2 - \langle x_n - f(x_n), x_n - z \rangle\right) \end{aligned} \quad (1.11)$$

rather than

$$\begin{aligned} & \frac{1}{2} \|x_{n+1} - z\|^2 - \frac{1}{2} \|x_n - z\|^2 + (1 - \omega)\omega(1 - \alpha_n) \|x_n - Tx_n\|^2 \\ & \leq \alpha_n \left( \alpha_n \|f(x_n) - x_n\|^2 - \langle x_n - f(x_n), x_n - z \rangle \right). \end{aligned} \quad (1.12)$$

Therefore, Theorem 1.1 above is valid for only  $\omega \in (0, 1/2)$  under the same technique.

The purpose of this paper is to simultaneously present a correction of the proof of Theorem 1.1 which is valid for all  $\omega \in (0, 1)$ , and extend his scheme to a wider class of mappings including average mappings, that is, mappings of the form  $(1 - \omega)I + \omega T$ . Our result is more general than Maingé's theorem. An example of a quasinonexpansive mapping which is not applied by Maingé's theorem but applied by our result is given.

## 2. Result

First, let us recall some lemmas which are needed for proving the main result.

**Lemma 2.1** (see [8, Lemma 2.3]). *Let  $\{s_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  a sequence of  $(0, 1)$  with  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\{\beta_n\}$  a sequence of nonnegative real numbers with  $\sum_{n=1}^{\infty} \beta_n < \infty$ , and  $\{\gamma_n\}$  a sequence of real numbers with  $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$ . Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n \quad (2.1)$$

for all  $n \in \mathbb{N}$ . Then  $\lim_{n \rightarrow \infty} s_n = 0$ .

The following nice result was proved by Maingé (see [7, Lemma 2.1]).

**Lemma 2.2.** *Let  $\{s_n\}$  be a sequence of nonnegative real numbers. If there exists a subsequence  $\{s_{n_j}\}$  of  $\{s_n\}$  such that  $s_{n_j} < s_{n_j+1}$  for all  $j \in \mathbb{N}$ , then there exists a subsequence  $\{s_{m_k}\}$  of  $\{s_n\}$  such that*

$$s_{m_k} \leq s_{m_k+1}, \quad s_k \leq s_{m_k+1} \quad (2.2)$$

for all  $k \in \mathbb{N}$ .

For a closed convex subset  $C$  of a Hilbert space  $H$ , the metric projection  $P_C : H \rightarrow C$  is defined for each  $x \in H$  as the unique element  $P_C x \in C$  such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}. \quad (2.3)$$

It is well known that (see, e.g., [9]) for  $x \in H$  and  $y \in C$

$$y = P_C x \iff \langle x - y, y - z \rangle \geq 0, \quad \forall z \in C. \quad (2.4)$$

For  $x, y \in H$ , the following inequality is known as the *subdifferential inequality*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (2.5)$$

A mapping  $T : C \rightarrow C$  is said to be *strongly quasinonexpansive* [10] if it is quasinonexpansive and  $z_n - Tz_n \rightarrow 0$  whenever  $\{z_n\}$  is a bounded sequence in  $C$  such that  $\lim_{n \rightarrow \infty} (\|z_n - p\| - \|Tz_n - p\|) = 0$  for some  $p \in \text{Fix}(T)$ . It is known that every metric projection is strongly quasinonexpansive.

We are now ready to present our main result.

**Theorem 2.3.** *Let  $C$  be a closed convex subset of a Hilbert space  $H$  and  $T : C \rightarrow C$  is a strongly quasinonexpansive mapping such that  $I - T$  is demiclosed at zero. Suppose that  $f : C \rightarrow C$  is a contraction. Let  $\{x_n\}$  be a sequence in  $C$  defined by*

$$\begin{aligned} x_1 &= x \in C \quad \text{arbitrarily chosen,} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \end{aligned} \tag{2.6}$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying

$$\begin{aligned} \text{(C1)} \quad & \lim_{n \rightarrow \infty} \alpha_n = 0, \\ \text{(C2)} \quad & \sum_{n=1}^{\infty} \alpha_n = \infty. \end{aligned}$$

Then the sequence  $\{x_n\}$  converges to an element  $z \in \text{Fix}(T)$  and the following inequality holds

$$\langle f(z) - z, q - z \rangle \leq 0 \tag{2.7}$$

for all  $q \in \text{Fix}(T)$ .

Before we give the proof, we note that  $\text{Fix}(T)$  is closed and convex (see [11] for more general setting). Hence the mapping  $P_{\text{Fix}(T)} \circ f : C \rightarrow C$  is a contraction. Then it follows from the well-known Banach's contraction principle that there exists a unique element  $z \in C$  such that  $z = P_{\text{Fix}(T)} \circ f(z)$ . In particular,  $z \in \text{Fix}(T)$  and  $\langle f(z) - z, q - z \rangle \leq 0$  for all  $q \in \text{Fix}(T)$ .

Let us assume that  $\|f(x) - f(y)\| \leq \alpha \|x - y\|$  for all  $x, y \in C$  where  $\alpha$  is a real number in  $[0, 1)$ .

**Lemma 2.4.** *The sequence  $\{x_n\}$  is bounded.*

*Proof.* We consider the following inequality:

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|f(x_n) - z\| + (1 - \alpha_n) \|Tx_n - z\| \\ &\leq \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| + (1 - \alpha_n) \|Tx_n - z\| \\ &\leq (\alpha_n \alpha + 1 - \alpha_n) \|x_n - z\| + \alpha_n \|f(z) - z\| \\ &= (1 - \alpha_n(1 - \alpha)) \|x_n - z\| + \alpha_n(1 - \alpha) \frac{\|f(z) - z\|}{1 - \alpha} \\ &\leq \max \left\{ \|x_n - z\|, \frac{\|f(z) - z\|}{1 - \alpha} \right\}. \end{aligned} \tag{2.8}$$

By induction, we conclude that the sequence  $\{\|x_n - z\|\}$  is bounded and hence so is the sequence  $\{x_n\}$ .  $\square$

**Lemma 2.5.** *The following inequality holds for all  $n \in \mathbb{N}$ :*

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \alpha \|x_n - z\| \|x_{n+1} - z\| + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle. \quad (2.9)$$

*Proof.* It follows from the subdifferential inequality that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(f(x_n) - z) + (1 - \alpha_n)(Tx_n - z)\|^2 \\ &\leq (1 - \alpha_n)^2 \|Tx_n - z\|^2 + 2\alpha_n \langle f(x_n) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \langle f(x_n) - f(z), x_{n+1} - z \rangle \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \alpha \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle. \end{aligned} \quad (2.10)$$

□

**Lemma 2.6.** *If there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\liminf_{k \rightarrow \infty} (\|x_{n_{k+1}} - z\| - \|x_{n_k} - z\|) \geq 0$ , then  $\limsup_{k \rightarrow \infty} \langle f(z) - z, x_{n_{k+1}} - z \rangle \leq 0$ .*

*Proof.* First, we note that  $\alpha_{n_k} \rightarrow 0$  and let us consider the following inequality:

$$\begin{aligned} 0 &\leq \liminf_{k \rightarrow \infty} (\|x_{n_{k+1}} - z\| - \|x_{n_k} - z\|) \\ &\leq \liminf_{k \rightarrow \infty} (\alpha_{n_k} \|f(x_{n_k}) - z\| + (1 - \alpha_{n_k}) \|Tx_{n_k} - z\| - \|x_{n_k} - z\|) \\ &= \liminf_{k \rightarrow \infty} (\|Tx_{n_k} - z\| - \|x_{n_k} - z\|) \\ &\leq \limsup_{k \rightarrow \infty} (\|Tx_{n_k} - z\| - \|x_{n_k} - z\|) \\ &\leq 0. \end{aligned} \quad (2.11)$$

This implies that  $\lim_{k \rightarrow \infty} (\|x_{n_k} - z\| - \|Tx_{n_k} - z\|) = 0$ . Since  $T$  is a strongly quasinonexpansive mapping,  $x_{n_k} - Tx_{n_k} \rightarrow 0$ . In particular,  $x_{n_k} - x_{n_{k+1}} \rightarrow 0$ . Because  $\{x_{n_k}\}$  is bounded, so there exists a subsequence  $\{x_{n_{k_l}}\}$  of  $\{x_{n_k}\}$  such that  $x_{n_{k_l}} \rightarrow q$  and

$$\lim_{l \rightarrow \infty} \langle f(z) - z, x_{n_{k_l}} - z \rangle = \limsup_{k \rightarrow \infty} \langle f(z) - z, x_{n_k} - z \rangle. \quad (2.12)$$

It follows from the demiclosedness of  $I - T$  at zero that  $q \in \text{Fix}(T)$ . Then

$$\lim_{l \rightarrow \infty} \langle f(z) - z, x_{n_{k_l}} - z \rangle = \langle f(z) - z, q - z \rangle \leq 0. \quad (2.13)$$

Hence  $\limsup_{k \rightarrow \infty} \langle f(z) - z, x_{n_{k+1}} - z \rangle = \limsup_{k \rightarrow \infty} \langle f(z) - z, x_{n_k} - z \rangle \leq 0$ , as desired. □

*Proof of Theorem 2.3.* Let us consider the following two cases.

*Case 1.* There exists an  $N \in \mathbb{N}$  such that  $\|x_{n+1} - z\| \leq \|x_n - z\|$  for all  $n \geq N$ . It follows then that  $\lim_{n \rightarrow \infty} \|x_n - z\|$  exists and hence  $\liminf_{n \rightarrow \infty} (\|x_{n+1} - z\| - \|x_n - z\|) = 0$ . This implies that  $\limsup_{n \rightarrow \infty} \langle f(z) - z, x_{n+1} - z \rangle \leq 0$ . By Lemma 2.5, for all  $n \geq N$ ,

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \alpha \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\leq (1 - 2\alpha_n + 2\alpha_n \alpha) \|x_n - z\|^2 + \alpha_n^2 \|x_n - z\|^2 \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &= (1 - 2\alpha_n(1 - \alpha)) \|x_n - z\|^2 \\ &\quad + 2\alpha_n(1 - \alpha) \left( \frac{\alpha_n \|x_n - z\|^2}{2(1 - \alpha)} + \frac{\langle f(z) - z, x_{n+1} - z \rangle}{1 - \alpha} \right). \end{aligned} \tag{2.14}$$

Notice that  $\sum_{n=N}^{\infty} 2\alpha_n(1 - \alpha) = \infty$  and

$$\limsup_{n \rightarrow \infty} \left( \frac{\alpha_n \|x_n - z\|^2}{2(1 - \alpha)} + \frac{\langle f(z) - z, x_{n+1} - z \rangle}{1 - \alpha} \right) \leq 0. \tag{2.15}$$

By Lemma 2.1, we have  $\lim_{n \rightarrow \infty} \|x_n - z\|^2 = 0$ .

*Case 2.* There exists a subsequence  $\{\|x_{n_j} - z\|\}$  of  $\{\|x_n - z\|\}$  such that  $\|x_{n_j} - z\| < \|x_{n_j+1} - z\|$  for all  $j \in \mathbb{N}$ . In this case, it follows from Lemma 2.2 that there exists a subsequence  $\{\|x_{m_k} - z\|\}$  of  $\{\|x_n - z\|\}$  such that

$$\|x_{m_k} - z\| \leq \|x_{m_k+1} - z\|, \quad \|x_k - z\| \leq \|x_{m_k+1} - z\| \tag{2.16}$$

for all  $k \in \mathbb{N}$ . It follows from  $\liminf_{k \rightarrow \infty} (\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \geq 0$  that  $\limsup_{k \rightarrow \infty} \langle f(z) - z, x_{m_k+1} - z \rangle \leq 0$ . Moreover, by Lemma 2.5, we have

$$\begin{aligned} \|x_{m_k+1} - z\|^2 &\leq (1 - \alpha_{m_k})^2 \|x_{m_k} - z\|^2 + 2\alpha_{m_k} \alpha \|x_{m_k} - z\| \|x_{m_k+1} - z\| \\ &\quad + 2\alpha_{m_k} \langle f(z) - z, x_{m_k+1} - z \rangle \\ &\leq (1 - \alpha_{m_k})^2 \|x_{m_k+1} - z\|^2 + 2\alpha_{m_k} \alpha \|x_{m_k+1} - z\|^2 \\ &\quad + 2\alpha_{m_k} \langle f(z) - z, x_{m_k+1} - z \rangle. \end{aligned} \tag{2.17}$$

In particular, it follows that

$$(2 - \alpha_{m_k} - 2\alpha) \|x_{m_k+1} - z\|^2 \leq 2 \langle f(z) - z, x_{m_k+1} - z \rangle. \tag{2.18}$$

This implies that

$$(2 - 2\alpha) \limsup_{k \rightarrow \infty} \|x_{m_{k+1}} - z\|^2 \leq \limsup_{k \rightarrow \infty} 2 \langle f(z) - z, x_{m_{k+1}} - z \rangle \leq 0. \quad (2.19)$$

Hence

$$\limsup_{k \rightarrow \infty} \|x_k - z\|^2 \leq \limsup_{k \rightarrow \infty} \|x_{m_{k+1}} - z\|^2 = 0. \quad (2.20)$$

Then  $\lim_{k \rightarrow \infty} \|x_k - z\|^2 = 0$ . This completes the proof.  $\square$

*Remark 2.7.* If  $C$  is a convex subset of a Hilbert space and  $T : C \rightarrow C$  is a quasinonexpansive mapping, then the mapping  $T_\omega := (1 - \omega)I + \omega T$  is strongly quasinonexpansive whenever  $\omega \in (0, 1)$  (see [10]). This means that Maingé's result is included in ours as a special case.

*Remark 2.8.* There is a strongly quasinonexpansive mapping  $S$  such that  $S$  is not of the form  $(1 - \omega)I + \omega T$  where  $\alpha \in (0, 1/2)$  and  $T$  is a quasinonexpansive mapping. This means that there is an example which is beyond the scope of Maingé's result (see Remark 1.3, Theorem 1.1 with his old proof is valid for only  $\alpha \in (0, 1/2)$ ).

*Example 2.9.* Let  $A = \{(x, x) : x \in \mathbb{R}\}$ . It is clear that  $A$  is a closed and convex subset of  $\mathbb{R}^2$ . Notice that  $S := P_A$  is a strongly quasinonexpansive mapping and  $(0, 0) \in \text{Fix}(S)$ . Suppose that  $S = (1 - \omega)I + \omega T$  where  $\omega \in (0, 1/2)$  and  $T$  is a quasinonexpansive mapping. Then, by Proposition 1.2, we have

$$\langle (2, 0) - S(2, 0), (2, 0) - (0, 0) \rangle \geq \frac{\omega}{2} \|(2, 0) - T(2, 0)\|^2 = \frac{1}{2\omega} \|(2, 0) - S(2, 0)\|^2. \quad (2.21)$$

It is easy to see that  $S(2, 0) = (1, 1)$ . In particular,

$$2 = \langle (2, 0) - (1, 1), (2, 0) \rangle \geq \frac{1}{2\omega} \|(2, 0) - (1, 1)\|^2 = \frac{1}{\omega}. \quad (2.22)$$

That is  $\omega \geq 1/2$ , a contradiction.

