รายงานวิจัยฉบับสมบูรณ์

โครงการ : ตัวดำเนินการที่สัมพันธ์กับดำเนินการเฮล์มโฮล์ซและตัวดำเนินการไคลน์–กอร์ดอน

The Operator Related to the Helmholtz and Klein-Gordon Operator

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

โดย

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มหาวิทยาลัยขอนแก่น

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หัวหน้าโครงการวิจัยผู้รับทุน : ผศ.ดร.คำสิงห์ นนเลาพล
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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และมหาวิทยาลัยขอนแก่นที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ในการทำงานวิจัยค้นคว้าครั้งนี้

ศาสตราจารย์อำนวย ขนันไทย นักวิจัยที่ปรึกษาให้กับโครงการนี้ที่อบรมสั่งสอน ถ่ายทอด ความรู้ด้านต่าง ๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

ผศ.ดร.คำสิงห์ นนเลาพล
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บทคัดย่อ: ในรายงานวิจัยนี้ เราได้ศึกษาผลเฉลยของตัวดำเนินการไดมอนด์ไคลน์-กอร์ดอนและสังวัตนาการ และเรายังได้ศึกษาผลการแปลงฟูเรียร์ของแก่นกลางไดมอนด์ไคลน์-กอร์ดอนซึ่งเป็นผลเฉลยของตัวดำเนินการไดมอนด์ไคลน์-กอร์ดอนอีกด้วย ผลลัพธ์ที่ได้นั้นพัฒนาและครอบคลุมผลงานวิจัยของนักคณิตศาสตร์ท่านอื่น

นอกจากนี้ เรายังได้ศึกษาผลเฉลยทั่วไปของตัวดำเนินการ $\square_c^k$ และตัวดำเนินการอัลตราไฮเพอร์โบลิกเบสเซลซึ่งทั้งสองตัวดำเนินการมีความสัมพันธ์กับตัวดำเนินการอัลตราไฮเพอร์โบลิก และเรายังได้ศึกษาการผกผันของแก่นกลางเบสเซลอัลตราไฮเพอร์โบลิกและแก่นกลางเบสเซลไดมอนด์ของมาร์เซล รีซส์อีกด้วย

คำหลัก: ดิสตริบิวชันไดแรค-เดลตา; ดิสตริบิวชันเทมเพอร์; ตัวดำเนินการอัลตราไฮเพอร์โบลิก; ตัวดำเนินการอัลตราไฮเพอร์โบลิก-เบสเซล; ตัวดำเนินการไดมอนด์ไคลน์-กอร์ดอน; แก่นกลางเบสเซลไดมอนด์ของมาร์เซล รีซส์; ผลการแปลงฟูเรียร์

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Abstract: In this project, we study the solution of the diamond Klein-Gordon operator and its convolution. We also study the Fourier transform of the diamond Klein-Gordon kernel which is the solution of the diamond Klein-Gordon operator. Our results improve and include the corresponding known results studied by another authors.

Moreover, we study the general solution of the operator $\Box_c^k$ and the ultrahyperbolic Bessel operator that both operator related to the ultrahyperbolic operator. And we also study the inverse Bessel ultrahyperbolic and Bessel diamond kernel of Marcel Riesz.

Keywords: Dirac-delta distribution; tempered distribution; ultrahyperbolic operator; ultrahyperbolic Bessel operator; diamond Klein-Gordon operator; Bessel diamond kernel of Marcel Riesz; Fourier transform

Contents

Chapter 1

Executive Summary

Generalized functions or distributions have of late been commanding constantly expanding interest in several different branches of mathematics. In somewhat nonrigorous form, they have already long been used in essence by physicists and opened up a new area of mathematical research, which in turn proved an impetus in the development of a number of mathematical disciplines, such as operational calculus, transformation theory, functional analysis, ordinary and partial differential equations.

Distributions close relation to the solutions of differential equations. In the case of the ordinary differential equation $Lu = 0$ with constant coefficients, every solution is the classical solution. The matter is quite different for partial differential equations. The solutions in a similar situation may now include generalized functions. For instance, $\partial u/\partial x_1 = 0$, in $\mathbb{R}^2$, has among its solutions the generalized function $\delta(x_2)$,

$$\langle \delta(x_2), \phi(x_1, x_2) \rangle = \int_{-\infty}^{\infty} \phi(x_1, 0) dx_1,$$

where ϕ is a test function.

Our aim is to find the solution of the partial differential equation

$$Lu(x) = s(x), \tag{1.0.1}$$

where L is partial differential operator and $s(x)$ is an arbitrary known distribution. It is well known that the *elementary* or *fundamental solution* is the solution for $s(x) = \delta(x)$.

A distribution $u(x)$ is a solution of (1.0.1) if for every test function $\phi(x)$, we have

$$\langle Lu(x), \phi(x) \rangle = \langle s(x), \phi(x) \rangle. \tag{1.0.2}$$

In searching for a solution $u(x)$ of differential equation (1.0.1) we may have the following situations :

(1) The solution $u(x)$ is a sufficiently smooth function, so that the operation in (1.0.1) can be performed in the classical sense and the resulting equation is an identity. Then $u(x)$ is the *classical solution*.

(2) The solution $u(x)$ is not sufficiently smooth function, so that the operation in (1.0.1) can not be performed, but it satisfy (1.0.2) as distribution. It is then a *weak solution*.

(3) The solution $u(x)$ is a singular distribution and satisfy (1.0.2). It is then a *distributional solution*.

All these solution are call *generalized solution*.

A purpose of this research is to find the solution of the partial differential equation that the solution is a weak solution or distributional solution.

Chapter 2

Main Results

2.1 Convolution equation

In the paper A1, we study the distribution $e^{\alpha x}\diamondsuit_c^k\delta$ where $\diamondsuit_c^k$ is the operator which related to the diamond type operator iterated k-times, δ is the Dirac-delta distribution, $x = (x_1, x_2, \ldots, x_n)$ is a variable and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a constant and both are the points in the n-dimensional Euclidean spaces $\mathbb{R}^n$.

At first, the properties of $e^{\alpha x}\diamondsuit_c^k\delta$ are studied and later we study the application of $e^{\alpha x}\diamondsuit_c^k\delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x}\diamondsuit_c^k\delta) * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r\diamondsuit_c^r\delta,$$

where $u(x)$ is the generalized function and C_r is a constant. We found that its solution related to the diamond kernel of Marcel Riesz and moreover, the type of solutions such as the ordinary functions or the tempered distributions depending on k, m and α.

2.2 General solution of some operators

We published two papers in this topic (see Appendices A3 and A5). We study the general solution of equation $\Box_c^k u(x) = f(x)$, where $\Box_c^k$ is the operator which related to the ultra-hyperbolic type operator iterated k-times and is defined by

$$\Box_c^k = \left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k,$$

$p + q = n, n$ is the dimension of $\mathbb{R}^n, f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

The paper A5 changes operator in above equation from the operator $\Box_c^k$ to ultra-hyperbolic Bessel operator. We prove a theorem for this new equation about distributional solution.

2.3 Nonlinear oplus heat equation

Consider the nonlinear equation of the form

$$\frac{\partial}{\partial t}u(x,t) - c^2 \oplus^k u(x,t) = f(x,t,u(x,t)),$$

where $\oplus^k$ is the oplus operator iterated k-times, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function of the form $(x,t)=(x_1,x_2,\dots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is the given generalized function, k is a positive integer and c is a positive constant.

In the paper A4, we proved that under the suitable conditions for f, u and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n\times(0,\infty)$. Moreover, if we put $k=1$ and $q=0$ we obtain the solution of nonlinear equation related to the heat equation.

2.4 Inverse of some kernel

In the paper A6, we define the Bessel ultra-hyperbolic Marcel Riesz operator on the function f by

$$U^\alpha(f) = R_\alpha^B * f,$$

where R_α^B is Bessel ultra-hyperbolic kernel of Marcel Riesz, $\alpha\in\mathbb{C}$, the symbol $*$ designates as the convolution, and $f\in\mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions. Our purpose of this paper is to obtained the operator $E^\alpha = (U^\alpha)^{-1}$ such that if $U^\alpha(f)=\varphi$, then $E^\alpha\varphi = f$.

The above equation is inspired by the equation introduced and studied by M. A. Aguirre [M. A. Aguirre, The inverse ultrahyperbolic Marcel Riesz kernel, Le Matematiche 54(1) (1999), 55-66.]. In fact, the equation is included in above equation if we let $|\nu|=0$. See Appendix A6.

The paper A7 changes the kernel in above equation from the Bessel ultra-hyperbolic kernel of Marcel Riesz to the Bessel diamond kernel of Marcel Riesz.

2.5 Diamond Klein-Gordon operator

In the paper A8, we introduced the diamond Klein-Gordon operator iterated k times, which is defined by

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k,$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$, for all $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, $m\geq 0$ and non-negative integers k. And we studied the fundamental solution of the operator $(\diamondsuit + m^2)^k$, to which we will refer as the diamond Klein-Gordon kernel. Moreover, we also studied the convolution of this kernel.

In the paper A2, we also studied the Fourier transform of the diamond Klein-Gordon kernel which is appeared in the paper A8.

Appendix

A1 **Kamsing Nonlaopon**, The convolution equation related to the diamond kernel of Macel Riesz, Far East Journal of Mathematical Sciences, 45(2010), no. 1, 121–134.

A2 Apisit Lunnaree and **Kamsing Nonlaopon**, On the Fourier transform of the diamond Klein-Gordon kernel, International Journal of Pure and Applied Mathematics, 61(2011), no. 1, 85–97.

A3 Jirapa Papakhao and **Kamsing Nonlaopon**, On the general solution of the operator $\square_c^k$ related to the ultra-hyperbolic operator, Applied Mathematical Sciences, 5(2011), no. 53, 2629–2638.

A4 Sasiwimon Khumdeeboon and **Kamsing Nonlaopon**, On the nonlinear oplus heat equation related to the spectrum, International Journal of Nonlinear Science, 11(2011), no. 1, 100–108.

A5 Rattapan Damkengpan and **Kamsing Nonlaopon**, On the general solution of the ultrahyperbolic Bessel operator, Mathematical Problems in Engineering, Volume 2011, Article ID 579645, 10 pages, doi:10.1155/2011/579645 (**Impact Factor 0.777**)

A6 Darunee Maneetus and **Kamsing Nonlaopon**, On the inversion of Bessel ultrahyperbolic kernel of Marcel Riesz, Abstract and Applied Analysis, Volume 2011, Article ID 419157, 13 pages, doi:10.1155/2011/419157 (**Impact Factor 1.422**)

A7 Tipasiri Salao and **Kamsing Nonlaopon**, On the inverse Bessel diamond kernel of Marcel Riesz, Integral Transforms and Special Functions, iFirst, 2012, 1-12, doi:10.1080/10652469.2012.671312 (**Impact Factor 0.835**)

A8 **Kamsing Nonlaopon**, Apisit Lunnaree and Amnuay Kananthai, On the solution of the n-dimensional diamond Klein-Gordon operator and its convolution, Far East Journal of Mathematical Sciences, 63(2012), no.2, 203–220.

A1. **Kamsing Nonlaopon**, The convolution equation related to the diamond kernel of Macel Riesz, Far East Journal of Mathematical Sciences, 45(2010), no. 1, 121--134.

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Volume 45, Number 1, 2010, Pages 121-134
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THE CONVOLUTION EQUATION RELATED TO THE DIAMOND KERNEL OF MARCEL RIESZ

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Abstract

In this paper, we study the distribution $e^{\alpha x}\Diamond_c^k\delta$, where $\Diamond_c^k$ is the operator which related to the diamond type operator iterated k-times, δ is the Dirac-delta distribution, $x = (x_1, x_2, ..., x_n)$ is a variable and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a constant and both are the points in the n-dimensional Euclidean spaces $\mathbb{R}^n$.

At first, the properties of $e^{\alpha x}\Diamond_c^k\delta$ are studied and later we study the application of $e^{\alpha x}\Diamond_c^k\delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x}\Diamond_c^k\delta) * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r\Diamond_c^r\delta,$$

where $u(x)$ is the generalized function and C_r is a constant. We found that its solution is related to the diamond kernel of Marcel Riesz and

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moreover, the type of solutions such as the ordinary functions or the tempered distributions depending on k, m and α.

1. Introduction

Gelfand and Shilov [2] have first introduced the elementary solution of the n-dimensional classical diamond operator. Trione [15] has shown that the n-dimensional ultra-hyperbolic equation has $u(x) = R_{2k}(x)$ as a unique elementary solution. Later, Tellez [14] has proved that $R_{2k}(x)$ exists only for case p is odd with $p + q = n$. Kananthai [4, 7, 9] has proved the convolutions and Fourier transformation of the diamond kernel of Marcel Riesz, and has shown that the solution of the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of the $\lozenge^k u(x) = \delta$.

Kananthai [8] has shown that the solution of the convolution form $u(x) = R_{2k, c_1}(x) * R_{2k, c_2}(x)$ is a unique elementary solution of the equation $\square_{c_1}^k \square_{c_2}^k u(x) = \delta$, where $\square_{c_1}^k$ and $\square_{c_2}^k$ are the operators which related to the ultra-hyperbolic type operator iterated k-times and in particular, if $k = p = 1$ with $x_1 = t$ (times), c_1 and c_2 are velocities, then $u(x) = R_{2, c_1}(x) * R_{2, c_2}(x)$ is the elementary solution of the elastic wave equation of fourth order.

Bupasiri and Nonlaopon [1] have studied the weak solutions of compound ultra-hyperbolic equation

$$\sum_{r=0}^{m} C_r \square_c^k u(x) = f(x), \tag{1.1}$$

where $\square_c^k$ is the operator which related to the ultra-hyperbolic type operator iterated k-times and defined by

$$\square_c^k = \left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \quad p + q = n. \tag{1.2}$$

Kananthai [6] has studied the properties of the distribution $e^{\alpha x}\Box^k\delta$ for solving the elementary solution of the equation of the ultra-hyperbolic type by using the convolution method.

Kananthai [5] has studied the application of $e^{\alpha x}\Box^k\delta$ for solving the solutions of the convolution equation which related to the ultra-hyperbolic equation.

Sasopa and Nonlaopon [11] have studied the application of $e^{\alpha x}\Box_c^k\delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x}\Box_c^k\delta) * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r\Box_c^r\delta, \tag{1.3}$$

which related to the ultra-hyperbolic type operator iterated k-times.

Furthermore, Kananthai [3] has studied the properties of the distribution $e^{\alpha x}\Diamond^k\delta$ and the application of $e^{\alpha x}\Diamond^k\delta$ for solving the solutions of the convolution equation which related to the diamond kernel of Marcel Riesz.

Sritanratana and Kananthai [13] have studied the product of the nonlinear diamond operators related to the elastic wave equation and also introduced the operator $\Diamond_c^k$ which related to the diamond operator.

In this paper, we study the properties of $e^{\alpha x}\Diamond_c^k\delta$ and the application of $e^{\alpha x}\Diamond_c^k\delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x}\Diamond_c^k\delta) * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r\Diamond_c^r\delta, \tag{1.4}$$

where $\Diamond_c^k$ is the operator which related to the diamond type operator iterated k-times and is defined by

$$\Diamond_c^k = \left[\frac{1}{c^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}\right)^2\right]^k, \tag{1.5}$$

where $u(x)$ is the generalized function and C_r is a constant. We found that its solution related to the diamond kernel of Marcel Riesz and moreover, the type of solutions such as the ordinary functions or the tempered distributions depending on k, m and α.

Before going to that point, the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be the point of the Euclidean space $\mathbb{R}^n$ and the function $S_{\gamma, c}(x)$ be defined by

$$S_{\gamma, c}(x) = \frac{X^{(\gamma - n)/2}}{P_n(\gamma)}, \tag{2.1}$$

where γ is a complex number,

$$P_n(\gamma) = \frac{2^{\gamma} \pi^{n/2} \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)} \text{ and } X = c^2(x_1^2 + x_2^2 + \cdots + x_p^2) + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2. \tag{2.2}$$

By putting $q = 0$ and $c = 1$ in (2.2), (2.1) reduces to

$$S_{\gamma}(x) = 2^{-\gamma} \pi^{-n/2} \Gamma\left(\frac{n-\gamma}{2}\right) \frac{|x|^{\gamma - n}}{\Gamma\left(\frac{\gamma}{2}\right)},$$

where $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. The function $S_{\gamma}(x)$ is precisely called the *elliptic kernel of Marcel-Riesz*. Now $S_{\gamma}(x)$ is an ordinary function for $\operatorname{Re}(\gamma) \geq n$ and is a distribution of γ for $\operatorname{Re}(\gamma) < n$.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be the point of the Euclidean space $\mathbb{R}^n$, write

$$V = c^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.3}$$

where $p + q = n$ and the interior of the forward cone is defined by

$$\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } V > 0\}.$$

For any complex number γ, define

$$R_{\gamma,c}(x) = \begin{cases} \dfrac{V^{(\gamma-n)/2}}{K_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where

$$K_n(\gamma) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\gamma-n}{2}\right)\Gamma\left(\frac{1-\gamma}{2}\right)\Gamma(\gamma)}{\Gamma\left(\frac{2+\gamma-p}{2}\right)\Gamma\left(\frac{p-\gamma}{2}\right)}. \tag{2.5}$$

The function $R_\gamma(x) = R_{\gamma,1}(x)$ is introduced by Nozaki [10]. It is well known that such function is an ordinary function if $\text{Re}(\gamma) \geq n$ and is a distribution of γ if $\text{Re}(\gamma) < n$.

By putting $p = c = 1$ in (2.3) and (2.5), and using the Legendre's duplication of $\Gamma(z)$, (2.4) reduces to

$$M_\gamma(x) = \begin{cases} \dfrac{V^{(\gamma-n)/2}}{H_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases}$$

where

$$H_n(\gamma) = 2^{\gamma-1}\pi^{(n-2)/2}\Gamma\left(\frac{2+\gamma-n}{2}\right)\Gamma\left(\frac{\gamma}{2}\right) \quad \text{and} \quad V = x_1^2 - x_2^2 - \cdots - x_n^2.$$

The function $M_\gamma(x)$ is precisely called the *hyperbolic kernel of Marcel Riesz*.

The proof of the following Lemmas 2.1 and 2.2 is given in [4].

Lemma 2.1. *The functions* $S_{\gamma,c}(x)$ *and* $R_{\gamma,c}(x)$ *defined by* (2.1) *and* (2.4), *respectively*, *are homogeneous distributions of order* $\gamma - n$ *and are also a tempered distribution*.

Lemma 2.2 (The convolution of a tempered distribution)**.** *The convolution* $S_{\gamma,c}(x) * R_{\gamma,c}(x)$ *exists and is also a tempered distribution.*

Lemma 2.3. *Given the equation*

$$\Diamond_c^k u(x) = \delta, \tag{2.6}$$

where $\Diamond_c^k$ *is defined by* (1.5), k *is a nonnegative integer and* δ *is the Dirac-delta distribution*. *Then* $u(x) = (-1)^k S_{2k,c}(x) * R_{2k,c}(x)$ *is the unique elementary solution of* (2.6), *where* $S_{2k,c}(x)$ *and* $R_{2k,c}(x)$ *are defined by* (2.1) *and* (2.4), *respectively*, *with* $\gamma = 2k$. *Moreover*, $u(x)$ *is a tempered distribution*.

The proof of this lemma is given in [9].

3. The Properties of $e^{\alpha x}\Diamond_c^k\delta$

Lemma 3.1. *The following equality holds*:

$$e^{\alpha x}\Diamond_c^k\delta = L^k\delta, \tag{3.1}$$

where L is the partial differential operator of diamond type and ia is defined by

$$\begin{aligned}
L \equiv \Diamond_c &+ \sum_{r=1}^{n}\alpha_r^2\Box_c - 2\sum_{r=1}^{n}\sum_{i=1}^{r}\left(\alpha_r\frac{\partial^3}{\partial x_i^2\partial x_r} + \alpha_i\frac{\partial^3}{\partial x_i\partial x_r^2}\right) \\
&+ 2\sum_{r=1}^{n}\sum_{j=p+1}^{p+q}\left(\alpha_r\frac{\partial^3}{\partial x_j^2\partial x_r} + \alpha_i\frac{\partial^3}{\partial x_j\partial x_r^2}\right) \\
&+ 4\left(\sum_{r=1}^{n}\sum_{i=1}^{p}\alpha_r\alpha_j\frac{\partial^2}{\partial x_i\partial x_r} - \sum_{r=1}^{n}\sum_{j=p+1}^{p+q}\alpha_r\alpha_j\frac{\partial^2}{\partial x_j\partial x_r}\right) \\
&- 2\sum_{r=1}^{n}\alpha_r^2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial x_j}\right) \\
&+ \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\Delta_c - 2\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\sum_{r=1}^{n}\alpha_r\frac{\partial}{\partial x_r} \\
&+ \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\sum_{r=1}^{n}\alpha_r^2,
\end{aligned} \tag{3.2}$$

where

$$\Delta_c = \left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) \tag{3.3}$$

and $\Box_c$ *is defined by* (1.2) *with* $k = 1$. *Actually*, $\Diamond_c = \Box_c \Delta_c$ *and* $e^{\alpha x} \Diamond_c^k \delta$ *is a tempered distribution of order* $4k$.

