NUMERICAL SOLUTION OF DIFFERENTIAL EQUATION: WAVELET APPLICATIONS

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ABSTRACT

This dissertation introduces the Haar wavelet-based method for solving two dimensional boundary value problems in terms of partial differential equations. The convergence analysis showed that the method was of order $O((1/2^{\tilde{J}+1})^2)$ in accuracy, and the analysis result was verified by two numerical examples of Poisson and Helmholtz equations.

KEY WORDS: WAVELET/ PARTIAL DIFFERENTIAL EQUATION
NUMERICAL ANALYSIS/ CONVERGENCE ANALYSIS.

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การประยุกต์เวฟเล็ตในการหาผลเฉลยเชิงตัวเลขของสมการเชิงอนุพันธ์
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บทคัดย่อ

วิทยานิพนธ์เล่มนี้แนะนำวิธีการหาผลเฉลยเชิงตัวเลขโดยอาศัยเวฟเล็ตต์ของฮาร์ในการแก้ปัญหาค่าขอบเขตของสมการเชิงอนุพันธ์ย่อยสองตัวแปร การวิเคราะห์การลู่เข้าแสดงให้เห็นว่าวิธีการหาคำตอบข้างต้นนั้นมีอันดับการลู่เข้าเป็น $O((1/2^{J+1})^2)$ นอกจากนี้ผลจากการวิเคราะห์ดังกล่าวได้ถูกยืนยันความถูกต้องโดยใช้ตัวอย่างการคำนวณเชิงตัวเลขจากสมการปัวซ์ซง และสมการเฮลม์โฮลทซ์

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CHAPTER I

INTRODUCTION AND REVIEW OF LITERATURES

1.1 Introduction

In 1982, wavelets were first inroduced by Morlet *et al.* [1, 2] in seismic signal processing. In 1984 Grossman *et al.* [3] introduced wavelets in context of quantum physics with a physical meaning as a small wave or an oscillation that decays quickly. Many types of wavelet have been proposed such as Haar, Shannon, Daubechies, Morlet, and so on. In the past, wavelets used to be called as Fourier function, Haar function, Basis function, etc. In order to provide a wavelet basis, multiresolution analysis is used for generating the orthogonal wavelet bases from the mother wavelets. Currently, wavelets have been applied in several research fields such as signal and image processing [4, 5, 6],data compression [7, 8, 9], solving the integral and differential equations [10, 11, 12, 13], etc.

In this dissertation, we are interested in wavelets in the context of convergence analysis of numerical solutions of partial differential equations (PDEs) proposed by Lepik [14]. Many numerical methods based on wavelets for solving PDEs have been proposed.

Firstly, during the 1990s, wavelets have been applied for solving PDEs numerically by Bertoluzza [15] and Beykin and Keiser [17, 16]. Li and Chen [18] categorized the wavelet-based methods as wavelet weighted residual method (WWRM) including wavelet Galerkin method (WGM) and wavelet collocation method (WCM); wavelet finite element method (WFEM); wavelet boundary element (WBE); wavelet meshless method (WMM); and other wavelet-based numerical methods. Recently, there is an effective and simple wavelet-based method for solving differential equation in which the highest derivative is approximated by wavelet series in the work of Chen [19], Hsiao [19, 20] and Lepik [21, 22, 24, 23, 14]. There are several methods to solve the numerical solution of PDEs such as finite-element, finite-difference, and finite volume methods. Wavelet-based methods have also been introduced for solving PDEs.

Initially, in 1997, Chen [19] and Hsiao [19, 20] presented an operational matrix of integration based on Haar wavelets and a procedure for applying the matrix in order to analyze lumped and distributed-parameter dynamical systems. They recommended to expand the highest derivative appearing in the differential equation into the Haar wavelet series. The other derivatives and the solution function are then calculated through integration. All derivatives and the solution function are substituted into the ODE system. After that the ODE system is discretized by the collocation method and then becomes a linear system of algebraic equations in order to calculate the wavelet coefficients. The technique can be interpreted from the incremental and multiresolution viewpoint. By increasing the multiresolution parameter m, the accuracy of solution can be improved. Lepik adapted the method of Chen and Hsiao [19] to solve various types of differential equations such as nonlinear ODEs [21], evolution equations [22], integral equations [23], the higher order ODEs [24] and PDEs [21, 14]. In 2011, Lepik [14] proposed a procedure to solve PDEs by using the two-dimensional Haar wavelet and claimed that the proposed method was mathematically simple and computationally efficient for solving the diffusion and Poisson equations. The main feature is to expand highest derivative into the 2-dimensional Haar wavelet series. The details of this method are going to be introduced in Chapter III.

Although many wavelet methods have been proposed, little on convergence analysis has been published. Siraj-ul-Islam *et al.* [25, 26] proved the convergence of the Haar wavelet series. Majak *et al.* [27, 28] published a convergence theorem for solving ODEs using the wavelet method proposed by Chen [19] and Hsiao [19, 20]. In 2015, Vijesh and Kumar [29] adapted the work of Lakshmikantham *et al.* [30] to provide an existence and uniqueness of the solution of semi linear parabolic initial value problem (SPIBVP) and the convergence of the generalised quasilinearlization method. They also provided the explanation of the extension of Haar and Legendre wavelet collocation methods in combination with quasilinearization for SPIBVP. Since the method of Lepik [14] is simple and efficient, we perform a convergence analysis on this method when applied to solve 2D PDEs with boundary value problems.

The convergence theorem which is presented here only holds for PDEs with boundary value problems. The validity of the convergence theorem is verified by two numerical examples, the Poisson and Helmholtz equations, which describe the numerous scientific problems.

This dissertation is organized into five chapters. Chapter I is an introduction. Chapter II introduces necessary theoretical backgrounds including a general definition of wavelet, Haar wavelet, the orthogonal property of Haar wavelets and the mean value theorem. In Chapter III, we review the numerical method for solving two dimensional partial differential equation with boundary value problem by using two dimensional Haar wavelet and provide an implementation of two-dimensional partial differential equation with boundary conditions. In Chapter IV, we derive the convergence theorem of the method introduced in Chapter III and describe how to estimate an error from numerical results. Finally, conclusions and numerical results will be presented in Chapter V.

CHAPTER II

THEORETICAL BACKGROUNDS

The essential backgrounds for analysing the convergence theorem of the Haar wavelet method solving 2D PDEs with boundary conditions and introduces the basic knowledge of Haar wavelet and its relevant properties are reviewed in this chapter.

2.1 Preliminaries

This section will introduce the basic definitions and theorems supporting the ideas in other sections. Definitions 2.1-2.6 and Theorem 2.7 can help to understand the basis ideas of wavelets such as multiresolution analysis and orthogonal property. Definitions 2.2-2.3 are expoited for evaluating the numerical errors. Theorems 2.8 and 2.9 are used in the process analysing the convergence theorem.

Definition 2.1 (Linear operator)**.** Let V and W be two linear spaces. An operator $L : V \to W$ is *linear* if

$$L(v+u) = L(v) + L(u), \qquad \forall v, u \in V, \qquad \text{(Addition)}, \tag{2.1}$$

$$L(\alpha v) = \alpha L(v), \qquad \forall v \in V,\ \forall \alpha \in \mathbb{R}, \qquad \text{(Scalar Multiplication)}. \tag{2.2}$$

Definition 2.2. The space $L^2[a,b]$ is the vector space of measurable functions that are square-integrable on $[a,b]$ under the usual addition for $f, g : [a,b] \to \mathbb{R}$,

$$(f+g)(t) = f(t) + g(t), \qquad t \in [a,b] \subset \mathbb{R}, \tag{2.3}$$

and scalar multiplication

$$(\alpha f)(t) = \alpha f(t), \qquad t \in [a,b] \subset \mathbb{R},\ \alpha \in \mathbb{R}, \tag{2.4}$$

and with norm defined by

$$\|f\|_2 = \left\{ \int\limits_a^b |f(t)|^2 \, \mathrm{d}t \right\}^{1/2} < \infty. \tag{2.5}$$

Definition 2.3. The space $l^2[a,b]$ is the normed space of a sequence of numbers $x = \{\xi_j\}_{j\in\mathbb{N}} = (\xi_1, \xi_2, \xi_3, \ldots)$, where $\xi_j \in [a,b] \subset \mathbb{R}$, with norm given by

$$\|x\| = \left\{ \sum_{j\in\mathbb{N}} |\xi_j|^2 \right\}^{1/2} < \infty. \tag{2.6}$$

Definition 2.4 (Support)**.** Given a function v on Ω, its support is defined by

$$\text{supp } v = \overline{\{x \in \Omega | v(x) \neq 0\}}. \tag{2.7}$$

Note that v has a *compact support* if a support of v is a proper subset of Ω.

Definition 2.5 (Orthogonal basis)**.** Suppose V is a n-dimensional vector space with an inner product denoted by $< \cdot, \cdot >$. A basis $\{b_i\}_{i\leq n\in\mathbb{N}}$ of V is an *orthogonal basis* if

$$< b_i, b_j >= 0 \qquad \text{when} \quad i \neq j. \tag{2.8}$$

Definition 2.6 (Direct sum)**.** A vector space X is said to be the *direct sum* of two subspaces Y and Z of X, i.e.,

$$X = Y \oplus Z, \tag{2.9}$$

if each $x \in X$ has a unique representation

$$x = y + z, \qquad y \in Y,\ z \in Z. \tag{2.10}$$

Theorem 2.7 (Direct sum)**.** *Let Y be any closed subspace of a Hibert space H and let $Y^\perp$ be the orthogonal complement of Y, i.e., the set of all vectors orthogonal to Y defined by*

$$Y^\perp = \{z \in H | z \perp Y\}, \tag{2.11}$$

then

$$H = Y \oplus Y^\perp. \tag{2.12}$$

Theorem 2.8 (Mean value theorem)**.** *If a function $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}x} f(c) = \frac{f(b) - f(a)}{b - a}. \tag{2.13}$$

Theorem 2.9 (Mean value theorem for definite integrals)**.** *Let $f(x)$ and $g(x)$ be continuous real-valued functions, and suppose that $g(x) \geq 0$ on an interval $[a, b]$, then there exists a point $c \in (a, b)$ such that*

$$\int\limits_a^b f(x)g(x)\mathrm{d}x = f(c)\int\limits_a^b g(x)\mathrm{d}x. \tag{2.14}$$

Particularly, when $g(x) = 1$, (2.14) reduces to

$$\int\limits_a^b f(x)\mathrm{d}x = f(c)(b-a). \tag{2.15}$$

2.2 Big O and Little o Notations

Definition 2.10 (Big O Notation)**.** Let $f(n)$ and $g(n)$ be functions defined on a domain $D \subseteq \mathbb{R}$. We denote that $f(n) = \mathcal{O}(g(n))$ if there exist positive constants c and n_0 such that

$$0 \leq f(n) \leq cg(n), \qquad \text{for all} \quad n \geq n_0. \tag{2.16}$$

Definition 2.11 (Little o Notation)**.** Let $f(n)$ and $g(n)$ be functions defined on a domain $D \subseteq \mathbb{R}$. We denote that $f(n) = o(g(n))$ for any positive constant c, there exists positive constant n_0 such that

$$0 \leq f(n) < cg(n), \qquad \text{for all} \quad n \geq n_0. \tag{2.17}$$

These two notations provide a coarse method of comparing two functions, such as $f(n)$ and $g(n)$, and are also frequently used when both functions converges to zero. If $f(n)$, $g(n)$ and $f(n) = \mathcal{O}(g(n))$, then $f(n)$ converges to zero at least as rapidly as $g(n)$. If $f(n) = o(g(n))$, then $f(n)$ converges to zero more rapidly than $g(n)$.

The Big $\mathcal{O}$ notation is used for funtions defined on $\mathbb{R}$. We will see an example that $\sin(x)$ expanded by using Taylor's series:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots. \tag{2.18}$$

(2.18) can be rewritten as

$$\sin(x) = x - \frac{1}{3!}x^3 + \mathcal{O}(x^5) \qquad \text{when} \quad x \to 0. \tag{2.19}$$

(2.19) implies that there exists a neighborhood of 0 and a constant C such that on that neighborhood

$$\left|\sin(x) - x + \frac{1}{3!}x^3\right| \leq \left|x^5\right|. \tag{2.20}$$

An equation in the form:

$$f(x) = \mathcal{O}(g(x)) \qquad \text{when} \quad x \to 0, \tag{2.21}$$

implies that there exist constants r and C so that

$$|f(x)| \leq C\,|g(x)|, \tag{2.22}$$

whenever $x \geq r$. For example, when $x \gg 1$, we can see that $\ln x \leq x$, so we have $\ln x = \mathcal{O}(x)$.

Generally, we write

$$f(x) = \mathcal{O}(g(x)) \qquad \text{when} \quad x \to x_0, \tag{2.23}$$

when there exist a constant C and a neighborhood x_0 such that $|f(x)| \leq C\,|g(x)|$ in that neighborhood. Similarly,

$$f(x) = o(g(x)) \qquad \text{when} \quad x \to x_0, \tag{2.24}$$

implies that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{2.25}$$

2.3 Order of Convergence

In this section, we introduce some special terminology describing the rapidity of the convergence of sequence.