### 3. Conclusion

We propose a viscosity approximation process for approximation of a fixed point of a quasinonexpansive mapping. This not only corrects Maingé's result but also essentially improves his result to a more general relaxation.

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RESEARCH

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# Another weak convergence theorems for accretive mappings in banach spaces

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## Abstract

We present two weak convergence theorems for inverse strongly accretive mappings in Banach spaces, which are supplements to the recent result of Aoyama et al. [Fixed Point Theory Appl. (2006), Art. ID 35390, 13pp.].

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**Keywords:** weak convergence theorem, accretive mapping, Banach space

## 1. Introduction

Let  $E$  be a real Banach space with the dual space  $E^*$ . We write  $\langle x, x^* \rangle$  for the value of a functional  $x^* \in E^*$  at  $x \in E$ . The *normalized duality mapping* is the mapping  $J : E \rightarrow 2E^*$  given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (x \in E).$$

In this paper, we assume that  $E$  is *smooth*, that is,  $\lim_{t \rightarrow 0} \frac{\|x+tx\| - \|x\|}{t}$  exists for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$ . This implies that  $J$  is single-valued and we do consider the singleton  $Jx$  as an element in  $E^*$ . For a closed convex subset  $C$  of a (smooth) Banach space  $E$ , the *variational inequality problem* for a mapping  $A : C \rightarrow E$  is the problem of finding an element  $u \in C$  such that

$$\langle Au, J(v - u) \rangle \geq 0 \quad \text{for all } v \in C.$$

The set of solutions of the problem above is denoted by  $S(C, A)$ . It is noted that if  $C = E$ , then  $S(C, A) = A^{-1}0 := \{x \in E : Ax = 0\}$ . This problem was studied by Stampacchia (see, for example, [1,2]). The applicability of the theory has been expanded to various problems from economics, finance, optimization and game theory.

Gol'shtein and Tret'yakov [3] proved the following result in the finite dimensional space  $\mathbb{R}^N$ .

**Theorem 1.1.** *Let  $\alpha > 0$ , and let  $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be an  $\alpha$ -inverse strongly monotone mapping, that is,  $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$  for all  $x, y \in \mathbb{R}^N$ . Suppose that  $\{x_n\}$  is a sequence in  $\mathbb{R}^N$  defined iteratively by  $x_1 \in \mathbb{R}^N$  and*

$$x_{n+1} = x_n - \lambda_n Ax_n,$$

where  $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ . If  $A^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges to some element of  $A^{-1}0$ .

The result above was generalized to the framework of Hilbert spaces by Iiduka et al. [4]. Note that every Hilbert space is uniformly convex and 2-uniformly smooth (the related

definitions will be given in the next section). Aoyama et al. [[5], Theorem 3.1] proved the following result.

**Theorem 1.2.** *Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space with the uniform smoothness constant  $K$ , and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$ , let  $\alpha > 0$  and let  $A : C \rightarrow E$  be an  $\alpha$ -inverse strongly accretive mapping with  $S(C, A) \neq \emptyset$ . Suppose that  $\{x_n\}$  is iteratively defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where  $\{\alpha_n\} \subset [b, c] \subset (0, 1)$  and  $\{\lambda_n\} \subset [a, \alpha/K^2] \subset (0, \alpha/K^2]$ . Then,  $\{x_n\}$  converges weakly to some element of  $S(C, A)$ .

Motivated by the result of Aoyama et al., we prove two more convergence theorems for  $\alpha$ -inverse strongly accretive mappings in a Banach space, which are supplements to Theorem 1.2 above. The first one is proved without the presence of the uniform convexity, while the last one is proved in uniformly convex space with some different control conditions on the parameters.

The paper is organized as follows: In Section 2, we collect some related definitions and known fact, which are referred in this paper. The main results are presented in Section 3. We start with some common tools in proving the main results in Section 3.1. In Section 3.2, we prove the first weak convergence theorem without the presence of uniform convexity. The second theorem is proved in uniformly convex Banach spaces in Section 3.3.

## 2. Definitions and related known fact

Let  $E$  be a real Banach space. If  $\{x_n\}$  is a sequence in  $E$ , we denote *strong convergence* of  $\{x_n\}$  to  $x \in E$  by  $x_n \rightarrow x$  and *weak convergence* by  $x_n \rightharpoonup x$ . Denote by  $\omega_w(\{x_n\})$  the set of weakly sequential limits of the sequence  $\{x_n\}$ , that is,  $\omega_w(\{x_n\}) = \{p : \text{there exists a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ such that } x_{n_k} \rightharpoonup p\}$ . It is known that if  $\{x_n\}$  is a bounded sequence in a reflexive space, then  $\omega_w(\{x_n\}) \neq \emptyset$ .

The space  $E$  is said to be *uniformly convex* if for each  $\varepsilon \in (0, 2)$  there exists  $\delta > 0$  such that for any  $x, y \in U := \{z \in E : \|z\| = 1\}$

$$\|x - y\| \geq \varepsilon \text{ implies } \|x + y\|/2 \leq 1 - \delta.$$

The following result was proved by Xu.

**Lemma 2.1** ([6]). *Let  $E$  be a uniformly convex Banach space, and let  $r > 0$ . Then, there exists a strictly increasing, continuous and convex function  $g : [0, 2r] \rightarrow [0, \infty)$  such that  $g(0) = 0$  and*

$$\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|)$$

for all  $\alpha \in [0, 1]$  and  $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ .

The space  $E$  is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all  $x, y \in U$ . The norm of  $E$  is said to be *Fréchet differentiable* if for each  $x \in U$ , the limit (2.1) is attained uniformly for  $y \in U$ .

Let  $C$  be a nonempty subset of a smooth Banach space  $E$  and  $\alpha > 0$ . A mapping  $A : C \rightarrow E$  is said to be  $\alpha$ -inverse strongly accretive if

$$\langle Ax - Ay, J(x - y) \rangle \geq \alpha \|Ax - Ay\|^2 \tag{2.2}$$

for all  $x, y \in C$ . It follows from (2.2) that  $A$  is  $\frac{1}{\alpha}$ -Lipschitzian, that is,

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\| \quad \text{for all } x, y \in C.$$

A Banach space  $E$  is 2-uniformly smooth if there is a constant  $c > 0$  such that  $\rho_E(\tau) \leq c\tau^2$  for all  $\tau > 0$  where

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2} (\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in U \right\}.$$

In this case, we say that a real number  $K > 0$  is a 2-uniform smoothness constant of  $E$  if the following inequality holds for all  $x, y \in E$ :

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\|Ky\|^2.$$

Note that every 2-uniformly smooth Banach space has the Fréchet differentiable norm and hence it is reflexive.

The following observation extracted from Lemma 2.8 of [5] plays an important role in this paper.

**Lemma 2.2.** *Let  $C$  be a nonempty closed convex subset of a 2-uniformly smooth Banach space  $E$  with a 2-uniform smoothness constant  $K$ . Suppose that  $A : C \rightarrow E$  is an  $\alpha$ -inverse strongly accretive mapping. Then, the following inequality holds for all  $x, y \in C$  and  $\lambda \in \mathbb{R}$ :*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(K^2\lambda - \alpha)\|Ax - Ay\|^2,$$

where  $I$  is the identity mapping. In particular, if  $\lambda \in [0, \frac{\alpha}{K^2}]$ , then  $I - \lambda A$  is nonexpansive, that is,  $\|(I - \lambda A)x - (I - \lambda A)y\| \leq \|x - y\|$  for all  $x, y \in C$ .

Let  $C$  be a subset of a Banach space  $E$ . A mapping  $Q : E \rightarrow C$  is said to be:

- (i) sunny if  $Q(Qx + t(x - Qx)) = Qx$  for all  $t \geq 0$ ;
- (ii) a retraction if  $Q^2 = Q$ .

It is known that a retraction  $Q$  from a smooth Banach space  $E$  onto a nonempty closed convex subset  $C$  of  $E$  is sunny and nonexpansive if and only if  $\langle x - Qx, J(Qx - y) \rangle \geq 0$  for all  $x \in E$  and  $y \in C$ . In this case,  $Q$  is uniquely determined. Using this result, Aoyama et al. obtained the following result. Recall that, for a mapping  $T : C \rightarrow E$ , the set of fixed points of  $T$  is denoted by  $F(T)$ , that is,  $F(T) = \{x \in C : x = Tx\}$ .

**Lemma 2.3** ([5]). *Let  $C$  be a nonempty closed convex subset of a smooth Banach space*

*$E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$ , and let  $A : C \rightarrow E$  be a mapping. Then, for each  $\lambda > 0$ ,*

$$S(C, A) = F(Q_C(I - \lambda A)).$$

The space  $E$  is said to satisfy *Opial's condition* if

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

whenever  $x_n \rightarrow x \in E$  and  $y \in E$  satisfy  $x \neq y$ . The following results are known from theory of nonexpansive mappings. It should be noted that Opial's condition and the Fréchet differentiability of the norm are independent in uniformly convex space setting.

**Lemma 2.4** ([7], [8]). *Let  $C$  be a nonempty closed convex subset of a Banach space.  $E$ . Suppose that  $E$  is uniformly convex or satisfies Opial's condition. Suppose that  $T$  is a nonexpansive mapping of  $C$  into itself. Then,  $I - T$  is demiclosed at zero, that is, if  $\{x_n\}$  is a sequence in  $C$  such that  $x_n \rightarrow p$  and  $x_n - Tx_n \rightarrow 0$ , then  $p = Tp$ .*

**Lemma 2.5** ([9]). *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm. Suppose that  $\{T_n\}_{n=1}^\infty$  is a sequence of nonexpansive mappings of  $C$  into itself with  $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ . Let  $x \in C$  and  $S_n = T_n T_{n-1} \cdot \dots \cdot T_1$  for all  $n \geq 1$ . Then, the set*

$$\bigcap_{n=1}^\infty \overline{\text{co}}\{S_m x : m \geq n\} \cap \bigcap_{n=1}^\infty F(T_n)$$

*consists of at most one element, where  $\overline{\text{co}}D$  is the closed convex hull of  $D$ .*

The following two lemmas are proved in the absence of uniform convexity, and they are needed in Section 3.2.

**Lemma 2.6** ([10]). *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space and  $\{\alpha_n\}$  be a real sequence in  $[0, 1]$  such that  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ . Suppose that  $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n$  for all  $n \geq 1$ . If  $\limsup_{n \rightarrow \infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$ , then  $x_n - y_n \rightarrow 0$ .*

**Lemma 2.7** ([11]). *Let  $\{z_n\}$  and  $\{w_n\}$  be sequences in a Banach space and  $\{\alpha_n\}$  be a real sequence in  $[0, 1]$ . Suppose that  $z_{n+1} = \alpha_n z_n + (1 - \alpha_n)w_n$  for all  $n \geq 1$ . If the following properties are satisfied:*

- (i)  $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$  and  $\liminf_{n \rightarrow \infty} \alpha_n > 0$ ;
- (ii)  $\lim_{n \rightarrow \infty} ||z_n|| = d$  and  $\limsup_{n \rightarrow \infty} ||w_n|| \leq d$ ;
- (iii) the sequence  $\{\sum_{i=1}^n (1 - \alpha_i)w_i\}$  is bounded;

*then  $d = 0$ .*

We also need the following simple but interesting results.

**Lemma 2.8** ([12]). *Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of nonnegative real numbers.*

*If  $\sum_{n=1}^\infty b_n < \infty$  and  $a_{n+1} \leq a_n + b_n$  for all  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} a_n$  exists.*

**Lemma 2.9** ([13]). *Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of nonnegative real numbers. If*

$$\sum_{n=1}^\infty a_n b_n < \infty \text{ and } \sum_{n=1}^\infty a_n b_n < \infty, \text{ then } \liminf_{n \rightarrow \infty} b_n = 0.$$

### 3. Main results

From now on, we assume that

- $E$  is 2-uniformly smooth Banach space with a 2-uniform smoothness constant  $K$ ;

- $C$  is a nonempty closed convex subset of  $E$ ;
- $Q_C$  is a sunny nonexpansive retraction from  $E$  onto  $C$ ;
- $A : C \rightarrow E$  is an  $\alpha$ -inverse strongly accretive mapping with  $S(C, A) \neq \emptyset$  and  $\alpha > 0$ .

Suppose that  $\{x_n\}$  is iteratively defined by

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{\lambda_n\} \subset (0, \frac{\alpha}{K^2}]$ . For convenience, we write  $y_n \equiv Q_C(x_n - \lambda_n A x_n)$ .

### 3.1. Some properties of the sequence $\{x_n\}$ for weak convergence theorems

We start with some propositions, which are the common tools for proving the main results in the next two subsections.

**Proposition 3.1.** *If  $p \in S(C, A)$ , then  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists, and hence, the sequences  $\{x_n\}$  and  $\{A x_n\}$  are both bounded.*

*Proof.* Let  $p \in S(C, A)$ . By the nonexpansiveness of  $Q_C(I - \lambda_n A)$  for all  $n \geq 1$  and Lemma 2.3, we have

$$\|y_n - p\| = \|Q_C(I - \lambda_n A)x_n - (Q_C(I - \lambda_n A)p)\| \leq \|x_n - p\|$$

for all  $n \geq 1$ . This implies that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(y_n - p)\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|y_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| = \|x_n - p\| \end{aligned}$$

for all  $n \geq 1$ . Therefore,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists, and hence, the sequence  $\{x_n\}$  is bounded. Since  $A$  is  $\frac{1}{\alpha}$ -Lipschitzian, we have  $\{A x_n\}$  is bounded. The proof is finished.

**Proposition 3.2.** *The following inequality holds:*

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\|$$

for all  $n \geq 1$ .

