



## รายงานวิจัยฉบับสมบูรณ์

โครงการ : วาไรตี้ย่อยวิภันัยในพีชคณิตทั่วไป

Fuzzy Subvarieties in Universal Algebra

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

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และสำนักงานกองทุนสนับสนุนการวิจัย

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# กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ในการทำงานวิจัยครั้งนี้

ศาสตราจารย์ฉวีวรรณ รัตนประเสริฐ และ Prof. Dr. Klaus Denecke นักวิจัยที่ปรึกษาให้กับโครงการนี้ที่ให้คำแนะนำ และให้ความรู้ต่าง ๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ และเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น รวมถึงผู้ช่วยเหลือในโครงการวิจัยในครั้งนี้

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**บทคัดย่อ:** ในรายงานวิจัยนี้ เราได้นิยามวาไรตี้ย่อยวิกษณัยของวาไรตี้  $V$  ที่กำหนดให้ และได้แสดงว่าแลตทิซของวาไรตี้ย่อยทั้งหมดของ  $V$  เป็นแลตทิซย่อยแบบสมบูรณ์ของแลตทิซของวาไรตี้ย่อยวิกษณัยของวาไรตี้  $V$  เราได้นิยามน้ำหนักของสมาชิกของวาไรตี้  $V$  และได้แสดงว่าแลตทิซของน้ำหนักของวาไรตี้ทั้งหมดของ  $V$  สมสัณฐาน (isomorphic) กับแลตทิซของวาไรตี้ย่อยวิกษณัยของวาไรตี้  $V$  ยิ่งกว่านั้นถ้า  $V$  เป็น  $M$ -โซลิดวาไรตี้ เราได้นิยาม  $M$ -โซลิดวาไรตี้ย่อยวิกษณัยของ  $V$  และได้พิสูจน์ว่าแลตทิซของ  $M$ -โซลิดวาไรตี้ย่อยวิกษณัยของ  $V$  เป็นแลตทิซย่อยแบบสมบูรณ์ของแลตทิซของวาไรตี้ย่อยวิกษณัยของวาไรตี้  $V$

**คำหลัก:** วาไรตี้ (Varieties), วาไรตี้ย่อยวิกษณัย (Fuzzy subvarieties),  $M$ -โซลิดวาไรตี้ ( $M$ -solid varieties),  $M$ -โซลิดวาไรตี้ย่อยวิกษณัย (Fuzzy  $M$ -solid subvarieties)

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**Abstract:** We define the concept of fuzzy subvarieties of a variety  $V$  and show that the lattice of all subvarieties of  $V$  can be embedded into the lattice of all fuzzy subvarieties of  $V$ . We consider the weights over a variety  $V$  and show that the lattice of all weights over  $V$  and the lattice of all fuzzy subvarieties of  $V$  are isomorphic. Moreover, we define the concept of fuzzy  $M$ -solid subvarieties of a given  $M$ -solid variety and show that the lattice of all fuzzy  $M$ -solid subvarieties is a complete sublattice of the lattice of all fuzzy subvarieties of the same  $M$ -solid variety.

**Keywords:** Varieties, Fuzzy subvarieties,  $M$ -solid varieties, Fuzzy  $M$ -solid subvarieties

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# Chapter 1

## Executive Summary

The concept of a fuzzy subset was established by L.A. Zadeh (in 1965) and was defined as a function from a nonempty set  $X$  to the real unit interval  $[0, 1]$ . Then fuzzification was applied to many algebraic structures, for example, to groups, to rings, to universal algebras, etc. The first application was fuzzy groups by Rosenfeld (in 1971) who inspired to many others. In 1991, V. Murali defined fuzzy subalgebras of a given algebra in the universal algebra sense, studied the lattice of all fuzzy subalgebras of a given algebra and showed that it forms a complete lattice. In 2002, T. Kuraoka and N. Y. Suzuki defined the notion of weights over a given (universal) algebra and showed that the lattice of all weights over an algebra forms a complete sublattice which is isomorphic to the lattice of all fuzzy subalgebras of the algebra. It was natural to apply fuzzification to classes of algebras, particularly in varieties. In this report, we define a notion of a fuzzy subvariety of a given variety in Universal algebra. In an analogous way of studying fuzzy subalgebras, we investigate the lattice of all fuzzy subvarieties of a given variety and prove that the lattice of all subvarieties of the variety can be embedded as a complete lattice into the lattice of all fuzzy subvarieties of the variety. We also define the notion of weights over a given variety and find that the lattice of all weights over the variety and the lattice of all fuzzy subvarieties of the variety are isomorphic as complete lattices. Moreover, we define the concept of a fuzzy  $M$ -solid subvariety of a given  $M$ -solid variety and show that the lattice of all fuzzy  $M$ -solid subvarieties is a complete sublattice of the lattice of all fuzzy subvarieties of the same  $M$ -solid variety.

# Chapter 2

## Main Results

We submitted a paper in the journal "Computers and Mathematics with Applications" (see Appendix A1). Let us summarize all results of this research in the following sections.

### 2.1 Lattices of Fuzzy subvarieties

A class  $V$  of algebras of type  $\tau$  is called a *variety* if it is closed under taking of homomorphic images (**H**), subalgebras (**S**) and direct products (**P**). Let  $K \subseteq \text{Alg}(\tau)$ . We denote by  $\mathbf{H}(K)$  the class of all homomorphic images of algebras in  $K$ , by  $\mathbf{S}(K)$  the class of all subalgebras of algebras in  $K$  and by  $\mathbf{P}(K)$  the class of all direct products of algebras in  $K$ . For a given variety of a same type  $\tau$  we define the notion of fuzzy subvarieties as following:

A mapping  $\mu : V \rightarrow [0, 1]$  is called a *fuzzy subvariety* of  $V$  if the following properties are satisfied:

- (i)  $\forall \mathcal{B} \in V \forall \mathcal{A} \in \mathbf{H}(\mathcal{B}), \mu(\mathcal{A}) \geq \mu(\mathcal{B})$ ,
- (ii)  $\forall \mathcal{B} \in V \forall \mathcal{A} \in \mathbf{S}(\mathcal{B}), \mu(\mathcal{A}) \geq \mu(\mathcal{B})$  and
- (iii)  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu(\mathcal{A}_j) \mid j \in J\}$ .

We denote by  $\text{FS}(V)$  the set of all fuzzy subvarieties of  $V$ . Then we have the following results:

- The lattice of all fuzzy subvarieties of  $V$  denoted by  $\mathcal{FS}(V) := (\text{FS}(V); \wedge, \vee)$  forms a complete lattice which has the least and the greatest elements, say  $\mathbf{0}, \mathbf{1}$ , respectively where  $\mathbf{0}(\mathcal{A}) = 0, \mathbf{1}(\mathcal{A}) = 1$  for all  $\mathcal{A} \in V$ .
- The lattice  $\mathcal{L}(V)$  of all subvarieties of  $V$  can be embedded into the lattice  $\mathcal{FS}(V)$  as complete lattices.

## 2.2 Fuzzy subvarieties and weights

Let  $K$  be a subclass of a given variety  $V$  of type  $\tau$ . We denote by  $\langle K \rangle_{var}$  the subvariety of  $V$  generated by  $K$ . We define weights in the analogous way of weights over  $V$  as following.

Let  $V$  be a variety of type  $\tau$  and  $Sub(V)$  be the class of all subvarieties of  $V$ . A mapping  $w : Sub(V) \rightarrow [0, 1]$  is called a *weight* over  $V$  if for any family  $\{K_i \mid i \in I\}$  of  $Sub(V)$ ,

$$w\left(\bigvee_{i \in I} K_i\right) = \inf_{i \in I} w(K_i).$$

We denote by  $W(V)$  the set of all weights over  $V$ . Then we have the following results:

- The lattices  $(FS(V), \leq)$  and  $(W(V), \leq)$  are isomorphic as complete lattices.

## 2.3 Fuzzy $M$ - solid subvarieties

In this section we assume that  $V$  is an  $M$ -solid variety of type  $\tau$  and then we give the notion of fuzzy  $M$ -solid subvarieties. First of all we start with the concept of hypersubstitutions introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991.

A mapping  $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$  which takes each  $n_i$ -ary operation symbol to an  $n_i$ -ary term is called a *hypersubstitution* of type  $\tau$ . Each hypersubstitution  $\sigma$  induces a map  $\hat{\sigma}$  on the set of all terms which is defined by

- (i)  $\hat{\sigma}[x] := x$  if  $x \in X$  is a variable,
- (ii)  $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$  for composite terms  
 $f_i(t_1, \dots, t_{n_i})$ .

On the set  $Hyp(\tau)$  of all hypersubstitutions of type  $\tau$ , we define a binary operation  $\circ_h : Hyp(\tau) \times Hyp(\tau) \rightarrow Hyp(\tau)$  by  $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$  where  $\circ$  is the usual composition of functions. Then together with the identity element  $\sigma_{id}$ , mapping each  $f_i$  to the term  $f_i(x_1, \dots, x_{n_i})$ , we obtain a monoid  $(Hyp(\tau); \circ_h, \sigma_{id})$ .

Let  $\mathcal{A} = (A, (f_i^A)_{i \in I})$  be an algebra of type  $\tau$  and  $\sigma \in Hyp(\tau)$ . The algebra  $\mathcal{A} := (A, (\sigma(f_i)^A)_{i \in I})$  is called *derived algebra* determined by  $\mathcal{A}$  and  $\sigma$ .

For the class  $K \subseteq Alg(\tau)$  and for a submonoid  $M \subseteq Hyp(\tau)$  we define

$$\chi_M^A[K] := \{\sigma(\mathcal{A}) \mid \sigma \in M \text{ and } \mathcal{A} \in K\}.$$

A variety  $V$  of type  $\tau$  is called  *$M$ -solid* if  $\chi_M^A[V] = V$ . Now we give the notion of  $M$ -solid subvariety of a given  $M$ -solid variety.

Let  $M$  be a submonoid of  $Hyp(\tau)$  and let  $V$  be an  $M$ -solid variety of type  $\tau$ . A fuzzy subset  $\mu$  of  $V$  is called a *fuzzy  $M$ -solid subvariety* if

- (1)  $\mu$  is a fuzzy subvariety of  $V$ , and
- (2)  $\forall \sigma \in M \forall \mathcal{A} \in V (\mu(\sigma(\mathcal{A})) \geq \mu(\mathcal{A}))$ .