Let $[x_n]$ be a sequence of real numbers tending to limit x^*. The rate of convergence is said by the following sentences:
at least *linear* if there exist a constant $0 < C < 1$ and an integer N such that

$$|x_{n+1} - x^*| \leq C\,|x_n - x^*| \qquad \text{when} \quad n \geq N, \tag{2.26}$$

at least *superlinear* if there exist a sequence $\epsilon_n > 0$ tending to zero and an integer N such that

$$|x_{n+1} - x^*| \leq \epsilon_n\,|x_n - x^*| \qquad \text{when} \quad n \geq N, \tag{2.27}$$

at least *quadratic* if there exist a constant $c > 0$ and an integer N such that

$$|x_{n+1} - x^*| \leq C\,|x_n - x^*|^2 \qquad \text{when} \quad n \geq N, \tag{2.28}$$

the rate of convergence is of *order* α if there exist constants $c > 0$ and α and an integer N such that

$$|x_{n+1} - x^*| \leq C\,|x_n - x^*|^\alpha \qquad \text{when} \quad n \geq N. \tag{2.29}$$

2.4 Wavelet

The original construction of a class of wavelets by using dilation and translation of a function $\psi(t)$ was introduced by Grossman and Morlet [3], i.e.,

$$\psi_{a,b}(t) = \left|a^{-1/2}\right| \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R} \quad \text{and} \quad a \neq 0, \tag{2.30}$$

where a and b are dilation and translation parameters, respectively. Each wavelet construction is based on the multiresolution analysis that we mention in the next section.

2.5 Multiresolution Analysis

We can construct wavelet families to form an orthonormal basis by *Multiresolution analysis*, so we can derive a wavelet function $\Psi(t)$ such that $\Psi(2^m t - n)$, where $(m, n) \in \mathbb{Z}^2$, is an orthogonal basis of $L^2(\mathbb{R})$. Chen *et al.* [31] introduce the concept of a reflnable set which led to a set-theoretic multiresolution analysis and establish a construction of multiscale piecewise polynomial functions and its corresponding multiscale collocation functionals. Describing the defnition of a multiresolution analysis and that of a reflnable set helps us more comprehensible in the context of wavelets.

Definition 2.12. The function $\phi(t)$ is a multiresolution analysis of $L^2(\mathbb{R})$ if there exists a nested sequence of closed subspaces $\{V_m\}_{m\in\mathbb{Z}}$ such that

1. There exists $\phi(t) \in V_0$ such that
$$\{\phi(t-n) : n \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_0, \tag{2.31}$$
2. $V_m \subset V_{m+1}$ for all $m \in \mathbb{Z}$, i.e., $\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots,$ (2.32)
3. $f(t) \in V_m$ if and only if $f(2t) \in V_{m+1}$ for all $m \in \mathbb{Z}$, (2.33)
4. $\displaystyle\bigcap_{m=-\infty}^{\infty} V_m = \{0\}\,,$ (2.34)
5. $\displaystyle\overline{\bigcup_{m=-\infty}^{\infty} V_m} = L^2(\mathbb{R}).$ (2.35)

The function $\phi(t)$ is called a *scaling function.* We define W_m to be the orthogonal complement of V_m in V_{m+1}, i.e.,

$$V_{m+1} = V_m \oplus W_m, \qquad m \in \mathbb{Z} \tag{2.36}$$

and

$$W_m \perp W_{m'} \qquad \text{if} \quad m \neq m'. \tag{2.37}$$

Since V_m is a nested subspace it follows that

$$V_M = V_m \oplus \bigoplus_{n=0}^{M-m-1} W_{m+n} \qquad \text{for} \quad m < M, \tag{2.38}$$

and that

$$L^2(\mathbb{R}) = V_0 \oplus \bigoplus_{m=0}^{\infty} W_m \qquad \text{for} \quad m < M. \tag{2.39}$$

From the multiresolution analysis, there exists another function $\psi(t)$ in W_0 such that $\{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal basis for W_0. The function $\psi(t)$ is called a *wavelet generator.*

Definition 2.13 (Refinable set)**.** A subset V of X is *refinable* relative to the mapping Φ if $V \subseteq \Phi(V)$.

By following multiresolution analysis, we have many possible kinds of wavelets such as Haar, Shannon, Daubechies, Morlet, and Mexican hat. In this dissertation, we focus on only Haar wavelet.

2.6 Haar Wavelet

In this dissertation, the Haar wavelet family is defined in the same way as Lepik [14], Chen and Hsiao [19].

Definition 2.14 (Haar wavelet)**.** Consider the interval $[A, B]$, where A and B are given constants. Let $M = 2^J$ and $m = 2^j$ where J is the maximum level of resolution and $j = 0, 1, \ldots, J$ is a dilation parameter. $k = 0, 1, \ldots, m-1$ is the translation parameter. When $i > 1$, the ith *Haar wavelet* function is defined as

$$h_i(x) = \begin{cases} 1, & x \in [\xi_1(i), \xi_2(i)], \\ -1, & x \in [\xi_2(i), \xi_3(i)], \\ 0, & \text{elsewhere}, \end{cases} \tag{2.40}$$

where

$$\begin{aligned} i &= 2^j + k + 1, & \xi_1(i) &= A + 2k\mu\Delta x, \\ \xi_2(i) &= A + (2k+1)\mu\Delta x, & \xi_3(i) &= A + 2(k+1)\mu\Delta x, \\ \mu &= M/m, \qquad \text{and} & \Delta x &= (B-A)/(2M). \end{aligned}$$

The index i is the wavelet number. The case $i = 1$ corresponds to the *scaling function* of Haar wavelet which is defined as

$$h_1(x) = 1, \quad x \in [A, B]. \tag{2.41}$$

In this definition, $h_2(x)$ is called the *Haar mother wavelet.*

2.7 Riemann-Liouville Integral

The method is proposed by Lepik [14] requires *Riemann-Liouville integral* to approximate the solution. According to Podlubny [33], the Riemann-Liouville integral is defined by,

$$\begin{aligned} {}_A I_x^\alpha f(x) &= \int_A^x \cdots \int_A^x f(t) \mathrm{d}t^\alpha \quad \text{(integrating } \alpha \text{ times)} \\ &= \frac{1}{\Gamma(\alpha)} \int_A^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \end{aligned} \tag{2.42}$$

where Γ is the Gamma function and A is an arbitrary. In this dissertation, Riemann-Liouville integral (2.42) is required to solve n th-order PDEs. See more details of this integral is in [33]. We define a function $p_{\alpha,i}(x)$ as

$$p_{\alpha,i}(x) = \int_A^x \ldots \int_A^x h_i(t)\mathrm{d}t^\alpha \quad \text{(integrating } \alpha \text{ times)}. \tag{2.43}$$

Applying the Riemann-Liouville integral for (2.43) yields

$$p_{\alpha,i}(x) = \frac{1}{(\alpha-1)!}\int_A^x (x-t)^{\alpha-1} h_i(t)\mathrm{d}t. \tag{2.44}$$

When $\alpha = 0$, we have

$$p_{0,i}(x) = h_i(x). \tag{2.45}$$

When $i = 1$, substituting (2.41) in (2.44) yields

$$p_{\alpha,1}(x) = \frac{1}{(\alpha-1)!}\int_A^x (x-t)^{\alpha-1}\mathrm{d}t = \frac{1}{\alpha!}(x-A)^\alpha. \tag{2.46}$$

The following results are from substituting (2.40) in (2.44) when $i = 2^j + k + 1 > 1$. When $x < \xi_1(i)$ and $i > 1$, we obviously obtain

$$p_{\alpha,i}(x) = \int_0^x (x-t)^{\alpha-1}(0)\mathrm{d}t = 0. \tag{2.47}$$

When $x \in [\xi_1(i), \xi_2(i)]$ and $i > 1$, we obtain

$$p_{\alpha,1}(x) = \frac{1}{(\alpha-1)!}\left[\int_0^{\xi_1(i)} (x-t)^{\alpha-1}(0)\mathrm{d}t + \int_{\xi_1(i)}^x (x-t)^{\alpha-1}\mathrm{d}t\right] = \frac{1}{\alpha!}[x-\xi_1(i)]^\alpha. \tag{2.48}$$

When $x \in [\xi_2(i), \xi_3(i)]$ and $i > 1$, we obtain

$$\begin{aligned} p_{\alpha,1}(x) &= \frac{1}{(\alpha-1)!}\left[\int_0^{\xi_1(i)} (x-t)^{\alpha-1}(0)\mathrm{d}t + \int_{\xi_1(i)}^{\xi_2(i)} (x-t)^{\alpha-1}\mathrm{d}t - \int_{\xi_2(i)}^x (x-t)^{\alpha-1}\mathrm{d}t\right] \\ &= \frac{1}{(\alpha-1)!}\left\{\frac{1}{\alpha}[x-\xi_1(i)]^\alpha - \frac{1}{\alpha}[x-\xi_1(i)]^\alpha - \frac{1}{\alpha}[x-\xi_2(i)]^\alpha\right\} \\ &= \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_1(i)]^\alpha\right\}. \end{aligned} \tag{2.49}$$

When $x > \xi_3(i)$ and $i > 1$, we obtain

$$\begin{aligned}
p_{\alpha,1}(x) &= \frac{1}{(\alpha-1)!}\left[\int\limits_{0}^{\xi_1(i)} (x-t)^{\alpha-1}(0)\mathrm{d}t + \int\limits_{\xi_1(i)}^{\xi_2(i)} (x-t)^{\alpha-1}\mathrm{d}t\right. \\
&\qquad\qquad \left. - \int\limits_{\xi_2(i)}^{\xi_3(i)} (x-t)^{\alpha-1}\mathrm{d}t + \int\limits_{\xi_3(i)}^{x} (x-t)^{\alpha-1}(0)\mathrm{d}t\right] \\
&= \frac{1}{(\alpha-1)!}\left\{\frac{1}{\alpha}[x-\xi_1(i)]^\alpha - \frac{1}{\alpha}[x-\xi_1(i)]^\alpha + \frac{1}{\alpha}[x-\xi_3(i)]^\alpha - \frac{1}{\alpha}[x-\xi_2(i)]^\alpha\right\} \\
&= \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_1(i)]^\alpha + [x-\xi_3(i)]^\alpha\right\}. \qquad (2.50)
\end{aligned}$$

From (2.47)-(2.50), we conclude that

$$p_{\alpha,i}(x) = \begin{cases} 0, & x < \xi_1(i), \\ \frac{1}{\alpha!}\left[x-\xi_1(i)\right]^\alpha, & x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2(x-\xi_2(i))^\alpha\right\}, & x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2\left[x-\xi_2(i)\right]^\alpha + \left[x-\xi_3(i)\right]^\alpha\right\}, & x > \xi_3(i), \end{cases} \qquad (2.51)$$

where $i = 2^j + k + 1$.

2.8 Haar Wavelet Orthogonal property

In order to analyse the convergence theorem, the orthogonal property of Haar wavelets defined in Definition 2.14 is needed. By following the multiresolution analysis procedure, Haar wavelets defined in Definition 2.14 have an orthogonal property such that

$$\int\limits_{A}^{B} h_i(x)h_{i'}(x)\mathrm{d}x = \begin{cases} B-A, & i = i' = 1, \\ (B-A)/2^j, & i = i' > 1, \\ 0, & i \neq i', \end{cases} \qquad (2.52)$$

where $i = 2^j + k + 1$, $i' = 2^{j'} + k' + 1$, and i', j', k' are defined in the same way as i, j, k, respectively.

2.9 Haar Wavelet Expansion and its Error

In this dissertation, the interest method does not require 1D Haar wavelet. However, introducing an 1D Haar wavelet expansion is beneficial to understand that of 2D wavelet. Since Haar wavelet orthogonal basis, constructed by multiresolution analysis, is in $L^2[A,B]$, any function $f(x) \in L^2[A,B]$ can be expanded into a Haar wavelet series via

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \tag{2.53}$$

where $a_i = 2^j \int_A^B f(x) h_i(x) \mathrm{d}x$. The expansion in (2.53) can be rewritten as

$$f(x) = a_1 h_1(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} h_{2^j+k+1}(x), \tag{2.54}$$

where $i = 1, \ldots, 2^j + k + 1$, and approximated at resolution J as

$$f_J(x) = a_1 h_1(x) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} a_{2^j+k+1} h_{2^j+k+1}(x). \tag{2.55}$$

By using (2.54) and (2.55), we can define the error of approximation as

$$\begin{aligned} E_J(x) &= |f(x) - f_J(x)| \\ &= \left| \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} h_{2^j+k+1}(x) \right|. \end{aligned} \tag{2.56}$$

CHAPTER III
METHODOLOGY

In this chapter, we introduce the method proposed by Lepik [14] for solving 2D PDEs with boundary conditions using 2D Haar wavelets.