*Proof.* Since  $Q_C(I - \lambda_{n+1} A)$  and  $Q_C$  are nonexpansive, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|Q_C(I - \lambda_{n+1} A)x_{n+1} - Q_C(I - \lambda_n A)x_n\| \\ &\leq \|Q_C(I - \lambda_{n+1} A)x_{n+1} - Q_C(I - \lambda_{n+1} A)x_n\| \\ &\quad + \|Q_C(I - \lambda_{n+1} A)x_n - Q_C(I - \lambda_n A)x_n\| \\ &\leq \|x_{n+1} - x_n\| + \|(I - \lambda_{n+1} A)x_n - (I - \lambda_n A)x_n\| \\ &= \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\|. \end{aligned}$$

□

**Proposition 3.3.** *Suppose that  $E$  is a reflexive Banach space such that either it is uniformly convex or it satisfies Opial's condition. Suppose that  $\{x_n\}$  is a bounded sequence of  $C$  satisfying  $x_n - Q_C(I - \lambda_n A)x_n \rightarrow 0$  and  $\{\lambda_n\} \subset [a, \frac{\alpha}{K^2}] \subset (0, \frac{\alpha}{K^2}]$ .*

*Then,  $\{x_n\}$  converges weakly to some element of  $S(C, A)$ .*

*Proof.* Suppose that  $E$  is a uniformly convex Banach space or a reflexive Banach space satisfying Opial's condition. Then,  $\omega_w(\{x_n\}) \neq \emptyset$ . We first prove that  $\omega_w(\{x_n\}) \subset S(C, A)$ . To see this, let  $z \in \omega_w(\{x_n\})$ . Passing to a subsequence, if necessary, we assume that there exists a subsequence  $\{n_k\}$  of  $\{n\}$  such that  $x_{n_k} \rightharpoonup z$  and  $\lambda_{n_k} \rightarrow \lambda \in [a, \frac{\alpha}{\alpha^2}]$ . We observe that

$$\begin{aligned} \|x_{n_k} - Q_C(I - \lambda A)x_{n_k}\| &\leq \|x_{n_k} - \gamma_{n_k}\| + \|\gamma_{n_k} - Q_C(I - \lambda A)x_{n_k}\| \\ &\leq \|x_{n_k} - \gamma_{n_k}\| + \|(I - \lambda_{n_k}A)x_{n_k} - (I - \lambda A)x_{n_k}\| \\ &= \|x_{n_k} - \gamma_{n_k}\| + |\lambda_{n_k} - \lambda| \|Ax_{n_k}\|. \end{aligned}$$

This implies that  $x_{n_k} - Q_C(I - \lambda A)x_{n_k} \rightarrow 0$ . By the nonexpansiveness of  $Q_C(I - \lambda A)$ , Lemmas 2.3 and 2.4, we obtain that  $z \in F(Q_C(I - \lambda A)) = S(C, A)$ . Hence  $\omega_w(\{x_n\}) \subset S(C, A)$ .

We next prove that  $\omega_w(\{x_n\})$  is exactly a singleton in the following cases.

Case 1:  $E$  is uniformly convex. We follow the idea of Aoyama et al. [5] in this case. For any  $n \geq 1$ , we define a nonexpansive mapping  $T_n : C \rightarrow C$  by

$$T_n = \alpha_n I + (1 - \alpha_n)Q_C(I - \lambda_n A).$$

We get that  $x_{n+1} = T_n T_{n-1} \cdots T_1 x_1$  for all  $n \geq 1$ . It follows from Lemma 2.3 that  $S(C, A) = \bigcap_{n=1}^{\infty} F(Q_C(I - \lambda_n A)) \subset \bigcap_{n=1}^{\infty} F(T_n)$ . Applying Lemma 2.5, since every 2-uniformly smooth Banach space has Fréchet differentiable norm, gives

$$\bigcap_{n=1}^{\infty} \overline{\text{co}}\{x_m : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n)$$

consists of at most one element. But we know that

$$\emptyset \neq \omega_w(\{x_n\}) \subset \bigcap_{n=1}^{\infty} \overline{\text{co}}\{x_m : m \geq n\} \cap S(C, A) \subset \bigcap_{n=1}^{\infty} \overline{\text{co}}\{x_m : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

Therefore,  $\omega_w(\{x_n\})$  is a singleton.

Case 2:  $E$  satisfies Opial's condition. Suppose that  $p$  and  $q$  are two different elements of  $\omega_w(\{x_n\})$ . There are subsequences  $\{x_{n_k}\}$  and  $\{x_{m_j}\}$  of  $\{x_n\}$  such that

$$x_{n_k} \rightharpoonup p \quad \text{and} \quad x_{m_j} \rightharpoonup q.$$

Since  $p$  and  $q$  also belong to  $S(C, A)$ , both limits  $\lim_{n \rightarrow \infty} \|x_n - p\|$  and  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exist. Consequently, by Opial's condition,

$$\begin{aligned} \lim_{k \rightarrow \infty} \|x_{n_k} - p\| &< \lim_{k \rightarrow \infty} \|x_{n_k} - q\| = \lim_{j \rightarrow \infty} \|x_{m_j} - q\| \\ &< \lim_{j \rightarrow \infty} \|x_{m_j} - p\| = \lim_{k \rightarrow \infty} \|x_{n_k} - p\|. \end{aligned}$$

This is a contradiction. Hence,  $\omega_w(\{x_n\})$  is a singleton, and the proof is finished.  $\square$

**Remark 3.4.** There exists a reflexive Banach space such that it satisfies Opial's condition but it is not uniformly convex. In fact, we consider  $E = \mathbb{R}^2$  with the norm  $\|(x, y)\| = |x| + |y|$  for all  $(x, y) \in \mathbb{R}^2$ . Note that  $E$  is finite dimensional, and hence it is reflexive and satisfies Opial's condition. To see that  $E$  is not uniformly convex, let  $x = (1, 0)$  and  $y = (0, 1)$ , it follows that  $\|x - y\| = \|(1, -1)\| = 2$  and  $\|x + y\|/2 = \|(1/2, 1/2)\| = 1 \notin 1 - \delta$  for all  $\delta > 0$ .

### 3.2. Convergence results without uniform convexity

In this subsection, we make use of Lemmas 2.6 and 2.7 to show that  $x_n - y_n \rightarrow 0$  under the additional restrictions on the sequences  $\{\alpha_n\}$  and  $\{\lambda_n\}$ .

**Proposition 3.5.** *Suppose that  $\{\alpha_n\} \subset [c, d] \subset (0, 1)$  and  $\lambda_{n+1} - \lambda_n \rightarrow 0$ . Then,  $x_n - y_n \rightarrow 0$ .*

*Proof.* We will apply Lemma 2.6. Let us rewritten the iteration as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n.$$

It follows from Proposition 3.1 that  $\{x_n\}$  and  $\{Ax_n\}$  are bounded. Then,  $\{y_n\} = \{(I - \lambda_n A)x_n\}$  is bounded. Since  $\lambda_{n+1} - \lambda_n \rightarrow 0$ , it is a consequence of Proposition 3.2 that

$$\limsup_{n \rightarrow \infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq \limsup_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| ||Ax_n|| = 0.$$

Since all the requirements of Lemma 2.6 are satisfied,  $x_n - y_n \rightarrow 0$ .  $\square$

**Proposition 3.6.** *Suppose that  $\{\alpha_n\}$  and  $\{\lambda_n\}$  satisfy the following properties:*

- (i)  $\{\alpha_n\} \subset [c, 1) \subset (0, 1)$  and  $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$ ;
- (ii)  $\frac{\lambda_{n+1} - \lambda_n}{1 - \alpha_n} \rightarrow 0$  and  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ .

*Then,  $x_n - y_n \rightarrow 0$ .*

*Proof.* We will apply Lemma 2.7. From the iteration, we have

$$z_{n+1} = \alpha_n z_n + (1 - \alpha_n) w_n,$$

where  $z_n \equiv x_n - y_n$  and  $w_n \equiv \frac{y_n - y_{n+1}}{1 - \alpha_n}$ . Using Proposition 3.2, we obtain

$$\begin{aligned} ||z_{n+1}|| &\leq \alpha_n ||z_n|| + ||y_n - y_{n+1}|| \\ &\leq \alpha_n ||z_n|| + ||x_{n+1} - x_n|| + |\lambda_{n+1} - \lambda_n| ||Ax_n|| \\ &= \alpha_n ||z_n|| + (1 - \alpha_n) ||z_n|| + |\lambda_{n+1} - \lambda_n| ||Ax_n|| \\ &= ||z_n|| + |\lambda_{n+1} - \lambda_n| ||Ax_n||. \end{aligned}$$

It follows from  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| ||Ax_n|| < \infty$  and Lemma 2.8 that  $d := \lim_{n \rightarrow \infty} ||z_n||$  exists. We next prove that  $\limsup_{n \rightarrow \infty} ||w_n|| \leq d$ . Again, by Proposition 3.2, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} ||w_n|| &= \limsup_{n \rightarrow \infty} \frac{||y_n - y_{n+1}||}{1 - \alpha_n} \\ &\leq \lim_{n \rightarrow \infty} ||z_n|| + \limsup_{n \rightarrow \infty} \frac{|\lambda_{n+1} - \lambda_n|}{1 - \alpha_n} ||Ax_n|| = d. \end{aligned}$$

Finally, for all  $n \geq 1$ , we have

$$\sum_{i=1}^n (1 - \alpha_i) w_i = \sum_{i=1}^n (y_i - y_{i+1}) = y_1 - y_{n+1}.$$

Hence, the sequence  $\{\sum_{i=1}^n (1 - \alpha_i) w_i\}$  is bounded. It follows then that  $d = 0$ .  $\square$

We now have the following weak convergence theorems without uniform convexity.

**Theorem 3.7.** *Let  $E$  be a 2-uniformly smooth Banach space satisfying Opial's condition. Let  $C$  be a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and  $A : C \rightarrow E$  be an  $\alpha$ -inverse strongly accretive mapping with  $S(C, A) \neq \emptyset$  and  $\alpha > 0$ . Suppose that  $\{x_n\}$  is iteratively defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{\lambda_n\} \subset [a, \frac{\alpha}{K^2}] \subset (0, \frac{\alpha}{K^2}]$  satisfy one of the following conditions:

- (i)  $\{\alpha_n\} \subset [c, d] \subset (0, 1)$  and  $\lambda_{n+1} - \lambda_n \rightarrow 0$ ;
- (ii)  $\{\alpha_n\} \subset [c, 1) \subset (0, 1)$ ,  $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$ ,  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ , and  $\frac{\lambda_{n+1} - \lambda_n}{1 - \alpha_n} \rightarrow 0$ .

Then,  $\{x_n\}$  converges weakly to an element in  $S(C, A)$ .

*Proof.* Note that every 2-uniformly smooth Banach space is reflexive. The result follows from Propositions 3.3, 3.5 and 3.6.  $\square$

**Remark 3.8.** Conditions (i) and (ii) in Theorem 3.7 are not comparable.

(1) If  $\alpha_n \equiv \frac{1}{2}$  and  $\{\lambda_n\}$  is a sequence in  $(0, \frac{\alpha}{K^2}]$  such that  $\lambda_n - \lambda_{n+1} \rightarrow 0$  and  $0 < \liminf_{n \rightarrow \infty} \lambda_n < \limsup_{n \rightarrow \infty} \lambda_n < 1$ , then  $\{\alpha_n\}$  and  $\{\lambda_n\}$  satisfy condition (i) but fail condition (ii).

(2) If  $\alpha_n \equiv \frac{n}{n+1}$  and  $\lambda_n \equiv \lambda \in (0, \frac{\alpha}{K^2}]$ , then  $\{\alpha_n\}$  and  $\{\lambda_n\}$  satisfy condition (ii) but fail condition (i).

**Remark 3.9.** Note that the Opial property and uniform convexity are independent. Theorem 3.7 is a supplementary to Theorem 3.1 of Aoyama et al. [5].

### 3.3. Convergence results in uniformly convex spaces

In this subsection, we prove two more convergence theorems in uniformly convex spaces, which are also a supplementary to Theorem 3.1 of Aoyama et al. [5]. Let us start with some propositions.

**Proposition 3.10.** Assume that  $E$  is a uniformly convex Banach space. Suppose that  $\{\alpha_n\}$  and  $\{\lambda_n\}$  satisfy the following properties:

- (i)  $\{\lambda_n\} \subset [a, \alpha/K^2] \subset (0, \alpha/K^2]$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$  and  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ .

Then,  $x_n - y_n \rightarrow 0$ .

*Proof.* Let  $p \in S(C, A)$ . Note that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists and hence both  $\{x_n\}$  and  $\{y_n\}$  are bounded. By the uniform convexity of  $E$  and Lemma 2.1, there exists a continuous and strictly increasing function  $g$  such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(y_n - p)\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|y_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - y_n\|) \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - y_n\|) \\ &= \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)g(\|x_n - y_n\|) \end{aligned}$$

for all  $n \geq 1$ . Hence, for each  $m \geq 1$ , we have

$$\sum_{n=1}^m \alpha_n(1 - \alpha_n)g(\|x_n - y_n\|) \leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2.$$

In particular,  $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)g(\|x_n - y_n\|) < \infty$ . It follows from  $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$  and Lemma 2.9 that  $\liminf_{n \rightarrow \infty} g(\|x_n - y_n\|) = 0$ . By the properties of the function  $g$ , we get that  $\liminf_{n \rightarrow \infty} \|x_n - y_n\| = 0$ . Finally, we show that  $\lim_{n \rightarrow \infty} \|x_n - y_n\|$  actually exists. To see this, we consider the following estimate obtained directly from Proposition 3.2:

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &\leq \|x_{n+1} - y_n\| + \|y_n - y_{n+1}\| \\ &\leq \alpha_n \|x_n - y_n\| + \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|Ax_n\| \\ &= \alpha_n \|x_n - y_n\| + (1 - \alpha_n) \|x_n - y_n\| + |\lambda_{n+1} - \lambda_n| \|Ax_n\| \\ &= \|x_n - y_n\| + |\lambda_{n+1} - \lambda_n| \|Ax_n\|. \end{aligned}$$

The assertion follows since  $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| \|Ax_n\| < \infty$  and Lemma 2.8.  $\square$

Let us recall the concept of strongly nonexpansive sequences introduced by Aoyama et al. (see [14]). A sequence of nonexpansive mappings  $\{T_n\}$  of  $C$  is called a *strongly nonexpansive sequence* if  $x_n - y_n - (T_n x_n - T_n y_n) \rightarrow 0$  whenever  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \rightarrow 0$ . It is noted that if  $\{T_n\}$  is a constant sequence, then this property reduces to the concept of strongly nonexpansive mappings studied by Bruck and Reich [15].

**Proposition 3.11.** *Assume that  $E$  is a uniformly convex Banach space and  $\{\lambda_n\} \subset (0, b] \subset (0, \alpha/K^2)$ . Then,  $\{Q_C(I - \lambda_n A)\}$  is a strongly nonexpansive sequence.*

*Proof.* Notice first that  $Q_C$  is a strongly nonexpansive mapping (see [16,17]). Next, we prove that  $\{I - \lambda_n A\}$  is a strongly nonexpansive sequence and then the assertion follows. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|(I - \lambda_n A)x_n - (I - \lambda_n A)y_n\| \rightarrow 0$ . It follows from Lemma 2.2 that

$$\begin{aligned} &\frac{2(\alpha - K^2 b)}{b} \|\lambda_n A x_n - \lambda_n A y_n\|^2 \\ &\leq \frac{2(\alpha - K^2 \lambda_n)}{\lambda_n} \|\lambda_n A x_n - \lambda_n A y_n\|^2 \\ &= 2\lambda_n(\alpha - K^2 \lambda_n) \|Ax_n - Ay_n\|^2 \\ &\leq \|x_n - y_n\|^2 - \|(I - \lambda_n A)x_n - (I - \lambda_n A)y_n\|^2 \rightarrow 0. \end{aligned}$$

In particular,  $\lambda_n A x_n - \lambda_n A y_n \rightarrow 0$  and hence

$$x_n - y_n - ((I - \lambda_n A)x_n - (I - \lambda_n A)y_n) = \lambda_n A x_n - \lambda_n A y_n \rightarrow 0.$$

**Proposition 3.12.** *Assume that  $E$  is a uniformly convex Banach space. Suppose that  $\alpha_n \equiv 0$  and  $\{\lambda_n\} \subset (0, b] \subset (0, \alpha/K^2)$ . Then,  $x_n - y_n \rightarrow 0$ .*

*Proof.* Let us rewritten the iteration as follows:

$$x_{n+1} = Q_C(I - \lambda_n A)x_n \quad (n \geq 1).$$

Let  $p \in S(C, A)$ . Notice that  $p = Q_C(I - \lambda_n A)p$  for all  $n \geq 1$ . Then,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists, and hence,

$$\|x_n - p\| - \|Q_C(I - \lambda_n A)x_n - p\| = \|x_n - p\| - \|x_{n+1} - p\| \rightarrow 0.$$

It follows from the preceding proposition that

$$x_n - Q_C(I - \lambda_n A)x_n = (x_n - p) - (Q_C(I - \lambda_n A)x_n - p) \rightarrow 0.$$

□

We now obtain the following weak convergence theorems in uniformly convex spaces.

