



รายงานวิจัยฉบับสมบูรณ์

โครงการ : ผลเฉลยขั้นมูลฐานและสมบัติต่าง ๆ ของตัวดำเนินการที่
สัมพันธ์กับดำเนินการไดมอนด์

The Elementary Solutions and Properties of the
Operators Related to the Diamond Operatos
(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ในการทำงานวิจัย คำนคว้าครั้งนี้

ศาสตราจารย์อำนาจ ขันนไทย นักวิจัยที่ปรึกษาให้กับโครงการนี้ที่อบรมสั่งสอน ถ่ายทอด ความรู้ ด้านต่าง ๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับ ของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

ดร.คำสิงห์ นนเลาพล
หัวหน้าโครงการวิจัย

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บทคัดย่อ: ในรายงานวิจัยนี้ เราได้ศึกษาผลเฉลยแบบอ่อนของสมการเชิงประกอบที่สัมพันธ์กับตัวดำเนินการอัลตราไฮเพอร์โบลิกและตัวดำเนินการอัลตราไฮเพอร์โบลิก-เบสเซล เรายังได้พิสูจน์ความเป็นไปได้โดยตรงของผลเฉลยแบบอ่อนของสมการดังกล่าวอีกด้วย ผลลัพธ์ที่ได้นั้นพัฒนาและครอบคลุมผลงานวิจัยของนักคณิตศาสตร์ท่านอื่น

นอกจากนี้ เรายังได้ศึกษาการวางนัยทั่วไปของสมการความร้อนอัลตราไฮเพอร์โบลิกแบบไม่เชิงเส้นและบางสมการไม่เชิงเส้นที่สัมพันธ์กับตัวดำเนินการลาปลาซและตัวดำเนินการลาปลาซ-เบสเซล

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Abstract: In this project, we study the weak solutions of the compound equations related to the ultra-hyperbolic and ultra-hyperbolic Bessel operators. We also prove the uniqueness of the weak solution of such equations. Our results improve and include the corresponding known results studied by another authors.

Moreover, we study the generalized nonlinear ultra-hyperbolic heat equation and some nonlinear equation related to Laplace and Laplace-Bessel operators.

Keywords: Dirac-delta distribution; tempered distribution; ultra-hyperbolic operator; ultra-hyperbolic Bessel operator; Laplace operator; Laplace-Bessel operator; Fourier transform

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Chapter 1

Executive Summary

Generalized functions or distributions have of late been commanding constantly expanding interest in several different branches of mathematics. In somewhat nonrigorous form, they have already long been used in essence by physicists and opened up a new area of mathematical research, which in turn proved an impetus in the development of a number of mathematical disciplines, such as operational calculus, transformation theory, functional analysis, ordinary and partial differential equations.

Distributions close relation to the solutions of differential equations. In the case of the ordinary differential equation $Lu = 0$ with constant coefficients, every solution is the classical solution. The matter is quite different for partial differential equations. The solutions in a similar situation may now include generalized functions. For instance, $\partial u / \partial x_1 = 0$, in \mathbb{R}^2 , has among its solutions the generalized function $\delta(x_2)$,

$$\langle \delta(x_2), \phi(x_1, x_2) \rangle = \int_{-\infty}^{\infty} \phi(x_1, 0) dx_1,$$

where ϕ is a test function.

Our aim is to find the solution of the partial differential equation

$$Lu(x) = s(x), \tag{1.0.1}$$

where L is partial differential operator and $s(x)$ is an arbitrary known distribution. It is well known that the *elementary* or *fundamental solution* is the solution for $s(x) = \delta(x)$.

A distribution $u(x)$ is a solution of (1.0.1) if for every test function $\phi(x)$, we have

$$\langle Lu(x), \phi(x) \rangle = \langle s(x), \phi(x) \rangle. \tag{1.0.2}$$

In searching for a solution $u(x)$ of differential equation (1.0.1) we may have the following situations :

- (1) The solution $u(x)$ is a sufficiently smooth function, so that the operation in (1.0.1) can be performed in the classical sense and the resulting equation is an identity. Then $u(x)$ is the *classical solution*.
- (2) The solution $u(x)$ is not sufficiently smooth function, so that the operation in (1.0.1) can not be performed, but it satisfy (1.0.2) as distribution. It is then a *weak solution*.
- (3) The solution $u(x)$ is a singular distribution and satisfy (1.0.2). It is then a *distributional solution*.

All these solution are call *generalized solution*.

A purpose of this research is to find the solution of the partial differential equation that the solution is a weak solution or distributional solution.

Chapter 2

Main Results

2.1 Weak solutions of the compound equations

We published two paper in this topics (see Appendices A1 and A5). We study the compound equation related to the ultra-hyperbolic operator of the form

$$\sum_{r=0}^m C_r \square_c^r u(x) = f(x),$$

where \square_c^r is the operator which related to the ultra-hyperbolic type operator iterated r -times, f is a given generalized function, u is an unknown function, $x \in \mathbb{R}^n$ and C_r is a constant.

The above compound equation is inspired by the compound ultra-hyperbolic equation introduced and studied by Kananthai and Nonlaopon [A. Kananthai, K. Nonlaopon, On the weak solution of the compound ultrahyperbolic equation, CMU J. 1(3) (2002), 209-214.]. In fact, the compound ultra-hyperbolic equation is included in above compound equation if we let $c = 1$. See Appendix A1.

The paper A5 changes operator in above equation from ultra-hyperbolic operator to ultra-hyperbolic Bessel operator. We prove a theorem for this new equation about distributional solution.

2.2 Nonlinear equations

2.2.1 Nonlinear ultra-hyperbolic heat equation

Consider the nonlinear equation of the form

$$\frac{\partial}{\partial t} u(x, t) - c^2 \square^k u(x, t) = f(x, t, u(x, t)),$$

where \square^k is the ultra-hyperbolic operator iterated k -times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space \mathbb{R}^n , $(x, t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, k is a positive integer and c is a positive constant.

In the paper A2, we prove that under the suitable conditions for f, u and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n \times (0, \infty)$. Moreover, if we put $k = 1$ and $q = 0$ we obtain the solution of nonlinear equation related to the heat equation.

2.2.2 Nonlinear Laplace and Laplace-Bessel operators

We study the solution of nonlinear equation

$$\Delta^k u(x) = f(x, \Delta^{k-1} u(x)),$$

where Δ^k is the Laplace operator iterated k -times, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a given function. In the paper A3, we found that the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\Delta^{k-1} u(x)$. Moreover, we study the solution of nonlinear equation

$$\Delta_B^k u(x) = f(x, \Delta_B^{k-1} u(x)),$$

where Δ_B^k is the Bessel-Laplace operator iterated k -times and $x \in \mathbb{R}_n^+$.

2.3 Convolution equation

In the paper A4, we study the distribution $e^{\alpha x} \square_c^k \delta$, where \square_c^k is the operator which related to the ultra-hyperbolic type operator iterated k -times, δ is the Dirac-delta distribution, $x = (x_1, x_2, \dots, x_n)$ is a variable and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a constant and both are the points in the n -dimensional Euclidean spaces \mathbb{R}^n .

At first, the properties of $e^{\alpha x} \square_c^k \delta$ are studied and later we study the application of $e^{\alpha x} \square_c^k \delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x} \square_c^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \square_c^r \delta,$$

where $u(x)$ is the generalized function and C_r is a constant. We found that its solution related to the ultra-hyperbolic equation and moreover, the type of solutions of the convolution equation, such as the ordinary functions, the tempered distributions or the singular distributions depending on k, m and α .

Appendix

- A1 Sudprathai Bupasiri and **Kamsing Nonlaopon**, On the weak solutions of compound equations related to the ultra-hyperbolic operators, *Far East Journal of Applied Mathematics*, 35(2009), no. 1, 129–139.
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ON THE WEAK SOLUTIONS OF COMPOUND EQUATIONS RELATED TO THE ULTRA-HYPERBOLIC OPERATORS

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Abstract

In this article, we have studied compound equations related to the ultra-hyperbolic operator of the form

$$\sum_{r=0}^m C_r \square_c^r u(x) = f(x),$$

where \square_c^r is the operator which related to the ultra-hyperbolic type operator iterated r -times, f is a given generalized function, u is an unknown function, $x \in \mathbb{R}^n$ and C_r is a constant. In this work, we study the weak solution $u(x)$ of the above equation and moreover, such a solution is unique.

1. Introduction

Gelfand and Shilov [2] have first introduced the elementary solution of the n -dimensional classical diamond operator. Trione [9] has shown

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that the n -dimensional ultra-hyperbolic equation has $u(x) = R_{2k,1}(x)$ as a unique elementary solution. Later, Tellez [8] has proved that $R_{2k,1}(x)$ exists only for case p is odd with $p + q = n$.

Kanantjai and Nonlaopon [4] have studied the weak solution of the compound ultra-hyperbolic equation. Sarikaya and Yildirim [6] have studied the weak solution of the compound Bessel ultra-hyperbolic equation.

Kanantjai [3] has showed that the solution of the convolution form $u(x) = R_{2k,c_1}(x) * R_{2k,c_2}(x)$ is a unique elementary solution of the equation $\square_{c_1}^k \square_{c_2}^k u(x) = \delta$, where $\square_{c_1}^k$ and $\square_{c_2}^k$ are the operators which related to the ultra-hyperbolic type operator iterated k -times and δ is the Dirac-delta distribution, and in particular, if $k = p = 1$ with $x_1 = t$ (times), c_1 and c_2 are velocities then $u(x) = R_{2,c_1}(x) * R_{2,c_2}(x)$ is the elementary solution of the elastic wave equation of fourth order. Sritanratana and Kanantjai [7] have studied the product of the nonlinear diamond operators related to the elastic wave and also introduced the ultra-hyperbolic operator \square_c^k .

In this article, we will consider the operator which related to the ultra-hyperbolic type operator iterated k -times with $x \in \mathbb{R}^n$

$$\square_c^k = \left(\frac{1}{c^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k \quad (1.1)$$

and equation

$$\square_c^k u(x) = f(x), \quad (1.2)$$

where $u(x)$ and f are some generalized functions.

We will develop equation (1.2) to the form

$$\sum_{r=0}^m C_r \square_c^r u(x) = f(x), \quad (1.3)$$

and by convention $\square_c^0 u(x) = u(x)$. In finding the solutions of (1.3), we use the properties of convolutions for the generalized functions.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional space \mathbb{R}^n ,

$$V = c^2(x_1^2 + x_2^2 + \dots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad (2.1)$$

where $p + q = n$, the *interior of forward cone* defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, V > 0\}$. For any complex number α , define

$$R_{\alpha, c}(x) = \begin{cases} \frac{V^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \quad (2.2)$$

where

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2 + \alpha - n}{2}\right) \Gamma\left(\frac{1 - \alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{\alpha - p + 2}{2}\right) \Gamma\left(\frac{p - \alpha}{2}\right)}. \quad (2.3)$$

The function $R_\alpha(x) = R_{\alpha, 1}(x)$ is introduced by Nozaki [5]. It is well known that such function is ordinary function if $Re(\alpha) \geq n$ and is distribution of α if $Re(\alpha) < n$.

By putting $p = c = 1$ in (2.1) and (2.3), and using the Legendre's duplication of $\Gamma(z)$, then (2.2) reduces to

$$M_\alpha(x) = \begin{cases} \frac{V^{(\alpha-n)/2}}{H_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+, \end{cases}$$

where

$$H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha - n + 2}{2}\right) \Gamma\left(\frac{\alpha}{2}\right), \quad V = c^2 x_1^2 - x_2^2 - \dots - x_n^2.$$

The function $M_\alpha(x)$ is precisely called the *hyperbolic kernel* of Marcel Riesz.

Lemma 2.1. $R_{\alpha,c}(x)$ is a homogeneous distribution of order $(\alpha - n)$.

In particular, it is a tempered distribution.

