รายงานวิจัยฉบับสมบูรณ์

โครงการ "การลู่เข้าของแบบแผนเวียนบังเกิดจุดตรึงในปริภูมิบานาค"

Convergences of fixed point iteration schemes in Banach spaces

(ทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่)

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา
และสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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บทคัดย่อ: ในรายงานวิจัยนี้ เราได้แนะนำและศึกษาแบบแผนเวียนบังเกิดต่างๆ สำหรับการส่งแบบ asymptotically nonexpansive และ nonexpansive การส่งแบบ relatively quasi-nonexpansive และการส่งแบบ Lipschitzian ในปริภูมิบานาคและฮิลเบิร์ต เราได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบอ่อนและแบบเข้มของแต่ละแบบแผนเวียนบังเกิดอีกด้วย ผลลัพธ์ที่ได้นั้นพัฒนาและครอบคลุมผลงานของนักคณิตศาสตร์จำนวนมาก นอกจากนี้เรายังได้แก้ไขผลงานที่คลุมเครือของผลงานที่มีมาก่อนหน้านี้

คำหลัก: asymptotically nonexpansive และ nonexpansive, การส่งแบบ relatively quasi-nonexpansive, การส่งแบบ Lipschitzian, ปริภูมิบานาคและฮิลเบิร์ต, ทฤษฎีบทการลู่เข้าแบบอ่อนและแบบเข้ม

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Abstract: We introduce and study many iteration schemes for various kinds of mappings, including asymptotically nonexpansive and nonexpansive mappings, relatively quasi-nonexpansive mappings, and Lipschitzian mappings in Banach spaces and Hilbert spaces. We also prove several weak and and strong convergence theorems of each iteration. Our results improve and include the corresponding known results studied by many authors. Moreover, we correct some doubtful results appeared in the literatures.

Keywords: asymptotically nonexpansive and nonexpansive mappings, relatively quasi-nonexpansive mapping, Lipschitzian mapping, Banach space, Hilbert space, weak and strong convergence theroems

กิตติกรรมประกาศ

ผู้วิจัยขอขอบพระคุณ

สำนักงานคณะกรรมการการอุดมศึกษา (สกอ.) และ สำนักงานกองทุนสนับสนุนการวิจัย (สกว.) ที่ได้ให้โอกาสผู้วิจัยได้รับทุนพัฒนาศักยภาพในการทำงานวิจัยของอาจารย์รุ่นใหม่ ในการทำงานวิจัยค้นคว้าครั้งนี้

ศาสตราจารย์ ดร. สุเทพ สวนใต้ ที่ตอบรับเป็นนักวิจัยที่ปรึกษาให้กับโครงการนี้ และยังให้คำแนะนำที่ดีต่าง ๆ มาโดยตลอด

ศาสตราจารย์ ดร. สมพงษ์ ธรรมพงษา ที่อบรมสั่งสอน ถ่ายทอด ความรู้ด้านต่างๆ จนผู้วิจัยสามารถทำงานวิจัยและค้นคว้าได้

คณะผู้ประเมิน (referee) ของวารสารวิชาการต่าง ๆ ที่ได้ให้คำแนะนำ ตลอดทั้งปรับปรุงต้นฉบับของบทความที่ส่งไปเพื่อตีพิมพ์ในวารสารนั้น ๆ

คณาจารย์ นักศึกษาระดับบัณฑิตศึกษาและเจ้าหน้าที่ฝ่ายสนับสนุน ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยขอนแก่น ได้ร่วมศึกษาวิจัยและช่วยเหลือโครงการวิจัยในครั้งนี้

Chapter 1

Executive Summary

The most important and powerful tools guaranteeing the existence of a solution of a given nonlinear system are Banach's contraction principle and Schauder's fixed point theorem. The application of the first theorem requires only the completeness of a space but a rather heavy assumption on a mapping, i.e., a contraction. For the latter one, it requires a weak assumption on a mapping, i.e., a continuous one, while its domain is compact and convex. In recent years, many mathematicians have investigated several new fixed point theorems requiring only that the mapping and its domain be weaker than the contraction and the compactness, respectively.

Let C be a nonempty convex subset of a Banach space X. A mapping $T : C \to C$ is said to be

(1) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(2) *asymptotically nonexpansive* if there exists a real sequence $\{k_n\}_{n=1}^{\infty}$ such that $\lim_n k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and all natural numbers n;

(3) *asymptotically nonexpansive in the intermediate sense* if T is uniformly

continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left\{ \|T^n x - T^n y\| - \|x-y\| \right\} \le 0;$$

(4) *asymptotically nonexpansive type* if T^N is continuous for some natural number N and

$$\limsup_{n\to\infty} \sup_{y\in C} \left\{ \|T^n x - T^n y\| - \|x-y\| \right\} \le 0$$

for all $x \in C$.

It is easy to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) provided that C is bounded.

Most of the proofs of fixed point theorem of such mappings are nonconstructive, that is, the existence of fixed points for such mapppings does not come together with effective algorithms by which they can be approximated. Moreover, it is possible that the simple Picard's iteration scheme no longer converge to a fixed point of mappings which are a generalization of a contraction. In 1953, W. R. Mann proposed a new iteration scheme and its generalization was introduced later in 1974 by S. Ishikawa. These two iteration schemes are widely studied since then. In 1991, J. Schu modified Mann- and Ishikawa-type iteration schemes for asymptotically nonexpansive mappings. The modified iterations of Schu are also used to approximate fixed points of mappings which are asymptotically nonexpansive in the intermediate sense and of asymptotically nonexpansive type.

A purpose of this research is to introduce and study a new iteration scheme for a class of mappings. Morever, we prove several weak and strong convergence theorems of such schemes.

Chapter 2

Main Results

2.1 A generalized Noor's iteration

We published two paper in this topics (see Appendices A1 and A2). We introduce the so-called *a generalized Noor's iteration* and *a generalized Noor's iteration with errors*, respectively: For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + \gamma_n T^n x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ are appropriate sequences in $[0, 1]$; and

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n + t_n = a_n + b_n +$

$c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

The above generalized Noor's iteration is inspired by the iteration introduced and studied by Suantai. In fact, Suantai's iteration is included in a generalized Noor's iteration if we let $\gamma_n \equiv 0$. The paper A1 presents a complementary result to the paper of Nilsrakoo and the author [W. Nilsrakoo, S. Saejung, A new threestep fixed point iteration scheme for asymptotically nonexpansive mappings, Appl. Math. Comput. 181 (2006) 10261034] as well as corrects the gaps of Yao and Noor's paper [Yao and Noor, Convergence of three-step iterations for asymptotically nonexpansive mappings, Appl. Math. Comput., 187 (2007) 883-892].

The paper A2 proposes the "with-errors" version of a generalized Noor's iteration. We prove several convergence theorems for this new iteration.

2.2 A semigroup of nonexpansive mappings

Besides, asymptotically nonexpansive mappings, we are interested in a semigroup of nonexpansive mappings, usually called a nonexpansive semigroup. Let $\{T(t) : t \geq 0\}$ be a family of mappings from a closed convex subset C of a real Hilbert space H into itself. We call it a *nonexpansive semigroup on* C if the following conditions are satisfied:

(SG1) $T(0)x = x$ for all $x \in C$;

(SG2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

(SG3) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(SG4) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$.

There are many results concerning the approximation of common fixed points of a nonexpansive semigroup. Most of them utilize the Bochner integral while our result does not. In the paper A3, we consider the following iterations:

$$\begin{aligned}
&x_0 \in H \text{ taken arbitrary;}\\
&C_1 = C;\\
&x_1 = P_{C_1}(x_0);\\
&y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)x_n;\\
&C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\};\\
&x_{n+1} = P_{C_{n+1}}(x_0)
\end{aligned}$$

where $\{\alpha_n\} \subset [0,a] \subset [0,1)$, $\liminf_n t_n = 0$, $\limsup_n t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$; and

$$\begin{aligned}
&x_0 \in C \text{ taken arbitrary,}\\
&y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)x_n;\\
&C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\};\\
&Q_n = \{z \in C : \langle x_n - x_0, z - x_n\rangle \ge 0\};\\
&x_{n+1} = P_{C_n \cap Q_n}(x_0)
\end{aligned}$$

where $\{\alpha_n\} \subset [0,a] \subset [0,1)$, $\liminf_n t_n = 0$, $\limsup_n t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$. We also correct the strong convergence theorem recently proved by He and Chen [Strong convergence theorems of the CQ method for nonexpansive semigroups, Fixed Point Theory and Applications, vol. 2007, Article ID 59735, 8 pages, 2007].

2.3 A countable family of mappings

2.3.1 Relatively quasi-nonexpansive mappings

Let T be a mapping from a closed convex subset C of a Banach space E into E and let $\mathrm{F}(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{\mathrm{F}}(T)$. We say that the mapping T is *relatively nonexpansive* if the following conditions are satisfied:

(R1) $\mathrm{F}(T) \neq \varnothing$;

(R2) $\phi(p, Tx) \leq \phi(p, x)$ for each $x \in C, p \in \mathrm{F}(T)$;

(R3) $\mathrm{F}(T) = \widehat{\mathrm{F}}(T)$.

If T is satisfies (R1) and (R2), then T is called *relatively quasi-nonexpansive.*

Let $\{T_n\}$ be a sequence of relatively quasi-nonexpansive mappings from C into E such that $\bigcap_{n=0}^{\infty} \mathrm{F}(T_n)$ is nonempty, and let $\{x_n\}$ be a sequence in C defined as follows:

$$\begin{aligned}
&x_0 \in C \text{ and } C_{-1} = Q_{-1} = C,\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n),\\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\},\\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots,
\end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. In the paper A4, we prove that under certain condition imposed on the family $\{T_n\}$ the iteration above converges strongly to a common fixed point of the family.

2.3.2 Lipschitzian mappings

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be *L-Lipschitzian (or, simply Lipschitzian)* if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

We use the following two iterations to approximate a common fixed point of a countable family of Lipschitzian mappings $\{T_n\}$: (see the paper A6)

Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_n x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$; and let $\{x_n\}$ be a sequence in C defined as follows:

$$\begin{aligned}
&x_0 \in C \quad \text{is arbitrary,} \\
&y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\
&C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots,
\end{aligned}$$

where T_n is L_n-Lipschitzian and

$$\theta_n = (1-\alpha_n)(L_n^2 - 1)(\operatorname{diam} C)^2 \to 0 \text{ as } n \to \infty.$$

2.4 A pertured Ishikawa's iteration

Inspired by Ishikawa's iteration, we introduce the following iteration to obtain weak and strong convergence theorems for three nonexpansive mappings S, T, R from a closed convex subset C of a uniformly convex Banach space E: The sequence $\{x_n\}$, defined by

$$\begin{aligned}
&x_0 \in C \\
&y_n = a'_n R x_n + b'_n T x_n + c'_n v_n \\
&x_{n+1} = a_n R x_n + b_n S y_n + c_n u_n, \qquad n \geq 1,
\end{aligned}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1], $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

It is easy to see that if R is an identity mapping and $c_n = c'_n \equiv 0$, then the iteration above reduces to the original Ishikawa's iteration.

In the paper A5, we not only prove several convergence theorems of this iteration but also point out that the condition recently studied by Rafiq [Convergence of an iterative scheme due to Agarwal et al. Rostock. Math. Kolloq. 61 (2006), 95–105] is not correct. An example of our iteration is presented as well.

2.5 An equillibrium problem

In the paper A7, we establish an iterative scheme by means of Mann's method and Moudafi's method to find a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Let C be a closed and convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C.$$

The set of solutions of the above problem is denoted by $\mathrm{EP}(F)$. Numerous problems in physics, optimization, and economics reduce to find $\mathrm{EP}(F)$ for some suitable bifunction F. For example, given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. The main result of this paper is the following:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

and let S be a nonexpansive mapping of C into H such that $\mathrm{F}(S) \cap \mathrm{EP}(F) \neq \varnothing$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$y_n = \alpha_n f(x_n) + (1 - \alpha_n)Su_n,$$
$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ with

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

and $\{r_n\}$ is a sequence in $(0, \infty)$ with

(r1) $\liminf_{n\to\infty} r_n > 0$, and

(r2) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathrm{F}(S)\cap\mathrm{EP}(F)} f(z)$.

Appendix

A1 Weerayuth Nilsrakoo and **Satit Saejung**, A reconsideration on convergence of three-step iterations for asymptotically nonexpansive mappings, Appl. Math. Comput. 190 (2007), no. 2, 1472-1478. (impact factor(2006): 0.816)

A2 Weerayuth Nilsrakoo, **Satit Saejung**, Generalized Noor iterations with errors for asymptotically nonexpansive mappings, Rostock. Math. Kolloq. 62(2007), 7185. (no impact factor)

A3 **Satit Saejung**, Strong convergence theorems for nonexpansive semigroups without Bochner integrals, Fixed Point Theory and Appl. Volume 2008 (2008), Article ID 745010, 7 pages. (impact factor: coming in 2008)

A4 Weerayuth Nilsrakoo and **Satit Saejung**, Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings, Fixed Point Theory and Appl., Volume 2008 (2008), Article ID 312454, 19 pages. (impact factor: coming in 2008)

A5 Duruni Boobchari and **Satit Saejung**, Weak and strong convergence of a scheme with errors for three nonexpansive mappings, Rostock. Math. Kolloq. 63(2007), 2535. (no impact factor)

A6 Weerayuth Nilsrakoo and **Satit Saejung**, Weak and strong convergence theorems for countable Lipschitzian mappings and its applications, Nonlinear Anal., to appear. (impact factor(2006): 0.677)

A7 Weerayuth Nilsrakoo and **Satit Saejung**, Equilibrium problems and Moudafi's viscosity approximation methods in Hilbert spaces, Dynamics Cont. Discrete Implus. Syst. Ser. A, to appear. (impact factor(2006): 0.244)

A1. Weerayuth Nilsrakoo and **Satit Saejung**, A reconsideration on convergence of three-step iterations for asymptotically nonexpansive mappings, Appl. Math. Comput. 190 (2007), no. 2, 1472-1478. (impact factor(2006): 0.816)

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A reconsideration on convergence of three-step iterations for asymptotically nonexpansive mappings

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Abstract

We illustrate that the control conditions of the main convergence theorems of Yao and Noor [Convergence of three-step iterations for asymptotically nonexpansive mappings, Appl. Math. Comput. in press] are incorrect. We also provide new control conditions which are complementary to Nilsrakoo and Saejung's results [W. Nilsrakoo, S. Saejung, A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings, Appl. Math. Comput. 181 (2006) 1026–1034].

Keywords: Asymptotically nonexpansive mapping; Uniformly convex Banach space; Three-step iteration; Convergence theorem

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geqslant 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geqslant 1$. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leqslant L\|x - y\|,$$

for all $x, y \in C$ and all $n \geqslant 1$.

Since Schu's results [1,2], the modified Mann and Ishikawa iterative schemes have been studied extensively by various authors to approximate fixed points of asymptotically nonexpansive mappings (see [1–6]).

In 2002, Xu and Noor [7] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in a Banach space. Glowinski and Le Tallec [8] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [8] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [9] studied the

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convergence analysis of three-step schemes of Glowinski and Le Tallec [8] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus, we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences. The authors of the present paper [11] defined a new three-step iterative schemes and gave some strong convergence theorems for asymptotically nonexpansive mappings. The scheme studied in the paper is defined as follows.

Let C be a nonempty convex subset of a real Banach space X and $T : C \to C$ be a mapping.

Algorithm 1. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_nT^nx_n + (1-a_n)x_n, \\ y_n &= b_nT^nz_n + c_nT^nx_n + (1-b_n-c_n)x_n, \\ x_{n+1} &= \alpha_nT^ny_n + \beta_nT^nz_n + \gamma_nT^nx_n + (1-\alpha_n-\beta_n-\gamma_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n+c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n+\beta_n+\gamma_n\}$ are appropriate sequences in $[0,1]$. The iterative scheme (1) is called the *three-step mean value iterative scheme*. If $\gamma_n \equiv 0$, then Algorithm 1 reduces to the *modified Noor iterative scheme*, defined by Suantai [10].

If $c_n = \beta_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to Algorithm 2:

Algorithm 2. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_nT^nx_n + (1-a_n)x_n, \\ y_n &= b_nT^nz_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nT^ny_n + (1-\alpha_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{2}$$

where $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ are appropriate sequences in $[0,1]$. The iterative scheme (2) is called the *Noor iterative scheme*, defined by Xu and Noor [7].

If $a_n = c_n = \beta_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to Algorithm 3:

Algorithm 3. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= b_nT^nx_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_nT^ny_n + (1-\alpha_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{3}$$

where $\{b_n\}$ and $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

Similarly, if $b_n = c_n = \alpha_n = \gamma_n \equiv 0$ then Algorithm 1 reduces to Algorithm 3′:

Algorithm 3′. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_nT^nx_n + (1-a_n)x_n, \\ x_{n+1} &= \beta_nT^nz_n + (1-\beta_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{4}$$

where $\{a_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0,1]$.

Let us note that schemes (3) and (4) are called the *modified Ishikawa iterative scheme*, defined by Schu [1,2]. For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$.

Recently, Yao and Noor [12] prove the following two results.

Lemma 1 ([12], Lemma 4). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$, $\limsup_n\|x_n\| \leqslant d$, $\limsup_n\|y_n\| \leqslant d$, $\limsup_n\|z_n\| \leqslant d$, and $\lim_n\|\alpha_nx_n+\beta_ny_n+\gamma_nz_n\| = d$, where $d \geqslant 0$. If*

$$0 < \liminf_n \alpha_n < \liminf_n(\alpha_n+\beta_n) \leqslant \limsup_n(\alpha_n+\beta_n) < 1,$$

then

$$\lim_n \|x_n - y_n\| = \lim_n \|y_n - z_n\| = \lim_n \|z_n - x_n\| = 0.$$

Lemma 2 ([12], Lemma 8). *Let X be a uniformly convex Banach space and C be a nonempty convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ (i.e., $F(T) := \{x \in C : x = Tx\} \neq \emptyset$) and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 and the parameters satisfy one of the following*:

(i) $0 < \liminf_n \alpha_n < \liminf_n(\alpha_n + \beta_n) < \liminf_n(\alpha_n + \beta_n + \gamma_n) \leqslant \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$;
(ii) $0 < \liminf_n \alpha_n < \liminf_n(\alpha_n + \beta_n) \leqslant \limsup_n(\alpha_n + \beta_n) < 1$, $\limsup_n c_n < 1$ *and* $\lim_n \gamma_n = 0$;
(iii) $0 < \liminf_n \beta_n < \liminf_n(\beta_n + \gamma_n) \leqslant \limsup_n(\beta_n + \gamma_n) < 1$ *and* $\lim_n \alpha_n = 0$;
(iv) $0 < \liminf_n \alpha_n$ *and* $0 < \liminf_n b_n < \liminf_n(b_n + c_n) \leqslant \limsup_n(b_n + c_n) < 1$;
(v) $0 < \liminf_n(\alpha_n b_n k_n + \beta_n)$ *and* $0 < \liminf_n a_n \leqslant \limsup_n a_n < 1$.

Then $\lim_n \|Tx_n - x_n\| = 0$.

Remark 3. In the proof of Lemma 1, the conclusion (4) is not correct. Moreover, the assumption

$$0 < \liminf_n \alpha_n < \liminf_n(\alpha_n + \beta_n) \leqslant \limsup_n(\alpha_n + \beta_n) < 1,$$

is *not enough* to guarantee that

$$\lim_n \|x_n - y_n\| = \lim_n \|y_n - z_n\| = 0,$$

as the following example shows.

Example. Let $X = \mathbb{R}$ with the usual norm and let $\alpha_{2n-1} = 1/3$, $\alpha_{2n} = 1/6$, $\beta_{2n-1} = 0$, $\beta_{2n} = 1/6$ and $\gamma_n = 2/3$ for all $n \in \mathbb{N}$. Then

$$0 < \liminf_n \alpha_n < \liminf_n(\alpha_n + \beta_n) \leqslant \limsup_n(\alpha_n + \beta_n) < 1.$$

Suppose that $x_n = z_n = 1$ and $y_{2n-1} = 1/2$, $y_{2n} = 1$ for all $n \in \mathbb{N}$. Then

$$\limsup_n |x_n| = \limsup_n |y_n| = \limsup_n |z_n| = 1$$

and

$$\lim_n |\alpha_n x_n + \beta_n y_n + \gamma_n z_n| = 1.$$

But

$$\lim_n |x_n - y_n| \neq 0 \quad \text{and} \quad \lim_n |y_n - z_n| \neq 0.$$

To present a correction of Lemma 1, we need Schu's lemma.

Lemma 4 [2]. *Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < b \leqslant \lambda_n \leqslant c < 1$ for all $n \geqslant 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_n \|x_n\| \leqslant d$, $\limsup_n \|y_n\| \leqslant d$ and $\lim_n \|\lambda_n x_n + (1 - \lambda_n) y_n\| = d$. Then $\lim_n \|x_n - y_n\| = 0$.*

Lemma 5. *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\limsup_n \|x_n\| \leqslant d$, $\limsup_n \|y_n\| \leqslant d$, $\limsup_n \|z_n\| \leqslant d$, and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.*

Proof. We may assume without loss of generality that $\alpha_n > 0$ and $\beta_n > 0$ for all $n \in \mathbb{N}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that

$$\lim_k \left\| \frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}} y_{n_k} \right\| = \liminf_n \left\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \right\|.$$

Then

$$\begin{aligned} d = \liminf_k \|\alpha_{n_k} x_{n_k} + \beta_{n_k} y_{n_k} + \gamma_{n_k} z_{n_k}\| &\leqslant \liminf_k \left((\alpha_{n_k}+\beta_{n_k}) \left\| \frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}} y_{n_k} \right\| + \gamma_{n_k}\|z_{n_k}\| \right) \\ &\leqslant \liminf_k (\alpha_{n_k}+\beta_{n_k}) \left\| \frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}} y_{n_k} \right\| + \limsup_k \gamma_{n_k}\|z_{n_k}\| \\ &\leqslant \liminf_k (\alpha_{n_k}+\beta_{n_k}) \liminf_n \left\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \right\| + d \limsup_k \gamma_{n_k}. \end{aligned}$$

This implies that

$$\liminf_k (\alpha_{n_k}+\beta_{n_k}) d = (1 - \limsup_k \gamma_{n_k}) d \leqslant \liminf_k (\alpha_{n_k}+\beta_{n_k}) \liminf_n \left\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \right\|.$$

Since $\liminf_n(\alpha_n+\beta_n) \geqslant \liminf_n \alpha_n > 0$, it follows that

$$d \leqslant \liminf_n \left\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \right\| \leqslant \limsup_n \left\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \right\| \leqslant d.$$

We now observe that

$$\liminf_n \frac{\alpha_n}{\alpha_n+\beta_n} \geqslant \liminf_n \alpha_n > 0 \quad \text{and} \quad \liminf_n \frac{\beta_n}{\alpha_n+\beta_n} \geqslant \liminf_n \beta_n > 0.$$

By Lemma 4, we have $\lim_n\|x_n - y_n\| = 0$. This completes the proof. □

We are in the position to present a complementary result of [11].

Lemma 6 ([11], Lemma 2). *Let X be a real Banach space and C be a nonempty convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1, then $\lim_n\|x_n - p\|$ exists for any $p \in F(T)$.*

Lemma 7 ([11], Lemma 6). *Let X be a real Banach space and C be a nonempty convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\lim_n k_n = 1$ and, $\{x_n\}$ be a sequence defined in C by Algorithm 1. If $\lim_n\|T^n x_n - x_n\| = 0$, then $\lim_n\|Tx_n - x_n\| = 0$.*

Lemma 8. *Let X be a uniformly convex Banach space and C be a nonempty convex subset of X. Let $T: C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 and the parameters satisfy one of the following control conditions:*

($C1$) $\liminf_n \alpha_n > 0$ *and one of the following holds*:
- (a) $\limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$ *and* $\limsup_n(b_n+c_n) < 1$;
- (b) $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$ *and* $\limsup_n c_n < 1$;
- (c) $0 < \liminf_n b_n \leqslant \limsup_n(b_n+c_n) < 1$ *and* $\limsup_n a_n < 1$;
- (d) $0 < \liminf_n c_n \leqslant \limsup_n(b_n+c_n) < 1$;

($C2$) $0 < \liminf_n \beta_n \leqslant \limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$ *and* $\limsup_n a_n < 1$;
($C3$) $0 < \liminf_n \gamma_n \leqslant \limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$;
($C4$) $0 < \liminf_n(\alpha_n b_n + \beta_n)$ *and* $0 < \liminf_n a_n \leqslant \limsup_n a_n < 1$.