**Theorem 3.13.** *Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space. Let  $C$  be a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and  $A : C \rightarrow E$  be an  $\alpha$ -inverse strongly accretive mapping with  $S(C, A) \neq \emptyset$  and  $\alpha > 0$ . Suppose that  $\{x_n\}$  is iteratively defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{\lambda_n\} \subset [a, \frac{\alpha}{K^2}] \subset (0, \frac{\alpha}{K^2}]$  satisfy one of the following conditions:

- (i)  $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$  and  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ ;
- (ii)  $\alpha_n \equiv 0$  and  $\{\lambda_n\} \subset [a, b] \subset (0, \alpha/K^2)$ .

Then,  $\{x_n\}$  converges weakly to an element in  $S(C, A)$ .

*Proof.* The result follows from Propositions 3.3, 3.10 and 3.12. □

**Remark 3.14.** It is easy to see that conditions (i) and (ii) in Theorem 3.13 are not comparable.

**Remark 3.15.** Compare Theorem 3.13 to Theorem 1.2 of Aoyama et al., our result is a supplementary to their result. It is noted that, for example, our iteration scheme with  $\alpha_n \equiv 0$  and  $\lambda_n \equiv \alpha/(\alpha/K^2)$  is simpler than the one in Theorem 1.2.

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#### Authors' contributions

All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

#### Competing interests

The authors declare that they have no competing interests.

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# On Maingé's Approach for Hierarchical Optimization Problems

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**Abstract** This paper aims at investigating an iterative method for solving a system of variational inequalities with fixed-point set constraints. Our scheme can be regarded as a more general variant of the algorithm proposed by Maingé. Strong convergence results are established in the setting of Hilbert spaces. We propose an alternative analysis that allows us to relax some assumption imposed in his paper for convergence of the considered method. As a complementary result, we show how to adapt these processes to the case when the constraints involve operators belonging to the class of hemi-contractive mappings; this goes beyond the scope of Maingé's result.

**Keywords** Fixed points · Hierarchical optimization problems · Nonexpansive mappings

## 1 Introduction

The concept of variational inequalities plays an important role in various kinds of problems in pure and applied sciences (see, for example, [1–10]). Moreover, the rapid development and the prolific growth of the theory of variational inequalities have been made by many researchers. For solving some certain variational inequalities, it is sometimes recast as fixed-point problems with appropriate projections. However, this is getting difficult when the projection cannot be computed explicitly. To overcome this drawback, we replace the projection by some suitable mapping with a nice

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fixed-point set. This strategy also suggests an effective approximation process and it has been widely investigated and developed in the literature [10–14]. In this paper, we investigate a more general variant of the scheme proposed by Maingé [14] for a system of variational inequalities with fixed-point set constraints. Our analysis allows us to remove some restriction on the parameters as was the case in his paper [14]. The idea of this analysis is taken from the other recent result of Maingé [15]. As a complementary result, we show how to adapt the considered algorithm to the case when the involved operators belong to the class of hemi-contractive mappings, which is beyond the scope of Maingé's result.

The paper is organized as follows. In Sect. 2, we collect together definitions and some preliminaries that pertain to the paper with corresponding references. Our main result is presented in Sect. 3 as Theorem 3.2. As a consequence, we deduce the convergence result of Maingé with a weaker assumption in Corollary 3.1. Finally, we summarize our results in Sect. 4.

## 2 Preliminaries

Throughout, let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . For a closed convex subset  $C$  of  $H$ , the (metric) projection  $P_C : H \rightarrow C$  is defined for each  $x \in H$  as the unique element  $P_C x \in C$  such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

**Lemma 2.1** [8] *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$ . Then, for all  $x \in H$  and  $y \in C$ ,  $y = P_C x$  if and only if  $\langle y - x, z - y \rangle \geq 0$  for all  $z \in C$ .*

For a mapping  $T : H \rightarrow H$ , the fixed-point set of  $T$  is denoted by  $\text{Fix}(T)$ , that is,  $\text{Fix}(T) = \{x \in H : x = Tx\}$ . In this paper, we are interested in a system of variational inequalities with fixed-point set constraints. More precisely, let  $T_1, T_2, F : H \rightarrow H$  be mappings such that  $\text{Fix}(T_1)$  and  $\text{Fix}(T_2)$  are nonempty, closed, and convex. The class of hierarchical optimization problems treated by Maingé [14] consists in finding  $(x^*, y^*) \in \text{Fix}(T_1) \times \text{Fix}(T_2)$  such that, for given positive real numbers  $\rho$  and  $\eta$ , the following two inequalities hold:

$$\begin{aligned} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in \text{Fix}(T_1), \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle &\geq 0, \quad \forall y \in \text{Fix}(T_2). \end{aligned} \tag{1}$$

In the important case when  $T_1$  and  $T_2$  are the nearest point projections onto the closed and convex sets  $K_1$  and  $K_2$ , respectively, it is clear that  $\text{Fix}(T_1) = K_1$ ,  $\text{Fix}(T_2) = K_2$  and (1) becomes the problem of finding  $(x^*, y^*) \in K_1 \times K_2$  such that

$$\begin{aligned} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in K_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle &\geq 0, \quad \forall y \in K_2. \end{aligned} \tag{2}$$

This system forms a more general problem originated from Nash equilibrium points and it was treated from a theoretical viewpoint in [1–5]. In particular, if  $K_1 = K_2 = K$  and  $\eta = 0, \rho > 0$ , then (2) reduces to the *classic variational inequality* (see [6, 7, 16, 17]), that is, the problem of finding  $x^* \in K$  such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in K.$$

In (1), it is worth noting that, since  $\text{Fix}(T_1)$  and  $\text{Fix}(T_2)$  are (nonempty) closed and convex, the projections  $P_{\text{Fix}(T_1)}$  and  $P_{\text{Fix}(T_2)}$  from  $H$  onto  $\text{Fix}(T_1)$  and  $\text{Fix}(T_2)$ , respectively, are well defined and the problem above is equivalent to the problem of finding  $(x^*, y^*) \in \text{Fix}(T_1) \times \text{Fix}(T_2)$  such that

$$\begin{aligned} x^* &= P_{\text{Fix}(T_1)}[y^* - \rho F(y^*)], \\ y^* &= P_{\text{Fix}(T_2)}[x^* - \eta F(x^*)]. \end{aligned} \tag{3}$$

However, in practice, the both sets  $\text{Fix}(T_1)$  and  $\text{Fix}(T_2)$  (and hence the two projections) are not given explicitly.

Inspired by the method studied by Yamada et al. (see also [10, 11]), Maingé [14] recently proposed an algorithm to solve this problem, which can be summarized as follows. (The related definitions will be given after this theorem.)

**Theorem 2.1** *Let  $T_1, T_2 : H \rightarrow H$  be  $\beta$ -demicontractive mappings, where  $\beta \in [0, 1[$ , such that  $I - T_1$  and  $I - T_2$  are demiclosed at zero. Let  $F : H \rightarrow H$  be a  $\mu$ -Lipschitzian and  $r$ -strongly monotone mapping. Let  $\{x_n\}$  and  $\{y_n\}$  be generated by*

$$\begin{aligned} x_0, y_0 &\in H \\ v_n &= x_n - \gamma_n [\rho F(y_n) + x_n - y_n], \\ w_n &= y_n - \gamma_n [\eta F(x_n) + y_n - x_n], \\ x_{n+1} &= [(1 - \xi_1)I + \xi_1 T_1](v_n), \\ y_{n+1} &= [(1 - \xi_2)I + \xi_2 T_2](w_n), \end{aligned} \tag{4}$$

where  $I$  is the identity mapping and the following conditions hold:

- (a)  $\xi_1, \xi_2 \in ]0, \frac{1-\beta}{2}[$ ;
- (b)  $\{\gamma_n\} \subset [0, 1[$ ,  $\lim_{n \rightarrow \infty} \gamma_n = 0$  and  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ;
- (c)  $\rho, \eta \in ]0, \frac{2r}{\mu^2}[$ .

Then the sequence  $\{(x_n, y_n)\}$  converges strongly to the unique solution  $(x^*, y^*)$  of the hierarchical optimization problem (1).

Let us recall some definitions of mappings involved in our study.

**Definition 2.1** A mapping  $T : H \rightarrow H$  is called

- $\mu$ -Lipschitzian iff

$$\|Tx - Ty\| \leq \mu \|x - y\| \quad \text{for all } x, y \in H;$$

- a contraction iff it is  $\alpha$ -Lipschitzian with  $\alpha \in [0, 1[$ , and in this case, we also say that  $T$  is a contraction with the coefficient  $\alpha$ ;
- $r$ -strongly monotone iff

$$\langle Tx - Ty, x - y \rangle \geq r \|x - y\|^2 \quad \text{for all } x, y \in H;$$

- quasi-nonexpansive iff  $\text{Fix}(T) \neq \emptyset$  and

$$\|Tx - q\| \leq \|x - q\| \quad \text{for all } (x, q) \in H \times \text{Fix}(T);$$

- $\beta$ -demicontractive [18, 19] iff  $\text{Fix}(T) \neq \emptyset$  and  $T$  satisfies

$$\langle x - Tx, x - q \rangle \geq \frac{1 - \beta}{2} \|x - Tx\|^2 \quad \text{for all } (x, q) \in H \times \text{Fix}(T)$$

(the inequality above is equivalent to

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \beta \|x - Tx\|^2;$$

- strongly quasi-nonexpansive [20] iff  $T$  is quasi-nonexpansive and

$$x_n - Tx_n \rightarrow 0$$

whenever  $\{x_n\}$  is a bounded sequence in  $H$  and  $\|x_n - q\| - \|Tx_n - q\| \rightarrow 0$  for some  $q \in \text{Fix}(T)$ .

It is clear from the preceding definitions that every quasi-nonexpansive mapping is 0-demicontractive. Note that if  $T : H \rightarrow H$  is a mapping with a nonempty fixed-point set  $\text{Fix}(T)$ , then

$$T \text{ is quasi-nonexpansive} \iff \langle x - Tx, x - q \rangle \geq \frac{1}{2} \|x - Tx\|^2 \\ \text{for all } (x, q) \in H \times \text{Fix}(T).$$

As shown in the following proposition, the classes of quasi-nonexpansive mappings and of demicontractive mappings are closely related. To find a fixed point of a  $\beta$ -demicontractive mapping, it suffices to find a fixed point of a quasi-nonexpansive mapping.

**Proposition 2.1** Let  $D : H \rightarrow H$  be a mapping and  $T = (1 - \omega)I + \omega D$  where  $\omega > 0$ . Then the following statements hold:

- $\text{Fix}(T) = \text{Fix}(D)$ .
- If  $D$  is a  $\beta$ -demicontractive mapping with  $\beta \in [0, 1[$  and  $\omega \in ]0, 1 - \beta]$ , then  $T$  is quasi-nonexpansive.

(c) If  $T$  is a quasi-nonexpansive mapping and  $\omega \in ]0, 1[$ , then  $D$  is  $\beta$ -demicontractive where  $\beta \in [1 - \omega, 1[$ .

*Proof* (a) It is obvious that  $\text{Fix}(D) = \text{Fix}(T)$ .

(b) This is Remark 2.1 of [14].

(c) Assume that  $T$  is quasi-nonexpansive and  $\omega \in ]0, 1[$ . Let  $(x, q) \in H \times \text{Fix}(T)$ . Then

$$\langle x - Dx, x - q \rangle = \frac{1}{\omega} \langle x - Tx, x - q \rangle \geq \frac{1}{2\omega} \|x - Tx\|^2 = \frac{\omega}{2} \|x - Dx\|^2.$$

Hence  $D$  is  $\beta$ -demicontractive if

$$\frac{\omega}{2} \geq \frac{1 - \beta}{2}, \quad \text{that is, } \beta \geq 1 - \omega. \quad \square$$

We denote the strong and weak convergence of a sequence  $\{x_n\}$  in  $H$  to an element  $x \in H$  by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$ , respectively. We also recall some known results with the corresponding references.

**Lemma 2.2** [8] For  $x, y \in H$  and  $\omega \in \mathbb{R}$ , we have the following statements:

- $|\langle x, y \rangle| \leq \|x\| \|y\|$ ;
- $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$  (the subdifferential inequality);
- $\|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - \omega(1 - \omega)\|x - y\|^2$ .

**Lemma 2.3** [15] Let  $\{a_n\}$  be a sequence of real numbers such that there exists a subsequence  $\{a_{m_j}\}$  of  $\{a_n\}$  such that  $a_{m_j} < a_{m_{j+1}}$  for all  $j \in \mathbb{N}$ . Then there exists a nondecreasing sequence  $\{n_k\}$  of  $\mathbb{N}$  such that  $\lim_{k \rightarrow \infty} n_k = \infty$  and the following properties are satisfied by all (sufficiently large) number  $k \in \mathbb{N}$ :

$$a_{n_k} \leq a_{n_{k+1}} \quad \text{and} \quad a_k \leq a_{n_{k+1}}.$$

In fact,  $n_k$  is the largest number  $n$  in the set  $\{1, \dots, k\}$  such that  $a_n < a_{n+1}$  holds.

**Lemma 2.4** [21, 22] Let  $\{a_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  a sequence in  $]0, 1[$  with  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\{b_n\}$  a sequence of nonnegative real numbers with  $\sum_{n=1}^{\infty} b_n < \infty$  and  $\{\gamma_n\}$  a sequence of real numbers with  $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$ . Suppose that the following inequality

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \gamma_n + b_n$$

holds for all  $n \in \mathbb{N}$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

The following result is a consequence of Lemmas 2.3 and 2.4. It plays an important role in this paper.

**Lemma 2.5** Let  $\{a_n\} \subset [0, \infty[$ ,  $\{\alpha_n\} \subset [0, 1[$ ,  $\{b_n\} \subset ] - \infty, \infty[$ ,  $\hat{\alpha} \in [0, 1[$  be such that

- $\{a_n\}$  is a bounded sequence;
- $a_{n+1} \leq (1 - \alpha_n)^2 a_n + 2\alpha_n \widehat{\alpha} \sqrt{a_n} \sqrt{a_{n+1}} + \alpha_n b_n$  for all  $n \in \mathbb{N}$ ;
- whenever  $\{a_{n_k}\}$  is a subsequence of  $\{a_n\}$  satisfying  $\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0$ , it follows that  $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$ ;
- $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof* Let us consider the following two cases.