Proof. Let $\varphi \in \mathscr{D}$ be the space of testing functions, infinitely differentiable with compact supports and $\mathscr{D}'$ be the space of distributions. Now

$$\langle e^{\alpha x} \Diamond_c \delta, \varphi(x) \rangle = \langle \delta, \Diamond_c e^{\alpha x} \varphi(x) \rangle$$

for $e^{\alpha x} \Diamond_c \delta \in \mathscr{D}'$. By computing directly, we obtain

$$\Diamond_c e^{\alpha x} \varphi(x) = e^{\alpha x} T \varphi(x), \tag{3.4}$$

where T is the partial differential operator of the form (3.2) and is defined by

$$\begin{aligned} T \equiv \Diamond_c &+ \sum_{r=1}^{n} \alpha_r^2 \Box_c + 2 \sum_{r=1}^{n} \sum_{i=1}^{r} \left(\alpha_r \frac{\partial^3}{\partial x_i^2 \partial x_r} + \alpha_i \frac{\partial^3}{\partial x_i \partial x_r^2} \right) \\ &+ 2 \sum_{r=1}^{n} \sum_{j=p+1}^{p+q} \left(\alpha_r \frac{\partial^3}{\partial x_j^2 \partial x_r} + \alpha_i \frac{\partial^3}{\partial x_j \partial x_r^2} \right) \\ &+ 4 \left(\sum_{r=1}^{n} \sum_{i=1}^{p} \alpha_r \alpha_j \frac{\partial^2}{\partial x_i \partial x_r} - \sum_{r=1}^{n} \sum_{j=p+1}^{p+q} \alpha_r \alpha_j \frac{\partial^2}{\partial x_j \partial x_r} \right) \\ &+ 2 \sum_{r=1}^{n} \alpha_r^2 \left(\sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} \right) \\ &+ \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \Delta_c + 2 \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \sum_{r=1}^{n} \alpha_r \frac{\partial}{\partial x_r} \\ &+ \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \sum_{r=1}^{n} \alpha_r^2. \end{aligned} \tag{3.5}$$

Thus

$$\langle \delta, \Diamond_c e^{\alpha x}\varphi(x)\rangle = \langle \delta, e^{\alpha x}T\varphi(x)\rangle = T\varphi(0).$$

By the properties of δ and its partial derivatives with the linear differential operator T, we obtain $T\varphi(0) = \langle L\delta, \varphi(x)\rangle$, where L is defined by (3.2). It follows that $e^{\alpha x}\Diamond_c \delta = L\delta$. Now

$$\underbrace{(e^{\alpha x}\Diamond_c\delta) * (e^{\alpha x}\Diamond_c\delta) * \cdots * (e^{\alpha x}\Diamond_c\delta)}_{k\text{-times}} = \underbrace{(L\delta) * (L\delta) * \cdots * (L\delta)}_{k\text{-times}},$$

we have $e^{\alpha x}(\delta * \Diamond_c^k \delta) = \delta * (L^k\delta)$. Thus $e^{\alpha x}(\delta * \Diamond_c^k\delta) = L^k\delta$. It follows that, for any k, we obtain (3.1). Since δ and its partial derivatives have compact support, hence by Schwartz [12], $L^k\delta$ are tempered distributions and $L^k\delta$ has order $4k$. It follows that $e^{\alpha x}\Diamond_c^k\delta$ is a tempered distribution of order $4k$ by (3.1). This completes the proofs. □

Lemma 3.2 (Boundedness property)**.** *For every* $\varphi \in \mathscr{D}$, *the space of testing functions, and* $e^{\alpha x}\Diamond_c^k\delta \in \mathscr{D}'$, *the space of distributions, then* $|\langle e^{\alpha x}\Diamond_c^k\delta, \varphi(x)\rangle| \leq M$, *where* M *is a constant.*

Proof. We have $\langle e^{\alpha x}\Diamond_c^k\delta, \varphi(x)\rangle = \langle \Diamond_c^k\delta, e^{\alpha x}\varphi(x)\rangle$ for every $\varphi(x) \in \mathscr{D}$ and $e^{\alpha x}\Diamond_c^k\delta \in \mathscr{D}'$. So

$$\langle e^{\alpha x}\Diamond_c^k\delta, e^{\alpha x}\varphi(x)\rangle = \langle \Diamond_c^{k-1}\delta, \Diamond_c e^{\alpha x}\varphi(x)\rangle = \langle \Diamond_c^{k-1}\delta, e^{\alpha x}T\varphi(x)\rangle,$$

where T is defined by (3.5). By keeping on operating $\Diamond_c$ with $k-1$ times, we obtain

$$\langle e^{\alpha x}\Diamond_c^k\delta, \varphi(x)\rangle = \langle \delta, e^{\alpha x}T^k\varphi(x)\rangle = T^k\varphi(0).$$

Since $\varphi \in \mathscr{D}$, so $\varphi(0)$ is bounded and also $T^k\varphi(0)$ is bounded. It follows that

$$|\langle e^{\alpha x}\Diamond_c^k\delta, \varphi(x)\rangle| = |T^k\varphi(0)| \leq M.$$

□

4. The Application of $e^{\alpha x}\Diamond_c^k\delta$

Given $u(x)$ is a distribution and by Lemma 3.1, we have

$$(e^{\alpha x}\Diamond_c^k\delta) * u(x) = L^k u(x),$$

where L^k is the operator defined by (3.2) and is iterated k-times with $L^0 u(x) = u(x)$.

Theorem 4.1. *Given the linear partial differential equation of the form*

$$(e^{\alpha x}\Diamond_c^k\delta) * u(x) = \delta. \tag{4.1}$$

Then $u(x) = e^{\alpha x}(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$ *is an elementary solution of* (4.1) *or the diamond kernel of Marcel Riesz of* (4.1), *where* $S_{2k,c}(x)$ *and* $R_{2k,c}(x)$ *are defined by* (2.1) *and* (2.4), *respectively*, *with* $\gamma = 2k$.

Proof. Convolving both sides of (4.1) by $e^{\alpha x}(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$, we obtain

$$e^{\alpha x}(-1)^k S_{2k,c}(x) * R_{2k,c}(x) * [(e^{\alpha x}\Diamond_c^k\delta) * u(x)] = e^{\alpha x}(-1)^k S_{2k,c}(x) * R_{2k,c}(x) * \delta,$$

or

$$e^{\alpha x}[\Diamond_c^k(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] * u(x) = e^{\alpha x}(-1)^k S_{2k,c}(x).$$

Since $\Diamond_c^k(-1)^k S_{2k,c}(x) * R_{2k,c}(x) = \delta$ by Lemma 2.3, we have

$$(e^{\alpha x}\delta) * u(x) = e^{\alpha x}(-1)^k S_{2k,c}(x) * R_{2k,c}(x).$$

It follows that $u(x) = e^{\alpha x}(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$. □

Theorem 4.2. *Given the convolution equation*

$$(e^{\alpha x}\Diamond_c^k\delta) * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r\Diamond_c^k\delta, \tag{4.2}$$

where $\Diamond_c^k$ *is the operator which related to the diamond type operator iterated k-times and defined by*

$$\Diamond_c^k = \left[\frac{1}{c^4} \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right)^2 \right]^k,$$

where $p + q = n$ *is the dimension of the space* $\mathbb{R}^n$, *the variable* $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, *the constant* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$, δ *is the Dirac-delta distribution with* $\Diamond_c^0 \delta = \delta$, $\Diamond_c^1 \delta = \Diamond_c \delta$ *and* C_r *is a constant. Then the type of solution* $u(x)$ *of* (4.2) *depends on k, m and* α *as the following cases*:

(1) *If* $m < k$ *and* $m = 0$, *then* (4.2) *has the solution*

$$u(x) = e^{\alpha x}[C_0(-1)^k S_{2k,c}(x) * R_{2k,c}(x)]$$

which is an ordinary function for $2k \geq n$ *with any* α *and is a tempered distribution for* $2k < n$ *and for some* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *with* $\alpha_i < 0$ $(i = 1, 2, ..., n)$.

(2) *If* $0 < m < k$ *and r have run from* 1, *then the solution of* (4.2) *is*

$$u(x) = e^{\alpha x}\left[\sum_{r=1}^{m} C_r(-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x) \right]$$

which is an ordinary function for $2k - 2r \geq n$ *for all r with any* α *and is a tempered distribution if* $2k - 2r < n$ *for all r and some* α *with* $\alpha_i < 0$ $(i = 1, 2, ..., n)$.

(3) *If* $m \geq k$ *and for any* α, *suppose that* $k \leq m \leq M$ *and r have run from k, then* (4.2) *has* $u(x) = e^{\alpha x} \sum_{r=k}^{M} C_r \Diamond_c^{r-k} \delta$ *as a solution which is the singular distribution*.

Proof. (1) For $m < k$ and $m = 0$, then (4.2) becomes

$$(e^{\alpha x} \Diamond_c^k \delta) * u(x) = e^{\alpha x} C_0 \delta = C_0 \delta.$$

By Theorem 4.1, we obtain

$$u(x) = e^{\alpha x}[C_0(-1)^k S_{2k,c}(x) * R_{2k,c}(x)].$$

Now, by (2.1) and (2.4), $S_{2k,c}(x)$ and $R_{2k,c}(x)$ are ordinary functions, respectively, for $2k \geq n$. It follows that $e^{\alpha x}[C_0(-1)^k S_{2k,c}(x) * R_{2k,c}(x)]$ is an ordinary function for $2k \geq n$ with any α. If $2k < n$, then $S_{2k,c}(x)$ and $R_{2k,c}(x)$ are the analytic functions except at the origin and by Lemma 2.1, $S_{2k,c}(x)$ and $R_{2k,c}(x)$ are tempered distributions and by Lemma 2.2, $(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$ exists and is a tempered distribution. Now, for some $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with $\alpha_i < 0$ $(i = 1, 2, ..., n)$, we have $e^{\alpha x}$ is a slow growth function and also its partial derivative is a slow growth. It follows that $e^{\alpha x}[C_0(-1)^k S_{2k,c}(x) * R_{2k,c}(x)]$ is also a tempered distribution.

(2) For $0 < m < k$ and r have run from 1, we have

$$(e^{\alpha x}\Diamond_c^k\delta) * u(x) = e^{\alpha x}[C_1\Diamond_c\delta + C_2\Diamond_c^2\delta + \cdots + C_m\Diamond_c^m\delta].$$

Convolving both sides by $e^{\alpha x}[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)]$, we obtain

$$u(x) = e^{\alpha x}[C_1\Diamond_c((-1)^k S_{2k,c}(x) * R_{2k,c}(x)) + C_2\Diamond_c^2((-1)^k S_{2k,c}(x) * R_{2k,c}(x))$$

$$+ \cdots + C_m\Diamond_c^m((-1)^k S_{2k,c}(x) * R_{2k,c}(x))],$$

by Theorem 4.1. By Lemma 2.3, if $\Diamond_c^k((-1)^k S_{2k,c}(x) * R_{2k,c}(x)) = \delta$, then

$$\Diamond_c^{k-r}\Diamond_c^r((-1)^k S_{2k,c}(x) * R_{2k,c}(x)) = \delta$$

for $r < k$. Convolving both sides by $(-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)$, we obtain

$$((-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)) * \Diamond_c^{k-r}\Diamond_c^r((-1)^k S_{2k,c}(x) * R_{2k,c}(x))$$

$$= ((-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)) * \delta$$

$$= (-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x),$$

or

$$\Diamond_c^{k-r}((-1)^{k-r} S_{2k-2r,\,c}(x) * R_{2k-2r,\,c}(x)) * \Diamond_c^{r}((-1)^{k} S_{2k,\,c}(x) * R_{2k,\,c}(x))$$

$$= (-1)^{k-r} S_{2k-2r,\,c}(x) * R_{2k-2r,\,c}(x).$$

Thus,

$$\Diamond_c^{r}((-1)^{k} S_{2k,c}(x) * R_{2k,c}(x)) = (-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)$$

for $r < k$. It follows that

$$u(x) = e^{\alpha x}[C_1(-1)^{k-1} S_{2k-2,\,c}(x) * R_{2k-2,\,c}(x) + C_2(-1)^{k-2} S_{2k-4,\,c}(x) * R_{2k-4,\,c}(x)$$

$$+ \cdots + C_m(-1)^{k-m} S_{2k-2m,\,c}(x) * R_{2k-2m,\,c}(x)],$$

or

$$u(x) = e^{\alpha x}\left[\sum_{r=1}^{m} C_r(-1)^{k-r} S_{2k-2r,\,c}(x) * R_{2k-2r,\,c}(x)\right].$$

Similarly, as in the case (1),

$$e^{\alpha x}\left[\sum_{r=1}^{m} C_r(-1)^{k-r} S_{2k-2r,\,c}(x) * R_{2k-2r,\,c}(x)\right]$$

is the ordinary function if $2k - 2r \geq n$ for any r, α, and is a tempered distribution if $2k - 2r < n$ for all r and some α with $\alpha_i < 0$ $(i = 1, 2, ..., n)$.

(3) For $m \geq k$ and for any α, suppose that $k \leq m \leq M$ and r have run from k, we have

$$(e^{\alpha x}\Diamond_c^{k}\delta) * u(x) = e^{\alpha x}[C_k\Diamond_c^{k}\delta + C_{k+1}\Diamond_c^{k+1}\delta + \cdots + C_M\Diamond_c^{M}\delta].$$

Convolving both sides by $e^{\alpha x}[(-1)^k S_{2k,\,c}(x) * R_{2k,\,c}(x)]$, we obtain

$$u(x) = e^{\alpha x}[C_k\Diamond_c^{k}((-1)^k S_{2k,\,c}(x) * R_{2k,\,c}(x)) + C_{k+1}\Diamond_c^{k+1}((-1)^k S_{2k,\,c}(x) * R_{2k,\,c}(x))$$

$$+ \cdots + C_M\Diamond_c^{M}((-1)^k S_{2k,\,c}(x) * R_{2k,\,c}(x))],$$

by Theorem 4.1. By Lemma 2.3, we have

$$\Diamond_c^m((-1)^k S_{2k,c}(x) * R_{2k,c}(x)) = \Diamond_c^{m-k}\Diamond_c^k((-1)^k S_{2k,c}(x) * R_{2k,c}(x)) = \Diamond_c^{m-k}\delta$$

for $k \leq m \leq M$. Thus, we obtain

$$u(x) = e^{\alpha x}[C_k\delta + C_{k+1}\Diamond_c\delta + C_{k+2}\Diamond_c^2\delta + \cdots + C_M\Diamond_c^{M-k}\delta],$$

or

$$u(x) = e^{\alpha x}\sum_{r=k}^{M} C_r\Diamond_c^{r-k}\delta.$$

Now, by (3.1) and (3.2), we have

$$e^{\alpha x}\Diamond_c^{r-k}\delta = L^{r-k} + \text{(the term of lower order of partial derivative of } \delta)$$

for $k \leq r \leq M$ and since all terms of the right-hand side of above equation are singular distributions, it follows that

$$u(x) = e^{\alpha x}\sum_{r=k}^{M} C_r\Diamond_c^{r-k}\delta$$

is the singular distribution. This completes the proof.

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ON THE FOURIER TRANSFORM OF THE DIAMOND KLEIN-GORDON KERNEL

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Abstract: In this article, the operator $(\diamondsuit + m^2)^k$ is introduced and named as the diamond Klein-Gordon operator iterated k-times and is defined by

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right]^k ,$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$, for $x=(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, m is a nonnegative real number and k is a nonnegative integer. In this work, we study the fundamental solution of operator $(\diamondsuit + m^2)^k$ and this fundamental solution is called the diamond Klein-Gordon kernel. Then, we study the Fourier transform of the diamond Klein-Gordon kernel and also the Fourier transform of their convolution.

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1. Introduction

The operator $\diamondsuit^k$ has been first introduced by A. Kananthai [7], is named as the diamond operator iterated k-times, and is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \quad p+q=n, \tag{1.1}$$

where n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the form $\diamondsuit^k = \Box^k \triangle^k = \triangle^k \Box^k$, where the operator $\triangle^k$ is Laplacian iterated k-times, and is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k, \tag{1.2}$$

and the operator $\Box^k$ is the ultra-hyperbolic operator iterated k-times, and is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{1.3}$$

In 1997, A. Kananthai [7] has shown that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x) \right) = \delta, \tag{1.4}$$

where the function $R^H_{2k}(x)$ is defined by (2.1) and $R^e_{2k}(x)$ is defined by (2.6), with $\alpha = 2k$. The fundamental solution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is called the diamond kernel of Marcel Riesz. Moreover, A. Kananthai [2] has proved the convolution equation related to the diamond kernel of Marcel Riesz.

Next, A. Kananthai [3] has proved the convolutions of the diamond kernel of Marcel Riesz and he has also studied the linear equation (see [4])

$$\diamondsuit^k u(x) = f(x). \tag{1.5}$$

This equation is the generalization of the ultra-hyperbolic equation and it can be applied to the wave equation. The solution of an equation (1.5) is $u(x) = (-1)^k M_{2k,2k}(x) * f(x)$, where

$$M_{2k,2k}(x) = R^e_{2k}(x) * R^H_{2k}(x). \tag{1.6}$$

Later, A. Kananthai [6] has proved the nonlinear diamond operator are related to the n-dimensional wave equation. M.A. Tellez and A. Kananthai [12] have proved the convolution product of the distributional families related to the diamond operator. Moreover, A. Kananthai [5] has studied Fourier transform and convolutions of the diamond kernel of Marcel Riesz and also the Fourier transform of their convolution.

In 2004, H. Yıldırım et al [17] have first introduced the operator $\diamondsuit_B^k$ that is name as diamond Bessel operator iterated k-times, and is defined by

$$\diamondsuit_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k, \tag{1.7}$$

$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$. In addition, they have studied the fundamental solution of the equation $\diamondsuit_B^k u(x) = \delta$, and this solution is called the Bessel diamond kernel of Riesz. Moreover, they have studied Fourier-Bessel transform and convolutions of the Bessel diamond kernel of Riesz and also the Fourier-Bessel transform of their convolution. Later, M.Z. Sarikaya and H. Yıldırım [10, 16] have studied the Bessel diamond and the nonlinear Bessel diamond operator related to the Bessel wave equation and B-convolution of the Bessel diamond kernel of Riesz.

Furthermore, S. E. Trione [14] has studied the fundamental solution of the ultra-hyperbolic Klein-Gordon operator iterated k-times, and is defined by

$$(\Box + m^2)^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2 \right)^k. \tag{1.8}$$

The fundamental solution of the operator $(\Box + m^2)^k$ is $W_{2k}(x,m)$, and is defined by

$$W_{2k}(x,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\Gamma(k)} (m^2)^r R_{2k+2r}^H(x), \tag{1.9}$$

where $R_{2k+2r}^H(x)$ is defined by (2.1) with $\alpha = 2k + 2r$. Next, Tellez [11] has studied the convolution product of $W_\alpha(x,m) * W_\beta(x,m)$ where α and β are any complex parameters. In addition, S.E. Trione [15] has studied the fundamental $(P \pm i0)^\lambda$-ultrahyperbolic solution of the Klein-Gordon operator iterated k-times and studied the convolution of such fundamental solution.

Later, K. Nonlaopon et al [8] have introduced the operator $(\diamondsuit + m^2)^k$ that is named as the diamond Klein-Gordon operator iterated k-times, and is defined by

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right]^k , \tag{1.10}$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, m is a nonnegative real number and k is a nonnegative integer. In this work, we study the fundamental solution of operator $(\diamondsuit+m^2)^k$ and this fundamental solution is called the diamond Klein-Gordon kernel. Then, we study the Fourier transform of the diamond Klein-Gordon kernel and also the Fourier transform of their convolution.

2. Preliminaries

Definition 2.1. Let $x=(x_1,x_2,\ldots,x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$, denoted by

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 + \cdots + x_{p+q}^2,$$

the nondegenerated quadratic form, $p+q=n$ is the dimension of the space $\mathbb{R}^n$. Let $\Gamma_+ = \{x\in\mathbb{R}^n : x_1>0 \text{ and } u>0\}$ be the interior of forward cone and let $\overline{\Gamma}_+$ denote its closure. For any complex number α, define the function

$$R_\alpha^H(x) = \begin{cases} \frac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x\in\Gamma_+, \\ 0, & \text{for } x\notin\Gamma_+, \end{cases} \tag{2.1}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}. \tag{2.2}$$

The function $R_\alpha^H(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [9]. It is well known that $R_\alpha^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha)\geq n$ and is a distribution of α if $\mathrm{Re}(\alpha)<n$. Let supp $R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$ and suppose that supp $R_\alpha^H(x)\subset\overline{\Gamma}_+$, that is, supp $R_\alpha^H(x)$ is compact.

By putting $p=1$ in $R_{2k}^H(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.3}$$

we obtain

$$I_\alpha^H(x) = \frac{v^{(\alpha-n)/2}}{H_n(\alpha)}, \tag{2.4}$$

and $v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha+2-n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{2.5}$$

$I_\alpha^H(x)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$, then the function $R_\alpha^e(x)$ denote the elliptic kernel of Marcel Riesz, and is defined by

$$R_\alpha^e(x) = \frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{2.6}$$

where

$$W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \tag{2.7}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

By (2.1) and (2.2) with $q = 0$, then $u^{(\alpha-n)/2}$ reduces to $\omega_p^{(\alpha-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$ and $K_n(\alpha)$ reduces to

$$K_p(\alpha) = \frac{\pi^{(p-1)/2}\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((p-\alpha)/2)}.$$

By using the Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.8}$$

and

$$\Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \pi \sec(\pi z). \tag{2.9}$$

We obtain

$$K_p(\alpha) = \frac{1}{2}\sec\left(\frac{\pi\alpha}{2}\right) W_p(\alpha), \tag{2.10}$$

where $W_p(\alpha)$ is defined by (2.7) with $n = p$. Thus, for $q = 0$,

$$R_\alpha^H(x) = \frac{u^{(\alpha-p)/2}}{K_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right)\frac{u^{(\alpha-p)/2}}{W_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) R_\alpha^e(x), \tag{2.11}$$

Thus, in case of $\alpha = 2k$,

$$R^H_{2k}(x) = 2(-1)^k R^e_{2k}(x), \tag{2.12}$$

for $q = 0$ and $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$.

The proof of the following lemma is given in [7] and [3].

Lemma 2.1. *$R^e_\alpha(x)$ and $R^H_\alpha(x)$ are the tempered distributions.*

From S.E. Trione [13], $R^H_{2k}(x)$ is the fundamental solution of the operator $\Box^k$, that is

$$\Box^k \left(R^H_{2k}(x)\right) = \delta. \tag{2.13}$$

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is the fundamental solution of the operator $\triangle^k$ (see [1]). That is,

$$\triangle^k((-1)^k R^e_{2k}(x)) = \delta. \tag{2.14}$$

It can be shown that $R^H_{-2k}(x) = \Box^k\delta$ and $R^e_{-2k}(x) = (-1)^k\triangle^k\delta$ for k is a nonnegative integer (see [13, 12]).

Let K be a compact set and $K \subset \overline{\Gamma}_+$ where $\overline{\Gamma}_+$ is defined as in the beginning. Choose the support of $R^H_{2k}(x)$ such that it is equal to K, then supp $R^H_{2k}(x)$ is compact (closed and bounded). So the convolution

$$(-1)^k R^e_{2k}(x) * R^H_{2k}(x) \tag{2.15}$$

exists and is a tempered distribution from lemma 2.1.

Lemma 2.2. *The convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the diamond operator iterated k-times, that is*

$$\diamondsuit^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta. \tag{2.16}$$

The proof of this Lemma is given in [7] and [12].

It can be shown that $R^e_{-2k}(x) * R^H_{-2k}(x) = (-1)^k \diamondsuit^k \delta(x)$, for k is a nonnegative integer.

Definition 2.3. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$, the function $T_\alpha(x, m)$ is defined by

$$T_\alpha(x, m) = \sum_{r=0}^{\infty} \binom{-\alpha/2}{r} (m^2)^r (-1)^{\alpha/2+r} R^e_{\alpha+2r}(x) * R^H_{\alpha+2r}(x), \tag{2.17}$$

where α is a complex parameter, m is a nonnegative real number, $R^H_{\alpha+2r}(x)$ and $R^e_{\alpha+2r}(x)$ are defined by (2.1) and (2.6), respectively.

From the definition of $T_\alpha(x,m)$ and by putting $\alpha = -2k$, we have

$$T_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r (-1)^{-k+r} R^e_{2(-k+r)}(x) * R^H_{2(-k+r)}(x).$$

Since the operator $(\diamondsuit + m^2)^k$ defined by (1.10) is linearly continuous and has 1-1 mapping of this possess its own inverses. From Lemma 2.2, we obtain

$$T_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r \diamondsuit^{k-r}\delta = (\diamondsuit + m^2)^k \delta. \tag{2.18}$$

By putting $k = 0$ in (2.18), we have $T_0(x,m) = \delta$. By putting $\alpha = 2k$ into (2.22), we have

$$\begin{aligned} T_{2k}(x,m) = &\binom{-k}{0} (m^2)^0 (-1)^{k+0} R^e_{2k+0}(x) * R^H_{2k+0}(x) \\ &+ \sum_{r=1}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x). \end{aligned} \tag{2.19}$$

The second summand of the right-hand member of (2.19) vanishes for $m = 0$ and then, we have

$$T_{2k}(x, m = 0) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x),$$

which is the fundamental solution of the diamond operator.

Definition 2.4. Let $f(x) \in L^1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \mathcal{F}f(x)) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx, \tag{2.20}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1\,dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \widehat{f}(\xi)\,d\xi. \tag{2.21}$$

If $f(x)$ is distribution with compact supports by [18], the equation (2.20) can be written as

$$\widehat{f}(\xi) = \mathcal{F}f(x) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i\xi\cdot x} \rangle. \tag{2.22}$$

The proof of the following Lemmas 2.3 and 2.4 are given in [8].