We denote by  $FMS(V)$  the set of all fuzzy  $M$ -solid subvarieties of  $V$ . Then we have the following results:

- The lattice  $\mathcal{FMS}(V) := (FMS(V), \wedge, \vee)$  of all fuzzy  $M$ -solid subvarieties of  $V$  is a complete lattice.
- The lattice  $\mathcal{FMS}(V) := (FMS(V), \wedge, \vee)$  of all fuzzy  $M$ -solid subvarieties of  $V$  is a complete sublattice of  $\mathcal{FS}(V) := (FS(V), \wedge, \vee)$  of all fuzzy subvarieties of  $V$ .
- For any two submonoids  $M_1$  and  $M_2$  of the monoid  $Hyp(\tau)$  with  $M_1 \subseteq M_2$  and  $M_1, M_2$  solid variety  $V$ , the lattice  $\mathcal{FM}_2\mathcal{S}(V)$  is a complete sublattice of the lattice  $\mathcal{FM}_1\mathcal{S}(V)$ .

# Appendix

A1 **Bundit Pibaljommee**, Fuzzy Subvarieties in Universal Algebra, Computers and Mathematics with Applications, submitted June 2010 (impact factor(2010): 1.192)

A1: **Bundit Pibaljommee**, Fuzzy Subvarieties in Universal Algebra, Computers and Mathematics with Applications, submitted 2010 (impact factor(2006): 1.192)

# Fuzzy Subvarieties in Universal Algebra

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## Abstract

In this paper, we define the concept of fuzzy subvarieties of a variety  $V$  and show that the lattice of all subvarieties of  $V$  can be embedded into the lattice of all fuzzy subvarieties of  $V$ . We consider the weights over a variety  $V$  and show that the lattice of all weights over  $V$  and the lattice of all fuzzy subvarieties of  $V$  are isomorphic. Moreover, we define the concept of fuzzy  $M$ -solid subvarieties of a given  $M$ -solid variety and show that the lattice of all fuzzy  $M$ -solid subvarieties is a complete sublattice of the lattice of all fuzzy subvarieties of the same  $M$ -solid variety.

*Keywords:* Varieties, Fuzzy subalgebras, Fuzzy subvarieties,  $M$ -solid varieties, Fuzzy  $M$ -solid subvarieties.

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## 1. Introduction

The concept of a fuzzy subset was established by L.A. Zadeh ([1]) and was defined as a function from a nonempty set  $X$  to the real unit interval  $[0, 1]$ . Then fuzzification was applied to many algebraic structures, for example, to groups, to rings, to universal algebras, etc. The first application was fuzzy groups by Rosenfeld ([2]) who inspired to many others. In [3], V. Murali defined fuzzy subalgebras of a given algebra in the universal algebra sense, studied the lattice of all fuzzy subalgebras of a given algebra and showed that it forms a complete lattice. In [4], T. Kuraoka and N. Y. Suzuki defined the notion of weights over a given (universal) algebra and showed that the lattice of all weights over an algebra forms a complete sublattice

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which is isomorphic to the lattice of all fuzzy subalgebras of the algebra. It was natural to apply fuzzification to classes of algebras. Mordeson ([5]) introduced first the concept of fuzzy algebraic varieties in ring theory. In [6], B. Šešelja defined the notion of a  $P$ -fuzzy subvariety of a given variety, where  $P$  is a partially ordered set. For our purpose, we applied the concept of  $P$ -fuzzy subvariety to define a notion of a fuzzy subvariety of a given variety in Universal algebra. In an analogous way of studying fuzzy subalgebras, we investigate the lattice of all fuzzy subvarieties of a given variety and prove that the lattice of all subvarieties of the variety can be embedded as a complete lattice into the lattice of all fuzzy subvarieties of the variety. We also define the notion of weights over a given variety and find that the lattice of all weights over the variety and the lattice of all fuzzy subvarieties of the variety are isomorphic as complete lattices. Moreover, we define the concept of a fuzzy  $M$ -solid subvariety of a given  $M$ -solid variety and show that the lattice of all fuzzy  $M$ -solid subvarieties is a complete sublattice of the lattice of all fuzzy subvarieties of the same  $M$ -solid variety.

## 2. Preliminaries

Let  $\tau = (n_i)_{i \in I}$  be a type of algebras with operation symbols  $(f_i)_{i \in I}$  where  $f_i$  is an  $n_i$ -ary operation. An algebra of type  $\tau$  is an ordered pair  $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i \in I})$ , where  $A$  is a non-empty set and  $(f_i^{\mathcal{A}})_{i \in I}$  is a sequence of operations on  $A$  indexed by a non-empty index set  $I$  such that to each  $n_i$ -ary operation symbol  $f_i$  there is a corresponding  $n_i$ -ary operation  $f_i^{\mathcal{A}}$  on  $A$ . The set  $A$  is called the *universe* of  $\mathcal{A}$  and the sequence  $(f_i^{\mathcal{A}})_{i \in I}$  is called the *sequence of fundamental operations* of  $\mathcal{A}$ . We often write  $\mathcal{A}$  instead of  $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i \in I})$ . We denote by  $\text{Alg}(\tau)$  the class of all algebras of type  $\tau$ . A class  $V$  of algebras of type  $\tau$  is called a *variety* if it is closed under taking of homomorphic images (**H**), subalgebras (**S**) and direct products (**P**). Let  $K \subseteq \text{Alg}(\tau)$ . We denote by  $\mathbf{H}(K)$  the class of all homomorphic images of algebras in  $K$ , by  $\mathbf{S}(K)$  the class of all subalgebras of algebras in  $K$  and by  $\mathbf{P}(K)$  the class of all direct products of algebras in  $K$ . Instead of  $\mathbf{H}(\{\mathcal{A}\})$ ,  $\mathbf{S}(\{\mathcal{A}\})$  and  $\mathbf{P}(\{\mathcal{A}\})$ , we write  $\mathbf{H}(\mathcal{A})$ ,  $\mathbf{S}(\mathcal{A})$  and  $\mathbf{P}(\mathcal{A})$ , respectively for every algebra  $\mathcal{A}$  in  $\text{Alg}(\tau)$ . Clearly,  $\text{Alg}(\tau)$  and the class of all one-element algebras  $\mathcal{T}(\tau)$  are varieties which are called *trivial varieties*.

Let  $X_n := \{x_1, \dots, x_n\}$  be a finite set of variables,  $W_\tau(X_n)^2$  be the set of all  $n$ -ary terms of type  $\tau$ , and let  $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ ,  $X :=$

$\{x_1, \dots, x_n, \dots\}$  be the set of all terms of type  $\tau$ . Then we denote by  $\mathcal{F}_\tau(X)$  the absolutely free algebra;  $\mathcal{F}_\tau(X) := (W_\tau(X_n); (\bar{f}_i)_{i \in I})$  with  $\bar{f}_i : (t_1, \dots, t_{n_i}) \mapsto f_i(t_1, \dots, t_{n_i})$ .

Let  $\tau$  be a type. An *equation* of type  $\tau$  is a pair  $(s, t)$  from  $W_\tau(X)$ ; such pairs are commonly written as  $s \approx t$ .

An equation  $s \approx t$  is an *identity* of an algebra  $\mathcal{A}$ , denoted by  $\mathcal{A} \models s \approx t$  if  $s^{\mathcal{A}} = t^{\mathcal{A}}$ , where  $s^{\mathcal{A}}$  and  $t^{\mathcal{A}}$  are the term operations induced by terms  $s$  and  $t$  on  $\mathcal{A}$ .

A class  $K$  of algebras satisfies an equation  $s \approx t$ , denoted by  $K \models s \approx t$ , if for every  $\mathcal{A} \in K$ ,  $\mathcal{A} \models s \approx t$ .

Let  $\Sigma$  be a set of equations of type  $\tau$ . A class  $K$  of algebras of type  $\tau$  is said to satisfy  $\Sigma$ , denoted by  $K \models \Sigma$ , if  $K \models s \approx t$  for every  $s \approx t \in \Sigma$ .

Clearly,  $\models := \{(\mathcal{A}, s \approx t) \mid \mathcal{A} \in \text{Alg}(\tau), s \approx t \in W_\tau(X)^2\} \subseteq \text{Alg}(\tau) \times W_\tau(X)^2$  is a relation between the class  $\text{Alg}(\tau)$  and the set  $W_\tau(X)^2$ . Therefore, the relation  $\models$  gives a Galois connection  $(\text{Mod}, \text{Id})$  between the class  $\text{Alg}(\tau)$  and the set  $W_\tau(X)^2$  defined by

$$\begin{aligned} \text{Id}K &:= \{s \approx t \in W_\tau(X)^2 \mid K \models s \approx t\}, \\ \text{Mod}\Sigma &:= \{\mathcal{A} \in \text{Alg}(\tau) \mid \mathcal{A} \models \Sigma\}. \end{aligned}$$

By the properties of Galois connections, we obtain

**Theorem 1.** *Let  $K, K_1, K_2 \subseteq \text{Alg}(\tau)$  and  $\Sigma, \Sigma_1, \Sigma_2 \subseteq W_\tau(X)^2$ . Then*

- (i)  $K \subseteq \text{ModId}K$  and  $\Sigma \subseteq \text{IdMod}\Sigma$ ;
- (ii)  $K_1 \subseteq K_2 \Rightarrow \text{Id}K_2 \subseteq \text{Id}K_1$  and  $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \text{Mod}\Sigma_2 \subseteq \text{Mod}\Sigma_1$ ;
- (iii)  $\text{ModId}$  and  $\text{IdMod}$  are closure operators on  $\text{Alg}(\tau)$  and  $W_\tau(X)^2$ , respectively;
- (iv) the closed sets under  $\text{IdMod}$  are exactly the sets of the form  $\text{Id}K$ ,  $K \in \text{Alg}(\tau)$ , and the closed sets under  $\text{ModId}$  are exactly the sets of the form  $\text{Mod}\Sigma$ ,  $\Sigma \subseteq W_\tau(X)^2$ .

A class  $K$  of algebras of type  $\tau$  is called an *equational class* if there is a set  $\Sigma \subseteq W_\tau(X)^2$  such that  $K = \text{Mod}\Sigma$ .