*Case 1:* There exists an  $n_0 \in \mathbb{N}$  such that  $a_{n+1} \leq a_n$  for all  $n \geq n_0$ . We may assume that  $\alpha_n < \frac{1}{2(1-\widehat{\alpha})}$  for all  $n \geq n_0$ . It follows then that  $\liminf_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$  and hence  $\limsup_{n \rightarrow \infty} b_n \leq 0$ . Furthermore, for all  $n \geq n_0$ ,

$$\begin{aligned} a_{n+1} &\leq (1 - \alpha_n)^2 a_n + 2\alpha_n \widehat{\alpha} \sqrt{a_n} \sqrt{a_{n+1}} + \alpha_n b_n \\ &\leq (1 - \alpha_n)^2 a_n + 2\alpha_n \widehat{\alpha} a_n + \alpha_n b_n \\ &= (1 - 2\alpha_n(1 - \widehat{\alpha})) a_n + 2\alpha_n(1 - \widehat{\alpha}) \left( \frac{\alpha_n a_n + b_n}{2(1 - \widehat{\alpha})} \right). \end{aligned}$$

Notice that  $\sum_{n=n_0}^{\infty} 2\alpha_n(1 - \widehat{\alpha}) = \infty$  and

$$\limsup_{n \rightarrow \infty} \frac{\alpha_n a_n + b_n}{2(1 - \widehat{\alpha})} = \limsup_{n \rightarrow \infty} \frac{b_n}{2(1 - \widehat{\alpha})} \leq 0.$$

It follows from Lemma 2.4 that  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Case 2:* There exists a subsequence  $\{a_{m_j}\}$  of  $\{a_n\}$  such that  $a_{m_j} < a_{m_{j+1}}$  for all  $j \in \mathbb{N}$ . From Lemma 2.3, there exists a nondecreasing sequence  $\{n_k\}$  of  $\mathbb{N}$  such that  $\lim_{k \rightarrow \infty} n_k = \infty$  and the following inequalities hold for all  $k \in \mathbb{N}$ :

$$a_{n_k} \leq a_{n_{k+1}} \quad \text{and} \quad a_k \leq a_{n_{k+1}}.$$

By discarding the repeated terms of  $\{n_k\}$  but still denoted by  $\{n_k\}$ , we can view  $\{a_{n_k}\}$  as a subsequence of  $\{a_n\}$ . In this case, we have  $\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0$ , and hence  $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$ . Notice that

$$\begin{aligned} a_{n_{k+1}} &\leq (1 - \alpha_{n_k})^2 a_{n_k} + 2\alpha_{n_k} \widehat{\alpha} \sqrt{a_{n_k}} \sqrt{a_{n_{k+1}}} + \alpha_{n_k} b_{n_k} \\ &\leq (1 - \alpha_{n_k})^2 a_{n_{k+1}} + 2\alpha_{n_k} \widehat{\alpha} a_{n_{k+1}} + \alpha_{n_k} b_{n_k}. \end{aligned}$$

Consequently,

$$(2 - \alpha_{n_k} - 2\widehat{\alpha}) a_{n_{k+1}} \leq b_{n_k}.$$

In particular, for all sufficiently large  $k$ ,

$$(2 - \alpha_{n_k} - 2\widehat{\alpha}) a_k \leq (2 - \alpha_{n_k} - 2\widehat{\alpha}) a_{n_{k+1}} \leq b_{n_k}.$$

This implies that

$$(2 - 2\widehat{\alpha}) \lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} (2 - \alpha_{n_k} - 2\widehat{\alpha})a_k \leq \limsup_{k \rightarrow \infty} b_{n_k} \leq 0.$$

Hence  $\lim_{k \rightarrow \infty} a_k = 0$ . □

### 3 Results

First, we discuss the existence and uniqueness of solutions of some related hierarchical optimization problems.

**Theorem 3.1** *Let  $S_1, S_2 : H \rightarrow H$  be quasi-nonexpansive mappings and  $f, g : H \rightarrow H$  be contractions. Then there exists a unique element  $(x^*, y^*) \in \text{Fix}(S_1) \times \text{Fix}(S_2)$  such that the following two inequalities are satisfied:*

$$\begin{aligned} \langle x^* - f(y^*), x - x^* \rangle &\geq 0 \quad \forall x \in \text{Fix}(S_1), \\ \langle y^* - g(x^*), y - y^* \rangle &\geq 0 \quad \forall y \in \text{Fix}(S_2). \end{aligned} \tag{5}$$

*Proof* The proof is a consequence of the well-known Banach’s contraction principle but it is given here for the sake of completeness. It is known that both sets  $\text{Fix}(S_1)$  and  $\text{Fix}(S_2)$  are closed and convex, and hence the projections  $P_{\text{Fix}(S_1)}$  and  $P_{\text{Fix}(S_2)}$  are well defined. It is clear that the mapping

$$P_{\text{Fix}(S_1)} \circ f \circ P_{\text{Fix}(S_2)} \circ g$$

is a contraction. Hence, there exists a unique element  $x^* \in H$  such that

$$x^* = (P_{\text{Fix}(S_1)} \circ f \circ P_{\text{Fix}(S_2)} \circ g)x^*.$$

Put  $y^* = P_{\text{Fix}(S_2)}g(x^*)$ . Then  $y^* \in \text{Fix}(S_2)$  and  $x^* = P_{\text{Fix}(S_1)}f(y^*)$ .

Suppose that there is an element  $(\widehat{x}, \widehat{y}) \in \text{Fix}(S_1) \times \text{Fix}(S_2)$  such that the following two inequalities are satisfied:

$$\begin{aligned} \langle \widehat{x} - f(\widehat{y}), x - \widehat{x} \rangle &\geq 0 \quad \forall x \in \text{Fix}(S_1), \\ \langle \widehat{y} - g(\widehat{x}), y - \widehat{y} \rangle &\geq 0 \quad \forall y \in \text{Fix}(S_2). \end{aligned}$$

Then

$$\begin{aligned} \widehat{x} &= P_{\text{Fix}(S_1)}f(\widehat{y}), \\ \widehat{y} &= P_{\text{Fix}(S_2)}g(\widehat{x}). \end{aligned}$$

Hence,

$$\widehat{x} = (P_{\text{Fix}(S_1)} \circ f \circ P_{\text{Fix}(S_2)} \circ g)\widehat{x}.$$

This implies that  $\widehat{x} = x^*$  and hence  $\widehat{y} = y^*$ . This completes the proof. □

For mappings  $S_1, S_2, f, g : H \rightarrow H$ , we define two iterative sequences  $\{v_n\}$  and  $\{w_n\}$  by

$$\begin{aligned} v_0, w_0 &\in H, \\ v_{n+1} &= (1 - \alpha_n)S_1v_n + \alpha_n f(S_2w_n), \\ w_{n+1} &= (1 - \alpha_n)S_2w_n + \alpha_n g(S_1v_n), \end{aligned} \quad (6)$$

where  $\{\alpha_n\}$  is a sequence in  $]0, 1[$  satisfying  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ .

**Theorem 3.2** *Let  $S_1, S_2 : H \rightarrow H$  be strongly quasi-nonexpansive mappings such that both  $I - S_1, I - S_2$  are demiclosed at zero and let  $f, g : H \rightarrow H$  be contractions. Then the iterative sequences  $\{v_n\}$  and  $\{w_n\}$  generated by (6) converge to  $x^*$  and  $y^*$ , respectively, where  $(x^*, y^*)$  is the unique element in  $\text{Fix}(S_1) \times \text{Fix}(S_2)$  verifying (5).*

Recall that a mapping  $T : H \rightarrow H$  is demiclosed at zero [9] iff

$$Tx = 0 \quad \text{whenever} \quad x_n \rightharpoonup x \text{ and } Tx_n \rightarrow 0.$$

We split the proof of Theorem 3.2 into the following lemmas. For convenience, assume that  $f, g : H \rightarrow H$  are contractions with the coefficient  $\widehat{\alpha}$ .

**Lemma 3.1** *The sequences  $\{v_n\}$  and  $\{w_n\}$  are bounded.*

*Proof* It follows since  $S_1, S_2$  are quasi-nonexpansive mappings and  $f$  is a contraction with the coefficient  $\widehat{\alpha}$  that

$$\begin{aligned} \|v_{n+1} - x^*\| &\leq (1 - \alpha_n)\|S_1v_n - x^*\| + \alpha_n\|f(S_2w_n) - x^*\| \\ &\leq (1 - \alpha_n)\|v_n - x^*\| + \alpha_n\|f(S_2w_n) - f(y^*)\| + \alpha_n\|f(y^*) - x^*\| \\ &\leq (1 - \alpha_n)\|v_n - x^*\| + \alpha_n\widehat{\alpha}\|S_2w_n - y^*\| + \alpha_n\|f(y^*) - x^*\| \\ &\leq (1 - \alpha_n)\|v_n - x^*\| + \alpha_n\widehat{\alpha}\|w_n - y^*\| + \alpha_n\|f(y^*) - x^*\|. \end{aligned}$$

Similarly, we also have

$$\|w_{n+1} - y^*\| \leq (1 - \alpha_n)\|w_n - y^*\| + \alpha_n\widehat{\alpha}\|v_n - x^*\| + \alpha_n\|g(x^*) - y^*\|.$$

This implies that

$$\begin{aligned} &\|v_{n+1} - x^*\| + \|w_{n+1} - y^*\| \\ &\leq (1 - \alpha_n(1 - \widehat{\alpha}))(\|v_n - x^*\| + \|w_n - y^*\|) \\ &\quad + \alpha_n(1 - \widehat{\alpha}) \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1 - \widehat{\alpha}} \\ &\leq \max \left\{ \|v_n - x^*\| + \|w_n - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1 - \widehat{\alpha}} \right\}. \end{aligned}$$

By induction, we have

$$\begin{aligned} & \|v_n - x^*\| + \|w_n - y^*\| \\ & \leq \max \left\{ \|v_0 - x^*\| + \|w_0 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1 - \widehat{\alpha}} \right\} \end{aligned}$$

for all  $n \in \mathbb{N}$ . In particular,  $\{v_n\}$  and  $\{w_n\}$  are bounded. Consequently, the sequences  $\{S_1 v_n\}$  and  $\{S_2 w_n\}$  are bounded, also. □

**Lemma 3.2** *For each  $n \in \mathbb{N}$ , the following inequality holds:*

$$\begin{aligned} & \|v_{n+1} - x^*\|^2 + \|w_{n+1} - y^*\|^2 \\ & \leq (1 - \alpha_n)^2 (\|v_n - x^*\|^2 + \|w_n - y^*\|^2) \\ & \quad + 2\alpha_n \widehat{\alpha} (\|w_n - y^*\| \|v_{n+1} - x^*\| + \|v_n - x^*\| \|w_{n+1} - y^*\|) \\ & \quad + 2\alpha_n (\langle f(y^*) - x^*, v_{n+1} - x^* \rangle + \langle g(x^*) - y^*, w_{n+1} - y^* \rangle). \end{aligned}$$

*Proof* It follows from the subdifferential inequality that

$$\begin{aligned} \|v_{n+1} - x^*\|^2 & = \|(1 - \alpha_n)(S_1 v_n - x^*) + \alpha_n(f(S_2 w_n) - x^*)\|^2 \\ & \leq \|(1 - \alpha_n)(S_1 v_n - x^*)\|^2 + 2\langle \alpha_n(f(S_2 w_n) - x^*), v_{n+1} - x^* \rangle \\ & = (1 - \alpha_n)^2 \|S_1 v_n - x^*\|^2 + 2\alpha_n \langle f(S_2 w_n) - f(y^*), v_{n+1} - x^* \rangle \\ & \quad + 2\alpha_n \langle f(y^*) - x^*, v_{n+1} - x^* \rangle \\ & \leq (1 - \alpha_n)^2 \|v_n - x^*\|^2 + 2\alpha_n \|f(S_2 w_n) - f(y^*)\| \|v_{n+1} - x^*\| \\ & \quad + 2\alpha_n \langle f(y^*) - x^*, v_{n+1} - x^* \rangle \\ & \leq (1 - \alpha_n)^2 \|v_n - x^*\|^2 + 2\alpha_n \widehat{\alpha} \|w_n - y^*\| \|v_{n+1} - x^*\| \\ & \quad + 2\alpha_n \langle f(y^*) - x^*, v_{n+1} - x^* \rangle. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|w_{n+1} - y^*\|^2 & \leq (1 - \alpha_n)^2 \|w_n - y^*\|^2 + 2\alpha_n \widehat{\alpha} \|v_n - x^*\| \|w_{n+1} - y^*\| \\ & \quad + 2\alpha_n \langle g(x^*) - y^*, w_{n+1} - y^* \rangle. \end{aligned}$$

Combining the last two inequalities gives the result. □

**Lemma 3.3** *If there exists a subsequence  $\{n_k\}$  of  $\{n\}$  such that*

$$\liminf_{k \rightarrow \infty} (\|v_{n_{k+1}} - x^*\|^2 + \|w_{n_{k+1}} - y^*\|^2 - \|v_{n_k} - x^*\|^2 - \|w_{n_k} - y^*\|^2) \geq 0,$$

then

$$\limsup_{k \rightarrow \infty} (\langle f(y^*) - x^*, v_{n_{k+1}} - x^* \rangle + \langle g(x^*) - y^*, w_{n_{k+1}} - y^* \rangle) \leq 0.$$

*Proof* We first consider the following assertion:

$$\begin{aligned} 0 &\leq \liminf_{k \rightarrow \infty} (\|v_{n_{k+1}} - x^*\|^2 + \|w_{n_{k+1}} - y^*\|^2 - \|v_{n_k} - x^*\|^2 - \|w_{n_k} - y^*\|^2) \\ &\leq \liminf_{k \rightarrow \infty} ((1 - \alpha_{n_k}) \|S_1 v_{n_k} - x^*\|^2 + \alpha_{n_k} \|f(S_2 w_{n_k}) - x^*\|^2 \\ &\quad + (1 - \alpha_{n_k}) \|S_2 w_{n_k} - y^*\|^2 + \alpha_{n_k} \|g(S_1 v_{n_k}) - y^*\|^2 \\ &\quad - \|v_{n_k} - x^*\|^2 - \|w_{n_k} - y^*\|^2) \\ &= \liminf_{k \rightarrow \infty} (\|S_1 v_{n_k} - x^*\|^2 - \|v_{n_k} - x^*\|^2) + (\|S_2 w_{n_k} - y^*\|^2 - \|w_{n_k} - y^*\|^2) \\ &\leq \limsup_{k \rightarrow \infty} (\|S_1 v_{n_k} - x^*\|^2 - \|v_{n_k} - x^*\|^2) + (\|S_2 w_{n_k} - y^*\|^2 - \|w_{n_k} - y^*\|^2) \\ &\leq 0. \end{aligned}$$