Lemma 2.3. *Given the equation*

$$(\diamondsuit + m^2)^k u(x) = \delta, \tag{2.23}$$

where $(\diamondsuit + m^2)^k$ is the diamond Klein-Gordon operator, and is defined by

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k, \tag{2.24}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer, m is a nonnegative real number and δ is the Dirac-delta distribution. Then we obtain

$$T_{2k}(x,m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x) \tag{2.25}$$

is the fundamental solution of the operator $(\diamondsuit + m^2)^k$, defined by (3.1), where $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by (2.1) and (2.6), respectively. Moreover, $u(x) = T_{2k}(x,m)$ is tempered distribution.

Lemma 2.4. *Let $T_{2k}(x,m)$ be the diamond Klein-Gordon kernel is defined by (3.2), then $T_{2k}(x,m)$ is a tempered distribution and can be expressed by*

$$T_{2k}(x,m) = T_{2k-2v}(x,m) * T_{2v}(x,m)$$

where v is nonnegative integer and $v < k$. Moreover, if we put $l = k - v$ and $h = v$, then we obtain

$$T_{2l}(x,m) * T_{2h}(x,m) = T_{2l+2h}(x,m) \qquad \text{for } l + h = k.$$

Lemma 2.5. (The Fourier Transform of $(\diamondsuit^k + m^2)^k\delta$) *Let $\|\xi\| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{1/2}$ for $\xi \in \mathbb{R}^n$. Then*

$$\left|\mathcal{F}(\diamondsuit^k + m^2)^k \delta\right| \leq \frac{1}{(2\pi)^{n/2}} (\|\xi\|^2 + m^2)^k. \tag{2.26}$$

That is, $\mathcal{F}(\diamondsuit^k + m^2)^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by the inverse Fourier transformation

$$(\diamondsuit^k + m^2)^k \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \cdots + \xi_{p+q}^2\right)^2 + m^2\right]^k.$$

Proof. From the Fourier transform (2.22), we have

$$\mathcal{F}(\diamondsuit^k + m^2)^k\delta = \frac{1}{(2\pi)^{n/2}}\left\langle(\diamondsuit^k + m^2)^k\delta, e^{-i\xi\cdot x}\right\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta, (\diamondsuit^k + m^2)^k e^{-i\xi\cdot x}\right\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta, \left[\left(\xi_1^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k e^{-i\xi\cdot x}\right\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k.$$

Now

$$\left|\mathcal{F}(\diamondsuit^k + m^2)^k\delta\right|$$
$$=\frac{1}{(2\pi)^{n/2}}\left[\left|\xi_1^2+\cdots+\xi_n^2\right|\left|\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2\cdots-\xi_{p+q}^2\right|+m^2\right]^k$$
$$\leq\frac{1}{(2\pi)^{n/2}}\left[\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|^2+m^2\right]^k$$
$$=\frac{1}{(2\pi)^{n/2}}(\|\xi\|^2+m^2)^k,$$

where $\|\xi\| = \left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^{1/2}$, $\xi_i (i = 1, 2, \ldots, n) \in \mathbb{R}$. Hence we obtain (2.26) and $\mathcal{F}(\diamondsuit^k + m^2)^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is 1-1 transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.21) we have

$$(\diamondsuit^k + m^2)^k\delta$$
$$=\frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k.$$

That completes the proof. □

3. Main Results

Theorem 3.1.

$$\mathcal{F}T_{2k}(x,m) = \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k}$$

and

$$|\mathcal{F}T_{2k}(x,m)| \leq \frac{1}{(2\pi)^{n/2}}M \quad \textit{for a large } \xi_i \in \mathbb{R}, \tag{3.1}$$

where M is a constant. That is, $\mathcal{F}$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distributions.

Proof. By Lemma 2.3,

$$(\diamondsuit^k + m^2)^k T_{2k}(x,m) = \delta,$$

or

$$(\diamondsuit^k + m^2)^k \delta * T_{2k}(x,m) = \delta. \tag{3.2}$$

If we applied the Fourier transform on both sides of (3.2), then we obtain

$$\mathcal{F}\left((\diamondsuit^k + m^2)^k \delta * T_{2k}(x,m)\right) = \mathcal{F}\delta = \frac{1}{(2\pi)^{n/2}}.$$

By (2.22), we have

$$\frac{1}{(2\pi)^{n/2}}\left\langle (\diamondsuit^k + m^2)^k \delta * T_{2k}(x,m), e^{-i\xi\cdot x}\right\rangle = \frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\frac{1}{(2\pi)^{n/2}}\left\langle (\diamondsuit^k + m^2)^k \delta, \left\langle T_{2k}(y,m), e^{-i\xi\cdot(x+y)}\right\rangle\right\rangle = \frac{1}{(2\pi)^{n/2}},$$

$$\frac{1}{(2\pi)^{n/2}}\left\langle T_{2k}(y,m), e^{-i\xi\cdot y}\right\rangle \left\langle (\diamondsuit^k + m^2)^k \delta, e^{-i\xi\cdot x}\right\rangle = \frac{1}{(2\pi)^{n/2}},$$

$$\mathcal{F}T_{2k}(x,m)(2\pi)^{n/2}\mathcal{F}\left((\diamondsuit^k + m^2)^k \delta\right) = \frac{1}{(2\pi)^{n/2}}.$$

By Lemma 2.5, we obtain

$$\mathcal{F}T_{2k}(x,m)\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k = \frac{1}{(2\pi)^{n/2}}.$$

It follows that

$$\mathcal{F}T_{2k}(x,m) = \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^k}.$$

Now,

$$\mathcal{F}T_{2k}(x,m) = \frac{1}{\left(\left|\xi_1^2+\cdots+\xi_n^2\right|\left|\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2\right|+m^2\right)^k},$$

where $\xi \in (\xi_1, \xi_2, \ldots, \xi_n) \in \Gamma_+$ with Γ_+ defined by Definition 2.1. Then $(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots \xi_{p+q}^2) > 0$ and for a large ξ_i and a large k, the right-hand side of (3.2) tends to zero. It follows that it is bounded by positive constant say M, that is, we obtain (3.1) as required and also by (3.1), $\mathcal{F}$ is continuous on the space $\mathcal{S}'$ of the tempered distribution. □

Theorem 3.2.

$$\mathcal{F}\left(T_{2k}(x,m) * T_{2l}(x,m)\right) = 2\pi^{n/2}\mathcal{F}[T_{2k}(x,m)]\mathcal{F}[T_{2l}(x,m)]$$
$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^{k+l}},$$

where k and l are nonnegative integers and $\mathcal{F}$ is bounded and continuous on the spaces $\mathcal{S}'$ of tempered distribution.

Proof. From Lemma 2.4, we have

$$T_{2k}(x,m) * T_{2l}(x,m) = T_{2k+2l}(x,m), \tag{3.3}$$

where k and l are nonnegative integers. Taking Fourier transform on both sides of (3.3) and using Theorem 3.1, we obtain

$$\mathcal{F}\left(T_{2k}(x,m) * T_{2l}(x,m)\right) = \mathcal{F}\left(T_{2k+2l}(x,m)\right)$$
$$= \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^{k+l}},$$
$$= \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^{k}}$$
$$\times \frac{(2\pi)^{n/2}}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^2 + m^2\right]^{l}},$$
$$= 2\pi^{n/2}\mathcal{F}\left[T_{2k}(x,m)\right]\mathcal{F}\left[T_{2l}(x,m)\right].$$

Since $T_{2k+2l}(x,m) \in \mathcal{S}'$, the space of tempered distribution, and by Theorem 3.1 we obtain that $\mathcal{F}$ is bounded and continuous on $\mathcal{S}'$. □

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On the General Solution of the Operator $\Box_c^k$ Related to the Ultra-Hyperbolic Operator

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Abstract

In this paper, we study the general solution of equation $\Box_c^k u(x) = f(x)$, where $\Box_c^k$ is the operator which related to the ultra-hyperbolic type operator iterated k-times and is defined by

$$\Box_c^k = \left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k ,$$

$p+q=n, n$ is the dimension of $\mathbb{R}^n$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Mathematics Subject Classification: 35D05, 35J05, 46F10.

Keywords: Tempered distribution, Ultra-hyperbolic type operator, Hyperbolic kernel of Marcel Riesz, Dirac-delta distribution

1 Introduction

I. M. Gel'fand and G. E. Shilov [4] have introduced the fundamental solution of the n-dimensional ultra-hyperbolic operator. Next, S. E. Trione [15] has shown that the n-dimensional ultra-hyperbolic equation has $u(x) = R_{2k,1}(x)$ as a unique fundamental solution. Later, M. A. Tellez [14] has proved that $R_{2k,1}(x)$ exists only for the case p is odd with $p+q=n$. A wealth of some effective works on studying some properties of the fundamental solution of the

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n-dimensional ultra-hyperbolic operator have been presented by Kananthai [6, 7, 8, 9].

In 1999, A. Kananthai [8] has showed that the solution of the convolution form $u(x) = R_{2k,c_1}(x) * R_{2k,c_2}(x)$ is an unique fundamental solution of the equation $\Box_{c_1}^k \Box_{c_2}^k u(x) = \delta$, where $\Box_{c_1}^k$ and $\Box_{c_2}^k$ are the operators which related to the ultra-hyperbolic type operator iterated k-times and in particular if $k = p = 1$ with $x_1 = t$(times), c_1 and c_2 are velocity then $u(x) = R_{2,c_1}(x) * R_{2,c_2}(x)$ is the fundamental solution of the elastic wave equation of fourth order.

Next, G. Sritanratana and A. Kananthai [13] have studied the product of the nonlinear diamond operators related to the elastic wave equation and have also introduced the ultra-hyperbolic operator $\Box_c^k$.

Moreover, S. Bupasiri and K. Nonlaopon [1] have studied the weak solutions of compound ultra-hyperbolic equation

$$\sum_{r=0}^{m} C_r \Box_c^k u(x) = f(x), \tag{1}$$

which related to the ultra-hyperbolic type operator iterated k-times.

Later, P. Sasopa and K. Nonlaopon [11] have studied the properties of the distribution $e^{\alpha x}\Box_c^k \delta$ and the application of $e^{\alpha x}\Box_c^k \delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x}\Box_c^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^{m} C_r \Box_c^r \delta, \tag{2}$$

where $u(x)$ is an unknown generalized function, δ is the Dirac-delta distribution, $\alpha = (\alpha_1, \dots, \alpha_n)$ and C_r are the constants.

In this paper, we study the general solution of equation $\Box_c^k u(x) = f(x)$, where $\Box_c^k$ is the operator which related to the ultra-hyperbolic type operator iterated k-time and is defined by

$$\Box_c^k = \left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \tag{3}$$

$p + q = n, n$ is the dimension of $\mathbb{R}^n$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Before going to that point, the following definitions and some concepts are needed.

2 Preliminary Notes

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be the point of the Euclidean space $\mathbb{R}^n$. Denote the nondegenerated quadratic form by

$$v = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{4}$$

where $p + q = n$. The interior of the forward cone is defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$. For any complex number α, we define

$$R_{\alpha,c}(x) = \begin{cases} \frac{v^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{5}$$

where the constant $K_n(\alpha)$ is given by formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{\alpha-p+2}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{6}$$

The function $R_\alpha(x) = R_{\alpha,1}(x)$ is introduced by Y. Nozaki [10]. It is well known that such function is an ordinary function if $Re(\alpha) \geq n$ and is the distribution of α if $Re(\alpha) < n$.

By putting $p = c = 1$ in (4), (5) and (6), and using the Legendre's duplication of $\Gamma(z)$;

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right),$$

the formula (5) is reduced to

$$M_\alpha(x) = \begin{cases} \frac{v^{(\alpha-n)/2}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{7}$$

where

$$H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1}\Gamma\left(\frac{\alpha-n+2}{2}\right)\Gamma\left(\frac{\alpha}{2}\right) \quad \text{and} \quad v = x_1^2 - x_2^2 - \cdots - x_n^2.$$

Note that the function $M_\alpha(x)$ is precisely called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$, then the function $S_\alpha(x)$ denote the elliptic kernel of Marcel Riesz and is defined by

$$S_\alpha(x) = \frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{8}$$

where α is complex parameter, $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$,

$$W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})} \tag{9}$$

and n is the dimension of $\mathbb{R}^n$.

It can be shown that $S_{-2k}(x) = (-1)^k \triangle^k \delta(x)$, where $\triangle^k$ is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k .$$

It follows that $S_0(x) = \delta(x)$, see [3]. Moreover, we obtain $(-1)^k S_{2k}(x)$ is the fundamental solution of the operator $\triangle^k$, see [4]. That is,

$$\triangle^k((-1)^k S_{2k}(x)) = \delta. \tag{10}$$

By (5) and (6) with $q = 0$ and $c = 1$, then $v^{(\alpha-p)/2}$ is reduced to $\omega_p^{(\alpha-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2 \in \mathbb{R}^p, p$ is even and $K_n(\alpha)$ is reduced to

$$K_p(\alpha) = \frac{\pi^{(p-1)/2} \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{p-\alpha}{2}\right)}.$$

By using the formula

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma\left(z + \frac{1}{2}\right),$$

and

$$\Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) = \pi \sec(\pi z),$$

we obtain

$$K_p(\alpha) = \frac{1}{2} \sec\left(\frac{\pi\alpha}{2}\right) W_p(\alpha),$$

where $W_p(\alpha)$ is defined by (9) with $n = p$. Thus, for $q = 0$,

$$R_{\alpha,1}(x) = R_\alpha(x) = \frac{v^{(\alpha-p)/2}}{K_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) \frac{\omega^{(\alpha-p)/2}}{W_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) S_\alpha(x),$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$. Thus, if $\alpha = 2k$, then

$$R_{2k,1}(x) = R_{2k}(x) = 2(-1)^k S_{2k}(x) \tag{11}$$

for $q = 0, c = 1$ and $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even.

The proof of the following Lemma is given in [1].

Lemma 2.3. *Given the equation*

$$\Box_c^k u(x) = \delta, \tag{12}$$

where $\Box_c^k$ *is defined by (3),* k *is a nonnegative integer and* δ *is the Dirac-delta distribution. Then* $u(x) = R_{2k,c}(x)$ *is the unique fundamental solution of (12), where* $R_{2k,c}(x)$ *is defined by (5) with* $\alpha = 2k$.

Lemma 2.4. *Given* P *is a hyper-surface then*

$$P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0,$$

where $\delta^{(k)}$ *is the Dirac-delta distribution with* k *derivatives.*

The proof of this Lemma is given in [4].

Lemma 2.5. *Given the equation*

$$\Box_c^k u(x) = 0, \tag{13}$$

where $\Box_c^k$ *is defined by (3) and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*, then*

$$u(x) = [R_{2(k-1),c}(x)]^{(m)},$$

defined by (5) with m*-derivatives, as a solution of (13) with* $m = \frac{n-4}{2}$, $n \geq 4$ *and* n *is an even dimension.*

Proof. We first show that the generalized function $\delta^{(m)}(c^2r^2 - s^2)$, where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, $p + q = n$, is a solution of the equation

$$\Box_c u(x) = 0, \tag{14}$$

and $\Box_c$ is defined by (3) with $k = 1$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Now for $1 \leq i \leq p$, we have

$$\frac{\partial}{\partial x_i}\delta^{(m)}(c^2r^2 - s^2) = 2c^2x_i\delta^{(m+1)}(c^2r^2 - s^2),$$
$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(c^2r^2 - s^2) = 2c^2\delta^{(m+1)}(c^2r^2 - s^2) + 4c^4x_i^2\delta^{(m+2)}(c^2r^2 - s^2).$$

Thus, we have

$$\begin{aligned}
\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(c^2r^2 - s^2) &= 2p\delta^{(m+1)}(c^2r^2 - s^2) + 4c^2r^2\delta^{(m+2)}(c^2r^2 - s^2) \\
&= 2p\delta^{(m+1)}(c^2r^2 - s^2) + 4(c^2r^2 - s^2)\delta^{(m+2)}(c^2r^2 - s^2) \\
&\quad + 4s^2\delta^{(m+2)}(c^2r^2 - s^2) \\
&= 2p\delta^{(m+1)}(c^2r^2 - s^2) - 4(m+2)\delta^{(m+1)}(c^2r^2 - s^2) \\
&\quad + 4s^2\delta^{(m+2)}(c^2r^2 - s^2) \\
&= [2p - 4(m+2)]\delta^{(m+1)}(c^2r^2 - s^2) + 4s^2\delta^{(m+2)}(c^2r^2 - s^2)
\end{aligned}$$

by applying Lemma 2.4 with $P = r^2 - s^2$.

Similarly, we have

$$\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\delta^{(m)}(c^2r^2-s^2) = [-2q+4(m+2)]\delta^{(m+1)}(c^2r^2-s^2)+4c^2r^2\delta^{(m+2)}(c^2r^2-s^2)$$

by applying Lemma 2.4 with $P = r^2 - s^2$.

Thus, we have

$$\begin{aligned}\Box_c\delta^{(m)}(c^2r^2 - s^2) &= \frac{1}{c^2}\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\delta^{(m)}(c^2r^2 - s^2) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\delta^{(m)}(c^2r^2 - s^2)\\ &= [2(p + q) - 8(m + 2)]\delta^{(m+1)}(c^2r^2 - s^2)\\ &\quad - 4(c^2r^2 - s^2)\delta^{(m+2)}(c^2r^2 - s^2)\\ &= [2n - 8(m + 2)]\delta^{(m+1)}(c^2r^2 - s^2) + 4(m + 2)\delta^{(m+1)}(c^2r^2 - s^2)\\ &= [2n - 4(m + 2)]\delta^{(m+1)}(c^2r^2 - s^2)\end{aligned}$$

by applying Lemma 2.4 with $P = r^2 - s^2$.

If $2n - 4(m + 2) = 0$, we have

$$\Box_c\delta^{(m)}(c^2r^2 - s^2) = 0.$$

That is, $u(x) = \delta^{(m)}(c^2r^2 - s^2)$ is a solution of (14) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is an even dimension. Now $\Box_c^k u(x) = 0$ can be written in the form

$$\Box(\Box^{k-1}u(x)) = 0.$$

From (13), we have

$$\Box^{k-1}u(x) = \delta^{(m)}(c^2r^2 - s^2)$$

with $m = \frac{n-4}{2}$, $n \geq 4$ and n is an even dimension.

Convolving both sides of the above equation with the function $R_{2(k-1),c}(x)$, we obtain

$$\begin{aligned}R_{2(k-1),c}(x) * \Box_c^{k-1}u(x) &= R_{2(k-1),c}(x) * \delta^{(m)}(c^2r^2 - s^2)\\ \Box_c^{k-1}[R_{2(k-1),c}(x)] * u(x) &= [R_{2(k-1),c}(x)]^{(m)}\\ \delta * u(x) &= [R_{2(k-1),c}(x)]^{(m)}\\ u(x) &= [R_{2(k-1),c}(x)]^{(m)}\end{aligned}$$

by Lemma 2.3.

It follows that $u(x) = [R_{2(k-1),c}(x)]^{(m)}$ is a solution of (13) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is an even dimension. This completes the proof. $\Box$

3 Main Results

Theorem 3.1. *Given the equation*

$$\square_c^k u(x) = f(x), \tag{15}$$

where $\square_c^k$ *is the operator which related to the ultra-hyperbolic type iterated* k*-times, and is defined by (3),* $f(x)$ *is a generalized function,* $u(x)$ *is an unknown generalized function,* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *the* n*-dimensional Euclidean space and* n *is even, then (15) has the general solution*

$$u(x) = [R_{2(k-1),c}(x)]^{(m)} + R_{2k,c}(x) * f(x), \tag{16}$$

where $[R_{2(k-1),c}(x)]^{(m)}$ *is a function defined by (5) with* m*-derivatives.*

Proof. Convolving both sides of equation (15) with $R_{2k,c}(x)$, we obtain

$$R_{2k,c}(x) * \square_c^k u(x) = R_{2k,c}(x) * f(x).$$

By Lemma 2.3, we have

$$\square_c^k (R_{2k,c}(x)) * u(x) = \delta * u(x) = R_{2k,c}(x) * f(x).$$

So, we obtain that

$$u(x) = R_{2k,c}(x) * f(x) \tag{17}$$

is the solution of (15).

For a homogeneous equation $\square_c^k u(x) = 0$, we have a solution

$$u(x) = [R_{2(k-1),c}(x)]^{(m)}$$

by Lemma 2.5. Thus the general solution of (15) is

$$u(x) = [R_{2(k-1),c}(x)]^{(m)} + R_{2k,c}(x) * f(x).$$

This completes the proof. □

In particular, if $q = 0$ and $c = 1$, then equation (15) becomes the Laplace equation

$$\triangle^k u(x) = f(x), \tag{18}$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even. From (11) and (17), we have

$$u(x) = 2(-1)^k S_{2k}(x) * f(x)$$

is the solution of (18), where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and p is even.

From (16), we obtain that the general solution of the Laplace equation is

$$u(x) = (-1)^{k-1}[S_{2(k-1)}(x)]^{(m)} + 2(-1)^k S_{2k}(x) * f(x), \tag{19}$$

for $x = (x_1, x_2, \ldots , x_p) \in \mathbb{R}^p$ and p is even.

Now consider the case for the wave equation. By putting $c = 1$, equation (15) becomes the ultra-hyperbolic equation

$$\Box^k V(x) = f(x), \tag{20}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated k-times, and is defined by (3) with $c = 1$, $V(x)$ is an unknown generalized function and $f(x)$ is a generalized function. From (17), we obtain that

$$V(x) = R_{2k}(x) * f(x)$$

is a solution of equation (20), where $R_{2k}(x) = R_{2k,1}(x)$ defined by (5).

And from (16), we obtain that the general solution of the ultra-hyperbolic equation is

$$V(x) = [R_{2(k-1)}(x)]^{(m)} + R_{2k}(x) * f(x). \tag{21}$$

Moreover, if we put $k = p = 1$ and $x_1 = t$(times), then equation (20) is reduced to the wave equation

$$\Box V(x) = \left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2} \right) V(x) = f(x), \tag{22}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the wave operator.

Thus, we obtain $V(x) = M_2(x) * f(x)$ as a solution of the wave equation, since $R_2(x)$ becomes $M_2(x)$, where $M_2(x)$ is the ultra-hyperbolic kernel of Marcel Riezs, and is defined by (7). And from (16), we obtain that the general solution of wave equation is

$$V(x) = \delta^{(m)}(x) + M_2(x) * f(x),$$

where $\delta^{(m)}(x)$ is a solution of equation

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2} \right) V(x) = 0. \tag{23}$$

Now we put $v = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ and $s^2 = x_2^2 + x_3^2 + \cdots + x_n^2$. By [2], we obtain that

$$V(x,t) = \delta^{(m)}(t^2 - s^2)$$

is the solution of (23) with the initial conditions $V(x,0) = 0$ and $\frac{\partial V(x,0)}{\partial t} = (-1)^m 2\pi^{m+1}\delta(x)$ at $t = 0$ and $x = (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$.

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On the Nonlinear Oplus Heat Equation Related to the Spectrum

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Abstract: In this article, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t}u(x,t) - c^2 \oplus^k u(x,t) = f(x,t,u(x,t)),$$

where $\oplus^k$ is the oplus operator iterated k-times, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function of the form $(x,t) = (x_1,x_2,\dots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function, k is a positive integer and c is a positive constant.

On the suitable conditions for f, u and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$. Moreover, if we put $k=1$ and $q=0$ we obtain the solution of nonlinear equation related to the heat equation.