3.1 The System of 2-Dimensional Partial Differential Equation

Consider the linear PDE,

$$\sum_{\gamma=0}^{\Gamma}\sum_{\lambda=0}^{\Lambda}D_{\gamma\lambda}(x,y)\frac{\partial^{\gamma+\lambda}}{\partial x^{\gamma}\partial y^{\lambda}}u(x,y)=f(x,y);\quad (x,y)\in\Omega \quad \text{and} \quad \partial\Omega=\sigma, \tag{3.1}$$

where Γ and Λ are given constants; and $D_{\gamma\lambda}(x,y)$ and $f(x,y)$ are given functions. The quantities Γ and Λ can be determined respectively from the maximum order of the x and y derivatives appearing in the linear system (3.1). By simplifying the system, the domain Ω is considered as a rectangular domain $[A_1,B_1]\times[A_2,B_2]$. In addition, any 2D domain can be transformed to a rectangular domain via conformal mapping. The intervals $[A_1,B_1]$ and $[A_2,B_2]$ are divided into $2M_1\left(=2^{J+1}\right)$ and $2M_2(=2^{J'+1})$ parts of equal length, respectively.

3.2 The Approximation of Function

If (3.1) holds for $(\partial^{\Gamma+\Lambda}/\partial x^{\Gamma}\partial y^{\Lambda})u(x,y)\in L^2(\Omega)$, we can expand such that

$$\begin{aligned}\frac{\partial^{\Gamma+\Lambda}}{\partial x^{\Gamma}\partial y^{\Lambda}}u(x,y)&=\sum_{i=1}^{\infty}\sum_{i'=1}^{\infty}a_{ii'}h_i(x)h_{i'}(y)\\&=\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{j'=J'+1}^{\infty}\sum_{k'=0}^{2^{j'}-1}a_{2^j+k+1,2^{j'}+k'+1}h_{2^j+k+1}(x)h_{2^{j'}+k'+1}(y),\end{aligned} \tag{3.2}$$

where $a_{ii'}$ are wavelet coefficients; $h_i(x)$ and $h_{i'}(y)$ are Haar functions; $i=2^j+k+1$; and $i'=2^{j'}+k'+1$. Then the solution $u(x,y)$ can be analytically calculated by taking the integrals in (2.44) Γ and Λ times with respect to x and y, respectively. In this process, the unknown functions can be analytically computed from the boundary conditions σ. The solution $u(x,y)$ will appear in the form:

$$u(x,y) = \sum_{i=1}^{\infty}\sum_{i'=1}^{\infty} a_{ii'} p_{\Gamma,i}(x) p_{\Lambda,i'}(y) + \Psi(x,y), \tag{3.3}$$

which can be approximated by

$$u_{JJ'}(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'} p_{\Gamma,i}(x) p_{\Lambda,i'}(y) + \Psi(x,y), \tag{3.4}$$

where $p_{\Gamma,i}(x)$, $p_{\Lambda,i'}(y)$ are functions defined in (2.46) and (2.51), and $\Psi(x,y)$ is a function satisfying the boundary conditions σ. The other derivatives can be directly determined by taking the derivatives of $u(x,y)$.

3.3 The Determination of Collocation Points

In order to obtain a system of linear equations for calculating $a_{ii'}$, a collocation point (x_r, y_s) is needed and can be defined by

$$(x_r, y_s) = \left(\frac{(2r-1)(B_1-A_1)}{4M_1}, \frac{(2s-1)(B_2-A_2)}{4M_2}\right), \tag{3.5}$$

where $r = 1, \ldots, 2M_1$ and $s = 1, \ldots, 2M_2$.

3.4 The Matrix Equation of the System

By substituting the approximate solution (3.4) and its derivatives at the collocation point (3.5) into (3.1), we obtain the system of linear equations

$$\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'} R_{ii'rs} = f(x_r, y_s). \tag{3.6}$$

The wavelet coefficients $a_{ii'}$ can be calculated from (3.6). However, dealing with a fourth-order matrix equation is definitely complicated. For convenience, we transform the fourth-order matrix equation to a second-order matrix equation by introducing new indices, namely,

$$\beta_1 = 2M_2(i-1) + i' \qquad \text{and} \qquad \beta_2 = 2M_2(r-1) + s. \tag{3.7}$$

Then, the equation (3.6) can be written as

$$\sum_{\beta_1=1}^{4M_1M_2} a_{\beta_1} R_{\beta_1\beta_2} = F_{\beta_2}, \tag{3.8}$$

where $F_{\beta_2} = f(x_r, y_s)$; a_{β_1} and F_{β_2} are $4M_1M_2$-dimensional row vectors; and $R_{\beta_1\beta_2}$ is a $4M_1M_2 \times 4M_1M_2$-dimensional matrix.

After solving (3.8) for a_{β_1}, the wavelet coefficient $a_{ii'}$ can be restored from (3.7). we substitute $a_{ii'}$ back into (3.4) to obtain the solution. We can see more detail about the method in the next section.

3.5 The second-order 2-Dimensional Partial Differential Equation

In this disseration, we show an implementation of this method by considering a second-order 2-Dimensional Partial Differential Equation

$$\sum_{\gamma=0}^{2}\sum_{\lambda=0}^{2} D_{\gamma\lambda}(x,y)\frac{\partial^{\gamma+\lambda}}{\partial x^\gamma \partial y^\lambda}u(x,y) = f(x,y), \tag{3.9}$$

where $(x,y) \in [0,1]^2$, with the boundary conditions: $u(x,0) = \Gamma_1(x)$, $u(0,y) = \Gamma_2(y)$, $u(x,1) = \Gamma_3(x)$ and $u(1,y) = \Gamma_4(y)$. Note that

$$\left.\begin{aligned} u(0,0) = \Gamma_1(0) = \Gamma_2(0), \quad u(0,1) = \Gamma_2(1) = \Gamma_3(0), \\ u(1,0) = \Gamma_1(1) = \Gamma_4(0), \quad u(1,1) = \Gamma_3(1) = \Gamma_4(1). \end{aligned}\right\} \tag{3.10}$$

Then, we approximate $u_{xxyy}(x,y)$ as

$$\frac{\partial^4}{\partial x^2 \partial y^2}u(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}h_i(x)h_{i'}(y). \tag{3.11}$$

The following procedure is the same as Lepik [14]. By integrating (3.11) twice each with respect to y and x, we have that

$$\frac{\partial^2}{\partial x^2}u(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}h_i(x)p_{2,i'}(y) + y\phi_1^{(2)}(x) + \phi_2^{(2)}(x), \tag{3.12}$$

$$\frac{\partial^2}{\partial y^2}u(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)h_{i'}(y) + x\psi_1^{(2)}(y) + \psi_2^{(2)}(y), \tag{3.13}$$

respectively. Integrating (3.12) and (3.13) twice each with respect to x and y yields

$$u_1(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(y) + y\phi_1(x) + \phi_2(x) + x\phi_3(y) + \phi_4(y), \tag{3.14}$$

$$u_2(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(y) + x\psi_1(y) + \psi_2(y) + y\psi_3(x) + \psi_4(x), \tag{3.15}$$

respectively. Then we evaluate $\phi_3(y)$, $\phi_4(y)$, $\psi_3(x)$ and $\psi_4(x)$. Since

$$u_1(x,y) - u_2(x,y) = 0, \qquad \forall(x,y), \tag{3.16}$$

we have

$$y(\phi_1(x) - \psi_3(x)) + (\phi_2(x) - \psi_4(x)) + x(\phi_3(y) - \psi_1(y)) + (\phi_4(y) - \psi_2(y)) = 0. \tag{3.17}$$

(3.17) enables to choose

$$\phi_1(x) = \psi_3(x), \qquad \phi_2(x) = \psi_4(x), \qquad \phi_3(y) = \psi_1(y), \qquad \phi_4(y) = \psi_2(y), \tag{3.18}$$

and then results in

$$u(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(y) + y\phi_1(x) + \phi_2(x) + x\psi_1(y) + \psi_2(y). \tag{3.19}$$

Incorporating the boundary conditions yields the following results. When $u(x,0) = \Gamma_1(x)$, we have

$$\left.\begin{aligned} \phi_2(x) &= -x\psi_1(0) - \psi_2(0) + \Gamma_1(x), \\ \phi_2(0) &= -\psi_2(0) + \Gamma_1(0), \\ \phi_2(1) &= -\psi_1(0) - \psi_2(0) + \Gamma_1(1). \end{aligned}\right\} \tag{3.20}$$

When $u(0,y) = \Gamma_2(y)$, we have

$$\left.\begin{aligned} \psi_2(y) &= -y\phi_1(0) - \phi_2(0) + \Gamma_2(y), \\ \psi_2(0) &= -\phi_2(0) + \Gamma_1(0), \\ \psi_2(1) &= -\phi_1(0) - \phi_2(0) + \Gamma_2(1). \end{aligned}\right\} \tag{3.21}$$

since $\Gamma_1(0) = \Gamma_2(0)$. When $u(x,1) = \Gamma_3(x)$, we have

$$\phi_1(x) = -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(1) - \phi_2(x) - x\psi_1(1) - \psi_2(1) + \Gamma_3(x). \tag{3.22}$$

Then,

$$\left.\begin{aligned} \phi_1(x) =& -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(1) + x[\psi_1(0) - \psi_1(1)] \\ &+ \psi_2(0) - \psi_2(1) + \Gamma_3(x) - \Gamma_1(x), \\ \phi_1(0) =& \psi_2(0) - \psi_2(1) + \Gamma_2(1) - \Gamma_1(0), \\ \phi_1(1) =& -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(1) + [\psi_1(0) - \psi_1(1)] \\ &+ \psi_2(0) - \psi_2(1) + \Gamma_3(1) - \Gamma_1(1), \end{aligned}\right\} \tag{3.23}$$

since $\Gamma_2(1) = \Gamma_3(0)$. when $u(1, y) = \Gamma_4(y)$

$$\psi_1(y) = -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(y) - \psi_2(y) - y\phi_1(1) - \phi_2(1) + \Gamma_4(y). \tag{3.24}$$

Then,

$$\left.\begin{aligned}\psi_1(y) =& -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(y) + y[\phi_1(0) - \phi_1(1)]\\ &+ \phi_2(0) - \phi_2(1) + \Gamma_4(y) - \Gamma_2(y),\\ \psi_1(0) =& \phi_2(0) - \phi_2(1) + \Gamma_1(1) - \Gamma_1(0),\\ \psi_1(1) =& -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(1) + [\phi_1(0) - \phi_1(1)]\\ &+ \phi_2(0) - \phi_2(1) + \Gamma_3(1) - \Gamma_2(1),\end{aligned}\right\} \tag{3.25}$$

since $\Gamma_1(1) = \Gamma_4(0)$, $\Gamma_1(0) = \Gamma_2(0)$, and $\Gamma_3(1) = \Gamma_4(1)$. Substituting (3.20) and (3.21) into (3.19) yields

$$\begin{aligned}u(x, y) =& \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(y) + y[\phi_1(x) - \phi_1(0)]\\ &+ x[\psi_1(y) - \psi_1(0)] - \phi_2(0) - \psi_2(0) + \Gamma_1(x) + \Gamma_2(y),\\ =& \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(y) + y[\phi_1(x) - \phi_1(0)]\\ &+ x[\psi_1(y) - \psi_1(0)] - \Gamma_1(0) + \Gamma_1(x) + \Gamma_2(y).\end{aligned} \tag{3.26}$$

From (3.23), we have

$$\begin{aligned}\phi_1(x) - \phi_1(0) =& -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(x)p_{2,i'}(1) + x[\psi_1(0) - \psi_1(1)]\\ &+ \Gamma_3(x) - \Gamma_1(x) - \Gamma_2(1) + \Gamma_1(0),\end{aligned} \tag{3.27}$$

$$\begin{aligned}\phi_1(0) - \phi_1(1) =& \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(1) - \psi_1(0) + \psi_1(1)\\ &+ \Gamma_1(1) - \Gamma_1(0) + \Gamma_2(1) - \Gamma_3(1).\end{aligned} \tag{3.28}$$

From (3.25), we have

$$\begin{aligned}\psi_1(y) - \psi_1(0) =& -\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(y) + y[\phi_1(0) - \phi_1(1)]\\ &+ \Gamma_4(y) - \Gamma_2(y) - \Gamma_1(1) + \Gamma_1(0)],\end{aligned} \tag{3.29}$$

$$\begin{aligned}\psi_1(0) - \psi_1(1) =& \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}p_{2,i}(1)p_{2,i'}(1) - \phi_1(0) + \phi_1(1)\\ &+ \Gamma_1(1) - \Gamma_1(0) + \Gamma_2(1) - \Gamma_3(1).\end{aligned} \tag{3.30}$$