Then $\lim_n\|Tx_n - x_n\| = 0$.

Proof. If one of (C1-a), (C2) or (C3) holds, we obtain the conclusion (see [11, Lemma 5]). Let $p \in F(T)$. By Lemma 6, let $\lim_n \|x_n - p\| = d$ for some $d \geqslant 0$. It follows from (1) that

$$d = \lim_n \|x_{n+1} - p\| = \lim_n \|\alpha_n(T^n y_n - p) + \beta_n(T^n z_n - p) + \gamma_n(T^n x_n - p) + (1 - \alpha_n - \beta_n - \gamma_n)(x_n - p)\|. \quad (5)$$

We first observe that

$$\|z_n - p\| \leqslant a_n \|T^n x_n - p\| + (1 - a_n)\|x_n - p\| \leqslant (1 + a_n(k_n - 1))\|x_n - p\|, \quad (6)$$

and

$$\begin{aligned} \|y_n - p\| &\leqslant b_n \|T^n z_n - p\| + c_n \|T^n x_n - p\| + (1 - b_n - c_n)\|x_n - p\| \\ &\leqslant b_n k_n \|z_n - p\| + (c_n k_n + (1 - b_n - c_n))\|x_n - p\| \\ &\leqslant (1 + (b_n + a_n b_n k_n + c_n)(k_n - 1))\|x_n - p\|. \end{aligned} \quad (7)$$

Since $\lim_n k_n = 1$, we have

$$\begin{aligned} &\limsup_n \|T^n x_n - p\| \leqslant \limsup_n k_n \|x_n - p\| = d, \\ &\limsup_n \|T^n y_n - p\| \leqslant \limsup_n k_n \|y_n - p\| = \limsup_n \|y_n - p\| \leqslant d, \end{aligned} \quad (8)$$

and

$$\limsup_n \|T^n z_n - p\| \leqslant \limsup_n k_n \|z_n - p\| = \limsup_n \|z_n - p\| \leqslant d. \quad (9)$$

We prove (C1-b). From (5) and Lemma 5, we have

$$\lim_n \|T^n y_n - x_n\| = 0 = \lim_n \|T^n z_n - x_n\|.$$

Notice that

$$\begin{aligned} \|T^n x_n - x_n\| &\leqslant \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \leqslant k_n \|x_n - y_n\| + \|T^n y_n - x_n\| \\ &\leqslant k_n (b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\|) + \|T^n y_n - x_n\|. \end{aligned}$$

Since $\lim_n k_n = 1$ and $\limsup_n c_n < 1$,

$$\lim_n \|T^n x_n - x_n\| = 0.$$

We now prove (C1-c). From (1) and (6), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant \alpha_n k_n \|y_n - p\| + \beta_n k_n \|z_n - p\| + \gamma_n k_n \|x_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| \\ &\leqslant \alpha_n k_n \|y_n - p\| + \beta_n k_n (1 + a_n(k_n - 1))\|x_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n + \gamma_n k_n)\|x_n - p\|, \end{aligned}$$

that is

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n k_n} + \frac{\|x_n - p\|}{k_n} \leqslant \|y_n - p\| + \frac{(\beta_n + a_n \beta_n k_n + \gamma_n)(k_n - 1)}{\alpha_n k_n}\|x_n - p\|.$$

Then

$$d = \lim_n \|x_n - p\| \leqslant \liminf_n \|y_n - p\|.$$

This implies from (8) that

$$d = \lim_n \|y_n - p\| = \lim_n \|b_n(T^n z_n - p) + c_n(T^n x_n - p) + (1 - b_n - c_n)(x_n - p)\|. \quad (10)$$

By Lemma 5, we have

$$\lim_n \|T^n z_n - x_n\| = 0.$$

Using (1),

$$\|T^n x_n - x_n\| \leqslant \|T^n z_n - T^n x_n\| + \|T^n z_n - x_n\| \leqslant k_n\|z_n - x_n\| + \|T^n z_n - x_n\|$$
$$= k_n a_n\|T^n x_n - x_n\| + \|T^n z_n - x_n\|.$$

Since $\lim_n k_n = 1$ and $\limsup_n a_n < 1$,

$$\lim_n \|T^n x_n - x_n\| = 0.$$

We prove (C1-d). By (10) and Lemma 5, we have

$$\lim_n \|T^n x_n - x_n\| = 0.$$

We finally prove (C4). From (1) and (7), we have

$$\|x_{n+1} - p\| \leqslant \alpha_n k_n\|y_n - p\| + \beta_n k_n\|z_n - p\| + \gamma_n k_n\|x_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\|$$
$$\leqslant (\alpha_n b_n k_n^2 + \beta_n k_n)\|z_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n + \gamma_n k_n)\|x_n - p\|$$
$$+ (\alpha_n c_n k_n^2 + \alpha_n k_n(1 - b_n - c_n))\|x_n - p\|,$$

which implies

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n b_n k_n^2 + \beta_n k_n} + \frac{\|x_n - p\|}{k_n} \leqslant \|z_n - p\| + \frac{(\alpha_n + \alpha_n c_n k_n + \gamma_n)(k_n - 1)}{\alpha_n b_n k_n^2 + \beta_n k_n}\|x_n - p\|.$$

Notice that

$$0 < \liminf_n(\alpha_n b_n + \beta_n) \iff 0 < \liminf_n(\alpha_n b_n k_n + \beta_n).$$

Hence

$$d = \lim_n \|x_n - p\| \leqslant \liminf_n \|z_n - p\|.$$

This implies from (9) that

$$d = \lim_n \|z_n - p\| = \lim_n \|a_n(T^n x_n - p) + (1 - a_n)(x_n - p)\|.$$

By Lemma 5, we have

$$\lim_n \|T^n x_n - x_n\| = 0.$$

The conclusion of Lemma 8 can be obtained from Lemma 7 immediately. This completes the proof. □

Theorem 9. *Let X, C, T and $\{x_n\}$ be as in Lemma 8. If T is a completely continuous mapping, then $\{x_n\}$ converges strongly to a fixed point of T.*

Recall that a Banach space X is said to satisfy Opial's condition [13] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Theorem 10. *Let X, C, T and $\{x_n\}$ be as in Lemma 8. If X satisfies Opial's condition, then $\{x_n\}$ converges weakly to a fixed point of T.*

Remark 11

(i) With a slight modification, the complete continuity imposed on T can be replaced by a more general assumption, e.g., condition (A) with respect to the sequence $\{x_n\}$ (see [11, Theorems 7]).
(ii) $0 < \liminf_n(\alpha_n b_n + \beta_n) \iff 0 < \liminf_n(\alpha_n b_n k_n + \beta_n)$.

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References

[1] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991) 407–413.
[2] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153–159.
[3] S.C. Bose, Weak convergence to the fixed point of an asymptotically nonexpansive map, Proc. Am. Math. Soc. 68 (1978) 305–308.
[4] K.K. Tan, H.K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Am. Math. Soc. 122 (1994) 733–739.
[5] S.S. Chang, On the approximation problem of fixed points for asymptotically nonexpansive mappings, Indian J. Pure Appl. Math. 32 (2001) 1297–1307.
[6] Z. Luchuan, Iterative approximation of fixed points of (asymptotically) nonexpansive mappings, Appl. Math. J. Chin. Univ. Ser. B 16 (2001) 402–408.
[7] B.L. Xu, M.A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 (2002) 444–453.
[8] R. Glowinski, P. Le Tallec, Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics, SIAM, Philadelphia, 1989.
[9] S. Haubruge, V.H. Nguyen, J.J. Strodiot, Convergence analysis and applications of the Glowinski–Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators, J. Opt. Theory Appl. 97 (1998) 645–673.
[10] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 311 (2005) 506–517.
[11] W. Nilsrakoo, S. Saejung, A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings, Appl. Math. Comput. 181 (2006) 1026–1034.
[12] Y. Yao, M. A. Noor, Convergence of three-step iterations for asymptotically nonexpansive mappings, Appl. Math. Comput., in press doi:10.1016/j.amc.2006.09.008.
[13] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Am. Math. Soc. 73 (1967) 591–597.

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Generalized Noor iterations with errors for asymptotically nonexpansive mappings

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ABSTRACT. In the present paper, we define and study a new three-step iterative schemes with errors. Several strong convergence theorems of this scheme are established for asymptotically nonexpansive mappings. Our results extend and improve the recent ones announced by Osilike and Aniagbosor, Cho et.al, Liu and Kang, Nammanee et al., and many others.

KEY WORDS. asymptotically nonexpansive mapping, uniformly convex Banach space, Mann-type iteration, Ishikawa-type iteration, Noor-type iteration

1 Introduction

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

In 2002, Xu and Noor [10] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in a Banach space. Glowinski and Le Tallec [2] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [2] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Haubruge, Nguyen and Strodiot [3] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [2] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences. In 2004, Cho, Zhou, and Guo [1], and Liu and Kang [4] extended the preceding scheme to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Recently, Nammanee, Noor and Suantai [5] defined a three-step iterative scheme with errors which is an extension of schemes in [1] and [4] iterations and gave some weak and strong convergence theorems for asymptotically nonexpansive mappings in a uniformly convex Banach space. The authors of the present paper [6] defined a new three-step iterative schemes and gave some strong convergence theorems for asymptotically nonexpansive mappings. Inspired by the preceding iteration scheme, we define a new iteration scheme with errors as follows.

Let C be a nonempty convex subset of a real Banach space X and $T : C \to C$ be a mapping.
Algorithm 1 For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (1) is called the *three-step mean value iterative scheme with errors*.

If $\beta_n \equiv 0$, then Algorithm 1 reduces to
Algorithm 2 [5] For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$,

$\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (2) is called the *modified Noor iterative scheme with errors.*

If $\beta_n = \delta_n = b_n \equiv 0$, then Algorithm 1 reduces to

Algorithm 3 [1, 4] For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + t_n w_n, \end{aligned} \tag{3}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \gamma_n + t_n = a_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (3) is called the *Noor iterative scheme with errors.*

2 Auxiliary Lemmas

For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$. In the sequel, we shall need the following lemmas.

Lemma 1 ([7], Lemma 1) *Let $\{a_n\}, \{b_n\}$ and $\{\lambda_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \lambda_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 2 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ (i.e., $F(T) := \{x \in C : x = Tx\} \neq \varnothing$) and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence defined by Algorithm 1 with the restrictions that $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$. Then we have the following conclusions.*

(i) $\lim_n \|x_n - p\|$ *exists for any $p \in F(T)$.*

(ii) $\lim_n d(x_n, F(T))$ *exists, where $d(x, F(T))$ denotes the distance from x to the fixed-point set $F(T)$.*

Proof: Let $p \in F(T)$. We note that $\{u_n - p\}$, $\{v_n - p\}$, and $\{w_n - p\}$ are bounded sequences in C. Let

$$L = \sup\{k_n : n \geq 1\} \text{ and } M = \sup\{\|u_n - p\|, \|v_n - p\|, \|w_n - p\| : n \geq 1\}.$$

By using (1), we have

$$\begin{aligned} \|z_n - p\| &\leq a'_n\|x_n - p\| + b'_n\|T^n x_n - p\| + r_n\|u_n - p\| \\ &\leq (1 - b'_n)\|x_n - p\| + b'_n k_n\|x_n - p\| + Mr_n \\ &\leq (1 + b'_n(k_n - 1))\|x_n - p\| + Mr_n \\ &\leq k_n\|x_n - p\| + Mr_n, \end{aligned} \tag{1}$$

$$\begin{aligned} \|y_n - p\| &\leq a_n\|x_n - p\| + b_n\|T^n x_n - p\| + c_n\|T^n z_n - p\| + s_n\|v_n - p\| \\ &\leq (1 - b_n - c_n)\|x_n - p\| + b_n k_n\|x_n - p\| + c_n k_n\|z_n - p\| + Ms_n \\ &\leq (1 + (b_n + c_n + c_n k_n)(k_n - 1))\|x_n - p\| + M(s_n + c_n r_n k_n) \\ &\leq (1 + (L + 2)(k_n - 1))\|x_n - p\| + M(s_n + Lc_n r_n), \end{aligned} \tag{2}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + \beta_n\|T^n x_n - p\| + \gamma_n\|T^n y_n - p\| \\ &\quad + \delta_n\|T^n z_n - p\| + t_n\|w_n - p\| \\ &\leq (1 - \beta_n - \gamma_n - \delta_n)\|x_n - p\| + \beta_n k_n\|x_n - p\| \\ &\quad + \gamma_n k_n\|y_n - p\| + \delta_n k_n\|z_n - p\| + Mt_n \\ &\leq (1 + (\beta_n + \gamma_n + \gamma_n k_n(L + 2) + \delta_n(k_n + 1))(k_n - 1))\|x_n - p\| \\ &\quad + M(t_n + \gamma_n k_n s_n + L\gamma_n k_n c_n r_n + \delta_n k_n r_n) \\ &\leq (1 + (L^2 + 3L + 3)(k_n - 1))\|x_n - p\| \\ &\quad + M(t_n + L\gamma_n s_n + L^2\gamma_n c_n r_n + L\delta_n r_n). \end{aligned}$$

By assumption, the conclusions of the lemma follow from Lemma 1. This completes the proof. □

We also need the following lemma proved by Schu [8].

Lemma 3 *Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < b \leq \lambda_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_n \|x_n\| \leq a$, $\limsup_n \|y_n\| \leq a$ and $\lim_n \|\lambda_n x_n + (1 - \lambda_n)y_n\| = a$ for some $a \geq 0$. Then $\lim_n \|x_n - y_n\| = 0$.*

By Schu's Lemma, we have the following lemma.

Lemma 4 *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$, $\limsup_n \|x_n\| \le a$, $\limsup_n \|y_n\| \le a$, $\limsup_n \|z_n\| \le a$, and $\lim_n \|\alpha_n x_n+\beta_n y_n+\gamma_n z_n\| = a$, where $a \ge 0$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.*

Proof: We may assume without loss of generality that $\alpha_n > 0$ and $\beta_n > 0$ for all $n \in \mathbb{N}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that

$$\lim_k \left\| \frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}} y_{n_k} \right\| = \liminf_n \left\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \right\|.$$

Then

$$\begin{aligned} a &= \liminf_k \|\alpha_{n_k} x_{n_k} + \beta_{n_k} y_{n_k} + \gamma_{n_k} z_{n_k}\| \\ &\le \liminf_k \Big((\alpha_{n_k}+\beta_{n_k}) \Big\| \frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}} y_{n_k} \Big\| + \gamma_{n_k}\|z_{n_k}\| \Big) \\ &\le \liminf_k (\alpha_{n_k}+\beta_{n_k}) \Big\| \frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}} y_{n_k} \Big\| + \limsup_k \gamma_{n_k}\|z_{n_k}\| \\ &\le \liminf_k (\alpha_{n_k}+\beta_{n_k}) \liminf_n \Big\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \Big\| + a \limsup_k \gamma_{n_k}. \end{aligned}$$

This implies that

$$\begin{aligned} &\liminf_k (\alpha_{n_k}+\beta_{n_k}) a \\ &= (1 - \limsup_k \gamma_{n_k}) a \\ &\le \liminf_k (\alpha_{n_k}+\beta_{n_k}) \liminf_n \Big\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \Big\|. \end{aligned}$$

Since $\liminf_n (\alpha_n+\beta_n) \ge \liminf_n \alpha_n + \liminf_n \beta_n > 0$, it follows that

$$a \le \liminf_n \Big\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \Big\| \le \limsup_n \Big\| \frac{\alpha_n}{\alpha_n+\beta_n} x_n + \frac{\beta_n}{\alpha_n+\beta_n} y_n \Big\| \le a.$$

We now observe that

$$\liminf_n \frac{\alpha_n}{\alpha_n+\beta_n} \ge \liminf_n \alpha_n > 0 \quad \text{and} \quad \liminf_n \frac{\beta_n}{\alpha_n+\beta_n} \ge \liminf_n \beta_n > 0.$$

By Lemma 3, we have $\lim_n \|x_n - y_n\| = 0$. This completes the proof. □

The following lemmas are the important ingredients for proving our main results in the next section.

Lemma 5 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty* *fixed-point set* $F(T)$ *and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{x_n\}$ *be a sequence defined by Algorithm 1 with the restrictions that* $\sum_{n=1}^{\infty} t_n < \infty$*,* $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$*,* $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n r_n < \infty$*. Then we have the following assertions.*

(i) *If* $0 < \liminf_n \gamma_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ *and* $\limsup_n(b_n + c_n) < 1$*, then* $\lim_n \|T^n x_n - x_n\| = 0$.

(ii) *If* $0 < \liminf_n \delta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ *and* $\limsup_n b'_n < 1$*, then* $\lim_n \|T^n x_n - x_n\| = 0$.

(iii) *If* $0 < \liminf_n \beta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$*, then* $\lim_n \|T^n x_n - x_n\| = 0$.

Proof: Let $p \in F(T)$. By Lemma 2, we have $\lim_n \|x_n - p\| = a$ for some $a \geq 0$. Since $\lim_n t_n = 0$,

$$\begin{aligned} a &= \lim_n \|x_{n+1} - p\| \\ &= \lim_n \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(T^n x_n - p) + \gamma_n(T^n y_n - p) \\ &\quad + \delta_n(T^n z_n - p) + t_n(w_n - x_n)\| \\ &= \lim_n \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(T^n x_n - p) \\ &\quad + \gamma_n(T^n y_n - p) + \delta_n(T^n z_n - p)\|. \end{aligned} \tag{3}$$

We first observe that

$$\limsup_n \|T^n x_n - p\| \leq \limsup_n k_n\|x_n - p\| = a. \tag{4}$$

To prove (i), let $\{m_j\}$ be a subsequence of $\{n\}$. We show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \|T^{n_k} y_{n_k} - x_{n_k}\| = 0$.
As $\liminf_n \gamma_n > 0$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, and $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$, $\lim_n s_n = c_n r_n = 0$. By using (2), we have

$$\limsup_j \|T^{m_j} y_{m_j} - p\| \leq \limsup_j k_{m_j}\|y_{m_j} - p\| \leq a. \tag{5}$$

If $\liminf_j \delta_{m_j} > 0$, then $\lim_j r_{m_j} = 0$. By (1), we gives

$$\limsup_j \|T^{m_j} z_{m_j} - p\| \leq \limsup_j k_{m_j}\|z_{m_j} - p\| \leq a. \tag{6}$$

It follows from (3)-(6) and Lemma 4 that

$$\lim_j \|T^{m_j} y_{m_j} - x_{m_j}\| = 0.$$

On the other hand, if $\liminf_j \delta_{m_j} = 0$, then we may extract a subsequence $\{\delta_{n_k}\}$ of $\{\delta_{m_j}\}$ so that $\lim_k \delta_{n_k} = 0$, it follows that

$$\lim_k \delta_{n_k}\|x_{n_k} - p\| = 0 = \lim_k \delta_{n_k}\|T^{n_k} z_{n_k} - p\|.$$

This together with (3) gives

$$\begin{aligned} a = \lim_k \|&(1 - \beta_{n_k} - \gamma_{n_k})(x_{n_k} - p) \\ &+ \beta_{n_k}(T^{n_k} x_{n_k} - p) + \gamma_{n_k}(T^{n_k} y_{n_k} - p)\|. \end{aligned} \tag{7}$$

It follows from (4), (5), (7), and Lemma 4 that

$$\lim_k \|T^{n_k} y_{n_k} - x_{n_k}\| = 0.$$

By double extract subsequence principle,

$$\lim_n \|(x_n - p) - (T^n y_n - p)\| = \lim_n \|T^n y_n - x_n\| = 0. \tag{8}$$

It follows that $\lim_n \|T^n y_n - p\| = a$. Also

$$a = \liminf_n \|T^n y_n - p\| \le \liminf_n k_n\|y_n - p\| = \liminf_n \|y_n - p\|.$$

From (2), we gives $\limsup_n \|y_n - p\| \le a$, so that $\lim_n \|y_n - p\| = a$.
Next we prove that

$$\lim_n \|T^n x_n - x_n\| = 0, \tag{9}$$

let $\{\ell_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{\ell_j\}$ such that $\lim_k \|T^{n_k} x_{n_k} - x_{n_k}\| = 0$. Since $\lim_n s_n = 0$,

$$\begin{aligned} a &= \lim_j \|y_{\ell_j} - p\| \\ &= \lim_j \|(1 - b_{\ell_j} - c_{\ell_j})(x_{\ell_j} - p) + b_{\ell_j}(T^{\ell_j} x_{\ell_j} - p) \\ &\quad + c_{\ell_j}(T^{\ell_j} z_{\ell_j} - p) + s_{\ell_j}(v_{\ell_j} - x_{\ell_j})\| \\ &= \lim_j \|(1 - b_{\ell_j} - c_{\ell_j})(x_{\ell_j} - p) + b_{\ell_j}(T^{\ell_j} x_{\ell_j} - p) + c_{\ell_j}(T^{\ell_j} z_{\ell_j} - p)\|. \end{aligned}$$

If $\liminf_j c_{\ell_j} > 0$, by Lemma 4 and $\limsup_n (b_n + c_n) < 1$, then

$$\lim_j \|T^{\ell_j} z_{\ell_j} - x_{\ell_j}\| = 0. \tag{10}$$

On the other hand, if $\liminf_j c_{\ell_j} = 0$, then we may extract a subsequence $\{c_{n_k}\}$ of $\{c_{\ell_j}\}$ so that $\lim_k c_{n_k} = 0$, it follows that

$$\lim_k c_{n_k}\|T^{n_k} z_{n_k} - x_{n_k}\| = 0. \tag{11}$$

By using (1), we have

$$\begin{aligned}\|T^{n_k}x_{n_k}-x_{n_k}\| &\le \|T^{n_k}x_{n_k}-T^{n_k}y_{n_k}\|+\|T^{n_k}y_{n_k}-x_{n_k}\|\\ &\le k_{n_k}\|x_{n_k}-y_{n_k}\|+\|T^{n_k}y_{n_k}-x_{n_k}\|\\ &\le k_{n_k}b_{n_k}\|T^{n_k}x_{n_k}-x_{n_k}\|+k_{n_k}c_{n_k}\|T^{n_k}z_{n_k}-x_{n_k}\|\\ &\quad +k_{n_k}s_{n_k}\|v_{n_k}-x_{n_k}\|+\|T^{n_k}y_{n_k}-x_{n_k}\|.\end{aligned}$$

This together with (8), (10), and (11) gives

$$\lim_k (1-k_{n_k}b_{n_k})\|T^{n_k}x_{n_k}-x_{n_k}\|=0.$$

As $\liminf_n(1-k_nb_n)=1-\limsup_n b_n\ge 1-\limsup_n(b_n+c_n)>0$, we have

$$\lim_k \|T^{n_k}x_{n_k}-x_{n_k}\|=0.$$

By double extract subsequence principle, we obtain (9) and the proof of (i) is finished.

By using a similar method, it can be shown that (ii) is satisfied.

(iii) To show that

$$\lim_n \|T^n x_n-x_n\|=0, \tag{12}$$

let $\{m_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k\|T^{n_k}x_{n_k}-x_{n_k}\|=0$. We consider the following cases.

Case 1: $\liminf_j \gamma_{m_j}>0$.

Subcase 1.1: $\liminf_j \delta_{m_j}>0$. Then we obtain (3)-(6). It follows from Lemma 4 that $\lim_j\|T^{m_j}x_{m_j}-x_{m_j}\|=0$.