Proof. We need to show that $R_{\alpha,c}(x)$ satisfies Euler equation

$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{\alpha,c}(x) = (\alpha - n)R_{\alpha,c}(x).$$

Now,

$$\begin{aligned} & \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{\alpha,c}(x) \\ &= \frac{1}{K_n(\alpha)} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{(\alpha-n)/2} \\ &= \frac{1}{K_n(\alpha)} (\alpha - n) (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{(\alpha-n-2)/2} \\ & \quad \times (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2) \\ &= \frac{1}{K_n(\alpha)} (\alpha - n) (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{(\alpha-n)/2} \\ &= \frac{(\alpha - n)V^{(\alpha-n)/2}}{K_n(\alpha)} = (\alpha - n)R_{\alpha,c}(x). \end{aligned}$$

Hence $R_{\alpha,c}(x)$ is a homogeneous distribution of order $(\alpha - n)$.

Donoghue [1] proved that every homogeneous distribution is a tempered distribution. So, $R_{\alpha,c}(x)$ is a tempered distribution. This completes the proof.

Lemma 2.2. Given equation

$$\square_c^k u(x) = \delta(x), \tag{2.4}$$

where \square_c^k is defined by (1.1), $x \in \mathbb{R}^n$ and δ is the Dirac-delta distribution, then we obtain $u(x) = R_{2k,c}(x)$ as a unique elementary solution of (2.4), where $R_{2k,c}(x)$ is defined by (2.2) with $\alpha = 2k$.

Proof. See [9].

Lemma 2.3 (The convolutions of tempered distributions)

- (a) $(\square_c^k \delta) * u(x) = \square_c^k u(x)$, where u is any tempered distribution.
- (b) Let $R_{2k,c}(x)$ and $R_{2m,c}(x)$ be defined by (2.2), then $R_{2k,c}(x) * R_{2m,c}(x)$ exists and is a tempered distribution.
- (c) Let $R_{2k,c}(x)$ and $R_{2m,c}(x)$ be defined by (2.2), then $R_{2k,c}(x) * R_{2m,c}(x) = R_{2k+2m,c}(x)$, where k and m are nonnegative integers.
- (d) Let $R_{2k,c}(x)$ and $R_{2m,c}(x)$ be defined by (2.2) and if $R_{2k,c}(x) * R_{2m,c}(x) = \delta$, then $R_{2k,c}(x)$ is an inverse of $R_{2m,c}(x)$ in the convolution algebra denoted by $R_{2k,c}(x) = R_{2m,c}^{*-1}(x)$, moreover, $R_{2m,c}^{*-1}(x)$ is unique.

Proof.

- (a) First, we consider the case $k = 1$, now

$$\square_c \delta(x) = \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \delta(x)}{\partial x_i^2} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(x)}{\partial x_j^2} \right), \quad p + q = n,$$

and let $\varphi(x)$ be a testing function in the Schwarts space S . By the definition of convolution, we have

$$\begin{aligned} & \langle \square_c \delta(x) * u(x), \varphi(x) \rangle \\ &= \langle u(x), \langle \square_c \delta(x), \varphi(x + y) \rangle \rangle \\ &= \left\langle u(x), \left\langle \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \delta(y)}{\partial x_i^2} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(y)}{\partial x_j^2} \right), \varphi(x + y) \right\rangle \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle u(x), \left\langle \delta(y), \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \varphi(x+y)}{\partial x_i^2} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \varphi(x+y)}{\partial x_j^2} \right) \right\rangle \right\rangle \\
&= \left\langle u(x), \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \varphi(x)}{\partial x_i^2} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \varphi(x)}{\partial x_j^2} \right) \right\rangle \\
&= \left\langle \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 u(x)}{\partial x_i^2} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 u(x)}{\partial x_j^2} \right), \varphi(x) \right\rangle \\
&= \langle \square_c u(x), \varphi(x) \rangle. \tag{2.5}
\end{aligned}$$

It follows that

$$\square_c \delta(x) * u(x) = \square_c u(x).$$

Similarly, for any k , we can show that

$$\square_c^k \delta(x) * u(x) = \square_c^k u(x).$$

(b) Since $R_{2k,c}(x)$ and $R_{2m,c}(x)$ are tempered distributions by Lemma 2.1. Now, chosen $\text{supp } R_{2k,c}(x) = K \subset \bar{\Gamma}_+$, where K is a compact set and $\bar{\Gamma}_+$ designates Γ_+ closure. Then $R_{2k,c}(x)$ is a tempered distribution with compact support. By [1], $R_{2k,c}(x) * R_{2m,c}(x)$ exists and is a tempered distribution.

(c) From equation $\square_c^{k+m} u(x) = \delta$, we obtain $u(x) = R_{2k+2m,c}(x)$ by Lemma 2.2. For any m is a nonnegative integer, we write

$$\square_c^{k+m} u(x) = \square_c^k \square_c^m u(x) = \delta,$$

then by Lemma 2.2, we have the following equality:

$$\square_c^m u(x) = R_{2k,c}(x).$$

Convolving both the sides by $R_{2m,c}(x)$, we obtain

$$R_{2m,c}(x) * \square_c^m u(x) = R_{2k,c}(x) * R_{2m,c}(x)$$

or

$$\square_c^m R_{2m,c}(x) * u(x) = R_{2k,c}(x) * R_{2m,c}(x).$$

Then from Lemma 2.2, we have the following equality:

$$\delta * u(x) = R_{2k,c}(x) * R_{2m,c}(x).$$

It follows that

$$u(x) = R_{2k,c}(x) * R_{2m,c}(x).$$

From the fact that $u(x) = R_{2k+2m,c}(x)$, we obtain $R_{2k,c}(x) * R_{2m,c}(x) = R_{2k+2m,c}(x)$.

(d) Since $R_{2k,c}(x)$ and $R_{2m,c}(x)$ are tempered distributions with compact supports, thus $R_{2k,c}(x)$ and $R_{2m,c}(x)$ are the elements of space of convolution algebra u' of distribution. Now, $R_{2k,c}(x) * R_{2m,c}(x) = \delta$, then by Zemanian [10] it is shown that $R_{2k,c}(x) = R_{2m,c}^{*-1}(x)$ is a unique inverse.

Lemma 2.4. *Let $R_{2k,c}(x)$ and $K_n(2k)$ be defined by (2.2) and (2.3).*

Then

$$(a) K_n(2k+2) = 2k(2k+2-n)K_n(2k).$$

(b) $\square_c^k R_{2m,c}(x) = R_{2m-2k,c}(x)$, where k and m are nonnegative integers.

$$(c) R_{-2k,c}(x) = \square_c^k \delta(x), \text{ where } k \text{ is a nonnegative integer.}$$

Proof.

(a) From (2.3), we have

$$\begin{aligned} & K_n(2k+2) \\ &= \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2k+4-n}{2}\right) \Gamma\left(\frac{-1-2k}{2}\right) \Gamma(2k+2)}{\Gamma\left(\frac{4+2k-p}{2}\right) \Gamma\left(\frac{p-2k-2}{2}\right)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi^{(n-1)/2} \left(\frac{2k+2-n}{2}\right) \Gamma\left(\frac{2k+2-n}{2}\right) \left(\frac{-2}{1+2k}\right) \Gamma\left(\frac{1-2k}{2}\right) (2k)(2k+1)\Gamma(2k)}{\left(\frac{2+2k-p}{2}\right) \Gamma\left(\frac{2+2k-p}{2}\right) \left(\frac{2}{p-2k-2}\right) \Gamma\left(\frac{p-2k}{2}\right)} \\
&= 2k(2k+2-n)K_n(2k).
\end{aligned}$$

(b) By Lemma 2.3 (c), we have

$$\delta * R_{2m,c}(x) = R_{2k,c}(x) * R_{2m-2k,c}(x),$$

$$\square_c^k R_{2k,c}(x) * R_{2m,c}(x) = R_{2k,c}(x) * R_{2m-2k,c}(x),$$

$$R_{2k,c}(x) * \square_c^k R_{2m,c}(x) = R_{2k,c}(x) * R_{2m-2k,c}(x)$$

and

$$\square_c^k R_{2m,c}(x) = R_{2m-2k,c}(x).$$

(c) For $m = k$, by Lemma 2.4 (b), we have

$$\square_c^m R_{2m,c}(x) = R_{0,c}(x), \quad R_{0,c} = \delta.$$

For $m = 0$, by Lemma 2.4 (b), we have

$$\square_c^k R_{0,c}(x) = R_{-2k,c}(x) \text{ or } \square_c^k \delta = R_{-2k,c}(x).$$

3. Main Results

Theorem 3.1. *Given the compound equation related to the ultra-hyperbolic operator of the form*

$$\sum_{r=0}^m C_r \square_c^r u(x) = f(x), \quad (3.1)$$

where \square_c^r is the operator which related to the ultra-hyperbolic type operator iterated r -times, f is a given generalized function, u is an unknown function, $x \in \mathbb{R}^n$ and C_r is a constant. Then (3.1) has a unique

weak solution

$$u(x) = f(x) * R_{2m,c}(x) * (C_m R_{0,c}(x) + w(x) R_{2,c}(x))^{*-1}, \quad (3.2)$$

where

$$\begin{aligned} w(x) = & C_{m-1} + C_{m-2} \frac{V}{2(4-n)} + C_{m-3} \frac{V^2}{2 \cdot 4(4-n)(6-n)} + \dots \\ & + C_0 \frac{V^{m-1}}{2 \cdot 4 \cdot 6 \dots 2(m-1)(4-n)(6-n) \dots (2m-n)} \end{aligned} \quad (3.3)$$

and V defined by (2.1) and $(C_m R_{0,c}(x) + w(x) R_{2,c}(x))^{*-1}$ is an inverse of $C_m R_{0,c}(x) + w(x) R_{2,c}(x)$.

Proof. By Lemma 2.3 (a), equation (3.1) can be written as

$$(C_m \square_c^m \delta + C_{m-1} \square_c^{m-1} \delta + \dots + C_1 \square_c \delta + C_0 \delta) * u(x) = f(x).$$

Convolving both the sides by $R_{2m,c}(x)$ defined by (2.2), we obtain

$$\begin{aligned} [C_m \square_c^m R_{2m,c}(x) + C_{m-1} \square_c^{m-1} R_{2m,c}(x) + \dots + C_1 \square_c R_{2m,c}(x) \\ + C_0 R_{2m,c}(x)] * u(x) = f(x) * R_{2m,c}(x). \end{aligned}$$

By Lemma 2.2 and Lemma 2.4 (b), we obtain

$$\begin{aligned} [C_m \delta + C_{m-1} R_{2,c}(x) + C_{m-2} R_{4,c}(x) + \dots + C_1 R_{2(m-1),c}(x) \\ + C_0 R_{2m,c}(x)] * u(x) = f(x) * R_{2m,c}(x). \end{aligned} \quad (3.4)$$

By Lemma 2.4 (a), we obtain

$$R_{4,c}(x) = \frac{V^{(4-n)/2}}{K_n(4)} = \frac{V^{(2-n)/2} \cdot V}{2(2+2-n)K_n(2)} = R_{2,c}(x) \cdot \frac{V}{2(4-n)}.$$

Similarly,

$$R_{6,c}(x) = R_{2,c}(x) \cdot \frac{V^2}{2 \cdot 4(4-n)(6-n)}$$

$$R_{8,c}(x) = R_{2,c}(x) \cdot \frac{V^3}{2 \cdot 4 \cdot 6(4-n)(6-n)(8-n)}$$

⋮

$$R_{2m,c}(x) = R_{2,c}(x) \cdot \frac{V^{m-1}}{2 \cdot 4 \cdot 6 \cdots 2(m-1)(4-n)(6-n)(8-n) \cdots (2m-n)}.$$

Thus, we obtain the function $w(x)$ of (3.3). Now, $w(x)$ is continuous and infinitely differentiable in classical sense for n is odd. Since $R_{2,c}(x)$ is a tempered distribution with compact support, hence $w(x)R_{2,c}(x)$ also is tempered distribution with compact support and so $C_m R_{0,c}(x) + w(x)R_{2,c}(x)$. By Lemma 2.3 (d), $C_m R_{0,c}(x) + w(x)R_{2,c}(x)$ has a unique inverse denoted by

$$(C_m R_{0,c}(x) + w(x)R_{2,c}(x))^{*-1}.$$

Now, (3.4) can be written as

$$(C_m R_{0,c}(x) + w(x)R_{2,c}(x)) * u(x) = f(x) * R_{2m,c}(x), \quad R_{0,c} = \delta.$$

Convolving both the sides by $(C_m R_{0,c}(x) + w(x)R_{2,c}(x))^{*-1}$, we have

$$u(x) = f(x) * R_{2m,c}(x) * (C_m R_{0,c}(x) + w(x)R_{2,c}(x))^{*-1}.$$

Since $R_{2m,c}(x)$ is a unique by Lemma 2.2 and $(C_m R_{0,c}(x) + w(x)R_{2,c}(x))^{*-1}$ also is a unique by Lemma 2.3 (d), it follows that $u(x)$ is a unique weak solution of (3.1) with odd-dimensional n . This completes the proof.