Keywords: heat kernel; Dirac-delta distribution; Fourier transform; spectrum

1 Introduction

The problem of existence of solutions of various differential equations has received a great deal of attention during the last few years. For example, H. Ni and F. X. Lin [15, 16] studied Duffing and Riccati differential equations and obtained the existence of almost periodic solutions; Z. B. Fan et al. [1, 2] obtained the existence of mild solutions to nonlocal neutral functional differential, integrodifferential equations and nonlocal Cauchy problem; K. Nonlaopon and A. Kananthai [20] have studied the solution of nonlinear equation $\triangle^k u(x) = f(x, \triangle^{k-1}u(x))$ and found that the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\triangle^{k-1}u(x)$.

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t) \tag{1}$$

with the initial condition $u(x,0) = f(x)$, where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

denotes the Laplace operator and $(x,t) = (x_1,x_2,\dots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, we can obtain the solution by

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n} f(x-y)e^{-|x|^2/4c^2t}dy$$

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or the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x),$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} e^{-|x|^2/4c^2 t} \tag{2}$$

and the symbol $*$ designates as the classical convolution.

K. Nonlaopon and A. Kananthai [17-19] have studied the ultra-hyperbolic heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box^k u(x,t) \tag{3}$$

with the initial condition $u(x,0) = f(x)$, where $\Box^k$ is the ultra-hyperbolic operator iterated k-times, and is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant. The solution of (3) can be expressed in the form

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) \exp\left(c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k + i(\xi, y) \right) d\xi dy \tag{4}$$

or the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x),$$

where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left(c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k + i(\xi, y) \right) d\xi, \tag{5}$$

which is so called ultra-hyperbolic heat kernel and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed, $t > 0$.

Next, A. Kananthai and K. Nonlaopon [5] have studied the nonlinear ultra-hyperbolic heat equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Box^k u(x,t) = f(x,t,u(x,t)). \tag{6}$$

On the suitable conditions for f, u and for the spectrum of the heat kernel, it can found unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$.

A. Kananthai [4] has studied diamond heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Diamond u(x,t) \tag{7}$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}^n$ of the n-dimensional Euclidean space. The operator $\Diamond$ is first introduced by Kananthai [3] and named the diamond operator defined by

$$\Diamond = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2,$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$. The solution of (7) can be expressed in the classical convolution form $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right) + i(\xi, x) \right] d\xi, \tag{8}$$

which is so called diamond heat kernel and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed, $t>0$.

Furthermore, A. Lunnaree and K. Nonlaopon [14] have studied generalized diamond heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \diamondsuit^k u(x,t) \tag{9}$$

with the initial condition $u(x,0)=f(x)$, where $\diamondsuit^k$ is the diamond operator iterated k-times and is defined by

$$\diamondsuit^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k,$$

k is a positive integer, $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$. The solution of (9) can be expressed in the classical convolution form $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^p \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k + i(\xi,x)\right] d\xi, \tag{10}$$

and $\Omega \in \mathbb{R}^n$ is spectrum of the $E(x,t)$ for any fixed, $t>0$.

Next, G. Sritanratana and A. Kananthai [23, 24] have studied the nonlinear equation

$$\frac{\partial}{\partial t}u(x,t) - c^2\diamondsuit^k u(x,t) = f(x,t,u(x,t)). \tag{11}$$

On the suitable conditions for f, u and for the spectrum of the diamond heat kernel, it can found unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$.

The operator $\oplus^k$ has been studied first by A. Kananthai et al. [6] and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k, \tag{12}$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$ and k is a positive integer. Next, A. Kananthai et al. [7] have studied the fundamental solution of the operator $\oplus^k$ related to wave equation and Laplacian. And, A. Kananthai and S. Suantai [8-10] have studied the convolution product, Fourier transform and inversion of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$. Moreover, J. Tariboon and A. Kananthai [25] have studied the Green function of the operator $(\oplus + m^2)^k$.

A. Liangprom and K. Nonlaopon [13] have studied the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \oplus^k u(x,t) \tag{13}$$

with the initial condition $u(x,0)=f(x)$, for $x\in\mathbb{R}^n$ of the n-dimensional Euclidean space, where operator $\oplus^k$ is named the oplus operator iterated k-times, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k, \tag{14}$$

$p+q=n$ is the dimension of space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1,x_2,\ldots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant. The solution of (13) can be expressed in the classical convolution form $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^p \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi,x)\right] d\xi. \tag{15}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed, $t > 0$. The function $E(x,t)$ is called the oplus heat kernel or the fundamental solution of (13).

In this article, we extend (13) to be the general of the nonlinear form

$$\frac{\partial}{\partial t} u(x,t) - c^2 \oplus^k u(x,t) = f(x,t,u(x,t)) \tag{16}$$

for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$ and with the following conditions on u and f as follows:

(1) $u(x,t) \in \mathcal{C}^{8k}(\mathbb{R}^n)$ for any $t > 0$, where $\mathcal{C}^{8k}(\mathbb{R}^n)$ is the space of all functions on $\mathbb{R}^n$ with continuous derivatives at least up to order $8k$.

(2) f satisfies the Lipchitz condition, that is

$$|f(x,t,u) - f(x,t,w)| \leq A|u-w|$$

where A is constant and $0 < A < 1$.

(3)

$$\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))| dx dt < \infty$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t \in (0,\infty)$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Under such conditions of f, u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution in the compact subset of $\mathbb{R}^n, t \in (0,\infty)$ and $E(x,t)$ is an fundamental solution defined by (21).

2 Preliminaries

Definition 1 *Let $f(x) \in \mathrm{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by*

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx, \tag{17}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1\, dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{18}$$

Definition 2 *The spectrum of the kernel $E(x,t)$, which is defined in (15), is the bounded support of the Fourier transform $\widehat{E}(\xi,t)$ for any fixed, $t > 0$.*

Definition 3 *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and denote by*

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by Definition 2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E}(\xi,t)$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E}(\xi,t) = \begin{cases} \dfrac{1}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\left(\displaystyle\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\displaystyle\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \right] & \text{for } \xi \in \Omega, \\ 0 & \text{for } \xi \notin \Omega. \end{cases} \tag{19}$$

Lemma 1 *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t} - c^2 \oplus^k, \tag{20}$$

where $\oplus^k$ is oplus operator iterated k-times, and is given by

$$\oplus^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k,$$

$p+q=n$ is the dimension of $\mathbb{R}^n$, k is a positive integer, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi \tag{21}$$

as a fundamental solution of (20) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $E(x,t)$, where is the kernel or the fundamental solution of operator L and δ is the Dirac-delta distribution. Thus, we have

$$\frac{\partial}{\partial t} E(x,t) - c^2 \oplus^k E(x,t) = \delta(x)\delta(t).$$

Applying the Fourier transform, which is defined by (17), to the both sides of the above equation, considering $\widehat{\delta}(x) = 1/(2\pi)^{n/2}$, we obtain

$$\frac{\partial}{\partial t} \widehat{E}(\xi,t) - c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \widehat{E}(\xi,t) = \frac{1}{(2\pi)^{n/2}} \delta(t).$$

Thus, we get

$$\widehat{E}(\xi,t) = \frac{H(t)}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \right],$$

where $H(t)$ is the Heaviside function, because $H(t) = 1$ holds for $t > 0$.

Therefore,

$$\widehat{E}(\xi,t) = \frac{1}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \right]$$

which has been already by (19). Thus from (18), we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi.$$

Thus, we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi,$$

where Ω is the spectrum of $E(x,t)$ and $t > 0$. ■

Definition 4 *Let us extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting*

$$\widehat{E}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n}\displaystyle\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi,x)\right] d\xi & \text{for } t>0, \\ 0 & \text{for } t \le 0. \end{cases}$$

3 Main Results

In this section, we will state our main results and give their proofs.

Theorem 2 *The kernel $E(x,t)$ defined by (21) has the following properties:*

(1) $E(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *the space of all continuous functions with infinitely differentiable.*

(2) $\left(\dfrac{\partial}{\partial t} - c^2 \oplus^k\right) E(x,t) = 0$ *for* $t > 0$.

(3)

$$|E(x,t)| \le \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}, \quad \text{for} \quad t>0,$$

where $M(t)$ is a function of $t>0$ in the spectrum Ω and Γ denote the gamma function. Thus $E(x,t)$ is bounded for any fixed, $t>0$.

(4) $\lim_{t\to 0} E(x,t) = \delta$.

Proof.
(1) From (21), and

$$\frac{\partial^n}{\partial x^n}E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n}\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi,x)\right] d\xi,$$

we have $E(x,t) \in \mathcal{C}^\infty$ for $x \in \mathbb{R}^n$, $t>0$.
(2) From $u(x,t) = E(x,t) * f(x)$, we have following equality for $f(x) = \delta(x)$ by Fourier transformation

$$u(x,t) = E(x,t).$$

Then by direct computation, we obtain

$$\left(\frac{\partial}{\partial t} - c^2\oplus^k\right)E(x,t) = 0.$$

(3) Since

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi,x)\right] d\xi,$$

then we obtain

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k\right] d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p \quad \text{and} \quad \xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q},$$

where $\sum_{i=1}^{p}\omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$. Thus

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(r^8 - s^8\right)^k\right] r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \le r \le R$ and $0 \le s \le T$, where R and T are constants. Thus, we obtain

$$\begin{aligned}|E(x,t)| &\le \frac{\Omega_p\,\Omega_q}{(2\pi)^n}\int_0^R\int_0^T \exp\left[c^2 t\left(r^8 - s^8\right)^k\right] r^{p-1}s^{q-1}\,ds\,dr\\ &= \frac{\Omega_p\,\Omega_q}{(2\pi)^n}M(t)\quad \text{for any fixed } t>0 \text{ in the spectrum } \Omega\\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\end{aligned} \tag{22}$$

where

$$M(t) = \int_0^R\int_0^T \exp\left[c^2 t\left(r^8 - s^8\right)^k\right] r^{p-1}s^{q-1}\,ds\,dr \tag{23}$$

is a function of t, $\Omega_p = \frac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$ and $\Omega_q = \frac{2\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.

(4) By (21), we have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi,x)\right] d\xi.$$

Since $E(x,t)$ exists, then

$$\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{i(\xi,x)}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}d\xi = \delta(x),\ \text{for } x \in \mathbb{R}^n,$$

see [11, p. 64, eq.4]. ■

Theorem 3 *Given the nonlinear equation*

$$\frac{\partial}{\partial t}u(x,t) - c^2 \oplus^k u(x,t) = f(x,t,u(x,t)) \tag{24}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, k is positive number and with the following conditions on u and f as follows:

(1) $u(x,t) \in \mathcal{C}^{8k}(\mathbb{R}^n)$ for any $t > 0$ where $\mathcal{C}^{8k}(\mathbb{R}^n)$ is the space of all functions on $\mathbb{R}^n$ with continuous derivatives at least up to order $8k$.

(2) f satisfies the Lipchitz condition, that is

$$|f(x,t,u) - f(x,t,w)| \le A|u-w|,$$

where A is constant and $0 < A < 1$.

(3)

$$\int_0^\infty\int_{\mathbb{R}^n}|f(x,t,u(x,t))|dxdt < \infty$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t \in (0,\infty)$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Then, for the spectrum of $E(x,t)$ we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{25}$$

as a unique solution of (24) for $x \in \Omega_0$ where $x \in \Omega_0$ is an compact subset of $\mathbb{R}^n, 0 \leq t \leq T$ with T is constant and $E(x,t)$ is a fundamental solution defined by (21) and also $u(x,t)$ is bounded.

In particular, if we put $k = 1$ and $q = 0$ in (24) then (24) reduces to the nonlinear equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle^4 u(x,t) = f(x,t,u(x,t))$$

which is related to the heat equation.

Proof. Convolving both sides of (24) with $E(x,t)$ and then we obtain the solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

or

$$u(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r,s) f(x-r, t-s, u(x-r,t-s)) dr ds$$

where $E(r,s)$ is given by Definition 4.

We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned} |u(x,t)| &\leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))| dr ds \\ &\leq \frac{2^{2-n}}{\pi^{n/2}} \frac{NM(t)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \end{aligned}$$

by the condition (3) and (22) where

$$N = \int_0^{\infty} \int_{\mathbb{R}^n} |f(x,t,u(x,t)) dx dt.$$

Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$.

To show that $u(x,t)$ is unique, suppose there is another solution $w(x,t)$ of equation (24). Let the operator

$$L = \frac{\partial}{\partial t} - c^2 \oplus^k$$

then (24) can be written in the form

$$Lu(x,t) = f(x,t,u(x,t)).$$

Thus

$$Lu(x,t) - Lw(x,t) = f(x,t,u(x,t)) - f(x,t,w(x,t)).$$

By the condition (2) of the Theorem,

$$|Lu(x,t) - Lw(x,t)| \leq A|u(x,t) - w(x,t)|. \tag{26}$$

Let $\Omega_0 \times (0,T]$ be compact subset of $\mathbb{R}^n \times (0,\infty)$ and $L : \mathcal{C}^{8k}(\Omega_0) \to \mathcal{C}^{8k}(\Omega_0)$ for $0 \leq t \leq T$

Now $(\mathcal{C}^{8k}(\Omega_0), \|\cdot\|)$ is a Banach space where $u(x,t) \in \mathcal{C}^{8k}(\Omega_0)$ for $0 \leq t \leq T$, $\|\cdot\|$ given by

$$\|u(x,t)\| = \sup_{x \in \Omega_0} |u(x,t)|.$$

Then, from (26) with $0 < A < 1$, the operator L is a contraction mapping on $\mathcal{C}^{8k}(\Omega_0)$. Since $(\mathcal{C}^{8k}(\Omega_0), \|\cdot\|)$ is a Banach space and $L : \mathcal{C}^{8k}(\Omega_0) \to C^{8k}(\Omega_0)$ is a contraction mapping on $\mathcal{C}^{8k}(\Omega_0)$, by Contraction Theorem, see [12, p. 300], we obtain the operator L has a fixed point and has uniqueness property. Thus $u(x,t) = w(x,t)$. It follows that the solution $u(x,t)$ of (24) is unique for $u(x,t) \in \Omega_0 \times (0,T]$, where $u(x,t)$ is defined by (25).

In particular, if we put $k = 1$ and $q = 0$ in (24) then (24) reduces to the nonlinear equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle^4 u(x,t) = f(x,t,u(x,t))$$

which has solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

where $E(x,t)$ is defined by (21) with $k = 1$ and $q = 0$. That is complete of proof. ■

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Research Article

On the General Solution of the Ultrahyperbolic Bessel Operator

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We study the general solution of equation $\square_{B,c}^k u(x) = f(x)$, where $\square_{B,c}^k$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by $\square_{B,c}^k = [(1/c^2)(B_{x_1} + B_{x_2} + \cdots + B_{x_p}) - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})]^k$, $p + q = n$, n is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ _x > 0, \ldots, x_n > 0\}$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0\ (i = 1, 2, \ldots, n)$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant, and $x \in \mathbb{R}_n^+$.

1. Introduction

The n-dimensional ultrahyperbolic operator $\square^k$ iterated k-times is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

where $p + q = n$, n is the dimension of space $\mathbb{R}^n$, and k is a nonnegative integer.

Consider the linear differential equation of the form

$$\square^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel'fand and Shilov [1] first introduced the fundamental solution of (1.2), which is a complicated form. Later, Trione [2] has shown that the generalized function $R_{2k}(x)$, defined by (2.8) with $|v| = 0$, is a unique fundamental solution of (1.2) and Téllez [3] also proved that $R_{2k}(x)$ exists only in the case when p is odd with n odd or even and $p+q = n$. A wealth of some effective works on the fundamental solution of the n-dimensional classical ultrahyperbolic operator have, presented by Kananthai and Sritanratana [4–9].

In 2004, Yildirim et al. [10] have introduced the Bessel ultrahyperbolic operator iterated k-times with $x \in \mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_1 > 0, \ldots, x_n > 0\}$,

$$\Box_B^k = \left(B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}}\right)^k, \tag{1.3}$$

where $p + q = n$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$ [11], k is a nonnegative integer, and n is the dimension of $\mathbb{R}_n^+$. They also have studied the fundamental solution of Bessel ultrahyperbolic operator.

In 2007, Sarikaya and Yildirim [12] have studied the weak solution of the compound Bessel ultrahyperbolic equation and also studied the Bessel ultrahyperbolic heat equation [13].

In 2009, Saglam et al. [14] have developed the operator of (1.3), defined by (1.6), and it is called the ultrahyperbolic Bessel operator iterated k-times. They have also studied the product of the ultrahyperbolic Bessel operator related to elastic waves.

Next, Srisombat and Nonlaopon [15] have studied the weak solution of

$$\Box_{B,c}^k u(x) = f(x), \tag{1.4}$$

where $u(x)$ and $f(x)$ are some generalized functions. They have developed (1.4) into the form

$$\sum_{k=0}^{m} C_k \Box_{B,c}^k u(x) = f(x), \tag{1.5}$$

which is called the compound ultrahyperbolic Bessel equation. In finding the solution of (1.5), they have used the properties of B-convolution for the generalized functions.

The purpose of this study is to find the general solution of equation $\Box_{B,c}^k u(x) = f(x)$, where $\Box_{B,c}^k$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by

$$\Box_{B,c}^k = \left[\frac{1}{c^2}\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right) - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)\right]^k \tag{1.6}$$

$p + q = n$, n is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ \ x_1 > 0, \ldots, x_n > 0\}$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0$ $(i = 1, 2, \ldots, n)$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant, and $x \in \mathbb{R}_n^+$.

2. Preliminaries

Let T_x^y be the generalized shift operator acting on the function φ, according to the law [11, 16]:

$$T_x^y\varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1y_1\cos\theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_ny_n\cos\theta_n}\right) \times \left(\prod_{i=1}^n \sin^{2v_i-1}\theta_i\right) d\theta_1\cdots d\theta_n, \tag{2.1}$$

where $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^n (\Gamma(v_i+1)/\Gamma(1/2)\Gamma(v_i))$. We remark that this shift operator is closely connected to the Bessel differential operator [11]:

$$\begin{gathered} \frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}, \\ U(x,0) = f(x), \\ U_y(x,0) = 0. \end{gathered} \tag{2.2}$$

The convolution operator is determined by the T_x^y as follows:

$$(f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.3}$$

The convolution (2.3) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x > 0$, and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^n x_i^{2v_i}\right) dx < \infty, \tag{2.4}$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.5}$$

(d) From (c), we have the following equality for $g(x) = 1$:

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.6}$$

(e) $(f * g)(x) = (g * f)(x)$.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional space $\mathbb{R}_n^+$. Denote the nondegenerated quadratic form by

$$V = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.7}$$

where $p + q = n$. The interior of the forward cone is defined by $\Gamma_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}_n^+ : x_i > 0,\ i = 1, \ldots, n$ and $V > 0\}$, where $\overline{\Gamma}_+$ designates its closure. For any complex number α, we define

$$R_{\alpha,c}^H(x) = \begin{cases} \dfrac{V^{(\alpha-n-2|v|)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.8}$$

where

$$K_n(\alpha) = \frac{\pi^{(n+2|v|-1)/2}\Gamma((2+\alpha-n-2|v|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p-2|v|)/2)\Gamma((p+2|v|-\alpha)/2)}. \tag{2.9}$$

The function $R_{\alpha,c}^H(x)$ is introduced by [10, 12, 17, 18]. It is well known that $R_{\alpha,c}^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is the distribution of α if $\mathrm{Re}(\alpha) < n$. Let supp $R_{\alpha,c}^H(x) \subset \overline{\Gamma}_+$, where supp $R_{\alpha,c}^H(x)$ denotes the support of $R_{\alpha,c}^H(x)$.

By putting $p = c = 1$ into (2.7), (2.8), and (2.9), and using the Legendre's duplication of $\Gamma(z)$,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.10}$$

the formula (2.8) is reduced to

$$M_\alpha^H(x) = \begin{cases} \dfrac{V^{((\alpha-n-2|v|)/2)}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.11}$$

where $V = x_1^2 - x_2^2 - \cdots - x_n^2$ and

$$H_n(\alpha) = \pi^{(n+2|v|-1)/2}2^{\alpha-1}\Gamma\left(\frac{2+\alpha-n-2|v|}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{2.12}$$

Note that the function $M_\alpha^H(x)$ is precisely the Bessel hyperbolic kernel of Marcel Riesz.

Lemma 2.2. *Given the equation*

$$\square_{B,c}^k u(x) = \delta(x), \tag{2.13}$$

where $\Box_{B,c}^{k}$ *is defined by* (1.6) *and* $x \in \mathbb{R}_n^+$, *then we obtain* $u(x) = R_{2k,c}^{H}(x)$ *as a fundamental solution of* (2.13), *where* $R_{2k,c}^{H}(x)$ *is defined by* (2.8).

The proof of this Lemma is given in [14].

Lemma 2.3. *The B-convolutions of tempered distributions.*

(a) $(\Box_{B,c}^{k}\delta) * u(x) = \Box_{B,c}^{k} u(x)$, *where* $u(x)$ *is any tempered distribution.*

(b) *Let* $R_{2k,c}^{H}(x)$ *and* $R_{2m,c}^{H}(x)$ *be defined by* (2.8); *then* $R_{2k,c}^{H}(x) * R_{2m,c}^{H}(x)$ *exists and is a tempered distribution.*

(c) *Let* $R_{2k,c}^{H}(x)$ *and* $R_{2m,c}^{H}(x)$ *be defined by* (2.8); *then* $R_{2k,c}^{H}(x) * R_{2m,c}^{H}(x) = R_{2k+2m,c}^{H}(x)$, *where* k *and* m *are nonnegative integers.*

The proof of this Lemma is given in [15].

Lemma 2.4. *Given that* P *is a hypersurface*

$$P\delta^{(m)}(P) + mP\delta^{(m-1)}(P) = 0, \tag{2.14}$$

where $\delta^{(m)}$ *is the Dirac-delta distribution with* m *derivatives.*

The proof of this Lemma is given in [1].

Lemma 2.5. *Given the equation*

$$\Box_{B,c}^{k} u(x) = 0, \tag{2.15}$$

where $\Box_{B,c}^{k}$ *is the ultrahyperbolic Bessel operator iterated* k*-times, as defined by* (1.6), *and* $x \in \mathbb{R}_n^+$, *then*

$$u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)}, \tag{2.16}$$

defined by (2.8) *with* m *derivatives, as a solution of* (2.15) *with* $m = ((n+2|v|-4)/2)$, $n+2|v| \geq 4$ *and* n *is an even dimension.*

Proof. We first show that the generalized function $\delta^{(m)}(c^2r^2 - s^2)$, where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$, $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, $p + q = n$, is a solution of

$$\Box_{B,c} u(x) = 0, \tag{2.17}$$

and $\Box_{B,c}$ is defined by (1.6) with $k = 1$ and $x \in \mathbb{R}_n^+$. Now for $1 \leq i \leq p$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2 - s^2\right) &= 2c^2x_i\delta^{(m+1)}\left(c^2r^2 - s^2\right),\\
\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2 - s^2\right) &= 2c^2\delta^{(m+1)}\left(c^2r^2 - s^2\right) + 4c^4x_i^2\delta^{(m+2)}\left(c^2r^2 - s^2\right).
\end{aligned} \tag{2.18}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{c^2}\sum_{i=1}^{p}\left[\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2-s^2\right)+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2-s^2\right)\right]\\
&\quad=2p\delta^{(m+1)}\left(c^2r^2-s^2\right)+4c^2r^2\delta^{(m+2)}\left(c^2r^2-s^2\right)+4\left|v'\right|\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&\quad=\left(2p+4\left|v'\right|\right)\delta^{(m+1)}\left(c^2r^2-s^2\right)+4\left(c^2r^2-s^2\right)\delta^{(m+2)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&\quad=\left(2p+4\left|v'\right|\right)\delta^{(m+1)}\left(c^2r^2-s^2\right)-4(m+2)\delta^{(m+1)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&\quad=\left[2p+4\left|v'\right|-4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)
\end{aligned}\tag{2.19}$$

by applying Lemma 2.4 with $P=c^2r^2-s^2$, where $|v'|=v_1+v_2+\cdots+v_p$.