(3.28) and (3.30) implies

$$\begin{aligned}\phi_1(0)-\phi_1(1)+\psi_1(0)-\psi_1(1)=&\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2}a_{ii'}p_{2,i}(1)p_{2,i'}(1)\\&+\Gamma_1(1)-\Gamma_1(0)+\Gamma_2(1)-\Gamma_3(1).\end{aligned}\tag{3.31}$$

Substituting (3.27) and (3.29) into (3.26) yields

$$\begin{aligned}u(x,y)=&\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2}a_{ii'}\left[p_{2,i}(x)p_{2,i'}(y)-yp_{2,i}(x)p_{2,i'}(1)-xp_{2,i}(1)p_{2,i'}(y)\right]\\&+y\left[\Gamma_3(x)-\Gamma_1(x)-\Gamma_2(1)+\Gamma_1(0)\right]+x\left[\Gamma_4(y)-\Gamma_2(y)-\Gamma_1(1)+\Gamma_1(0)\right]\\&+xy[\phi_1(0)-\phi_1(1)+\psi_1(0)-\psi_1(1)]+\Gamma_1(x)+\Gamma_2(y)-\Gamma_1(0).\end{aligned}\tag{3.32}$$

By substituting (3.31) into (3.32), we have the approximate solution that

$$\begin{aligned}u(x,y)=&\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2}a_{ii'}\left[p_{2,i}(x)p_{2,i'}(y)-yp_{2,i}(x)p_{2,i'}(1)\right.\\&\left.-xp_{2,i}(1)p_{2,i'}(y)+xyp_{2,i}(1)p_{2,i'}(1)\right]\\&+y\left[\Gamma_3(x)-\Gamma_1(x)-\Gamma_2(1)+\Gamma_1(0)\right]+x\left[\Gamma_4(y)-\Gamma_2(y)-\Gamma_1(1)+\Gamma_1(0)\right]\\&+xy[\Gamma_1(1)-\Gamma_1(0)+\Gamma_2(1)-\Gamma_3(1)]+\Gamma_1(x)+\Gamma_2(y)-\Gamma_1(0),\end{aligned}\tag{3.33}$$

and its derivatives:

$$\begin{aligned}
u_x(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}\left[p_{1,i}(x)p_{2,i'}(y) - yp_{1,i}(x)p_{2,i'}(1)\right.\\
&\left.-p_{2,i}(1)p_{2,i'}(y) + yp_{2,i}(1)p_{2,i'}(1)\right]\\
&+ y\left[\Gamma_3'(x) - \Gamma_1'(x)\right] + \left[\Gamma_4(y) - \Gamma_2(y) - \Gamma_1(1) + \Gamma_1(0)\right]\\
&+ y[\Gamma_1(1) - \Gamma_1(0) + \Gamma_2(1) - \Gamma_3(1)] + \Gamma_1'(x), \qquad (3.34)\\
u_y(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}\left[p_{2,i}(x)p_{1,i'}(y) - p_{2,i}(x)p_{2,i'}(1)\right.\\
&\left.-xp_{2,i}(1)p_{1,i'}(y) + xp_{2,i}(1)p_{2,i'}(1)\right]\\
&+ \left[\Gamma_3(x) - \Gamma_1(x) - \Gamma_2(1) + \Gamma_1(0)\right] + x\left[\Gamma_4'(y) - \Gamma_2'(y)\right]\\
&+ x[\Gamma_1(1) - \Gamma_1(0) + \Gamma_2(1) - \Gamma_3(1)] + \Gamma_2'(y), \qquad (3.35)\\
u_{xy}(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}\left[p_{1,i}(x)p_{1,i'}(y) - p_{1,i}(x)p_{2,i'}(1)\right.\\
&\left.-p_{2,i}(1)p_{1,i'}(y) + p_{2,i}(1)p_{2,i'}(1)\right]\\
&+ \left[\Gamma_3'(x) - \Gamma_1'(x)\right] + \left[\Gamma_4'(y) - \Gamma_2'(y)\right]\\
&+ [\Gamma_1(1) - \Gamma_1(0) + \Gamma_2(1) - \Gamma_3(1)], \qquad (3.36)\\
u_{xx}(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}h_i(x)\left[p_{2,i'}(y) - yp_{2,i'}(1)\right] + y\left[\Gamma_3''(x) - \Gamma_1''(x)\right] + \Gamma_1''(x), \qquad (3.37)\\
u_{yy}(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}\left[p_{2,i}(x) - xp_{2,i}(1)\right]h_{i'}(y) + x\left[\Gamma_4''(y) - \Gamma_2''(y)\right] + \Gamma_2''(y), \qquad (3.38)\\
u_{xxy}(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}h_i(x)\left[p_{1,i'}(y) - p_{2,i'}(1)\right] + \left[\Gamma_3''(x) - \Gamma_1''(x)\right], \qquad (3.39)\\
u_{xyy}(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}\left[p_{1,i}(x) - p_{2,i}(1)\right]h_{i'}(y) + \left[\Gamma_4''(y) - \Gamma_2''(y)\right], \qquad (3.40)\\
u_{xxyy}(x,y) = &\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}h_i(x)h_{i'}(y). \qquad (3.41)
\end{aligned}$$

By substituting the solution (3.33) and its derivatives (3.34)-(3.41) into the PDE (3.9), we will obtain the matrix equation. After solving the matrix equation, we have the wavelet coefficient $a_{ii'}$ corresponding to the solution (3.33).

3.6 Poisson Equation

For convenience, the Poisson equation in this context is introduced in the same boundary conditions as in [14] (See equation (26) in [14]) and considered as

$$\frac{\partial^2}{\partial x^2}u(x,y)+\frac{\partial^2}{\partial y^2}u(x,y)=f(x,y), \tag{3.42}$$

where $(x,y)\in[0,1]^2$, with the boundary conditions: $u(x,0)=u(0,y)=u(x,1)=0$ and $u(1,y)=g(y)$. Then, $u_{xxyy}(x,y)$ can be approximated as

$$\frac{\partial^4}{\partial x^2\partial y^2}u(x,y)=\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2}a_{ii'}h_i(x)h_{i'}(y). \tag{3.43}$$

Expoiting the result from Section 3.5 yields the solution $u(x,y)$ as

$$\begin{aligned}u(x,y)=&\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2}a_{ii'}\left[p_{2,i}(x)p_{2,i'}(y)-yp_{2,i}(x)p_{2,i'}(1)\right.\\&\left.-xp_{2,i}(1)p_{2,i'}(y)+xyp_{2,i}(1)p_{2,i'}(1)\right]+xg(y),\end{aligned} \tag{3.44}$$

and its derivatives:

$$\frac{\partial^2}{\partial x^2}u(x,y)=\sum_{i=1}^{\infty}\sum_{i'=1}^{\infty}a_{ii'}\left[h_i(x)p_{2,i'}(y)-h_i(x)yp_{2,i'}(1)\right], \tag{3.45}$$

$$\frac{\partial^2}{\partial y^2}u(x,y)=\sum_{i=1}^{\infty}\sum_{i'=1}^{\infty}a_{ii'}\left[p_{2,i}(x)h_{i'}(y)-xp_{2,i}(1)h_{i'}(y)\right]+xg''(y). \tag{3.46}$$

Substituting (3.45) and (3.46) into (3.42) yields the matrix equation in the form of (3.6) as

$$\sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2}a_{ii'}R_{ii'll'}=f_{ll'}, \tag{3.47}$$

where

$$\begin{aligned}R_{ii'll'}=&h_i(x_l)p_{2,i'}(y_{l'})+p_{2,i}(x_l)h_{i'}(y_{l'})-h_i(x_l)y_{l'}p_{2,i'}(1)-x_lp_{2,i}(1)h_{i'}(y_{l'}),\\f_{ll'}=&f(x_l,y_{l'})-x_lg''(y_{l'}).\end{aligned}$$

The numerical solution can be obtained by calculating $a_{ii'}$ and substituting back into (3.44).

3.7 Helmholtz Equation

In order to expoit the calculation of the Poisson equation for obtaining the general solution in the form (3.4), the Helmholtz equation is introduced as

$$\nabla^2u(x,y)+k^2u(x,y)=f(x,y), \tag{3.48}$$

where $(x, y) \in [0, 1]^2$, with the boundary conditions: $u(x, 0) = u(0, y) = u(x, 1) = 0$ and $u(1, y) = g(y)$. Then, let

$$\frac{\partial^4}{\partial x^2 \partial y^2} u(x, y) = \sum_{i=1}^{2M_1} \sum_{i'=1}^{2M_2} a_{ii'} h_i(x) h_{i'}(y). \tag{3.49}$$

Since the Helmholtz equation (3.48) has the same boundary condition as the Poisson equation in (3.42), by the same procedure, the general solution of the Helmholtz equation $u(x, y)$, derivatives $u_{xx}(x, y)$ and $u_{yy}(x, y)$ are the same as (3.44), (3.45) and (3.46), respectively. By substituting (3.44), (3.45) and (3.46) into (3.48), the matrix equation is in the form of (3.6) as

$$\sum_{i=1}^{2M_1} \sum_{i'=1}^{2M_2} a_{ii'} R_{ii'll'} = f_{ll'}, \tag{3.50}$$

where

$$\begin{aligned}
R_{ii'll'} =& h_i(x_l) p_{2,i'}(y_{l'}) + p_{2,i}(x_l) h_{i'}(y_{l'}) - h_i(x_l) y_{l'} p_{2,i'}(1) - x_l p_{2,i}(1) h_{i'}(y_{l'}) \\
&+ k^2 \left[p_{2,i}(x_l) p_{2,i'}(y_{l'}) - y_{l'} p_{2,i}(x_l) p_{2,i'}(1) \right. \\
&\left. - x_l p_{2,i}(1) p_{2,i'}(y_{l'}) + x_l y_{l'} p_{2,i}(1) p_{2,i'}(1) \right], \\
f_{ll'} =& f(x_l, y_{l'}) - x_l g^{(2)}(y_{l'}) - x_l k^2 g(y_{l'}).
\end{aligned}$$

The numerical solution can be obtained by the same procedure as in the case of the Poisson equation.

CHAPTER IV

CONVERGENCE ANALYSIS

This chapter consists of 2 sections. The section 4.1 rewrites the upper bound of $p_{\alpha,i}(x)$ from [27] in order to derive the convergence theorem more conveniently. The section 4.2 is the main result that shows the proof of the convergence theorem.

4.1 The Upper Bound of $p_{\alpha,i}(x)$

Theorem 4.1. *Suppose $p_{\alpha,i}(x)$ is defined as in (2.46) and (2.51) on [0,1], then the upper bound of $p_{\alpha,i}(x)$ is as follows:*

$$\begin{aligned} &p_{0,i}(x) \leq 1, \\ &p_{\alpha,1}(x) \leq \frac{1}{\alpha!}, \\ &p_{1,i}(x) \leq \frac{1}{2^{j+1}} && \textit{when} \quad i > 1, \\ &p_{\alpha,i}(x) < C(\alpha)\left(\frac{1}{2^{j+1}}\right)^2 && \textit{when} \quad \alpha \geq 2 \quad \textit{and} \quad i > 1, \end{aligned}$$

where

$$C(\alpha) = \frac{8}{3(\lfloor(\alpha+1)/2\rfloor!)^2} \quad \textit{and} \quad i = 2^j + k + 1 \quad \textit{for} \quad i < 1.$$

Proof. Suppose that $p_{\alpha,i}(x)$ is defined as in (2.46) and (2.51) on [0,1]. According to definition 2.14, we have $\xi_1(i) = 2k/2^{j+1}$, $\xi_2(i) = (2k+1)/2^{j+1}$ and $\xi_3(i) = 2(k+1)/2^{j+1}$, since $p_{\alpha,i}(x)$ is defined on [0,1]. Then, we have

$$\xi_2(i) - \xi_1(i) = \xi_3(i) - \xi_2(i) = \frac{1}{2^{j+1}}. \tag{4.1}$$

According to (2.45), $p_{0,i}(x)$ is a Haar wavelet defined in Definition 2.14, so it is obvious that

$$-1 \leq p_{0,i} = h_i(x) \leq 1, \quad \forall i. \tag{4.2}$$

We now find the upper bound for $p_{1,i}(x)$. By taking d/dx on $p_{1,1}(x)$ defined as (2.46) on $[0,1]$, we obtain

$$\frac{d}{dx}p_{1,1}(x) = 1, \tag{4.3}$$

so $p_{1,1}(x)$ is non-decreasing on $[0,1]$. Then,

$$p_{1,1}(x) \leq 1. \tag{4.4}$$

We then find the upper bound for $p_{1,i}$ when $i > 1$. Taking d/dx on $p_{1,i}(x)$ defined as (2.51) on $[0,1]$ yields

$$\frac{d}{dx}p_{1,i}(x) = 0, \qquad \text{when} \quad x \in [0, \xi_1(i)], \tag{4.5}$$

$$\frac{d}{dx}p_{1,i}(x) = 1 > 0, \qquad \text{when} \quad x \in [\xi_1(i), \xi_2(i)], \tag{4.6}$$

$$\frac{d}{dx}p_{1,i}(x) = -1 < 0, \qquad \text{when} \quad x \in [\xi_2(i), \xi_3(i)], \tag{4.7}$$

$$\frac{d}{dx}p_{1,i}(x) = 0, \qquad \text{when} \quad x \in [\xi_3(i), 1]. \tag{4.8}$$