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On the generalized nonlinear ultra-hyperbolic heat equation related to the spectrum

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Abstract. In this paper, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t} u(x, t) - c^2 \square^k u(x, t) = f(x, t, u(x, t))$$

where \square^k is the ultra-hyperbolic operator iterated k -times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space \mathbb{R}^n , $(x, t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, k is a positive integer and c is a positive constant.

On the suitable conditions for f , u and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n \times (0, \infty)$. Moreover, if we put $k = 1$ and $q = 0$ we obtain the solution of nonlinear equation related to the heat equation.

Mathematical subject classification: 35L30, 46F12, 32W30.

Key words: ultra-hyperbolic heat equation, the Dirac delta distribution, the spectrum, Fourier transform.

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1 Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Delta u(x, t) \quad (1.1)$$

with the initial condition

$$u(x, 0) = f(x)$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x, t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, and f is a continuous function, we obtain the solution

$$u(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|x-y|^2}{4c^2t}\right] f(y) dy \quad (1.2)$$

as the solution of (1.1).

Now, (1.2) can be written as $u(x, t) = E(x, t) * f(x)$ where

$$E(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left[-\frac{|x|^2}{4c^2t}\right]. \quad (1.3)$$

$E(x, t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$ and $t > 0$, see [1, p. 208–209].

Moreover, we obtain $E(x, t) \rightarrow \delta$ as $t \rightarrow 0$, where δ is the Dirac-delta distribution. We also have extended (1.1) to be the equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \square u(x, t) \quad (1.4)$$

where \square is *the ultra-hyperbolic operator*, defined by

$$\square = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right).$$

We obtain *the ultra-hyperbolic heat kernel*

$$E(x, t) = \frac{(i)^q}{(4c^2\pi t)^{n/2}} \exp\left[\frac{\sum_{i=1}^p x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2t}\right]$$

where $p + q = n$ is the dimension of the Euclidean space \mathbb{R}^n and $i = \sqrt{-1}$. For finding the kernel $E(x, t)$ see [4].

In this paper, we extend (1.4) to be the general of the nonlinear form

$$\frac{\partial}{\partial t} u(x, t) - c^2 \square^k u(x, t) = f(x, t, u(x, t)) \quad (1.5)$$

for $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and with the following conditions on u and f as follows,

- (1) $u(x, t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$ -derivatives.
- (2) f satisfies the Lipchitz condition, that is

$$|f(x, t, u) - f(x, t, w)| \leq A|u - w|$$

where A is constant and $0 < A < 1$.

- (3)

$$\int_0^\infty \int_{\mathbb{R}^n} |f(x, t, u(x, t))| dx dt < \infty$$

for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ and $u(x, t)$ is continuous function on $\mathbb{R}^n \times (0, \infty)$.

Under such conditions of f , u and for the spectrum of $E(x, t)$, we obtain the convolution

$$u(x, t) = E(x, t) * f(x, t, u(x, t))$$

as a unique solution in the compact subset of $\mathbb{R}^n \times (0, \infty)$ and $E(x, t)$ is an elementary solution defined by (2.5).

2 Preliminaries

Definition 2.1. Let $f(x) \in \mathbb{L}_1(\mathbb{R}^n)$ -the space of integrable function in \mathbb{R}^n . The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx \quad (2.1)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$ is the usual inner product in \mathbb{R}^n and $dx = dx_1 dx_2 \dots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \widehat{f}(\xi) d\xi. \quad (2.2)$$

Definition 2.2. *The spectrum of the kernel $E(x, t)$ defined by (2.5) is the bounded support of the Fourier transform $\widehat{E}(\xi, t)$ for any fixed $t > 0$.*

Definition 2.3. *Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be a point in \mathbb{R}^n and we write*

$$u = \xi_1^2 + \xi_2^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \dots - \xi_{p+q}^2, \quad p + q = n.$$

Denote by

$$\Gamma_+ = \{ \xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0 \}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+ .

Let Ω be spectrum of $E(x, t)$ defined by Definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E}(\xi, t)$ be the Fourier transform of $E(x, t)$ and define

$$\widehat{E}(\xi, t) = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k \right] & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \quad (2.3)$$

Lemma 2.1. *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t} - c^2 \square^k \quad (2.4)$$

where \square^k is the ultra-hyperbolic operator iterated k -times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of \mathbb{R}^n , $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$, k is a positive integer and c is a positive constant. Then we obtain

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k + i(\xi, x) \right] d\xi \quad (2.5)$$

as a elementary solution of (2.4) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $LE(x, t) = \delta(x, t)$ where $E(x, t)$ is the kernel or the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x, t) - c^2 \square^k E(x, t) = \delta(x)\delta(t).$$

Take the Fourier transform defined by (2.1) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t} \widehat{E(\xi, t)} - c^2 \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k \widehat{E(\xi, t)} = \frac{1}{(2\pi)^{n/2}} \delta(t).$$

Thus

$$\widehat{E(\xi, t)} = \frac{H(t)}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k \right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi, t)} = \frac{1}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k \right]$$

which has been already defined by (2.3). Thus

$$E(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \widehat{E(\xi, t)} d\xi = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(\xi, x)} \widehat{E(\xi, t)} d\xi$$

where Ω is the spectrum of $E(x, t)$. Thus from (2.3)

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k + i(\xi, x) \right] d\xi \quad \text{for } t > 0.$$

□

Definition 2.4. Let us extend $E(x, t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting

$$E(x, t) = \begin{cases} \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k + i(\xi, x) \right] d\xi & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

3 Main Results

Theorem 3.1. *The kernel $E(x, t)$ defined by (2.5) have the following properties:*

(1) $E(x, t) \in C^\infty$ -the space infinitely differentiable.

(2) $\left(\frac{\partial}{\partial t} - c^2 \square^k\right) E(x, t) = 0$ for $t > 0$.

(3)

$$|E(x, t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q}{2}\right)}, \quad \text{for } t > 0,$$

where $M(t)$ is a function of t in the spectrum Ω and Γ denote the Gamma function. Thus $E(x, t)$ is bounded for any fixed $t > 0$.

(4) $\lim_{t \rightarrow 0} E(x, t) = \delta$.

Proof.

(1) From (2.5), since

$$\frac{\partial^n}{\partial x^n} E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \frac{\partial^n}{\partial x^n} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k + i(\xi, x) \right] d\xi.$$

Thus $E(x, t) \in C^\infty$ for $x \in \mathbb{R}^n, t > 0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} - c^2 \square^k\right) E(x, t) = 0.$$

(3) We have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k + i(\xi, x) \right] d\xi.$$

$$|E(x, t)| \leq \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k \right] d\xi.$$

By changing to bipolar coordinates

$$\begin{aligned} \xi_1 &= r\omega_1, \xi_2 = r\omega_2, \dots, \xi_p = r\omega_p \quad \text{and} \\ \xi_{p+1} &= s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \dots, \xi_{p+q} = s\omega_{p+q} \end{aligned}$$

where $\sum_{i=1}^p \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$. Thus

$$|E(x, t)| \leq \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t (s^2 - r^2)^k \right] r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q$$

where $d\xi = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of surface area of the unit sphere in \mathbb{R}^p and \mathbb{R}^q respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x, t)$ and we suppose $0 \leq r \leq R$ and $0 \leq s \leq L$ where R and L are constants. Thus we obtain

$$\begin{aligned} |E(x, t)| &\leq \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^R \int_0^L \exp \left[c^2 t (s^2 - r^2)^k \right] r^{p-1} s^{q-1} ds dr \\ &= \frac{\Omega_p \Omega_q}{(2\pi)^n} M(t) \quad \text{for any fixed } t > 0 \text{ in the spectrum } \Omega \\ &= \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \end{aligned} \tag{3.1}$$

where

$$M(t) = \int_0^R \int_0^L \exp \left[c^2 t (s^2 - r^2)^k \right] r^{p-1} s^{q-1} ds dr \tag{3.2}$$

is a function of

$$t > 0, \quad \Omega_p = \frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})} \quad \text{and} \quad \Omega_q = \frac{2\pi^{q/2}}{\Gamma(\frac{q}{2})}.$$

Thus, for any fixed $t > 0$, $E(x, t)$ is bounded.

(4) By (2.5), we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^p \xi_i^2 \right)^k + i(\xi, x) \right] d\xi.$$

Since $E(x, t)$ exists, then

$$\begin{aligned} \lim_{t \rightarrow 0} E(x, t) &= \frac{1}{(2\pi)^n} \int_{\Omega} e^{i(\xi, x)} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi, x)} d\xi \\ &= \delta(x), \quad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

See [3, p. 396, Eq. (10.2.19b)]. □

Theorem 3.2. *Given the nonlinear equation*

$$\frac{\partial}{\partial t} u(x, t) - c^2 \square^k u(x, t) = f(x, t, u(x, t)) \quad (3.3)$$

for $(x, t) \in \mathbb{R}^n \times (0, \infty)$, k is positive number and with the following conditions on u and f as follows,

- (1) $u(x, t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$ -derivatives.
- (2) f satisfies the Lipchitz condition, that is

$$|f(x, t, u) - f(x, t, w)| \leq A|u - w|$$

where A is constant and $0 < A < 1$.

- (3)

$$\int_0^\infty \int_{\mathbb{R}^n} |f(x, t, u(x, t))| dx dt < \infty$$

for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ and $u(x, t)$ is continuous function on $\mathbb{R}^n \times (0, \infty)$.

Then, for the spectrum of $E(x, t)$ we obtain the convolution

$$u(x, t) = E(x, t) * f(x, t, u(x, t)) \quad (3.4)$$

as a unique solution of (3.3) for $x \in \Omega_0$ where Ω_0 is an compact subset of \mathbb{R}^n , $0 \leq t \leq T$ with T is constant and $E(x, t)$ is an elementary solution defined by (2.5) and also $u(x, t)$ is bounded.

In particular, if we put $k = 1$ and $q = 0$ in (3.3) then (3.3) reduces to the nonlinear heat equation.

Proof. Convolving both sides of (3.3) with $E(x, t)$ and then we obtain the solution

$$u(x, t) = E(x, t) * f(x, t, u(x, t))$$

or

$$u(x, t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r, s) f(x - r, t - s, u(x - r, t - s)) dr ds$$

where $E(r, s)$ is given by Definition 2.4.

We next show that $u(x, t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$. We have

$$\begin{aligned} |u(x, t)| &\leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r, s)| |f(x - r, t - s, u(x - r, t - s))| dr ds \\ &\leq \frac{2^{2-n} N.M(t)}{\pi^{n/2} \Gamma(\frac{p}{2}) \Gamma(\frac{q}{2})} \end{aligned}$$

by the condition (3) and (3.1) where

$$N = \int_0^{\infty} \int_{\mathbb{R}^n} |f(x, t, u(x, t))| dx dt.$$

Thus $u(x, t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$.

To show that $u(x, t)$ is unique, suppose there is another solution $w(x, t)$ of equation (3.3). Let the operator

$$L = \frac{\partial}{\partial t} - c^2 \square^k$$

then (3.3) can be written in the form

$$L u(x, t) = f(x, t, u(x, t)).$$

Thus

$$L u(x, t) - L w(x, t) = f(x, t, u(x, t)) - f(x, t, w(x, t)).$$

By the condition (2) of the Theorem,

$$|L u(x, t) - L w(x, t)| \leq A |u(x, t) - w(x, t)|. \quad (3.5)$$

Let $\Omega_0 \times (0, T]$ be compact subset of $\mathbb{R}^n \times (0, \infty)$ and $L: C^{(2k)}(\Omega_0) \rightarrow C^{(2k)}(\Omega_0)$ for $0 \leq t \leq T$.