This implies that

$$\lim_{k \rightarrow \infty} (\|S_1 v_{n_k} - x^*\|^2 - \|v_{n_k} - x^*\|^2) = \lim_{k \rightarrow \infty} (\|S_2 w_{n_k} - y^*\|^2 - \|w_{n_k} - y^*\|^2) = 0.$$

In fact, since the sequences  $\{\|S_1 v_{n_k} - x^*\| + \|v_{n_k} - x^*\|\}$  and  $\{\|S_2 w_{n_k} - y^*\| + \|w_{n_k} - y^*\|\}$  are bounded, we have

$$\lim_{k \rightarrow \infty} (\|S_1 v_{n_k} - x^*\| - \|v_{n_k} - x^*\|) = \lim_{k \rightarrow \infty} (\|S_2 w_{n_k} - y^*\| - \|w_{n_k} - y^*\|) = 0.$$

Since  $S_1$  and  $S_2$  are strongly quasi-nonexpansive,

$$S_1 v_{n_k} - v_{n_k} \rightarrow 0 \quad \text{and} \quad S_2 w_{n_k} - w_{n_k} \rightarrow 0.$$

Moreover, by the iteration scheme (6), we have  $v_{n_k} - v_{n_{k+1}} \rightarrow 0$  and  $w_{n_k} - w_{n_{k+1}} \rightarrow 0$ . It follows from the boundedness of  $\{v_{n_k}\}$  that there exists a subsequence  $\{v_{n_{k_l}}\}$  of  $\{v_{n_k}\}$  such that  $v_{n_{k_l}} \rightharpoonup v$  and

$$\begin{aligned} \lim_{l \rightarrow \infty} \langle f(y^*) - x^*, v_{n_{k_l}} - x^* \rangle &= \limsup_{k \rightarrow \infty} \langle f(y^*) - x^*, v_{n_k} - x^* \rangle \\ &= \limsup_{k \rightarrow \infty} \langle f(y^*) - x^*, v_{n_{k+1}} - x^* \rangle. \end{aligned}$$

Since  $I - S_1$  is demiclosed at zero, it follows that  $v \in F(S_1)$ . It follows from (5) that

$$\lim_{l \rightarrow \infty} \langle f(y^*) - x^*, v_{n_{k_l}} - x^* \rangle = \langle f(y^*) - x^*, v - x^* \rangle \leq 0.$$

Consequently,

$$\limsup_{k \rightarrow \infty} \langle f(y^*) - x^*, v_{n_{k+1}} - x^* \rangle \leq 0.$$

By using the same argument, we have

$$\limsup_{k \rightarrow \infty} \langle f(x^*) - y^*, w_{n_{k+1}} - y^* \rangle \leq 0.$$

Therefore, we obtain the desired inequality. □

We are now ready to prove Theorem 3.2.

*Proof of Theorem 3.2* We first observe that

$$\begin{aligned} & \|w_n - x^*\| \|v_{n+1} - x^*\| + \|v_n - y^*\| \|w_{n+1} - y^*\| \\ & \leq (\|w_n - x^*\|^2 + \|v_n - y^*\|^2)^{\frac{1}{2}} (\|w_{n+1} - x^*\|^2 + \|v_{n+1} - y^*\|^2)^{\frac{1}{2}}. \end{aligned}$$

Put

$$\begin{aligned} a_n & := \|v_n - x^*\|^2 + \|w_n - y^*\|^2, \\ b_n & := 2(\langle f(y^*) - x^*, v_{n+1} - x^* \rangle + \langle g(x^*) - y^*, w_{n+1} - y^* \rangle). \end{aligned}$$

Then we have the following statements:

- $\{a_n\}$  is a bounded sequence (Lemma 3.1);
- $a_{n+1} \leq (1 - \alpha_n)^2 a_n + 2\alpha_n \widehat{\alpha} \sqrt{a_n} \sqrt{a_{n+1}} + \alpha_n b_n$  for all  $n \in \mathbb{N}$  (Lemma 3.2);
- whenever  $\{a_{n_k}\}$  is a subsequence of  $\{a_n\}$  satisfying  $\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0$ , it follows that  $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$  (Lemma 3.3).

Hence, it follows from Lemma 2.5 that

$$\lim_{n \rightarrow \infty} (\|v_n - x^*\|^2 + \|w_n - y^*\|^2) = \lim_{n \rightarrow \infty} a_n = 0.$$

This means that  $\lim_{n \rightarrow \infty} \|v_n - x^*\| = \lim_{n \rightarrow \infty} \|w_n - y^*\| = 0$ , as desired. □

Using our Theorem 3.2, we can deduce Maingé’s result (see Theorem 2.1) with a more relaxed restriction on the parameters  $\xi_1$  and  $\xi_2$ . More precisely,  $\xi_1$  and  $\xi_2$  in our Corollary 3.1 can be chosen from the wider range  $]0, 1 - \beta[$  while his result requires the condition  $\xi_1, \xi_2 \in ]0, \frac{1-\beta}{2}[$ .

**Corollary 3.1** *Let  $T_1, T_2 : H \rightarrow H$  be  $\beta$ -demictractive mappings where  $\beta \in [0, 1[$ , such that  $I - T_1, I - T_2$  are demiclosed at zero. Let  $F : H \rightarrow H$  be a  $\mu$ -Lipschitzian*

and  $r$ -strongly monotone mapping. Let  $\{x_n\}$  and  $\{y_n\}$  be generated by

$$\begin{aligned} x_0, y_0 &\in H, \\ v_n &= x_n - \gamma_n[\rho F(y_n) + x_n - y_n], \\ w_n &= y_n - \gamma_n[\eta F(x_n) + y_n - x_n], \\ x_{n+1} &= [(1 - \xi_1)I + \xi_1 T_1](v_n), \\ y_{n+1} &= [(1 - \xi_2)I + \xi_2 T_2](w_n), \end{aligned} \tag{7}$$

where  $I$  is the identity mapping and the following conditions hold:

- (a)  $\xi_1, \xi_2 \in ]0, 1 - \beta[$ ;
- (b)  $\{\gamma_n\} \subset ]0, 1[$ ,  $\lim_{n \rightarrow \infty} \gamma_n = 0$  and  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ;
- (c)  $\rho, \eta \in ]0, \frac{2r}{\mu^2}[$ .

Then the sequence  $\{(x_n, y_n)\}$  converges strongly to the unique solution  $(x^*, y^*)$  of the hierarchical optimization problem (1).

Before proving the result, we mention the following probably known fact with the proof for the sake of completeness. The second assertion strengthens Remark 2.1 of Maingé [14].

**Proposition 3.1**

- (a) If  $F : H \rightarrow H$  is a  $\mu$ -Lipschitzian and  $r$ -strongly monotone mapping, then the mapping  $f := I - \rho F$  is a contraction provided that  $\rho \in ]0, \frac{2r}{\mu^2}[$  (see, for example, [10–13]).
- (b) If  $T : H \rightarrow H$  is a  $\beta$ -demicontractive mapping with  $\beta \in [0, 1[$ , then the mapping  $T_\omega := (1 - \omega)I + \omega T$  is strongly quasi-nonexpansive whenever  $\omega \in ]0, 1 - \beta[$ .

*Proof* (a) For  $x, y \in H$ , we have

$$\begin{aligned} \|f(x) - f(y)\|^2 &= \|(x - y) - \rho(Fx - Fy)\|^2 \\ &= \|x - y\|^2 - 2\rho\langle x - y, Fx - Fy \rangle + \rho^2 \|Fx - Fy\|^2 \\ &\leq (1 - 2\rho r + \rho^2 \mu^2) \|x - y\|^2 = \left(1 - \rho \mu^2 \left(\frac{2r}{\mu^2} - \rho\right)\right) \|x - y\|^2. \end{aligned}$$

(b) Let  $\omega \in ]0, 1 - \beta[$ . It follows from Proposition 2.1 that  $T_\omega$  is quasi-nonexpansive and  $\text{Fix}(T_\omega) = \text{Fix}(T)$ . Let  $\{z_n\}$  be a bounded sequence and  $\|z_n - p\| - \|T_\omega z_n - p\| \rightarrow 0$  for some  $p \in \text{Fix}(T_\omega) = \text{Fix}(T)$ . We show that  $\|T_\omega z_n - z_n\| \rightarrow 0$ . Note that

$$\begin{aligned} \|T_\omega z_n - p\|^2 &= \|z_n - p - \omega(z_n - Tz_n)\|^2 \\ &= \|z_n - p\|^2 - 2\omega\langle z_n - p, z_n - Tz_n \rangle + \omega^2 \|z_n - Tz_n\|^2 \\ &\leq \|z_n - p\|^2 + (-\omega(1 - \beta) + \omega^2) \|z_n - Tz_n\|^2 \end{aligned}$$

$$= \|z_n - p\|^2 - \omega(1 - \beta - \omega)\|z_n - Tz_n\|^2.$$

Since  $\omega(1 - \beta - \omega) > 0$  and  $\|z_n - p\|^2 - \|T_\omega z_n - p\|^2 \rightarrow 0$ , we get

$$\|T_\omega z_n - z_n\| = \omega\|Tz_n - z_n\| \rightarrow 0.$$

This completes the proof. □

*Proof of Corollary 3.1* From (7), we obtain

$$\begin{aligned} v_n &= (1 - \gamma_n)x_n + \gamma_n(1 - \rho F)y_n, \\ w_n &= (1 - \gamma_n)y_n + \gamma_n(1 - \eta F)x_n, \\ x_{n+1} &= [(1 - \xi_1)I + \xi_1 T_1](v_n), \\ y_{n+1} &= [(1 - \xi_2)I + \xi_2 T_2](w_n). \end{aligned}$$

Put

$$\begin{aligned} f &= I - \rho F, \\ g &= I - \eta F, \\ S_1 &= (1 - \xi_1)I + \xi_1 T_1, \\ S_2 &= (1 - \xi_2)I + \xi_2 T_2. \end{aligned}$$

Then  $\text{Fix}(S_i) = \text{Fix}(T_i)$  for  $i = 1, 2$  and

$$\begin{aligned} v_n &= (1 - \gamma_n)S_1 v_{n-1} + \gamma_n f(S_2 w_{n-1}), \\ w_n &= (1 - \gamma_n)S_2 w_{n-1} + \gamma_n g(S_1 v_{n-1}). \end{aligned}$$

Set  $\alpha_n = \gamma_{n+1}$ . Hence,

$$\begin{aligned} v_{n+1} &= (1 - \alpha_n)S_1 v_n + \alpha_n f(S_2 w_n), \\ w_{n+1} &= (1 - \alpha_n)S_2 w_n + \alpha_n g(S_1 v_n). \end{aligned}$$

It follows from Proposition 3.1 that both  $f, g$  are contractions and both  $S_1, S_2$  are strongly quasi-nonexpansive mappings. It is easy to see that  $I - S_1$  and  $I - S_2$  are demiclosed at zero. By applying our Theorem 3.2, the conclusion follows immediately. □

The limitation on the use of Maingé’s result and our Corollary 7 happens when dealing with  $\beta$ -demicontractive mappings where  $\beta = 1$ . We will modify the preceding construction of a strongly quasi-nonexpansive mappings from a 1-demicontractive mapping which is  $L$ -Lipschitzian by using Ishikawa’s idea (see [23]).

**Proposition 3.2** *Let  $T : H \rightarrow H$  be an  $L$ -Lipschitzian and 1-demicontractive mapping. Define the mappings  $S$  and  $U$  for some positive value  $\alpha$  by*

$$\begin{aligned} S &:= (1 - \alpha)I + \alpha T, \\ U &:= (1 - \alpha)I + \alpha TS. \end{aligned}$$

Then

$$\begin{aligned} \|Ux - q\|^2 &\leq \|x - q\|^2 + \alpha^2(\alpha^2 L^2 + 2\alpha - 1)\|x - Tx\|^2, \\ &\text{for all } (x, q) \in H \times \text{Fix}(T). \end{aligned}$$

In addition, if  $\alpha \in ]0, \frac{-1 + \sqrt{L^2 + 1}}{L^2}[$ , then  $U$  is a strongly quasi-nonexpansive mapping and  $\text{Fix}(U) = \text{Fix}(T)$ .

*Proof* Let  $(x, q) \in H \times \text{Fix}(T)$ . Note that

$$\begin{aligned} \|Ux - q\|^2 &= \|(1 - \alpha)(x - q) + \alpha(TSx - q)\|^2 \\ &= (1 - \alpha)\|x - q\|^2 + \alpha\|TSx - q\|^2 - \alpha(1 - \alpha)\|TSx - x\|^2. \end{aligned} \quad (8)$$

Since  $T$  is 1-demicontractive, we have

$$\|TSx - q\|^2 \leq \|Sx - q\|^2 + \|Sx - TSx\|^2. \quad (9)$$

Next we estimate two terms on the right of the preceding inequality:

$$\begin{aligned} \|Sx - q\|^2 &= \|(1 - \alpha)(x - q) + \alpha(Tx - q)\|^2 \\ &= (1 - \alpha)\|x - q\|^2 + \alpha\|Tx - q\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2 \\ &\leq \|x - q\|^2 + \alpha\|x - Tx\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2 \\ &= \|x - q\|^2 + \alpha^2\|x - Tx\|^2; \end{aligned} \quad (10)$$

$$\begin{aligned} \|Sx - TSx\|^2 &= \|(1 - \alpha)(x - TSx) + \alpha(Tx - TSx)\|^2 \\ &= (1 - \alpha)\|x - TSx\|^2 + \alpha\|Tx - TSx\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2 \\ &\leq (1 - \alpha)\|x - TSx\|^2 + \alpha L^2\|x - Sx\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2 \\ &= (1 - \alpha)\|x - TSx\|^2 + \alpha L^2 \alpha^2 \|x - Tx\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2 \\ &= (1 - \alpha)\|x - TSx\|^2 + \alpha(L^2 \alpha^2 + \alpha - 1)\|x - Tx\|^2. \end{aligned} \quad (11)$$

From (9), (10), and (11), we obtain

$$\|TSx - q\|^2 \leq \|x - q\|^2 + \alpha(L^2 \alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + (1 - \alpha)\|x - TSx\|^2. \quad (12)$$

It follows from (8) and (12) that

$$\|Ux - q\|^2 \leq \|x - q\|^2 + \alpha^2(L^2\alpha^2 + 2\alpha - 1)\|Tx - x\|^2.$$

This proves the first assertion.

Finally, we prove the last assertion. Observe that  $\alpha^2(L^2\alpha^2 + 2\alpha - 1) < 0$  for all  $\alpha \in ]0, \frac{-1+\sqrt{L^2+1}}{L^2}[$ . It follows from the inequality obtained in the first part that  $\text{Fix}(U) = \text{Fix}(T)$ . Then it is clear that  $U$  is a quasi-nonexpansive mapping. To prove that  $U$  is a strongly quasi-nonexpansive mapping, let  $\{x_n\}$  be a bounded sequence in  $H$  such that  $\|x_n - q\| - \|Ux_n - q\| \rightarrow 0$ , for some  $q \in \text{Fix}(U) (= \text{Fix}(T))$ . From the last inequality, we have

$$0 \leq -\alpha^2(L^2\alpha^2 + 2\alpha - 1)\|Tx_n - x_n\|^2 \leq \|x_n - q\|^2 - \|Ux_n - q\|^2 \rightarrow 0.$$

This implies that  $x_n - Tx_n \rightarrow 0$  and so  $x_n - Sx_n = \alpha(x_n - Tx_n) \rightarrow 0$ . By the uniform continuity of  $T$ , we also get  $Tx_n - TSx_n \rightarrow 0$ . Then  $x_n - TSx_n \rightarrow 0$ , and hence  $x_n - Ux_n = \alpha(x_n - TSx_n) \rightarrow 0$ . The proof is finished.  $\square$

We now obtain the following result whose proof is omitted.