(4.5)-(4.8) imply that $p_{1,i}(x)$ is non-decreasing when $x \in [0, \xi_2(i)]$ and then non-increasing when $x \in [\xi_2(i), 1]$. By (2.51), $p_{1,i}(x)$ is continuous, so $p_{1,i}(x)$ is a maximum at $x = \xi_2(i)$. Then,

$$p_{1,i}(x) \leq \xi_2(i) - \xi_1(i) = \frac{1}{2^{j+1}} \quad \text{for} \quad \forall x \in [0,1] \quad \text{and} \quad \forall i > 1. \tag{4.9}$$

By taking d/dx on $p_{\alpha,1}(x)$, we obtain

$$\frac{d}{dx}p_{\alpha,1}(x) = \frac{1}{(\alpha-1)!}x^{\alpha-1} \geq 0, \tag{4.10}$$

so $p_{\alpha,1}(x)$ is non-decreasing over $x \in [0,1]$ and hence a maximum at $x = 1$. Therefore,

$$p_{\alpha,1}(x) \leq \frac{1}{\alpha!} \quad \text{when} \quad \alpha \geq 2. \tag{4.11}$$

Taking d/dx on $p_{\alpha,i}(x)$ when $\alpha \geq 2$ and $i > 1$ yields

$$\frac{d}{dx}p_{\alpha,i}(x) = 0, \qquad \text{when} \quad x \in [0, \xi_1(i)], \tag{4.12}$$

$$\frac{d}{dx}p_{\alpha,i}(x) = \frac{1}{(\alpha-1)!}\left[x - \xi_1(i)\right]^{\alpha-1} \geq 0, \qquad \text{when} \quad x \in [\xi_1(i), \xi_2(i)], \tag{4.13}$$

$$\frac{d}{dx}p_{\alpha,i}(x) = \frac{1}{(\alpha-1)!}\left\{\left[x - \xi_1(i)\right]^{\alpha-1} - 2\left[x - \xi_2(i)\right]^{\alpha-1}\right\}, \qquad \text{when} \quad x \in [\xi_2(i), \xi_3(i)], \tag{4.14}$$

$$\frac{d}{dx}p_{\alpha,i}(x) = \frac{1}{(\alpha-1)!}\left\{\left[x - \xi_1(i)\right]^{\alpha-1} - 2\left[x - \xi_2(i)\right]^{\alpha-1} + \left[x - \xi_3(i)\right]^{\alpha-1}\right\}, \qquad \text{when} \quad x \in [\xi_3(i), 1]. \tag{4.15}$$

In order to show that $p_{\alpha,i}(x)$ is non-decreasing for every $x \in [0,1]$ when $\alpha \geq 2$, we suppose that $(d/dx)p_{\alpha,i} \geq 0$ when $\alpha \geq 2$, $i > 1$ and $x \in [\xi_2(i), \xi_3(i)]$, so we have

$$[x - \xi_1(i)]^{\alpha-1} \geq 2[x - \xi_2(i)]^{\alpha-1}. \tag{4.16}$$

Since $[x-\xi_1(i)], [x-\xi_2(i)] \geq 0$ and $\alpha \geq 2$, we have

$$[x-\xi_1(i)] \geq 2^{\frac{1}{\alpha-1}}[x-\xi_2(i)], \tag{4.17}$$

$$\left[2^{1/(\alpha-1)}-1\right]\xi_2(i)+[\xi_2(i)-\xi_1(i)] \geq 2^{1/(\alpha-1)}x-x. \tag{4.18}$$

By rearranging (4.18) and applying (4.1), we obtain

$$x \leq \xi_2(i)+[\xi_3(i)-\xi_2(i)]\left(\frac{1}{2^{1/(\alpha-1)}-1}\right). \tag{4.19}$$

Then, $p_{\alpha,i}(x)$ defined in (2.51) when $x \in [\xi_2(i), \xi_3(i)]$ is non-decreasing when $x \leq \xi_2(i)+[\xi_3(i)-\xi_2(i)](1/(2^{1/(\alpha-1)}-1))$. Since $0 < 2^{1/(\alpha-1)}-1 \leq 1$,

$$\begin{aligned} \xi_3(i) =& \xi_2(i)+[\xi_3(i)-\xi_2(i)] && (4.20)\\ \leq& \xi_2(i)+[\xi_3(i)-\xi_2(i)]\left(\frac{1}{2^{1/(\alpha-1)}-1}\right), && (4.21)\end{aligned}$$

so $p_{\alpha,i}(x)$ is non-decreasing when $x \in [\xi_2(i), \xi_3(i)]$ and $\alpha \geq 2$. In the subinterval $x \in [\xi_3(i), 1]$, by using the binomial expansion, $p_{\alpha,i}$ can be rewritten as

$$\begin{aligned} p_{\alpha,i}(x) =& \frac{1}{\alpha!}\left\{\sum_{l=0}^{\alpha}\binom{\alpha}{l}[x-\xi_2(i)]^{\alpha-l}[\xi_2(i)-\xi_1(i)]^l-2[x-\xi_2(i)]^{\alpha}\right.\\ &\left.+\sum_{l=0}^{\alpha}\binom{\alpha}{l}[x-\xi_2(i)]^{\alpha-l}[\xi_2(i)-\xi_3(i)]^l\right\}. \end{aligned} \tag{4.22}$$

Then,

$$\begin{aligned} p_{\alpha,i}(x) =& \frac{1}{\alpha!}\left\{[x-\xi_2(i)]^{\alpha}+[x-\xi_2(i)]^{\alpha-1}[\xi_2(i)-\xi_1(i)]\right.\\ &+\sum_{l=0}^{\alpha}\binom{\alpha}{l}[x-\xi_2(i)]^{\alpha-l}[\xi_2(i)-\xi_1(i)]^l-2[x-\xi_2(i)]^{\alpha}\\ &+[x-\xi_2(i)]^{\alpha}+[x-\xi_2(i)]^{\alpha-1}[\xi_3(i)-\xi_2(i)]\\ &\left.+\sum_{l=0}^{\alpha}\binom{\alpha}{l}[x-\xi_2(i)]^{\alpha-l}[\xi_2(i)-\xi_3(i)]^l\right\}. \end{aligned} \tag{4.23}$$

Substituting (4.1) into (4.23) and rearranging yield

$$p_{\alpha,i}(x) = \frac{1}{\alpha!}\sum_{l=2}^{\alpha}\left\{\binom{\alpha}{l}[x-\xi_2(i)]^{\alpha-l}\left[\left(\frac{1}{2^{j+1}}\right)^l+\left(-\frac{1}{2^{j+1}}\right)^l\right]\right\}, \tag{4.24}$$

where $\alpha \geq 2$ and $i > 1$. The term $[(1/(2^{j+1}))^l+(-1/(2^{j+1}))^l]$ from (4.24) equals zero when l is odd and larger than zero when l is even. Thus

$$\frac{d}{dx}p_{\alpha,i}(x) = \frac{2}{\alpha!}\sum_{l=2}^{\alpha}\binom{\alpha}{l}\frac{(\alpha-l)[x-\xi_2(i)]^{\alpha-l-1}}{(2^{j+1})^l} \geq 0, \tag{4.25}$$

when α, $i > 1$ and l is even. Hence, $p_{\alpha,i}(x)$ is non-decreasing when $x \in [\xi_3(i), 1]$, $\alpha \geq 2$ and $i > 1$. Thus, $p_{\alpha,i}(x)$ is non-decreasing for $x \in [0, 1]$ when $\alpha \geq 2$ and $i > 1$, and has a maximum at $x = 1$.

Finally, we find the upper bound of $p_{\alpha,i}(x)$ defined by (2.51) at $x = 1$ when $\alpha \geq 2$ and $i > 1$. It is obvious that

$$\frac{\alpha!}{l!\,(\alpha - l)!} = \binom{\alpha}{l} \leq \binom{\alpha}{\overline{\alpha}} = \frac{\alpha!}{\overline{\alpha}!\,(\alpha - \overline{\alpha})!} = \frac{\alpha!}{(\overline{\alpha}!)^2}, \tag{4.26}$$

where $\alpha \geq l$ and $\overline{\alpha} = \lfloor(\alpha + 1)/2\rfloor$. By investigating the width of the subinterval $[\xi_2(i), 1]$, we have

$$[x - \xi_2(i)]^{\alpha - l} \leq [1 - \xi_2(i)]^{\alpha - l} < 1, \tag{4.27}$$

where $\alpha - l \geq 0$ and $x \in [\xi_3(i), 1]$. By considering (4.26) and (4.27), $p_{\alpha,i}(x)$ in (4.24) is bounded by the following:

$$\begin{aligned} p_{\alpha,i}(x) &\leq \frac{2}{(\overline{\alpha}!)^2} \sum_{l=2}^{\alpha} \left(\frac{1}{2^{j+1}}\right)^l \\ &= \frac{2}{(\overline{\alpha}!)^2} \sum_{l=2}^{\alpha} \left(\frac{1}{2^j}\right)^l \left(\frac{1}{2}\right)^l, \end{aligned} \tag{4.28}$$

where l is even. Since $(1/2^j)^2 > (1/2^j)^3 > \ldots > (1/2^j)^l > \ldots$, (4.28) becomes

$$\begin{aligned} p_{\alpha,i}(x) &< \frac{8}{(\overline{\alpha}!)^2} \left(\frac{1}{2^{j+1}}\right)^2 \sum_{l=2}^{\alpha} \left(\frac{1}{2}\right)^l \\ &= \frac{8}{(\overline{\alpha}!)^2} \left(\frac{1}{2^{j+1}}\right)^2 \sum_{\rho=1}^{\lfloor\alpha/2\rfloor} \left(\frac{1}{2}\right)^{2\rho} \\ &= \frac{8}{(\overline{\alpha}!)^2} \left(\frac{1}{2^{j+1}}\right)^2 \left[\sum_{\rho=0}^{\lfloor\alpha/2\rfloor} \left(\frac{1}{2}\right)^{2\rho} - 1\right] \\ &< \frac{8}{(\overline{\alpha}!)^2} \left(\frac{1}{2^{j+1}}\right)^2 \left[\sum_{\rho=0}^{\infty} \left(\frac{1}{2}\right)^{2\rho} - 1\right] \\ &= \frac{8}{(\overline{\alpha}!)^2} \left(\frac{1}{2^{j+1}}\right)^2 \left(\frac{4}{3} - 1\right) \\ &= \frac{8}{3\,(\overline{\alpha}!)^2} \left(\frac{1}{2^{j+1}}\right)^2. \end{aligned} \tag{4.29}$$

Therefore,

$$p_{\alpha,i}(x) < C(\alpha) \left(\frac{1}{2^{j+1}}\right)^2, \forall x \in [0, 1], \tag{4.30}$$

where $C(\alpha) = 8/(3(\lfloor(\alpha + 1)/2\rfloor!)^2)$, $\alpha \geq 2$ and $i > 1$. The proof is complete. □

4.2 Convergence Theorem

Definition 4.2. According to Lepik[14], the solution of a 2D PDE of a boundary value problem is

$$u(x,y) = \sum_{i=1}^{\infty}\sum_{i'=1}^{\infty} a_{ii'} p_{\Gamma,i}(x) p_{\Lambda,i'}(y) + \Psi(x,y), \tag{4.31}$$

and (4.31) can be approximated with the maximum level of resolution J and J' as

$$u_{JJ'}(x,y) = \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'} p_{\Gamma,i}(x) p_{\Lambda,i'}(y) + \Psi(x,y), \tag{4.32}$$

where $M_1 = 2^J$ and $M_2 = 2^{J'}$. Then, the error of the approximation at the maximal level of resolution J and J' is defined as

$$E_{JJ'} = \left| u(x,y) - u_{JJ'}(x,y) \right|. \tag{4.33}$$

Definition 4.3. By Definition 4.2, the L^2-norm of the error of the approximation at the maximum level of resolution J and J' can be defined as

$$\left\| E_{JJ'}(x,y) \right\|_2 = \left\{ \iint_{\mathbb{D}} \left[E_{JJ'}(x,y) \right]^2 \mathrm{d}x\mathrm{d}y \right\}^{1/2}, \tag{4.34}$$

where $(x,y) \in \mathbb{D}$. Then, $\left\| E_{JJ'}(x,y) \right\|_2$ is called that L^2*-norm of the error.*