Now $(C^{(2k)}(\Omega_0), \|\cdot\|)$ is a Banach space where $u(x, t) \in C^{(2k)}(\Omega_0)$ for $0 \leq t \leq T$, $\|\cdot\|$ given by

$$\|u(x, t)\| = \sup_{x \in \Omega_0} |u(x, t)|.$$

Then, from (3.5) with $0 < A < 1$, the operator L is a contraction mapping on $C^{(2k)}(\Omega_0)$. Since $(C^{(2k)}(\Omega_0), \|\cdot\|)$ is a Banach space and $L: C^{(2k)}(\Omega_0) \rightarrow C^{(2k)}(\Omega_0)$ is a contraction mapping on $C^{(2k)}(\Omega_0)$, by Contraction Theorem, see [3, p. 300], we obtain the operator L has a fixed point and has uniqueness property. Thus $u(x, t) = w(x, t)$. It follows that the solution $u(x, t)$ of (3.3) is unique for $(x, t) \in \Omega_0 \times (0, T]$ where $u(x, t)$ is defined by (3.4).

In particular, if we put $k = 1$ and $q = 0$ in (3.3) then (3.3) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t} u(x, t) - c^2 \Delta u(x, t) = f(x, t, u(x, t))$$

which has solution

$$u(x, t) = E(x, t) * f(x, t, u(x, t))$$

where $E(x, t)$ is defined by (2.5) with $k = 1$ and $q = 0$. That is complete of proof. \square

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On the Generalized Nonlinear Laplace and Laplace-Bessel Operator

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Abstract. In this paper, we study the solution of nonlinear equation $\Delta^k u(x) = f(x, \Delta^{k-1} u(x))$ where Δ^k is the Laplace operator iterated k -times, $x \in \mathbb{R}^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a given function. It is found that the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\Delta^{k-1} u(x)$. Moreover, we study the solution of nonlinear equation $\Delta_B^k u(x) = f(x, \Delta_B^{k-1} u(x))$ where Δ_B^k is the Laplace-Bessel operator iterated k -times and $x \in \mathbb{R}_n^+$.

Keywords: Laplace-Bessel operator; Bessel ultra-hyperbolic operator; Dirac-delta distribution

1 Introduction

Soliton theory is one of the most important aspect in nonlinearity, which is widely applied in many natural sciences such as chemistry, biology, mathematics, communication and physics. For finding some new exact solutions of nonlinear equations, a wealth of some effective works have been presented [1, 2, 14]. However, a weak solution has been studied too and there are many different definitions of weak solution, appropriate for different classes of equations. One of the most important is based on the notation of distributions [3, 4, 11, 12]. In this work, we study solution of nonlinear equation $\Delta^k u(x) = f(x, \Delta^{k-1} u(x))$ and $\Delta_B^k u(x) = f(x, \Delta_B^{k-1} u(x))$ in the sense of distributions, weak solution.

R. Courant and D. Hilbert [5] have studied the nonlinear equation of the form $\Delta u(x) = f(x, u(x))$ with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open set in \mathbb{R}^n , $\partial\Omega$ denotes the boundary of Ω and Δ is the Laplace operator, defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \quad (1)$$

They found that the solution $u(x)$ of such equation is unique under the condition $|f(x, u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $u(x) = 0$ for all $x \in \partial\Omega$.

A. Kananthai [10] has first introduced the operator \diamond^k and is named the diamond operator iterated k -times and is defined by

$$\diamond^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right]^k, \quad (2)$$

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$p + q = n$ and moreover, he has studied elementary solution of the n -dimensional diamond operator.

Next, A. Kananthai [7–9] has studied the convolution equation related to the diamond kernel of Marcel Riesz, the convolution of the diamond kernel of Marcel Riesz and the general solution of the equation $\diamond^k u(x) = f(x)$.

Later, G. Sritanratana and A. Kananthai [17] have studied the nonlinear equation of the form

$$\diamond^k u(x) = f(x, \Delta^{k-1} \square^k u(x)) \quad (3)$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where \square^k is the ultra-hyperbolic operator iterated k -times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \quad (4)$$

They found that the solution $u(x)$ of (3) which is unique under the condition $|f(x, \Delta^{k-1} \square^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\Delta^{k-1} \square^k u(x) = 0$ for all $x \in \partial\Omega$.

H. Yildirim et al. [18] have introduced the Bessel diamond operator iterated k -times with $x \in \mathbb{R}_n^+ = \{x : x = (x_1, x_2, \dots, x_n), x_1 > 0, \dots, x_n > 0\}$,

$$\diamond_B^k = \left[(B_{x_1} + \cdots + B_{x_p})^2 - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})^2 \right]^k \quad (5)$$

where $p + q = n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}$ where $2v_i = 2\beta_i + 1$, $\beta_i > -\frac{1}{2}$ [13], k is nonnegative integer and n is the dimension of \mathbb{R}_n^+ and studied the elementary solution of this operator. Moreover, they have studied the Fourier-Bessel transform of the elementary solution.

Next, M. Z. Sarikaya and H. Yildirim [15, 16] have studied the B -convolution of the Bessel diamond kernel of Riesz and the nonlinear equation of the form

$$\diamond_B^k u(x) = f(x, \Delta_B^{k-1} \square_B^k u(x)) \quad (6)$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where \square_B^k is the Bessel ultra-hyperbolic operator iterated k -times, defined by

$$\square_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}})^k, \quad (7)$$

Ω is an open set in \mathbb{R}_n^+ and $\partial\Omega$ denotes the boundary of Ω . They found that the solution $u(x)$ of (6) which is unique under the condition $|f(x, \Delta_B^{k-1} \square_B^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\Delta_B^{k-1} \square_B^k u(x) = 0$ for all $x \in \partial\Omega$.

In this work, we will study the nonlinear equation of the form

$$\Delta^k u(x) = f(x, \Delta^{k-1} u(x)) \quad (8)$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open set in \mathbb{R}^n and $\partial\Omega$ denotes the boundary of Ω . We will find the solution $u(x)$ of (8) which is unique under the condition $|f(x, \Delta^{k-1} u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\Delta^{k-1} u(x) = 0$ for all $x \in \partial\Omega$. Moreover, we will study the nonlinear equation of the form

$$\Delta_B^k u(x) = f(x, \Delta_B^{k-1} u(x)) \quad (9)$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of \mathbb{R}_n^+ and $\partial\Omega$ denotes the boundary of Ω . We will find the solution $u(x)$ of (9) which is unique under the condition $|f(x, \Delta_B^{k-1} u(x))| \leq M$ where M is a constant for all $x \in \Omega$ and the boundary condition $\Delta_B^{k-1} u(x) = 0$ for all $x \in \partial\Omega$.

2 Preliminaries

In this section, we give some notations and definitions.

Definition 2.1 Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathbb{R}^n and the function $R_\alpha^e(x)$ be defined by

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{10}$$

where $W_n(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma((n-\alpha)/2)$, α is a complex parameter and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$.

It can be shown that $R_{-2k}^e(x) = (-1)^k \Delta^k \delta$ where Δ^k is defined by (1) and δ is the Dirac-delta distribution. It follows that $R_0^e(x) = \delta$, see ([6], p.74).

Moreover, the function $(-1)^k R_{2k}^e(x)$ an elementary solution of the Laplace operator iterated k -times Δ^k , that is

$$\Delta^k \left((-1)^k R_{2k}^e(x) \right) = \delta, \tag{11}$$

see ([10], Lemma 2.4).

The function $E(x) = -S_2(x)$ as defined by (14) is an elementary solution of the Laplace-Bessel operator

$$\Delta_B = \sum_{i=1}^n B_{x_i} = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i} \right), \tag{12}$$

that is, $\Delta_B E(x) = \delta$ where $x \in \mathbb{R}_n^+$.

The operator \diamond_B^k can be expressed as the product of the operators \square_B^k and Δ_B^k , that is

$$\diamond_B^k = \left[\left(\sum_{i=1}^p B_{x_i} \right)^2 - \left(\sum_{i=p+1}^{p+q} B_{x_i} \right)^2 \right]^k = \left[\sum_{i=1}^p B_{x_i} - \sum_{i=p+1}^{p+q} B_{x_i} \right]^k \left[\sum_{i=1}^p B_{x_i} + \sum_{i=p+1}^{p+q} B_{x_i} \right]^k = \square_B^k \Delta_B^k.$$

Denoted by T^y the generalized shift operator acting according to the law [13]

$$T_x^y \varphi(x) = C_v^* \int_0^\pi \dots \int_0^\pi \varphi \left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos \theta_1}, \dots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos \theta_n} \right) \times \left(\prod_{i=1}^n \sin^{2v_i-1} \theta_i \right) d\theta_1 \dots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+$, $C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [13]

$$\frac{d^2 U}{dx^2} + \frac{2v}{x} \frac{dU}{dx} = \frac{d^2 U}{dy^2} + \frac{2v}{y} \frac{dU}{dy}, \quad U(x, 0) = f(x), \quad U_y(x, 0) = 0.$$

The convolution operator determined by the T^y is as follows:

$$(f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy. \tag{13}$$

Convolution (13) known as a B -convolution. We note the following properties of the B -convolution and the generalized shift operator.

- (a) $T_x^y \cdot 1 = 1$;
- (b) $T_x^0 \cdot f(x) = f(x)$;

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for all $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^n x_i^{2v_i} \right) dx < \infty$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$.

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

The proof of the following lemmas can be seen in [18].

Lemma 2.1 Given the equation $\Delta_B u(x) = \delta$ for $x \in \mathbb{R}_n^+$, where Δ_B is the Laplace-Bessel operator defined by (12). Then $E(x) = -S_2(x)$ is an elementary solution of the operator Δ_B where

$$E(x) = -S_2(x) = -\frac{2^{n+2|v|-4} \Gamma\left(\frac{n+2|v|-2}{2}\right)}{\prod_{i=1}^n 2^{v_i-\frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right)} |x|^{2-n-2|v|}. \quad (14)$$

Lemma 2.2 Given the equation $\Delta_B^k u(x) = \delta$ for $x \in \mathbb{R}_n^+$, where Δ_B^k is the Laplace-Bessel operator iterated k -times defined by (12). Then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator Δ_B^k where

$$S_{2k}(x) = \frac{2^{n+2|v|-4k} \Gamma\left(\frac{n+2|v|-2k}{2}\right)}{\prod_{i=1}^n 2^{v_i-\frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \Gamma(k)} |x|^{2k-n-2|v|}. \quad (15)$$

The proof of the following lemmas can be seen in [17].

Lemma 2.3 Given the equation

$$\Delta^k u(x) = 0, \quad (16)$$

where Δ is defined by (1) and $x \in \mathbb{R}^n$, then $u(x) = (-1)^{k-1} (R_{2(k-1)}^e(x))^{(m)}$ as a solution of (16) where m is a nonnegative integer with $m = (n-4)/2$, $n \geq 4$ and n is even and $(R_{2(k-1)}^e(x))^{(m)}$ is a function defined by (10) with m derivatives and $\alpha = 2(k-1)$.

Lemma 2.4 Given the equation

$$\Delta u(x) = f(x, u(x)), \quad (17)$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open set in \mathbb{R}^n and $\partial\Omega$ is the boundary of Ω , assume that f is bounded, that is $|f(x, u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (17).

The proof of the following lemmas can be seen in [16].

Lemma 2.5 Given the equation

$$\Delta_B^k u(x) = 0, \quad (18)$$

where Δ_B is defined by (12) and $x \in \mathbb{R}_n^+$, then $u(x) = (-1)^{k-1} (S_{2(k-1)}(x))^{(m)}$ as a solution of (18) where m is a nonnegative integer with $m = (n+2|v|-4)/2$, $n+2|v| \geq 4$ and n is even and $(S_{2(k-1)}(x))^{(m)}$ is a function defined by (15) with m derivatives.

Lemma 2.6 Given the equation

$$\Delta_B u(x) = f(x, u(x)), \quad (19)$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open set in \mathbb{R}_n^+ and $\partial\Omega$ is the boundary of Ω , assume that f is bounded, that is $|f(x, u)| \leq M$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (19).

3 Main Results

In this section, we will state our main results and give their proofs.