A class  $\Sigma$  of equations of type  $\tau$  is called an *equational theory* if there is a class  $K$  of algebras of type  $\tau$  such that  $\Sigma = \text{Id}K$ .

Let  $Y$  be a non-empty set of variables, and  $K$  be a class of algebras of type  $\tau$ . The algebra  $\mathcal{F}_K(Y) := \mathcal{F}_\tau(Y)/\text{Id}K$  is called the  *$K$ -free algebra over  $Y$*  or the *free algebra with respect to  $K$*  generated by  $Y/\text{Id}K$ .

**Theorem 2.** ([7]) *Let  $K \subseteq \text{Alg}(\tau)$ . Then the following conditions are equivalent:*

- (i)  $K$  is a variety of type  $\tau$ ;
- (ii)  $K$  is an equational class, i.e.  $K = \text{Mod}\Sigma$  for some  $\Sigma \subseteq W_\tau(X)^2$ ;
- (iii)  $K = \mathbf{HSP}(\{\mathcal{F}_K(X_n) \mid n \in \mathbb{N}^+\}) = \mathbf{HSP}(\{\mathcal{F}_K(X)\})$ .

Note that  $\text{Alg}(\tau) = \text{Mod}\{x \approx x\}$  and  $\mathcal{T}(\tau) = \{x \approx y\}$ . Let  $V$  be a variety of type  $\tau$ . We denote by  $\text{Sub}(V)$  the class of all subvarieties of  $V$ . It is known ([8]) that the lattice  $\mathcal{L}(V) := (\text{Sub}(V); \wedge, \vee)$  is a complete lattice, where

$$\begin{aligned} \bigwedge \{W_i \in \text{Sub}(V) \mid i \in I\} &= \bigcap_{i \in I} W_i \quad \text{and} \\ \bigvee \{W_i \in \text{Sub}(V) \mid i \in I\} &= \bigcap \{W \in \text{Sub}(V) \mid W \supseteq \bigcup_{i \in I} W_i\} \\ &= \text{ModId}(\bigcup_{i \in I} W_i) \end{aligned}$$

for every family  $\{W_j \mid j \in J\}$  of subvarieties of  $V$ .

### 3. Fuzzy subvarieties

In this section, we define fuzzy subvarieties of a given variety  $V$  by extending the concept of  $P$ -fuzzy subvariety introduced by B. Šešelja in 1998 and prove that the lattice of subvarieties of  $V$  is embedded into the lattice of all fuzzy subvarieties of  $V$ . Before we define the concept of fuzzy subvariety, we want to mention the concept of  $P$ -fuzzy subvariety.

**Definition 3.** ([6]) Let  $(P; \leq)$  be a poset and  $V$  be a variety. A mapping  $\mu : V \rightarrow P$  is called *poset valued set* ( $P$ -fuzzy set) of  $V$  and the poset valued  $\mu$  is called a  $P$ -fuzzy subvariety of  $V$  if for all  $\mathcal{A}, \mathcal{B} \in V$  such that

- $\mathcal{B}$  is a subalgebra of  $\mathcal{A}$ , or
- $\mathcal{B}$  is a homomorphic image of  $\mathcal{A}$ , or
- $\mathcal{B} = \mathcal{A}^2$ ,

then  $\mu(\mathcal{B}) \geq \mu(\mathcal{A})$ .

Now we consider for  $(P; \leq)$  the real interval  $[0, 1]$  with the usual order of the reals. Then we define the concept of fuzzy subvariety in the following way.

**Definition 4.** Let  $V$  be a variety of type  $\tau$ . A mapping  $\mu : V \rightarrow [0, 1]$  is called a fuzzy subvariety of  $V$  if the following properties are satisfied:

- (i)  $\forall \mathcal{B} \in V \forall \mathcal{A} \in \mathbf{H}(\mathcal{B}), \mu(\mathcal{A}) \geq \mu(\mathcal{B}),$
- (ii)  $\forall \mathcal{B} \in V \forall \mathcal{A} \in \mathbf{S}(\mathcal{B}), \mu(\mathcal{A}) \geq \mu(\mathcal{B})$  and
- (iii)  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu(\mathcal{A}_j) \mid j \in J\}.$

We denote by  $\text{FS}(V)$  the set of all fuzzy subvarieties of  $V$ . By the definition above, we note that if  $\mathcal{A}, \mathcal{B} \in V$  such that  $\mathcal{A}$  is isomorphic to  $\mathcal{B}$ , then  $\mu(\mathcal{A}) = \mu(\mathcal{B})$ .

Since every one element algebra of type  $\tau$  is contained in every variety of type  $\tau$  and by Theorem 2, we have the following remark.

**Remark 1.** Let  $V$  be a variety of type  $\tau$  and  $\mu$  a fuzzy subvariety of  $V$ . Then the following conditions hold.

- (i) Every one-element algebra in  $V$  has the same image with respect to  $\mu$  and it is the greatest value, i.e. if  $\mathcal{A} \in V$  with  $|\mathcal{A}| = 1$ , then  $\mu(\mathcal{A}) \geq \mu(\mathcal{B})$  for all  $\mathcal{B} \in V$ .
- (ii)  $\mu(F_V(X)) \leq \mu(\mathcal{A})$  for all  $\mathcal{A} \in V$ .

**PROOF.** (i) Let  $\mathcal{A}$  be a one element algebra in  $V$ . Then  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$  for all  $\mathcal{B} \in V$ . We get  $\mu(\mathcal{A}) \geq \mu(\mathcal{B})$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be one element algebras in  $V$ . Then  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$  and  $\mathcal{B} \in \mathbf{H}(\mathcal{A})$ . This implies  $\mu(\mathcal{A}) \geq \mu(\mathcal{B})$  and  $\mu(\mathcal{B}) \geq \mu(\mathcal{A})$ . Hence,  $\mu(\mathcal{A}) = \mu(\mathcal{B})$ .

(ii) By Theorem 2,  $V = \mathbf{HSP}(F_V(X))$  and then we have :

$$\begin{aligned}
 \mathcal{A} \in V = \mathbf{HSP}(F_V(X)) &\Rightarrow \mathcal{A} \in \mathbf{H}(\mathcal{B}), \text{ for some } \mathcal{B} \in \mathbf{SP}(F_V(X)) \\
 &\Rightarrow \mu(\mathcal{A}) \geq \mu(\mathcal{B}) \\
 \mathcal{B} \in \mathbf{SP}(F_V(X)) &\Rightarrow \exists \mathcal{C} \in \mathbf{P}(F_V(X)) \text{ and } \mathcal{B} \in \mathbf{S}(\mathcal{C}) \\
 &\Rightarrow \mu(\mathcal{B}) \geq \mu(\mathcal{C}) \\
 &\Rightarrow \mu(\mathcal{B}) \geq \mu(F_V(X)).
 \end{aligned}$$

Therefore,  $\mu(\mathcal{A}) \geq \mu(F_V(X))$ .

**Definition 5.** Let  $\mu$  be a fuzzy subset of  $V$  and  $t \in [0, 1]$ . The set  $\mu_t := \{\mathcal{A} \in V \mid \mu(\mathcal{A}) \geq t\}$  is called a level subset of  $V$ .

The next proposition is equivalent to the definition of a fuzzy subvariety.

**Proposition 6.** *Let  $V$  be a variety of type  $\tau$ ,  $\mu : V \rightarrow [0, 1]$  a fuzzy subset of  $V$  and  $t \in [0, 1]$ . Then  $\mu$  is a fuzzy subvariety of  $V$  iff for all  $t \in [0, 1]$  the level subset  $\mu_t \neq \emptyset$  is a subvariety of  $V$ .*

PROOF. ( $\Rightarrow$ ): Assume that  $\mu$  is a fuzzy subvariety of  $V$  and  $t \in [0, 1]$ . We want to show that  $\mu_t = \{\mathcal{A} \in V \mid \mu(\mathcal{A}) \geq t\}$  is a subvariety of  $V$ .

(1) Let  $\mathcal{A} \in \mathbf{H}(\mu_t)$ . We have:

$$\begin{aligned} \mathcal{A} \in \mathbf{H}(\mu_t) &\Rightarrow \exists \mathcal{B} \in \mu_t, \mathcal{A} \in \mathbf{H}(\mathcal{B}) \\ &\Rightarrow \mu(\mathcal{A}) \geq \mu(\mathcal{B}) \geq t \\ &\Rightarrow \mathcal{A} \in \mu_t. \end{aligned}$$

(2) Let  $\mathcal{A} \in \mathbf{S}(\mu_t)$ . We have

$$\begin{aligned} \mathcal{A} \in \mathbf{S}(\mu_t) &\Rightarrow \exists \mathcal{B} \in \mu_t, \mathcal{A} \in \mathbf{S}(\mathcal{B}) \\ &\Rightarrow \mu(\mathcal{A}) \geq \mu(\mathcal{B}) \geq t \\ &\Rightarrow \mathcal{A} \in \mu_t. \end{aligned}$$

(3) Let  $\{\mathcal{A}_j \mid j \in J\}$  be a family of algebras in  $\mu_t$ . We have  $\mu(\mathcal{A}_j) \geq t$ , for all  $j \in J$ . Then  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu(\mathcal{A}_j) \mid j \in J\} \geq t$ . Therefore,

$$\prod_{j \in J} \mathcal{A}_j \in \mathbf{P}(\mu_t).$$

( $\Leftarrow$ ): Let  $\mu$  be a fuzzy subset of  $V$ . Assume that for every  $t \in [0, 1]$ ,  $\mu_t$  is a subvariety of  $V$ . We have to show that  $\mu$  is a fuzzy subvariety of  $V$ .

(1) Let  $\mathcal{A}, \mathcal{B} \in V$  such that  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$ . We have to show that  $\mu(\mathcal{A}) \geq \mu(\mathcal{B})$ . We choose  $t = \mu(\mathcal{B})$ . Then  $\mathcal{B} \in \mu_t$ . Since  $\mu_t$  is a subvariety of  $V$  and  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$ , we have  $\mathcal{A} \in \mu_t$ . This implies  $\mu(\mathcal{A}) \geq t = \mu(\mathcal{B})$ .