Subcase 1.2: $\liminf_j\delta_{m_j}=0=\lim_k\delta_{n_k}$, where $\{\delta_{n_k}\}\subset\{\delta_{m_j}\}$. Then we obtain (7), and so

$$\lim_k \|T^{n_k}x_{n_k}-x_{n_k}\|=0.$$

Case 2: $\liminf_j\gamma_{m_j}=0$. Choose $\{\gamma_{\ell_k}\}\subset\{\gamma_{m_j}\}$ such that $\lim_k\gamma_{\ell_k}=0$, it follows that

$$\lim_k \gamma_{\ell_k}\|x_{\ell_k}-p\|=0=\lim_k\gamma_{\ell_k}\|T^{\ell_k}y_{\ell_k}-p\|.$$

This together with (3) gives

$$a=\lim_k \|(1-\beta_{\ell_k}-\delta_{\ell_k})(x_{\ell_k}-p)+\beta_{\ell_k}(T^{\ell_k}x_{\ell_k}-p)+\delta_{\ell_k}(T^{\ell_k}z_{\ell_k}-p)\|. \tag{13}$$

Subcase 2.1: $\liminf_k\delta_{\ell_k}>0$. By (1), we have $\limsup_k\|T^{\ell_k}z_{\ell_k}-p\|\le a$. It follows from (4), (13) and Lemma 4,

$$\lim_k \|T^{\ell_k}x_{\ell_k}-x_{\ell_k}\|=0.$$

Subcase 2.2: $\liminf_k \delta_{\ell_k} = 0 = \lim_i \delta_{n_i}$, where $\{\delta_{n_i}\} \subset \{\delta_{\ell_k}\}$. It follows that

$$\lim_i \delta_{n_i} \|T^{n_i} z_{n_i} - p\| = 0.$$

This together with (13) gives

$$a = \lim_i \|(1 - \beta_{n_i})(x_{n_i} - p) + \beta_{n_i}(T^{n_i} x_{n_i} - p)\|.$$

It follows from Lemma 3, $\lim_i \|T^{n_i} x_{n_i} - x_{n_i}\| = 0$. By double extract subsequence principle, we obtain (12). This completes the proof. □

Lemma 6 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\lim_n k_n = 1$ and, $\{x_n\}$ be a sequence defined in C by Algorithm 1 with the restrictions that $\lim_n t_n = \lim_n \gamma_n s_n = \lim_n \gamma_n c_n r_n = \lim_n \delta_n r_n = 0$. If $\lim_n \|T^n x_n - x_n\| = 0$, then $\lim_n \|Tx_n - x_n\| = 0$.*

Proof: Using (1), we have

$$\begin{aligned}
\|T^n z_n - x_n\| &\leq \|T^n z_n - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leq k_n \|z_n - x_n\| + \|T^n x_n - x_n\|, \\
&\leq (b'_n k_n + 1)\|T^n x_n - x_n\| + r_n k_n \|u_n - x_n\|,
\end{aligned}$$

$$\begin{aligned}
\|T^n y_n - x_n\| &\leq \|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leq k_n \|y_n - x_n\| + \|T^n x_n - x_n\|, \\
&\leq b_n k_n \|T^n x_n - x_n\| + c_n k_n \|T^n z_n - x_n\| \\
&\quad + s_n k_n \|v_n - x_n\| + \|T^n x_n - x_n\| \\
&\leq (b_n k_n + c_n b'_n k_n^2 + c_n k_n + 1)\|T^n x_n - x_n\| \\
&\quad + s_n k_n \|v_n - x_n\| + c_n r_n k_n^2 \|u_n - x_n\|,
\end{aligned}$$

and so

$$\begin{aligned}
&\|x_{n+1} - T^n x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
&\le (1+k_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
&\le \beta_n(1+k_n)\|T^n x_n - x_n\| + \gamma_n(1+k_n)\|T^n y_n - x_n\| \\
&\quad + \delta_n(1+k_n)\|T^n z_n - x_n\| + t_n(1+k_n)\|w_n - x_n\| + \|T^n x_n - x_n\| \\
&\le \beta_n(1+k_n)\|T^n x_n - x_n\| \\
&\quad + \gamma_n(1+k_n)(b_n k_n + c_n b'_n k_n^2 + c_n k_n + 1)\|T^n x_n - x_n\| \\
&\quad + \gamma_n s_n k_n(1+k_n)\|v_n - x_n\| + \gamma_n c_n r_n(1+k_n)k_n^2\|u_n - x_n\| \\
&\quad + \delta_n(1+k_n)(b'_n k_n + 1)\|T^n x_n - x_n\| + \delta_n r_n(1+k_n)k_n\|u_n - x_n\| \\
&\quad + t_n(1+k_n)\|w_n - x_n\| + \|T^n x_n - x_n\| \to 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\
&\le \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^n x_{n+1} - x_{n+1}\| \to 0,
\end{aligned}$$

which implies $\lim_n \|Tx_n - x_n\| = 0$. This completes the proof. □

3 Main results

In this section, we establish several strong convergence theorems of the three-step mean value iterative scheme with errors for asymptotically nonexpansive mappings.

Theorem 7 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \le \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$,

(ii) $\limsup_n(b_n + c_n) < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Let $\{x_n\}$ be a given sequence in C. Recall that a mapping $T : C \to C$ with the nonempty fixed-point set $F(T)$ in C satisfies *Condition (A) with respect to the sequence* $\{x_n\}$ ([9]) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x_n, F(T))) \leq \|x_n - Tx_n\|, \text{ for all } n \geq 1.$$

Proof. By Lemma 5(i) and Lemma 6, we have

$$\lim_n \|Tx_n - x_n\| = 0.$$

Let f be a nondecreasing function corresponding to Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\| \to 0,$$

and so

$$d(x_n, F(T)) \to 0.$$

Therefore, the conclusion of the theorem follows exactly from [6]. This completes the proof. □

Remark 8 Suppose we rewrite our scheme by treating the additional terms as error terms in the sense of Xu [11] in this way: $x_1 \in C$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + c_n T^n z_n + (b_n + s_n)\Big(\frac{b_n}{b_n + s_n} T^n x_n + \frac{s_n}{b_n + s_n} v_n\Big), \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + (\beta_n + \delta_n + t_n) \\ &\quad \times \Big(\frac{\beta_n}{\beta_n + \delta_n + t_n} T^n x_n + \frac{\delta_n}{\beta_n + \delta_n + t_n} T^n z_n + \frac{t_n}{\beta_n + \delta_n + t_n} w_n\Big), \end{aligned}$$

for all $n \geq 1$. To obtain a strong convergence theorem by Theorem 2.4 of [1], we are restricted to the following

$$\sum_{n=1}^{\infty} (\beta_n + \delta_n + t_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + s_n) < \infty,$$

from which $\lim_n \beta_n = \lim_n \delta_n = \lim_n b_n = 0$, $\sum_{n=1}^{\infty} s_n < \infty$, and $\sum_{n=1}^{\infty} t_n < \infty$. But our Theorem 7 still gives the result for more general restriction. For example, our result is applicable to the case of $\beta_n = \delta_n = b_n = 1/4$ and $s_n = t_n = 1/2^n$.

Consequently, we obtain the following corollaries. When $\beta_n \equiv 0$, we have

Corollary 9 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with* *the nonempty fixed-point set and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{x_n\}$ *be a sequence in* C *defined by Algorithm 2 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \leq \limsup_n(\gamma_n + \delta_n) < 1$,

(ii) $\limsup_n(b_n + c_n) < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T *satisfies Condition (A) with respect to the sequence* $\{x_n\}$*, then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

When $\beta_n = \delta_n = b_n \equiv 0$ in Theorem 7, we also have

Corollary 10 *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed convex subset of* X*. Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{x_n\}$ *be a sequence in* C *defined by Algorithm 3 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \leq \limsup_n \gamma_n < 1$,

(ii) $\limsup_n c_n < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$.

If T *satisfies Condition (A) with respect to the sequence* $\{x_n\}$*, then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Remark 11

1. Corollary 9 extends and improves Theorem 2.3 of [5] in the following ways:
   - (i) The condition $\liminf_n c_n > 0$ is removed.
   - (ii) The restriction $\sum_{n=1}^{\infty} r_n < \infty$ is weakened and replaced by $\sum_{n=1}^{\infty} c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.
   - (iii) The complete continuity imposed on T is replaced by the more general Condition (A) with respect to $\{x_n\}$ (see also [1, Corollary 2.5]).
2. Corollary 10 extends and improves Theorem 2.4 of [1]. The restriction $\sum_{n=1}^{\infty} r_n < \infty$ is weakened and replaced by $\sum_{n=1}^{\infty} c_n r_n < \infty$.

3. Corollary 10 also extends and improves Theorem 3.2 of [4] in the following ways:

   (i) The semi-compactness imposed on T is weakened by assuming that T satisfies Condition (A) with respect to $\{x_n\}$ [1, Corollary 2.5].

   (ii) The condition $\lim_n c_n = 0$ is weakened and replaced by $\limsup_n c_n < 1$.

Next, as consequences of Lemma 5(ii), (iii) and Lemma 6, we have the following theorems.

Theorem 12 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \delta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$,

(ii) $\limsup_n b'_n < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n s_n < \infty$, $\sum_{n=1}^{\infty} r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 13 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \beta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ *and*

(ii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 14 By using the same ideas and techniques, we can also discuss the weak convergence for asymptotically nonexpansive mappings with errors and thereby improve the corresponding results obtained by Cho, Zhou and Guo [1], Liu and Kang [4], and Nammanee, Noor and Suantai [5].

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References

[1] **Cho, Y. J., Zhou, H. Y.**, and **Guo, G.** : *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings.* Comput. Math. Appl. **47**, 707–717 (2004)

[2] **Glowinski, R.**, and **Le Tallec, P.** : *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics.* SIAM Studies in Applied Mathematics, 9. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (1989)

[3] **Haubruge, S., Nguyen, V. H.**, and **Strodiot, J. J.** : *Convergence analysis and applications of the Glowinski-Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators.* J. Optim. Theory Appl. **97**, 645–673 (1998)

[4] **Liu, Z. Q.**, and **Kang, S. M.** : *Weak and strong convergence for fixed points of asymptotically non-expansive mappings.* Acta Math. Sin. (Engl. Ser.) **20**, 1009–1018 (2004)

[5] **Nammanee, K., Noor, M. A.**, and **Suantai, S.** : *Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings.* J. Math. Anal. Appl. **314**, 320–334 (2006)

[6] **Nilsrakoo, W.**, and **Saejung, S.** : *A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings.* Appl. Math. Comp. **181**, 1026–1034 (2006)

[7] **Osilike, M. O.**, and **Aniagbosor, S. C.** : *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings.* Math. Comput. Modelling. **32**, 1181–1191 (2000)

[8] **Schu, J.** : *Weak and strong convergence of fixed points of asymptotically nonexpansive mappings.* Bull. Austral. Math. Soc. **43**, 153–159 (1991)

[9] **Senter, H. F.**, and **Dotson, W. G., Jr.** : *Approximating fixed points of nonexpansive mappings.* Proc. Amer. Math. Soc. **44**, 375–380 (1974)

[10] **Xu, B. L.**, and **Noor, M. A.** : *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces.* J. Math. Anal. Appl. **267**, 444–453 (2002)

[11] **Xu, Y. :** *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. **224**, 91–101 (1998)

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Research Article

Strong Convergence Theorems for Nonexpansive Semigroups without Bochner Integrals

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We prove a convergence theorem by the new iterative method introduced by Takahashi et al. (2007). Our result does not use Bochner integrals so it is different from that by Takahashi et al. We also correct the strong convergence theorem recently proved by He and Chen (2007).

1. Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $\{T(t) : t \geq 0\}$ be a family of mappings from a subset C of H into itself. We call it a nonexpansive semigroup on C if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in C$;

(2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

(3) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(4) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$.

Motivated by Suzuki's result [1] and Nakajo-Takahashi's results [2], He and Chen [3] recently proved a strong convergence theorem for nonexpansive semigroups in Hilbert spaces by hybrid method in the mathematical programming. However, their proof of the main result ([3, Theorem 2.3]) is very questionable. Indeed, the existence of the subsequence $\{s_j\}$ such that (2.16) of [3] are satisfied, that is,

$$s_j \longrightarrow 0, \quad \frac{\|x_j - T(s_j)x_j\|}{s_j} \longrightarrow 0, \tag{1.1}$$

needs to be proved precisely. So, the aim of this short paper is to correct He-Chen's result and also to give a new result by using the method recently introduced by Takahashi et al.

We need the following lemma proved by Suzuki [4, Lemma 1].

Lemma 1.1. *Let $\{t_n\}$ be a real sequence and let τ be a real number such that $\liminf_n t_n \leq \tau \leq \limsup_n t_n$. Suppose that either of the following holds:*

(i) $\limsup_n (t_{n+1} - t_n) \leq 0$, *or*

(ii) $\liminf_n (t_{n+1} - t_n) \geq 0$.

Then τ is a cluster point of $\{t_n\}$. Moreover, for $\varepsilon > 0$, $k, m \in \mathbb{N}$, there exists $m_0 \geq m$ such that $|t_j - \tau| < \varepsilon$ for every integer j with $m_0 \leq j \leq m_0 + k$.

2. Results

2.1. The shrinking projection method

The following method is introduced by Takahashi et al. in [5]. We use this method to approximate a common fixed point of a nonexpansive semigroup without Bochner integrals as was the case in [5, Theorem 4.4].

Theorem 2.1. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C with a nonempty common fixed point F, that is, $F = \cap_{t\geq 0} F(T(t)) \neq \varnothing$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following scheme:*

$$
\begin{aligned}
x_0 &\in H \text{ taken arbitrary}, \\
C_1 &= C, \\
x_1 &= P_{C_1}(x_0), \\
y_n &= \alpha_n x_n + (1-\alpha_n) T(t_n) x_n, \\
C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned}
\tag{2.1}
$$

where $\{\alpha_n\} \subset [0, a] \subset [0, 1)$, $\liminf_n t_n = 0$, $\limsup_n t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. Then $x_n \to P_F(x_0)$.

Proof. It is well known that F is closed and convex. We first show that the iterative scheme is well defined. To see that each C_n is nonempty, it suffices to show that $F \subset C_n$. The proof is by induction. Clearly, $F \subset C_1$. Suppose that $F \subset C_k$. Then, for $z \in F \subset C_k$,

$$
\begin{aligned}
\|y_k - z\| &\leq \alpha_k \|x_k - z\| + (1-\alpha_k)\|T(t_k)x_k - z\| \\
&\leq \alpha_k \|x_k - z\| + (1-\alpha_k)\|x_k - z\| \\
&= \|x_k - z\|.
\end{aligned}
\tag{2.2}
$$

That is, $z \in C_{k+1}$ as required.

Notice that

$$
\widehat{C}_n := \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}
\tag{2.3}
$$

is convex since

$$\|y_n - z\| \leq \|x_n - z\| \Longleftrightarrow 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2. \tag{2.4}$$

This implies that each subset $C_n = C \cap \widehat{C}_1 \cap \cdots \cap \widehat{C}_{n-1}$ is convex. It is also clear that C_n is closed. Hence the first claim is proved.

Next, we prove that $\{x_n\}$ is bounded. As $x_n = P_{C_n}(x_0)$,

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall z \in C_n. \tag{2.5}$$

In particular, for $z \in F \subset C_n$ for all $n \in \mathbb{N}$, the sequence $\{x_n - x_0\}$ is bounded and hence so is $\{x_n\}$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. As $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n}(x_0)$,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n. \tag{2.6}$$

Moreover, since the sequence $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} \|x_n - x_0\| \text{ exists.} \tag{2.7}$$

Note that

$$\langle x_0 - x_n, x_n - v\rangle \geq 0 \quad \forall v \in C_n. \tag{2.8}$$

In particular, since $x_{n+k} \in C_{n+k} \subset C_n$ for all $k \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+k} - x_n\|^2 &= \|x_{n+k} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+k} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+k} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{2.9}$$

It then follows from the existence of $\lim_n \|x_n - x_0\|^2$ that $\{x_n\}$ is a Cauchy sequence. In fact, for $\varepsilon > 0$, there exists a natural number N such that, for all $n \geq N$,

$$\left| \|x_n - x_0\|^2 - a \right| < \frac{\varepsilon}{2}, \tag{2.10}$$

where $a = \lim_n \|x_n - x_0\|^2$. In particular, if $n \geq N$ and $k \in \mathbb{N}$, then

$$\begin{aligned} \|x_{n+k} - x_n\|^2 &\leq \|x_{n+k} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\leq a + \frac{\varepsilon}{2} - \left(a - \frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned} \tag{2.11}$$

Moreover,

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{2.12}$$

We now assume that $x_n \to p$ for some $p \in C$. Now since $\alpha_n \leq a < 1$ for all $n \in \mathbb{N}$ and $x_{n+1} \in C_n$,

$$\begin{aligned}\|x_n - T(t_n)x_n\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\leq \frac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\leq \frac{2}{1-a}\|x_{n+1} - x_n\| \longrightarrow 0.\end{aligned} \tag{2.13}$$

The last convergence follows from (2.12). We choose a sequence $\{t_{n_k}\}$ of positive real number such that

$$t_{n_k} \longrightarrow 0, \quad \frac{1}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| \longrightarrow 0. \tag{2.14}$$

We now show that how such a special subsequence can be constructed. First we fix $\delta > 0$ such that

$$\liminf_n t_n = 0 < \delta < \limsup_n t_n. \tag{2.15}$$

From (2.13), there exists $m_1 \in \mathbb{N}$ such that $\|T(t_n)x_n - x_n\| < 1/3^2$ for all $n \geq m_1$. By Lemma 1.1, $\delta/2$ is a cluster point of $\{t_n\}$. In particular, there exists $n_1 > m_1$ such that $\delta/3 < t_{n_1} < \delta$. Next, we choose $m_2 > n_1$ such that $\|T(t_n)x_n - x_n\| < 1/4^2$ for all $n \geq m_2$. Again, by Lemma 1.1, $\delta/3$ is a cluster point of $\{t_n\}$ and this implies that there exists $n_2 > m_2$ such that $\delta/4 < t_{n_2} < \delta/2$. Continuing in this way, we obtain a subsequence $\{n_k\}$ of $\{n\}$ satisfying

$$\|T(t_{n_k})x_{n_k} - x_{n_k}\| < \frac{1}{(k+2)^2}, \quad \frac{\delta}{k+2} < t_{n_k} < \frac{\delta}{k} \quad \forall k \in \mathbb{N}. \tag{2.16}$$

Consequently, (2.14) is satisfied.

We next show that $p \in F$. To see this, we fix $t > 0$,

$$\begin{aligned}&\|x_{n_k} - T(t)p\| \\ &\leq \sum_{j=0}^{[t/t_{n_k}]-1} \|T(jt_{n_k})x_{n_k} - T((j+1)t_{n_k})x_{n_k}\| \\ &\quad + \left\|T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)x_{n_k} - T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)p\right\| + \left\|T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)p - T(t)p\right\| \\ &\leq \left[\frac{t}{t_{n_k}}\right]\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - p\| + \left\|T\left(t - \left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)p - p\right\| \\ &\leq \frac{t}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - p\| + \sup\{\|T(s)p - p\| : 0 \leq s \leq t_{n_k}\}.\end{aligned} \tag{2.17}$$

As $x_{n_k} \to p$ and (2.14), we have $x_{n_k} \to T(t)p$ and so $T(t)p = p$.

Finally, we show that $p = P_F(x_0)$. Since $F \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}(x_0)$,

$$\|x_{n+1} - x_0\| \leq \|q - x_0\| \quad \forall n \in \mathbb{N}, \; q \in F. \tag{2.18}$$

But $x_n \to p$; we have

$$\|p - x_0\| \le \|q - x_0\| \quad \forall q \in F. \tag{2.19}$$

Hence $p = P_F(x_0)$ as required. This completes the proof. □

2.2. The hybrid method

We consider the iterative scheme computing by the hybrid method (some authors call the CQ-method). The following result is proved by He and Chen [3]. However, the important part of the proof seems to be overlooked. Here we present the correction under some additional restriction on the parameter $\{t_n\}$.

Theorem 2.2. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \ge 0\}$ be a nonexpansive semigroup on C with a nonempty common fixed point F, that is, $F = \cap_{t\ge 0} F(T(t)) \ne \varnothing$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following scheme:*

$$\begin{gathered} x_0 \in C \text{ taken arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T(t_n) x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - x_0, z - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{gathered} \tag{2.20}$$

where $\{\alpha_n\} \subset [0,a] \subset [0,1)$, $\liminf_n t_n = 0$, $\limsup_n t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. Then $x_n \to P_F(x_0)$.

Proof. For the sake of clarity, we give the whole sketch proof even though some parts of the proof are the same as [3]. To see that the scheme is well defined, it suffices to show that both C_n and Q_n are closed and convex, and $C_n \cap Q_n \ne \varnothing$ for all $n \in \mathbb{N}$. It follows easily from the definition that C_n and Q_n are just the intersection of C and the half-spaces, respectively,

$$\begin{aligned} \widehat{C}_n &:= \left\{z \in H : 2\langle x_n - y_n, z\rangle \le \|x_n\|^2 - \|y_n\|^2\right\}, \\ \widehat{Q}_n &:= \{z \in H : \langle x_n - x_0, z - x_n \rangle \ge 0\}. \end{aligned} \tag{2.21}$$

As in the proof of the preceding theorem, we have $F \subset C_n$ for all $n \in \mathbb{N}$. Clearly, $F \subset C = Q_1$. Suppose that $F \subset Q_k$ for some $k \in \mathbb{N}$, we have $p \in C_k \cap Q_k$. In particular, $\langle x_{k+1} - x_0, p - x_{k+1}\rangle \ge 0$, that is, $p \in Q_{k+1}$. It follows from the induction that $F \subset Q_n$ for all $n \in \mathbb{N}$. This proves the claim.

We next show that $x_n - T(t_n)x_n \to 0$. To see this, we first prove that

$$x_{n+1} - x_n \longrightarrow 0. \tag{2.22}$$

As $x_{n+1} \in Q_n$ and $x_n = P_{Q_n}(x_0)$,

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \quad \forall n \in \mathbb{N}. \tag{2.23}$$

For fixed $z \in F$. It follows from $F \subset Q_n$ for all $n \in \mathbb{N}$ that

$$\|x_n - x_0\| \le \|z - x_0\| \quad \forall n \in \mathbb{N}. \tag{2.24}$$

This implies that sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - x_0\| \text{ exists.} \tag{2.25}$$

Notice that

$$\langle x_{n+1} - x_n, x_n - x_0 \rangle \ge 0. \tag{2.26}$$

This implies that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0. \end{aligned} \tag{2.27}$$

It then follows from $x_{n+1} \in C_n$ that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$ and hence

$$\begin{aligned} \|T(t_n)x_n - x_n\| &= \frac{1}{\alpha_n}\|y_n - x_n\| \\ &\le \frac{1}{\alpha_n}\left(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\right) \longrightarrow 0. \end{aligned} \tag{2.28}$$

As in Theorem 2.1, we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that

$$x_{n_k} \xrightarrow{w} p \in C, \; t_{n_k} \longrightarrow 0, \quad \frac{1}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| \longrightarrow 0. \tag{2.29}$$

Consequently, for any $t > 0$,

$$\|x_{n_k} - T(t)p\| \le \frac{t}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - p\| + \sup\left\{\|T(s)p - p\| : 0 \le s \le t_{n_k}\right\}. \tag{2.30}$$

This implies that

$$\limsup_{k\to\infty}\|x_{n_k} - T(t)p\| \le \limsup_{k\to\infty}\|x_{n_k} - p\|. \tag{2.31}$$

In virtue of Opial's condition of H, we have $p = T(t)p$ for all $t > 0$, that is, $p \in F$. Next, we observe that

$$\|x_0 - P_F(x_0)\| \le \|x_0 - p\| \le \liminf_{k\to\infty}\|x_0 - x_{n_k}\| \le \limsup_{k\to\infty}\|x_0 - x_{n_k}\| \le \|x_0 - P_F(x_0)\|. \tag{2.32}$$

This implies that

$$\lim_{k\to\infty}\|x_0 - x_{n_k}\| = \|x_0 - P_F(x_0)\| = \|x_0 - p\|. \tag{2.33}$$

Consequently,

$$x_{n_k} \longrightarrow P_F(x_0) = p. \tag{2.34}$$

Hence the whole sequence must converge to $P_F(x_0) = p$, as required. □

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References

[1] T. Suzuki, "On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces," *Proceedings of the American Mathematical Society*, vol. 131, no. 7, pp. 2133–2136, 2003.

[2] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.

[3] H. He and R. Chen, "Strong convergence theorems of the CQ method for nonexpansive semigroups," *Fixed Point Theory and Applications*, vol. 2007, Article ID 59735, 8 pages, 2007.