Theorem 3.1 Consider the nonlinear equation

$$\Delta^k u(x) = f(x, \Delta^{k-1} u(x)), \tag{20}$$

where Δ^k is the Laplace operator iterated k -times, defined by (1), let f be defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, where Ω is an open set in \mathbb{R}^n and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Let f be a bounded function, that is for all $x \in \Omega$,

$$|f(x, \Delta^{k-1} u(x))| \leq N \tag{21}$$

and the boundary condition for $x \in \partial\Omega$ be

$$\Delta^{k-1} u(x) = 0. \tag{22}$$

Then we obtain

$$u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * W(x) \tag{23}$$

as a solution of (20) with the boundary condition

$$u(x) = (-1)^{k-2} (R_{2(k-2)}^e(x))^{(m)} \quad \text{for } x \in \partial\Omega$$

where $m = (n - 4)/2, k = 2, 3, \dots$ and $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $R_{2(k-2)}^e(x)$ is given by (10) with $\alpha = 2(k - 2)$.

Proof We have

$$\Delta^k u(x) = \Delta(\Delta^{k-1} u(x)) = f(x, \Delta^{k-1} u(x)). \tag{24}$$

Since $u(x)$ has continuous derivatives up to order $2k$ for $k = 1, 2, 3, \dots$, we can assume for all $x \in \Omega$

$$\Delta^{k-1} u(x) = W(x). \tag{25}$$

Thus, (24) can be written in the form

$$\Delta^k u(x) = \Delta W(x) = f(x, W(x)), \tag{26}$$

by (21), for all $x \in \Omega$

$$|f(x, W(x))| \leq N \tag{27}$$

and by (22), $W(x) = 0$ or for all $x \in \partial\Omega$

$$\Delta^{k-1} u(x) = 0. \tag{28}$$

Thus by Lemma 2.4 there exists a unique solution $W(x)$ of (26) which satisfies (27). Convolving both sides of (25) by $(-1)^{k-1} R_{2(k-1)}^e(x)$, we obtain

$$(-1)^{k-1} R_{2(k-1)}^e(x) * \Delta^{k-1} u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * W(x),$$

or

$$\Delta^{k-1} \left((-1)^{k-1} R_{2(k-1)}^e(x) \right) * u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * W(x),$$

or

$$\delta * u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * W(x).$$

Thus

$$u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * W(x) \tag{29}$$

as required. We consider for $x \in \partial\Omega$

$$\Delta^{k-1} u(x) = 0.$$

By Lemma 2.3, we have

$$u(x) = (-1)^{k-2} (R_{2(k-2)}^e(x))^{(m)}$$

where $m = (n - 4)/2, n \geq 4$ and n is even and $k = 2, 3, \dots$

This complete the proof. □

Theorem 3.2 Consider the nonlinear equation

$$\Delta_B^k u(x) = f(x, \Delta_B^{k-1} u(x)), \quad (30)$$

where Δ_B^k is the Laplace-Bessel operator iterated k -times, defined by (12), let f be defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, where Ω is an open set in \mathbb{R}_n^+ and $\partial\Omega$ denotes the boundary of Ω and n is even with $n + 2|v| \geq 4$. Let f be a bounded function, that is for all $x \in \Omega$,

$$|f(x, \Delta_B^{k-1} u(x))| \leq M \quad (31)$$

and the boundary condition for $x \in \partial\Omega$ be

$$\Delta_B^{k-1} u(x) = 0. \quad (32)$$

Then we obtain

$$u(x) = (-1)^{k-1} S_{2(k-1)}(x) * W(x) \quad (33)$$

as a solution of (30) with the boundary condition

$$u(x) = (-1)^{k-2} (S_{2(k-2)}(x))^{(m)} \quad \text{for } x \in \partial\Omega$$

where $m = (n + 2|v| - 4)/2, k = 2, 3, \dots$ and $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $S_{2(k-2)}(x)$ is given by (15).

The proof of this Theorem is similar to the proof of Theorem 3.1, by using Lemma 2.5, 2.6 and B -convolution.

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THE CONVOLUTION EQUATION RELATED TO THE ULTRA-HYPERBOLIC EQUATIONS

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Abstract

In this paper, we study the distribution $e^{\alpha x} \square_c^k \delta$, where \square_c^k is the operator related to the ultra-hyperbolic type operator iterated k -times, δ is the Dirac delta distribution, $x = (x_1, x_2, \dots, x_n)$ is a variable and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a constant and both are the points in the n -dimensional Euclidean space \mathbb{R}^n .

At first, the properties of $e^{\alpha x} \square_c^k \delta$ are studied and later we study the application of $e^{\alpha x} \square_c^k \delta$ for solving the solutions of the convolution

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equation

$$(e^{\alpha x} \square_c^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \square_c^r \delta,$$

where $u(x)$ is the generalized function and C_r is a constant. We found that its solution is related to the ultra-hyperbolic equation and moreover, the type of solutions of the convolution equation, such as the ordinary functions, the tempered distributions or the singular distributions depending on k , m and α .

1. Introduction

Gelfand and Shilov [2] have introduced the elementary solution of the n -dimensional ultra-hyperbolic operator. Next, Trione [11] has shown that the n -dimensional ultra-hyperbolic equation has $u(x) = R_{2k,1}(x)$ as a unique elementary solution. Later, Téllez [10] has proved that $R_{2k,1}(x)$ exists only for case p is odd with $p + q = n$.

Kanantjai [5] has showed that the solution of the convolution form $u(x) = R_{2k,c_1}(x) * R_{2k,c_2}(x)$ is a unique elementary solution of the equation $\square_{c_1}^k \square_{c_2}^k u(x) = \delta$, where $\square_{c_1}^k$ and $\square_{c_2}^k$ are the operators related to the ultra-hyperbolic type operator iterated k -times and in particular if $k = p = 1$ with $x_1 = t$ (times), c_1 and c_2 are velocity, then $u(x) = R_{2,c_1}(x) * R_{2,c_2}(x)$ is the elementary solution of the elastic wave equation of fourth order.

Sritanratana and Kanantjai [9] have studied the product of the nonlinear diamond operators related to the elastic wave equation and also introduced the ultra-hyperbolic operator \square_c^k .

Bupasiri and Nonlaopon [1] have studied the weak solutions of compound ultra-hyperbolic equation

$$\sum_{r=0}^m C_r \square_c^k u(x) = f(x), \quad (1.1)$$

which is related to the ultra-hyperbolic type operator iterated k -times.

Kanantjai [3] has studied the properties of the distribution $e^{\alpha x} \square^k \delta$ for solving the elementary solution of the equation of the ultra-hyperbolic type by using the convolution method.

Kanantjai [4] has studied the properties of the distribution $e^{\alpha x} \diamond^k \delta$ and the application of $e^{\alpha x} \diamond^k \delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x} \diamond^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \diamond^r \delta, \tag{1.2}$$

where \diamond^k is the diamond operator iterated k -times and is defined by

$$\diamond^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \tag{1.3}$$

$p + q = n$ is the dimension of the Euclidean space \mathbb{R}^n .

Kanantjai [6] has studied the application of the distribution $e^{\alpha x} \square^k \delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x} \square^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \square^r \delta, \tag{1.4}$$

which is related to the ultra-hyperbolic equation.

In this paper, we study the properties of the distribution $e^{\alpha x} \square_c^k \delta$ and the application of $e^{\alpha x} \square_c^k \delta$ for solving the solutions of the convolution equation

$$(e^{\alpha x} \square_c^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \square_c^r \delta, \tag{1.5}$$

where \square_c^k is the operator related to the ultra-hyperbolic type operator iterated k -times and defined by

$$\square_c^k = \left(\frac{1}{c^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \tag{1.6}$$

$u(x)$ is the generalized function and C_r is a constant. We found that its solution is

related to the ultra-hyperbolic equation and moreover, the type of solutions of the convolution equation, such as the ordinary functions, the tempered distributions or the singular distributions depending on k , m and α .

Before going to that point, the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be the point of the Euclidean space \mathbb{R}^n , write

$$V = c^2(x_1^2 + x_2^2 + \dots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad (2.1)$$

where $p + q = n$ and the interior of the forward cone is defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } V > 0\}$. For any complex number γ , define

$$R_{\gamma, c}(x) = \begin{cases} V^{(\gamma-n)/2} K_n(\gamma) & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \quad (2.2)$$

where

$$K_n(\gamma) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2+\gamma-n}{2}\right) \Gamma\left(\frac{1-\gamma}{2}\right) \Gamma(\gamma)}{\Gamma\left(\frac{\gamma-p+2}{2}\right) \Gamma\left(\frac{p-\gamma}{2}\right)}. \quad (2.3)$$

The function $R_\gamma(x) = R_{\gamma, 1}(x)$ is introduced by Nozaki [7]. It is well known that such function is an ordinary function if $\text{Re}(\gamma) \geq n$ and is a distribution of γ if $\text{Re}(\gamma) < n$.

By putting $p = c = 1$ in (2.1) and (2.3), and using the Legendre's duplication of $\Gamma(z)$, (2.2) reduces to

$$M_\gamma(x) = \begin{cases} V^{(\gamma-n)/2} H_n(\gamma) & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases}$$

where

$$H_n(\gamma) = \pi^{(n-2)/2} 2^{\gamma-1} \Gamma\left(\frac{\gamma-n+2}{2}\right) \Gamma\left(\frac{\gamma}{2}\right) \text{ and } V = x_1^2 - x_2^2 - \dots - x_n^2.$$

The function $M_\gamma(x)$ is precisely called the *hyperbolic kernel* of Marcel Riesz.

The proofs of the following Lemmas 2.1, 2.2 and 2.3 are given in [1].

Lemma 2.1. *The function $R_{\gamma,c}(x)$ is a homogeneous distribution of order $\gamma - n$ and is also a tempered distribution.*

Lemma 2.2. *Given the equation*

$$\square_c^k u(x) = \delta, \tag{2.4}$$

where \square_c^k is defined by (1.6), k is a nonnegative integer and δ is the Dirac delta distribution. Then $u(x) = R_{2k,c}(x)$ is the unique elementary solution of (2.4), where $R_{2k,c}(x)$ is defined by (2.2) with $\gamma = 2k$.

Lemma 2.3. *Let $R_{\gamma,c}(x)$ and $R_{\beta,c}(x)$ be defined by (2.2), and γ, β are positive even numbers with $\gamma + \beta = 2k$, where k is a nonnegative integer. Then $R_{\gamma,c}(x) * R_{\beta,c}(x) = R_{\gamma+\beta,c}(x)$.*

3. Properties of $e^{\alpha x} \square_c^k \delta$

First, we shall consider the distribution $e^{\alpha x} \square_c^k \delta$ with $k = 1$.

Lemma 3.1. *The distribution $e^{\alpha x} \square_c \delta$ has the following properties:*

Properties 3.1. *For the operator \square_c^k with $k = 1$ and is defined by (1.6), then*

$$e^{\alpha x} \square_c \delta = \square_c \delta - 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial \delta}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \delta}{\partial x_j} \right) + \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \delta \tag{3.1}$$

and $e^{\alpha x} \square_c \delta$ is a tempered distribution of order 2 with support $\{0\}$.