Similarly, we have

$$\begin{aligned}
&\sum_{i=p+1}^{p+q}\left[\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2-s^2\right)+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2-s^2\right)\right]\\
&\quad=\left[-\left(2q+4\left|v''\right|\right)+4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4c^2r^2\delta^{(m+2)}\left(c^2r^2-s^2\right)
\end{aligned}\tag{2.20}$$

by applying Lemma 2.4 with $P=c^2r^2-s^2$, where $|v''|=v_{p+1}+v_{p+2}+\cdots+v_{p+q}$.

Thus, we have

$$\begin{aligned}
\square_{B,c}\delta^{(m)}\left(c^2r^2-s^2\right)&=\frac{1}{c^2}\sum_{i=1}^{p}\left[\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right]\delta^{(m)}\left(c^2r^2-s^2\right)\\
&\quad-\sum_{i=p+1}^{p+q}\left[\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right]\delta^{(m)}\left(c^2r^2-s^2\right)\\
&=\left[2\left(p+q+2|v|\right)-8(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&\quad-4\left(c^2r^2-s^2\right)\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&=\left[2(n+2|v|)-8(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4(m+2)\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&=\left[2(n+2|v|)-4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)
\end{aligned}\tag{2.21}$$

by applying Lemma 2.4 with $P=c^2r^2-s^2$, where $|v|=|v'|+|v''|$.

If $[2(n+2|v|)-4(m+2)]=0$, we obtain

$$\square_{B,c}\delta^{(m)}\left(c^2r^2-s^2\right)=0.\tag{2.22}$$

That is, $u(x) = \delta^{(m)}(c^2r^2 - s^2)$ is a solution of (2.15) with $m = (n+2|v|-4)/2$, $n+2|v| \geq 4$, and n is an even dimension. Now $\Box_{B,c}^{k} u(x) = 0$ can be written in the form

$$\Box_{B,c}\left(\Box_{B,c}^{k-1} u(x)\right) = 0. \tag{2.23}$$

From (2.17), we have

$$\Box_{B,c}^{k-1} u(x) = \delta^{(m)}\left(c^2r^2 - s^2\right) \tag{2.24}$$

with $m = (n+2|v|-4)/2, n+2|v| \geq 4$, and n being an even dimension. By Lemma 2.3(a), we can write (2.24) in the from

$$\Box_{B,c}^{k-1} \delta * u(x) = \delta^{(m)}\left(c^2r^2 - s^2\right). \tag{2.25}$$

B-convolving both sides of the above equation with the function $R_{2(k-1),c}^{H}(x)$, we obtain

$$\begin{gathered} R_{2(k-1),c}^{H}(x) * \Box_{B,c}^{k-1} \delta * u(x) = R_{2(k-1),c}^{H}(x) * \delta^{(m)}\left(c^2r^2 - s^2\right), \\ \Box_{B,c}^{k-1}\left[R_{2(k-1),c}^{H}(x)\right] * u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)}, \\ \delta * u(x) = u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)}, \end{gathered} \tag{2.26}$$

by Lemma 2.2.

It follows that $u(x) = [R_{2(k-1),c}^{H}(x)]^{(m)}$ is a solution of (2.15) with $m = (n+2|v|-4)/2$, $n+2|v| \geq 4$ and n is an even dimension.

The generalized function $\delta^{(m)}(c^2r^2 - s^2)$ mentioned in Lemma 2.5 has been also studied on the aspect of multiplicative product, distributional product and applications, for more details, see [19–23]. □

3. Main Result

Theorem 3.1. *Given the equation*

$$\Box_{B,c}^{k} u(x) = f(x), \tag{3.1}$$

where $\Box_{B,c}^{k}$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by (1.6), *$f(x)$ is a generalized function, $u(x)$ is an unknown generalized function, $x \in \mathbb{R}_n^{+}$, and n is an even, then* (3.1) *has the general solution*

$$u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)} + R_{2k,c}^{H}(x) * f(x), \tag{3.2}$$

where $[R_{2k,c}^{H}(x)]^{(m)}$ is a function defined by (2.8) *with m derivatives.*

Proof. B-convolving both sides of (3.1) with $R^H_{2k,c}(x)$, we obtain

$$R^H_{2k,c}(x) * \left(\square^k_{B,c} u(x)\right) = R^H_{2k,c}(x) * f(x). \tag{3.3}$$

By Lemma 2.2, we have

$$\square^k_{B,c}\left(R^H_{2k,c}(x)\right) * u(x) = \delta * u(x) = R^H_{2k,c}(x) * f(x). \tag{3.4}$$

So, we obtain that

$$u(x) = R^H_{2k,c}(x) * f(x) \tag{3.5}$$

is the solution of (3.1).

For a homogeneous equation $\square^k_{B,c} u(x) = 0$, we have a solution

$$u(x) = \left[R^H_{2(k-1),c}(x)\right]^{(m)} \tag{3.6}$$

by Lemma 2.5. Thus the general solution of (3.1) is

$$u(x) = \left[R^H_{2(k-1),c}(x)\right]^{(m)} + R^H_{2k,c}(x) * f(x). \tag{3.7}$$

This completes the proof. □

By putting $c = 1$, (3.1) becomes the Bessel ultrahyperbolic equation

$$\square^k_B w(x) = f(x), \tag{3.8}$$

where $\square^k_B$ is the Bessel ultrahyperbolic operator iterated k-times, and is defined by (1.3), $f(x)$ is a generalized function and $w(x)$ is an unknown generalized function. From (3.5) we have that

$$w(x) = R^H_{2k}(x) * f(x) \tag{3.9}$$

is a solution of (3.8), where $R^H_{2k}(x) = R^H_{2k,1}(x)$ defined by (2.8).

From (3.2), we obtain that the general solution of the Bessel ultrahyperbolic equation is

$$w(x) = \left[R^H_{2(k-1)}(x)\right]^{(m)} + R^H_{2k}(x) * f(x). \tag{3.10}$$

Moreover, if we put $k = 1,\ p = 1$ and $x_1 = t$(times), then (3.8) is reduced to the Bessel wave equation

$$\square_B w(x) = \left(B_t - \sum_{i=2}^{n} B_{x_i}\right) w(x) = f(x), \tag{3.11}$$

where

$$\square_B = B_t - \sum_{i=2}^{n} B_{x_i} \tag{3.12}$$

is the Bessel wave operator and $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$.

Thus, we obtain $w(x) = M_2(x) * f(x)$ as a solution of the Bessel wave equation, since $R_2^H(x)$ becomes $M_2^H(x)$, where $M_2^H(x)$ is the Bessel ultrahyperbolic kernel of Marcel Riesz, and is defined by (2.11) with $\alpha = 2$. And from (3.2), we obtain the general solution of Bessel wave equation as

$$w(x) = \delta^{(m)}(x) + M_2^H(x) * f(x), \tag{3.13}$$

where $\delta^{(m)}(x)$ is a solution of

$$\left(B_t - \sum_{i=2}^{n} B_{x_i}\right) w(x) = 0. \tag{3.14}$$

Now we put $V = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ and $s^2 = x_2^2 + x_3^2 + \cdots + x_n^2$. By [24], we obtain that

$$w(x,t) = \delta^{(m)}\left(t^2 - s^2\right) \tag{3.15}$$

is the solution of (3.14) with the initial conditions $w(x,0) = 0$ and $\partial w(x,0)/\partial t = (-1)^m 2\pi^{m+1}\delta(x)$ at $t = 0$ and $x = (x_2, x_3, \ldots, x_n) \in \mathbb{R}_{n-1}^{+}$.

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Research Article

On the Inversion of Bessel Ultrahyperbolic Kernel of Marcel Riesz

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We define the Bessel ultrahyperbolic Marcel Riesz operator on the function f by $U^\alpha(f) = R_\alpha^B * f$, where R_α^B is Bessel ultrahyperbolic kernel of Marcel Riesz, $\alpha \ldots \mathbb{C}$, the symbol $*$ designates as the convolution, and $f \in \mathcal{S}, \mathcal{S}$ is the Schwartz space of functions. Our purpose in this paper is to obtain the operator $E^\alpha = (U^\alpha)^{-1}$ such that, if $U^\alpha(f) = \varphi$, then $E^\alpha \varphi = f$.

1. Introduction

The n-dimensional ultrahyperbolic operator $\Box^k$ iterated k times is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and k is a nonnegative integer.

Consider the linear differential equation in the form of

$$\Box^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel′fand and Shilov [1] have first introduced the fundamental solution of (1.2), which is a complicated form. Later, Trione [2] has shown that the generalized function $R_{2k}^H(x)$, defined by (2.6) with $\gamma = 2k$, is the unique fundamental solution of (1.2) and Téllez [3] has also proved that $R^H_{2k}(x)$ exists only when $n = p + q$ with odd p.

Next, Kananthai [4] has first introduced the operator $\Diamond^k$ called the diamond operator iterated k times, which is defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{1.3}$$

where $n = p + q$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \ldots, x_n)$, and k is a nonnegative integer. The operator $\Diamond^k$ can be expressed in the form

$$\Diamond^k = \triangle^k \Box^k = \Box^k \triangle^k, \tag{1.4}$$

where $\Box^k$ is defined by (1.1), and

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{1.5}$$

is the Laplace operator iterated k times. On finding the fundamental solution of this product, Kananthai uses the convolution of functions which are fundamental solutions of the operators $\Box^k$ and $\triangle^k$. He found that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the operator $\Diamond^k$, that is,

$$\Diamond^k \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta(x), \tag{1.6}$$

where $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by (2.6) and (2.11), respectively with $\gamma = 2k$ and $\delta(x)$ is the Dirac delta distribution. The fundamental solution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is called the diamond kernel of Marcel Riesz. A wealth of some effective works on the diamond kernel of Marcel Riesz have been presented by Kananthai [5–10].

In 1978, Domínguez and Trione [11] have introduced the distributional functions $H_\alpha(P \pm i0, n)$ which are causal (anticausal) analogues of the elliptic kernel of Riesz [12]. Next, Cerutti and Trione [13] have defined the causal (anticausal) generalized Marcel Riesz potentials of order α, $\alpha \in \mathbb{C}$, by

$$R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi, \tag{1.7}$$

where $\varphi \in \mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions [14] and $H_\alpha(P \pm i0, n)$ is given by

$$H_\alpha(P \pm i0, n) = \frac{e^{\mp \alpha\pi i/2} e^{\pm q\pi i/2} \Gamma((n-\alpha)/2)(P \pm i0)^{(\alpha-n)/2}}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}. \tag{1.8}$$

Here, P is defined by

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{1.9}$$

where q is the number of negative terms of the quadratic form P. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \to 0} \left(P \pm i\epsilon |x|^2 \right)^\lambda, \tag{1.10}$$

where $\epsilon > 0$, $\lambda \in \mathbb{C}$, and $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$; see [1]. They have also studied the inverse operator of R^α, denoted by $(R^\alpha)^{-1}$, such that, if $f = R^\alpha \varphi$, then $(R^\alpha)^{-1} f = \varphi$.

Later, Aguirre [15] has defined the ultrahyperbolic Marcel Riesz operator M^α of the function f by

$$M^\alpha(f) = R_\alpha^H * f, \tag{1.11}$$

where R_α^H is defined by (2.6) and $f \in \mathcal{S}$. He has also studied the operator $N^\alpha = (M^\alpha)^{-1}$ such that, if $M^\alpha(f) = \varphi$, then $N^\alpha \varphi = f$.

Let us consider the diamond kernel of Marcel Riesz $K_{\alpha,\beta}(x)$ introduced by Kananthai in [6], which is given by the convolution

$$K_{\alpha,\beta}(x) = R_\alpha^e * R_\beta^H, \tag{1.12}$$

where R_α^e is elliptic kernel defined by (2.11) and R_β^H is the ultrahyperbolic kernel defined by (2.6). Tellez and Kananthai [16] have proved that $K_{\alpha,\beta}(x)$ exists and is in the space of rapidly decreasing distributions. Moreover, they have also shown that the convolution of the distributional families $K_{\alpha,\beta}(x)$ relates to the diamond operator.

Later, Maneetus and Nonlaopon [17] have defined the diamond Marcel Riesz operator of order (α, β) of the function f by

$$M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} * f, \tag{1.13}$$

where $K_{\alpha,\beta}$ is defined by (1.12), $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{S}$. They have also studied the operator $N^{(\alpha,\beta)} = [M^{(\alpha,\beta)}]^{-1}$ such that, if $M^{(\alpha,\beta)}(f) = \varphi$, then $N^{(\alpha,\beta)}\varphi = f$.

In this paper, we define the Bessel ultrahyperbolic Marcel Riesz operator of order α of the function f by

$$U^\alpha(f) = R_\alpha^B * f, \tag{1.14}$$

where $\alpha \in \mathbb{C}$ and $f \in \mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions. Our aim in this paper is to obtain the operator $E^\alpha = (U^\alpha)^{-1}$ such that, if $U^\alpha(f) = \varphi$, then $E^\alpha \varphi = f$.

Before we proceed to our main theorem, the following definitions and concepts require some clarifications.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in the n-dimensional Euclidean space $\mathbb{R}^n$. Let

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

be the nondegenerated quadratic form, where $p + q = n$ is the dimension of $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : u > 0 \text{ and } x_i > 0\,(i = 1, 2, \ldots, p)\}$ be the interior of a forward cone, and let $\overline{\Gamma}_+$ denote its closure. For any complex number γ, we define

$$R_\gamma^B(x) = \begin{cases} \dfrac{u^{(\gamma-2|\nu|-n)/2}}{K_n^{|\nu|}(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where

$$K_n^{|\nu|}(\gamma) = \frac{\pi^{(n-1+2|\nu|)/2}\Gamma((2+\gamma-n-2|\nu|)/2)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((2+\gamma-p-2|\nu|)/2)\Gamma((p-\gamma)/2)}, \tag{2.3}$$

$2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$ and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$, see [18–20].

The function $R_\gamma^B(x)$ is called the Bessel ultrahyperbolic kernel and was introduced by Aguirre [21]. It is well known that $R_\gamma^B(x)$ is an ordinary function if $\mathrm{Re}(\gamma - 2|\nu|) \geq n$ and is a distribution of $(\gamma - 2|\nu|)$ if $\mathrm{Re}(\gamma - 2|\nu|) < n$. Let $\mathrm{supp} R_\gamma^B(x)$ denote the support of $R_\gamma^B(x)$ and suppose that $\mathrm{supp} R_\gamma^B(x) \subset \overline{\Gamma}_+$ (i.e., $\mathrm{supp} R_\gamma^B(x)$ is compact).

Letting $\gamma = 2k$ in (2.2) and (2.3), we obtain

$$R_{2k}^B(x) = \frac{u^{(2k-n-2|\nu|)/2}}{K_n(2k)}, \tag{2.4}$$

where

$$K_n(2k) = \frac{\pi^{(n-1+2|\nu|)/2}\Gamma((2+2k-n-2|\nu|)/2)\Gamma((1-2k)/2)\Gamma(2k)}{\Gamma((2+2k-p-2|\nu|)/2)\Gamma((p-2k)/2)}. \tag{2.5}$$

By putting $|\nu| = 0$ in (2.2) and (2.3), then formulae (2.2) and (2.3) reduce to

$$R_\gamma^H(x) = \begin{cases} \dfrac{u^{(\gamma-n)/2}}{K_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.6}$$

$$K_n(\gamma) = \frac{\pi^{(n-1)/2}\Gamma((\gamma-n)/2+1)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((\gamma-p)/2+1)\Gamma((p-\gamma)/2)}. \tag{2.7}$$

The function $R_\gamma^H(x)$ is called the ultrahyperbolic kernel of Marcel Riesz and was introduced by Nozaki [22]. It is well known that $R_\gamma^H(x)$ is an ordinary function if $\mathrm{Re}(\gamma) \geq n$ and is a distribution of γ if $\mathrm{Re}(\gamma) < n$. Let $\mathrm{supp}R_\gamma^H(x)$ denote the support of $R_\gamma^H(x)$ and suppose that $\mathrm{supp}R_\gamma^H(x) \subset \overline{\Gamma}_+$ (i.e., $\mathrm{supp}R_\gamma^H(x)$ is compact).

By putting $p = 1$ in $R_{2k}^H(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.8}$$

we obtain

$$I_\gamma^H(x) = \frac{v^{(\gamma-n)/2}}{H_n(\gamma)} \tag{2.9}$$

and $v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\gamma) = \pi^{(n-2)/2}2^{\gamma-1}\Gamma\left(\frac{\gamma+2-n}{2}\right)\Gamma\left(\frac{\gamma}{2}\right). \tag{2.10}$$

The function $I_\gamma^H(x)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$. The elliptic kernel of Marcel Riesz is defined by

$$R_\gamma^e(x) = \frac{\omega^{(\gamma-n)/2}}{W_n(\gamma)}, \tag{2.11}$$

where n is the dimension of $\mathbb{R}^n$, $\gamma \in \mathbb{C}$, and

$$W_n(\gamma) = \frac{\pi^{n/2}2^\gamma\Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}. \tag{2.12}$$

Note that $n = p + q$. By putting $q = 0$ (i.e., $n = p$) in (2.6) and (2.7), we can reduce $u^{(\gamma-n)/2}$ to $\omega_p^{(\gamma-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$, and reduce $K_n(\gamma)$ to

$$K_p(\gamma) = \frac{\pi^{(p-1)/2}\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((p-\gamma)/2)}. \tag{2.13}$$

Using Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.14}$$

and

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi\sec(\pi z), \tag{2.15}$$

we obtain

$$K_p(\gamma) = \frac{1}{2}\sec\left(\frac{\gamma\pi}{2}\right)W_p(\gamma). \tag{2.16}$$

Thus, for $q = 0$, we have

$$R_\gamma^H(x) = \frac{u^{(\gamma-p)/2}}{K_p(\gamma)} = 2\cos\left(\frac{\gamma\pi}{2}\right)\frac{u^{(\gamma-p)/2}}{W_p(\gamma)} = 2\cos\left(\frac{\gamma\pi}{2}\right)R_\gamma^e(x). \tag{2.17}$$

In addition, if $\gamma = 2k$ for some nonnegative integer k, then

$$R_{2k}^H(x) = 2(-1)^k R_{2k}^e(x). \tag{2.18}$$

The proofs of Lemma 2.3 are given in [2].

Lemma 2.3. *The function $R_\alpha^H(x)$ has the following properties:*

(i) $R_0^H(x) = \delta(x)$;

(ii) $R_{-2k}^H(x) = \Box^k\delta(x)$;

(iii) $\Box^k R_\alpha^H(x) = R_{\alpha-2k}^H(x)$;

(iv) $\Box^k R_{2k}^H(x) = \delta(x)$.

Lemma 2.4. *If $|\nu| \neq 0$, then*

$$R_\gamma^B(x) = h_{\gamma,p,|\nu|}R_{\gamma-2|\nu|}^H(x), \tag{2.19}$$

where $R_\gamma^B(x)$ and $R_{\gamma-2|\nu|}^H(x)$ are defined by (2.2) *and* (2.6), *respectively, and*

$$h_{\gamma,p,|\nu|} = \frac{\Gamma((1-\gamma)/2+|\nu|)\Gamma(\gamma-2|\nu|)\Gamma((p-\gamma)/2)}{\pi^{|\nu|}\Gamma((p-\gamma)/2+|\nu|)\Gamma((1-\gamma)/2)\Gamma(\gamma)}. \tag{2.20}$$

Proof. We get (2.19) by computing directly from definition of $R_\gamma^B(x)$ and $R_{\gamma-2|\nu|}^H(x)$. □

The proof of the following lemma is given in [23].

Lemma 2.5 (the convolutions of $R_\alpha^H(x)$). *(i) If p is odd, then*

$$R_\alpha^H(x) * R_\beta^H(x) = R_{\alpha+\beta}^H(x) + A_{\alpha,\beta}, \tag{2.21}$$

where

$$A_{\alpha,\beta} = -\frac{i}{2}\frac{\sin(\alpha\pi/2)\sin(\beta\pi/2)}{\sin((\alpha+\beta)\pi/2)}\left[H^{+}_{\alpha+\beta} - H^{-}_{\alpha+\beta}\right], \tag{2.22}$$

$$H^{\pm}_{\alpha+\beta} = H_{\alpha+\beta}(P \pm i0, n) \tag{2.23}$$

as defined by (1.8).

(ii) If p is even, then

$$R^{H}_{\alpha}(x) * R^{H}_{\beta}(x) = B_{\alpha,\beta}R^{H}_{\alpha+\beta}(x), \tag{2.24}$$

where

$$B_{\alpha,\beta} = \frac{\cos(\alpha\pi/2)\cos(\beta\pi/2)}{\cos((\alpha+\beta)\pi/2)}. \tag{2.25}$$

Lemma 2.6 (the convolutions of $R^{B}_{\alpha}(x)$). *(i) If p is odd, then*

$$R^{B}_{\alpha}(x) * R^{B}_{\beta}(x) = h_{\alpha,p,|\nu|}h_{\beta,p,|\mu|}\left(R^{H}_{\alpha+\beta-2(|\nu|+|\mu|)} + A_{\alpha-2|\nu|,\beta-2|\mu|}\right), \tag{2.26}$$

where $R^{H}_{\alpha}(x)$ *and* $A_{\alpha-2|\nu|,\beta-2|\mu|}$ *are defined by* (2.6) *and* (2.22)*, respectively.*

(ii) If p is even, then

$$R^{B}_{\alpha}(x) * R^{B}_{\beta}(x) = h_{\alpha,p,|\nu|}h_{\beta,p,|\mu|}\left(B_{\alpha-2|\nu|,\beta-2|\mu|}R^{H}_{\alpha+\beta-2(|\nu|+|\mu|)}\right), \tag{2.27}$$

where $B_{\alpha-2|\nu|,\beta-2|\mu|}$ *is defined by* (2.25).

The proof of this lemma can be easily seen from Lemmas 2.4, 2.5 and [23].

3. The Convolution $R^{B}_{\alpha}(x) * R^{B}_{\beta}(x)$ When $\beta = -\alpha$

We will now consider the property of $R^{B}_{\alpha}(x) * R^{B}_{\beta}(x)$ when $\beta = -\alpha$.

From (2.26) and (2.27), we immediately obtain the following properties.

(1) If p is odd and q is even, then

$$R^{B}_{\alpha}(x) * R^{B}_{\beta}(x) = h_{\alpha,p,|\nu|}h_{\beta,p,|\mu|}\left(R^{H}_{\alpha+\beta-2(|\nu|+|\mu|)} + A_{\alpha-2|\nu|,\beta-2|\mu|}\right), \tag{3.1}$$

where $R^{H}_{\alpha}(x)$ and $A_{\alpha-2|\nu|,\beta-2|\mu|}$ are defined by (2.6) and (2.22), respectively.