[4] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[5] W. Takahashi, Y. Takeuchi, and R. Kubota, "Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 276–286, 2007.

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Research Article

Strong Convergence to Common Fixed Points of Countable Relatively Quasi-Nonexpansive Mappings

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We prove that a sequence generated by the monotone CQ-method converges strongly to a common fixed point of a countable family of relatively quasi-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space. Our result is applicable to a wide class of mappings.

1. Introduction

Let E be a real Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to E$ be a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C. \tag{1.1}$$

We denote by $\mathrm{F}(T)$ the set of fixed points of T, that is, $\mathrm{F}(T) = \{x \in C : x = Tx\}$. A mapping T is said to be quasi-nonexpansive if $\mathrm{F}(T) \neq \varnothing$ and

$$\|Tx - y\| \le \|x - y\| \quad \forall x \in C,\ y \in \mathrm{F}(T). \tag{1.2}$$

It is easy to see that if T is nonexpansive with $\mathrm{F}(T) \neq \varnothing$, then it is quasi-nonexpansive. There are many methods for approximating fixed points of a quasi-nonexpansive mapping. In 1953, Mann [1] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.3}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [2]. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence (see [3, 4]). Attempts to modify the Mann iteration method (1.3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [5] proposed the following modification of Mann iteration method (1.3) for a nonexpansive mapping T from C into itself in a Hilbert space:

$$\begin{aligned} x_0 &\in C \text{ is arbitrary,} \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n &= \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \}, \\ Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.4}$$

where P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H and prove that the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$. A projection onto intersection of two halfspaces is computed by solving a linear system of two equations with two unknowns (see [6, Section 3]).

Recently, Su and Qin [7] modified iteration (1.4), so-called the monotone CQ method for nonexpansive mapping, as follows:

$$\begin{aligned} x_0 &\in C \text{ is arbitrary,} \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_0 &= \{ z \in C : \| y_0 - z \| \leq \| x_0 - z \| \}, \\ Q_0 &= C, \\ C_n &= \{ z \in C_{n-1} \cap Q_{n-1} : \| y_n - z \| \leq \| x_n - z \| \}, \\ Q_n &= \{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.5}$$

and prove that the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

We now recall some definitions concerning relatively quasi-nonexpansive mappings and what have been proved until now. Let E be a real smooth Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the pairing between E and E^*. The normalized duality mapping J from E to E^* is defined by

$$Jx = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \quad \text{where } x \in E. \tag{1.6}$$

The reader is directed to [8] (and its review [9]), where the properties on the duality mapping and several related topics are presented. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x, y \in E. \tag{1.7}$$

Let T be a mapping from C into E. A point p in C is said to be an *asymptotic fixed point* of T [10] if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{\mathrm{F}}(T)$. We say that the mapping T is *relatively nonexpansive* if the following conditions are satisfied:

(R1) $\mathrm{F}(T) \neq \varnothing$;

(R2) $\phi(p, Tx) \leq \phi(p, x)$ for each $x \in C,\ p \in \mathrm{F}(T)$;

(R3) $\mathrm{F}(T) = \widehat{\mathrm{F}}(T)$.

If T satisfies (R1) and (R2), then T is called *relatively quasi-nonexpansive*.

Several articles have appeared providing method for approximating fixed points of relatively quasi-nonexpansive mappings [11–16]. Matsushita and Takahashi [12] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \prod_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \tag{1.8}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0,1]$, T is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T. Moreover, Matsushita and Takahashi [13] proposed the following modification of iteration (1.8):

$$\begin{aligned}
&x_0 \in C \text{ is arbitrary},\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n),\\
&C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
&x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{1.9}$$

and prove that the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathrm{F}(T)}x_0$.

Recently, Kohsaka and Takahashi [11] extended iteration (1.8) to obtain a weak convergence theorem for common fixed points of a finite family of relatively nonexpansive mapping $\{T_i\}_{i=1}^m$ by the following iteration:

$$x_{n+1} = \prod_C J^{-1}\left(\sum_{i=1}^m w_{n,i}(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})JT_i x_n)\right), \quad n = 1, 2, \ldots, \tag{1.10}$$

where $\alpha_{n,i} \subset [0,1]$ and $w_{n,i} \subset [0,1]$ with $\sum_{i=1}^m w_{n,i} = 1$ for all $n \in \mathbb{N}$.

Employing the ideas of Su and Qin [7], and of Aoyama et al. [17], we modify iterations (1.5), (1.8)–(1.10) to obtain strong convergence theorems for common fixed points of countable relatively quasi-nonexpansive mappings in a Banach space. Consequently, we obtain strong convergence theorems for quasi-nonexpansive mappings in a Hilbert space without using demiclosedness principle. Moreover, we introduce a new certain condition for an infinite family of mappings which is inspired by Aoyama et al. [17], and we also show how to generate a corresponding sequence of mappings satisfying our condition.

2. Preliminaries

Throughout the paper, let E be a real Banach space. We say that E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \text{ imply } \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x-y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{2.2}$$

It is known that if E is uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.3) is attained uniformly for $x \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.3) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

We also know the following properties (see, e.g., [18] for details).

(a) E (E^*, resp.) is uniformly convex if and only if E^* (E, resp.) is uniformly smooth.

(b) $J(x) \neq \varnothing$ for each $x \in E$.

(c) If E is reflexive, then J is a mapping of E onto E^*.

(d) If E is strictly convex, then $J(x) \cap J(y) = \varnothing$ for all $x \neq y$.

(e) If E is smooth, then J is single valued.

(f) If E has a Fréchet differentiable norm, then J is norm to norm continuous.

(g) If E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of E.

(h) If E is a Hilbert space, then J is the identity operator.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x, y \in E. \tag{2.4}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2 \quad \forall x, y \in E. \tag{2.5}$$

Moreover, we know the following results.

Lemma 2.1 (see [13, Remark 2.1]). *Let E be a strictly convex and smooth Banach space, then $\phi(x,y)=0$ if and only if $x=y$.*

Lemma 2.2 (see [11, Lemma 2.5]). *Let E be a uniformly convex and smooth Banach space and let $r>0$. Then there exists a continuous, strictly increasing, and convex function $g:[0,2r]\to[0,\infty)$ such that $g(0)=0$ and*

$$g(\|x-y\|)\leq\phi(x,y) \tag{2.6}$$

for all $x,y\in B_r=\{z\in E:\|z\|\leq r\}$.

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex, and smooth. It is known that [19] for any $x\in E$, there exists a unique point $x^*\in C$ such that

$$\phi(x^*,x)=\min_{y\in C}\phi(y,x). \tag{2.7}$$

Following Alber [20], we denote such an x^* by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that in a Hilbert space, the mapping Π_C coincides with the metric projection P_C. Concerning the generalized projection, the following are well known.

Lemma 2.3 (see [19, Proposition 4]). *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x\in E$. Then*

$$x^*=\prod_C x\Longleftrightarrow\langle x^*-y,Jx-Jx^*\rangle\geq 0\quad\text{for each } y\in C. \tag{2.8}$$

Lemma 2.4 (see [19, Proposition 5]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E, and let $x\in E$. Then*

$$\phi\left(y,\prod_C x\right)+\phi\left(\prod_C x,x\right)\leq\phi(y,x)\quad\text{for each } y\in C. \tag{2.9}$$

Dealing with the generalized projection from E onto the fixed point set of a relatively quasi-nonexpansive mapping, we get the following result.

Lemma 2.5. *Let E be a strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a relatively quasi-nonexpansive mapping from C into E. Then $\mathrm{F}(T)$ is closed and convex.*

Proof. The proof of [13, Proposition 2.4] does not invoke condition (R3) at all. So the conclusion holds for relatively quasi-nonexpansive mappings as well. □

Let C be a subset of a Banach space E and let $\{T_n\}$ be a family of mappings from C into E. For a subset B of C, we say that

(i) $(\{T_n\},B)$ satisfies condition AKTT if

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|:z\in B\}<\infty; \tag{2.10}$$

(ii) $(\{T_n\}, B)$ satisfies condition *AKTT if

$$\sum_{n=1}^{\infty} \sup\left\{\left\|JT_{n+1}z - JT_n z\right\| : z \in B\right\} < \infty. \tag{2.11}$$

Aoyama et al. [17, Lemma 3.2] prove the following result which is very useful in our main result.

Lemma 2.6. *Let C be a nonempty subset of a Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfying condition AKTT, then there exists a mapping $\widetilde{T} : B \to E$ such that*

$$\widetilde{T}x = \lim_{n\to\infty} T_n x \quad \forall x \in B \tag{2.12}$$

and $\lim_{n\to\infty} \sup\{\|\widetilde{T}z - T_n z\| : z \in B\} = 0$.

Inspired by the preceding lemma, we have the following result.

Lemma 2.7. *Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of E, and let $\{T_n\}$ be a sequence of mappings from C into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfying condition *AKTT, then there exists a mapping $\widehat{T} : B \to E$ such that*

$$\widehat{T}x = \lim_{n\to\infty} T_n x \quad \forall x \in B \tag{2.13}$$

and $\lim_{n\to\infty} \sup\{\|J\widehat{T}z - JT_n z\| : z \in B\} = 0$.

Proof. For $x \in B$, we show that $\{JT_n x\}$ is a Cauchy sequence in E^*. Let $\varepsilon > 0$. By the condition *AKTT of $(\{T_n\}, B)$, there exists $l_0 \in \mathbb{N}$ such that

$$\sum_{n=l_0}^{\infty} \sup\left\{\left\|JT_{n+1}z - JT_n z\right\| : z \in B\right\} < \varepsilon. \tag{2.14}$$

In particular, if $k > l \geq l_0$, then

$$\begin{aligned} \left\|JT_k x - JT_l x\right\| &\leq \sum_{n=l}^{k-1} \sup\left\{\left\|JT_{n+1}z - JT_n z\right\| : z \in B\right\} \\ &\leq \sum_{n=l_0}^{\infty} \sup\left\{\left\|JT_{n+1}z - JT_n z\right\| : z \in B\right\} < \varepsilon. \end{aligned} \tag{2.15}$$

Hence, $\{JT_n x\}$ is a Cauchy sequence in E^*. It follows then that $\lim_{n\to\infty} JT_n x$ exists for all $x \in B$. Moreover, it is noted that the convergence is uniform on B. Since E is reflexive and strictly convex, J is bijective and we can define a mapping $\widehat{T}$ from B into E such that

$$\widehat{T}x = J^{-1}\left(\lim_{n\to\infty} JT_n x\right) \quad \forall x \in B. \tag{2.16}$$

Since E has a Fréchet differentiable norm, J is norm-to-norm continuous and hence

$$\widehat{T}x = J^{-1}J\left(\lim_{n\to\infty} T_n x\right) = \lim_{n\to\infty} T_n x \quad \forall x \in B. \tag{2.17}$$

This completes the proof. □

Combining Lemmas 2.6 and 2.7, we obtain a crucial tool for our main result.

Lemma 2.8. *Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of E, and let $\{T_n\}$ be a sequence of mappings from C into E. Suppose that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Then there exists a mapping $T : C \to E$ such that*

$$Tx = \lim_{n\to\infty} T_n x \quad \forall x \in C. \tag{2.18}$$

Proof. To see that T is well defined, we suppose that $(\{T_n\}, \{x\})$ satisfies condition AKTT and condition *AKTT. Then, by Lemmas 2.6 and 2.7, there exist $\widetilde{T}$ and $\widehat{T}$ such that $\widetilde{T}x = \lim_{n\to\infty} T_n x = \widehat{T}x$. □

Lemma 2.9 (see [11, Lemma 3.2]). *Let E be a reflexive, strictly convex, and smooth Banach space, let $z \in E$, and let $\{t_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in E such that*

$$\phi\left(z, J^{-1}\left(\sum_{i=1}^m t_i J x_i\right)\right) = \sum_{i=1}^m t_i \phi(z, x_i), \tag{2.19}$$

then $x_1 = x_2 \cdots = x_m$.

Lemma 2.10. *Let E be a strictly convex Banach space and let $\{t_n\} \subset (0,1)$ with $\sum_{n=1}^\infty t_n = 1$. If $\{x_n\}$ is a sequence in E such that $\sum_{n=1}^\infty t_n x_n$ and $\sum_{n=1}^\infty t_n \|x_n\|^2$ converge, and*

$$\left\|\sum_{n=1}^\infty t_n x_n\right\|^2 = \sum_{n=1}^\infty t_n \|x_n\|^2, \tag{2.20}$$

then $\{x_n\}$ is a constant sequence.

Proof. Suppose that $x_i \neq x_j$ for some $i, j \in \mathbb{N}$. Then, by the strict convexity of E,

$$\left\|\frac{t_i}{t_i+t_j} x_i + \frac{t_j}{t_i+t_j} x_j\right\|^2 < \frac{t_i}{t_i+t_j}\|x_i\|^2 + \frac{t_j}{t_i+t_j}\|x_j\|^2. \tag{2.21}$$

It follows that

$$\begin{aligned} \left\|\sum_{n=1}^\infty t_n x_n\right\|^2 &= \left\|(t_i+t_j)\left(\frac{t_i}{t_i+t_j} x_i + \frac{t_j}{t_i+t_j} x_j\right) + \sum_{n\neq i,j} t_n x_n\right\|^2 \\ &\leq (t_i+t_j)\left\|\frac{t_i}{t_i+t_j} x_i + \frac{t_j}{t_i+t_j} x_j\right\|^2 + \sum_{n\neq i,j} t_n\|x_n\|^2 \\ &< (t_i+t_j)\left(\frac{t_i}{t_i+t_j}\|x_i\|^2 + \frac{t_j}{t_i+t_j}\|x_j\|^2\right) + \sum_{n\neq i,j} t_n\|x_n\|^2 \\ &= \sum_{n=1}^\infty t_n \|x_n\|^2. \end{aligned} \tag{2.22}$$

This is a contradiction. □

3. Main results

In this section, we establish strong convergence theorem for finding common fixed points of a countable family of relatively quasi-nonexpansive mappings in a Banach space.

This theorem generalizes a recent theorem by Su et al. [21, Theorem 3.1]. It is noted that relative quasi-nonexpansiveness considered in the paper and hemirelative nonexpansiveness of [21] are the same. We do prefer the former name because in a Hilbert space setting, relatively quasi-nonexpansive mappings are just quasi-nonexpansive.

Recall that an operator T in a Banach space is *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of relatively quasi-nonexpansive mappings from C into E such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty and let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{aligned} &x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\ &y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n), \\ &C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \\ &Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ &x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.1}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into E defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.*

Proof. We first note that each C_n and Q_n are closed and convex. This follows since $\phi(z, y_n) \le \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Jy_n \rangle \le \|x_n\|^2 - \|y_n\|^2. \tag{3.2}$$

It is clear that $\bigcap_{n=0}^{\infty} F(T_n) \subset C = C_{-1} \cap Q_{-1}$. Next, we show that

$$\bigcap_{n=0}^{\infty} F(T_n) \subset C_n \cap Q_n \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Suppose that $\bigcap_{n=0}^{\infty} F(T_n) \subset C_{k-1} \cap Q_{k-1}$ for some $k \in \mathbb{N} \cup \{0\}$. Let $p \in \bigcap_{n=0}^{\infty} F(T_n)$. Then

$$\begin{aligned} \phi(p, y_k) &= \phi(p, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J T_k x_k)) \\ &= \|p\|^2 - 2\langle p, \alpha_k J x_k + (1-\alpha_k) J T_k x_k \rangle + \|\alpha_k J x_k + (1-\alpha_k) J T_k x_k\|^2 \\ &\le \|p\|^2 - 2\alpha_k \langle p, Jx_k \rangle - 2(1-\alpha_k)\langle p, JT_k x_k \rangle + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_k x_k\|^2 \\ &= \alpha_k (\|p\|^2 - 2\langle p, Jx_k \rangle + \|x_k\|^2) + (1-\alpha_k)\left(\|p\|^2 - 2\langle p, JT_k x_k \rangle + \|T_k x_k\|^2\right) \\ &= \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, T_k x_k) \\ &\le \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, x_k) \\ &= \phi(p, x_k). \end{aligned} \tag{3.4}$$

This implies that $\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \subset C_k$. From $x_k = \Pi_{C_{k-1} \cap Q_{k-1}} x_0$ and by Lemma 2.3, we have

$$\langle x_k - z, Jx_0 - Jx_k \rangle \geq 0 \quad \text{for each } z \in C_{k-1} \cap Q_{k-1}. \tag{3.5}$$

In particular,

$$\langle x_k - p, Jx_0 - Jx_k \rangle \geq 0 \quad \text{for every } p \in \bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \tag{3.6}$$

and hence $\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \subset Q_k$. It follows that

$$\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \subset C_k \cap Q_k. \tag{3.7}$$

By induction, (3.3) holds. This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n and Lemma 2.3 that $x_n = \Pi_{Q_n} x_0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.8}$$

Therefore, $\phi(x_n, x_0)$ is nondecreasing. Using $x_n = \Pi_{Q_n} x_0$ and Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0) \tag{3.9}$$

for all $p \in \bigcap_{n=0}^{\infty} \mathrm{F}(T_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $\phi(x_n, x_0)$ is bounded. So

$$\lim_{n \to \infty} \phi(x_n, x_0) \text{ exists.} \tag{3.10}$$

In particular, by (2.5), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies that $\{x_n\}$ is bounded. Noticing again that $x_n = \Pi_{Q_n} x_0$, and for any positive integer k, we have $x_{n+k} \in Q_{n+k-1} \subset Q_n$. By Lemma 2.4,

$$\phi(x_{n+k}, x_n) = \phi\left(x_{n+k}, \prod_{Q_n} x_0 \right) \leq \phi(x_{n+k}, x_0) - \phi\left(\prod_{Q_n} x_0, x_0 \right) = \phi(x_{n+k}, x_0) - \phi(x_n, x_0). \tag{3.11}$$

Using Lemma 2.2, we have, for m, n with $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0), \tag{3.12}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$. Then the properties of the function g yield that $\{x_n\}$ is a Cauchy sequence in C, so there exists $w \in C$ such that $x_n \to w$. In view of $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$ and the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.13}$$

It follows that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{3.14}$$

By using Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.15}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_{n+1}-Jy_n\|=\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=0. \tag{3.16}$$

On the other hand, we have, for each $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}\|Jx_{n+1}-Jy_n\|&=\|Jx_{n+1}-(\alpha_nJx_n+(1-\alpha_n)JT_nx_n)\|\\&=\|(1-\alpha_n)(Jx_{n+1}-JT_nx_n)-\alpha_n(Jx_n-Jx_{n+1})\|\\&\geq(1-\alpha_n)\|Jx_{n+1}-JT_nx_n\|-\alpha_n\|Jx_n-Jx_{n+1}\|,\end{aligned} \tag{3.17}$$

and hence

$$\|Jx_{n+1}-JT_nx_n\|\leq\frac{1}{1-\alpha_n}\|Jx_{n+1}-Jy_n\|+\frac{\alpha_n}{1-\alpha_n}\|Jx_n-Jx_{n+1}\|. \tag{3.18}$$

From (3.16) and $\limsup_{n\to\infty}\alpha_n<1$, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1}-JT_nx_n\|=0. \tag{3.19}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_{n+1}-T_nx_n\|=\lim_{n\to\infty}\|J^{-1}(Jx_{n+1})-J^{-1}(JT_nx_n)\|=0. \tag{3.20}$$

It follows from (3.15) that

$$\|x_n-T_nx_n\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-T_nx_n\|\longrightarrow 0 \tag{3.21}$$

and so

$$\lim_{n\to\infty}\|Jx_n-JT_nx_n\|=0. \tag{3.22}$$

Case 1. $(\{T_n\},\{x_n\})$ satisfies condition AKTT. We apply Lemma 2.6 to get

$$\begin{aligned}\|x_n-Tx_n\|&\leq\|x_n-T_nx_n\|+\|T_nx_n-Tx_n\|\\&\leq\|x_n-T_nx_n\|+\sup\{\|T_nz-Tz\|:z\in\{x_n\}\}\longrightarrow 0.\end{aligned} \tag{3.23}$$

Case 2. $(\{T_n\}, \{x_n\})$ satisfies condition *AKTT. It follows from Lemma 2.7 that

$$\begin{aligned} \|Jx_n - JTx_n\| &\leq \|Jx_n - JT_nx_n\| + \|JT_nx_n - JTx_n\| \\ &\leq \|Jx_n - JT_nx_n\| + \sup\{\|JT_nz - JTz\| : z \in \{x_n\}\} \longrightarrow 0. \end{aligned} \tag{3.24}$$

Hence,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_n) - J^{-1}(JTx_n)\| = 0. \tag{3.25}$$

From both cases, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.26}$$

Since T is closed and $x_n \to w$, we have $w \in \mathrm{F}(T)$. Furthermore, by (3.9),

$$\phi(w, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(p, x_0) \quad \forall p \in \mathrm{F}(T). \tag{3.27}$$

Hence, $w = \Pi_{\mathrm{F}(T)}x_0$. □

Corollary 3.2 (see [21, Theorem 3.1]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let T be a closed relatively quasi-nonexpansive mapping from C into E such that $\mathrm{F}(T)$ is nonempty and let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{aligned} &x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\ &y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ &C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ &Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ &x_{n+1} = \prod_{C_n\cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.28}$$

where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty}\alpha_n < 1$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{F}(T)}x_0$.

Remark 3.3. If, in Theorem 3.1, T_n is continuous for each $n \in \mathbb{N}$, then the mapping T is continuous and closed.

In our main theorem, we assume that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. As in [17], we can generate a sequence $\{T_n\}$ of relatively quasi-nonexpansive mappings satisfying such an assumption by using convex combination of a given sequence $\{S_k\}$ of relatively quasi-nonexpansive mappings with a nonempty common fixed point set.