Theorem 4.4. *Given* $\Gamma, \Lambda \geq 2$*, assume that*

$$K(x,y) = \frac{\partial^{(\Gamma+\Lambda)}}{\partial x^{\Gamma} \partial y^{\Lambda}} u(x,y) \in L^2(\mathbb{R}^2) \tag{4.35}$$

is a continuous function on $[0,1]^2$ *and can be approximated as*

$$\begin{aligned} K(x,y) &= \frac{\partial^{(\Gamma+\Lambda)}}{\partial x^{\Gamma} \partial y^{\Lambda}} u(x,y) \\ &\approx \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'} h_i(x) h_{i'}(y), \end{aligned} \tag{4.36}$$

where $M_1 = 2^J$ *and* $M_2 = 2^{J'}$*. For all* $(x,y) \in [0,1]^2$*, there exists* $\omega \geq 0$ *such that*

$$\left| K(x,y) \right|, \left| \frac{\partial}{\partial x} K(x,y) \right|, \left| \frac{\partial}{\partial y} K(x,y) \right|, \left| \frac{\partial^2}{\partial x \partial y} K(x,y) \right| \leq \omega. \tag{4.37}$$

Then the Haar wavelet method, based on [14], is convergent, and it yields the following properties for a boundary value problem on domain $[0,1]^2$ *of 2D PDE. Let* $\left\| E_{JJ'}(x,y) \right\|_2$

be the L^2*-norm of the error at the maximum level of resolution* J *and* J' *defined by Definition 4.3.*

$$\|E_{JJ'}(x,y)\|_2 < G(\Gamma,\Lambda)\left(\frac{1}{2^{\tilde{J}+1}}\right)^2, \tag{4.38}$$

or

$$\|E_{JJ'}(x,y)\|_2 = \mathcal{O}\left(\left(\frac{1}{2^{\tilde{J}+1}}\right)^2\right) \quad \textit{when} \quad \Gamma,\Lambda \geq 2, \tag{4.39}$$

where

$$\begin{aligned} G(\Gamma,\Lambda) =& \left\{\frac{1}{36}\frac{[C(\Lambda)]^2}{(\Gamma!)^2} + \frac{1}{36}\frac{[C(\Gamma)]^2}{(\Lambda!)^2} + \frac{1}{1296}[C(\Gamma)]^2[C(\Lambda)]^2\right. \\ &\left. + \frac{1}{8}\frac{C(\Gamma)C(\Lambda)}{\Gamma!\Lambda!} + \frac{1}{108}\frac{[C(\Gamma)]^2 C(\Lambda)}{\Lambda!} + \frac{1}{108}\frac{C(\Gamma)[C(\Lambda)]^2}{\Gamma!}\right\}^{\frac{1}{2}} \omega, \\ C(\Gamma) =& \frac{8}{3}\frac{1}{(\lfloor(\Gamma+1)/2\rfloor!)^2} \quad \textit{for} \quad n \geq 2, \quad \textit{and} \quad \tilde{J} = \min\{J, J'\}. \end{aligned}$$

Proof. According to Lepik[14], the solution of 2D PDEs is in the form

$$u(x,y) = \sum_{i=1}^{\infty}\sum_{i'=1}^{\infty} a_{ii'}p_{\Gamma,i}(x)p_{\Lambda,i'}(y) + \Psi(x,y), \tag{4.40}$$

where $\Psi(x,y)$ is a function determined by the boundary conditions, and (4.40) can be rewritten as

$$\begin{aligned} u(x,y) =& a_{11}p_{\Gamma,1}(x)p_{\Lambda,1}(y) + \Psi(x,y) \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} a_{2^j+k+1,1}p_{\Gamma,2^j+k+1}(x)p_{\Lambda,1}(y) \\ &+ \sum_{j'=0}^{\infty}\sum_{k'=0}^{2^{j'}-1} a_{1,2^{j'}+k'+1}p_{\Gamma,1}(x)p_{\Lambda,2^{j'}+k'+1}(y) \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\sum_{j'=0}^{\infty}\sum_{k'=0}^{2^{j'}-1} a_{2^j+k+1,2^{j'}+k'+1}p_{\Gamma,2^j+k+1}(x)p_{\Lambda,2^{j'}+k'+1}(y), \end{aligned} \tag{4.41}$$

where $i = 2^j + k + 1$ and $i' = 2^{j'} + k' + 1$. Then, the approximate solution at the maximum level of resolution J and J' relevant to $p_{\Gamma,i}(x)$ and $p_{\Lambda,i'}(y)$, repectively, is

$$\begin{aligned}u_{JJ'}(x,y) =& a_{11}p_{\Gamma,1}(x)p_{\Lambda,1}(y) + \Psi(x,y)\\ &+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1} a_{2^j+k+1,1}p_{\Gamma,2^j+k+1}(x)p_{\Lambda,1}(y)\\ &+\sum_{j'=0}^{J'}\sum_{k'=0}^{2^{j'}-1} a_{1,2^{j'}+k'+1}p_{\Gamma,1}(x)p_{\Lambda,2^{j'}+k'+1}(y)\\ &+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}\sum_{j'=0}^{J'}\sum_{k'=0}^{2^{j'}-1} a_{2^j+k+1,2^{j'}+k'+1}p_{\Gamma,2^j+k+1}(x)p_{\Lambda,2^{j'}+k'+1}(y), \qquad (4.42)\end{aligned}$$

where $i = 2^j + k + 1$ and $i' = 2^{j'} + k' + 1$. By Definition 4.2, the error of approximation $E_{JJ'}(x,y) = |u(x,y) - u_{JJ'}(x,y)|$ with respect to (4.41) and (4.32) can be written as

$$E_{JJ'}(x,y) = \left|T_1 + T_2 + T_{T_3}\right|, \qquad (4.43)$$

where

$$\begin{aligned}T_1 &= \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} a_{2^j+k+1,1}p_{\Gamma,2^j+k+1}(x)p_{\Lambda,1}(y)\\ T_2 &= \sum_{j'=J'+1}^{\infty}\sum_{k'=0}^{2^{j'}-1} a_{1,2^{j'}+k'+1}p_{\Gamma,1}(x)p_{\Lambda,2^{j'}+k'+1}(y)\\ T_3 &= \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{j'=J'+1}^{\infty}\sum_{k'=0}^{2^{j'}-1} a_{2^j+k+1,2^{j'}+k'+1}p_{\Gamma,2^j+k+1}(x)p_{\Lambda,2^{j'}+k'+1}(y),\end{aligned}$$

where $i = 2^j + k + 1$ and $i' = 2^{j'} + k' + 1$. From Definition 4.3, the L^2-norm of the error $\|E_{JJ'}(x,y)\|_2$ can be written as

$$\|E_{JJ'}(x,y)\|_2^2 = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6, \qquad (4.44)$$

where

$$\begin{aligned}
\mathcal{D}_1 =& \sum_{j,k}\sum_{r,s} a_{2^j+k+1,1}a_{2^r+s+1,1}\mathcal{I}^{(2)}_{\Gamma;j,k;r,s}\mathcal{I}^{(1)}_{\Lambda}, && (4.45)\\
\mathcal{D}_2 =& \sum_{j',k'}\sum_{r',s'} a_{1,2^{j'}+k'+1}a_{1,2^{r'}+s'+1}\mathcal{I}^{(1)}_{\Gamma}\mathcal{I}^{(2)}_{\Lambda;j',k';r',s'}, && (4.46)\\
\mathcal{D}_3 =& \sum_{j,k}\sum_{r,s}\sum_{j',k'}\sum_{r',s'} a_{2^j+k+1,2^{j'}+k'+1}a_{2^r+s+1,2^{r'}+s'+1}\mathcal{I}^{(2)}_{\Gamma;j,k;r,s}\mathcal{I}^{(2)}_{\Lambda;j',k';r',s'}, && (4.47)\\
\mathcal{D}_4 =& 2\sum_{j,k}\sum_{r',s'} a_{2^j+k+1,1}a_{1,2^{r'}+s'+1}\mathcal{I}^{(3)}_{\Gamma;j,k}\mathcal{I}^{(3)}_{\Lambda;r',s'}, && (4.48)\\
\mathcal{D}_5 =& 2\sum_{j,k}\sum_{r,s}\sum_{r',s'} a_{2^j+k+1,1}a_{2^r+s+1,2^{r'}+s'+1}\mathcal{I}^{(2)}_{\Gamma;j,k;r,s}\mathcal{I}^{(3)}_{\Lambda;r',s'}, && (4.49)\\
\mathcal{D}_6 =& 2\sum_{r,s}\sum_{j',k'}\sum_{r',s'} a_{1,2^{j'}+k'+1}a_{2^r+s+1,2^{r'}+s'+1}\mathcal{I}^{(3)}_{\Gamma;r,s}\mathcal{I}^{(2)}_{\Lambda;j',k';r',s'}, && (4.50)
\end{aligned}$$

where

$$\mathcal{I}^{(1)}_{\Gamma} = \int_0^1 [p_{\Gamma,1}(x)]^2\,\mathrm{d}x, \tag{4.51}$$

$$\mathcal{I}^{(2)}_{\Gamma;j,k;r,s} = \int_0^1 \left[p_{\Gamma,2^j+k+1}(x)p_{\Gamma,2^r+s+1}(x)\right]\mathrm{d}x, \tag{4.52}$$

$$\mathcal{I}^{(3)}_{\Gamma;j,k} = \int_0^1 \left[p_{\Gamma,2^j+k+1}(x)p_{\Gamma,1}(x)\right]\mathrm{d}x, \tag{4.53}$$

where $j = J+1,\dots,\infty$; $k = 0,1,\dots,2^j-1$; $r = J+1,\dots,\infty$; $s = 0,1,\dots,2^r-1$; $j' = J'+1,\dots,\infty$; $k' = 0,1,\dots,2^{j'}-1$; $r' = J'+1,\dots,\infty$; $s' = 0,1,\dots,2^{r'}-1$; and r and r' are dilation parameters, and s, s' are translation parameters as j, j', k and k', respectively.

The next step is to find the upper bound of $a_{ii'}$ by considering

$$K(x,y) = \frac{\partial^{(\Gamma+\Lambda)}}{\partial x^{\Gamma}\partial y^{\Lambda}}u(x,y) \approx \sum_{i=1}^{2M_1}\sum_{i'=1}^{2M_2} a_{ii'}h_i(x)h_{i'}(y). \tag{4.54}$$

Since the problem of interest is a boundary value problem on domain $[0,1]^2$, both $p_{\Gamma,i}(x)$ and $p_{\Lambda,i}(y)$ are defined on $[0,1]$. Then, by Definition 2.14, we have

$$\xi_2(i)-\xi_1(i) = \xi_3(i)-\xi_2(i) = \frac{1}{2^{j+1}}. \tag{4.55}$$

Since the domain is $[0,1]^2$, (2.52) becomes

$$\int_0^1 h_i(x)h_{i'}(x)x = \begin{cases} 1 & \text{when } i = i' = 1,\\ 2^{-j} & \text{when } i = i' > 1,\\ 0 & \text{when } i \neq i'. \end{cases} \tag{4.56}$$

Then, we need to find the upper bound of a_{11}. By applying the orthogonal property (4.56) in (4.54) and setting $i, i' = 1$, we see that

$$a_{11} = \int_0^1 \int_0^1 K(x,y) h_1(x) h_1(y) \mathrm{d}x \mathrm{d}y = \int_0^1 \int_0^1 K(x,y) \mathrm{d}x \mathrm{d}y. \tag{4.57}$$

Applying the mean value theorem for integrals with repect to x and y to (4.57), we find

$$a_{11} = \int_0^1 K(\delta, y) \mathrm{d}y = K(\delta, \tilde{\delta}) \leq \left| K(\delta, \tilde{\delta}) \right| \leq \omega, \tag{4.58}$$

where $\delta, \tilde{\delta} \in [0,1]$. We then find the upper bound of $a_{i,1}$ when $i > 1$. By letting $\varepsilon_3 \in (0,1)$, $\varepsilon_4 \in (\xi_1(i), \xi_2(i))$, $\varepsilon_5 \in (\xi_2(i), \xi_3(i))$, $\varepsilon_6 \in (\varepsilon_4, \varepsilon_5)$ and $i = 2^j + k + 1$, where $\xi_1(i)$, $\xi_2(i)$, $\xi_3(i)$ are defined by Definition 2.14, it yields

$$\begin{aligned} |\varepsilon_4 - \varepsilon_5| &< |\varepsilon_4 - \xi_2(i)| + |\xi_2(i) - \varepsilon_5| \\ &< |\xi_1(i) - \xi_2(i)| + |\xi_2(i) - \xi_3(i)| = \frac{1}{2^j}, \end{aligned} \tag{4.59}$$

where $i = 2^j + k + 1$. Applying the orthogonal property (4.56) for $i > 1$ and $i' = 1$ to (4.54) yields

$$a_{i,1} = 2^j \int_0^1 \int_0^1 K(x,y) h_i(x) h_1(y) \mathrm{d}x \mathrm{d}y = 2^j \int_0^1 \int_0^1 K(x,y) h_i(x) \mathrm{d}y \mathrm{d}x. \tag{4.60}$$