(2) If p and q are both odd, then

$$R_\alpha^B(x) * R_\beta^B(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|} \left(R^H_{\alpha+\beta-2(|\nu|+|\mu|)} + A_{\alpha-2|\nu|,\beta-2|\mu|} \right). \tag{3.2}$$

(3) If p is even and q is odd, then

$$R_\alpha^B(x) * R_\beta^B(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|} \left(\frac{\cos((\alpha-2|\nu|)\pi/2) \cdot \cos((\beta-2|\mu|)\pi/2)}{\cos((\alpha+\beta-2(|\nu|+|\mu|))\pi/2)} R^H_{\alpha+\beta-2(|\nu|+|\mu|)} \right). \tag{3.3}$$

(4) If p and q are both even, then

$$R_\alpha^B(x) * R_\beta^B(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|} \left(\frac{\cos((\alpha-2|\nu|)\pi/2) \cdot \cos((\beta-2|\mu|)\pi/2)}{\cos((\alpha+\beta-2(|\nu|+|\mu|))\pi/2)} R^H_{\alpha+\beta-2(|\nu|+|\mu|)} \right). \tag{3.4}$$

Moreover, it follows from (2.22) that

$$\begin{aligned} A_{\alpha-2|\nu|,-(\alpha-2|\nu|)} &= \lim_{\beta-2|\mu| \to -(\alpha-2|\nu|)} A_{\alpha-2|\nu|,\beta-2|\mu|} \\ &= -\frac{i}{2} \lim_{\gamma \to 0} \frac{\sin((\alpha-2|\nu|)\pi/2) \sin((\gamma-(\alpha-2|\nu|))\pi/2)}{\sin(\gamma\pi/2)} \left[H_\gamma^+ - H_\gamma^- \right] \\ &= -\frac{i}{2} \lim_{\gamma \to 0} \frac{\sin((\alpha-2|\nu|)\pi/2) \sin((\gamma-(\alpha-2|\nu|))\pi/2)}{\sin(\gamma\pi/2)} \cdot \lim_{\gamma \to 0} \left[H_\gamma^+ - H_\gamma^- \right], \end{aligned} \tag{3.5}$$

where $\gamma = \alpha + \beta - 2(|\nu| + |\mu|)$.

On the other hand, using (2.23) and (1.8), we have

$$\begin{aligned} \lim_{\gamma \to 0} \left[H_\gamma^+ - H_\gamma^- \right] &= \frac{\Gamma(n/2)}{\pi^{n/2}} \Bigg[\lim_{\gamma \to 0} e^{-\gamma\pi i/2} e^{q\pi i/2} \frac{(P+i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \\ &\qquad - \lim_{\gamma \to 0} e^{\gamma\pi i/2} e^{-q\pi i/2} \frac{(P-i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \Bigg] \\ &= \frac{\Gamma(n/2)}{\pi^{n/2}} \Bigg[\lim_{\gamma \to 0} e^{-\gamma\pi i/2} e^{q\pi i/2} \cdot \frac{\operatorname{Res}_{\beta=-n/2}(P+i0)^\beta}{\operatorname{Res}_{\beta=-n/2}\Gamma(\beta+n/2)} \\ &\qquad - \lim_{\gamma \to 0} e^{\gamma\pi i/2} e^{-q\pi i/2} \cdot \frac{\operatorname{Res}_{\beta=-n/2}(P-i0)^\beta}{\operatorname{Res}_{\beta=-n/2}\Gamma(\beta+n/2)} \Bigg]. \end{aligned} \tag{3.6}$$

Now, taking n as an odd integer, we obtain

$$\operatorname*{Res}_{\lambda=-n/2-k}(P \pm i0)^{\lambda} = \frac{e^{\pm q\pi i/2}\pi^{n/2}}{2^{2k}k!\Gamma(n/2+k)}\Box^{k}\delta(x), \tag{3.7}$$

where $\Box^k$ is defined by (1.1), $p+q=n$, and k is nonnegative integer; see [24, 25]. If p and q are both even, then

$$\operatorname*{Res}_{\lambda=-n/2-k}(P \pm i0)^{\lambda} = \frac{e^{\pm q\pi i/2}\pi^{n/2}}{2^{2k}k!\Gamma(n/2+k)}\Box^{k}\delta(x). \tag{3.8}$$

Nevertheless, if p and q are both odd, then

$$\operatorname*{Res}_{\lambda=-n/2-k}(P \pm i0)^{\lambda} = 0. \tag{3.9}$$

Therefore, we have

$$\begin{aligned}\lim_{\gamma\to 0}\left[H_{\gamma}^{+}-H_{\gamma}^{-}\right] &= \frac{\Gamma(n/2)}{\pi^{n/2}}\cdot\frac{\pi^{n/2}}{\Gamma(n/2)}\left[\lim_{\gamma\to 0}e^{-\gamma\pi i/2}-\lim_{\gamma\to 0}e^{\gamma\pi i/2}\right]\delta(x)\\ &= \lim_{\gamma\to 0}\left[-2i\sin(\gamma\pi/2)\right]\delta(x).\end{aligned} \tag{3.10}$$

From (3.6) and (3.9), we have

$$\lim_{\gamma\to 0}\left[H_{\gamma}^{+}-H_{\gamma}^{-}\right] = 0 \tag{3.11}$$

if p and q are both odd (n even).

Applying (3.10) and (3.11) into (3.5), we have

$$\begin{aligned}A_{\alpha-2|\nu|,-\alpha+2|\nu|} &= -\frac{i}{2}\lim_{\gamma\to 0}\frac{\sin((\alpha-2|\nu|)\pi/2)\sin((\gamma-(\alpha-2|\nu|))\pi/2)}{\sin(\gamma\pi/2)}\cdot\lim_{\gamma\to 0}\left[-2i\sin(\gamma\pi/2)\right]\delta(x)\\ &= \sin^{2}((\alpha-2|\nu|)\pi/2)\delta(x)\end{aligned} \tag{3.12}$$

if p is odd and q is even and

$$A_{\alpha-2|\nu|,-\alpha+2|\nu|} = 0 \tag{3.13}$$

if p and q are both odd.

From (3.1)—(3.4) and using Lemmas 2.3, and 2.6 and formulae (3.12) and (3.13), if p is odd and q is even, then we obtain

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left(R_0^H + A_{\alpha-2|\nu|,-\alpha+2|\nu|} \right) \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left[\delta(x) + \sin^2((\alpha-2|\nu|)\pi/2)\delta(x) \right] \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left[1 + \sin^2((\alpha-2|\nu|)\pi/2) \right] \delta(x). \end{aligned} \tag{3.14}$$

If p and q are both odd, then

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left(R_0^H + A_{\alpha-2|\nu|,-\alpha+2|\nu|} \right) \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \delta(x). \end{aligned} \tag{3.15}$$

If p is even and q is odd, then

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \frac{\cos((\alpha-2|\nu|)\pi/2)\cos((-\alpha+2|\nu|)\pi/2)}{\cos((\alpha-\alpha-2|\nu|+2|\nu|)\pi/2)} R_0^H \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \cos^2((\alpha-2|\nu|)\pi/2)\delta(x). \end{aligned} \tag{3.16}$$

Finally, if p and q are both even, then

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \frac{\cos((\alpha-2|\nu|)\pi/2)\cos((-\alpha+2|\nu|)\pi/2)}{\cos((\alpha-\alpha-2|\nu|+2|\nu|)\pi/2)} R_0^H \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \cos^2((\alpha-2|\nu|)\pi/2)\delta(x). \end{aligned} \tag{3.17}$$

4. The Main Theorem

Let $M^\alpha(f)$ be the Bessel ultrahyperbolic Marcel Riesz operator of order α of the function f, which is defined by

$$U^\alpha(f) = R_\alpha^B * f, \tag{4.1}$$

where R_α^B is defined by (2.2), $\alpha \in \mathbb{C}$, and $f \in \mathcal{S}$.

Recall that our objective is to obtain the operator $E^\alpha = (U^\alpha)^{-1}$ such that, if $U^\alpha(f) = \varphi$, then $E^\alpha \varphi = f$ for all $\alpha \in \mathbb{C}$.

We are now ready to state our main theorem.

Theorem 4.1. *If* $U^\alpha(f) = \varphi$ *(where* $U^\alpha(f)$ *is defined by* (4.1) *and* $f \in \mathcal{S}$*), then* $E^\alpha\varphi = f$ *such that*

$$E^\alpha = (U^\alpha)^{-1} = \begin{cases} \dfrac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}R^B_{-\alpha} & \text{if } p \text{ is odd and } q \text{ is even}, \\ \dfrac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}R^B_{-\alpha} & \text{if } p \text{ and } q \text{ are both odd}, \\ \dfrac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)R^B_{-\alpha} & \text{if } p \text{ is even with} (\alpha-2|\nu|)/2 \neq 2s+1 \end{cases} \tag{4.2}$$

for any nonnegative integer s.

Proof. By (4.1), we have

$$U^\alpha(f) = R^B_\alpha * f = \varphi, \tag{4.3}$$

where R^B_α is defined by (2.2), $\alpha \in \mathbb{C}$, and $f \in \mathcal{S}$. If p is odd and q is even, then, in view of (3.14), we obtain

$$\begin{aligned} &\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}R^B_{-\alpha} * \left(R^B_\alpha * f\right) \\ &\quad= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}\left(R^B_{-\alpha} * R^B_\alpha\right) * f \\ &\quad= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1} \\ &\qquad\times\left\{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]\delta(x)\right\} * f \\ &\quad= \delta * f = f. \end{aligned} \tag{4.4}$$

Hence,

$$\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}R^B_{-\alpha} = (U^\alpha)^{-1} = \left(R^B_\alpha\right)^{-1} \tag{4.5}$$

for all $\alpha \in \mathbb{C}$.

Similarly, if both p and q are odd, then, by (3.15), we obtain

$$\begin{aligned} \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}R^B_{-\alpha} * \left(R^B_\alpha * f\right) &= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left(R^B_{-\alpha} * R^B_\alpha\right) * f \\ &= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}\delta(x) * f \\ &= f. \end{aligned} \tag{4.6}$$

Hence,

$$\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}R^B_{-\alpha} = (U^\alpha)^{-1} = \left(R^B_\alpha\right)^{-1} \tag{4.7}$$

for all $\alpha \in \mathbb{C}$.

Finally, if p is even, then, by (3.16) and (3.17), we have

$$\begin{aligned}
&\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)R^B_{-\alpha} * \left(R^B_\alpha * f\right) \\
&\quad= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)\left(R^B_{-\alpha} * R^B_\alpha\right) * f \\
&\quad= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)\left\{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}\cos^2((\alpha-2|\nu|)\pi/2)\delta(x)\right\} * f \\
&\quad= \delta * f = f,
\end{aligned} \tag{4.8}$$

provided that $(\alpha-2|\nu|)/2 \neq 2s+1$ for any nonnegative integer s.

Hence,

$$\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)R^B_{-\alpha} = (U^\alpha)^{-1} = \left(R^B_\alpha\right)^{-1} \tag{4.9}$$

for all $\alpha \in \mathbb{C}$ with $(\alpha-2|\nu|)/2 \neq 2s+1$ for any nonnegative integer s.

In this conclusion, formulae (4.5), (4.7), and (4.9) are the desired results, and this completes the proof. □

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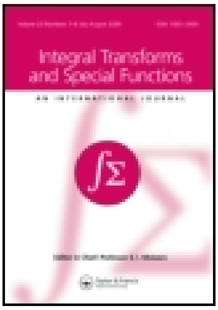

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On the inverse Bessel diamond kernel of Marcel Riesz

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On the inverse Bessel diamond kernel of Marcel Riesz

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In this paper, we define the Bessel diamond kernel of Marcel Riesz $K^B_{\alpha,\beta}$ and the Bessel diamond Marcel Riesz operator of order (α, β) on the function f by

$$U^{(\alpha,\beta)}(f) = K^B_{\alpha,\beta} * f,$$

where $\alpha, \beta \in \mathbb{C}$, the symbol $*$ designates the convolution, and $f \in \mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions. In this paper, we aim to study the convolution of $K^B_{\alpha,\beta}$ and obtain the operator $E^{(\alpha,\beta)} = [U^{(\alpha,\beta)}]^{-1}$ such that if $U^{(\alpha,\beta)}(f) = \varphi$, then $E^{(\alpha,\beta)}\varphi = f$.

Keywords: Bessel diamond kernel of Marcel Riesz; Bessel diamond operator; Dirac delta distribution

AMS Subject Classifications: 46F10; 46F12

1. Introduction

The n-dimensional ultra-hyperbolic operator $\Box^k$ iterated k times is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and k is a non-negative integer.

Consider the linear differential equation in the form of

$$\Box^k u(x) = f(x), \tag{2}$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel'fand and Shilov [6] were the first to introduce the fundamental solution of (2), which is a complicated form. Later, Trione [24] showed that the generalized function $R_{2k}(x)$, defined by (21) with $\alpha = 2k$, is the unique fundamental solution of (2) and Tellez [20] also proved that $R_{2k}(x)$ exists only when $n = p + q$ with odd p.

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Later, Kananthai [7] was the first to introduce the operator $\diamondsuit^k$ called the diamond operator iterated k times, which is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{3}$$

where $n = p + q$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \ldots, x_n)$, and k is a non-negative integer. The operator $\diamondsuit^k$ can be expressed in the form

$$\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k, \tag{4}$$

where $\Box^k$ is defined by (1), and

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{5}$$

is the Laplace operator iterated k times. On finding the fundamental solution of this product, Kananthai used the convolution of functions which are fundamental solutions of the operators $\Box^k$ and $\triangle^k$. He found that the convolution $(-1)^k S_{2k}(x) * R_{2k}(x)$ is the fundamental solution of the operator $\diamondsuit^k$, that is,

$$\diamondsuit^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta(x), \tag{6}$$

where $R_{2k}(x)$ and $S_{2k}(x)$ are defined by (21) and (28), respectively, with $\alpha = 2k$, and $\delta(x)$ is the Dirac delta distribution. The fundamental solution $(-1)^k S_{2k}(x) * R_{2k}(x)$ is called the diamond kernel of Marcel Riesz. A wealth of effective works on the diamond kernel of Marcel Riesz were presented by Kananthai [8–12] and Sritanratana and Kananthai [19].

In 1978, Dominguez and Trione [5] introduced the distributional functions $H_\alpha(P \pm i0, n)$, which are causal (anti-causal) analogues of the elliptic kernel of Riesz [16]. Later, Cerutti and Trione [3] defined the causal (anti-causal) generalized Marcel Riesz potentials of order α, $\alpha \in \mathbb{C}$, by

$$R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi, \tag{7}$$

where $\varphi \in \mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions [18], and $H_\alpha(P \pm i0, n)$ is given by

$$H_\alpha(P \pm i0, n) = \frac{e^{\mp \alpha\pi i/2}\, e^{\pm q\pi i/2}\, \Gamma((n-\alpha)/2)(P \pm i0)^{(\alpha-n)/2}}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}. \tag{8}$$

Here, P is defined by

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{9}$$

where q is the number of negative terms of the quadratic form P. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \to 0}(P \pm i\epsilon|x|^2)^\lambda, \tag{10}$$

where $\epsilon > 0$, $\lambda \in \mathbb{C}$, and $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, see [6]. They also studied the inverse operator of R^α, denoted by $(R^\alpha)^{-1}$, such that if $f = R^\alpha \varphi$, then $(R^\alpha)^{-1} f = \varphi$.

Later, Aguirre [1] defined the ultra-hyperbolic Marcel Riesz operator M^α on the function f by

$$M^\alpha(f) = R_\alpha * f, \tag{11}$$

where R_α is defined by (21) and $f \in \mathcal{S}$. He also studied the operator $N^\alpha = (M^\alpha)^{-1}$ such that if $M^\alpha(f) = \varphi$, then $N^\alpha \varphi = f$.

Let us consider the diamond kernel of Marcel Riesz $K_{\alpha,\beta}(x)$ introduced by Kananthai [10], which is given by the convolution

$$K_{\alpha,\beta}(x) = S_\alpha(x) * R_\beta(x), \tag{12}$$

where $S_\alpha(x)$ is an elliptic kernel defined by (28) and $R_\beta(x)$ is the ultra-hyperbolic kernel defined by (21). Tellez and Kananthai [23] proved that $K_{\alpha,\beta}(x)$ exists and is in the space of rapidly decreasing distributions. Moreover, they also showed that the convolution of the distributional families $K_{\alpha,\beta}(x)$ relates to the diamond operator.

Later, Maneetus and Nonlaopon [13] defined the diamond Marcel Riesz operator of order (α, β) on the function f by

$$M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} * f, \tag{13}$$

where $K_{\alpha,\beta}$ is defined by (12), $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{S}$. They also studied the operator $N^{(\alpha,\beta)} = [M^{(\alpha,\beta)}]^{-1}$ such that if $M^{(\alpha,\beta)}(f) = \varphi$, then $N^{(\alpha,\beta)}\varphi = f$. Moreover, they defined the Bessel ultra-hyperbolic Marcel Riesz operator of order α on the function f by

$$U^\alpha(f) = R^B_\alpha * f, \tag{14}$$

where R^B_α is the Bessel ultra-hyperbolic kernel of Marcel Riesz defined by (17), $\alpha \in \mathbb{C}$, and $f \in \mathcal{S}$; see [14], for more details. In addition, they studied the operator $E^\alpha = (U^\alpha)^{-1}$ such that if $U^\alpha(f) = \varphi$, then $E^\alpha\varphi = f$.

In this paper, we define the Bessel diamond kernel of Marcel Riesz by

$$K^B_{\alpha,\beta}(x) = S^B_\alpha(x) * R^B_\beta(x), \tag{15}$$

where $R^B_\beta(x)$ and $S^B_\alpha(x)$ are defined by (17) and (26), respectively, and the Bessel diamond Marcel Riesz operator of order (α, β) on the function f by

$$U^{(\alpha,\beta)}(f) = K^B_{\alpha,\beta} * f, \tag{16}$$

where $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{S}$. In this paper, we aim to study the convolution of $K^B_{\alpha,\beta}$ and obtain the operator $E^{(\alpha,\beta)} = [U^{(\alpha,\beta)}]^{-1}$ such that if $U^{(\alpha,\beta)}(f) = \varphi$, then $E^{(\alpha,\beta)}\varphi = f$.

Before we proceed to our main theorem, the following definitions and some concepts require some clarifications.

2. Preliminaries

DEFINITION 2.1 *Let $x = (x_1, x_2, \ldots, x_n)$ be a point in the n-dimensional Euclidean space $\mathbb{R}^n$. Let*

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$$

be the non-degenerated quadratic form, where $p + q = n$ is the dimension of $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : u > 0$ and $x_i > 0,\ (i = 1 = 1, 2, \ldots, p)\}$ be the interior of forward cone and let $\bar{\Gamma}_+$ denote its closure. For any complex number α, we define

$$R^B_\alpha(x) = \begin{cases} \dfrac{u^{(\alpha-2|\nu|-n)/2}}{K_n^{|\nu|}(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{17}$$

where

$$K_n^{|\nu|}(\alpha) = \frac{\pi^{(n-1+2|\nu|)/2}\Gamma((2+\alpha-n-2|\nu|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p-2|\nu|)/2)\Gamma((p-\alpha)/2)}, \tag{18}$$

$2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$, $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$, *see* [4,17,25].

The function $R_\alpha^B(x)$ is called the Bessel ultra-hyperbolic kernel and was introduced by Aguirre [2]. *It is well known that $R_\alpha^B(x)$ is an ordinary function if $Re(\alpha - 2|\nu|) \geq n$ and is a distribution of $(\alpha - 2|\nu|)$ if $Re(\alpha - 2|\nu|) < n$. Let $suppR_\alpha^B(x)$ denote the support of $R_\alpha^B(x)$ and suppose that $suppR_\alpha^B(x) \subset \bar{\Gamma}_+$ (i.e. $suppR_\alpha^B(x)$ is compact).*

By letting $\alpha = 2k$ in (17) and (18), we obtain

$$R_{2k}^B(x) = \frac{u^{(2k-n-2|\nu|)/2}}{K_n^{|\nu|}(2k)}, \tag{19}$$

where

$$K_n^{|\nu|}(2k) = \frac{\pi^{(n-1+2|\nu|)/2}\Gamma((2+2k-n-2|\nu|)/2)\Gamma((1-2k)/2)\Gamma(2k)}{\Gamma((2+2k-p-2|\nu|)/2)\Gamma((p-2k)/2)}. \tag{20}$$

By putting $|\nu| = 0$ in (17) and (18), formulae (17) and (18) reduce to

$$R_\alpha(x) = \begin{cases} \dfrac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{21}$$

and

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((\alpha-n)/2+1)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((\alpha-p)/2+1)\Gamma((p-\alpha)/2)}. \tag{22}$$

The function $R_\alpha(x)$ is called *the ultra-hyperbolic kernel of Marcel Riesz* and was introduced by Nozaki [15]. It is well known that $R_\alpha(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is a distribution of α if $\mathrm{Re}(\alpha) < n$. Suppose that $\mathrm{supp}R_\alpha(x) \subset \bar{\Gamma}_+$ (i.e. $\mathrm{supp}R_\alpha(x)$ is compact).

By putting $p = 1$ in $R_{2k}(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + \tfrac{1}{2}), \tag{23}$$

we obtain

$$I_\alpha^H(x) = \frac{v^{(\alpha-n)/2}}{H_n(\alpha)}, \tag{24}$$

and $v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1}\Gamma\left(\frac{\alpha+2-n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{25}$$

The function $I_\alpha^H(x)$ is called *the hyperbolic kernel of Marcel Riesz*.

Definition 2.2 *Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$. The Bessel elliptic kernel of Marcel Riesz is defined by*

$$S_\alpha^B(x) = \frac{\omega^{(\alpha-2|\nu|-n)/2}}{W_n^{|\nu|}(\alpha)}, \tag{26}$$

if α is a complex parameter, $\alpha \neq 2|\nu| - 2j, j = 1, 2, \ldots,$ and

$$W_n^{|\nu|}(\alpha) = \frac{\Gamma(\alpha/2)\prod_{i=1}^{n} 2^{\nu_i - 1/2}\Gamma(\nu_i + 1/2)}{2^{n+2|\nu|-2\alpha}\Gamma((n-\alpha)/2 + |\nu|)}. \tag{27}$$

The elliptic kernel of Marcel Riesz is defined by

$$S_\alpha(x) = \frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{28}$$

where

$$W_n(\alpha) = \frac{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}. \tag{29}$$

It is well known that $S_\alpha(x)$ is an ordinary function for $Re(\alpha) \geq n$ and is a distribution of α for $Re(\alpha) < n$.

Note that $n = p + q$. By putting $q = 0$ (i.e. $n = p$) in (21) and (22), we can reduce $u^{(\alpha-n)/2}$ to $\omega^{(\alpha-p)/2}$, where $\omega = x_1^2 + x_2^2 + \cdots + x_p^2$, and reduce $K_n(\alpha)$ to

$$K_p(\alpha) = \frac{\pi^{(p-1)/2}\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((p-\alpha)/2)}.$$

Using Legendre's duplication formula,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + \tfrac{1}{2}) \tag{30}$$

and

$$\Gamma(\tfrac{1}{2} + z)\Gamma(\tfrac{1}{2} - z) = \pi \sec(\pi z), \tag{31}$$

we obtain

$$K_p(\alpha) = \tfrac{1}{2} \sec(\alpha\pi/2)\, W_p(\alpha). \tag{32}$$

Thus, for $q = 0$, we have

$$R_\alpha(x) = \frac{u^{(\alpha-p)/2}}{K_p(\alpha)} = 2\cos\left(\frac{\alpha\pi}{2}\right)\frac{u^{(\alpha-p)/2}}{W_p(\alpha)} = 2\cos\left(\frac{\alpha\pi}{2}\right)S_\alpha(x). \tag{33}$$

In addition, if $\alpha = 2k$ for some non-negative integer k, then

$$R_{2k}(x) = 2(-1)^k S_{2k}(x). \tag{34}$$

The proofs of Lemmas 2.3 and 2.4 are given in [23].