Let $\{\beta_n^k\}$ be a family of positive real numbers with indices $n,k \in \mathbb{N} \cup \{0\}$ with $k \leq n$ such that

(i) $\sum_{k=0}^{n} \beta_n^k = 1$ for every $n \in \mathbb{N} \cup \{0\}$;

(ii) $\lim_{n\to\infty} \beta_n^k = \beta^k > 0$ for every $k \in \mathbb{N} \cup \{0\}$; and

(iii) $\sum_{n=0}^{\infty} \sum_{k=0}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. For a sequence $\{S_k\}_{k=1}^{\infty}$ of continuous relatively quasi-nonexpansive mappings with a common fixed point and S_0 is the identity mapping, we define a sequence $\{T_n\}$ of mappings from C into E by

$$T_n x = J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k x\right) \tag{3.29}$$

for $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. We note that

$$\bigcap_{k=0}^{\infty} F(S_k) \subset \bigcap_{k=0}^{n} F(S_k) \subset F(T_n) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.30}$$

For $n \in \mathbb{N} \cup \{0\}$, let $p \in \bigcap_{k=0}^{n} F(S_k)$. Then

$$\begin{aligned}
\phi(p, T_n x) &= \phi\left(p, J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k x\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \sum_{k=0}^{n} \beta_n^k J S_k x \right\rangle + \left\|\sum_{k=0}^{n} \beta_n^k J S_k x\right\|^2 \\
&\leq \|p\|^2 - 2\sum_{k=0}^{n} \beta_n^k \langle p, J S_k x\rangle + \sum_{k=0}^{n} \beta_n^k \|S_k x\|^2 \\
&= \sum_{k=0}^{n} \beta_n^k \phi(p, S_k x) \\
&\leq \phi(p, x)
\end{aligned} \tag{3.31}$$

for all $x \in C$. Then, for all $z \in F(T_n)$ and fix $q \in \bigcap_{k=0}^{\infty} F(S_k)$,

$$\phi(q, z) = \phi(q, T_n z) = \phi\left(q, J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k z\right)\right) \leq \sum_{k=0}^{n} \beta_n^k \phi(q, S_k z) \leq \phi(q, z), \tag{3.32}$$

that is,

$$\phi\left(q, J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k z\right)\right) = \sum_{k=0}^{n} \beta_n^k \phi(q, S_k z) = \phi(q, z). \tag{3.33}$$

By Lemma 2.9, we have $z = S_0 z = S_1 z = \cdots = S_n z$. So

$$\mathrm{F}(T_n) \subset \bigcap_{k=0}^{n} \mathrm{F}(S_k) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.34}$$

This implies that

$$\mathrm{F}(T_n) = \bigcap_{k=0}^{n} \mathrm{F}(S_k) \quad \forall n \in \mathbb{N} \cup \{0\}, \tag{3.35}$$

and so

$$\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) = \bigcap_{k=0}^{\infty} \mathrm{F}(S_k) \neq \varnothing. \tag{3.36}$$

Then, by (3.31), we have that $\{T_n\}$ is a sequence of relatively quasi-nonexpansive mappings. Let B be a bounded subset of C and let $p \in \bigcap_{k=0}^{\infty} \mathrm{F}(S_k)$. By (2.5), we have

$$\left(\|S_k x\| - \|p\|\right)^2 \leq \phi(p, S_k x) \leq \phi(p, x) \leq \left(\|x\| + \|p\|\right)^2, \tag{3.37}$$

and hence

$$\|S_k x\| \leq 2\|p\| + \sup\{\|z\| : z \in B\} \tag{3.38}$$

for all $x \in B$ and $k \in \mathbb{N} \cup \{0\}$. Let $M = \sup\{\|S_k x\| : x \in B,\ k \in \mathbb{N} \cup \{0\}\}$. For $x \in B$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\|JT_{n+1}x - JT_n x\| &= \left\|\sum_{k=0}^{n+1} \beta_{n+1}^k J S_k x - \sum_{k=0}^{n} \beta_n^k J S_k x\right\| \\
&\leq \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| \|J S_k x\| + \beta_{n+1}^{n+1} \|J S_k x\| \\
&= \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| \|S_k x\| + \left(1 - \sum_{k=0}^{n} \beta_{n+1}^k\right) \|S_k x\| \\
&\leq \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| M + \left(\sum_{k=0}^{n} \beta_n^k - \sum_{k=0}^{n} \beta_{n+1}^k\right) M \\
&\leq 2M \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right|.
\end{aligned} \tag{3.39}$$

Therefore,

$$\sup\{\|JT_{n+1}x - JT_nx\| : x \in B\} \leq 2M\sum_{k=0}^{n} |\beta_{n+1}^k - \beta_n^k|. \tag{3.40}$$

It follows from (iii) that

$$\sum_{n=0}^{\infty} \sup\{\|JT_{n+1}x - JT_nx\| : x \in B\} \leq 2M\sum_{n=0}^{\infty}\sum_{k=0}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty. \tag{3.41}$$

By Lemma 2.7, we can define a mapping T by

$$Tx = \lim_{n\to\infty} T_nx, \quad \forall x \in C. \tag{3.42}$$

Using the same argument presented in the proof of [17, pages 2357-2358], we have

$$\lim_{n\to\infty}\sum_{k=0}^{n} |\beta_n^k - \beta^k| = 0, \qquad \sum_{k=0}^{\infty}\beta^k = 1. \tag{3.43}$$

For each $x \in C$, the series $\sum_{k=0}^{\infty}\beta^k JS_kx$ converges absolutely and

$$\begin{aligned}
\left\|JTx - \sum_{k=0}^{\infty}\beta^k JS_kx\right\| &= \lim_{n\to\infty}\left\|JT_nx - \sum_{k=0}^{\infty}\beta^k JS_kx\right\| \\
&= \lim_{n\to\infty}\left\|\sum_{k=0}^{n}\beta_n^k JS_kx - \sum_{k=0}^{\infty}\beta^k JS_kx\right\| \\
&\leq \lim_{n\to\infty}\left(\sum_{k=0}^{n}|\beta_n^k - \beta^k|\|JS_kx\| + \sum_{k=n+1}^{\infty}\beta^k\|JS_kx\|\right) \\
&\leq \lim_{n\to\infty}\sum_{k=0}^{n}|\beta_n^k - \beta^k|\|S_kx\| + \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\beta^k\|S_kx\| \\
&\leq \lim_{n\to\infty}\sum_{k=0}^{n}|\beta_n^k - \beta^k|M + \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\beta^k M = 0.
\end{aligned} \tag{3.44}$$

This implies that

$$Tx = J^{-1}\left(\sum_{k=0}^{\infty}\beta^k JS_kx\right) \quad \forall x \in C. \tag{3.45}$$

It is obvious that

$$\bigcap_{k=0}^{\infty} \mathrm{F}(S_k) \subset \mathrm{F}(T). \tag{3.46}$$

Let $z \in \mathrm{F}(T)$ and fix $p \in \bigcap_{k=0}^{\infty} \mathrm{F}(S_k)$. Then

$$\begin{aligned}
\phi(p,z) = \phi(p,Tz) &= \phi\left(p, J^{-1}\left(\sum_{k=0}^{\infty} \beta^k J S_k z\right)\right) \\
&= \lim_{n\to\infty} \phi\left(p, J^{-1}\left(\sum_{k=0}^{n} \beta^k J S_k z\right)\right) \\
&= \lim_{n\to\infty} \left(\|p\|^2 - 2\left\langle p, \sum_{k=0}^{n} \beta^k J S_k z\right\rangle + \left\|\sum_{k=0}^{n} \beta^k J S_k z\right\|^2\right) \\
&\le \lim_{n\to\infty} \left(\|p\|^2 - 2\left\langle p, \sum_{k=0}^{n} \beta^k J S_k z\right\rangle + \sum_{k=0}^{n} \beta^k \|J S_k z\|^2\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \beta^k \|p\|^2 - 2\sum_{k=0}^{n} \beta^k \langle p, J S_k z\rangle + \sum_{k=0}^{n} \beta^k \|S_k z\|^2\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \beta^k \phi(p, S_k z) + \sum_{k=n+1}^{\infty} \beta^k \|p\|^2\right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \beta^k \phi(p, S_k z) \\
&= \sum_{k=0}^{\infty} \beta^k \phi(p, S_k z) \\
&\le \sum_{k=0}^{\infty} \beta^k \phi(p, z) \\
&= \phi(p,z).
\end{aligned} \tag{3.47}$$

It follows that

$$\left\|\sum_{k=0}^{\infty} \beta^k J S_k z\right\|^2 = \sum_{k=0}^{\infty} \beta^k \|J S_k z\|^2. \tag{3.48}$$

By the strict convexity of E^* and Lemma 2.10,

$$J S_k z = J S_0 z = J z \quad \forall k \in \mathbb{N}. \tag{3.49}$$

Since J is one to one,

$$S_k z = S_0 z = z \quad \forall k \in \mathbb{N}. \tag{3.50}$$

So $z \in \bigcap_{k=0}^{\infty} \mathrm{F}(S_k)$. Therefore,

$$\mathrm{F}(T) \subset \bigcap_{k=0}^{\infty} \mathrm{F}(S_k). \tag{3.51}$$

This together with (3.36) and (3.46) gives

$$F(T) = \bigcap_{n=0}^{\infty} F(T_n) = \bigcap_{k=0}^{\infty} F(S_k). \tag{3.52}$$

Hence, we obtain that $\{T_n\}$ satisfies all the conditions of our main theorem. Now, we have the following result.

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\{\beta_n^k\}$ be a family of positive real numbers with indices n, $k \in \mathbb{N} \cup \{0\}$ with $k \le n$ such that*

(i) $\sum_{k=0}^{n} \beta_n^k = 1$ *for every* $n \in \mathbb{N} \cup \{0\}$;

(ii) $\lim_{n\to\infty} \beta_n^k = \beta^k > 0$ *for every* $k \in \mathbb{N} \cup \{0\}$;

(iii) $\sum_{n=0}^{\infty} \sum_{k=0}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

Let $\{S_k\}$ be a sequence of continuous relatively quasi-nonexpansive mappings with a common fixed point and let S_0 be the identity operator, one defines a sequence $\{T_n\}$ of relatively quasi-nonexpansive mappings from C into E by

$$T_n x = J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k x\right) \tag{3.53}$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then the sequence $\{x_n\}$ in C defined by (3.1) *converges strongly to* $\Pi_{\bigcap_{k=0}^{\infty} F(S_k)} x_0$.

4. Deduced theorems

In Hilbert spaces, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same. We obtain the following result.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings from C into E such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty and let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{aligned}
&x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\
&y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \le \|x_n - z\|\}, \\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies condition AKTT. Let T be the mapping from C into E defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

Proof. Since J is an identity operator, we have

$$\phi(x,y) = \|x-y\|^2, \tag{4.2}$$

for every $x,y \in H$. Therefore,

$$\|T_n x - p\| \le \|x-p\| \Longleftrightarrow \phi(p,T_n x) \le \phi(p,x) \tag{4.3}$$

for every $x \in C$ and $p \in \mathrm{F}(T_n)$. Hence, T_n is quasi-nonexpansive if and only if T_n is relatively quasi-nonexpansive. Then, by Theorem 3.1, we obtain the result. □

Corollary 4.2 (see [22, Theorem 2.1]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a closed quasi-nonexpansive mapping from C into E such that $\mathrm{F}(T)$ is nonempty and let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{aligned}
&x_0 \in C, \quad C_{-1} = Q_{-1} = C,\\
&y_n = \alpha_n x_n + (1-\alpha_n) T x_n,\\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \le \|x_n - z\|\},\\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0,1,2,\ldots,
\end{aligned} \tag{4.4}$$

where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(T)} x_0$.

We give an example of a countable family of quasi-nonexpansive mappings which are not nonexpansive but satisfy all the requirements of our main theorem.

Example 4.3. Let $E = \mathbb{R}$ with the usual norm. For $n \in \mathbb{N}$, we define a mapping T_n on $\mathbb{R}$ by

$$T_n x = \begin{cases} 0 & \text{if } x \le \dfrac{1}{n^2},\\[2mm] \dfrac{1}{n^2} & \text{if } x > \dfrac{1}{n^2}, \end{cases} \tag{4.5}$$

for all $x \in \mathbb{R}$. Then $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \mathrm{F}(T_n) = \{0\}$ and

$$|T_n x - 0| \le |x - 0| \quad \forall x \in \mathbb{R}. \tag{4.6}$$

So $\{T_n\}$ is a sequence of quasi-nonexpansive mappings. Let $z \in \mathbb{R}$, then

$$|T_{n+1} z - T_n z| = \begin{cases} 0 & \text{if } z \le \dfrac{1}{(n+1)^2},\\[2mm] \dfrac{1}{n^2} & \text{if } \dfrac{1}{(n+1)^2} < z \le \dfrac{1}{n^2},\\[2mm] \dfrac{1}{n^2} - \dfrac{1}{(n+1)^2} & \text{if } z > \dfrac{1}{n^2}, \end{cases} \tag{4.7}$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \sup\{|T_{n+1}z - T_n z| : z \in \mathbb{R}\} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \tag{4.8}$$

We now define a mapping T on $\mathbb{R}$ by

$$Tx = \lim_{n\to\infty} T_n x = 0 \quad \forall x \in \mathbb{R}. \tag{4.9}$$

Hence, the sequence $\{T_n\}$ satisfies all conditions in our main result. We also note that each T_n is neither nonexpansive nor relatively nonexpansive. Actually, T_n above fails to have the condition (R3). Let $\{x_m\}$ be a sequence define by $x_m = 1/n^2 + 1/m$. Then

$$x_m \longrightarrow \frac{1}{n^2}, \quad x_m - T_n x_m = \frac{1}{m} \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{4.10}$$

This implies that $1/n^2 \in \widehat{\mathrm{F}}(T_n)$ and $1/n^2 \notin \mathrm{F}(T_n)$.

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References

[1] W. R. Mann, "Mean value methods in iteration," Proceedings of the American Mathematical Society, vol. 4, no. 3, pp. 506–510, 1953.
[2] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," Journal of Mathematical Analysis and Applications, vol. 67, no. 2, pp. 274–276, 1979.
[3] H. H. Bauschke, E. Matoušková, and S. Reich, "Projection and proximal point methods: convergence results and counterexamples," Nonlinear Analysis: Theory, Methods & Applications, vol. 56, no. 5, pp. 715–738, 2004.
[4] A. Genel and J. Lindenstrauss, "An example concerning fixed points," Israel Journal of Mathematics, vol. 22, no. 1, pp. 81–86, 1975.
[5] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," Journal of Mathematical Analysis and Applications, vol. 279, no. 2, pp. 372–379, 2003.
[6] M. V. Solodov and B. F. Svaiter, "Forcing strong convergence of proximal point iterations in a Hilbert space," Mathematical Programming, Series A, vol. 87, no. 1, pp. 189–202, 2000.
[7] Y. Su and X. Qin, "Monotone CQ iteration processes for nonexpansive semigroups and maximal monotone operators," to appear in *Nonlinear Analysis: Theory, Methods & Applications*.
[8] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.
[9] S. Reich, "Review: geometry of Banach spaces, duality mappings and nonlinear problems," Bulletin of the American Mathematical Society, vol. 26, no. 2, pp. 367–370, 1992.
[10] S. Reich, "A weak convergence theorem for the alternating method with Bregman distances," in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 313–318, Dekker, New York, NY, USA, 1996.
[11] F. Kohsaka and W. Takahashi, "Block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces," Fixed Point Theory and Applications, vol. 2007, Article ID 21972, 18 pages, 2007.

[12] S.-Y. Matsushita and W. Takahashi, "Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces," Fixed Point Theory and Applications, vol. 2004, no. 1, pp. 37–47, 2004.
[13] S.-Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," Journal of Approximation Theory, vol. 134, no. 2, pp. 257–266, 2005.
[14] S.-Y. Matsushita and W. Takahashi, "An iterative algorithm for relatively nonexpansive mappings by hybrid method and applications," in Nonlinear Analysis and Convex Analysis, pp. 305–313, Yokohama Publishers, Yokohama, Japan, 2004.
[15] S. Plubtieng and K. Ungchittrakool, "Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space," Journal of Approximation Theory, vol. 149, no. 2, pp. 103–115, 2007.
[16] X. Qin and Y. Su, "Strong convergence theorems for relatively nonexpansive mappings in a Banach space," Nonlinear Analysis: Theory, Methods & Applications, vol. 67, no. 6, pp. 1958–1965, 2007.
[17] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," Nonlinear Analysis: Theory, Methods & Applications, vol. 67, no. 8, pp. 2350–2360, 2007.
[18] W. Takahashi, Nonlinear Functional Analysis, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, Japan, 2000.
[19] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," SIAM Journal on Optimization, vol. 13, no. 3, pp. 938–945, 2002.
[20] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Dekker, New York, NY, USA, 1996.
[21] Y. Su, D. Wang, and M. Shang, "Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings," to appear in *Fixed Point Theory and Applications*.
[22] Y. Su, D. Wang, and M. Shang, "Strong convergence of monotone hybrid algorithm for quasi-nonexpansive mappings," International Journal of Mathematical Analysis, vol. 1, pp. 1235–1241, 2007.

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Weak and strong convergence of a scheme with errors for three nonexpansive mappings

ABSTRACT. We establish weak and strong convergence theorems of modified Ishikawa iteration with errors with respect to three nonexpansive mappings. We improve and extend many results due to Khan and Fukhar-ud-din, Tamura and Takahashi and many authors. We also point out that an additional condition imposed in Rafiq's paper does not make sense.

KEY WORDS. nonexpansive mapping, Ishikawa iteration, uniformly convex space, Opial's condition, condition (A'')

1 Introduction

Nonexpansive mappings have been widely and extensively studied by many authors in many aspects. One is to approximate a common fixed point of nonexpansive mappings by means of an iteratively constructed sequence.

Let C be a nonempty convex subset of a normed space E and $R,\ S,\ T : C \to C$ be three mappings. Xu [13] introduced the following iterative scheme,

(a) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in [0,1] such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $c_n \equiv 0$, i.e.,

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + (1 - a_n) T x_n, \quad n \geq 1, \end{cases} \tag{2}$$

where $\{a_n\}$ is a sequence in [0,1].

(b) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1] satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is called the Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme if $c_n \equiv 0 \equiv c'_n$, i.e.,

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + (1 - a'_n) T x_n \\ x_{n+1} = a_n x_n + (1 - a_n) T x_n, \quad n \geq 1, \end{cases} \tag{4}$$

where $\{a_n\}, \{a'_n\}$ are sequences in [0,1].

A generalization of Mann and Ishikawa iterative schemes was given by Das and Debata [3] and Takahashi and Tamura [11]. This scheme dealt with two mappings:

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + (1 - a'_n) T x_n \\ x_{n+1} = a_n x_n + (1 - a_n) S y_n, \quad n \geq 1, \end{cases} \tag{5}$$

(c) The sequence $\{x_n\}$, defined by

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{6}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1] satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is studied by S. H. Khan and H. Fukhar-ud-din [4].

Inspired by [4] and [5], we generalize the scheme (6) to three nonexpansive mappings with errors as follows:

(d) The sequence $\{x_n\}$, defined by

$$\begin{cases} x_0 \in C \\ y_n = a'_n R x_n + b'_n T x_n + c'_n v_n \\ x_{n+1} = a_n R x_n + b_n S y_n + c_n u_n, \qquad n \geq 1, \end{cases} \tag{7}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1], $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

2 Preliminaries

Let E be a Banach space and let C be a nonempty closed convex subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to be satisfy Opial's condition [7] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta_E(\varepsilon)$ of convexity of E by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for every $\varepsilon > 0$.

A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ it follows that $x \in C$ and $Tx = y$.

Next we state the following useful lemmas.

Lemma 1 ([9]) *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2 ([12], Lemma 1) *Let $\{s_n\}, \{t_n\}$ be two nonnegative real sequences satisfying*

$$s_{n+1} \leq s_n + t_n \quad \textit{for all} \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 3 ([1]) *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

3 Main results

In this section, we shall prove the weak and strong convergence theorems of the iteration scheme to a common fixed point of the nonexpansive mappings R, S and T. Let $F(T)$ denote the set of all fixed points of T.

Lemma 4 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $R,\ S,\ T : C \to C$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (7) with $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(R) \cap F(S) \cap F(T) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(R) \cap F(S) \cap F(T)$.*

Proof: Assume that $F(R) \cap F(S) \cap F(T) \neq \emptyset$. Let $p \in F(R) \cap F(S) \cap F(T)$. Since $S,\ T,\ R$ are nonexpansive mappings, we have

$$\begin{aligned} \|y_n - p\| &= \|a'_n Rx_n + b'_n Tx_n + c'_n v_n - p\| \\ &\le a'_n \|Rx_n - p\| + b'_n \|Tx_n - p\| + c'_n \|v_n - p\| \\ &\le a'_n \|x_n - p\| + b'_n \|x_n - p\| + c'_n \|v_n - p\| \\ &= (a'_n + b'_n) \|x_n - p\| + c'_n \|v_n - p\| \\ &= (1 - c'_n) \|x_n - p\| + c'_n \|v_n - p\| \\ &\le \|x_n - p\| + c'_n \|v_n - p\| \end{aligned} \tag{8}$$

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n Rx_n + b_n Sy_n + c_n u_n - p\| \\ &\le a_n \|Rx_n - p\| + b_n \|Sy_n - p\| + c_n \|u_n - p\| \\ &\le a_n \|y_n - p\| + b_n \|x_n - p\| + c_n \|u_n - p\| \\ &\le a_n (\|x_n - p\| + c'_n \|v_n - p\|) + b_n \|x_n - p\| + c_n \|u_n - p\| \\ &= a_n \|x_n - p\| + a_n c'_n \|v_n - p\| + b_n \|x_n - p\| + c_n \|u_n - p\| \\ &= (a_n + b_n) \|x_n - p\| + a_n c'_n \|v_n - p\| + c_n \|u_n - p\| \\ &\le \|x_n - p\| + c'_n \|v_n - p\| + c_n \|u_n - p\| \end{aligned} \tag{9}$$

By Lemma 2, $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 5 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S, T, R : C \to C$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (7) with $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$ and $0 < \delta \le b_n, b'_n \le 1 - \delta < 1$. If $F(R) \cap F(S) \cap F(T) \neq \emptyset$ and*

$$\|x - Sy\| \le \|Rx - Sy\| \quad \text{for all } x, y \in C, \tag{10}$$

then

$$\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Rx_n - x_n\| = 0$$

for all $p \in F(R) \cap F(S) \cap F(T)$.

Proof: From Lemma 4, we get $\lim_{n\to\infty}\|x_n-p\|$ exists. Let $\lim_{n\to\infty}\|x_n-p\|=c$. Then if $c=0$, we are done. Assume that $c>0$. Next, we want to show that $\lim_{n\to\infty}\|Sy_n-Rx_n\|=0$. We note that $\{u_n-Rx_n-p\}$ is a bounded sequence, so $\lim_{n\to\infty}c_n\|u_n-Rx_n-p\|=0$. Consider

$$\begin{aligned} c &= \lim_{n\to\infty}\|x_{n+1}-p\| \\ &= \lim_{n\to\infty}\|(1-b_n)Rx_n+b_nSy_n+c_nu_n-c_nRx_n-p\| \\ &= \lim_{n\to\infty}\|(1-b_n)(Rx_n-p)+b_n(Sy_n-p)+c_n(u_n-Rx_n-p)\| \\ &= \lim_{n\to\infty}\|(1-b_n)(Rx_n-p)+b_n(Sy_n-p)\| \end{aligned} \tag{11}$$

and from (8) we have

$$\limsup_{n\to\infty}\|Sy_n-p\|\le\limsup_{n\to\infty}\|y_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|+c'_n\|v_n-p\|=c \tag{12}$$

also,

$$\limsup_{n\to\infty}\|Rx_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|=c.$$

Using Lemma 1 and (11), we have

$$\lim_{n\to\infty}\|Sy_n-Rx_n\|=0. \tag{13}$$

It follows then that

$$\begin{aligned} \|Rx_n-x_n\| &\le \|Rx_n-Sy_n\|+\|Sy_n-x_n\| \\ &\le 2\|Rx_n-Sy_n\|\to 0, \end{aligned} \tag{14}$$

and hence

$$\|Sy_n-x_n\|\le\|Sy_n-Rx_n\|+\|Rx_n-x_n\|\to 0. \tag{15}$$

We are going to apply Lemma 1 again. To show that $\lim_{n\to\infty}\|y_n-p\|=c$, we observe that $\|x_n-p\|\le\|x_n-Sy_n\|+\|Sy_n-p\|\le\|x_n-Sy_n\|+\|y_n-p\|$ which implies that

$$c\le\liminf_{n\to\infty}\|y_n-p\|.$$

This together with (12) gives

$$\lim_{n\to\infty}\|y_n-p\|=c. \tag{16}$$

Finally, from (16) and the boundedness of the sequence $\{v_n-Rx_n-p\}$, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\|y_n-p\| \\ &= \lim_{n\to\infty}\|(1-b'_n)Rx_n+b'_nTx_n+c'_nv_n-c'_nRx_n-p\| \\ &= \lim_{n\to\infty}\|(1-b'_n)(Rx_n-p)+b'_n(Tx_n-p)+c'_n(v_n-Rx_n-p)\| \\ &= \lim_{n\to\infty}\|(1-b'_n)(Rx_n-p)+b'_n(Tx_n-p)\|. \end{aligned}$$

Moreover,

$$\limsup_{n\to\infty} \|Tx_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c,$$

and

$$\limsup_{n\to\infty} \|Rx_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c.$$

Applying Lemma 1, we get

$$\lim_{n\to\infty} \|Rx_n - Tx_n\| = 0. \tag{17}$$

Using (14) and (17), we get that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{18}$$

Consequently, using (13), (14), (18) and

$$\begin{aligned} \|x_n - Sx_n\| &\le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\le \|x_n - Sy_n\| + \|y_n - x_n\| \\ &\le \|x_n - Sy_n\| + a'_n\|Rx_n - x_n\| + b'_n\|Tx_n - x_n\| + c'_n\|v_n - x_n\| \\ &\le \|x_n - Sy_n\| + a'_n\|Rx_n - x_n\| + b'_n\|Tx_n - x_n\| + c'_n\|v_n - p\|, \end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{19}$$

This completes the proof. □

We first establish the weak convergence theorem of our iteration.

Theorem 6 *Let E be a uniformly convex Banach space satisfies the Opial's condition and C, S, T, R and $\{x_n\}$ be taken as in Lemma 5. If $F(R) \cap F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of S, T and R.*

Proof: Let $p \in F(R) \cap F(S) \cap F(T)$, then as proved in Lemma 4, we get $\lim_{n\to\infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(R) \cap F(S) \cap F(T)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{m_j}\}$ of $\{x_n\}$, respectively. By Lemma 11, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 3, therefore we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$ and $Rz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F(R) \cap F(S) \cap F(T)$. Next, we prove the uniqueness.