(2) Let  $\mathcal{A}, \mathcal{B} \in V$  such that  $\mathcal{A} \in \mathbf{S}(\mathcal{B})$ . We have to show that  $\mu(\mathcal{A}) \geq \mu(\mathcal{B})$ . We choose  $t = \mu(\mathcal{B})$ . Then  $\mathcal{B} \in \mu_t$ . Since  $\mu_t$  is a subvariety of  $V$  and  $\mathcal{A} \in \mathbf{S}(\mathcal{B})$ , we have  $\mathcal{A} \in \mu_t$ . This implies  $\mu(\mathcal{A}) \geq t = \mu(\mathcal{B})$ .

(3) Let  $\{\mathcal{A}_j \mid j \in J\}$  be a family of algebras in  $V$ . We have to show that  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu(\mathcal{A}_j) \mid j \in J\}$ . We choose  $t = \inf\{\mu(\mathcal{A}_j) \mid j \in J\}$ . Then  $\mu(\mathcal{A}_j) \geq t$ . It turns out  $\mathcal{A}_j \in \mu_t, \forall j \in J$ . Since  $\mu_t$  is a subvariety of  $V$ ,  $\prod_{j \in J} \mathcal{A}_j \in \mu_t$ . Thus  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq t$ . That is  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu(\mathcal{A}_j) \mid j \in J\}$ .

Let  $V$  be a variety and  $W \subseteq V$ . A function  $\mu_W : V \rightarrow [0, 1]$  defined by

$$\mu_W(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in W; \\ 0, & \text{if } \mathcal{A} \in V \setminus W \end{cases}$$

for all  $\mathcal{A} \in V$  is called a *characteristic function*.

**Proposition 7.** *Let  $W$  be a subclass of a variety  $V$ . Then the characteristic function  $\mu_W$  is a fuzzy subvariety of  $V$  iff  $W$  is a subvariety of  $V$ .*

PROOF. ( $\Rightarrow$ ): Assume that  $\mu_W$  is a fuzzy subvariety of  $V$ .

- (1) Let  $\mathcal{A} \in \mathbf{H}(W)$ . There exists an algebra  $\mathcal{B} \in W$  such that  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$ . Then  $\mu_W(\mathcal{A}) \geq \mu_W(\mathcal{B}) = 1$ . This implies  $\mu_W(\mathcal{A}) = 1$ . Hence,  $\mathcal{A} \in W$ .
- (2) For  $\mathcal{A} \in \mathbf{S}(W)$ , we can prove in the same way that  $\mathcal{A}$  belongs to  $W$ .
- (3) Let  $\{\mathcal{A}_j \mid j \in J\}$  be a family of algebras in  $W$ . By assumption,  $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu_W(\mathcal{A}_j) \mid \mathcal{A}_j \in W\} = 1$ . Thus  $\prod_{j \in J} \mathcal{A}_j \in W$ .

These show that  $W$  is a subvariety of  $V$ .

( $\Leftarrow$ ): Assume that  $W$  is a subvariety of  $V$  and  $\mathcal{A} \in V$ .

- (1) Let  $\mathcal{B} \in V$  and  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$ . We want to show that  $\mu_W(\mathcal{A}) \geq \mu_W(\mathcal{B})$ .  
**Case 1** : If  $\mathcal{B} \in W$ , then  $\mathcal{A} \in W$ . So, we have  $\mu_W(\mathcal{A}) = 1 \geq \mu_W(\mathcal{B})$ .  
**Case 2** : If  $\mathcal{B} \in V \setminus W$ , then  $\mu_W(\mathcal{B}) = 0$ . It follows  $\mu_W(\mathcal{A}) \geq \mu_W(\mathcal{B})$ .
- (2)  $\mu_W(\mathcal{A}) \geq \mu_W(\mathcal{B})$  for all  $\mathcal{B} \in V$  with  $\mathcal{A} \in \mathbf{S}(\mathcal{B})$ , can be shown in the same way.
- (3) Let  $\{\mathcal{A}_j \mid j \in J\}$  be a family of algebras in  $V$ .  
**Case 1** : If for all  $j \in J$ ,  $\mathcal{A}_j \in W$ , then  $\prod_{j \in J} \mathcal{A}_j \in W$ . Thus  $\mu_W(\prod_{j \in J} \mathcal{A}_j) = 1 \geq \inf\{\mu_W(\mathcal{A}_j) \mid j \in J\}$ .  
**Case 2** : If there exists  $j \in J$ ,  $\mathcal{A}_j \in V \setminus W$ , then  $\inf\{\mu_W(\mathcal{A}_j) \mid \mathcal{A}_j \in V\} = 0$ . Thus,  $\mu_W(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu_W(\mathcal{A}_j) \mid \mathcal{A}_j \in V\}$ .

At the end of this section, we present another example of fuzzy subvariety of a given variety.

**Example 8.** *Consider the variety of semigroups  $SEM = Mod\{x(yz) \approx (xy)z\}$ . We have that the varieties*

- $B := Mod\{x(yz) \approx (xy)z, x^2 \approx x\}$  of band semigroups and,
- $RB := Mod\{x(yz) \approx (xy)z, x^2 \approx x, xyz \approx xz\}$  of rectangular bands and
- $\mathcal{T} := \{\mathcal{A} \in SEM \mid |\mathcal{A}| = 1\}$  of the trivial variety

are subvarieties of the variety  $SEM$ . We define the mapping  $\mu : SEM \rightarrow [0, 1]$  by

$$\mu(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in \mathcal{T}; \\ 0.6, & \text{if } \mathcal{A} \in RB \setminus \mathcal{T}; \\ 0.5, & \text{if } \mathcal{A} \in B \setminus RB; \\ 0, & \text{otherwise,} \end{cases}$$

for every  $\mathcal{A} \in SEM$ .

Then  $\mu$  is a fuzzy subvariety of  $SEM$ , since every level subset of  $SEM$  is a subvariety of  $SEM$ .

#### 4. The lattice of fuzzy subvarieties

Let  $V$  be a variety of type  $\tau$  and  $\mu, \nu \in FS(V)$ . We define the order on  $FS(V)$  by  $\mu \leq \nu \Leftrightarrow \mu \subseteq \nu$  means  $\mu(\mathcal{A}) \leq \nu(\mathcal{A})$  for all  $\mathcal{A} \in V$ . We repeat the meaning of unions and intersections of fuzzy subsets of  $V$ . Let  $\mu$  and  $\nu$  be fuzzy subsets of  $V$ . We define

$$\begin{aligned} (\mu \cap \nu)(\mathcal{A}) &:= \min\{\mu(\mathcal{A}), \nu(\mathcal{A})\} \\ (\mu \cup \nu)(\mathcal{A}) &:= \max\{\mu(\mathcal{A}), \nu(\mathcal{A})\}. \end{aligned}$$

For arbitrary intersections and unions, we define

$$\begin{aligned} \left(\bigcap_{i \in I} \mu_i\right)(\mathcal{A}) &:= \inf\{\mu_i(\mathcal{A}) \mid i \in I\} \\ \left(\bigcup_{i \in I} \mu_i\right)(\mathcal{A}) &:= \sup\{\mu_i(\mathcal{A}) \mid i \in I\} \end{aligned}$$

for a family  $\{\mu_i \mid i \in I\}$  of fuzzy subsets of  $V$ . Then the following Lemma is easy to verify.

**Lemma 9.** *Let  $\{\mu_i \mid i \in I\}$  be a family of fuzzy subvarieties of  $V$ . Then  $\bigcap_{i \in I} \mu_i$  is a fuzzy subvariety of  $V$ .*

In general, the union of fuzzy subvarieties need not to be a fuzzy subvariety as the following example shows.

**Example 10.** Consider the variety of semigroups  $SEM = Mod\{x(yz) \approx (xy)z\}$ . We have that the varieties

$B := Mod\{x(yz) \approx (xy)z, x^2 \approx x\}$  of band semigroups and,

$C := Mod\{x(yz) \approx (xy)z, xy \approx yx\}$  of rectangular bands and ,

$\mathcal{T} := \{\mathcal{A} \in SEM \mid |\mathcal{A}| = 1\}$  of one-element semigroups

are subvarieties of the variety  $SEM$ . We define a fuzzy subvariety  $\nu : SEM \rightarrow [0, 1]$  by

$$\nu(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in \mathcal{T}; \\ 0.7, & \text{if } \mathcal{A} \in B \setminus \mathcal{T}; \\ 0, & \text{otherwise,} \end{cases}$$

for every  $\mathcal{A} \in SEM$  and define a fuzzy subvariety  $\mu : SEM \rightarrow [0, 1]$  by

$$\mu(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in \mathcal{T}; \\ 0.6, & \text{if } \mathcal{A} \in C \setminus \mathcal{T}; \\ 0, & \text{otherwise,} \end{cases}$$

for every  $\mathcal{A} \in SEM$ .

Then we have

$$(\nu \cup \mu)(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in \mathcal{T}; \\ 0.7, & \text{if } \mathcal{A} \in B \setminus \mathcal{T}; \\ 0.6, & \text{if } \mathcal{A} \in C \setminus B; \\ 0, & \text{otherwise.} \end{cases}$$

Then  $(\nu \cup \mu)_{0.6} = C \cup B$  which is not a subvariety of  $SEM$ . By Proposition 6, we obtain that  $\nu \cup \mu$  is not a fuzzy subvariety of  $SEM$ .

For a fuzzy subset  $\mu$  of  $V$ , we define the fuzzy subvariety generated by  $\mu$  by

$$\langle \mu \rangle_F := \bigcap \{ \nu \in FS(V) \mid \mu \subseteq \nu \}.$$

Now, we want to describe the construction of the fuzzy subvariety of  $V$  generated by a fuzzy subset of  $V$ .

**Lemma 11.** Let  $V$  be a variety of type  $\tau$  and  $\mu$  be a fuzzy subset of  $V$ . Define a mapping  $\nu : V \rightarrow [0, 1]$  by

$$\nu(\mathcal{A}) := \sup \{ t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}(\mu_t) \}$$

for every  $\mathcal{A} \in V$ . Then  $\nu = \langle \mu \rangle_F$ .

PROOF. Let  $\mathcal{A} \in V$  and let  $\mathcal{I}_{\mathcal{A}} = \{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}(\mu_t)\}$ .