Proof. Let $\varphi \in \mathcal{D}$ be the space of testing functions, infinitely differentiable with compact supports and \mathcal{D}' be the space of distributions. Now

$$\langle e^{\alpha x} \square_c \delta, \varphi(x) \rangle = \langle \delta, \square_c e^{\alpha x} \varphi(x) \rangle,$$

for $e^{\alpha x} \square_c \delta \in \mathcal{D}'$. By computing directly, we obtain

$$\begin{aligned} \square_c e^{\alpha x} \varphi(x) &= \frac{1}{c^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} (e^{\alpha x} \varphi(x)) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} (e^{\alpha x} \varphi(x)) \\ &= e^{\alpha x} \square_c \varphi(x) + 2e^{\alpha x} \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial \varphi(x)}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \varphi(x)}{\partial x_j} \right) \\ &\quad + e^{\alpha x} \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \varphi(x). \end{aligned} \quad (3.2)$$

Then

$$\begin{aligned} \langle \delta, \square_c e^{\alpha x} \varphi(x) \rangle &= \square_c \varphi(0) + 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial \varphi(0)}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \varphi(0)}{\partial x_j} \right) \\ &\quad + \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \varphi(0) \\ &= \left\langle \square_c \delta - 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial \delta}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial \delta}{\partial x_j} \right) \right. \\ &\quad \left. + \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) \delta, \varphi(x) \right\rangle. \end{aligned} \quad (3.3)$$

By equality of distributions, we obtain (3.1) as required. To show that $e^{\alpha x} \square_c \delta$ is a tempered distribution, from (3.1) δ , $\partial \delta / \partial x_i$, $\partial \delta / \partial x_j$ and $\square_c \delta$ have support $\{0\}$ which is compact, hence, by Schwartz [8], they are tempered distributions. From (3.1), it follows that $e^{\alpha x} \square_c \delta$ is also tempered distribution and by Zamanian [12, Theorem 3.5-2, p. 98] $e^{\alpha x} \square_c \delta$ is of order 2 with point support $\{0\}$. \square

Properties 3.2 (Boundedness property). *For every testing function $\varphi \in S$, the Schwartz space and $e^{\alpha x} \square_c \delta \in S'$, the space of tempered distributions, then $|\langle e^{\alpha x} \square_c \delta, \varphi \rangle| \leq CM$, where C and M are constant with*

$$M = \max \left\{ |\varphi(0)|, \left| \frac{\partial \varphi(0)}{\partial x_i} \right|, \left| \frac{\partial \varphi(0)}{\partial x_j} \right|, |\square_c \varphi(0)| \right\},$$

$$C = 1 + \frac{2}{c^2} \sum_{i=1}^p |\alpha_i| + 2 \sum_{j=p+1}^{p+q} |\alpha_j| + \frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 + \sum_{j=p+1}^{p+q} \alpha_j^2. \quad (3.4)$$

Proof. Since $\langle e^{\alpha x} \square_c \delta, \varphi(x) \rangle = \langle \delta, \square_c e^{\alpha x} \varphi(x) \rangle$, hence by (3.3), we have

$$|\langle e^{\alpha x} \square_c \delta, \varphi \rangle| \leq |\square_c \varphi(0)| + \frac{2}{c^2} \sum_{i=1}^p |\alpha_i| \left| \frac{\partial \varphi(0)}{\partial x_i} \right| + 2 \sum_{j=p+1}^{p+q} |\alpha_j| \left| \frac{\partial \varphi(0)}{\partial x_j} \right|$$

$$+ \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 + \sum_{j=p+1}^{p+q} \alpha_j^2 \right) |\varphi(0)|.$$

Let $M = \max \{ |\varphi(0)|, |\partial \varphi(0)/\partial x_i|, |\partial \varphi(0)/\partial x_j|, |\square_c \varphi(0)| \}$. Then

$$|\langle e^{\alpha x} \square_c \delta, \varphi \rangle| \leq \left(1 + \frac{2}{c^2} \sum_{i=1}^p |\alpha_i| + 2 \sum_{j=p+1}^{p+q} |\alpha_j| + \frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 + \sum_{j=p+1}^{p+q} \alpha_j^2 \right) M.$$

It follows that $|\langle e^{\alpha x} \square_c \delta, \varphi \rangle| \leq CM$, where C is defined by (3.4). □

Lemma 3.2. *Given $u(x)$ any distribution in the space S' , then*

$$(e^{\alpha x} \square_c \delta) * u(x) = \square_c u(x) - 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial u(x)}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial u(x)}{\partial x_j} \right)$$

$$+ \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right) u(x). \quad (3.5)$$

Proof. Convolving both sides of (3.1) by $u(x)$, we obtain (3.5) as required. If L

is the operator and is defined by

$$L \equiv \square_c - 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} \right) + \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right), \quad (3.6)$$

then (3.5) can be written as

$$(e^{\alpha x} \square_c \delta) * u(x) = Lu(x). \quad (3.7)$$

□

Lemma 3.3 (The generalization of Lemma 3.2).

$$(e^{\alpha x} \square_c^k \delta) * u(x) = L^k u(x), \quad (3.8)$$

where L^k is the operator defined by (3.6) and is iterated k -times with $L^0 u(x) = u(x)$.

Proof. We have $\langle e^{\alpha x} \square_c^k \delta, \varphi(x) \rangle = \langle \square_c^k \delta, e^{\alpha x} \varphi(x) \rangle$ for every $\varphi(x) \in \mathcal{D}$ and $e^{\alpha x} \square_c^k \delta \in \mathcal{D}'$. So

$$\langle \square_c^k \delta, e^{\alpha x} \varphi(x) \rangle = \langle \square_c^{k-1} \delta, \square_c e^{\alpha x} \varphi(x) \rangle = \langle \square_c^{k-1} \delta, e^{\alpha x} T \varphi(x) \rangle,$$

where T is the partial differential operator from (3.2) and is defined by

$$T \equiv \square_c + 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} \right) + \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right). \quad (3.9)$$

So

$$\begin{aligned} \langle \square_c^{k-1} \delta, e^{\alpha x} T \varphi(x) \rangle &= \langle \square_c^{k-2} \delta, \square_c e^{\alpha x} T \varphi(x) \rangle \\ &= \langle \square_c^{k-2} \delta, e^{\alpha x} T(T \varphi(x)) \rangle \\ &= \langle \square_c^{k-2} \delta, e^{\alpha x} T^2 \varphi(x) \rangle. \end{aligned}$$

By keeping on operating \square_c with $k - 2$ times, we obtain

$$\langle \square_c^{k-2} \delta, e^{\alpha x} T^2 \varphi(x) \rangle = \langle \delta, e^{\alpha x} T^k \varphi(x) \rangle = T^k \varphi(0),$$

where T^k is the operator of (3.9) iterated k -times. Now

$$T^k \varphi(0) = \langle \delta, T^k \varphi(x) \rangle = \langle L\delta, T^{k-1} \varphi(x) \rangle,$$

by the operator L in (3.6) and the derivative of distribution. Continuing this process, we obtain $T^k \varphi(0) = \langle L^k \delta, \varphi(x) \rangle$ or $\langle e^{\alpha x} \square_c^k \delta, \varphi(x) \rangle = \langle L^k \delta, \varphi(x) \rangle$. It follows that:

$$e^{\alpha x} \square_c^k \delta = L^k \delta. \tag{3.10}$$

Convolving both sides of (3.10) by distribution $u(x)$, we obtain (3.8). □

4. Main Results

Theorem 4.1. *Let L be the partial differential operator defined by*

$$L \equiv \square_c - 2 \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} \right) + \left(\frac{1}{c^2} \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right),$$

where this operator appears in (3.1). Now L is of ultra-hyperbolic type. Consider the equation

$$Lu(x) = \delta, \tag{4.1}$$

where $u(x)$ is any distribution in S' , then $u(x) = e^{\alpha x} R_{2,c}(x)$ is a unique elementary solution of (4.1), where $R_{2,c}(x)$ is defined by (2.2) with $\gamma = 2$.

Proof. From (3.1) and (4.1) we can write $(e^{\alpha x} \square_c \delta) * u(x) = Lu(x) = \delta$.

Convolving both sides by $e^{\alpha x} R_{2,c}(x)$, we have

$$(e^{\alpha x} R_{2,c}(x)) * (e^{\alpha x} \square_c \delta * u(x)) = e^{\alpha x} R_{2,c}(x) * \delta.$$

Then

$$e^{\alpha x} (R_{2,c}(x) * (\square_c \delta)) * u(x) = e^{\alpha x} R_{2,c}(x),$$

or equivalently,

$$(e^{\alpha x} \square_c R_{2,c}(x)) * u(x) = e^{\alpha x} R_{2,c}(x).$$

Because $\square_c R_{2,c}(x) = \delta$ by Lemma 2.2 with $k = 1$, we obtain $(e^{\alpha x} \delta) * u(x) =$

$e^{\alpha x} R_{2,c}(x)$. Since $e^{\alpha x} \delta = \delta$, $\delta * u(x) = e^{\alpha x} R_{2,c}(x)$. It follows that $u(x) = e^{\alpha x} R_{2,c}(x)$. We can check the solution $u(x)$ by computing directly from (4.1). \square

Theorem 4.2 (The generalization of Theorem 4.1). *From Lemma 3.3, consider*

$$(e^{\alpha x} \square_c^k \delta) * u(x) = \delta \quad (4.2)$$

or

$$L^k u(x) = \delta, \quad (4.3)$$

then $u(x) = e^{\alpha x} R_{2k,c}(x)$ is a unique elementary solution of (4.2) or (4.3).

Proof. We can prove by using equation (4.2) or (4.3) as well. If we start with equation (4.2), we convolve both sides of (4.2) by $e^{\alpha x} R_{2k,c}(x)$, then we obtain

$$(e^{\alpha x} R_{2k,c}(x)) * (e^{\alpha x} \square_c^k \delta * u(x)) = e^{\alpha x} R_{2k,c}(x) * \delta,$$

or $e^{\alpha x} (\square_c^k R_{2k,c}(x)) * u(x) = e^{\alpha x} R_{2k,c}(x)$. Since $\square_c^k R_{2k,c}(x) = \delta$ by Lemma 2.2, we have $(e^{\alpha x} \delta) * u(x) = e^{\alpha x} R_{2k,c}(x)$ or $u(x) = e^{\alpha x} R_{2k,c}(x)$ as required. Or if we use equation (4.3), we convolve both sides of (4.3) by $e^{\alpha x} R_{2,c}(x)$, then we obtain

$$e^{\alpha x} R_{2,c}(x) * L^k u(x) = e^{\alpha x} R_{2,c}(x) * \delta,$$

or $L(e^{\alpha x} R_{2,c}(x)) * L^{k-1} u(x) = e^{\alpha x} R_{2,c}(x)$. By Theorem 4.1, we obtain $L^{k-1} u(x) = e^{\alpha x} R_{2,c}(x)$. By keeping on convolving $e^{\alpha x} R_{2,c}(x)$ with $k-1$ times, we obtain

$$u(x) = e^{\alpha x} (R_{2,c}(x) * R_{2,c}(x) * \cdots * R_{2,c}(x)) = e^{\alpha x} R_{2k,c}(x),$$

by Lemma 2.3. \square

Theorem 4.3. *Given the convolution equation*

$$(e^{\alpha x} \square_c^k \delta) * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \square_c^r \delta, \quad (4.4)$$

where \square_c^k is the operator related to the ultra-hyperbolic type operator iterated

k-times and defined by

$$\square_c^k = \left(\frac{1}{c^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k,$$

where $p + q = n$ is the dimension of the space \mathbb{R}^n with p odd and q odd or p odd and q even, the variable $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the constant $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$, δ is the Dirac delta distribution with $\square_c^0 \delta = \delta$, $\square_c^1 \delta = \square_c \delta$ and C_r is a constant. Then the type of solution $u(x)$ of (4.4) depends on k , m and α as the following cases.

(1) If $m < k$ and $m = 0$, then the solution of (4.4) is $u(x) = C_0 e^{\alpha x} R_{2k,c}(x)$ which is the elementary solution of the operator \square_c^k . Now $R_{2k,c}(x)$ is defined by (2.2) with $\gamma = 2k$. If $2k \geq n$ and for any α , then $e^{\alpha x} R_{2k,c}(x)$ is the ordinary function. If $2k < n$ and for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i < 0$ ($i = 1, 2, \dots, n$), then $e^{\alpha x} R_{2k,c}(x)$ is a tempered distribution.

(2) If $0 < m < k$ and r run from 1, then the solution of (4.4) is $u(x) = e^{\alpha x} \sum_{r=1}^m C_r R_{2k-2r,c}(x)$ which is ordinary function for $2k - 2r \geq n$ with any arbitrary constant α and it is a tempered distribution if $2k - 2r < n$ for some α with $\alpha_i < 0$ ($i = 1, 2, \dots, n$).

(3) If $m \geq k$ and for any α , suppose that $k \leq m \leq M$ and r run from k , then (4.4) has $u(x) = e^{\alpha x} \sum_{r=k}^M C_r \square_c^{r-k} \delta$ as a solution which is only the singular distribution.