For this we suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{j\to\infty} \|x_{m_j} - z_2\| \\
&< \lim_{j\to\infty} \|x_{m_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(R) \cap F(S) \cap F(T)$. □

Our next goal is to prove a strong convergence theorem. Recall that a mapping $T : C \to C$ where C is a subset of E, is said to satisfy condition (A) ([10]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Senter and Dotson [10] approximated fixed points of nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [6] and Tan and Xu [12] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition (A) which is weaker than the requirement that d is demicompact.

Three mappings $R, S, T : C \to C$ where C is a subset of E, are said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{3}(\|x - Rx\| + \|x - Tx\| + \|x - Sx\|) \geq f(d(x, F))$$

for all $x \in C$ where $d(x, F) = \inf\{\|x - x^*\| : x^* \in F = F(R) \cap F(S) \cap F(T)\}$.

Note that condition (A'') reduces to condition (A) when $R = S = T$. We shall use condition (A'') instead of the compactness of C to study the strong convergence of $\{x_n\}$ defined in (7). It is noted that if $R = I$, then condition (A'') reduces to condition (A') of Khan and Fukhar-ud-din [4].

Theorem 7 *Let E be a uniformly convex Banach space and C, $\{x_n\}$ be taken as in Lemma 5. Let R, S, $T : C \to C$ be three mappings satisfying condition (A''). If $F(R) \cap F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of R, S and T.*

Proof: By Lemma 4, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(R) \cap F(S) \cap F(T)$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \geq 0$. If $c = 0$, we are done. Suppose that $c > 0$. By

Lemma 5, $\lim_{n\to\infty}\|Sx_n - x_n\| = \lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|Rx_n - x_n\| = 0$. Let $M = \sup\{\|v_n - x_n\|, \|u_n - x_n\| : n \in \mathbb{N}\}$. Moreover, by (9),

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\le \|x_n - p\| + c_n'\|v_n - p\| + c_n\|u_n - p\| \\
&\le \|x_n - p\| + c_n'\|v_n - x_n\| + c_n'\|x_n - p\| + c_n\|u_n - x_n\| + c_n\|x_n - p\| \\
&\le (1 + c_n' + c_n)\|x_n - p\| + c_n'\|v_n - x_n\| + c_n\|u_n - x_n\| \\
&\le (1 + c_n' + c_n)\|x_n - p\| + (c_n' + c_n)M. \qquad (20)
\end{aligned}$$

This implies that $d(x_{n+1}, F) \le (1+c_n'+c_n)d(x_n, F)+(c_n'+c_n)M$ and hence $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 2. By condition (A''),

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in E.

Let $\epsilon > 0$. We choose a positive integer N_1 such that

$$d(x_{N_1}, F) < \frac{\varepsilon}{4}. \qquad (21)$$

We next choose $q \in F$ such that

$$\|x_{N_1} - q\| < \frac{\varepsilon}{4}. \qquad (22)$$

By $\lim_{n\to\infty}\|x_n - q\|$ exists, the sequence $\{\|x_n - p\|\}$ is bounded. Let $K = \sup_{n\in\mathbb{N}}\{\|x_n - q\|, M\}$. Then from (20), we have

$$\|x_{n+1} - q\| \le \|x_n - q\| + (c_n' + c_n)K. \qquad (23)$$

Since $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c_n' < \infty$, there exists N_2 such that

$$\sum_{i=N_2}^{\infty} Q_i < \frac{\varepsilon}{4}, \qquad (24)$$

where $Q_i = (c_i + c_i')K$. We take $N = \max\{N_1, N_2\}$. Let $n \ge N$ and $m \ge 1$. It follows from (22), (23) and (24) that

$$\begin{aligned}\|x_{n+m}-x_n\| &\le \|x_{n+m}-p\|+\|p-x_n\| \\ &\le \|x_n-p\|+\|p-x_n\|+\sum_{i=n}^{n+m-1}Q_i \\ &= 2\|x_n-p\|+\sum_{i=n}^{n+m-1}Q_i \\ &\le 2\|x_N-p\|+2\sum_{i=N}^{n-1}Q_i+\sum_{i=n}^{n+m-1}Q_i \\ &\le 2\|x_N-p\|+2\sum_{i=N}^{n+m-1}Q_i \\ &\le 2\|x_N-p\|+2\sum_{i=N}^{\infty}Q_i \\ &\le 2(\frac{\varepsilon}{4})+2(\frac{\varepsilon}{4})=\varepsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in E. Since C is closed, $x_n \to x \in C$. By the continuities of S, R, T and (14), (18), (19), we get $Sx = Rx = Tx = x$. So $x \in F(R) \cap F(S) \cap F(T)$. This completes the proof. □

If R is the identity mapping, then (10) is automatically satisfied and we have the following.

Corollary 8 ([4],Theorem 1, Theorem 2) *Let E be a uniformly convex Banach space and C, S, T and $\{x_n\}$ be taken as in Theorem 7. Suppose that $F(S) \cap F(T) \neq \emptyset$. Then*

1. *If E has the Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of S and T,*

2. *If the mappings S and T satisfy condition (A'), then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Remark 1 Theorem 6 and Theorem 7 extend and improve Theorem 1 and Theorem 2 of [4] in the following ways:

1. the iteration methods in [4] are included as a special case of ours. Indeed, the identity mapping is replaced by the more general nonexpansive mapping,

2. the boundedness of C is not assumed as was the case in [4].

Remark 2 The following example [5, see Example 3.1] shows that our results extend substantially results in [4].

Example 9 Let E be the real line with the usual norm and let $C = [-1,1]$. Define $R, S, T : C \to C$ by

$$Rx = \begin{cases} x, & x \in [0,1] \\ -x, & x \in [-1,0) \end{cases}$$

$$Sx = \begin{cases} -\sin x, & x \in [0,1] \\ \sin x, & x \in [-1,0) \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{2}x, & x \in [0,1] \\ -\frac{1}{2}x, & x \in [-1,0). \end{cases}$$

Obviously, $F(R) \cap F(S) \cap F(T) = \{0\}$. Moreover, it is not hard to see that nonexpansive mappings R, S and T satisfy condition (A'').

Remark 3 Recently, Rafiq [8] introduced the following condition: two mappings $S, T : C \to C$ are said to satisfy (AU-N) if

$$\|Sx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

It the clear that if $S = T$, then (AU-N) is the definition of nonexpansive mappings. Unfortunately, if S and T satisfy (AU-N), then

$$\|Sx - Tx\| \leq \|x - x\| = 0 \quad \text{for all } x \in C,$$

from which $S = T$. This means (AU-N) is meaningless. Consequently, all results in [8] are just dealing with only one mapping.

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References

[1] **Browder, F. E. :** *Nonlinear operators and nonlinear equations of evolution in Banach spaces.* Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol. 18, Part 2, Chicago, Ill., 1968). 1–308. Amer. Math. Soc., Providence, R. I., (1976)

[2] **Chidume, C. E.**, and **Moore, C. :** *Fixed point iteration for pseudocontractive maps.* Proc. Amer. Math. Soc. **127(4)**, 1163–1170, (1999)

[3] **Das, G.**, and **Debata, J. P. :** *Fixed points of quasinonexpansive mappings.* Indian J. Pure Appl. Math. **17(11)**, 1263–1269, (1986)

[4] **Khan, S. H.**, and **Fukhar-ud-din, H. :** *Weak and strong convergence of a scheme with errors for two nonexpansive mappings.* Nonlinear Anal. **61(8)**, 1295–1301, (2005)

[5] **Liu, Z., Agarwal, R. P., Feng, C.**, and **Kang, S. M. :** *Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings.* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **44** , 83–96, (2005)

[6] **Maiti, M.**, and **Ghosh, M. K. :** *Approximating fixed points by Ishikawa iterates.* Bull. Austral. Math. Soc. **40(1)**, 113–117, (1989)

[7] **Opial, Z. :** *Weak convergence of the sequence of successive approximations for nonexpansive mappings.* Bull. Amer. Math. Soc. **73**, 591–597, (1967)

[8] **Rafiq, A. :** *Convergence of an iterative scheme due to Agarwal et al.* Rostock. Math. Kolloq. **61**, 95–105, (2006)

[9] **Schu, J. :** *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings.* Bull. Austral. Math. Soc. **43(1)**, 153–159, (1991)

[10] **Senter, H. F.**, and **Dotson, W. G. :** *Approximating fixed points of nonexpansive mappings.* Proc. Amer. Math. Soc. **44**, 375–380, (1974)

[11] **Takahashi, W.**, and **Tamura, T. :** *Convergence theorems for a pair of nonexpansive mappings.* J. Convex Anal. **5(1)**, 45–56, (1998)

[12] **Tan, K. K.**, and **Xu, H. K. :** *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process.* J. Math. Anal. Appl. **178(2)**, 301–308, (1993)

[13] **Xu, Y. :** *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. **224(1)**, 91–101, (1998)

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Weak and strong convergence theorems for countable Lipschitzian mappings and its applications

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Abstract

We use Mann's iteration and the hybrid method in mathematical programming to obtain weak and strong convergence to common fixed points of a countable family of Lipschitzian mappings. Finally, we apply our results to solve the equilibrium problems and variational inequalities for continuous monotone mappings.

MSC: 47H09; 47H10

Keywords: Fixed point theorem; Lipschitzian mapping; Nonexpansive mapping; Equilibrium problem; Variational inequality

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. A mapping $T : C \to C$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \le L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L-Lipschitzian. Clearly, if T is L_1-Lipschitzian and $L_1 < L_2$, then T is L_2-Lipschitzian. Throughout the paper, we assume that every Lipschitzian mapping is L-Lipschitzian with $L \ge 1$. If $L = 1$, then T is known as a nonexpansive mapping. We denote by $F(T)$ the set of fixed points of T. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [9]). We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x. There are many methods for approximating the fixed points of a nonexpansive mapping. In 1953, Mann [10] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved

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by Reich [19]. In an infinite-dimensional Hilbert space, Mann iteration could conclude only weak convergence [5]. Attempts to modify the Mann iteration method (1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [13] proposed the following modification of Mann iteration method (1):

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases} \tag{2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

In this paper, thanks to the condition introduced by Aoyama et al. [1], we establish weak and strong convergence theorem for finding common fixed points of a countable family of certain Lipschitzian mappings in a real Hilbert space. The additional condition imposed on Lipschitzian mappings is inspired by Kim and Xu's work [8]. Finally, we apply our results to solve the equilibrium problems and variational inequalities for continuous monotone mappings.

2. Preliminaries

Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{3}$$

and

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{4}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies

(1) the Opial's condition [17], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

(2) the Kadec–Klee property [6,21], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together implies $\|x_n - x\| \to 0$.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, z - y\rangle \ge 0 \quad \text{for all } y \in C.$$

We also need the following lemma (see [24], Lemma 1).

Lemma 1. *Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1 + b_n) a_n + c_n \quad \textit{for all } n \ge 1,$$

$\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 ([18], Lemma 2.2). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$. Then $\liminf_n b_n = 0$.*

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Lemma 3 (*[1], Lemma 3.2*). *Let C be a nonempty closed subset of a Banach space and let $\{T_n\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\to\infty} T_n y \quad \text{for all } y \in C.$$

Then $\lim_{n\to\infty} \sup\{\|T_n z - Tz\| : z \in C\} = 0$.

The following lemma is a generalization of [23, Lemma 3.2].

Lemma 4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and let $\{\delta_n\}$ be a sequence in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and*

$$\|x_{n+1} - y\| \le (1 + \delta_n)\|x_n - y\| \quad \text{for all } \ y \in C \text{ and } n \in \mathbb{N}.$$

Then the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

Proof. Let $z_n = P_C(x_n)$. We have

$$\begin{aligned}
\|z_{n+1} - z_n\|^2 &= \|z_{n+1} - x_{n+1}\|^2 + \|x_{n+1} - z_n\|^2 + 2\langle z_{n+1} - x_{n+1}, x_{n+1} - z_n\rangle \\
&= \|z_{n+1} - x_{n+1}\|^2 + \|x_{n+1} - z_n\|^2 \\
&\quad + 2\left(\langle z_{n+1} - x_{n+1}, x_{n+1} - z_{n+1}\rangle + \langle z_{n+1} - x_{n+1}, z_{n+1} - z_n\rangle\right) \\
&\le \|x_{n+1} - z_n\|^2 - \|x_{n+1} - z_{n+1}\|^2 \\
&\le (1 + \delta_n)^2\|x_n - z_n\|^2 - \|x_{n+1} - z_{n+1}\|^2. \qquad (5)
\end{aligned}$$

This means

$$\|x_{n+1} - z_{n+1}\| \le (1 + \delta_n)\|x_n - z_n\| \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 1, $\lim_{n\to\infty} \|x_n - z_n\|$ exists. To see that $\{z_n\}$ is a Cauchy sequence, we observe from (5) that

$$\begin{aligned}
\|z_{n+1} - z_n\|^2 &\le (1 + 2\delta_n + \delta_n^2)\|x_n - z_n\|^2 - \|x_{n+1} - z_{n+1}\|^2 \\
&= (\|x_n - z_n\|^2 - \|x_{n+1} - z_{n+1}\|^2) + (2\delta_n + \delta_n^2)\|x_n - z_n\|^2.
\end{aligned}$$

Since $\{\|x_n - z_n\|^2\}$ is bounded and $\sum_{n=1}^{\infty} \delta_n < \infty$, we have

$$\sum_{n=1}^{\infty} (2\delta_n + \delta_n^2)\|x_n - z_n\|^2 < \infty.$$

It then follows that

$$\sum_{n=1}^{\infty} \|z_{n+1} - z_n\|^2 < \infty.$$

In particular, $\{z_n\}$ is a Cauchy sequence. Hence, $\{z_n\}$ converges strongly to some $z \in C$. □

3. Weak convergence theorems

Theorem 5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of L_n-Lipschitzian mappings from C into itself with $\sum_{n=1}^{\infty}(L_n - 1) < \infty$ and let $\bigcap_{n=1}^{\infty} F(T_n)$ be nonempty. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Let $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C and T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then $\{x_n\}$ converges weakly to $w \in F(T)$. Moreover, $\lim_{n\to\infty} P_{F(T)} x_n = w$.

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Proof. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$, it follows from (4) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \\ &\le (1 + (1-\alpha_n)(L_n^2 - 1))\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \qquad (6)\end{aligned}$$

for all $n \in \mathbb{N}$. Hence

$$\|x_{n+1} - p\|^2 \le (1 + (1-\alpha_n)(L_n^2 - 1))\|x_n - p\|^2 \qquad (7)$$

for all $n \in \mathbb{N}$. By Lemma 1, we have that $\lim_{n\to\infty}\|x_n - p\|$ exists. Furthermore, from (6), we have

$$\alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\alpha_n)(L_n^2 - 1)M,$$

where $M = \sup\{\|x_n - p\|^2 : n \in \mathbb{N}\}$. Summing from 1 to m and tending to infinity for m, we have

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 < \infty.$$

Since $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and from Lemma 2, we have

$$\liminf_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

We next prove that the limit $\lim_{n\to\infty}\|x_n - T_n x_n\|$ actually exists. Since $\{x_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty} \sup\{\|T_n z - T_{n+1} z\| : z \in \{x_n\}\} < \infty. \qquad (8)$$

We compute

$$\begin{aligned}\|x_{n+1} - T_{n+1}x_{n+1}\| &\le \alpha_n\|x_n - T_{n+1}x_{n+1}\| + (1-\alpha_n)\|T_n x_n - T_{n+1}x_{n+1}\| \\ &\le \alpha_n\left(\|x_n - x_{n+1}\| + \|x_{n+1} - T_{n+1}x_{n+1}\|\right) \\ &\quad + (1-\alpha_n)\left(\|T_n x_n - T_n x_{n+1}\| + \|T_n x_{n+1} - T_{n+1}x_{n+1}\|\right) \\ &\le (1 + (1-\alpha_n)(L_n - 1))\|x_n - x_{n+1}\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\ &\quad + (1-\alpha_n)\|T_n x_{n+1} - T_{n+1}x_{n+1}\| \\ &= (1 + (1-\alpha_n)(L_n - 1))(1-\alpha_n)\|x_n - T_n x_n\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\ &\quad + (1-\alpha_n)\|T_n x_{n+1} - T_{n+1}x_{n+1}\|.\end{aligned}$$

It follows from $1 - \alpha_n > 0$ that

$$\begin{aligned}\|x_{n+1} - T_{n+1}x_{n+1}\| &\le (1 + (1-\alpha_n)(L_n - 1))\|x_n - T_n x_n\| + \|T_n x_{n+1} - T_{n+1}x_{n+1}\| \\ &\le (1 + (1-\alpha_n)(L_n - 1))\|x_n - T_n x_n\| + \sup\{\|T_n z - T_{n+1}z\| : z \in \{x_n\}\}.\end{aligned}$$

By Lemma 1 and (8), we have that $\lim_{n\to\infty}\|x_n - T_n x_n\|$ exists. Thus

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

We apply Lemma 3 to get

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\le \|x_n - T_n x_n\| + \sup\{\|T_n z - Tz\| : z \in \{x_n\}\} \to 0.\end{aligned}$$

From the definition of T and $\lim_{n\to\infty} L_n = 1$, we have that T is nonexpansive. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. So by the demiclosedness principle, $w \in F(T)$. To prove that $x_n \rightharpoonup w$, suppose that there exist $\{x_{m_j}\} \subset \{x_n\}$ and $w' \neq w$ such that $x_{m_j} \rightharpoonup w'$. So, we have $w' \in F(T)$. It follows that

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$$\begin{aligned}\lim_{n\to\infty}\|x_n - w\| &= \lim_{i\to\infty}\|x_{n_i} - w\| < \lim_{i\to\infty}\|x_{n_i} - w'\| \\ &= \lim_{j\to\infty}\|x_{m_j} - w'\| < \lim_{j\to\infty}\|x_{m_j} - w\| \\ &= \lim_{n\to\infty}\|x_n - w\|,\end{aligned}$$

arriving at a contradiction. Hence $x_n \rightharpoonup w \in F(T)$. Finally we prove that $\lim_{n\to\infty} z_n = w$, where $z_n = P_{F(T)}x_n$ for each $n \in \mathbb{N}$. By (7) and Lemma 4, there is $w_0 \in F(T)$ such that $z_n \to w_0$. From $z_n = P_{F(T)}x_n$ and $w \in F(T)$, we have

$$\langle x_n - z_n, z_n - w\rangle \geq 0, \qquad \text{for all } n \in \mathbb{N}.$$

It follows from $z_n \to w_0$ and $x_n \rightharpoonup w$ that

$$\langle w - w_0, w_0 - w\rangle \geq 0$$

and then $w_0 = w$. This completes the proof. □

Setting $L_n \equiv 1$ in Theorem 5, we have the following result.

Corollary 6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Let $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C and T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then $\{x_n\}$ converges weakly to $w \in F(T)$. Moreover, $\lim_{n\to\infty} P_{F(T)}x_n = w$.

Remark 7. Corollary 6 is also a direct consequence of [14, Theorem 3.1].

As in [1, Theorem 4.1], we can generate a sequence $\{T_n\}$ of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C by using convex combination of a general sequence $\{S_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{\alpha_n\}$ be a sequence in $[0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Let $\{\beta_n^k\}$ be a family of positive real numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that*

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for every* $n \in \mathbb{N}$;
(ii) $\lim_{n\to\infty} \beta_n^k > 0$ *for every* $k \in \mathbb{N}$, *and*
(iii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

Let $\{S_k\}$ be a sequence of nonexpansive mappings from C into itself with a common fixed point. Then a sequence $\{x_n\}$ in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\sum_{k=1}^{n} \beta_n^k S_k x_n \quad \text{for all } n \in \mathbb{N},$$

converges weakly to $w \in \bigcap_{n=1}^{\infty} F(S_k)$. Moreover, $\lim_{n\to\infty} P_{\bigcap_{k=1}^{\infty} F(S_k)}x_n = w$.

Corollary 9 (*[19]*). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1). If $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then $\{x_n\}$ converges weakly to $w \in F(T)$. Moreover, $\lim_{n\to\infty} P_{F(T)}x_n = w$.*

4. Strong convergence theorems

Inspired by Nakajo–Takahashi's paper [13] and Kim–Xu's paper [8], Mann type iteration in Section 3 is modified to obtain the strong convergence theorem as follows.

Theorem 10. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of L_n-Lipschitzian mappings from C into itself with $L_n \geq 1$ and let $\bigcap_{n=0}^{\infty} F(T_n)$ be nonempty. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases}$$

where

$$\theta_n = (1-\alpha_n)(L_n^2 - 1)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

Let $\sum_{n=0}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C and T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$, then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. We first prove that C_n and Q_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. We prove that C_n is convex. Since $\|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n$ is equivalent to

$$2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n,$$

it follows that C_n is convex. Next, we show that

$$F(T) \subset C_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{9}$$

Let $p \in F(T)$ and $n \in \mathbb{N} \cup \{0\}$. Then from

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_n x_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)L_n^2 \|x_n - p\|^2 \\ &= \|x_n - p\|^2 + (1-\alpha_n)(L_n^2 - 1)\|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \theta_n, \end{aligned}$$

we have $p \in C_n$. Therefore we obtain (9). Next, we show that

$$F(T) \subset Q_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{10}$$

We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Suppose that $F(T) \subset Q_n$. Then $\emptyset \neq F(T) \subset C_n \cap Q_n$ and there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. Then

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0$$

for each $z \in C_n \cap Q_n$. In particular,

$$\langle x_{n+1} - p, x_0 - x_{n+1}\rangle \geq 0$$

for each $p \in F(T)$. It follows that $F(T) \subset Q_{n+1}$ and hence (10) holds. This together with (9) gives

$$F(T) \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Now the sequence $\{x_n\}$ is well defined. It follows from the definition of Q_n that $x_n = P_{Q_n} x_0$. Therefore

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \text{for all } z \in Q_n \text{ and all } n \in \mathbb{N} \cup \{0\}.$$

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Let $z \in F(T) \subset Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\{\|x_n - x_0\|\}$ is nondecreasing and bounded. So $\lim_{n\to\infty} \|x_n - x_0\|$ exists. This implies that $\{x_n\}$ is bounded. Since $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have $\langle x_n - x_{n+1}, x_0 - x_n \rangle \ge 0$. It follows from (3) that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{11}$$

Since $x_{n+1} \in C_n$, $\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$ which implies that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n} \to 0.$$

From $\limsup_{n\to\infty} \alpha_n < 1$, we get

$$\begin{aligned}
\|x_n - T_n x_n\| &= \frac{1}{1-\alpha_n} \|y_n - x_n\| \\
&\le \frac{1}{1-\alpha_n} (\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\|) \to 0.
\end{aligned}$$

We apply Lemma 3 to get

$$\begin{aligned}
\|x_n - T x_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - T x_n\| \\
&\le \|x_n - T_n x_n\| + \sup\{\|T_n z - T z\| : z \in \{x_n\}\} \to 0.
\end{aligned} \tag{12}$$

Finally, we show that $x_n \to w$, where $w = P_{F(T)} x_0$. Since $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w'$. Since $I - T$ is demiclosed and from (12), we have $w' \in F(T)$. Since $x_n = P_{Q_n} x_0$ and $w \in F(T) \subset Q_n$, we have

$$\|x_n - x_0\| \le \|w - x_0\|.$$

It follows from $w = P_{F(T)} x_0$ and the lower semicontinuity of the norm that

$$\|w - x_0\| \le \|w' - x_0\| \le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|w - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec–Klee property of H, we obtain that

$$\lim_{k\to\infty} x_{n_k} = w' = w.$$

Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_{F(T)} x_0$. □

It is not difficult to see from the proof above that the boundedness of C can be discarded if $\{T_n\}$ is a sequence of nonexpansive mappings. So we immediately obtain the following corollary.

Corollary 11. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Assume that $\{\alpha_n\}$ is a sequence in* $[0, 1)$

with $\limsup_{n\to\infty}\alpha_n < 1$. *Let* $\{x_n\}$ *be a sequence in* C *defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases}$$

Let $\sum_{n=0}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C *and* T *be a mapping of* C *into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$, *then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Remark 12. Corollary 11 is a consequence of [15, Theorem 3.1(iii)].