**Corollary 3.2** *Let  $T_1, T_2 : H \rightarrow H$  be 1-demicontractive and  $L$ -Lipschitzian mappings such that  $I - T_1, I - T_2$  are demiclosed at zero. Let  $F : H \rightarrow H$  be a  $\mu$ -Lipschitzian and  $r$ -strongly monotone mapping. Let  $\{x_n\}$  and  $\{y_n\}$  be generated by*

$$\begin{aligned} x_0, y_0 &\in H, \\ v_n &= x_n - \gamma_n[\rho F(y_n) + x_n - y_n], \\ w_n &= y_n - \gamma_n[\eta F(x_n) + y_n - x_n], \\ x_{n+1} &= U_1(v_n), \\ y_{n+1} &= U_2(w_n), \end{aligned}$$

where  $U_i = (1 - \xi_i)I + \xi_i T_i ((1 - \xi_i)I + \xi_i T_i)$  and the following conditions hold:

- (a)  $\xi_1, \xi_2 \in ]0, \frac{-1+\sqrt{L^2+1}}{L^2}[$ ;
- (b)  $\{\gamma_n\} \subset ]0, 1[$ ,  $\lim_{n \rightarrow \infty} \gamma_n = 0$  and  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ;
- (c)  $\rho, \eta \in ]0, \frac{2r}{\mu^2}[$ .

Then the sequence  $\{(x_n, y_n)\}$  converges strongly to the unique solution  $(x^*, y^*)$  of the hierarchical optimization problem (1).

Finally, we remark that our scheme is a *genuine generalization* of Maingé’s result because there is a 1-demicontractive and Lipschitzian mapping, which is not a  $\beta$ -demicontractive mapping for all  $\beta \in [0, 1[$ . The following example was introduced by Chidume and Mutangadura [24].

*Example 3.1* Let  $H = \mathbb{R}^2$ ,  $B = \{x \in H : \|x\| \leq 1\}$ ,  $B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$ , and  $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$ . For  $x = (a, b) \in H$ , let  $x^\perp = (b, -a)$ . Define the mapping  $T : B \rightarrow B$  by

$$Tx = \begin{cases} x + x^\perp & \text{if } x \in B_1; \\ \frac{x}{\|x\|} - x + x^\perp & \text{if } x \in B_2. \end{cases}$$

It is easy to see that  $\text{Fix}(T) = \{0\}$ . It was proved in [24] that  $T$  is 5-Lipschitzian and

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2 \quad (13)$$

for all  $x, y \in B$ . In particular,  $T$  is 1-demicontractive. Moreover, the inequality (13) becomes an equality whenever  $x \in B_1$  and  $y = 0$ , that is,

$$\|Tx\|^2 = \|x + x^\perp\|^2 = \|x\|^2 + \|x^\perp\|^2 = \|x\|^2 + \|x - Tx\|^2.$$

This shows that  $T$  cannot be a  $\beta$ -demicontractive mapping where  $\beta \in [0, 1[$ .

## 4 Conclusion

Inspired by the iterative scheme for hierarchical optimization proposed by Maingé, we establish a more general variant of his scheme to obtain a strong convergence theorem on Hilbert space setting. With this new technique, the range of some parameters as was the case in Maingé's result is relaxed. We also deduce another strong convergence theorem for a class of mappings, which is beyond the scope of Maingé's result.

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# Revising an iterative algorithm for finding a common solution of a system of variational inequalities for two inverse strongly accretive operators

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**Abstract** In this paper, we give a short and simple proof of the recent result of Katchang and Kumam (Positivity 15:281–295, 2011). In our proof, we do not assume the uniform convexity of a space.

**Keywords** Fixed point · General system of variational inequalities · Inverse strongly accretive operator · Nonexpansive mapping · Strong convergence

**Mathematics Subject Classification** 47H05 · 47H10 · 47J25

## 1 Introduction

Let  $E$  be a real Banach space with the dual space  $E^*$  and  $\langle \cdot, \cdot \rangle$  denote the pairing between  $E$  and  $E^*$ . The *duality mapping*  $J : E \rightarrow 2^{E^*}$  is the point-to-set mapping defined by

$$Jx = \{j \in E^* : \langle x, j \rangle = \|x\|^2 = \|j\|^2\} \quad (\text{for all } x \in E).$$

In 2011, inspired by Verma's work [25] and the recent work of Qin et al. [14], Katchang and Kumam [11] introduced the so-called *general system of variational inequalities* for operators  $A, B : C \rightarrow E$  where  $C$  is a closed convex subset of  $E$  and  $\lambda, \mu$  are two

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positive real numbers, that is, the problem of finding  $(\widehat{x}, \widehat{y}) \in C \times C$  such that, for each  $x \in C$ , the following inequalities

$$\begin{cases} \langle \lambda A\widehat{y} + \widehat{x} - \widehat{y}, j(x - \widehat{x}) \rangle \geq 0; \\ \langle \mu B\widehat{x} + \widehat{y} - \widehat{x}, j(x - \widehat{y}) \rangle \geq 0 \end{cases} \quad (1)$$

holds for some  $j(x - \widehat{x}) \in J(x - \widehat{x})$  and  $j(x - \widehat{y}) \in J(x - \widehat{y})$ . The set of solutions of (1) is denoted by  $\text{GSVIP}(C; A, B; \lambda, \mu)$ .

*Remark 1.1* If  $B = 0$ , then  $\langle \widehat{y} - \widehat{x}, j(x - \widehat{y}) \rangle \geq 0$  for all  $x \in C$ . This implies that  $-\|\widehat{x} - \widehat{y}\|^2 = \langle \widehat{y} - \widehat{x}, j(\widehat{x} - \widehat{y}) \rangle \geq 0$ , and so  $\widehat{x} = \widehat{y}$ . Hence the general system of variational inequalities (1) becomes the (classical) variational inequality problem, that is, the problem of finding  $\widehat{x} \in C$  such that

$$\langle A\widehat{x}, j(x - \widehat{x}) \rangle \geq 0 \quad \text{for all } x \in C. \quad (2)$$

This problem is interesting and have been extensively studied by many authors because it includes various problems in many branches in mathematics and sciences, for example, convex optimization problems, complementarity problems, etc. Relations between convex functions and variational inequalities were considered by Rockafellar [19] and Moreau [13]. The fact that complementarity problems can be deduced from variational inequalities was given by Karamardian [10].

Inspired by the iteration methods of Qin et al. [14], Katchang and Kumam [11] proposed the following method for approximating a common element of the solution set of the general system of variational inequalities  $\text{GSVIP}(C; A, B; \lambda, \mu)$ , where  $A, B : C \rightarrow E$  are certain types of accretive operators, and the fixed point set of a nonexpansive mapping  $S : C \rightarrow C$ ;

$$\begin{cases} x_1 = x \in C \\ y_n = Q_C(x_n - \mu Bx_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n [\delta Sx_n + (1 - \delta)Q_C(y_n - \lambda Ay_n)], \quad (n \geq 1) \end{cases} \quad (3)$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying certain conditions,  $\delta$  is a constant in  $(0, 1)$ ,  $f$  is a contraction of  $C$  into itself and  $Q_C$  denotes the sunny nonexpansive retraction from  $E$  onto  $C$ .

The purpose of this paper to present a short and simple proof of their recent result [11]. In our proof, we do not assume the uniform convexity of a space. Moreover, in the presence of uniform convexity, we can conclude the strong convergence by the simpler iteration scheme.

## 2 Definitions and related known fact

Let us recall some geometric properties which are used in this paper. A Banach space  $E$  is said to be

- *strictly convex* if whenever  $\|x\| = \|y\| = \frac{1}{2}\|x + y\| = 1$  it follows that  $x = y$ ;

- *smooth* if  $\lim_{t \rightarrow 0} \frac{1}{t}(\|x + ty\| - \|x\|)$  exists for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$ .

The uniform character of two properties above are related to the following moduli:

$$\rho_E(\tau) := \sup \left\{ \frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : \|x\| = \|y\| = 1 \right\} \quad (\tau \geq 0)$$

$$\delta_E(\varepsilon) := \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\} \quad (\varepsilon \in [0, 2]).$$

We say that  $E$  is

- *uniformly smooth* if  $\lim_{\tau \rightarrow 0} \frac{\rho_E(\tau)}{\tau} = 0$ ;
- *uniformly convex* if  $\delta_E(\varepsilon) > 0$  for all  $\varepsilon \in (0, 2)$ .

To deal with some estimates, we also need the stronger property than uniform smoothness and uniform convexity. A space  $E$  is

- *2-uniformly smooth* if there exists a constant  $k > 0$  such that  $\rho_E(\tau) \leq k\tau^2$  for all  $\tau > 0$ ;
- *2-uniformly convex* if there exists a constant  $c > 0$  such that  $\delta_E(\varepsilon) \geq c\varepsilon^2$  for all  $\varepsilon \in (0, 2)$ .

Let us recall some known fact concerning the geometric properties and the properties of the duality mapping (for more information, we refer to Cioranescu's book [9] and its review by Reich [17]):

- if  $E$  is smooth, then  $Jx$  is a singleton for all  $x \in E$ . In this case, we treat each singleton  $Jx$  as the element in  $Jx$  and we consider  $J$  as a single-valued mapping  $J : E \rightarrow E^*$ ;
- if  $E$  is reflexive, then  $E$  is smooth (strictly convex, resp.) if and only if  $E^*$  is strictly convex (smooth, resp.);
- $E$  is uniformly smooth (uniformly convex, resp.) if and only if  $E^*$  is uniformly convex (uniformly smooth, resp.);
- $E$  is 2-uniformly smooth (2-uniformly convex, resp.) if and only if  $E^*$  is 2-uniformly convex (2-uniformly smooth, resp.).

**Lemma 2.1** [3,4,27] *Let  $E$  be a Banach space. Then  $E$  is 2-uniformly smooth if and only if there exists a constant  $K > 0$  such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j_x \rangle + 2\|Ky\|^2$$

for all  $x, y \in E$  and  $j_x \in Jx$ . The smallest constant  $K$  such that the inequality above holds is called the 2-uniform smoothness constant of  $E$ .

Let  $C$  be a subset of a Banach space  $E$ . A mapping  $T : C \rightarrow C$  is said to be

- *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ ;
- a *contraction* if there is an  $\alpha \in [0, 1)$  such that  $\|Tx - Ty\| \leq \alpha\|x - y\|$  for all  $x, y \in C$ .

The set of *fixed points* of  $T$  is denoted by  $F(T)$ , that is,  $F(T) = \{x \in C : x = Tx\}$ . Let  $D$  be a subset of  $C$ . A mapping  $Q : C \rightarrow D$  is said to be

- a *retraction* if  $Q^2 = Q$ ;
- *sunny* if  $Q(Qx + t(x - Qx)) = Qx$  for all  $x \in C$  and  $t \geq 0$  such that  $Qx + t(x - Qx) \in C$ .

In a smooth Banach space, a retraction  $Q : C \rightarrow D$  is sunny and nonexpansive, where  $C$  is convex, if and only if  $\langle x - Qx, J(Qx - y) \rangle \geq 0$  for all  $x \in C$  and  $y \in D$  (see [15]). In this case,  $Q$  is uniquely determined. A subset  $D$  of  $C$  is said to be a *sunny nonexpansive retract* of  $C$  if there exists a sunny nonexpansive retraction from  $C$  onto  $D$ .

The following lemma is proved by Katchang and Kumam and it is a generalization of Lemma 2.1 of Ceng et al. [7].

**Lemma 2.2** [11, Lemma 3.2] *Let  $E$  be a smooth Banach space and let  $C$  be a closed convex subset and a sunny nonexpansive retract of  $E$ . Let  $A, B : C \rightarrow E$  be operators and  $(\hat{x}, \hat{y}) \in C \times C$ . Suppose that  $G : C \rightarrow C$  is defined by*

$$Gx = Q_C(I - \lambda A)Q_C(I - \mu B)x \quad \text{for all } x \in C \tag{4}$$

where  $I$  is the identity mapping,  $\lambda, \mu$  are positive real numbers and  $Q_C$  denotes the sunny nonexpansive retraction from  $E$  onto  $C$ . Then  $(\hat{x}, \hat{y}) \in \text{GSVIP}(C; A, B; \lambda, \mu)$  if and only if  $\hat{x}$  is a fixed point of the mapping  $G$  and  $\hat{y} = Q_C(I - \mu B)\hat{x}$ .

In order to find a solution of the general system of variational inequalities, it suffices to find a fixed point of the *resolvent mapping*  $G$ . From now on we denote the fixed point set of the resolvent mapping  $G$  by  $\Omega$ , that is,

$$F(G) = \{x : x = Gx\} = \Omega.$$

The following results are essentially needed for our main result.

**Lemma 2.3** [2, 18, 26] *Let  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  be sequences of nonnegative real numbers satisfying*

$$\gamma_{n+1} \leq (1 - \alpha_n)\gamma_n + \alpha_n\beta_n \quad \text{for all } n \geq 1.$$

*If  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and  $\lim_{n \rightarrow \infty} \beta_n = 0$ , then  $\lim_{n \rightarrow \infty} \gamma_n = 0$ .*

**Lemma 2.4** [5, Lemma 3] *Let  $E$  be a strictly convex Banach space and let  $C$  be a closed convex subset of  $E$ . Let  $T_1$  and  $T_2$  be two nonexpansive mappings from  $C$  into itself with a common fixed point. Defined a mapping  $T : C \rightarrow C$  by*

$$Tx = \delta T_1x + (1 - \delta)T_2x \quad \text{for all } x \in C$$

where  $\delta \in (0, 1)$ . Then  $T$  is nonexpansive and  $F(T) = F(T_1) \cap F(T_2)$ .

Let  $C$  be a subset of a smooth Banach space  $E$  and  $\beta > 0$ . An operator  $A : C \rightarrow E$  is said to be  $\beta$ -inverse strongly accretive if

$$\langle Ax - Ay, J(x - y) \rangle \geq \beta \|Ax - Ay\|^2 \quad \text{for all } x, y \in C. \quad (5)$$

**Lemma 2.5** [1, Lemma 2.8] *Let  $C$  be a closed convex subset of a 2-uniformly smooth Banach space  $E$  with the 2-uniform smoothness constant  $K$ . Suppose that  $A : C \rightarrow E$  is an  $\beta$ -inverse strongly accretive mapping. Then the following inequality holds for all  $x, y \in C$  and  $\lambda \in \mathbb{R}$ :*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(K^2\lambda - \beta)\|Ax - Ay\|^2,$$

where  $I$  is the identity mapping. In particular, if  $\lambda \in (0, \beta/K^2]$ , then  $I - \lambda A$  is nonexpansive, that is,  $\|(I - \lambda A)x - (I - \lambda A)y\| \leq \|x - y\|$  for all  $x, y \in C$ .