Proof. (1) For $m < k$ and $m = 0$, (4.4) becomes

$$(e^{\alpha x} \square_c^k \delta) * u(x) = C_0 e^{\alpha x} \delta = C_0 \delta$$

and by Theorem 4.2, we obtain $u(x) = C_0 e^{\alpha x} R_{2k,c}(x)$. Now $R_{2k,c}$ is defined by

(2.2) with $\gamma = 2k$, is the ordinary function for $2k \geq n$. It follows that $C_0 e^{\alpha x} R_{2k,c}(x)$ is an ordinary function for $2k \geq n$ with any α . If $2k < n$, then $R_{2k,c}(x)$ is an analytic function except at the origin and by Lemma 2.1, $R_{2k,c}(x)$ is tempered distribution and for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i < 0$ ($i = 1, 2, \dots, n$) we have $e^{\alpha x}$ as a slow growth function and also its partial derivative is a slow growth. It follows that $C_0 e^{\alpha x} R_{2k,c}(x)$ is a tempered distribution.

(2) For $0 < m < k$ and r run from 1, we can write (4.4) as

$$(e^{\alpha x} \square_c^k \delta) * u(x) = e^{\alpha x} [C_1 \square_c \delta + C_2 \square_c^2 \delta + \dots + C_m \square_c^m \delta].$$

Convolving both sides by $e^{\alpha x} R_{2k,c}(x)$ and by Lemma 2.2, we obtain

$$u(x) = e^{\alpha x} [C_1 \square_c R_{2k,c}(x) + C_2 \square_c^2 R_{2k,c}(x) + \dots + C_m \square_c^m R_{2k,c}(x)].$$

Now $\square_c^k R_{2k,c}(x) = \delta$, then $\square_c^{k-r} \square_c^r R_{2k,c}(x) = \delta$ for $r < k$. Convolving both sides by $R_{2k-2r,c}(x)$, we obtain

$$R_{2k-2r,c}(x) * \square_c^{k-r} \square_c^r R_{2k,c}(x) = R_{2k-2r,c}(x),$$

or

$$\square_c^{k-r} R_{2k-2r,c}(x) * \square_c^r R_{2k,c}(x) = R_{2k-2r,c}(x),$$

or

$$\square_c^r R_{2k,c}(x) = R_{2k-2r,c}(x)$$

for $r < k$. It follows that

$$u(x) = e^{\alpha x} [C_1 R_{2k-2,c}(x) + C_2 R_{2k-4,c}(x) + \dots + C_m R_{2k-2m,c}(x)],$$

or

$$u(x) = e^{\alpha x} \sum_{r=1}^m C_r R_{2k-2r,c}(x).$$

Similarly, as in the case (1), $e^{\alpha x} R_{2k-2r,c}(x)$ is the ordinary function for $2k - 2r$

$\geq n$ with any α . It follows that $u(x) = e^{\alpha x} \sum_{r=1}^m C_r R_{2k-2r,c}(x)$ is also the ordinary function with any α . For the case $2k - 2r < n$ and for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i < 0$ ($i = 1, 2, \dots, n$), we obtain $e^{\alpha x} R_{2k-2r,c}(x)$ is a tempered distribution.

It follows that $u(x) = e^{\alpha x} \sum_{r=1}^m C_r R_{2k-2r,c}(x)$ is also a tempered distribution.

(3) If $m \geq k$ and for any α , suppose that $k \leq m \leq M$ and r run from k , then (4.4) becomes

$$(e^{\alpha x} \square_c^k \delta) * u(x) = e^{\alpha x} [C_k \square_c^k \delta + C_{k+1} \square_c^{k+1} \delta + \dots + C_M \square_c^M \delta].$$

Convolving both sides by $e^{\alpha x} R_{2k,c}(x)$ and by Lemma 2.2, we have

$$u(x) = e^{\alpha x} [C_k \square_c^k R_{2k,c}(x) + C_{k+1} \square_c^{k+1} R_{2k,c}(x) + \dots + C_M \square_c^M R_{2k,c}(x)].$$

Now

$$\square_c^m R_{2k,c}(x) = \square_c^{m-k} \square_c^k R_{2k,c}(x) = \square_c^{m-k},$$

for $k \leq m \leq M$. So

$$\begin{aligned} u(x) &= e^{\alpha x} [C_k \delta + C_{k+1} \square_c \delta + C_{k+2} \square_c^2 \delta + \dots + C_M \square_c^{M-k} \delta] \\ &= e^{\alpha x} \sum_{r=k}^M C_r \square_c^{r-k} \delta. \end{aligned}$$

Now, by (3.6) and (3.10), we have

$$e^{\alpha x} \square_c^{r-k} \delta = \square_c^{r-k} \delta + (\text{the terms of lower order of partial derivative of } \delta),$$

for $k \leq r \leq M$ and since all terms of the right-hand side of above equation are singular distribution, it follows that

$$u(x) = e^{\alpha x} \sum_{r=k}^M C_r \square_c^{r-k} \delta$$

is only the singular distribution. That completes the proof. □

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On The Weak Solutions of The Compound Ultra-Hyperbolic Bessel Equation

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Abstract. In this article, we have studied the compound ultra-hyperbolic Bessel equation of the form

$$\sum_{r=0}^m C_r \square_{B,c}^r u(x) = f(x),$$

where $\square_{B,c}^r$ is the ultra-hyperbolic Bessel operator iterated r -times, f is a given generalized function, u is an unknown function, $x \in \mathbb{R}_n^+$ and C_r is a constant. In this work, we study the weak solution $u(x)$ of above the equation which is of the form ultra-hyperbolic Bessel operator and moreover, such a solution is unique.

Key words: Dirac-delta distribution; Ultra-hyperbolic Bessel operator; Tempered distribution.

2000 Mathematics Subject Classification: 46F10.

1. Introduction

I. M. Gelfand and G. E. Shilov [3] have first introduced the elementary solution of the n -dimensional classical ultra-hyperbolic operator. S. E. Trione [9] has shown that the n -dimensional ultra-hyperbolic equation has $u(x) = R_{2k}(x)$ as a unique elementary solution. Later, M. A. Tellez [8] has proved that $R_{2k}(x)$ exists only for case p is odd with $p + q = n$.

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H. Yildirim et al. [10] have introduced the Bessel ultra-hyperbolic type operator iterated k -times with $x \in \mathbb{R}_n^+ = \{x : x = (x_1, x_2, \dots, x_n), x_1 > 0, \dots, x_n > 0\}$,

$$(1.1) \quad \square_B^k = (B_{x_1} + B_{x_2} + \dots + B_{x_p} - B_{x_{p+1}} - \dots - B_{x_{p+q}})^k$$

where $p + q = n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}$ where $2v_i = 2\beta_i + 1$, $\beta_i > -\frac{1}{2}$ [5], k is nonnegative integer and n is the dimension of \mathbb{R}_n^+ and studied the elementary solution of this operator. Moreover, they have introduced the Bessel diamond operator and have studied the elementary solution of this operator and also the Fourier-Bessel transform of the elementary solution.

A. Kananthai and K. Nonlaopon [4] have studied the weak solution of the compound ultra-hyperbolic equation. Next, M. Z. Sarikaya and H. Yildirim [7] have studied the weak solution of the compound Bessel ultra-hyperbolic equation. Later, S. Bupasiri and K. Nonlaopon [1] have studied the weak solution of the compound equation related to the ultra-hyperbolic operator of the form

$$(1.2) \quad \sum_{r=0}^m C_r \square_c^r u(x) = f(x),$$

where \square_c^r is the operator which related to the ultra-hyperbolic type operator iterated r -times, defined by

$$(1.3) \quad \square_c^r = \left(\frac{1}{c^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^r.$$

A. Saglam et al. [6] have developed the operator of (1.1), defined by

$$(1.4) \quad \square_{B,c}^k = \left[\frac{1}{c^2} (B_{x_1} + B_{x_2} + \dots + B_{x_p}) - (B_{x_{p+1}} + \dots + B_{x_{p+q}}) \right]^k$$

and is called the ultra-hyperbolic Bessel operator iterated k -times. Moreover, they studied the product of the ultra-hyperbolic Bessel operator related to elastic waves.

In this article, we will consider the equation

$$(1.5) \quad \square_{B,c}^k u(x) = f(x)$$

where $u(x)$ and $f(x)$ are some generalized function.

We will develop the equation (1.5) to the form

$$(1.6) \quad \sum_{k=0}^m C_k \square_{B,c}^k u(x) = f(x),$$

which is called the compound ultra-hyperbolic Bessel equation and by convention $\square_{B,c}^0 u(x) = u(x)$. In finding the solutions of (1.6), we use the properties of convolutions for the generalized functions.

2. Preliminaries

Denoted by T^y the generalized shift operator acting according to the law [5]:

$$T_x^y \varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi \left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos \theta_1}, \dots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos \theta_n} \right) \\ \times \left(\prod_{i=1}^n \sin^{2v_i-1} \theta_i \right) d\theta_1 \cdots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+$, $C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [5]:

$$\frac{d^2 U}{dx^2} + \frac{2v}{x} \frac{dU}{dx} = \frac{d^2 U}{dy^2} + \frac{2v}{y} \frac{dU}{dy} \\ U(x, 0) = f(x), \\ U_y(x, 0) = 0.$$

The convolution operator determined by the T^y is as follows:

$$(2.1) \quad (f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

Convolution (2.1) known as a B -convolution. We note the following properties of the B -convolution and the generalized shift operator.

- (a) $T_x^y \cdot 1 = 1$.
- (b) $T_x^0 \cdot f(x) = f(x)$.
- (c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x > 0$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^n x_i^{2v_i} \right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

- (d) From (c), we have the following equality for $g(x) = 1$.

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

- (e) $(f * g)(x) = (g * f)(x)$.

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional space \mathbb{R}_n^+ ,

$$(2.2) \quad V = c^2 (x_1^2 + x_2^2 + \dots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2,$$

where $p+q = n$, the interior of forward cone defined by $\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \dots, x_n > 0 \text{ and } V > 0\}$. For any complex number α , define

$$(2.3) \quad R_{\alpha,c}^H(x) = \begin{cases} \frac{V^{\frac{\alpha-n-2|v|}{2}}}{K_{n,v}(x)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases}$$

where

$$(2.4) \quad K_{n,v}(\alpha) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+\alpha-2|v|}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p-2|v|}{2}\right) \Gamma\left(\frac{p+2|v|-\alpha}{2}\right)}.$$

Lemma 2.1. $R_{2k,c}^H(x)$ is a homogeneous distribution of order $(\alpha - n - 2|v|)$. In particular, it is a tempered distribution.

Proof. We need to show that $R_{\alpha,c}^H(x)$ satisfies the Euler equation

$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{\alpha,c}^H(x) = (\alpha - n - 2|v|) R_{\alpha,c}^H(x).$$

Now

$$\begin{aligned} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{\alpha,c}^H(x) &= \frac{1}{K_{n,v}(\alpha)} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (c^2(x_1^2 + \dots + x_p^2) - x_{p+1}^2 - \dots - x_{p+q}^2)^{\frac{\alpha-n-2|v|}{2}} \\ &= \frac{\alpha - n - 2|v|}{K_{n,v}(\alpha)} (c^2(x_1^2 + \dots + x_p^2) - x_{p+1}^2 - \dots - x_{p+q}^2)^{\frac{\alpha-n-2|v|-2}{2}} \\ &\quad \times (c^2(x_1^2 + \dots + x_p^2) - x_{p+1}^2 - \dots - x_{p+q}^2) \\ &= \frac{1}{K_{n,v}(\alpha)} (\alpha - n - 2|v|) (c^2(x_1^2 + \dots + x_p^2) - x_{p+1}^2 - \dots - x_{p+q}^2)^{\frac{\alpha-n-2|v|}{2}} \\ &= \frac{(\alpha - n - 2|v|) V^{\frac{\alpha-n-2|v|}{2}}}{K_{n,v}(\alpha)} \\ &= (\alpha - n - 2|v|) R_{\alpha,c}^H(x). \end{aligned}$$

Hence $R_{\alpha,c}^H(x)$ is a homogeneous distribution of order $(\alpha - n - 2|v|)$. W. F. Donoghue [2] prove that every homogeneous distribution is a tempered distribution. So $R_{\alpha,c}^H(x)$ is a tempered distribution. This is complete of proof. \square

Lemma 2.2. Given the equations

$$(2.5) \quad \square_{B,c}^k u(x) = \delta(x),$$

where $\square_{B,c}^k$ is defined by (1.4) and $x \in \mathbb{R}_n^+$, then we obtain $u(x) = R_{2k,c}^H(x)$ as an unique elementary solution of (2.5), where $R_{2k,c}^H(x)$ is defined by (2.3). The proof of this lemma is given in [6].