Applying the mean value theorem for integrals with repect to y and x to (4.60), we get

$$\begin{aligned} a_{i,1} =& 2^j \int_0^1 K(x, \tilde{\varepsilon}_3) h_i(x) \mathrm{d}x = 2^j \left\{ \int_{\xi_1(i)}^{\xi_2(i)} K(x, \tilde{\varepsilon}_3) \mathrm{d}x - \int_{\xi_2(i)}^{\xi_3(i)} K(x, \tilde{\varepsilon}_3) \mathrm{d}x \right\} \\ =& 2^j \left\{ \left[\xi_2(i) - \xi_1(i) \right] K(\varepsilon_4, \tilde{\varepsilon}_3) - \left[\xi_3(i) - \xi_2(i) \right] K(\varepsilon_5, \tilde{\varepsilon}_3) \right\}. \end{aligned} \tag{4.61}$$

Substituting (4.55) into (4.61) yields

$$a_{i,1} = 2^j \left\{ \left(\frac{1}{2^{j+1}} \right) K(\varepsilon_4, \tilde{\varepsilon}_3) - \left(\frac{1}{2^{j+1}} \right) K(\varepsilon_5, \tilde{\varepsilon}_3) \right\} = \frac{1}{2} \left[K(\varepsilon_4, \tilde{\varepsilon}_3) - K(\varepsilon_5, \tilde{\varepsilon}_3) \right]. \tag{4.62}$$

Applying the mean value theorem to $K(\varepsilon_4, \tilde{\varepsilon}_3)$ and $K(\varepsilon_5, \tilde{\varepsilon}_3)$ in (4.62) yields

$$a_{i,1} = \frac{1}{2} (\varepsilon_4 - \varepsilon_5) \frac{\partial}{\partial x} K(\varepsilon_6, \tilde{\varepsilon}_3). \tag{4.63}$$

(4.59) and (4.63) imply that

$$a_{i,1} \leq \frac{1}{2} |\varepsilon_4 - \varepsilon_5| \left| \frac{\partial}{\partial x} K(\varepsilon_6, \tilde{\varepsilon}_3) \right| < \frac{\omega}{2^{j+1}}. \tag{4.64}$$

Next, we find an upper bound of $a_{1,i'}$ when $i' > 1$ by the same procedure as finding $a_{i,1}$. By letting $\varepsilon_3 \in (0,1)$, $\tilde{\varepsilon}_4 \in (\xi_1(i'), \xi_2(i'))$, $\tilde{\varepsilon}_5 \in (\xi_2(i'), \xi_3(i'))$, $\tilde{\varepsilon}_6 \in (\tilde{\varepsilon}_4, \tilde{\varepsilon}_5)$ and $i' = 2^{j'} + k' + 1$, where $\xi_1(i')$, $\xi_2(i')$, $\xi_3(i')$ are defined by Definition 2.14, it yields

$$\begin{aligned}|\tilde{\varepsilon}_4 - \tilde{\varepsilon}_5| &< \left|\tilde{\varepsilon}_4 - \xi_2(i')\right| + \left|\xi_2(i') - \tilde{\varepsilon}_5\right| \\ &< \left|\xi_1(i') - \xi_2(i')\right| + \left|\xi_2(i') - \xi_3(i')\right| = \frac{1}{2^{j'}}. \end{aligned} \tag{4.65}$$

Applying the orthogonal property (4.56) for $i = 1$ and $i' > 1$ to (4.54) yields

$$a_{1,i'} = 2^{j'} \int_0^1 \int_0^1 K(x,y) h_1(x) h_{i'}(y) \mathrm{d}y \mathrm{d}x = 2^{j'} \int_0^1 \int_0^1 K(x,y) h_{i'}(y) \mathrm{d}y \mathrm{d}x, \tag{4.66}$$

where $i' = 2^{j'} + k' + 1$. Applying the mean value theorem for integrals with repect to x and then y to (4.66), we obtain

$$\begin{aligned} a_{1,i'} =& 2^{j'} \int_0^1 K(x,y) h_{i'}(y) \mathrm{d}y = 2^{j'} \left\{ \int_{\xi_1(i')}^{\xi_2(i')} K(x,y) \mathrm{d}y - \int_{\xi_2(i')}^{\xi_3(i')} K(x,y) \mathrm{d}y \right\} \\ =& 2^{j'} \left\{ \left[\xi_2(i') - \xi_1(i')\right] K(\varepsilon_3, \tilde{\varepsilon}_4) - \left[\xi_3(i') - \xi_2(i')\right] K(\varepsilon_3, \tilde{\varepsilon}_5) \right\}. \end{aligned} \tag{4.67}$$

Substituting (4.55) into (4.67) yields

$$\begin{aligned} a_{1,i'} =& 2^{j'} \left\{ \left(\frac{1}{2^{j'+1}}\right) K(\varepsilon_3, \tilde{\varepsilon}_4) - \left(\frac{1}{2^{j'+1}}\right) K(\varepsilon_3, \tilde{\varepsilon}_5) \right\} \\ =& \frac{1}{2} \left[K(\varepsilon_3, \tilde{\varepsilon}_4) - K(\varepsilon_3, \tilde{\varepsilon}_5)\right]. \end{aligned} \tag{4.68}$$

Applying the mean value theorem for $K(\varepsilon_3, \tilde{\varepsilon}_4)$ and $K(\varepsilon_3, \tilde{\varepsilon}_5)$ in (4.68) yields

$$a_{1,i'} = \frac{1}{2} (\tilde{\varepsilon}_4 - \tilde{\varepsilon}_5) \frac{\partial}{\partial y} K(\varepsilon_3, \tilde{\varepsilon}_6) \leq \frac{1}{2} |\tilde{\varepsilon}_4 - \tilde{\varepsilon}_5| \left| \frac{\partial}{\partial y} K(\varepsilon_3, \tilde{\varepsilon}_6) \right|. \tag{4.69}$$

(4.65) and (4.69) result in

$$a_{1,i'} < \left(\frac{1}{2^{j'+1}}\right) \omega, \tag{4.70}$$

where $i' = 2^{j'} + k' + 1$. Then, we find an upper bound of $a_{i,i'}$ when $i, i' > 1$. By letting $\varepsilon_1 \in (\xi_1(i), \xi_2(i))$, $\varepsilon_2 \in (\xi_2(i), \xi_3(i))$, $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, $\tilde{\varepsilon}_1 \in (\xi_1(i'), \xi_2(i'))$, $\tilde{\varepsilon}_2 \in (\xi_2(i'), \xi_3(i'))$ and $\tilde{\varepsilon} \in (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)$, where $\xi_1(i)$, $\xi_2(i)$, $\xi_3(i)$, $\xi_1(i')$, $\xi_2(i')$, $\xi_3(i')$ are defined by Definition 2.14, $i = 2^j + k + 1$ and $i' = 2^{j'} + k' + 1$, it yields

$$\begin{aligned}|\varepsilon_1 - \varepsilon_2| &< |\varepsilon_1 - \xi_2(i)| + |\xi_2(i) - \varepsilon_2| \\ &< |\xi_1(i) - \xi_2(i)| + |\xi_2(i) - \xi_3(i)| = \frac{1}{2^j}, \end{aligned} \tag{4.71}$$

$$\begin{aligned}|\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2| &< \left|\tilde{\varepsilon}_1 - \xi_2(i')\right| + \left|\xi_2(i') - \tilde{\varepsilon}_2\right| \\ &< \left|\xi_1(i') - \xi_2(i')\right| + \left|\xi_2(i') - \xi_3(i')\right| = \frac{1}{2^{j'}}. \end{aligned} \tag{4.72}$$

Applying the orthogonal property (4.56) for $i, i' > 1$ to (4.54) yields

$$\begin{aligned} a_{i,i'} &= 2^{j+j'} \int_0^1 \int_0^1 K(x,y) h_i(x) h_{i'}(y) \mathrm{d}x \mathrm{d}y \\ &= 2^{j+j'} \int_0^1 \left\{ \int_{\xi_1(i)}^{\xi_2(i)} K(x,y) h_{i'}(y) \mathrm{d}x - \int_{\xi_2(i)}^{\xi_3(i)} K(x,y) h_{i'}(y) \mathrm{d}x \right\} \mathrm{d}y, \end{aligned} \tag{4.73}$$

where $i = 2^j + k + 1$ and $i' = 2^{j'} + k' + 1$. Applying the mean value theorem for integrals with respect to x to (4.73), we have

$$a_{i,i'} = 2^{j+j'} \int_0^1 \left\{ [\xi_2(i) - \xi_1(i)] K(\varepsilon_1, y) h_{i'}(y) - [\xi_3(i) - \xi_2(i)] K(\varepsilon_2, y) h_{i'}(y) \right\} \mathrm{d}y. \tag{4.74}$$

Substituting (4.55) into (4.74) yields

$$a_{i,i'} = 2^{j'-1} \int_0^1 \left[K(\varepsilon_1, y) - K(\varepsilon_2, y) \right] h_{i'}(y) \mathrm{d}y. \tag{4.75}$$

Applying the mean value theorem for $K(\varepsilon_1, y)$ and $K(\varepsilon_2, y)$ in (4.75) yields

$$\begin{aligned} a_{i,i'} &= 2^{j'-1} \int_0^1 \left[(\varepsilon_1 - \varepsilon_2) \frac{\partial}{\partial x} K(\varepsilon, y) h_{i'}(y) \right] \mathrm{d}y \\ &= 2^{j'-1} (\varepsilon_1 - \varepsilon_2) \left\{ \int_{\xi_1(i')}^{\xi_2(i')} \frac{\partial}{\partial x} K(\varepsilon, y) \mathrm{d}y - \int_{\xi_2(i')}^{\xi_3(i')} \frac{\partial}{\partial x} K(\varepsilon, y) \mathrm{d}y \right\}. \end{aligned} \tag{4.76}$$

Applying the mean value theorem for integrals with respect to y to (4.76) yields

$$a_{i,i'} = 2^{j'-1} (\varepsilon_1 - \varepsilon_2) \left\{ [\xi_2(i') - \xi_1(i')] \frac{\partial}{\partial x} K(\varepsilon, \tilde{\varepsilon}_1) - [\xi_3(i') - \xi_2(i')] \frac{\partial}{\partial x} K(\varepsilon, \tilde{\varepsilon}_2) \right\}. \tag{4.77}$$

Substituting (4.55) into (4.77) yields

$$\begin{aligned} a_{i,i'} &= 2^{j'-1} (\varepsilon_1 - \varepsilon_2) \left[\left(\frac{1}{2^{j'+1}} \right) \frac{\partial}{\partial x} K(\varepsilon, \tilde{\varepsilon}_1) - \left(\frac{1}{2^{j'+1}} \right) \frac{\partial}{\partial x} K(\varepsilon, \tilde{\varepsilon}_2) \right] \\ &= \frac{(\varepsilon_1 - \varepsilon_2)}{4} \left[\frac{\partial}{\partial x} K(\varepsilon, \tilde{\varepsilon}_1) - \frac{\partial}{\partial x} K(\varepsilon, \tilde{\varepsilon}_2) \right]. \end{aligned} \tag{4.78}$$

Applying the mean value theorem for $(\partial/\partial x) K(\varepsilon, \tilde{\varepsilon}_1)$ and $(\partial/\partial x) K(\varepsilon, \tilde{\varepsilon}_2)$ in (4.78) yields

$$a_{i,i'} = \frac{1}{4} (\varepsilon_1 - \varepsilon_2) (\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2) \frac{\partial^2}{\partial x \partial y} K(\varepsilon, \tilde{\varepsilon}). \tag{4.79}$$

Taking the inequalities(4.71) and (4.72) into (4.79) yields

$$\begin{aligned} a_{i,i'} &\leq \frac{1}{4}\left|\varepsilon_1-\varepsilon_2\right|\left|\tilde{\varepsilon}_1-\tilde{\varepsilon}_2\right|\left|\frac{\partial^2}{\partial x\partial y}K(\varepsilon,\tilde{\varepsilon})\right| \\ &\leq \left(\frac{1}{2^{j+1}}\right)\left(\frac{1}{2^{j'+1}}\right)\omega, \end{aligned} \tag{4.80}$$

where $i = 2^j + k + 1$ and $i' = 2^{j'} + k' + 1$.