Corollary 13. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{\alpha_n\}$ *be a sequence in* $[0, 1)$ *such that* $\limsup_{n\to\infty}\alpha_n < 1$. *Let* $\{\beta_n^k\}$ *be a family of positive real numbers with indices* $n, k \in \mathbb{N}$ *with* $k \le n$ *such that*

(i) $\sum_{k=0}^{n}\beta_n^k = 1$ *for every* $n \in \mathbb{N}\cup\{0\}$;
(ii) $\lim_{n\to\infty}\beta_n^k > 0$ *for every* $k \in \mathbb{N}\cup\{0\}$, *and*
(iii) $\sum_{n=0}^{\infty}\sum_{k=0}^{n}|\beta_{n+1}^k - \beta_n^k| < \infty$.

Let $\{S_k\}$ *be a sequence of nonexpansive mappings from* C *into itself with a common fixed point. Then a sequence* $\{x_n\}$ *in* C *defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{k=0}^{n}\beta_n^k S_k x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases}$$

converges strongly to $P_{\bigcap_{n=1}^{\infty}F(S_k)}x_0$.

Corollary 14 (*[13], Theorem 3.4*). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* T *be a nonexpansive mapping from* C *into itself such that* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence in* C *defined by* (2). *If* $\{\alpha_n\}$ *is a sequence in* $[0, 1)$ *with* $\limsup_{n\to\infty}\alpha_n < 1$, *then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

5. Applications

5.1. Equilibrium problems

Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \ge 0 \quad \text{for all } y \in C. \tag{13}$$

The set of solutions of (13) is denoted by $\mathrm{EP}(f)$. Numerous problems in physics, optimization, and economics can be reduced to find a solution of (13). Some methods have been proposed to solve the equilibrium problem (see [2,4,11,22]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $\mathrm{EP}(f)$ is nonempty and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [2]):

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for any $x, y \in C$;

(A3) f is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} f(tz + (1-t)x, y) \le f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [2, Corollary 1] and [3, Lemma 2.12], we have the following lemma.

Lemma 15. *Let C be a nonempty closed convex subset of a real Hilbert space H, let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *and let $r > 0$ and $x \in H$. Then there exists a unique $x^* \in C$ such that*

$$f(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \ge 0 \quad \text{for all } y \in C.$$

Moreover, let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(ii) $F(T_r) = \mathrm{EP}(f)$*;*

(iii) $\mathrm{EP}(f)$ *is closed and convex.*

Using Theorem 5, we have the following theorem.

Theorem 16. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4) *and $\mathrm{EP}(f) \ne \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) u_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges weakly to $w \in \mathrm{EP}(f)$. Moreover, $\lim_{n\to\infty} P_{\mathrm{EP}(f)} x_n = w$.

Proof. In order to apply Theorem 5, we first prove that

$$\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} < \infty \tag{14}$$

for any bounded subset B of C. Let B be a bounded subset of C. For $n \in \mathbb{N}$ and $z \in B$, let $z_n = T_{r_n} z$. Then

$$f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - z\rangle \ge 0 \quad \text{for all } y \in C \tag{15}$$

and

$$f(z_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - z_{n+1}, z_{n+1} - z\rangle \ge 0 \quad \text{for all } y \in C. \tag{16}$$

Putting $y = z_{n+1}$ in (15) and $y = z_n$ in (16), we have

$$f(z_n, z_{n+1}) + \frac{1}{r_n}\langle z_{n+1} - z_n, z_n - z\rangle \ge 0$$

and

$$f(z_{n+1}, z_n) + \frac{1}{r_{n+1}}\langle z_n - z_{n+1}, z_{n+1} - z\rangle \ge 0.$$

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So, from (A2) we have

$$\left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z}{r_{n+1}} \right\rangle \geq 0$$

and hence

$$\left\langle z_{n+1} - z_n, z_n - z - \frac{r_n}{r_{n+1}}(z_{n+1} - z) \right\rangle \geq 0.$$

Thus,

$$\langle z_{n+1} - z_n, z_n - z_{n+1} \rangle + \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right)(z_{n+1} - z) \right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \|z_{n+1} - z_n\|^2 &\leq \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right)(z_{n+1} - z) \right\rangle \\ &\leq \|z_{n+1} - z_n\| \left|1 - \frac{r_n}{r_{n+1}}\right| \|z_{n+1} - z\|, \end{aligned}$$

and hence

$$\begin{aligned} \|T_{r_{n+1}} z - T_{r_n} z\| &= \|z_{n+1} - z_n\| \\ &\leq \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|T_{r_{n+1}} z - z\| \\ &\leq \frac{1}{b} |r_{n+1} - r_n| \|T_{r_{n+1}} z - z\|. \end{aligned} \tag{17}$$

By Lemma 15(ii), we have $\mathrm{EP}(f) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. Let $v \in \mathrm{EP}(f)$ and $M = \sup\{\|z - v\| : z \in B\}$. Then

$$\begin{aligned} \|T_{r_{n+1}} z - z\| &\leq \|T_{r_{n+1}} z - v\| + \|v - z\| \\ &= \|T_{r_{n+1}} z - T_{r_{n+1}} v\| + \|v - z\| \\ &\leq 2\|z - v\|. \end{aligned}$$

This together with (17) gives

$$\sup\{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} \leq \frac{2M}{b} |r_{n+1} - r_n|.$$

Since $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, we obtain (14). By Lemma 3, we define a mapping T by

$$Tx = \lim_{n \to \infty} T_{r_n} x \quad \text{for all } x \in C.$$

Finally we prove that

$$F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n}). \tag{18}$$

It easy to see that $\bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T)$. Let $w \in F(T)$. For $n \in \mathbb{N}$, let $w_n = T_{r_n} w$. Then

$$f(w_n, y) + \frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq 0 \quad \text{for all } y \in C.$$

By (A2), we have

$$\frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq f(y, w_n) \quad \text{for all } y \in C.$$

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Since $w_n \to w$ and from (A4), we have

$$0 \geq f(y, w) \quad \text{for all } y \in C.$$

Then, for $t \in (0, 1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y) \end{aligned}$$

or

$$f(ty + (1-t)w, y) \geq 0.$$

Letting $t \to 0^+$ and using (A3), we get

$$f(w, y) \geq 0 \quad \text{for all } y \in C$$

and hence $w \in \mathrm{EP}(f) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. Therefore we obtain (18). Applying Theorem 5, $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_{\mathrm{EP}(f)} x_n$. □

Using Corollary 11, we have the following theorem.

Theorem 17. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4) and $\mathrm{EP}(f) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and*

$$\begin{cases} f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{EP}(f)} x_0$.

5.2. *Convergence theorem for monotone mappings*

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A : C \to H$ be a mapping. The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The set of solutions of classical variational inequality is denoted by $\mathrm{VIP}(C, A)$. The variational inequality has been extensively studied in the literature (see [7,12,16,23,25–27] and the references therein). We recall that a mapping $A : C \to H$ is said to be:

(a) monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall u, v \in C;$$

(b) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2 \quad \forall u, v \in C;$$

(c) r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Au - Av, u - v\rangle \geq r\|u - v\|^2 \quad \forall u, v \in C;$$

(d) relaxed (γ, r)-cocoercive if there exist constant $\gamma, r > 0$ such that

$$\langle Au - Av, u - v\rangle \geq -\gamma\|Au - Av\|^2 + r\|u - v\|^2 \quad \forall u, v \in C;$$

(e) μ-Lipschitzian if there exists a constant $\mu > 0$ such that

$$\|Au - Av\| \le \mu\|u - v\| \quad \forall u, v \in C.$$

Remark 18. (1) Every α-inverse-strongly monotone mapping is monotone and $1/\alpha$-Lipschitzian.
(2) Every r-strongly monotone is monotone.
(3) Every relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping with $\gamma\mu^2 \le r$ is monotone.

Lemma 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H. Define a bifunction $f : C \times C \to \mathbb{R}$ as follows*

$$f(x, y) := \langle Ax, y - x\rangle \tag{19}$$

for all $x, y \in C$. Then f satisfies (A1)–(A4) *and* $\mathrm{VIP}(C, A) = \mathrm{EP}(f)$.

Proof. It easy to see that $\mathrm{VIP}(C, A) = \mathrm{EP}(f)$. We show that f satisfies (A1)–(A4).

(A1) Let $x \in C$. Then $f(x, x) = \langle Ax, x - x\rangle = 0$.

(A2) Let $x, y \in C$. Since A is monotone, we have $\langle Ax - Ay, x - y\rangle \ge 0$. Then

$$\begin{aligned} f(x, y) + f(y, x) &= \langle Ax, y - x\rangle + \langle Ay, x - y\rangle \\ &= \langle Ay - Ax, x - y\rangle \\ &= -\langle Ax - Ay, x - y\rangle \le 0. \end{aligned}$$

(A3) Let $x, y, z \in C$ and $t \in [0, 1]$. Since A is continuous, we have

$$\begin{aligned} \lim_{t\to 0^+} f(tz + (1 - t)x, y) &= \lim_{t\to 0^+} \langle A(tz + (1 - t)x), y - (tz + (1 - t)x)\rangle \\ &= \langle Ax, y - x\rangle = f(x, y). \end{aligned}$$

(A4) Let $x \in C$. Then

$$\begin{aligned} f(x, ty + (1 - t)z) &= \langle Ax, (ty + (1 - t)z - x)\rangle \\ &= \langle Ax, t(y - x)\rangle + \langle Ax, (1 - t)(z - x)\rangle \\ &= t\langle Ax, y - x\rangle + (1 - t)\langle Ax, z - x\rangle \\ &= tf(x, y) + (1 - t)f(x, z), \end{aligned}$$

for all $y, z \in C$ and $t \in [0, 1]$. So we have that $f(x, \cdot)$ is linear. Let $y \in C$ and $\{y_n\} \subset C$ such that $y_n \to y$. Then

$$\lim_{n\to\infty} f(x, y_n) = \lim_{n\to\infty} \langle Ax, y_n - x\rangle = \langle Ax, y - x\rangle = f(x, y).$$

Hence $f(x, \cdot)$ is continuous. □

Lemma 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H and let f be a bifunction defined by* (19). *Let $x \in H$, $u \in C$ and $r > 0$. Then*

$$f(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \ge 0, \quad \forall y \in C \quad \Leftrightarrow \quad u = P_C(x - rAu).$$

Proof. Let $y \in C$. Then

$$\begin{aligned} f(u, y) + \frac{1}{r}\langle y - u, u - x\rangle &= \langle Au, y - u\rangle + \left\langle y - u, \frac{u - x}{r}\right\rangle \\ &= \left\langle y - u, Au + \frac{u - x}{r}\right\rangle \\ &= \frac{1}{r}\langle y - u, u - (x - rAu)\rangle \end{aligned}$$

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and hence

$$\begin{aligned}
&f(u,y)+\frac{1}{r}\langle y-u,u-x\rangle \ge 0, \quad \forall y\in C\\
&\Leftrightarrow \langle y-u,u-(x-rAu)\rangle \ge 0, \quad \forall y\in C\\
&\Leftrightarrow u=P_C(x-rAu). \quad \square
\end{aligned}$$

Using Theorem 16, Lemmas 19 and 20, we have the following theorem.

Theorem 21. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C such that $\mathrm{VIP}(C,A)\neq\emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1\in C$ and*

$$\begin{cases} u_n=P_C(x_n-r_nAu_n)\\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)u_n, \end{cases}$$

for all $n\in\mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$ and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n>0$, and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$. Then $\{x_n\}$ converges weakly to $w\in\mathrm{VIP}(C,A)$. Moreover, $\lim_{n\to\infty}P_{\mathrm{VIP}(C,A)}x_n=w$.

Using Theorem 17, Lemmas 19 and 20, we have the following theorem.

Theorem 22. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C such that $\mathrm{VIP}(C,A)\neq\emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0\in C$ and*

$$\begin{cases} u_n=P_C(x_n-r_nAu_n)\\ y_n=\alpha_n x_n+(1-\alpha_n)u_n,\\ C_n=\{z\in C:\|y_n-z\|\le\|x_n-z\|\},\\ Q_n=\{z\in C:\langle x_n-z,x_0-x_n\rangle\ge 0\},\\ x_{n+1}=P_{C_n\cap Q_n}x_0, \quad n=0,1,2\ldots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty}\alpha_n<1$ and $\{r_n\}$ is a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n>0$, and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{VIP}(C,A)}x_0$.

Remark 23. (1) As Remark 18, we obtain a strong convergence theorem for α-inverse-strongly monotone mappings, r-strongly monotone and continuous mappings and relaxed (γ,r)-cocoercive and μ-Lipschitzian mappings with $\gamma\mu^2\le r$.

(2) Some weak and strong convergence theorems for monotone Lipschitzian mappings were established by several authors [7,12,16,23,25–27]. However, there is a monotone continuous mapping which is not Lipschitzian. Let $H=\mathbb{R}$ with the usual inner product and usual norm and let $C=[0,1]$. Define $A:C\to C$ by $Ax=1-(1-x^{2/3})^{3/2}$ for all $x\in C$. Then A is not Lipschitzian (see [20]). But A is monotonically increasing, so

$$\langle Ax-Ay,x-y\rangle=(Ax-Ay)(x-y)\ge 0 \quad \forall x,y\in C.$$

Hence A is monotone and continuous with $0\in\mathrm{VIP}(C,A)\neq\emptyset$. Therefore, Theorems 21 and 22 provide a new convergence theorem for a wider class of mappings.

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References

[1] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2006) 2350–2360.

[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student. 63 (1994) 123–145.
[3] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.
[4] S.D. Flam, A.S. Antipin, Equilibrium programming using proximal-link algorithms, Math. Program. 78 (1997) 29–41.
[5] A. Genel, J. Lindenstrass, An example concerning fixed points, Israel. J. Math. 22 (1975) 81–86.
[6] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.
[7] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mapping and inverse-strong monotone mappings, Nonlinear Anal. 61 (2005) 341–350.
[8] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 64 (2006) 1140–1152.
[9] W.A. Kirk, Fixed point theorem for mappings which do not increase distance, Amer. Math. Monthly. 72 (1965) 1004–1006.
[10] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.
[11] A. Moudafi, M. Thera, Proximal and dynamical approaches to equilibrium problems, in: Lecture note in Economics and Mathematical Systems, vol. 477, Springer-Verlag, New York, 1999, pp. 187–201.
[12] N. Nadezhkina, W. Takahashi, Weak convergence theorems by extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006) 191–201.
[13] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.
[14] K. Nakajo, K. Shimoji, W. Takahashi, On weak convergence by products of mappings in Hilbert spaces, Comm. Appl. Nonlinear Anal. 13 (2006) 27–50.
[15] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces, Taiwanese J. Math. 10 (2006) 339–360.
[16] M.A. Noor, Z. Huang, Three-step methods for nonexpansive mapping and variational inequalities, Appl. Math. Comput. 187 (2007) 680–685.
[17] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.
[18] E.U. Ofoedu, Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space, J. Math. Anal. Appl. 321 (2006) 722–728.
[19] S. Reich, Weak convergence theorems for nonexpansive mappings, J. Math. Anal. Appl. 67 (1979) 274–276.
[20] B.E. Rhoades, Comments on two fixed point iterative methods, J. Math. Anal. Appl. 56 (1976) 741–750.
[21] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
[22] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515.
[23] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417–428.
[24] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.
[25] Y. Yao, Y.C. Liou, J.C. Yao, An extragradient method for fixed point problems and variational inequality problems, J. Inequal. Appl. (2007), doi:10.1155/2007/38752. Article ID 38752, 12 pages.
[26] Y. Yao, J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, Appl. Math. Comput. 186 (2007) 1551–1558.
[27] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese J. Math. 10 (2006) 1293–1303.

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EQUILIBRIUM PROBLEMS AND MOUDAFI'S VISCOSITY APPROXIMATION METHODS IN HILBERT SPACES

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Abstract. We establish an iterative scheme by means of Mann's method and Moudafi's method to find a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. We prove a convergence theorem of our iteration under the weaker assumption as were the case in Takahashi and Takahashi's recent results. The new iteration considered in the paper is applied to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for continuous monotone mappings. Consequently, the corresponding results for α-inverse-strongly monotone mappings, r-strongly monotone mappings and relaxed (γ, r)-cocoercive mappings are obtained respectively. We also propose a slightly modified Mann-type iteration to obtain a strong convergence theorem for continuous pseudocontractive mappings.
Keywords. viscosity approximations method, equilibrium problem, variational inequality problem, nonexpansive mapping, monotone mapping, pseudocontractive mapping
AMS (MOS) subject classification: 47H09, 47H10, 47J25

1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $\mathrm{EP}(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). Some methods have been proposed to solve the equilibrium problem (see [1, 5, 13]). In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and they also proved a strong convergence theorem.

A mapping f of C into H is called a contraction if there exists $a \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq a\|x - y\| \quad \text{for all } x, y \in C$$

and a mapping S of C into H is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $\mathrm{F}(S)$ the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $\mathrm{F}(S)$ is nonempty (see [9], cf. also [22]).

There are many methods for approximating fixed points of a nonexpansive mapping. In 1953, Mann [11] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n$$

where the initial guess $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [17]. In an infinite-dimensional Hilbert space, Mann iteration can conclude only weak convergence [6].

Some attempts to construct iteration method so that strong convergence is guaranteed have recently been made.

For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in C$, let the sequence $\{x_n\}$ in C be iteratively defined by $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n. \tag{2}$$

The recursion formula (2) was first introduced in 1967 by Halpern [7]. In 1977, Lions [10] improved the result of Halpern by proving strong convergence of $\{x_n\}$ to a fixed point of S where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty} (\alpha_{n+1} - \alpha_n)/\alpha_{n+1}^2 = 0$.

Wittmann [24] proved a strong convergence theorem under the assumption of (C1), (C2) and

(C3′) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Xu [25, 26] considered the conditions (C1), (C2) and

(C3″) $\lim_{n\to\infty} (\alpha_{n+1} - \alpha_n)/\alpha_{n+1} = 0$.

He showed that condition (C3) and (C3′) are not comparable and that (C3′) and (C3″) are not comparable either. We note that (C3″) is weaker than (C3) by removing the square in the denominator so that the canonical choice of $\alpha_n = 1/(n+1)$ is possible.

Moudafi [12] introduced the viscosity approximation method as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sx_n. \tag{3}$$

If $f(x) = u \in C$ for all $x \in H$, then Moudafi's iteration reduces to Halpern iteration.

Chidume and Chidume [2], and, independently, Suzuki [20] introduced iteration similar to (2) as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(\lambda Sx_n + (1-\lambda)x_n) \tag{4}$$

for all $n \in \mathbb{N}$, where $u \in C$, $\lambda \in (0,1)$ and $\{\alpha_n\} \subset [0,1]$.

Recently, Yao, Chen and Zhou [27] introduced a modified Halpern iteration as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)y_n \end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

Setting $f(x) = u \in C$ for all $x \in C$, $\beta'_n = (1-\alpha_n)(1-\lambda)$ and $\alpha'_n = \alpha_n/(1-\beta'_n)$ for all $n \in \mathbb{N}$. Then we can rewrite (4) as follows:

$$\begin{aligned} y_n &= \alpha'_n f(x_n) + (1-\alpha'_n)Sx_n, \\ x_{n+1} &= \beta'_n x_n + (1-\beta'_n)y_n. \end{aligned}$$

Inspired by (3), Takahashi and Takahashi [21] introduce a new iteration by the Moudafi's viscosity method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. The purpose of the paper is to extend Takahashi–Takahashi's iteration by means of Mann's method and Moudafi's method. In section 3 we prove a convergence theorem of our iteration under the weaker assumption as were the case in [21]. The new iteration considered in the paper is applied to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for continuous monotone mappings. Consequently, the corresponding results for α-inverse-strongly monotone mappings, r-strongly monotone mappings and relaxed (γ, r)-cocoercive mappings are obtained respectively. We also propose a slightly modified Mann-type iteration to obtain a strong convergence theorem for continuous pseudocontractive mappings. It is known that there is a Lipschitzian pseudocontractive mapping with a unique fixed point for which every Mann sequence fails to converge [3].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. In a real Hilbert space H, we have

$$\|x+y\|^2 = \|x\|^2 - \|y\|^2 + 2\langle x+y, y\rangle \tag{6}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{7}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies the Opial's condition [16], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further more, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, z - y\rangle \ge 0 \quad \text{for all } y \in C.$$

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [1]):

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \le F(x,y);$$

(A4) $F(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma will be useful for proving the main result of this paper.

Lemma 1 ([1], Corollary 1). *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfies* (A1)-(A4) *and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \quad \textit{for all } y \in C.$$

Lemma 2 ([4], Lemma 2.12). *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies* (A1)-(A4). *For* $r > 0$ *and* $x \in H$, *defined a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \big\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\big\}$$

for all $x \in H$. *Then, the following hold:*

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $\mathrm{F}(T_r) = \mathrm{EP}(F)$;

(iv) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 3 ([19], Lemma 2.2). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space, and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with*

(B) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$ *for all* $n \in \mathbb{N}$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 4 ([25], Lemma 2.1). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers that satisfies that condition*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n$$

for all $n \in \mathbb{N}$, *where the sequences* $\{\gamma_n\}$ *in* $(0, 1)$ *and* $\{\delta_n\}$ *satisfy conditions:* $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\limsup_{n\to\infty} \delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

3 Equilibrium problems

In this section, we approximate a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4) *and let* S *be a nonexpansive mapping of* C *into* H *such that* $\mathrm{F}(S) \cap \mathrm{EP}(F) \neq \varnothing$. *Let* f *be a contraction of* H *into itself and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n \end{cases}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *with* (C1), (C2) *and* (B) *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$ *with*

(R1) $\liminf_{n\to\infty} r_n > 0$, *and*

(R2) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z = P_{\mathrm{F}(S)\cap\mathrm{EP}(F)} f(z)$.