First, we prove that  $\nu$  is a fuzzy subvariety of  $V$ . It is sufficient to prove that for all  $t \in \text{Im}(\nu)$ ,  $\nu_t$  is a subvariety of  $V$ . Let  $t \in \text{Im}(\nu)$  and  $t_n = t - \frac{1}{n}, n \in \mathbb{N} \setminus \{0\}$ . Let  $\mathcal{A} \in \nu_t$ . Then  $\nu(\mathcal{A}) \geq t$ , so  $\nu(\mathcal{A}) > t_n$  for all  $n \in \mathbb{N} \setminus \{0\}$ . Hence, there exists  $s \in \mathcal{I}_{\mathcal{A}}$  such that  $s > t_n$ , since if for all  $s \in \mathcal{I}_{\mathcal{A}}, s \leq t_n$ , then  $\nu(\mathcal{A}) = \sup \mathcal{I}_{\mathcal{A}} \leq t_n$ , contradiction. Thus,  $\mu_s \subseteq \mu_{t_n}$  and so,  $\mathcal{A} \in \mathbf{HSP}(\mu_s) \subseteq \mathbf{HSP}(\mu_{t_n})$ , for all  $n \in \mathbb{N} \setminus \{0\}$ . Therefore,  $\mathcal{A} \in \bigcap_{n \in \mathbb{N} \setminus \{0\}} \mathbf{HSP}(\mu_{t_n})$ . Conversely, if  $\mathcal{A} \in \bigcap_{n \in \mathbb{N} \setminus \{0\}} \mathbf{HSP}(\mu_{t_n})$ , then  $t_n \in \mathcal{I}_{\mathcal{A}}$ , for all  $n \in \mathbb{N} \setminus \{0\}$ . Then  $t_n = t - \frac{1}{n} \leq \sup \mathcal{I}_{\mathcal{A}} = \nu(\mathcal{A})$ , for all  $n \in \mathbb{N} \setminus \{0\}$ . Hence  $\nu(\mathcal{A}) \geq t$ , i.e.  $\mathcal{A} \in \nu_t$ . Then we have  $\nu_t = \bigcap_{n \in \mathbb{N} \setminus \{0\}} \mathbf{HSP}(\mu_{t_n})$ , which

means that  $\nu_t$  is subvariety of  $V$ . Therefore,  $\nu$  is a fuzzy subvariety of  $V$ .

Next step is to show that  $\mu \subseteq \nu$ . Let  $\mathcal{A} \in V$ . Since  $\mu(\mathcal{A}) \in \mathcal{I}_{\mathcal{A}}$ , we have  $\nu(\mathcal{A}) \geq \mu(\mathcal{A})$ .

Finally, we want to prove that for any fuzzy subvariety  $\alpha$  containing  $\mu$  we have  $\nu \subseteq \alpha$ . It is clear that for all  $t \in [0, 1]$ , we have that  $\mathbf{HSP}(\mu_t)$  is a subvariety of  $\alpha_t$ , since for every  $\mathcal{A} \in V, \mathcal{A} \in \mu_t$  implies  $t \leq \mu(\mathcal{A}) \leq \alpha(\mathcal{A})$ , i.e.  $\mathcal{A} \in \alpha_t$  and so,  $\mathbf{HSP}(\mu_t)$  is a subvariety of  $\alpha_t$ . Now, we prove that for every  $\mathcal{A} \in V, \nu(\mathcal{A}) \leq \alpha(\mathcal{A})$ . Let  $\mathcal{A} \in V$  and  $t \in \mathcal{I}_{\mathcal{A}}$ . Then  $\mathcal{A} \in \mathbf{HSP}(\mu_t) \subseteq \alpha_t$  implies  $\alpha(\mathcal{A}) \geq t$ . Therefore,  $\nu(\mathcal{A}) = \sup \mathcal{I}_{\mathcal{A}} \leq \alpha(\mathcal{A})$ .

Next, we consider the lattice of all fuzzy subvarieties of  $V$ . Let  $\{\mu_j \mid j \in J\}$  be a family of fuzzy subvarieties of  $V$ . We define the meet  $\wedge$  and the join  $\vee$  on  $\text{FS}(V)$  as follows:

$$\begin{aligned} \bigwedge_{j \in J} \mu_j &:= \bigcap_{j \in J} \mu_j \\ \bigvee_{j \in J} \mu_j &:= \bigcap \{ \mu \in \text{FS}(V) \mid \bigcup_{j \in J} \mu_j \subseteq \mu \} \end{aligned}$$

The following theorem is easy to verify.

**Theorem 12.** *The lattice of all fuzzy subvarieties of  $V$  denoted by  $\text{FS}(V) := (\text{FS}(V); \wedge, \vee)$  forms a complete lattice which has the least and the greatest elements, say  $\mathbf{0}, \mathbf{1}$ , respectively where  $\mathbf{0}(\mathcal{A}) = 0, \mathbf{1}(\mathcal{A}) = 1$  for all  $\mathcal{A} \in V$ .*

Using Proposition 7, we prove that the lattice of all subvarieties of a variety  $V$  can be embedded into the lattice of all fuzzy subvarieties of  $V$ .

**Theorem 13.** *The lattice  $\mathcal{L}(V)$  can be embedded into the lattice  $\mathcal{FS}(V)$ .*

PROOF. We define a mapping  $\varphi : \text{Sub}(V) \rightarrow \mathcal{FS}(V)$  by

$$\varphi(W) = \mu_W$$

for all subvariety  $W$  of  $V$ .

By Proposition 7,  $\varphi$  is an injective mapping. Let  $K_1, K_2$  be subvarieties of  $V$ . It is clear that  $\varphi(K_1 \wedge K_2) = \mu_{K_1} \wedge \mu_{K_2}$ . To show  $\varphi(K_1 \vee K_2) = \mu_{K_1} \vee \mu_{K_2}$ , we note that if  $t > 0$  then  $(\mu_{K_1} \cup \mu_{K_2})_t = K_1 \cup K_2$  and if  $t = 0$  then  $(\mu_{K_1} \cup \mu_{K_2})_t = V$ . Let  $\mathcal{A} \in V$ .

**Case 1.** If  $\mathcal{A} \notin K_1 \vee K_2$ , then  $\mu_{K_1 \vee K_2}(\mathcal{A}) = 0$ . By Lemma 11,

$$\begin{aligned} (\mu_{K_1} \vee \mu_{K_2})(\mathcal{A}) &= \sup\{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}((\mu_{K_1} \cup \mu_{K_2})_t)\} \\ &= \sup\{0\} = 0 \\ &= \mu_{K_1 \vee K_2}(\mathcal{A}). \end{aligned}$$

**Case 2.** If  $\mathcal{A} \in K_1 \vee K_2 = \mathbf{HSP}(K_1 \cup K_2)$  then we have  $\mu_{K_1 \vee K_2}(\mathcal{A}) = 1$ . Since  $(\mu_{K_1} \cup \mu_{K_2})_t = K_1 \cup K_2$  for all  $t > 0$ . Then we have  $\mathcal{A} \in \mathbf{HSP}((\mu_{K_1} \cup \mu_{K_2})_t)$  for all  $t > 0$  then  $\mathcal{A}$  always belongs to  $(\mu_{K_1} \cup \mu_{K_2})_0$ . So,  $(\mu_{K_1} \vee \mu_{K_2})(\mathcal{A}) = \sup\{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}((\mu_{K_1} \cup \mu_{K_2})_t)\} = \sup[0, 1] = 1$ . This completes the proof.

## 5. Fuzzy subvarieties and weights

Let  $K$  be a subclass of a given variety  $V$  of type  $\tau$ . We denote by  $\langle K \rangle_{\text{var}}$  the subvariety of  $V$  generated by  $K$ . It is well-known (see, e.g. [8] and [9]) that  $\langle K \rangle_{\text{var}} = \mathbf{HSP}(K)$ . In the case  $K = \{\mathcal{A}\}$ , we write  $\langle \mathcal{A} \rangle_{\text{var}}$  instead of  $\langle \{\mathcal{A}\} \rangle_{\text{var}}$ . We define weights over  $V$  in the analogous way of weights over algebras in [4].

**Definition 14.** *Let  $V$  be a variety of type  $\tau$  and  $\text{Sub}(V)$  be the class of all subvarieties of  $V$ . A mapping  $w : \text{Sub}(V) \rightarrow [0, 1]$  is called a weight over  $V$  if for any family  $\{K_i \mid i \in I\}$  of  $\text{Sub}(V)$ ,*

$$w\left(\bigvee_{i \in I} K_i\right) = \inf_{i \in I} w(K_i).$$

We denote by  $W(V)$  the set of all weights over  $V$ . Note that for a weight over  $\text{Sub}(V)$ , the following conditions hold:

(i)  $w(\langle \emptyset \rangle_{var}) = 1$ .

(ii) If  $K_1, K_2 \in Sub(V)$  and  $K_1 \subseteq K_2$ , then  $w(K_1) \geq w(K_2)$ .

The condition (i) is true by the standard definition:  $\inf_{i \in \emptyset} w(K_i) = 1$ . To prove the condition (ii), we have:

$$\begin{aligned} K_1 \subseteq K_2 &\Rightarrow K_1 \vee K_2 = K_2 \\ &\Rightarrow w(K_1 \vee K_2) = \inf\{w(K_1), w(K_2)\} = w(K_2) \\ &\Rightarrow w(K_1) \geq w(K_2). \end{aligned}$$

We define the order on  $W(V)$  by

$$w \leq w' :\Leftrightarrow w(K_1) \leq w(K_2), \forall K_1, K_2 \in Sub(V).$$

Then we obtain the following Lemma.

**Lemma 15.** *The lattice  $\mathfrak{W}(V) := (W(V); \leq)$  of all weights over  $V$  is a complete lattice.*

Let  $\mu$  be a fuzzy subvariety of a variety  $V$  and  $w$  a weight over  $V$ . We define a mapping

$$\mu^* : Sub(V) \rightarrow [0, 1]$$

by

$$\mu^*(K) = \inf_{\mathcal{A} \in K} \mu(\mathcal{A})$$

for every  $K \in Sub(V)$ .

We extend the domain of  $\mu^*$  to the power set of  $V$  by setting  $\mu^*(T) := \inf_{\mathcal{A} \in T} \mu(\mathcal{A})$  for every  $T \in V$ . Then we get  $\mu^*(T) = \mu^*(\langle T \rangle_{var})$  as the following lemma shows.