The next step is to determine the upper bound of the integrals in (4.51)-(4.53). By applying Theorem 4.1, we obtain

$$\mathcal{I}_{\Gamma}^{(1)} \leq \int_0^1 \frac{1}{(\Gamma!)^2}\mathrm{d}x = \frac{1}{(\Gamma!)^2}, \tag{4.81}$$

$$\mathcal{I}_{\Gamma;j,k;r,s}^{(2)} < \int_0^1 \frac{[C(\Gamma)]^2}{(2^{j+1})^2\,(2^{r+1})^2}\mathrm{d}x = \frac{[C(\Gamma)]^2}{(2^{j+1})^2\,(2^{r+1})^2}, \tag{4.82}$$

$$\mathcal{I}_{\Gamma;j,k}^{(3)} < \int_0^1 \frac{C(\Gamma)}{\Gamma!\,(2^{j+1})^2}\mathrm{d}x = \frac{C(\Gamma)}{\Gamma!\,(2^{j+1})^2}, \tag{4.83}$$

where $C(\Gamma) = 8/(3\,(\lfloor(\Gamma+1)/2\rfloor)^2)$ and $\Gamma \geq 2$. Before determining the upper bound of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$, we requires the following:

$$\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\left(\frac{1}{2^{j+1}}\right)^3 = \frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2. \tag{4.84}$$

To prove 4.84, we start with

$$\sum_{N}^{M} 1 = M+1-N, \quad \text{then} \quad \sum_{k=0}^{2^j-1} 1 = 2^j. \tag{4.85}$$

So that

$$\begin{aligned} &\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\left(\frac{1}{2^{j+1}}\right)^3 = \sum_{j=J+1}^{\infty}\left(\frac{1}{2^{j+1}}\right)^3 2^j = \frac{1}{8}\sum_{j=J+1}^{\infty}\left(\frac{1}{4}\right)^j \\ &= \frac{1}{8}\left[\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^j - \sum_{j=o}^{J}\left(\frac{1}{4}\right)^j\right] = \frac{1}{8}\left\{\frac{1}{1-(1/4)} - \frac{1-(1/4)^{J+1}}{1-(1/4)}\right\} = \frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2. \end{aligned} \tag{4.86}$$

The next step is to find an upper bound of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$. By applying the bounded value of $a_{ii'}$, (4.81)-(4.83) and (4.84) for (4.45)-(4.50), we obtain

$$\begin{aligned}\mathcal{D}_1 <& \sum_{j,k}\sum_{r,s}\frac{[C(\Gamma)]^2\,\omega^2}{(\Lambda!)^2\,(2^{j+1})^3\,(2^{r+1})^3}\\ =& \frac{[C(\Gamma)]^2\,\omega^2}{(\Lambda!)^2}\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right] \leq \frac{[C(\Gamma)]^2\,\omega^2}{36\,(\Lambda!)^2}\left(\frac{1}{2^{\tilde{J}+1}}\right)^4,\end{aligned} \tag{4.87}$$

$$\begin{aligned}\mathcal{D}_2 <& \sum_{j',k'}\sum_{r',s'}\frac{[C(\Lambda)]^2\,\omega^2}{(\Gamma!)^2\,(2^{j'+1})^3\,(2^{r'+1})^3}\\ =& \frac{[C(\Lambda)]^2\,\omega^2}{(\Gamma!)^2}\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right] \leq \frac{[C(\Lambda)]^2\,\omega^2}{36\,(\Gamma!)^2}\left(\frac{1}{2^{\tilde{J}+1}}\right)^4,\end{aligned} \tag{4.88}$$

$$\begin{aligned}\mathcal{D}_3 <& \sum_{j,k}\sum_{r,s}\sum_{j',k'}\sum_{r',s'}\frac{[C(\Gamma)]^2\,[C(\Lambda)]^2\,\omega^2}{(2^{j+1})^3\,(2^{r+1})^3\,(2^{j'+1})^3\,(2^{r'+1})^3}\\ =& [C(\Gamma)]^2\,[C(\Lambda)]^2\,\omega^2\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right]\\ \leq& \frac{[C(\Gamma)]^2\,[C(\Lambda)]^2\,\omega^2}{1296}\left(\frac{1}{2^{\tilde{J}+1}}\right)^8,\end{aligned} \tag{4.89}$$

$$\begin{aligned}\mathcal{D}_4 <& 2\sum_{j,k}\sum_{r',s'}\frac{C(\Gamma)C(\Lambda)\omega^2}{\Gamma!\Lambda!\,(2^{j+1})^3\,(2^{r'+1})^3}\\ =& \frac{2C(\Gamma)C(\Lambda)\omega^2}{\Gamma!\Lambda!}\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right] \leq \frac{C(\Gamma)C(\Lambda)\omega^2}{18\Gamma!\Lambda!}\left(\frac{1}{2^{\tilde{J}+1}}\right)^4,\end{aligned} \tag{4.90}$$

$$\begin{aligned}\mathcal{D}_5 <& 2\sum_{j,k}\sum_{r,s}\sum_{r',s'}\frac{[C(\Gamma)]^2\,C(\Lambda)\omega^2}{\Lambda!\,(2^{j+1})^3\,(2^{r+1})^3\,(2^{r'+1})^3}\\ =& \frac{2\,[C(\Gamma)]^2\,C(\Lambda)\omega^2}{\Lambda!}\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right]\\ \leq& \frac{[C(\Gamma)]^2\,C(\Lambda)\omega^2}{108\Lambda!}\left(\frac{1}{2^{\tilde{J}+1}}\right)^6,\end{aligned} \tag{4.91}$$

$$\begin{aligned}\mathcal{D}_6 <& 2\sum_{r,s}\sum_{j',k'}\sum_{r',s'}\frac{C(\Gamma)\,[C(\Lambda)]^2\,\omega^2}{\Gamma!\,(2^{r+1})^3\,(2^{j'+1})^3\,(2^{r'+1})^3}\\ =& \frac{2C(\Gamma)\,[C(\Lambda)]^2\,\omega^2}{\Gamma!}\left[\frac{1}{6}\left(\frac{1}{2^{J+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right]\left[\frac{1}{6}\left(\frac{1}{2^{J'+1}}\right)^2\right]\\ \leq& \frac{C(\Gamma)\,[C(\Lambda)]^2\,\omega^2}{108\Gamma!}\left(\frac{1}{2^{\tilde{J}+1}}\right)^6,\end{aligned} \tag{4.92}$$

where $j = J+1,\ldots,\infty$; $k = 0,1,\ldots,2^j-1$; $r = J+1,\ldots,\infty$; $s = 0,1,\ldots,2^r-1$; $j' = J'+1,\ldots,\infty$; $k' = 0,1,\ldots,2^{j'}-1$; $r' = J'+1,\ldots,\infty$; $s' = 0,1,\ldots,2^{r'}-1$; and

$\tilde{J} = \min\{J, J'\}$. By substituting (4.87)-(4.92) into (4.44), we have

$$\begin{aligned}
[\|E_{JJ'}(x,y)\|_2]^2 <& \frac{[C(\Gamma)]^2\,\omega^2}{36\,(\Lambda!)^2}\left(\frac{1}{2^{\tilde{J}+1}}\right)^4 + \frac{[C(\Lambda)]^2\,\omega^2}{36\,(\Gamma!)^2}\left(\frac{1}{2^{\tilde{J}+1}}\right)^4 \\
&+ \frac{[C(\Gamma)]^2\,[C(\Lambda)]^2\,\omega^2}{1296}\left(\frac{1}{2^{\tilde{J}+1}}\right)^8 + \frac{C(\Gamma)C(\Lambda)\omega^2}{18\Gamma!\Lambda!}\left(\frac{1}{2^{\tilde{J}+1}}\right)^4 \\
&+ \frac{[C(\Gamma)]^2\,C(\Lambda)\omega^2}{108\Lambda!}\left(\frac{1}{2^{\tilde{J}+1}}\right)^6 + \frac{C(\Gamma)\,[C(\Lambda)]^2\,\omega^2}{108\Gamma!}\left(\frac{1}{2^{\tilde{J}+1}}\right)^6. \qquad (4.93)
\end{aligned}$$

Hence,

$$\|E_{JJ'}(x,y)\|_2 < G(\Gamma,\Lambda)\left(\frac{1}{2^{\tilde{J}+1}}\right)^2, \qquad (4.94)$$

or

$$\|E_{JJ'}(x,y)\|_2 = \mathcal{O}\left(\left(\frac{1}{2^{\tilde{J}+1}}\right)^2\right) \quad \text{when} \quad \Gamma,\Lambda \geq 2, \qquad (4.95)$$

where

$$\begin{aligned}
G(\Gamma,\Lambda) = \Bigg\{ & \frac{[C(\Lambda)]^2}{36(\Gamma!)^2} + \frac{[C(\Gamma)]^2}{36(\Lambda!)^2} + \frac{[C(\Gamma)]^2\,[C(\Lambda)]^2}{1296} \\
& + \frac{C(\Gamma)C(\Lambda)}{18\Gamma!\Lambda!} + \frac{[C(\Gamma)]^2\,C(\Lambda)}{108\Lambda!} + \frac{C(\Gamma)\,[C(\Lambda)]^2}{108\Gamma!} \Bigg\}^{\frac{1}{2}} \omega,
\end{aligned}$$

and

$$C(\Gamma) = 8/(3\,(\lfloor(\Gamma+1)/2\rfloor!)^2) \quad \text{for} \quad \Gamma \geq 2.$$

The proof is complete. □

4.3 The Estimation of Error

We set the maximum level of resolution that $J = J'$, so $\tilde{J} = J = J'$, and $M_1 = M_2 = 2^{\tilde{J}}$. When estimating $E_{JJ'}(x,y)$, we consider at the collocation points x_r and y_s.

Then, we introduce $E_{\tilde{J}}(x,y)$ as the error of approximation $E_{JJ'}(x,y)$ when $J = J' = \tilde{J}$, By Theorem 4.4, we have the error of approximation at the maximal level of resolution $\tilde{J}$ as

$$\left\|E_{\tilde{J}}(x,y)\right\|_2 \sim \left(\frac{1}{2^{\tilde{J}+1}}\right)^2. \qquad (4.96)$$

Then, we have

$$\frac{\left\|E_{\tilde{J}}(x,y)\right\|_2}{\left\|E_{\tilde{J}+1}(x,y)\right\|_2} \sim 4, \quad \text{i.e.,} \quad \log_2\left(\frac{\left\|E_{\tilde{J}}(x,y)\right\|_2}{\left\|E_{\tilde{J}+1}(x,y)\right\|_2}\right) \sim 2. \tag{4.97}$$

Since the order of convergence from Theorem 4.4 is equal to 2, we can evaluate the order of convergence from

$$\text{order} = \log_2\left(\frac{\left\|E_{\tilde{J}}(x,y)\right\|_2}{\left\|E_{\tilde{J}+1}(x,y)\right\|_2}\right). \tag{4.98}$$

The error of approximation from the numerical result can be obtained by

$$\left(\left\|E_{\tilde{J}}(x,y)\right\|_2\right)_{\text{num}} = \sqrt{\frac{\sum_{\Omega_{\text{coll}}}\left(E_{\tilde{J}}(x_r,y_s)\right)^2}{n\left(\Omega_{\text{coll}}\right)}}, \tag{4.99}$$

where Ω_{coll} is the set of collocation points (x_r, y_s) and $n\left(\Omega_{\text{coll}}\right)$ is the number of collocation points.

CHAPTER V

CONCLUSIONS AND NUMERICAL RESULTS

Theorem 4.4 shows that the method [14] based on the two-dimensional Haar wavelet converges as the maximum level of resolution increases. The convergence analysis shows that the order of approximation is of 2 for boundary value problems. The numerical results can validate in 2 examples: Poisson and Helmholtz equations which are shown in Examples 5.1 and 5.2, respectively.

Example 5.1. *The Poisson equation*

$$\nabla^2 u(x,y) = 2\left[y^2(1-6x^2)(1-y^2)+x^2(1-6y^2)(1-x^2)\right], \tag{5.1}$$

with the boundary conditions that $u(x,0)=u(0,y)=u(x,1)=u(1,y)=0$*, has the exact solution* $u_{ex}(x,y)$ *in the form*

$$u_{ex}(x,y) = x^2y^2(1-x^2)(1-y^2). \tag{5.2}$$

Numerical errors from solving the Poisson equation using the Haar wavelet method are shown in Table 5.1.

Example 5.2. *The Helmholtz equation*

$$\nabla^2 u(x,y) + k^2 u(x,y) = (k^2-2\pi^2)\sin(\pi x)\sin(\pi y), \tag{5.3}$$

with the boundary conditions that $u(x,0)=u(0,y)=u(x,1)=u(1,y)=0$*, has the exact solution* $u_{ex}(x,y)$ *in the form of*

$$u_{ex}(x,y) = \sin(\pi x)\sin(\pi y). \tag{5.4}$$

Numerical errors from solving the Helmholtz equation using the Haar wavelet method (when $k=0.5$*) are shown in Table 5.2.*

Table 5.1: The order of convergence of the numerical solution of Poisson equation in Example 5.1

$\tilde{J}$	**Numerical error**	**Order**
1	6.19×10^{-04}	—
2	1.67×10^{-04}	1.8940
3	4.24×10^{-05}	1.9731
4	1.07×10^{-05}	1.9932
5	2.67×10^{-06}	1.9983

Table 5.2: The order of convergence of the numerical solution of Helmholtz equation in Example 5.2 when $k = 0.5$

$\tilde{J}$	**Numerical error**	**Order**
1	1.04×10^{-02}	—
2	2.63×10^{-03}	1.9840
3	6.59×10^{-04}	1.9969
4	1.65×10^{-04}	1.9993
5	4.12×10^{-05}	1.9998

The further study should be to deal with the other wavelets such as Legendre and Chebyshev wavelets.

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