Lemma 2.3 *The function $K_{\alpha,\beta}(x)$ has the following properties*:

(i) $K_{0,0}(x) = \delta(x)$;
(ii) $K_{-2k,-2k}(x) = (-1)^k \diamondsuit^k \delta(x)$;

(iii) $\diamondsuit^k(K_{\alpha,\beta}(x)) = (-1)^k K_{\alpha-2k,\beta-2k}(x)$;
(iv) $\diamondsuit^k(K_{2k,2k}(x)) = (-1)^k \delta(x)$; *and*
(v) $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \diamondsuit^k K_{\alpha,\beta}(x)$.

LEMMA 2.4 (*The convolutions of* $K_{\alpha,\beta}(x)$)

(i) *If p is odd, then*

$$K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'} + A_{\beta,\beta'}) * S_{\alpha+\alpha'}, \tag{35}$$

where R_β *and* S_α *are defined by* (21) *and* (28), *respectively.* $A_{\beta,\beta'}$ *is defined by*

$$A_{\beta,\beta'} = -\frac{\mathrm{i}}{2}\frac{\sin(\beta\pi/2)\sin(\beta'\pi/2)}{\sin((\beta+\beta')\pi/2)}[H^+_{\beta+\beta'} - H^-_{\beta+\beta'}] \tag{36}$$

and

$$H^\pm_\beta = H_\beta(P \pm \mathrm{i}0, n) \tag{37}$$

is defined by (8).

(ii) *If p is even, then*

$$K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = B_{\beta,\beta'} R_{\beta+\beta'} * S_{\alpha+\alpha'}, \tag{38}$$

where

$$B_{\beta,\beta'} = \frac{\cos(\beta\pi/2)\cos(\beta'\pi/2)}{\cos((\beta+\beta')\pi/2)}. \tag{39}$$

LEMMA 2.5 *If* $|\nu| \neq 0$, *then*

(i) $S^B_\alpha(x) = g_{\alpha,n,|\nu|} S_{\alpha-2|\nu|}(x)$, *where*

$$g_{\alpha,n,|\nu|} = \frac{\pi^{n/2} 2^{n-\alpha}\Gamma(\alpha/2 - |\nu|)}{\Gamma(\alpha/2)\prod_{i=1}^n 2^{\nu_i - 1/2}\Gamma(\nu_i + 1/2)}, \tag{40}$$

$S^B_\alpha(x)$ *and* $S_{\alpha-2|\nu|}(x)$ *are defined by* (26) *and* (28), *respectively.*

(ii) $R^B_\alpha(x) = h_{\alpha,p,|\nu|} R_{\alpha-2|\nu|}(x)$, *where*

$$h_{\alpha,p,|\nu|} = \frac{\Gamma((1-\alpha)/2 + |\nu|)\Gamma(\alpha - 2|\nu|)\Gamma((p-\alpha)/2)}{\pi^{|\nu|}\Gamma((p-\alpha)/2 + |\nu|)\Gamma((1-\alpha)/2)\Gamma(\alpha)}, \tag{41}$$

$R^B_\alpha(x)$ *and* $R_{\alpha-2|\nu|}(x)$ *are defined by* (17) *and* (21), *respectively.*

(iii) $K^B_{\alpha,\beta}(x) = g_{\alpha,n,|\nu|} h_{\beta,p,|\mu|} K_{\alpha-2|\nu|,\beta-2|\mu|}(x)$, *where* $K^B_{\alpha,\beta}(x)$, $K_{\alpha-2|\nu|,\beta-2|\mu|}(x)$, $g_{\alpha,n,|\nu|}$, *and* $h_{\beta,p,|\mu|}$ *are defined by* (15), (12), (40), *and* (41), *respectively.*

Proof We get (*i*) and (*ii*) by computing directly from the definition of $S^B_\alpha(x)$ and $S_{\alpha-2|\nu|}(x)$ and $R^B_\alpha(x)$ and $R_{\alpha-2|\nu|}(x)$. For (*iii*), by formulae (12) and (15) and (*i*) and (*ii*), we obtain

$$\begin{aligned} K^B_{\alpha,\beta}(x) &= S^B_\alpha(x) * R^B_\beta(x) = g_{\alpha,n,|\nu|} S_{\alpha-|2\nu|}(x) * h_{\beta,p,|\mu|} R_{\beta-|2\mu|}(x) \\ &= g_{\alpha,n,|\nu|} h_{\beta,p,|\mu|} K_{\alpha-2|\nu|,\beta-2|\mu|}(x). \end{aligned}$$

■

LEMMA 2.6 The convolutions of $K^B_{\alpha,\beta}(x)$

(i) *If p is odd, then*

$$K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x) = g_{\alpha,n,|\nu|}g_{\alpha',n,|\nu'|}h_{\beta,p,|\mu|}h_{\beta',p,|\mu'|} \times [(R_{\beta+\beta'-2(|\mu|+|\mu'|)} + A_{\beta-2|\mu|,\beta'-2|\mu'|}) * S_{\alpha+\alpha'-2(|\nu|+|\nu'|)}], \tag{42}$$

where R_β, S_α, *and* $A_{\beta,\beta'}$ *are defined by* (21), (28), *and* (36), *respectively.*

(ii) *If p is even, then*

$$K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x) = g_{\alpha,n,|\nu|}g_{\alpha',n,|\nu'|}h_{\beta,p,|\mu|}h_{\beta',p,|\mu'|}B_{\beta-2|\mu|,\beta'-2|\mu'|}K_{\beta+\beta'-2(|\mu|+|\mu'|)}, \tag{43}$$

where $B_{\beta-2|\mu|,\beta'-2|\mu'|}$ *is defined by* (39).

The proof of this lemma can be easily obtained from Lemmas 2.5(iii) and 2.4 and [23].

3. The convolution $K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x)$ when $\alpha' = -\alpha, \beta' = -\beta$

Now, we consider the property of $K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x)$ when $\alpha' = -\alpha$ and $\beta' = -\beta$.

From (42) and (43), we know that the following properties are valid:

(1) If p is odd and q is even, then

$$K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x) = g_{\alpha,n,|\nu|}g_{\alpha',n,|\nu'|}h_{\beta,p,|\mu|}h_{\beta',p,|\mu'|} \times [(R_{\beta+\beta'-2(|\mu|+|\mu'|)} + A_{\beta-2|\mu|,\beta'-2|\mu'|}) * S_{\alpha+\alpha'-2(|\nu|+|\nu'|)}], \tag{44}$$

where R_β, S_α, and $A_{\beta-2|\mu|,\beta'-2|\mu'|}$ are defined by (21), (28), and (36), respectively.

(2) If both p and q are odd, then

$$K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x) = g_{\alpha,n,|\nu|}g_{\alpha',n,|\nu'|}h_{\beta,p,|\mu|}h_{\beta',p,|\mu'|} \times [(R_{\beta+\beta'-2(|\mu|+|\mu'|)} + A_{\beta-2|\mu|,\beta'-2|\mu'|}) * S_{\alpha+\alpha'-2(|\nu|+|\nu'|)}]. \tag{45}$$

(3) If p is even and q is odd, then

$$K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x) = g_{\alpha,n,|\nu|}g_{\alpha',n,|\nu'|}h_{\beta,p,|\mu|}h_{\beta',p,|\mu'|} \times \left[\frac{\cos((\beta-2|\mu|)\pi/2)\cdot\cos((\beta'-2|\mu'|)\pi/2)}{\cos((\beta+\beta'-2(|\mu|+|\mu'|))\pi/2)}K_{\beta+\beta'-2(|\mu|+|\mu'|)}\right]. \tag{46}$$

(4) If both p and q are even, then

$$K^B_{\alpha,\beta}(x) * K^B_{\alpha',\beta'}(x) = g_{\alpha,n,|\nu|}g_{\alpha',n,|\nu'|}h_{\beta,p,|\mu|}h_{\beta',p,|\mu'|} \times \left[\frac{\cos((\beta-2|\mu|)\pi/2)\cdot\cos((\beta'-2|\mu'|)\pi/2)}{\cos((\beta+\beta'-2(|\mu|+|\mu'|))\pi/2)}K_{\beta+\beta'-2(|\mu|+|\mu'|)}\right]. \tag{47}$$

Moreover, it follows from (36) that

$$\begin{aligned} A_{\beta-2|\mu|,-(\beta-2|\mu|)} &= \lim_{\beta'-2|\mu'|\to-(\beta-2|\mu|)} A_{\beta-2|\mu|,\beta'-2|\mu'|} \\ &= -\frac{\mathrm{i}}{2}\lim_{\gamma\to 0}\frac{\sin((\beta-2|\mu|)\pi/2)\sin((\gamma-(\beta-2|\mu|))\pi/2)}{\sin(\gamma\pi/2)}[H_\gamma^+ - H_\gamma^-] \\ &= -\frac{\mathrm{i}}{2}\lim_{\gamma\to 0}\frac{\sin((\beta-2|\mu|)\pi/2)\sin((\gamma-(\beta-2|\mu|))\pi/2)}{\sin(\gamma\pi/2)}\cdot\lim_{\gamma\to 0}[H_\gamma^+ - H_\gamma^-], \end{aligned} \tag{48}$$

where $\gamma = \beta + \beta' - 2(|\mu| + |\mu'|)$.

On the other hand, using (37) and (8), we have

$$\begin{aligned} \lim_{\gamma\to 0}[H_\gamma^+ - H_\gamma^-] &= \frac{\Gamma(n/2)}{\pi^{n/2}}\left[\lim_{\gamma\to 0}\mathrm{e}^{-\gamma\pi\mathrm{i}/2}\,\mathrm{e}^{q\pi\mathrm{i}/2}\frac{(P+\mathrm{i}0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)}\right. \\ &\quad \left. - \lim_{\gamma\to 0}\mathrm{e}^{\gamma\pi\mathrm{i}/2}\,\mathrm{e}^{-q\pi\mathrm{i}/2}\frac{(P-\mathrm{i}0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)}\right] \\ &= \frac{\Gamma(n/2)}{\pi^{n/2}}\left[\lim_{\gamma\to 0}\mathrm{e}^{-\gamma\pi\mathrm{i}/2}\,\mathrm{e}^{q\pi\mathrm{i}/2}\cdot\frac{\operatorname*{Res}_{\beta=-n/2}(P+\mathrm{i}0)^{\beta}}{\operatorname*{Res}_{\beta=-n/2}\Gamma(\beta+n/2)}\right. \\ &\quad \left. - \lim_{\gamma\to 0}\mathrm{e}^{\gamma\pi\mathrm{i}/2}\,\mathrm{e}^{-q\pi\mathrm{i}/2}\cdot\frac{\operatorname*{Res}_{\beta=-n/2}(P-\mathrm{i}0)^{\beta}}{\operatorname*{Res}_{\beta=-n/2}\Gamma(\beta+n/2)}\right]. \end{aligned} \tag{49}$$

Now, taking n as an odd integer, we obtain

$$\operatorname*{Res}_{\lambda=-n/2-k}(P\pm\mathrm{i}0)^{\lambda} = \frac{\mathrm{e}^{\pm q\pi\mathrm{i}/2}\pi^{n/2}}{2^{2k}k!\Gamma(n/2+k)}\Box^k\delta(x), \tag{50}$$

where $\Box^k$ is defined by (1), $p + q = n$, and k is non-negative integer; see [21,22]. If both p and q are even, then

$$\operatorname*{Res}_{\lambda=-n/2-k}(P\pm\mathrm{i}0)^{\lambda} = \frac{\mathrm{e}^{\pm q\pi\mathrm{i}/2}\pi^{n/2}}{2^{2k}k!\Gamma(n/2+k)}\Box^k\delta(x). \tag{51}$$

Nevertheless, if both p and q are odd, then

$$\operatorname*{Res}_{\lambda=-n/2-k}(P\pm\mathrm{i}0)^{\lambda} = 0. \tag{52}$$

Therefore, we have

$$\begin{aligned} \lim_{\gamma\to 0}[H_\gamma^+ - H_\gamma^-] &= \frac{\Gamma(n/2)}{\pi^{n/2}}\cdot\frac{\pi^{n/2}}{\Gamma(n/2)}\left[\lim_{\gamma\to 0}\mathrm{e}^{-\gamma\pi\mathrm{i}/2} - \lim_{\gamma\to 0}\mathrm{e}^{\gamma\pi\mathrm{i}/2}\right]\delta(x) \\ &= \lim_{\gamma\to 0}\left[-2\mathrm{i}\sin\left(\frac{\gamma\pi}{2}\right)\right]\delta(x). \end{aligned} \tag{53}$$

From (49) and (52), we have

$$\lim_{\gamma\to 0}[H_\gamma^+ - H_\gamma^-] = 0 \tag{54}$$

if both p and q are odd (n even).

By applying (53) and (54) in (48), we have

$$\begin{aligned}
A_{\beta-2|\mu|,-\beta+2|\mu|} &= -\frac{\mathrm{i}}{2}\lim_{\gamma\to 0}\frac{\sin((\beta-2|\mu|)\pi/2)\sin((\gamma-(\beta-2|\mu|))\pi/2)}{\sin(\gamma\pi/2)} \\
&\quad\times \lim_{\gamma\to 0}\left[-2\mathrm{i}\sin\left(\frac{\gamma\pi}{2}\right)\right]\delta(x) \\
&= \sin^2\left(\frac{(\beta-2|\mu|)\pi}{2}\right)\delta(x) \\
&= \sin^2\left(\frac{\beta\pi}{2}\right)\delta(x) \qquad (55)
\end{aligned}$$

if p is odd and q is even, and

$$A_{\beta-2|\mu|,-\beta+2|\mu|} = 0 \qquad (56)$$

if both p and q are odd.

From (44) to (47) and using Lemmas 2.5 and 2.6 and formulae (55) and (56), if p is odd and q is even, then we obtain

$$\begin{aligned}
K^B_{\alpha,\beta}(x) * K^B_{-\alpha,-\beta}(x) &= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}(R_0 + A_{\beta-2|\mu|,-\beta+2|\mu|}) * S_0 \\
&= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\left[\delta(x) + \sin^2\left(\frac{\beta\pi}{2}\right)\delta(x)\right] * \delta(x) \\
&= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\left[1 + \sin^2\left(\frac{\beta\pi}{2}\right)\right]\delta(x). \qquad (57)
\end{aligned}$$

If both p and q are odd, then

$$\begin{aligned}
K^B_{\alpha,\beta}(x) * K^B_{-\alpha,-\beta}(x) &= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}(R_0 + A_{\beta-2|\mu|,-\beta+2|\mu|}) * S_0 \\
&= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}(R_0 * S_0) \\
&= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}K_{0,0} \\
&= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\delta(x). \qquad (58)
\end{aligned}$$

If p is even and q is odd, then

$$\begin{aligned}
&K^B_{\alpha,\beta}(x) * K^B_{-\alpha,-\beta}(x) \\
&\quad= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\frac{\cos((\beta-2|\mu|)\pi/2)\cos((-\beta+2|\mu|)\pi/2)}{\cos((\beta-\beta-2|\mu|+2|\mu|)\pi/2)}K_{0,0} \\
&\quad= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\cos^2\left(\frac{(\beta-2|\mu|)\pi}{2}\right)\delta(x) \\
&\quad= g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\cos^2\left(\frac{\beta\pi}{2}\right)\delta(x). \qquad (59)
\end{aligned}$$

Finally, if both p and q are even, then

$$
\begin{aligned}
&K^B_{\alpha,\beta}(x) * K^B_{-\alpha,-\beta}(x) \\
&\quad = g_{\alpha,n,|\nu|} g_{-\alpha,n,|\nu|} h_{\beta,p,|\mu|} h_{-\beta,p,|\mu|} \frac{\cos((\beta - 2|\mu|)\pi/2)\cos((-\beta + 2|\mu|)\pi/2)}{\cos((\beta - \beta - 2|\mu| + 2|\mu|)\pi/2)} K_{0,0} \\
&\quad = g_{\alpha,n,|\nu|} g_{-\alpha,n,|\nu|} h_{\beta,p,|\mu|} h_{-\beta,p,|\mu|} \cos^2\left(\frac{(\beta - 2|\mu|)\pi}{2}\right)\delta(x) \\
&\quad = g_{\alpha,n,|\nu|} g_{-\alpha,n,|\nu|} h_{\beta,p,|\mu|} h_{-\beta,p,|\mu|} \cos^2(\beta\pi/2)\,\delta(x). \qquad (60)
\end{aligned}
$$

4. Main results

Let $U^{(\alpha,\beta)}(f)$ be the Bessel diamond Marcel Riesz operator of order (α,β) on the function f, which is defined by

$$U^{(\alpha,\beta)}(f) = K^B_{\alpha,\beta} * f, \qquad (61)$$

where $K^B_{\alpha,\beta}$ is defined by (15), $\alpha,\beta \in \mathbb{C}$, and $f \in \mathcal{S}$.

Recall that our objective is to obtain the operator $E^{(\alpha,\beta)} = [U^{(\alpha,\beta)}]^{-1}$ such that if $U^{(\alpha,\beta)}(f) = \varphi$, then $E^{(\alpha,\beta)}\varphi = f$ for all $\alpha,\beta \in \mathbb{C}$.

We are now ready to state our main theorem.

THEOREM 4.1 *If $U^{(\alpha,\beta)}(f) = \varphi$ (where $U^{(\alpha,\beta)}(f)$ is defined by* (61) *and $f \in \mathcal{S}$), then $E^{(\alpha,\beta)}\varphi = f$ such that*

$$
\begin{aligned}
&E^{(\alpha,\beta)} = [U^{(\alpha,\beta)}]^{-1} \\
&\quad = \begin{cases}
\dfrac{1}{g_{\alpha,n,|\nu|} g_{-\alpha,n,|\nu|} h_{\beta,p,|\mu|} h_{-\beta,p,|\mu|}}\left[1 + \sin^2\left(\dfrac{\beta\pi}{2}\right)\right]^{-1} K^B_{-\alpha,-\beta}, & \text{if } p \text{ is odd and } q \text{ is even;} \\
\dfrac{1}{g_{\alpha,n,|\nu|} g_{-\alpha,n,|\nu|} h_{\beta,p,|\mu|} h_{-\beta,p,|\mu|}} K^B_{-\alpha,-\beta}, & \text{if both } p \text{ and } q \text{ are odd;} \\
\dfrac{1}{g_{\alpha,n,|\nu|} g_{-\alpha,n,|\nu|} h_{\beta,p,|\mu|} h_{-\beta,p,|\mu|}} \sec^2\left(\dfrac{\beta\pi}{2}\right) K^B_{-\alpha,-\beta}, & \text{if } p \text{ is even with } \dfrac{\beta}{2} \neq 2s+1
\end{cases}
\end{aligned}
$$

for any non-negative integer s.

Proof By (61), we have

$$U^{(\alpha,\beta)}(f) = K^B_{\alpha,\beta} * f = \varphi,$$

where $K^B_{\alpha,\beta}$ is defined by (15), $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{S}$. If p is odd and q is even, then, in view of (57), we obtain

$$\begin{aligned}
&\frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}}\left[1+\sin^2\left(\frac{\beta\pi}{2}\right)\right]^{-1} K^B_{-\alpha,-\beta} * (K^B_{\alpha,\beta} * f) \\
&\quad = \frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}}\left[1+\sin^2\left(\frac{\beta\pi}{2}\right)\right]^{-1} (K^B_{-\alpha,-\beta} * K^B_{\alpha,\beta}) * f \\
&= \frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}}\left[1+\sin^2\left(\frac{\beta\pi}{2}\right)\right]^{-1} \\
&\quad \times \left\{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\left[1+\sin^2\left(\frac{\beta\pi}{2}\right)\right]\delta(x)\right\} * f \\
&= \delta * f = f.
\end{aligned}$$

Hence,

$$\frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}}\left[1+\sin^2\left(\frac{\beta\pi}{2}\right)\right]^{-1} K^B_{-\alpha,-\beta} = [U^{(\alpha,\beta)}]^{-1} = (K^B_{\alpha,\beta})^{-1} \tag{62}$$

for all $\alpha, \beta \in \mathbb{C}$.

Similarly, if both p and q are odd, then by (58), we obtain

$$\begin{aligned}
&\frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} K^B_{-\alpha,-\beta} * (K^B_{\alpha,\beta} * f) \\
&\quad = \frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} (K^B_{-\alpha,-\beta} * K^B_{\alpha,\beta}) * f \\
&\quad = \frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\delta(x) * f \\
&\quad = \delta * f = f.
\end{aligned}$$

Hence,

$$\frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} K^B_{-\alpha,-\beta} = [U^{(\alpha,\beta)}]^{-1} = (K^B_{\alpha,\beta})^{-1} \tag{63}$$

for all $\alpha \in \mathbb{C}$.

Finally, if p is even, then by (59) and (60), we have

$$\begin{aligned}
&\frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} \sec^2\left(\frac{\beta\pi}{2}\right) K^B_{-\alpha,-\beta} * (K^B_{\alpha,\beta} * f) \\
&\quad = \frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} \sec^2\left(\frac{\beta\pi}{2}\right) (K^B_{-\alpha,-\beta} * K^B_{\alpha,\beta}) * f \\
&\quad = \frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}} \sec^2\left(\frac{\beta\pi}{2}\right) \\
&\qquad \times \left\{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}\cos^2\left(\frac{\beta\pi}{2}\right)\delta(x)\right\} * f \\
&\quad = \delta * f = f,
\end{aligned}$$

provided that $\beta/2 \neq 2s+1$ for any non-negative integer s.

Hence,

$$\frac{1}{g_{\alpha,n,|\nu|}g_{-\alpha,n,|\nu|}h_{\beta,p,|\mu|}h_{-\beta,p,|\mu|}}\sec^2\left(\frac{\beta\pi}{2}\right)K^B_{-\alpha,-\beta} = [U^{(\alpha,\beta)}]^{-1} = (K^B_{\alpha,\beta})^{-1} \tag{64}$$

for all $\alpha, \beta \in \mathbb{C}$ with $\beta/2 \neq 2s+1$ for any non-negative integer s.

In conclusion, formulae (62), (63), and (64) are the desired results, and this completes the proof. ■

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ON THE SOLUTION OF THE *n*-DIMENSIONAL DIAMOND KLEIN-GORDON OPERATOR AND ITS CONVOLUTION

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Abstract

In this article, we introduce the diamond Klein-Gordon operator iterated k times, which is defined by

$$(\Diamond + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k,$$

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where $p + q = n$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $m \geq 0$ and non-negative integers k. Our aim is study the fundamental solution of the operator $(\Diamond + m^2)^k$, to which we will refer as the diamond Klein-Gordon kernel. Moreover, we will study the convolution of this kernel.

1. Introduction

The n-dimensional ultra-hyperbolic operator ^{k} iterated k times is defined by

$$^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$, and k is a non-negative integer. We consider the linear differential equation of the form

$$^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions, and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gelfand and Shilov [1] have first introduced the fundamental solution of the operator (1.1), which was initially complicated. Later, Trione [2] has shown that the generalized function $R_{2k}^H(x)$ defined by (2.1) with $\alpha = 2k$ is the unique fundamental solution of (1.2) and Tellez [3] has also proved that $R_{2k}^H(x)$ exists only when $n = p + q$ with odd p.