Lemma 2.3.(The B -convolutions of tempered distributions)

- (a) $(\square_{B,c}^k \delta(x)) * u(x) = \square_{B,c}^k u(x)$, where u is any tempered distribution.
- (b) Let $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ be defined by (2.3), then $R_{2k,c}^H(x) * R_{2m,c}^H(x)$ exists and is a tempered distribution.
- (c) Let $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ be defined by (2.3), then $R_{2k,c}^H(x) * R_{2m,c}^H(x) = R_{2k+2m,c}^H(x)$, where k and m are nonnegative integer.
- (d) Let $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ be defined by (2.3) and if $R_{2k,c}^H(x) * R_{2m,c}^H(x) = \delta(x)$ then $R_{2k,c}^H(x)$ is an inverse of $R_{2m,c}^H(x)$ in the B -convolution algebra, denoted by $R_{2k,c}^H(x) = R_{2m,c}^{H*-1}(x)$, moreover $R_{2m,c}^{H*-1}(x)$ is unique.

Proof.

- (a) First, we consider the case $k = 1$, now

$$\square_{B,c} \delta(x) = \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \delta(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \delta(x)}{\partial x_i} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \delta(x)}{\partial x_j} \right), \quad p+q = n$$

and let $\varphi(x)$ be a testing function in the Schwartz space S . By the definition of B -convolution, we have

$$\begin{aligned} \langle \square_{B,c} \delta(x) * u(x), \varphi(x) \rangle &= \langle u(x), \langle \square_{B,c} \delta(x), \varphi(x+y) \rangle \rangle \\ &= \left\langle u(x), \left\langle \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \delta(y)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \delta(y)}{\partial x_i} \right) \right. \right. \\ &\quad \left. \left. - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(y)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \delta(y)}{\partial x_j} \right), \varphi(x+y) \right\rangle \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle u(x), \left\langle \delta(y), \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \varphi(x+y)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \varphi(x+y)}{\partial x_i} \right) \right. \right. \\
&\quad \left. \left. - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \varphi(x+y)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \varphi(x+y)}{\partial x_j} \right) \right\rangle \right\rangle \\
&= \left\langle u(x), \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \varphi(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \varphi(x)}{\partial x_i} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \varphi(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \varphi(x)}{\partial x_j} \right) \right\rangle \\
&= \left\langle \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 u(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial u(x)}{\partial x_i} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 u(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial u(x)}{\partial x_j} \right), \varphi(x) \right\rangle \\
(2.6) \quad &= \langle \square_{B,c} u(x), \varphi(x) \rangle.
\end{aligned}$$

It follow that

$$\square_{B,c} \delta(x) * u(x) = \square_{B,c} u(x).$$

Similarly for any k , we can show that

$$\square_{B,c}^k \delta(x) * u(x) = \square_{B,c}^k u(x).$$

(b) Since $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ are tempered distribution by Lemma 2.1. Now chosen $\text{supp } R_{2k,c}^H(x) = K \subset \bar{\Gamma}_+$, where K is a compact set and $\bar{\Gamma}_+$ designates of Γ_+ closure. Then $R_{2k,c}^H(x)$ is a tempered distribution with compact support. By [2], $R_{2k,c}^H(x) * R_{2m,c}^H(x)$ exists and is a tempered distribution.

(c) From equation $\square_{B,c}^{k+m} u(x) = \delta(x)$ we obtain $u(x) = R_{2k+2m,c}^H(x)$ by Lemma 2.2. For any m is a nonnegative integer, we write

$$\square_{B,c}^{k+m} u(x) = \square_{B,c}^k \square_{B,c}^m u(x) = \delta(x),$$

then by Lemma 2.2, we have the following equality

$$\square_{B,c}^m u(x) = R_{2k,c}^H(x).$$

B -convolving both sides by $R_{2m,c}^H(x)$ we obtain

$$R_{2m,c}^H(x) * \square_{B,c}^m u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x)$$

or

$$\square_{B,c}^m R_{2m,c}^H(x) * u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x).$$

Then from Lemma 2.2, we have the following equality

$$\delta(x) * u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x).$$

It follows that

$$u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x).$$

From the fact that $u(x) = R_{2k+2m,c}^H(x)$ we obtain $R_{2k,c}^H(x) * R_{2m,c}^H(x) = R_{2k+2m,c}^H(x)$.

(d) Since $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ are tempered distributions with compact supports, thus $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ are the elements of space of B -convolution algebra u' of distribution. Now $R_{2k,c}^H(x) * R_{2m,c}^H(x) = \delta(x)$ then by A. H. Zemanian [11] show that $R_{2k,c}^H(x) = R_{2m,c}^{H*-1}(x)$ is a unique inverse. \square

Lemma 2.4. Let $R_{2k,c}^H(x)$ and $K_{n,v}(2k)$, be defined by (2.3) and (2.4). Then

- (a) $K_{n,v}(2k+2) = 2k(2k+2-n-2|v|)K_{n,v}(2k)$,
- (b) $\square_{B,c}^k R_{2m,c}^H(x) = R_{2m-2k,c}^H(x)$, where k and m are nonnegative integer,
- (c) $R_{-2k,c}^H(x) = \square_{B,c}^k \delta(x)$, where k is a nonnegative integer.

Proof.

- (a) From (2.4), we have

$$\begin{aligned} K_{n,v}(2k+2) &= \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2k+2-n-2|v|}{2}\right) \Gamma\left(\frac{1-2k-2}{2}\right) \Gamma(2k+2)}{\Gamma\left(\frac{2+2k+2-p-2|v|}{2}\right) \Gamma\left(\frac{p+2|v|-2k-2}{2}\right)} \\ &= \frac{\pi^{\frac{n+2|v|-1}{2}} \frac{(2k+2-n-2|v|)}{2} \Gamma\left(\frac{2k+2-n-2|v|}{2}\right) \left(\frac{-2}{1+2k}\right) \Gamma\left(\frac{1-2k}{2}\right) (2k)(2k+1)\Gamma(2k)}{\frac{2k+2-p-2|v|}{2} \Gamma\left(\frac{2k+2-p-2|v|}{2}\right) \frac{2}{p+2|v|-2k-2} \Gamma\left(\frac{p+2|v|-2k}{2}\right)} \\ &= 2k(2k+2-n-2|v|)K_{n,v}(2k). \end{aligned}$$

- (b) By Lemma 2.3.(c), we have

$$\begin{aligned} \delta(x) * R_{2m,c}^H(x) &= R_{2k,c}^H(x) * R_{2m-2k,c}^H(x) \\ \square_{B,c}^k R_{2k,c}^H(x) * R_{2m,c}^H(x) &= R_{2k,c}^H(x) * R_{2m-2k,c}^H(x) \\ R_{2k,c}^H(x) * \square_{B,c}^k R_{2m,c}^H(x) &= R_{2k,c}^H(x) * R_{2m-2k,c}^H(x), \end{aligned}$$

and

$$\square_{B,c}^k R_{2m,c}^H(x) = R_{2m-2k,c}^H(x).$$

- (c) For $m = k$, by Lemma 2.4.(b) we have

$$\square_{B,c}^m R_{2m,c}^H(x) = R_{0,c}^H(x), \quad R_{0,c}^H(x) = \delta(x).$$

For $m = 0$, by Lemma 2.4.(b) we have

$$\square_{B,c}^k R_{0,c}^H(x) = R_{-2k,c}^H(x) \quad \text{or} \quad \square_{B,c}^k \delta(x) = R_{-2k,c}^H(x).$$

\square

3. Main Results

Theorem 3.1. Given the compound equation related to the ultra-hyperbolic Bessel operator of the form

$$(3.1) \quad \sum_{r=0}^m C_r \square_{B,c}^r u(x) = f(x),$$

where $\square_{B,c}^r$ is the ultra-hyperbolic Bessel operator iterated r -times defined by (1.4), f is a given generalized function, u is an unknown function, $x \in \mathbb{R}_n^+$, n is odd and C_r is a constant. Then (3.1) has a unique weak solution

$$(3.2) \quad u(x) = f(x) * R_{2m,c}^H(x) * (C_m R_{0,c}^H + w(x) R_{2,c}^H)^{*^{-1}},$$

where

$$(3.3) \quad w(x) = C_{m-1} + C_{m-2} \frac{V}{2(4-n-2|v|)} + C_{m-3} \frac{V^2}{2 \cdot 4(4-n-2|v|)(6-n-2|v|)} + \dots \\ + C_0 \frac{V^{m-1}}{2 \cdot 4 \cdot 6 \dots 2(m-1)(4-n-2|v|)(6-n-2|v|) \dots (2m-n-2|v|)}$$

and V defined by (2.2) and $(C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*^{-1}}$ is an inverse of $C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x)$.

Proof. By Lemma 2.3.(a), equation (3.1) can be written as

$$(C_m \square_{B,c}^m \delta(x) + C_{m-1} \square_{B,c}^{m-1} \delta(x) + \dots + C_1 \square_{B,c} \delta(x) + C_0 \delta(x)) * u(x) = f(x).$$

B -convolving both sides by $R_{2m,c}^H(x)$ defined by (2.3), we obtain

$$\left[C_m \square_{B,c}^m R_{2m,c}^H(x) + C_{m-1} \square_{B,c}^{m-1} R_{2m,c}^H(x) + \dots + C_1 \square_{B,c} R_{2m,c}^H(x) + C_0 R_{2m,c}^H(x) \right] * u(x) \\ = f(x) * R_{2m,c}^H(x).$$

By Lemma 2.2 and Lemma 2.4.(b), we obtain

$$(3.4) \quad \left[C_m \delta(x) + C_{m-1} R_{2,c}^H(x) + C_{m-2} R_{4,c}^H(x) + \dots + C_1 R_{2(m-1),c}^H(x) + C_0 R_{2m,c}^H(x) \right] * u(x) \\ = f(x) * R_{2m,c}^H(x).$$

By Lemma 2.4.(a), we obtain

$$R_{4,c}^H(x) = \frac{V^{\frac{4-n-2|v|}{2}}}{K_n(4)} = R_{2,c}^H(x) \cdot \frac{V}{2(4-n-2|v|)K_n(2)}.$$

Similarly,

$$R_{6,c}^H(x) = R_{2,c}^H(x) \cdot \frac{V^2}{2 \cdot 4(4-n-2|v|)(6-n-2|v|)} \\ R_{8,c}^H(x) = R_{2,c}^H(x) \cdot \frac{V^3}{2 \cdot 4 \cdot 6(4-n-2|v|)(6-n-2|v|)(8-n-2|v|)} \\ \vdots \\ R_{2m,c}^H(x) = R_{2,c}^H(x) \cdot \frac{V^{m-1}}{2 \cdot 4 \cdot 6 \dots 2(m-1)(4-n-2|v|)(6-n-2|v|) \dots (2m-n-2|v|)}.$$

Thus we obtain the function $w(x)$ of (3.3). Now $w(x)$ is continuous and infinitely differentiable in classical sense for n is odd. Since $R_{2,c}^H(x)$ is a tempered distribution with compact support, hence $w(x)R_{2,c}^H(x)$ also is tempered distribution with compact support and so $C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x)$. By Lemma 2.3.(d), $C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x)$ has a unique inverse denote by

$$(C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x))^{*-1}.$$

Now (3.4) can be written as

$$(C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x)) * u(x) = f(x) * R_{2m,c}^H(x), \quad R_{0,c}^H(x) = \delta(x).$$

B-convolving both sides by $(C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x))^{*-1}$, we have

$$u(x) = f(x) * R_{2m,c}^H(x) * (C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x))^{*-1}.$$

Since $R_{2m,c}^H(x)$ is a unique by Lemma 2.2 and $(C_m R_{0,c}^H(x) + w(x)R_{2,c}^H(x))^{*-1}$ also is a unique by Lemma 2.3.(d), it follows that $u(x)$ is a unique weak solution of (3.1) with odd-dimensional n . This completes the proof. \square

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