**Lemma 16.** *Let  $\mu$  be a fuzzy subvariety of  $V$  and  $T$  be a subclass of  $V$ . Then  $\mu^*(T) = \mu^*(\langle T \rangle_{var})$ .*

**PROOF.** Clearly,  $\mu^*(T) \geq \mu^*(\langle T \rangle_{var})$ , since  $T \subseteq \langle T \rangle_{var}$ . Suppose that  $\mu^*(T) > \mu^*(\langle T \rangle_{var})$ . There exists  $\mathcal{A}_0 \in \langle T \rangle_{var} \setminus T$  such that  $\mu^*(T) > \mu(\mathcal{A}_0)$ . Let  $\mathcal{A}_0 \in \langle T \rangle_{var} = \mathbf{HSP}(T)$ . There exists  $\mathcal{B} \in \mathbf{SP}(T)$  such that  $\mathcal{A}_0 \in \mathbf{H}(\mathcal{B})$ . Let  $\mathcal{B} \in \mathbf{SP}(T)$ . There exists  $\mathcal{C} \in \mathbf{P}(T)$  such that  $\mathcal{B} \in \mathbf{S}(\mathcal{C})$ . Let  $\mathcal{C} \in \mathbf{P}(T)$ .

There exists a class  $\{\mathcal{D}_j \mid j \in J\}$  of algebras in  $T$  such that  $\mathcal{C} \in \prod_{j \in J} \mathcal{D}_j$ . Let  $\mu$  is a fuzzy subvariety of  $V$ . We have

$$\mu(\mathcal{A}_0) \geq \mu(\mathcal{B}) \geq (\mathcal{C}) \geq \inf_{j \in J} \mu(\mathcal{D}_j).$$

Hence  $\mu^*(T) = \inf_{\mathcal{A} \in T} \mu(\mathcal{A}) > \mu(\mathcal{A}_0) \geq \mu(\mathcal{B}) \geq \mu(\mathcal{C}) \geq \inf_{j \in J} \mu(\mathcal{D}_j)$ , since  $\{\mathcal{D}_j \mid j \in J\} \subseteq T$ . This is a contradiction. Therefore,  $\mu^*(T) = \mu^*(\langle T \rangle_{var})$ .

Let  $w$  be a weight over a variety  $V$ . We define a mapping

$$w^\circ : V \rightarrow [0, 1]$$

by

$$w^\circ(\mathcal{A}) = w(\langle \mathcal{A} \rangle_{var})$$

for every  $\mathcal{A} \in V$ .

**Theorem 17.** *Let  $V$  be a variety of type  $\tau$ ,  $w$  be a weight over  $V$  and  $\mu$  be a fuzzy subvariety of  $V$ . Then the following conditions hold.*

- (i)  $\mu^*$  is a weight over  $V$ .
- (ii)  $w^\circ$  is a fuzzy subvariety of  $V$ .

PROOF. (i) Let  $\{K_i \mid i \in I\}$  be a family of subvarieties of  $V$ . By Lemma 16,  $\mu^*(\bigvee_{i \in I} K_i) = \mu^*(\mathbf{HSP}(\bigcup_{i \in I} K_i)) = \mu^*(\bigcup_{i \in I} K_i) = \inf_{\mathcal{A} \in \bigcup_{i \in I} K_i} \mu(\mathcal{A}) = \inf_{i \in I} \mu^*(K_i)$ .

This means  $\mu^*$  is a weight over  $V$ .

(ii) To show  $w^\circ$ , we let  $\mathcal{A}, \mathcal{B} \in V$  with  $\mathcal{A} \in \mathbf{H}(\mathcal{B})$ . Since  $\langle \mathcal{B} \rangle_{var} = \langle \mathbf{H}(\mathcal{B}) \rangle_{var}$ , we have  $w^\circ(\mathcal{B}) = w(\langle \mathcal{B} \rangle_{var}) = w(\langle \mathbf{H}(\mathcal{B}) \rangle_{var}) = w(\bigvee_{\mathcal{C} \in \mathbf{H}(\mathcal{B})} \langle \mathcal{C} \rangle_{var}) =$

$$\inf_{\mathcal{C} \in \mathbf{H}(\mathcal{B})} w(\langle \mathcal{C} \rangle_{var}) = \inf_{\mathcal{C} \in \mathbf{H}(\mathcal{B})} w^\circ(\mathcal{C}) \leq w^\circ(\mathcal{A}), \text{ since } \mathcal{A} \in \mathbf{H}(\mathcal{B}).$$

If  $\mathcal{A}, \mathcal{B} \in V$  with  $\mathcal{A} \in \mathbf{S}(\mathcal{B})$ , then we can similarly prove that  $w^\circ(\mathcal{B}) \leq w^\circ(\mathcal{A})$ .

Let  $\{\mathcal{A}_j \mid j \in J\} \subseteq V$ . We have:

$$\begin{aligned} \langle \prod_{j \in J} \mathcal{A}_j \rangle_{var} \subseteq \bigvee_{j \in J} \langle \mathcal{A}_j \rangle_{var} &\Rightarrow w^\circ(\prod_{j \in J} \mathcal{A}_j) = w(\langle \prod_{j \in J} \mathcal{A}_j \rangle_{var}) \geq w(\bigvee_{j \in J} \langle \mathcal{A}_j \rangle_{var}) \\ &= \inf_{j \in J} w(\langle \mathcal{A}_j \rangle_{var}) = \inf_{j \in J} w^\circ(\mathcal{A}_j). \end{aligned}$$

Therefore,  $w^\circ$  is a weight over  $V$ .

**Theorem 18.** *Let  $V$  be a variety of type  $\tau$ .*

- (1) *If  $\mu \in \text{FS}(V)$  and  $w \in W(V)$ , then  $\mu^{*\circ} = \mu$  and  $w^{\circ*} = w$ .*  
(2) *The lattices  $(\text{FS}(V), \leq)$  and  $(W(V), \leq)$  are isomorphic.*

PROOF. (1) Let  $\mathcal{A} \in V$ . We have  $\mu^{*\circ}(\mathcal{A}) = \mu^*(\langle \mathcal{A} \rangle_{\text{var}}) = \mu^*(\{\mathcal{A}\}) = \inf \mu(\mathcal{A}) = \mu(\mathcal{A})$ . Let  $K$  be a subvariety of  $V$ . We have  $w^{\circ*}(K) = \inf_{\mathcal{A} \in K} w^{\circ}(\mathcal{A}) = \inf_{\mathcal{A} \in K} w(\langle \mathcal{A} \rangle_{\text{var}}) = w^{\circ}(K)$ .

(2) Let  $\phi : \text{FS}(V) \rightarrow W(V)$  and  $\psi : W(V) \rightarrow \text{FS}(V)$  defined by  $\phi(\mu) = \mu^*$  and  $\psi(w) = w^{\circ}$  for every  $\mu \in \text{FS}(V)$  and  $w \in W(V)$ . By (1) it is clear that  $\phi$  is bijective and  $\psi$  is its inverse.

Next step is to show  $\phi$  and  $\psi$  are order-preserving. Let  $\mu_1, \mu_2 \in \text{FS}(V)$  with  $\mu_1 \leq \mu_2$ . Let  $K \in \text{Sub}(V)$ . We have

$$\mu_1^*(K) = \inf_{\mathcal{A} \in K} \mu_1(\mathcal{A}) \leq \inf_{\mathcal{A} \in K} \mu_2(\mathcal{A}) = \mu_2^*(K).$$

This implies  $\mu_1^* \leq \mu_2^*$ .

Finally, if  $w_1, w_2 \in W(V)$  and  $w_1 \leq w_2$  then

$$w_1^{\circ}(\mathcal{A}) = w_1(\langle \mathcal{A} \rangle_{\text{var}}) \leq w_2(\langle \mathcal{A} \rangle_{\text{var}}) = w_2^{\circ}(\mathcal{A}).$$

This means  $w_1^{\circ} \leq w_2^{\circ}$ . Therefore,  $\phi$  and  $\psi$  are order-preserving. By Theorem 2.3 in [8], we have that the lattices  $\mathcal{FS}(V)$  and  $\mathfrak{M}(V)$  are isomorphic.

By Theorem 13, the lattice of all subvarieties of a variety  $V$  can be embedded into the lattice of fuzzy subvarieties of a variety  $V$ , and by Theorem 18, we have:

**Corollary 19.** *The lattice  $\mathcal{L}(V)$  can be embedded into the lattice  $\mathfrak{M}(V)$ .*

At the end of this section, we give an example of the weight over a variety induced by a fuzzy subvariety of the variety.

**Example 20.** *Related to Example 8, we have the fuzzy subvariety  $\mu$  of SEM defined by*

$$\mu(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in \mathcal{T}; \\ 0.6, & \text{if } \mathcal{A} \in RB \setminus \mathcal{T}; \\ 0.5, & \text{if } \mathcal{A} \in B \setminus RB; \\ 0, & \text{otherwise,} \end{cases}$$

for every  $\mathcal{A} \in SEM$ . Then we have  $\mu^*(\mathcal{T}) = 1$ ,  $\mu^*(RB) = 0.6$  and  $\mu^*(B) = 0.5$ . Therefore, for all  $K \in Sub(V)$ ,

$$\mu^*(K) := \begin{cases} 1, & \text{if } K = \mathcal{T}; \\ 0.6, & \text{if } \mathcal{T} \subset K \subseteq RB; \\ 0.5, & \text{if } RB \subset K \subseteq B; \\ 0, & \text{otherwise.} \end{cases}$$

## 6. Fuzzy $M$ - solid subvarieties

In this section we assume that  $V$  is an  $M$ -solid variety of type  $\tau$  and then we give the notion of fuzzy  $M$ -solid subvarieties. First of all we start with the concept of hypersubstitutions introduced in [10] (see also in [11]).

**Definition 21.** A mapping  $\sigma : \{f_i | i \in I\} \rightarrow W_\tau(X)$  which takes each  $n_i$ -ary operation symbol to an  $n_i$ -ary term is called a hypersubstitution of type  $\tau$ . Each hypersubstitution  $\sigma$  induces a map  $\hat{\sigma}$  on the set of all terms which is defined by

- (i)  $\hat{\sigma}[x] := x$  if  $x \in X$  is a variable,
- (ii)  $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$  for composite terms  $f_i(t_1, \dots, t_{n_i})$ .