Kananthai [4] has first introduced the operator $\Diamond^k$ called the *diamond operator iterated k times*, which is defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \tag{1.3}$$

where $n = p + q$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, ..., x_n)$ and non-negative integers *k*. The operator $\Diamond^k$ can be expressed as

$$\Diamond^k = \Delta^k \; {}^k = \; {}^k \Delta^k, \tag{1.4}$$

where ^{k} is defined by (1.1), and

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k \tag{1.5}$$

is the Laplace operator iterated *k* times. On finding the fundamental solution of the operator $\Diamond^k$ in (1.4), Kananthai used the convolution of functions which are fundamental solutions of the operators ^{k} and Δ^k. He found that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is the fundamental solution of the operator $\Diamond^k$, that is,

$$\Diamond^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta, \tag{1.6}$$

where $R_{2k}^H(x)$ and $R_{2k}^e(x)$ are defined by (2.1) and (2.6), respectively $(\text{with } \alpha = 2k)$, and δ is the Dirac delta distribution. The fundamental solution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is called the *diamond kernel* of Marcel Riesz. Later in 1998, he has also studied the properties of the distribution $e^{\alpha x}\Diamond^k\delta$ and its application for solving the convolution equation

$$e^{\alpha x}\Diamond^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Diamond^r \delta, \tag{1.7}$$

where C_r is a constant. Recently, Nonlaopon gave some generalizations of his paper [5]; see [6] for more details.

Next, Kananthai [7, 8] has studied the Fourier transform and convolutions of the diamond kernel of Marcel Riesz. In [9], he has also studied the linear equation

$$\Diamond^k u(x) = f(x). \tag{1.8}$$

This equation is a generalization of the ultra-hyperbolic equation, which can be applied to the wave equation. It is found that the solution of an equation (1.8) is $u(x) = (-1)^k M_{2k,\,2k}(x) * f(x),$ where

$$M_{2k,\,2k}(x) = R_{2k}^{e}(x) * R_{2k}^{H}(x). \tag{1.9}$$

Later, Sritanratana and Kananthai [10] have studied the nonlinear diamond operator, which has some applications to the *n*-dimensional wave equation. Tellez and Kananthai [11] have studied the convolution product of the distributional families related to this operator.

In 1988, Trione [12] has studied the fundamental solution of the ultra-hyperbolic Klein-Gordon operator iterated *k* times, which is defined by

$$(\quad + m^2)^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right)^k. \tag{1.10}$$

The study has shown that the fundamental solution of the operator $(\quad + m^2)^k$ is given by

$$W_{2k}(x,\, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r!\,\Gamma(k)} (m^2)^r R_{2k+2r}^{H}(x), \tag{1.11}$$

where $R_{2k+2r}^{H}(x)$ is defined by (2.1) with $\alpha = 2k + 2r.$ Next, Tellez [13] has studied the convolution product of $W_\alpha(x,\, m) * W_\beta(x,\, m),$ where α and β are any complex numbers. In addition, Trione [14] has studied the integral representation of the kernel $W_\alpha(x,\, m)$ and the fundamental $(P \pm i0)^\lambda$ -ultra-hyperbolic solution of the Klein-Gordon operator iterated *k* times; see [15] for more details. Finally, Tariboon and Kananthai [16] have studied the fundamental solution or Green function of the operator $(\oplus + m^2)^k,$ which is defined by

$$(\oplus + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4 + m^2\right]^k, \tag{1.12}$$

for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $m \geq 0$, k is a non-negative integer and $p + q = n$ is the dimension of $\mathbb{R}^n$. In fact, the operator $(\oplus + m^2)^k$ can be seen to be related to the ultra-hyperbolic Klein-Gordon operator $(\quad + m^2)^k$ and the Helmholrz operator $(\Delta + m^2)$.

In this article, we introduce the diamond Klein-Gordon operator iterated k times, which is defined by

$$(\Diamond + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k, \tag{1.13}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $m \geq 0$ and non-negative integers k. Our aim is to study the fundamental solution of the operator $(\Diamond + m^2)^k$, to which we will refer as the diamond Klein-Gordon kernel. Moreover, the convolution of this kernel will be also studied at the end.

Before we proceed to our main results, the following definitions and concepts require clarifications.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point in the n-dimensional Euclidean space $\mathbb{R}^n$. Let

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$$

be the nondegenerated quadratic form associated to x, where $p + q = n$.

Then the interior of forward cone is defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0$ and $u > 0\}$, where $\overline{\Gamma}_+$ designated its closure. For any $\alpha \in \mathbb{C}$, we define

$$R_\alpha^H(x) = \begin{cases} \dfrac{u^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.1}$$

where

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}. \tag{2.2}$$

The function $R_\alpha^H(x)$ is called the *ultra-hyperbolic kernel* of Marcel Riesz and was first introduced by Nozaki [17]. It is well known that $R_\alpha^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$; and is a distribution of α otherwise. Let $\operatorname{supp} R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$, and suppose that $\operatorname{supp} R_\alpha^H(x) \subset \overline{\Gamma}_+$ (that is, $\operatorname{supp} R_\alpha^H(x)$ is compact). Putting $p = 1$ in $R_{2k}^H(x)$ and taking into account Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2), \tag{2.3}$$

we obtain

$$I_\alpha^H(x) = \frac{v^{(\alpha-n)/2}}{H_n(\alpha)}, \tag{2.4}$$

and $v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1}\Gamma((\alpha+2-n)/2)\Gamma(\alpha/2). \tag{2.5}$$

The function $I_\alpha^H(x)$ is called the *hyperbolic kernel* of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$, and let $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$. Then the elliptic kernel of Marcel Riesz is defined by

$$R_\alpha^e(x) = \frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{2.6}$$

where $\alpha \in \mathbb{C}$, and

$$W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}. \tag{2.7}$$

Note that $n = p + q$. By putting $q = 0$ (i.e., $n = p$) in (2.1) and (2.2), we can reduce $u^{(\alpha-n)/2}$ to $\omega_p^{(\alpha-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$. Moreover, $K_n(\alpha)$ can be reduced to

$$K_p(\alpha) = \frac{\pi^{(p-1)/2}\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((p-\alpha)/2)}.$$

Using the Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2), \tag{2.8}$$

and

$$\Gamma(1/2 + z)\Gamma(1/2 - z) = \pi \sec(\pi z), \tag{2.9}$$

we obtain

$$K_p(\alpha) = \frac{1}{2}\sec(\pi\alpha/2) W_p(\alpha). \tag{2.10}$$

Thus, if $q = 0$, then we have

$$R_\alpha^H(x) = \frac{u^{(\alpha-p)/2}}{K_p(\alpha)} = 2\cos(\pi\alpha/2)\frac{u^{(\alpha-p)/2}}{W_p(\alpha)} = 2\cos(\pi\alpha/2) R_\alpha^e(x). \tag{2.11}$$

In addition, if $\alpha = 2k$ for some non-negative integer *k*, then

$$R_{2k}^H(x) = 2(-1)^k R_{2k}^e(x). \tag{2.12}$$

The proofs of Lemma 2.1 and 2.2 are given in [4] and [8].

Lemma 2.1. *$R_\alpha^e(x)$ and $R_\alpha^H(x)$ are the tempered distributions.*

Lemma 2.2 (The convolution of tempered distributions)**.**

(a) *Let* $R_\alpha^e(x)$ *and* $R_\beta^e(x)$ *be elliptic kernels of Marcel Riesz, which are defined by* (2.6). *Then*

$$R_\alpha^e(x) * R_\beta^e(x) = R_{\alpha+\beta}^e(x), \tag{2.13}$$

for all complex numbers α, β.

(b) *Let* $R_\alpha^H(x)$ *and* $R_\beta^H(x)$ *be ultra-hyperbolic kernels of Marcel Riesz, which are defined by* (2.1). *Then*

$$R_\alpha^H(x) * R_\beta^H(x) = R_{\alpha+\beta}^H(x), \tag{2.14}$$

for all integers α, β *such that* α *or* β *is even*.

(c) *In particular, if* $R_\alpha^H(x) * R_\beta^H(x) = \delta$, *then* $R_\alpha^H(x)$ *is the unique inverse of* $R_\beta^H(x)$ $($*denoted by* $R_\beta^{H*-1}(x))$ *in the convolution algebra*.

From Trione [2], $R_{2k}^H(x)$ is the fundamental solution of the operator ^{k}, that is,

$$^{k}(R_{2k}^H(x)) = \delta. \tag{2.15}$$

Moreover, we obtain from [1] that $(-1)^k R_{2k}^e(x)$ is the fundamental solution of the operator Δ^k, that is,

$$\Delta^k((-1)^k R_{2k}^e(x)) = \delta. \tag{2.16}$$

It can be shown (see [2, 11]) that $R_{-2k}^H(x) = {}^{k}\delta$ and $R_{-2k}^e(x) = (-1)^k \Delta^k \delta$ for any non-negative integer k.

Before proving our main theorems, we will consider the convolution between $(-1)^k R_{2k}^e(x)$ and $R_{2k}^H(x)$ (defined by (2.1) with $\alpha = 2k$) for $k =$

$0, 1, 2, \ldots .$ For the case $2k \geq n,$ we find that $(-1)R^e_{2k}(x)$ and $R^H_{2k}(x)$ are analytic functions which are the ordinary functions, thus the convolution

$$(-1)^k R^e_{2k}(x) * R^H_{2k}(x) \tag{2.17}$$

exists. On the other hand, if $2k < n,$ then by Lemma 2.1 (with $\alpha = 2k$), we can see that $(-1)^k R^e_{2k}(x)$ and $R^H_{2k}(x)$ are tempered distributions.

Let K be a compact (i.e., closed and bounded) subset of $\overline{\Gamma}_+,$ where $\overline{\Gamma}_+$ is defined as in Definition 2.1. If we choose $\operatorname{supp} R^H_{2k}(x) = K,$ then $\operatorname{supp} R^H_{2k}(x)$ is compact. Hence the convolution

$$(-1)^k R^e_{2k}(x) * R^H_{2k}(x) \tag{2.18}$$

exists, and is a tempered distribution by Lemma 2.1.

For the proof of the following lemma, see [4] and [11].

Lemma 2.3. *The convolution* $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ *is the fundamental solution of the diamond operator iterated k times, that is,*

$$\Diamond^k ((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) = \delta. \tag{2.19}$$

It can be shown that $R^e_{-2k}(x) * R^H_{-2k}(x) = (-1)^k \Diamond^k \delta$ for any non-negative integer k.

Definition 2.3. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$. The function $T_\alpha(x, m)$ is defined by

$$T_\alpha(x, m) = \sum_{r=0}^{\infty} \binom{-\alpha/2}{r} (m^2)^r (-1)^{\alpha/2+r} R^e_{\alpha+2r}(x) * R^H_{\alpha+2r}(x), \tag{2.20}$$

where α is a complex parameter and $m \geq 0$. Here, $R^H_{\alpha+2r}(x)$ and $R^e_{\alpha+2r}(x)$ are defined by (2.1) and (2.6), respectively.

From the definition of $T_\alpha(x, m)$, by putting $\alpha = -2k$, we have

$$T_{-2k}(x, m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r (-1)^{-k+r} R^e_{2(-k+r)}(x) * R^H_{2(-k+r)}(x).$$

Since the operator $(\Diamond + m^2)^k$ defined by (1.13) is linearly continuous and has 1-1 mapping, it therefore possesses an inverse. By Lemma 2.3, we obtain

$$T_{-2k}(x, m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r \Diamond^{k-r} \delta = (\Diamond + m^2)^k \delta. \tag{2.21}$$

Substituting $k = 0$ in (2.21), we have $T_0(x, m) = \delta$. On the other hand, letting $\alpha = 2k$ in (2.20) yields

$$T_{2k}(x, m) = \binom{-k}{0} (m^2)^0 (-1)^{k+0} R^e_{2k+0}(x) * R^H_{2k+0}(x)$$

$$+ \sum_{r=1}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x). \tag{2.22}$$

Clearly, the second summand on the right-hand side on (2.22) vanishes when $m = 0$. Hence, we obtain

$$T_{2k}(x, 0) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x),$$

which is the fundamental solution of the diamond operator $\Diamond^k$.

3. Main Results

Theorem 3.1. *Consider the equation*

$$(\Diamond + m^2)^k u(x) = \delta, \tag{3.1}$$

where $(\Diamond + m^2)^k$ *is the diamond Klein-Gordon operator iterated k times, which is defined by*

$$(\Diamond + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right]^k, \tag{3.2}$$

with a non-negative integer k and the Dirac-delta distribution δ. *Then, we obtain*

$$T_{2k}(x, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x) \tag{3.3}$$

as the fundamental solution of the operator $(\Diamond + m^2)^k$, *where* $R^H_{2k}(x)$ *and* $R^e_{2k}(x)$ *are defined by* (2.1) *and* (2.6), *respectively*. *Moreover, we find that* $u(x) = T_{2k}(x, m)$ *is a tempered distribution*.

Proof. Since the operator $(\Diamond + m^2)^k$ is linearly continuous and has 1-1 mapping, it therefore possesses an inverse. Then by Lemma 2.3, we have

$$\begin{aligned} T_{2k}(x, m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2(k+r)}(x) * R^H_{2(k+r)}(x) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \Diamond^{-k-r} \delta \\ &= (\Diamond + m^2)^{-k} \delta. \end{aligned} \tag{3.4}$$

Applying the operator $(\Diamond + m^2)^k$ to both sides of (3.4), we obtain

$$(\Diamond + m^2)^k T_{2k}(x, m) = (\Diamond + m^2)^k (\Diamond + m^2)^{-k} \delta = (\Diamond + m^2)^0 \delta = \delta.$$

Thus,

$$T_{2k}(x, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x)$$

is a fundamental solution of the diamond Klein-Gordon operator iterated k

times. By Lemma 2.1, $R^e_{2k+2r}(x)$ and $R^H_{2k+2r}(x)$ are tempered distributions. Moreover, it follows from (2.18) and (2.20) that $T_{2k}(x, m)$ exists and is also a tempered distribution. This completes the proof. □

Theorem 3.2. *For* $0 < j < k$, *we have*

$$(\Diamond + m^2)^j T_{2k}(x, m) = T_{2k-2j}(x, m) \tag{3.5}$$

and for $k \leq l$,

$$(\Diamond + m^2)^l T_{2k}(x, m) = (\Diamond + m^2)^{l-k} \delta. \tag{3.6}$$

Proof. By Theorem 3.1, we readily know that $(\Diamond + m^2)^k T_{2k}(x, m) = \delta.$ From this, we can see easily that

$$(\Diamond + m^2)^{k-j}(\Diamond + m^2)^j T_{2k}(x, m) = \delta,$$

or equivalently,

$$(\Diamond + m^2)^{k-j}\delta * (\Diamond + m^2)^j T_{2k}(x, m) = \delta,$$

for $0 < j < k$. Convolving both sides by $T_{2k-2j}(x, m)$, we obtain

$$(\Diamond + m^2)^{k-j} T_{2k-2j}(x, m) * (\Diamond + m^2)^j T_{2k}(x, m) = T_{2k-2j}(x, m) * \delta,$$

or

$$\delta * (\Diamond + m^2)^j T_{2k}(x, m) = T_{2k-2j}(x, m),$$

by Theorem 3.1. It then follows that

$$(\Diamond + m^2)^j T_{2k}(x, m) = T_{2k-2j}(x, m) \text{ for } 0 < l < k$$

as required. Finally, for $k \leq l$, we obtain

$$\begin{aligned}(\Diamond + m^2)^l T_{2k}(x, m) &= (\Diamond + m^2)^{l-k}(\Diamond + m^2)^k T_{2k}(x, m)\\ &= (\Diamond + m^2)^{l-k}\delta\end{aligned}$$

by Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let* $f(x)$ *be a generalized function and* $u(x)$ *be an unknown generalized function. Then the equation*

$$(\Diamond + m^2)^k u(x) = f(x), \tag{3.7}$$

has the solution

$$u(x) = T_{2k}(x, m) * f(x), \tag{3.8}$$

where $T_{2k}(x, m)$ *is defined by* (3.3).

Proof. Convolving both sides of (3.7) by $T_{2k}(x, m)$ and applying Theorem 3.1, we obtain (3.8) as required. □

Theorem 3.4. *The type of solution of linear differential equation*

$$(\Diamond + m^2)^k u(x) = \sum_{j=0}^{l} c_j (\Diamond + m^2)^j \delta, \tag{3.9}$$

where $c_0, c_1, ..., c_l$ *are constants, depends on the relationship between the values of k and l as follows*:

(1) *If* $l < k$ *and* $l = 0,$ *then the solution of* (3.9) *is* $u(x) = c_0 T_{2k}(x, m),$ *which is the fundamental solution of the operator* $(\Diamond + m^2)^k.$ *Moreover, if* $2k \geq n,$ *then this solution is an ordinary function.*

(2) *If* $0 < l < k,$ *then the solution of* (3.9) *is*

$$u(x) = \sum_{j=1}^{l} c_j T_{2k-2j}(x, m),$$

which is an ordinary function if $2k - 2j \geq n;$ *and is a tempered distribution otherwise, for all* $j = 1, 2, ..., l.$

(3) *If* $l \geq k$ *and we suppose* $k \leq l \leq M$ *for some constant M, then* (3.9) *has the solution*

$$u(x) = \sum_{j=k}^{M} c_j (\Diamond + m^2)^{j-k} \delta,$$

which is only a singular distribution.

Proof. (1) For $l = 0,$ we have $(\Diamond + m^2)^k u(x) = c_0 \delta,$ and by Theorem 3.1, we obtain

$$u(x) = c_0 T_{2k}(x, m).$$

Now, $(-1)^k R^e_{2k+2r}(x)$ and $R^H_{2k+2r}(x)$ are the analytic functions for $2k + 2r \geq n.$ Furthermore, it follows from (2.17) and (2.20) that $T_{2k}(x, m)$ exists and is an analytic function. Hence, we can conclude that $T_{2k}(x, m)$ is the ordinary function whenever $2k + 2r \geq n.$

(2) Suppose that $0 < l < k$ and j runs from 1, we have

$$(\Diamond + m^2)^k u(x) = c_1(\Diamond + m^2)\delta + c_2(\Diamond + m^2)^2 \delta + \cdots + c_l(\Diamond + m^2)^l \delta.$$

Convolving both sides by $T_{2k}(x, m),$ we obtain

$$(\Diamond + m^2)^k T_{2k}(x, m) * u(x) = c_1(\Diamond + m^2) T_{2k}(x, m) + c_2(\Diamond + m^2)^2 T_{2k}(x, m)$$
$$+ \cdots + c_l(\Diamond + m^2)^l T_{2k}(x, m).$$

Hence, by Theorem 3.1 and Theorem 3.2, we have

$$u(x) = c_1 T_{2k-2}(x, m) + c_2 T_{2k-4}(x, m) + \cdots + c_l T_{2k-2l}(x, m)$$
$$= \sum_{j=1}^{l} c_j T_{2k-2j}(x, m).$$

Similarly, as in case (1), $u(x)$ is an ordinary function if $2k - 2j + 2r \geq n$ for all $r = 0, 1, \ldots,$ that is, $2k - 2j \geq n;$ otherwise, $u(x)$ is a tempered distribution.

(3) Suppose $k \leq l \leq M,$ for some constant M and j runs from k. Then

we have

$$(\Diamond + m^2)^k u(x) = c_k(\Diamond + m^2)^k \delta + c_{k+1}(\Diamond + m^2)^{k+1}\delta + \cdots + c_M(\Diamond + m^2)^M \delta.$$

Convolving both sides by $T_{2k}(x, m)$, we obtain

$$\begin{aligned}(\Diamond + m^2)^k T_{2k}(x, m) * u(x) = c_k(\Diamond + m^2)^k T_{2k}(x, m) \\ + c_{k+1}(\Diamond + m^2)^{k+1} T_{2k}(x, m) \\ + \cdots + c_M(\Diamond + m^2)^M T_{2k}(x, m).\end{aligned}$$

Again, by Theorem 3.1 and Theorem 3.2, it follows that

$$\begin{aligned}u(x) &= c_k\delta + c_{k+1}(\Diamond + m^2)\delta + c_{k+2}(\Diamond + m^2)^2\delta + \cdots + c_M(\Diamond + m^2)^{M-k}\delta \\ &= \sum_{j=k}^{M} c_j(\Diamond + m^2)^{j-k}\delta.\end{aligned}$$

Since $(\Diamond + m^2)^{j-k}\delta$ is a singular distribution, hence so is $u(x)$. This completes the proof. □

Theorem 3.5. *Let* $T_{2k}(x, m)$ *be the diamond Klein-Gordon kernel defined by* (3.3). *Then* $T_{2k}(x, m)$ *is a tempered distribution and can be expressed by*

$$T_{2k}(x, m) = T_{2k-2v}(x, m) * T_{2v}(x, m),$$

where v *is a non-negative integer and* $v < k$. *In particular, if we let* $l = k - v$ *and* $h = v$, *then we obtain*

$$T_{2l}(x, m) * T_{2h}(x, m) = T_{2l+2h}(x, m),$$

for all l, h *such that* $k = l + h$.

Proof. It follows from Theorem 3.1 that the function

$$T_{2k}(x, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x)$$

is a tempered distribution. Then for $v < k$, we have

$$T_{2k-2v}(x, m) * T_{2v}(x, m)$$

$$= \left[\sum_{r=0}^{\infty} \binom{-(k-v)}{r} (m^2)^r (-1)^{k-v+r} R^e_{2k-2v+2r}(x) * R^H_{2k-2v+2r}(x)\right]$$

$$* \left[\sum_{s=0}^{\infty} \binom{-v}{s} (m^2)^s (-1)^{v+s} R^e_{2v+2s}(x) * R^H_{2v+2s}(x)\right]$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (m^2)^{r+s} (-1)^{k+r+s} \binom{-(k-v)}{r}\binom{-v}{s} [(R^e_{2k-2v+2r}(x)$$

$$* R^H_{2k-2v+2r}(x)) * (R^e_{2v+2s}(x) * R^H_{2v+2s}(x))]. \tag{3.10}$$

Combining (3.10) with (2.13) and (2.14), this leads to

$$T_{2k-2v}(x, m) * T_{2v}(x, m)$$

$$= \sum_{n=0}^{\infty} (m^2)^n (-1)^{k+n} \left[\sum_{r=0}^{n} \binom{-(k-v)}{r}\binom{-v}{n-r}\right] R^e_{2k+2n}(x) * R^H_{2k+2n}(x). \tag{3.11}$$

Since

$$\sum_{r=0}^{n} \binom{-(k-v)}{r}\binom{-v}{n-r} = \binom{-k}{n}, \tag{3.12}$$

(3.11) becomes

$$T_{2k-2v}(x, m) * T_{2v}(x, m) = \sum_{n=0}^{\infty} (m^2)^n (-1)^{k+n} \binom{-k}{n} (R^e_{2k+2n}(x) * R^H_{2k+2n}(x)),$$

or equivalently,

$$T_{2k}(x, m) = T_{2k-2v}(x, m) * T_{2v}(x, m),$$

which proves the first part. Finally, by letting $l = k - v$ and $h = v$, we have

$$T_{2l+2h}(x, m) = T_{2l}(x, m) * T_{2h}(x)$$

as required. □

Theorem 3.6. *Let* $T_{2k}(x, m)$ *be the diamond Klein-Gordon kernel defined by* (3.3). *Then* $T_{2k}(x, m)$ *is an element of the space of convolution algebra. Moreover, there exists an inverse* $T_{2k}^{*-1}(x, m)$ *of* $T_{2k}(x, m)$ *such that*

$$T_{2k}(x, m) * T_{2k}^{*-1}(x, m) = T_{2k}^{*-1}(x, m) * T_{2k}(x, m) = \delta.$$

Proof. Since $T_{2k}(x, m)$ is a tempered distribution by Theorem 3.5, this means that the supports of $R_{2k}^{H}(x)$ and $R_{2k}^{e}(x)$ are compact. Thus, these supports are elements of the space of convolution algebra of distribution and so is $T_{2k}(x, m)$. By Lemma 2.3(c), there exists a unique inverse $T_{2k}^{*-1}(x, m)$ such that

$$T_{2k}(x, m) * T_{2k}^{*-1}(x, m) = T_{2k}^{*-1}(x, m) * T_{2k}(x, m) = \delta.$$

This completes the proof. □

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