**Lemma 2.6** [11, Lemma 3.1] *Let  $E$  be a 2-uniformly smooth Banach space with the 2-uniform smoothness constant  $K$  and let  $C$  be a closed convex subset and a sunny nonexpansive retract of  $E$ . Suppose that  $A, B : C \rightarrow E$  are  $\beta$ - and  $\gamma$ -inverse strongly accretive operators, respectively, and  $G$  is defined by formula (4). If  $\lambda \in (0, \beta/K^2]$  and  $\mu \in (0, \gamma/K^2]$ , then  $G$  is nonexpansive.*

Let  $C$  be a subset of a Banach space  $E$ . A mapping  $T : C \rightarrow C$  is said to be strongly nonexpansive [6] if  $T$  is nonexpansive and

$$x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$$

whenever  $\{x_n\}$  and  $\{y_n\}$  are a sequence in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0$ .

*Remark 2.7* Bruck and Reich [6, Proposition 1.1 and Theorem 1.3] showed that:

- The composition of two strongly nonexpansive mappings is again strongly nonexpansive.
- The convex combination of a nonexpansive mapping and a strongly nonexpansive mapping is strongly nonexpansive in a uniformly convex Banach space.

**Lemma 2.8** *Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smoothness constant  $K$  and let  $C$  be a closed convex subset and a sunny nonexpansive retract of  $E$ . Suppose that  $A, B : C \rightarrow E$  are  $\beta$ - and  $\gamma$ -inverse strongly accretive operators, respectively, and  $G$  is defined by the formula (4). If  $\lambda \in (0, \beta/K^2]$  and  $\mu \in (0, \gamma/K^2]$ , then  $G$  is a strongly nonexpansive mapping.*

*Proof* Notice first that  $Q_C$  is a strongly nonexpansive mapping (see [12, 16]) and  $G$  is nonexpansive by Lemma 2.6. It suffices to show that both  $I - \lambda A$  and  $I - \mu B$  are strongly nonexpansive mappings and then the assertion follows. Note that both  $I - \lambda A$  and  $I - \mu B$  are nonexpansive mappings by Lemma 2.5. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences

in  $C$  such that  $\{x_n - y_n\}$  is bounded and  $\|x_n - y_n\| - \|(I - \lambda A)x_n - (I - \lambda A)y_n\| \rightarrow 0$ . It follows from Lemma 2.5 that

$$2\lambda(\beta - K^2\lambda)\|Ax_n - Ay_n\|^2 \leq \|x_n - y_n\|^2 - \|(I - \lambda A)x_n - (I - \lambda A)y_n\|^2 \rightarrow 0.$$

Since  $\lambda \in (0, \beta/K^2)$ , we have  $Ax_n - Ay_n \rightarrow 0$  and hence

$$x_n - y_n - ((I - \lambda A)x_n - (I - \lambda A)y_n) = \lambda(Ax_n - Ay_n) \rightarrow 0.$$

Therefore, the mapping  $I - \lambda A$  is a strongly nonexpansive mapping. Similarly, since  $\mu \in (0, \gamma/K^2)$ , the mapping  $I - \mu B$  is strongly nonexpansive. The proof is finished.  $\square$

### 3 A recapitulation of Katchang and Kumam's result

#### 3.1 Convergence theorem without uniform convexity

The following theorem is known (see Chidume and Chidume [8] and Suzuki [23]).

**Theorem 3.1** *Let  $E$  be a uniformly smooth Banach space and let  $C$  be a closed convex subset of  $E$ . Suppose that  $T : C \rightarrow C$  is a nonexpansive mapping with  $F(T) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by the following scheme:*

$$\begin{cases} x_1 = u \in C \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad (n \geq 1) \end{cases} \quad (6)$$

where  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

1.  $\alpha_n + \beta_n + \gamma_n = 1$  for all  $n \geq 1$ ;
2.  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
3.  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ .

Then  $\{x_n\}$  converges strongly to  $Q_{F(T)}u$ , where  $Q_{F(T)}$  is the sunny nonexpansive retraction from  $C$  onto  $F(T)$ .

**Theorem 3.2** *Let  $E, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $Q_{F(T)}$  be as in Theorem 3.1. Suppose that  $f : C \rightarrow C$  is a contraction. Let  $\{x_n\}$  be a sequence generated by the following scheme:*

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n f(x) + \beta_n x_n + \gamma_n T x_n, \quad (n \geq 1). \end{cases} \quad (7)$$

Then  $\{x_n\}$  converges strongly to  $z$  satisfying  $Q_{F(T)}f(z) = z$ .

*Proof* We assume that  $f$  is a contraction with a coefficient  $\alpha \in [0, 1)$ . We follow the idea of Suzuki [24] in this theorem. Since  $f$  is a contraction, so is  $Q_{F(T)}f$ . Then

$Q_{F(T)}f$  has a unique fixed point, say  $z$ . That is  $z = Q_{F(T)}f(z)$ . Define a sequence  $\{y_n\}$  in  $C$  by

$$\begin{cases} y_1 = f(z) \in C \\ y_{n+1} = \alpha_n f(z) + \beta_n y_n + \gamma_n T y_n, \quad (n \geq 1). \end{cases} \quad (8)$$

Then by Theorem 3.1,  $\lim_{n \rightarrow \infty} y_n = z$ . For each  $n \geq 1$ , we have that

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &\leq \alpha_n \|f(x_n) - f(z)\| + \beta_n \|x_n - y_n\| + \gamma_n \|T x_n - T y_n\| \\ &\leq \alpha_n \alpha (\|x_n - y_n\| + \|y_n - z\|) + \beta_n \|x_n - y_n\| + \gamma_n \|x_n - y_n\| \\ &= (\alpha_n \alpha + \beta_n + \gamma_n) \|x_n - y_n\| + \alpha_n \alpha \|y_n - z\| \\ &= (1 - (1 - \alpha) \alpha_n) \|x_n - y_n\| + (1 - \alpha) \alpha_n \frac{\alpha \|y_n - z\|}{1 - \alpha}. \end{aligned}$$

It follows from Lemma 2.3 and  $\sum_{n=1}^{\infty} (1 - \alpha) \alpha_n = \infty$  that  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$  and hence the sequence  $\{x_n\}$  converges strongly to  $z$ . The proof is finished.  $\square$

We recapitulate the result given by Katchang and Kumam without assuming the uniform convexity condition.

**Theorem 3.3** *Let  $E$  be a strictly convex and 2-uniformly smooth Banach space with the 2-uniform smoothness constant  $K$  and let  $C$  be a closed convex subset and a sunny nonexpansive retract of  $E$ . Let  $A, B : C \rightarrow E$  be  $\beta$ - and  $\gamma$ -inverse strongly accretive operators, respectively, let  $S : C \rightarrow C$  be a nonexpansive mapping and let  $f : C \rightarrow C$  be a contraction. Assume that  $\{x_n\}$  is the sequence defined by*

$$\begin{cases} x_1 = x \in C \\ y_n = Q_C(x_n - \mu B x_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n [\delta S x_n + (1 - \delta) Q_C(y_n - \lambda A y_n)], \quad (n \geq 1) \end{cases} \quad (9)$$

where  $\lambda \in (0, \beta/K^2]$ ,  $\mu \in (0, \gamma/K^2]$  and  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

1.  $\alpha_n + \beta_n + \gamma_n = 1$  for all  $n \geq 1$ ;
2.  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
3.  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ .

If  $F(S) \cap \Omega \neq \emptyset$ , then  $\{x_n\}$  converges strongly to  $\hat{x} = Q_{F(S) \cap \Omega} f(\hat{x})$ , where  $Q_{F(S) \cap \Omega}$  is the sunny nonexpansive retraction from  $C$  onto  $F(S) \cap \Omega$ . Moreover, if  $\hat{y} = Q_C(\hat{x} - \mu B \hat{x})$ , then  $(\hat{x}, \hat{y}) \in \text{GSVIP}(C; A, B; \lambda, \mu)$ .

*Proof* Let  $G$  be the mapping defined by the formula (4) and put  $T := \delta S + (1 - \delta)G$ . By Lemma 2.6, the mapping  $G$  is nonexpansive, and hence so is  $T$ . Since  $F(S) \cap \Omega \neq \emptyset$ , it follows from Lemma 2.4 that  $F(T) = F(S) \cap \Omega$ . We rewrite the iteration scheme (9) as

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad (n \geq 1). \end{cases} \quad (10)$$

Consequently, by Theorem 3.2,  $\{x_n\}$  converges strongly to  $\hat{x} = Q_{F(T)}f(\hat{x}) = Q_{F(S)\cap\Omega}f(\hat{x})$ . Moreover,  $(\hat{x}, \hat{y}) \in \text{GSVIP}(C; A, B; \lambda, \mu)$  where  $\hat{y} = Q_C(\hat{x} - \mu B\hat{x})$  by Lemma 2.2. The proof is finished.  $\square$

*Remark 3.4* The proof of Theorem 3.3 is shorter and simpler than that of Katchang and Kumam (the proof of Theorem 3.4 of [11] contains 4 pages). Moreover, we do not assume the uniform convexity of a space.

*Remark 3.5* There exists a strictly convex and 2-uniformly smooth space which is not uniformly convex. The following example is suggested by Professor Simeon Reich and Professor Yoav Benyamini. For each  $n \in \mathbb{N}$ , suppose that  $E_n := (\mathbb{R}^2, \|\cdot\|_n)$  is a two-dimensional space with its modulus of convexity of power type 2 and its modulus of smoothness of power type  $q_n$  where  $q_n \downarrow 1$ . (Attaching to each side of the square  $[-1, 1]^2$  a part of a disk of radius  $R > 1$ , we obtain a convex set  $K$  which is 2-uniformly convex, but not smooth. We then smooth it by replacing each corner with a part of a small disk of radius  $r < 1$ , which is tangent to the boundary of  $K$ . The resulting set is still 2-uniformly convex (with estimates that do not depend on  $r$ ) and uniformly smooth, but its modulus of smoothness deteriorates as  $r \downarrow 0$ .) Next, we consider  $E := \ell^2(E_n) = \{\{x_n\} : x_n \in E_n \text{ for all } n \in \mathbb{N}, \sum_{n=1}^{\infty} \|x_n\|_n^2 < \infty\}$  and  $\|\{x_n\}\| := (\sum_{n=1}^{\infty} \|x_n\|_n^2)^{1/2}$  for all  $\{x_n\} \in E$ . Since the family  $\{E_n\}$  is not “uniformly” uniformly smooth, the space  $E$  is smooth but not uniformly smooth (for example see [22]). It was shown in [21] that  $E$  is 2-uniformly convex. Let  $F$  be the dual space of  $E$ . Then  $F$  is a strictly convex and 2-uniformly smooth space but it is not uniformly convex.

### 3.2 Convergence theorem with uniform convexity

In this subsection, we prove strong convergence theorems in uniformly convex spaces. Let us start with the recent result of the second author.

**Theorem 3.6** [20, Theorem 4] *Let  $E$  be a uniformly smooth Banach space and  $C$  be a closed convex subset of  $E$ . Suppose that  $T : C \rightarrow C$  is a strongly nonexpansive mapping with  $F(T) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by the following scheme:*

$$\begin{cases} x_1 = u \in C \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad (n \geq 1) \end{cases} \quad (11)$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to  $Q_{F(T)}u$ , where  $Q_{F(T)}$  is the sunny nonexpansive retraction from  $C$  onto  $F(T)$ .

Following Suzuki’s idea [24], we have also the following result whose proof is omitted.

**Theorem 3.7** *Let  $E, C, T, \{\alpha_n\}$  and  $Q_{F(T)}$  be as in Theorem 3.6. Suppose that  $f : C \rightarrow C$  is a contraction. Let  $\{x_n\}$  be a sequence generated by the following scheme:*

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n f(x) + (1 - \alpha_n)Tx_n, \quad (n \geq 1). \end{cases} \quad (12)$$

Then  $\{x_n\}$  converges strongly to  $z$  satisfying  $Q_{F(T)}f(z) = z$ .

**Theorem 3.8** Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smoothness constant  $K$  and let  $C$  be a closed convex subset and a sunny nonexpansive retract of  $E$ . Let  $A, B : C \rightarrow E$  be  $\beta$ - and  $\gamma$ -inverse strongly accretive operators, respectively, let  $S : C \rightarrow C$  be a nonexpansive mapping and let  $f : C \rightarrow C$  be a contraction. Assume that  $\{x_n\}$  is a sequence defined by the following scheme

$$\begin{cases} x_1 = x \in C \\ y_n = Q_C(x_n - \mu Bx_n) \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[\delta Sx_n + (1 - \delta)Q_C(y_n - \lambda Ay_n)], \quad (n \geq 1) \end{cases} \quad (13)$$

where  $\delta \in (0, 1)$ ,  $\lambda \in (0, \beta/K^2)$ ,  $\mu \in (0, \gamma/K^2)$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . If  $F(S) \cap \Omega \neq \emptyset$ , then  $\{x_n\}$  converges strongly to  $\hat{x} = Q_{F(S) \cap \Omega}f(\hat{x})$ , where  $Q_{F(S) \cap \Omega}$  is the sunny nonexpansive retraction from  $C$  onto  $F(S) \cap \Omega$ . Moreover, if  $\hat{y} = Q_C(\hat{x} - \mu B\hat{x})$ , then  $(\hat{x}, \hat{y}) \in \text{GSVIP}(C; A, B; \lambda, \mu)$ .

*Proof* Let  $G$  be the mapping defined by the formula (4) and put  $T := \delta S + (1 - \delta)G$ . By Lemma 2.8, the mapping  $G$  is strongly nonexpansive, and hence so is  $T$  by Remark 2.7. Notice that  $F(T) = F(S) \cap \Omega$  by Lemma 2.4. The iteration scheme (13) can be rewritten as

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad (n \geq 1). \end{cases} \quad (14)$$

It follows immediately from Theorem 3.7 that  $\{x_n\}$  converges strongly to  $\hat{x} = Q_{F(T)}f(\hat{x}) = Q_{F(S) \cap \Omega}f(\hat{x})$ . Moreover,  $(\hat{x}, \hat{y}) \in \text{GSVIP}(C; A, B; \lambda, \mu)$  where  $\hat{y} = Q_C(\hat{x} - \mu B\hat{x})$  by Lemma 2.2. The proof is finished.  $\square$

*Remark 3.9* The iteration scheme (13) is simpler than (3) proposed by Katchang and Kumam. Moreover, their Theorem 3.4 excludes the case  $\beta_n = 0$  for all  $n$  because of the assumption  $\liminf_{n \rightarrow \infty} \beta_n > 0$ . In other word, our Theorem 3.8 can be viewed as a supplement of Katchang and Kumam’s result.

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