On the set  $Hyp(\tau)$  of all hypersubstitutions of type  $\tau$ , we define a binary operation  $\circ_h : Hyp(\tau) \times Hyp(\tau) \rightarrow Hyp(\tau)$  by  $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$  where  $\circ$  is the usual composition of functions. Then together with the identity element  $\sigma_{id}$ , mapping each  $f_i$  to the term  $f_i(x_1, \dots, x_{n_i})$ , we obtain a monoid  $(Hyp(\tau); \circ_h, \sigma_{id})$ .

Let  $\mathcal{A} = (A, (f_i^A)_{i \in I})$  be an algebra of type  $\tau$  and  $\sigma \in Hyp(\tau)$ . The algebra  $\mathcal{A} := (A, (\sigma(f_i)^A)_{i \in I})$  is called *derived algebra* determined by  $\mathcal{A}$  and  $\sigma$ .

For the class  $K \subseteq Alg(\tau)$  and for a submonoid  $M \subseteq Hyp(\tau)$  we define

$$\chi_M^A[K] := \{\sigma(\mathcal{A}) \mid \sigma \in M \text{ and } \mathcal{A} \in K\}.$$

A variety  $V$  of type  $\tau$  is called  *$M$ -solid* if  $\chi_M^A[V] = V$ . Now we give the notion of  $M$ -solid subvariety of a given  $M$ -solid variety.

**Definition 22.** Let  $M$  be a submonoid of  $Hyp(\tau)$  and let  $V$  be an  $M$ -solid variety of type  $\tau$ . A fuzzy subset  $\mu$  of  $V$  is called a *fuzzy  $M$ -solid subvariety* if

- (1)  $\mu$  is a fuzzy subvariety of  $V$ , and  
(2)  $\forall \sigma \in M \forall \mathcal{A} \in V (\mu(\sigma(\mathcal{A})) \geq \mu(\mathcal{A}))$ .

We denote by  $\text{FMS}(V)$  the set of all fuzzy  $M$ -solid subvarieties of  $V$ .

**Proposition 23.** *Let  $V$  be an  $M$ -solid variety of type  $\tau$ ,  $\mu : V \rightarrow [0, 1]$  a fuzzy subset of  $V$  and  $t \in [0, 1]$ . Then  $\mu$  is a fuzzy  $M$ -solid subvariety of  $V$  iff for all  $t \in [0, 1]$  the level subset  $\mu_t \neq \emptyset$  is an  $M$ -solid subvariety of  $V$ .*

**PROOF.** ( $\Rightarrow$ ) : Assume that  $\mu$  is a fuzzy  $M$ -solid subvariety of  $V$  and  $t \in [0, 1]$ . Since  $V$  is a variety and by Proposition 6, we have that  $\mu_t$  is a subvariety of  $V$ . Let  $\sigma \in M$  and  $\mathcal{A} \in \mu_t$ . By assumption, we obtain  $\mu(\sigma(\mathcal{A})) \geq \mu(\mathcal{A}) \geq t$ . Hence,  $\sigma(\mathcal{A}) \in \mu_t$ . Altogether,  $\mu_t$  is an  $M$ -solid subvariety of  $V$ .

( $\Leftarrow$ ) : Assume that for every  $t \in [0, 1]$ ,  $\mu_t \neq \emptyset$  is an  $M$ -solid subvariety of  $V$ . Since  $V$  is a variety and by Proposition 6, we have  $\mu$  is a fuzzy subvariety of  $V$ . Let  $\sigma \in M$  and  $\mathcal{A} \in \mu_t$ . We choose  $t = \mu(\mathcal{A})$  and by assumption, we have  $\mu(\sigma(\mathcal{A})) \in \mu_t$ . This means  $\mu(\sigma(\mathcal{A})) \geq t = \mu(\mathcal{A})$ . Therefore,  $\mu$  is a fuzzy  $M$ -solid subvariety of  $V$ .

**Proposition 24.** *Let  $W$  be a subclass of an  $M$ -solid variety  $V$ . Then the characteristic function  $\mu_W$  is a fuzzy  $M$ -solid subvariety of  $V$  iff  $W$  is an  $M$ -solid subvariety of  $V$ .*

**PROOF.** ( $\Rightarrow$ ) : Assume that  $\mu_W$  is a fuzzy  $M$ -solid subvariety of  $V$ . By Proposition 7,  $W$  is a subvariety of  $V$ . Let  $\sigma \in M$  and  $\mathcal{A} \in W$ . We have  $\mu_W(\sigma(\mathcal{A})) \geq \mu_W(\mathcal{A}) = 1$ . Then  $\sigma(\mathcal{A}) \in W$ . Now we have that  $W$  is an  $M$ -solid subvariety of  $V$ .

( $\Leftarrow$ ) : Assume that  $W$  is an  $M$ -solid subvariety of  $V$ . By Proposition 7,  $\mu_W$  is a fuzzy subvariety of  $V$ . Let  $\mathcal{A} \in V$  and  $\sigma \in M$ . If  $\mathcal{A} \in W$  then by assumption, it turns out that  $\sigma(\mathcal{A}) \in W$  and  $\mu_W(\sigma(\mathcal{A})) = 1 = \mu_W(\mathcal{A})$ , i.e.  $\mu_W(\sigma(\mathcal{A})) \geq \mu_W(\mathcal{A})$ . If  $\mathcal{A} \notin W$  then  $\mu_W(\mathcal{A}) = 0$ . Hence,  $\mu_W(\sigma(\mathcal{A})) \geq \mu_W(\mathcal{A})$ . Now we obtain  $\mu_W$  a fuzzy  $M$ -solid subvariety of  $V$ .

**Example 25.** *Let  $\tau = (2)$  with a binary operation symbol  $\cdot$ ,  $X_2 := \{x_1, x_2\}$ ,  $W_{(2)}(X_2)$  the set of all binary terms of type  $\tau = (2)$  and  $\text{Hyp}(2)$  the monoid of all hypersubstitutions of type  $\tau = (2)$ . Let  $\text{Pre} := \{\sigma_t \in \text{Hyp}(2) \mid t \in W_{(2)}(X_2), t \neq x_1, x_2\}$ , where  $\sigma_t$  maps the operation symbols  $\cdot$  to the term  $t \in W_{(2)}(X_2)$ . It is easy to show that  $\text{Pre}$  is a submonoid of  $\text{Hyp}(2)$  (see*

in [12]). Consider the Pre-solid variety of commutative semigroups  $V_{PC}$ . In [13], it was proved that the complete lattice  $\mathcal{L}(V_{PC})$  is precisely

$$L(V_{PC}) = \{P_n \mid n \in \mathbb{N}\} \cup \{V_{PC}\},$$

where  $P_n := \text{Mod}\{(xy)z \approx x(yz), xy \approx yx, x^2 \approx y^2, x^2y = xy^2, x_0 \dots x_n \approx y_0 \dots y_n\}$  for  $n \in \mathbb{N}$ . Now, we define the mapping  $\mu : V_{PC} \rightarrow [0, 1]$  by

$$\mu(\mathcal{A}) := \begin{cases} 1, & \text{if } \mathcal{A} \in P_0 = \mathcal{T}; \\ \frac{1}{n+1}, & \text{if } \mathcal{A} \in P_n \setminus P_{n-1}; \\ 0, & \text{otherwise,} \end{cases}$$

for every  $\mathcal{A} \in V_{PC}$ . Then  $\mu$  is a fuzzy Pre-solid subvariety of  $V_{PC}$ . Because every level subset of  $V_{PC}$  is a Pre-solid subvariety of  $V_{PC}$ .

For a fuzzy subset  $\mu$  of an  $M$ -solid variety  $V$ , we define the fuzzy  $M$ -solid subvariety of  $V$  generated by  $\mu$  by

$$\langle \mu \rangle_F := \bigcap \{ \nu \in \text{FMS}(V) \mid \mu \subseteq \nu \},$$

Now, we want to describe the construction of the fuzzy  $M$ -solid subvariety of  $V$  generated by a fuzzy set of  $V$ . We refer a result of E. Graczyńska in [14] (see also in [12]) that for every subclass  $K$  of  $V$ ,  $\mathbf{HSP}(\chi_M^A[K])$  is the  $M$ -solid subvariety of  $V$  generated by  $K$ .

**Lemma 26.** Let  $V$  be an  $M$ -solid variety of type  $\tau$  and  $\mu$  be a fuzzy subset of  $V$ . Define a mapping  $\nu : V \rightarrow [0, 1]$  by

$$\nu(\mathcal{A}) := \sup \{ t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}(\chi_M^A[\mu_t]) \},$$

for every  $\mathcal{A} \in V$ . Then  $\nu = \langle \mu \rangle_F$ .

**PROOF.** Let  $\mathcal{A} \in V$  and let  $\mathcal{I}_\mathcal{A} = \{ t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}(\chi_M^A[\mu_t]) \}$ .

First, we prove that  $\nu$  is a fuzzy  $M$ -solid subvariety of  $V$ . It suffices to prove that for all  $t \in \text{Im}(\nu)$ ,  $\nu_t$  is an  $M$ -solid subvariety of  $V$ . Let  $t \in \text{Im}(\nu)$  and  $t_n = t - \frac{1}{n}$ ,  $n \in \mathbb{N} \setminus \{0\}$ . Let  $\mathcal{A} \in \nu_t$ . Then  $\nu(\mathcal{A}) \geq t$ , so  $\nu(\mathcal{A}) > t_n, \forall n \in \mathbb{N} \setminus \{0\}$ . Hence, there exists  $s \in \mathcal{I}_\mathcal{A}$  such that  $s > t_n$ , since if  $\forall s \in \mathcal{I}_\mathcal{A}, s \leq t_n$ , then  $\nu(\mathcal{A}) = \sup \mathcal{I}_\mathcal{A} \leq t_n$ . This gives a contradiction. Thus,  $\mu_s \subseteq \mu_{t_n}$  and so,  $\mathcal{A} \in \mathbf{HSP}(\chi_M^A[\mu_s]) \subseteq \mathbf{HSP}(\chi_M^A[\mu_{t_n}])$ , for all  $n \in \mathbb{N} \setminus \{0\}$ , since  $\chi_M^A$  is a closure operator. Therefore,  $\mathcal{A} \in \bigcap_{n \in \mathbb{N} \setminus \{0\}} \mathbf{HSP}(\chi_M^A[\mu_{t_n}])$ .