Proof. Let $v \in \mathrm{F}(S) \cap \mathrm{EP}(F)$. Then $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| \le \|T_{r_n} x_n - T_{r_n} v\| \le \|x_n - v\| \tag{8}$$

for all $n \in \mathbb{N}$. Put $K = \max\{\|x_1 - v\|, \frac{1}{1-a}\|f(v) - v\|\}$. We note here that $a \in (0,1)$. It is obvious that $\|x_1 - v\| \le K$. Suppose that $\|x_n - v\| \le K$. Then, by (8),

$$\begin{aligned}\|y_n - v\| &\le \alpha_n\|f(x_n) - v\| + (1-\alpha_n)\|Su_n - v\| \\ &\le \alpha_n\|f(x_n) - f(v)\| + \alpha_n\|f(v) - v\| + (1-\alpha_n)\|u_n - v\| \\ &\le \alpha_n a\|x_n - v\| + \alpha_n(1-a)\frac{1}{1-a}\|f(v) - v\| + (1-\alpha_n)\|x_n - v\| \\ &\le K,\end{aligned}$$

and so

$$\|x_{n+1} - v\| \le \beta_n\|x_n - v\| + (1-\beta_n)\|y_n - v\| \le K.$$

So, we have that $\|x_n - v\| \le K$ for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{Su_n\}$, $\{f(x_n)\}$ and $\{y_n\}$ are bounded. Next, we show that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{9}$$

Since $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \quad \text{for all } y \in C \tag{10}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0 \quad \text{for all } y \in C. \tag{11}$$

Putting $y = u_{n+1}$ in (10) and $y = u_n$ in (11), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0.$$

So, from (A2) we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \ge 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Thus,

$$\langle u_{n+1} - u_n, u_n - u_{n+1}\rangle + \langle u_{n+1} - u_n, u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \\ &= \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\big(\|x_{n+1} - x_n\| + \big|1 - \frac{r_n}{r_{n+1}}\big|\|u_{n+1} - x_{n+1}\|\big)\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L, \qquad (12)\end{aligned}$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Observe that

$$\begin{aligned}y_{n+1} - y_n &= \alpha_{n+1}f(x_{n+1}) + (1 - \alpha_{n+1})Su_{n+1} - \alpha_n f(x_n) - (1 - \alpha_n)Su_n \\ &= (\alpha_{n+1} - \alpha_n)f(x_{n+1}) + \alpha_n(f(x_{n+1}) - f(x_n)) \\ &\quad + (1 - \alpha_{n+1})(Su_{n+1} - Su_n) + (\alpha_n - \alpha_{n+1})Su_n.\end{aligned}$$

This together with (12) gives

$$\begin{aligned}&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &\quad\leq |\alpha_{n+1} - \alpha_n|(\|f(x_{n+1})\| + \|Su_n\|) + \alpha_n\|f(x_{n+1}) - f(x_n)\| \\ &\qquad + (1 - \alpha_{n+1})\|Su_{n+1} - Su_n\| - \|x_{n+1} - x_n\| \\ &\quad\leq |\alpha_{n+1} - \alpha_n|(\|f(x_{n+1})\| + \|Su_n\|) + (\alpha_n a - 1)\|x_{n+1} - x_n\| \\ &\qquad + (1 - \alpha_{n+1})\|u_{n+1} - u_n\| \\ &\quad\leq |\alpha_{n+1} - \alpha_n|(\|f(x_{n+1})\| + \|Su_n\|) + (\alpha_n a - \alpha_{n+1})\|x_{n+1} - x_n\| \\ &\qquad + (1 - \alpha_{n+1})\frac{1}{b}|r_{n+1} - r_n|L.\end{aligned}$$

Since $\{f(x_n)\}, \{Su_n\}$ and $\{x_n\}$ are bounded, (C1) and (R2), we obtain that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 3, we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{13}$$

Therefore

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|y_n - x_n\| = 0.$$

This proves (9). It follows from (12) and (R2) that $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Observe also that

$$y_n - x_n = \alpha_n(f(x_n) - x_n) + (1-\alpha_n)(Su_n - x_n).$$

This together with (13) and (C1), gives

$$\lim_{n\to\infty} \|Su_n - x_n\| = 0. \tag{14}$$

For $v \in \mathrm{F}(S) \cap \mathrm{EP}(F)$, we have

$$\begin{aligned}
\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\
&= \langle u_n - v, x_n - v\rangle \\
&= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2)
\end{aligned}$$

and hence

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|u_n - x_n\|^2. \tag{15}$$

Therefore, from (7) and (15), we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\|y_n - v\|^2 \\
&\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big(\alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|u_n - v\|^2\big) \\
&\le \beta_n\|x_n - v\|^2 + \alpha_n\|f(x_n) - v\|^2 + (1-\beta_n)\|u_n - v\|^2 \\
&\le \|x_n - v\|^2 + \alpha_n\|f(x_n) - v\|^2 - (1-\beta_n)\|u_n - x_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
&(1-\beta_n)\|u_n - x_n\|^2 \\
&\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
&\le \alpha_n\|f(x_n) - v\|^2 + \|x_{n+1} - x_n\|(\|x_n - v\| + \|x_{n+1} - v\|).
\end{aligned}$$

From (B), (C1) and (9), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

This together with (14) gives

$$\|Su_n - u_n\| \le \|Su_n - x_n\| + \|u_n - x_n\| \to 0.$$

Let $Q = P_{\mathrm{F}(S)\cap\mathrm{EP}(F)}$. Then Qf is a contraction of H into itself. In fact,

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\|$$

for all $x, y \in H$. Now, since H is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$ is an element of C. We next show

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \leq 0. \tag{16}$$

To this end, we choose a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{i\to\infty} \langle f(z) - z, x_{n_i} - z\rangle.$$

Since $\{u_{n_i}\}$ is bounded, we can assume without loss of generality that $u_{n_i} \rightharpoonup w$. Let us show that $w \in \mathrm{EP}(F)$. By using (10) and (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n) \quad \text{for all } y \in C$$

and so

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}) \quad \text{for all } y \in C.$$

From $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, $u_{n_i} \rightharpoonup w$ and (A4), we have

$$0 \geq F(y, w) \quad \text{for all } y \in C.$$

Then, for $t \in (0, 1]$ and $y \in C$,

$$\begin{aligned} 0 &= F(ty + (1-t)w, ty + (1-t)w) \\ &\leq tF(ty + (1-t)w, y) + (1-t)F(ty + (1-t)w, w) \\ &\leq tF(ty + (1-t)w, y) \end{aligned}$$

or

$$F(ty + (1-t)w, y) \geq 0.$$

Letting $t \to 0^+$ and using (A3), we get

$$F(w, y) \geq 0 \quad \text{for all } y \in C$$

and hence $w \in \mathrm{EP}(F)$. We shall show that $w \in \mathrm{F}(S)$. Assume that $w \notin \mathrm{F}(S)$. Since $u_{n_i} \rightharpoonup w$ and $Sw \neq w$ from Opial's condition we have

$$\begin{aligned} \liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\| \\ &\leq \liminf_{i\to\infty} \|Su_{n_i} - Sw\| + \lim_{i\to\infty} \|u_{n_i} - Su_{n_i}\| \\ &\leq \liminf_{i\to\infty} \|u_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So we get $w \in \mathrm{F}(S)$. Therefore $w \in \mathrm{F}(S) \cap \mathrm{EP}(F)$. It follows since $z = P_{\mathrm{F}(S)\cap\mathrm{EP}(F)}f(z)$ and $x_{n_i} \rightharpoonup w$ that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle \\ &= \lim_{i\to\infty}\langle f(z) - z, w - z\rangle \le 0.\end{aligned}$$

Now (16) is proved. Finally we prove that

$$\lim_{n\to\infty}\|x_n - z\| = 0. \tag{17}$$

From (8), we have

$$\begin{aligned}\|\beta_n(x_n - z)+&(1-\beta_n)(1-\alpha_n)(Su_n - z)\| \\ &\le \beta_n\|x_n - z\| + (1-\beta_n)(1-\alpha_n)\|Su_n - z\| \\ &\le \beta_n\|x_n - z\| + (1-\beta_n)(1-\alpha_n)\|u_n - z\| \\ &\le (1-(1-\beta_n)\alpha_n)\|x_n - z\|. \end{aligned}\tag{18}$$

We observe that

$$\begin{aligned}\langle x_{n+1} - z, f(x_n) - f(z)\rangle &\le a\|x_{n+1} - z\|\|x_n - z\| \\ &\le \frac{1}{2}a(\|x_{n+1} - z\|^2 + \|x_n - z\|^2)\end{aligned}\tag{19}$$

and

$$\begin{aligned}x_{n+1} - z &= \beta_n(x_n - z) + (1-\beta_n)(y_n - z) \\ &= \beta_n(x_n - z) + (1-\beta_n)(1-\alpha_n)(Su_n - z) + (1-\beta_n)\alpha_n(f(x_n) - z).\end{aligned}$$

Then, by (6), (18) and (19),

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\le \|\beta_n(x_n - z) + (1-\beta_n)(1-\alpha_n)(Su_n - z)\|^2 \\ &\quad + 2(1-\beta_n)\alpha_n\langle x_{n+1} - z, f(x_n) - z\rangle \\ &\le (1-(1-\beta_n)\alpha_n)^2\|x_n - z\|^2 + 2(1-\beta_n)\alpha_n\langle x_{n+1} - z, f(x_n) - f(z)\rangle \\ &\quad + 2(1-\beta_n)\alpha_n\langle x_{n+1} - z, f(z) - z\rangle \\ &\le (1-(1-\beta_n)\alpha_n)^2\|x_n - z\|^2 + a(1-\beta_n)\alpha_n(\|x_{n+1} - z\|^2 + \|x_n - z\|^2) \\ &\quad + 2(1-\beta_n)\alpha_n\langle x_{n+1} - z, f(z) - z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\le \frac{1-(2-a)(1-\beta_n)\alpha_n+(1-\beta_n)^2\alpha_n^2}{1-a(1-\beta_n)\alpha_n}\|x_n-z\|^2 \\
&\quad+\frac{2(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\langle f(z)-z, x_{n+1}-z\rangle \\
&=\Big(1-\frac{(1-a)(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\Big)\|x_n-z\|^2-\frac{(1-a)(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\|x_n-z\|^2 \\
&\quad+\frac{(1-\beta_n)^2\alpha_n^2}{1-a(1-\beta_n)\alpha_n}\|x_n-z\|^2+\frac{2(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\langle f(z)-z, x_{n+1}-z\rangle \\
&\le\Big(1-\frac{(1-a)(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\Big)\|x_n-z\|^2+\frac{(1-\beta_n)^2\alpha_n^2}{1-a(1-\beta_n)\alpha_n}\|x_n-z\|^2 \\
&\quad+\frac{2(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\langle f(z)-z, x_{n+1}-z\rangle \\
&\le\Big(1-\frac{(1-a)(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\Big)\|x_n-z\|^2+\frac{(1-\beta_n)^2\alpha_n^2}{1-a(1-\beta_n)\alpha_n}M \\
&\quad+\frac{2(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}\langle f(z)-z, x_{n+1}-z\rangle
\end{aligned}$$

where $M=\sup\{\|x_n-z\|^2 : n\in\mathbb{N}\}$. Put

$$\gamma_n=\frac{(1-a)(1-\beta_n)\alpha_n}{1-a(1-\beta_n)\alpha_n}$$

and

$$\delta_n=\frac{(1-\beta_n)\alpha_n}{1-a}M+\frac{2}{1-a}\langle f(z)-z, x_{n+1}-z\rangle.$$

Then

$$\|x_{n+1}-z\|^2\le(1-\gamma_n)\|x_n-z\|^2+\gamma_n\delta_n.$$

Then, by (C1), (C2) and (16),

$$\lim_{n\to\infty}\gamma_n=0,\quad \sum_{n=1}^{\infty}\gamma_n=\infty,\quad\text{and}\quad \limsup_{n\to\infty}\delta_n\le 0,$$

and we apply Lemma 4 to get (17). So we conclude that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z=P_{\mathrm{F}(S)\cap\mathrm{EP}(F)}f(z)$. □

Remark 6. The iterative scheme in Theorem 5 is slightly different from one in [21] but we can weaken and remove the restrictions on the sequences $\{\alpha_n\}$ and $\{r_n\}$ in [21, Theorem 3.2]. More precisely,

(1) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ is replaced by the weaker assumption that

$$\lim_{n\to\infty} (r_{n+1} - r_n) = 0$$

and the restriction (C3) on $\{\alpha_n\}$ is removed and replaced by the assumption on $\{\beta_n\}$;

(2) Takahashi–Takahashi's result is not applicable to the case $\alpha_{2n} = 1/2n$, $\alpha_{2n+1} = 0$, and $r_n = 1 + \alpha_n$ for all $n \in \mathbb{N}$. Clearly, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| = \infty$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| = \infty$ but $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Corollary 7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *such that* $\mathrm{EP}(F) \neq \varnothing$. *Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ *with* (C1), (C2) *and* (B) *and $\{r_n\}$ is a sequence in* $(0,\infty)$ *with* (R1) *and* (R2). *Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathrm{EP}(F)}f(z)$.*

Proof. Put $Sx = x$ for all $x \in C$ in Theorem 5. □

Corollary 8. *Let C be a nonempty closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of C into H such that* $\mathrm{F}(S) \neq \varnothing$. *Let f be a contraction of H into itself and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1 - \alpha_n)SP_C x_n, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n)y_n \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ *with* (C1), (C2) *and* (B). *Then $\{x_n\}$ converges strongly to $z = P_{\mathrm{F}(S)}f(z)$.*

Proof. Put $F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$. Then $u_n = P_C x_n$. □

The conjunction of (C1) and (C2) is a sufficient condition on the following iteration.

Corollary 9. *Let C be a nonempty closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of C into H such that* $\mathrm{F}(S) \neq \varnothing$.

Let f be a contraction of H into itself and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(\lambda S P_C x_n + (1-\lambda)x_n)$$

for all $n \in \mathbb{N}$, where $\lambda \in (0,1)$, $\{\alpha_n\} \subset [0,1]$ with (C1) *and* (C2). *Then $\{x_n\}$ converges strongly to $z = P_{\mathrm{F}(S)} f(z)$.*

Proof. Put $\beta'_n = (1-\alpha_n)(1-\lambda)$ and $\alpha'_n = \alpha_n/(1-\beta'_n)$ for all $n \in \mathbb{N}$. Then

$$y_n = \alpha'_n f(x_n) + (1-\alpha'_n) S P_C x_n,$$
$$x_{n+1} = \beta'_n x_n + (1-\beta'_n) y_n$$

and so

$$\lim_{n\to\infty} \beta'_n = 1-\lambda \in (0,1), \quad \lim_{n\to\infty} \alpha'_n = 0 \quad \text{and} \sum_{n=1}^{\infty} \alpha'_n = \infty.$$

□

4 Applications

4.1 Strong convergence theorem for monotone mappings

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A : C \to H$ be a mapping. The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The set of solutions of classical variational inequality is denoted by $\mathrm{VIP}(C, A)$. The variational inequality has been extensively studied in the literature (see [8, 14, 15, 23, 28, 29, 30] and the references therein). We recall that a mapping $A : C \to H$ is said to be:

(a) monotone if
$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall u, v \in C;$$

(b) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that
$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2 \quad \forall u, v \in C;$$

(c) r-strongly monotone if there exists a constant $r > 0$ such that
$$\langle Au - Av, u - v\rangle \geq r\|u - v\|^2 \quad \forall u, v \in C;$$

(d) relaxed (γ, r)-cocoercive if there exist constant γ, $r > 0$ such that
$$\langle Au - Av, u - v\rangle \geq -\gamma\|Au - Av\|^2 + r\|u - v\|^2 \quad \forall u, v \in C;$$

(e) μ-Lipschitzian if there exists a constant $\mu > 0$ such that

$$\|Au - Av\| \leq \mu\|u - v\| \quad \forall u, v \in C.$$

Remark 10. (1) Every α-inverse-strongly monotone mapping is monotone and $1/\alpha$-Lipschitzian.

(2) Every r-strongly monotone is monotone.

(3) Every relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping with $\gamma\mu^2 \leq r$ is monotone.

Lemma 11. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H. Define a bifunction $F : C \times C \to \mathbb{R}$ as follows:*

$$F(x, y) := \langle Ax, y - x\rangle \tag{20}$$

for all $x, y \in C$. Then F satisfies (A1)-(A4) *and* $\mathrm{VIP}(C, A) = \mathrm{EP}(F)$*.*

Proof. It easy to see that $\mathrm{VIP}(C, A) = \mathrm{EP}(F)$. We show that F satisfies (A1)-(A4).

(A1) Let $x \in C$. Then $F(x, x) = \langle Ax, x - x\rangle = 0$.

(A2) Let $x, y \in C$. Since A is monotone, we have $\langle Ax - Ay, x - y\rangle \geq 0$. Then

$$\begin{aligned} F(x, y) + F(y, x) &= \langle Ax, y - x\rangle + \langle Ay, x - y\rangle \\ &= \langle Ay - Ax, x - y\rangle \\ &= -\langle Ax - Ay, x - y\rangle \leq 0. \end{aligned}$$

(A3) Let $x, y, z \in C$ and $t \in [0, 1]$. Since A is continuous, we have

$$\begin{aligned} \lim_{t\to 0^+} F(tz + (1 - t)x, y) &= \lim_{t\to 0^+} \langle A(tz + (1 - t)x), y - (tz + (1 - t)x)\rangle \\ &= \langle Ax, y - x\rangle = F(x, y). \end{aligned}$$

(A4) Let $x \in C$. Then

$$\begin{aligned} F(x, ty + (1 - t)z) &= \langle Ax, (ty + (1 - t)z - x)\rangle \\ &= \langle Ax, t(y - x)\rangle + \langle Ax, (1 - t)(z - x)\rangle \\ &= t\langle Ax, y - x\rangle + (1 - t)\langle Ax, z - x\rangle \\ &= tF(x, y) + (1 - t)F(x, z), \end{aligned}$$

for all $y, z \in C$ and $t \in [0, 1]$. So we have $F(x, \cdot)$ is linear. Let $y \in C$ and $\{y_n\} \subset C$ such that $y_n \to y$. Then

$$\lim_{n\to\infty} F(x, y_n) = \lim_{n\to\infty} \langle Ax, y_n - x\rangle = \langle Ax, y - x\rangle = F(x, y).$$

Hence $F(x, \cdot)$ is continuous. □

Lemma 12. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H and let F be a bifunction defined by (20). Let $x \in H$, $u \in C$ and $r > 0$. Then*

$$F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall y \in C \Leftrightarrow u = P_C(x - rAu).$$

Proof. Let $y \in C$. Then

$$\begin{aligned} & F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \\ & = \langle Au, y - u\rangle + \langle y - u, \frac{u - x}{r}\rangle \\ & = \langle y - u, Au + \frac{u - x}{r}\rangle \\ & = \frac{1}{r}\langle y - u, u - (x - rAu)\rangle \end{aligned}$$

and hence

$$\begin{aligned} & F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \ \ \forall y \in C \\ & \Leftrightarrow \langle y - u, u - (x - rAu)\rangle \geq 0, \ \ \forall y \in C \\ & \Leftrightarrow u = P_C(x - rAu). \end{aligned}$$

□

Using Theorem 5, Lemmas 11 and 12, we have the following theorems.

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H and let S be a nonexpansive mapping of C into H such that $\mathrm{F}(S) \cap \mathrm{VIP}(C, A) \neq \varnothing$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} u_n = P_C(x_n - r_n A u_n) \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ with (C1), (C2) *and* (B) *and $\{r_n\}$ is a sequence in $(0, \infty)$ with* (R1) *and* (R2). *Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathrm{F}(S)\cap\mathrm{VIP}(C,A)} f(z)$.*

Corollary 14 (cf. [29], Theorem 3.1). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $\mathrm{F}(S) \cap \mathrm{VIP}(C, A) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = u \in C$ and*

$$\begin{cases} u_n = P_C(x_n - r_n A u_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - r_n A u_n) \end{cases} \tag{21}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, (C1), (C2) and (B) and $\{r_n\}$ is a sequence in $(0,\infty)$ with (R1) and (R2). Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathrm{F}(S)\cap \mathrm{VIP}(C,A)}(u)$.

Proof. Let $f(x) = u$ for all $x \in C$. Put $\alpha'_n = \alpha_n/(1-\beta_n)$ and $\beta'_n = \beta_n$ for all $n \in \mathbb{N}$. Then we can rewrite (21) as follows:

$$\begin{cases} u_n = P_C(x_n - r_n A u_n) \\ y_n = \alpha'_n f(x_n) + (1-\alpha'_n) S u_n, \\ x_{n+1} = \beta'_n x_n + (1-\beta'_n) y_n \end{cases}$$

and so

$$\lim_{n\to\infty} \alpha'_n = 0, \quad \sum_{n=1}^{\infty} \alpha'_n = \infty \quad \text{ and } 0 < \liminf_{n\to\infty} \beta'_n \le \limsup_{n\to\infty} \beta'_n < 1.$$

Therefore, by Theorem 13, the conclusion follows. □

Remark 15. (1) As Remark 10, we obtain a strong convergence theorem for α-inverse-strongly monotone mappings, r-strongly monotone mappings and relaxed (γ, r)-cocoercive and μ-Lipschitzian mappings with $\gamma\mu^2 \le r$.

(2) Some weak and strong convergence theorems for monotone Lipschitzian mappings were established by several authors [8, 14, 15, 23, 28, 29, 30]. However, there is a monotone continuous mapping which is not Lipschitzian. Let $H = \mathbb{R}$ with the usual inner product and usual norm and let $C = [0,1]$. Define $A : C \to C$ by $Ax = 1 - (1 - x^{2/3})^{3/2}$ for all $x \in C$. Then A is not Lipschitzian (see [18]). But A is monotonically increasing, so

$$\langle Ax - Ay, x - y\rangle = (Ax - Ay)(x-y) \ge 0 \quad \forall x, y \in C.$$

Hence A is monotone and continuous with $0 \in \mathrm{VIP}(C,A) \ne \varnothing$. Therefore, Theorem 13 provides a new convergence theorem for a wider class of mappings.

4.2 Strong convergence theorem for pseudocontractive mappings

A mapping $T : C \to C$ is called pseudocontractive if

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2 \quad \forall x, y \in C$$

and T is called strictly pseudocontractive if there exists k with $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2 \quad \forall x, y \in C.$$

If $k = 0$, then T is nonexpansive. It is obvious that any strictly pseudocontractive mapping is pseudocontractive, but the converse is not true ([18]). Moreover, if T is a pseudocontractive mapping, then $A := I - T$ is monotone and $\mathrm{VIP}(C, A) = F(T)$. Actually, we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \le \|x - y\|^2 + \|Ax - Ay\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle Ax - Ay, x - y\rangle.$$

Hence

$$\langle Ax - Ay, x - y\rangle \ge 0.$$

Using Corollary 14, we have the following theorem.

Theorem 16. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a continuous pseudocontractive mapping of C into itself such that* $\mathrm{F}(T) \neq \varnothing$. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = u \in C$ and*

$$\begin{cases} u_n = P_C(x_n - u_n + Tu_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n u_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ *with* (C1), (C2) *and* (B). *Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_{\mathrm{F}(T)}(u)$.*

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6 References

[1] E. Blum, W. Oettli, From optimization and varitinal inequalities to equilibrium problems, *Math. Student.* **63** (1994) 123–145.

[2] C. E. Chidume, C. O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, *J. Math. Anal. Appl.* **318** (2006) 288–295.

[3] C. E. Chidume, A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, *Proc. Amer. Math. Soc.* **129** (2001) 2359–2363.

[4] P. L. Combettes, S. A. Hirstoaga, Equilibrium progamming in Hilbert spaces, *J. Nonlinear Convex Anal.* **6** (2005) 117–136.

[5] S. D. Flam, A. S. Antipin, Equilibrium progamming using proximal-link algolithms, *Math. Program.* **78** (1997) 29–41.

[6] A. Genel, J. Lindenstrauss, An example concerning fixed points, *Isael. J. Math.* **22** (1975) 81–86.

[7] B. Halpren, Fixed points of nonexpansive maps, *Bull. Amer. Math. Soc.* **73** (1967) 957–961.

[8] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strong monotone mappings, *Nonlinear Anal.* **61** (2005) 341–350.

[9] W. A. Kirk, Fixed point theorem for mappings which do not increase distance, *Amer. Math. Monthly.* **72** (1965) 1004–1006.

[10] P.-L. Lions, Approximation de points fixes de contractions, *C. R. Acad. Sci. Paris Ser. A-B.* **284** (1977) A1357–A1359.

[11] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953) 506–510.

[12] A. Moudafi, Viscosity approximation methods for fixed problems, *J. Math. Anal. Appl.* **241** (2000) 46–55.

[13] L. D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constraint equilibria, *Nonlinear Anal.* **18** (1992) 1159–1166.

[14] N. Nadezhkina, W. Takahashi, Weak convergence theorems by extragradient method for nonexpansive mapping and monotone mappings, *J. Optim. Theory Appl.* **128** (2006) 191–201.

[15] M. A. Noor and Z. Huang, Three-step methods for nonexpansive mapping and variational inequalities. *Appl. Math. Comput.* **190** (2007) 1312–1321.

[16] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967) 591-597.

[17] S. Reich, Weak convergence theorems for nonexpansive mappings, *J. Math. Anal. Appl.* **67** (1979) 274–276.

[18] B. E. Rhoades, Comments on two fixed point iterative methods, *J. Math. Anal. Appl.* **56** (1976) 741–750.

[19] T. Suzuki, Strong convergence theorems for infinite of nonexpansive mappings in general Banach spaces, *Fixed Point Theory and Appl.* **2005** (2005) 103–123.

[20] T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* **135** (2007) 99–106.

[21] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.* **331** (2007) 506–515.

[22] W. Takahashi, Nonlinear functional analysis. Yokohama Publishers, Yokohama, 2000.

[23] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* **118** (2003) 417–428.

[24] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* **58** (1992) 486–491.

[25] H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.* **65** (2002) 109–113.

[26] H. K. Xu, Iterative algorthms for nonlinear operators, *J. London. Math. Soc.* **66** (2002) 240–256.

[27] Y. Yao, R. Chen, H. Y. Zhou, Strong convergence and control condition of modified Halpern iterations in Banach spaces, *Int. J. Math. Math. Sci.* 2006 Article ID 29728, page 1–10. (2006) doi:10.1155/ijmms 2006.29728.

[28] Y. Yao, Y. C. Liou, J. C. Yao, An extragradient method for fixed point problems and variational inequality problems, *J. Inequal. Appl.* (to appear).

[29] Y. Yao, J. C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* **186** (2007) 1551–1558.

[30] L. C. Zeng, J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.* **10** (2006) 1293–1303.

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