Conversely, if  $\mathcal{A} \in \bigcap_{n \in \mathbb{N} \setminus \{0\}} \mathbf{HSP}(\chi_M^A[\mu_{t_n}])$ , then  $t_n \in \mathcal{I}_{\mathcal{A}}, \forall n \in \mathbb{N} \setminus \{0\}$ . Then  $t_n = t - \frac{1}{n} \leq \sup \mathcal{I}_{\mathcal{A}} = \nu(\mathcal{A}), \forall n \in \mathbb{N} \setminus \{0\}$ . Hence  $\nu(\mathcal{A}) \geq t$ , i.e.  $\mathcal{A} \in \nu_t$ . Then we have  $\nu_t = \bigcap_{n \in \mathbb{N} \setminus \{0\}} \mathbf{HSP}(\chi_M^A[\mu_{t_n}])$ , which means that  $\nu_t$  is an  $M$ -solid subvariety of  $V$ . Therefore,  $\nu$  is a fuzzy  $M$ -solid subvariety of  $V$ .

Next step is to show that  $\mu \subseteq \nu$ . Let  $\mathcal{A} \in V$ . Since  $\mu(\mathcal{A}) \in \mathcal{I}_{\mathcal{A}}$ , we have  $\nu(\mathcal{A}) \geq \mu(\mathcal{A})$ .

Finally, we want to prove that for any fuzzy  $M$ -solid subvariety  $\delta$  of  $V$  containing  $\mu$ , we have  $\nu \subseteq \delta$ . It is clear that for all  $t \in [0, 1]$ , we have  $\mathbf{HSP}(\chi_M^A[\mu_t])$  is an  $M$  solid subvariety of  $\delta_t$ , since for every  $\mathcal{A} \in V, \mathcal{A} \in \mu_t$  implies  $t \leq \mu(\mathcal{A}) \leq \delta(\mathcal{A})$ , i.e.  $\mathcal{A} \in \delta_t$  and so,  $\mathbf{HSP}(\chi_M^A[\mu_t])$  is an  $M$ -solid subvariety of  $\delta_t$ . Now, we prove that for every  $\mathcal{A} \in V, \nu(\mathcal{A}) \leq \delta(\mathcal{A})$ . Let  $\mathcal{A} \in V$  and  $t \in \mathcal{I}_{\mathcal{A}}$ . Then  $\mathcal{A} \in \mathbf{HSP}(\chi_M^A[\mu_t]) \subseteq \delta_t$  implies  $\delta(\mathcal{A}) \geq t$ . Therefore,  $\nu(\mathcal{A}) = \sup \mathcal{I}_{\mathcal{A}} \leq \delta(\mathcal{A})$ , i.e.  $\nu \subseteq \mu$ .

The following lemma is easy to verify.

**Lemma 27.** *The lattice  $\mathcal{FMS}(V) := (\mathbf{FMS}(V), \wedge, \vee)$  of all fuzzy  $M$ -solid subvarieties of  $V$  is a complete lattice.*

**Lemma 28.** *Let  $M$  be a submonoid of  $\text{Hyp}(\tau)$ ,  $V$  an  $M$ -solid variety of type  $\tau$  and  $\{\mu_j \mid j \in J\} \subseteq \mathbf{FMS}(V)$ . Then  $(\bigcup_{j \in J} \mu_j)_t$  is closed under taking the operator  $\chi_M^A$ , i.e.  $\chi_M^A[(\bigcup_{j \in J} \mu_j)_t] = (\bigcup_{j \in J} \mu_j)_t$ .*

PROOF. Let  $\{\mu_j \mid j \in J\} \subseteq \mathbf{FMS}(V)$ . Let  $\mathcal{A} \in (\bigcup_{j \in J} \mu_j)_t$  and  $\sigma \in M$ . We have  $(\bigcup_{j \in J} \mu_j)(\mathcal{A}) = \sup\{\mu_j(\mathcal{A}) \mid j \in J\} \geq t$ . Since  $\mu_j(\sigma(\mathcal{A})) \geq \mu_j(\mathcal{A}), \forall j \in J$ , we have  $\sup\{\mu_j(\sigma(\mathcal{A})) \mid j \in J\} \geq \sup\{\mu_j(\mathcal{A}) \mid j \in J\} \geq t$ . Then  $(\bigcup_{j \in J} \mu_j)(\sigma(\mathcal{A})) \geq t, \forall \sigma \in M$ . This shows that  $\chi_M^A[(\bigcup_{j \in J} \mu_j)_t] = (\bigcup_{j \in J} \mu_j)_t$ .

**Theorem 29.** *The lattice  $\mathcal{FMS}(V) := (\mathbf{FMS}(V), \wedge, \vee)$  of all fuzzy  $M$ -solid subvarieties of  $V$  is a complete sublattice of  $\mathcal{FS}(V) := (\mathbf{FS}(V), \wedge, \vee)$  of all fuzzy subvarieties of  $V$ .*

PROOF. Obviously,  $\mathbf{FMS}(V) \subseteq \mathbf{FS}(V)$ . Let  $\{\mu_j \mid j \in J\} \subseteq \mathbf{FMS}(V)$ . It is clear that if  $\bigwedge_{j \in J} \mu_j \in \mathbf{FS}(V)$ , then  $\bigwedge_{j \in J} \mu_j \in \mathbf{FMS}(V)$ . Now we want to show

that if  $\bigvee_{j \in J} \mu_j \in \mathcal{FS}(V)$ , then  $\bigvee_{j \in J} \mu_j \in \mathcal{FMS}(V)$ . Let  $\mathcal{A} \in V$ . By Lemma 28 and Lemma 11, we obtain  $(\bigvee_{j \in J} \mu_j)(\mathcal{A}) = \sup\{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}((\bigcup_{j \in J} \mu_j)_t)\} = \sup\{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}(\chi_{M_1}^{\mathcal{A}}[(\bigcup_{j \in J} \mu_j)_t])\}$  for all  $\mathcal{A} \in V$ . By Lemma 26, we have  $\bigvee_{j \in J} \mu_j \in \mathcal{FMS}(V)$ . These show that  $\mathcal{FMS}(V)$  is a complete sublattice of  $\mathcal{FS}(V)$ .

It is very natural to ask for the relationship between the complete lattices  $\mathcal{FM}_1\mathcal{S} := (\mathcal{FM}_1\mathcal{S}, \wedge, \vee)$  and  $\mathcal{FM}_2\mathcal{S} := (\mathcal{FM}_2\mathcal{S}, \wedge, \vee)$  when  $M_1$  and  $M_2$  are both submonoids of  $\mathit{Hyp}(\tau)$  and  $M_1$  is a submonoid of  $M_2$ .

**Theorem 30.** *For any two submonoids  $M_1$  and  $M_2$  of the monoid  $\mathit{Hyp}(\tau)$  with  $M_1 \subseteq M_2$  and  $M_1, M_2$  solid variety  $V$ , the lattice  $\mathcal{FM}_2\mathcal{S}(V)$  is a complete sublattice of the lattice  $\mathcal{FM}_1\mathcal{S}(V)$ .*

**PROOF.** First, we show that  $\mathcal{FM}_2\mathcal{S}(V) \subseteq \mathcal{FM}_1\mathcal{S}(V)$ . Let  $\mu \in \mathcal{FM}_2\mathcal{S}(V), \sigma \in M_2$  and  $\mathcal{A} \in V$ . We have  $\mu(\sigma(\mathcal{A})) \geq \mu(\mathcal{A})$ . Since  $M_1 \subseteq M_2$ , we have  $\mu(\sigma(\mathcal{A})) \geq \mu(\mathcal{A})$  for all  $\sigma \in M_1$ . Hence,  $\mu \in \mathcal{FM}_1\mathcal{S}$ . Let  $\{\mu_j \mid j \in J\} \subseteq \mathcal{FM}_2\mathcal{S}(V)$ . It is clear that if  $\bigwedge_{j \in J} \mu_j \in \mathcal{FM}_1\mathcal{S}(V)$ , then  $\bigwedge_{j \in J} \mu_j \in \mathcal{FM}_2\mathcal{S}(V)$ .

Next, we want to show that if  $\bigvee_{j \in J} \mu_j \in \mathcal{FM}_1\mathcal{S}(V)$ , then  $\bigvee_{j \in J} \mu_j \in \mathcal{FM}_2\mathcal{S}(V)$ .

By Lemma 28 and  $\mathcal{FM}_2\mathcal{S}(V) \subseteq \mathcal{FM}_1\mathcal{S}(V)$ , we have

$$\chi_{M_2}^{\mathcal{A}}[(\bigcup_{j \in J} \mu_j)_t] = (\bigcup_{j \in J} \mu_j)_t = \chi_{M_1}^{\mathcal{A}}[(\bigcup_{j \in J} \mu_j)_t].$$

Let  $\mathcal{A} \in V$ . Assume that  $\bigvee_{j \in J} \mu_j \in \mathcal{FM}_1\mathcal{S}(V)$ . By Lemma 26, we have

$$\begin{aligned} (\bigvee_{j \in J} \mu_j)(\mathcal{A}) &= \sup\{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}_{\chi_{M_1}^{\mathcal{A}}}[(\bigcup_{j \in J} \mu_j)_t]\} \\ &= \sup\{t \in [0, 1] \mid \mathcal{A} \in \mathbf{HSP}_{\chi_{M_2}^{\mathcal{A}}}[(\bigcup_{j \in J} \mu_j)_t]\} \end{aligned}$$

This means  $(\bigvee_{j \in J} \mu_j) \in \mathcal{FM}_2\mathcal{S}(V)$ . Therefore, the lattice  $\mathcal{FM}_2\mathcal{S}(V)$  is a complete sublattice of the lattice  $\mathcal{FM}_1\mathcal{S}(V)$ .

## 7. Conclusion

We have shown that the lattice of all subvarieties of a given variety can be embedded as a complete lattice into the lattice of all fuzzy subvarieties of the variety. The lattice of all fuzzy subvarieties of a variety and the lattice of all weights over the variety coincide. We have defined the notion of fuzzy  $M$ -solid subvariety of a given  $M$ -solid variety and showed that the lattice of all fuzzy  $M$ -solid subvarieties of a given  $M$ -solid variety is a complete sublattice of the lattice of all fuzzy subvarieties of the  $M$ -solid variety. Every variety can be defined by a set of identities. But the connection between fuzzy subvarieties of a given variety and the set of all identities of the variety is still open.

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