รายงานวิจัยฉบับสมบูรณ์

โครงการ ดูอัลลิตีสำหรับโคชีวาไรตีที่ก่อกำเนิดโดยพีชคณิตอันดับพริมอล

โดย นางสาวรัตนา ศรีทัศน์

มิถุนายน 2558

สัญญาเลขที่ MRG5680113

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยศิลปากร

กิตติกรรมประกาศ

ดิฉันขอขอบพระคุณทางสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยศิลปากรเป็นอย่างสูง ที่ให้ทุนวิจัยแก่ดิฉันตามสัญญาเลขที่ MRG5680113 ขอบพระคุณ รศ. ดร. รณสรรพย์ ชินรัมย์ Prof. Dr. B. A. Davey และ Dr. Jane Pitkethly นักวิจัยพี่เลี้ยงของดิฉันเป็นอย่างสูงที่ให้ข้อมูลและคำแนะนำอันมีค่ายิ่งในการทำงานวิจัย รวมทั้งเป็นกำลังใจ และคอยสนับสนุนดิฉันตลอด 2 ปีที่ผ่านมา ถ้าปราศจากท่านเหล่านี้ ดิฉันคงไม่สามารถทำงานวิจัยชิ้นนี้ให้สำเร็จลุล่วงไปได้ด้วยดี

นอกจากนี้ดิฉันขอบคุณอาจารย์ทุกท่านในภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากรที่เป็นกำลังใจและช่วยเหลือดิฉันในด้านต่างๆ ด้วยดีเสมอมา

รัตนา ศรีทัศน์

บทคัดย่อ

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ชื่อโครงการ: ดูอัลลิตีสำหรับโคชีวาไรตีที่ก่อกำเนิดโดยพีชคณิตอันดับพริมอล

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ให้ $\mathbf{P}$ เป็นโครงสร้างเชิงความสัมพันธ์แบบจำกัด ซึ่งมีโพลีมอร์ฟิซึมแบบเนีย-อันนิมิตีระดับ $(k+1)$ แล้วทฤษฎีดูอัลลิตีบอกเราได้ว่า พีชคณิต $\underline{\mathbf{P}} = (P;\, \mathrm{Clo}\, \mathbf{P})$ ซึ่งตัวดำเนินการของมันเป็นโพลีมอร์ฟิซึมของ $\mathbf{P}$ เป็นพีชคณิตแบบดูไลซาเบิล โดยที่อัลเตอร์ อีโกสามารถสร้างจากออบทรัคชันซึ่งถูกแนะนำโดยซาโดริ เราแสดงว่าในกรณีที่ $\mathbf{P}$ เป็นเซตอันดับ (และดังนั้น $\underline{\mathbf{P}}$ เป็นพีชคณิตอันดับพริมอล) ดูอัลลิตีที่เราได้จากการศึกษาข้างต้นจะเป็นดูอัลลิตีแบบเข้ม เฟนซ์เป็นเซตอันดับซึ่งแผนภาพของอันดับเป็นทางเดินสลับขึ้นลง เราแสดงได้ว่าดูอัลลิตีที่ได้จะเป็นดูอับลิตีแบบเหมาะที่สุด

พิจารณาเฟนซ์จำกัด $\mathbf{P}$ เราแสดงได้ว่ากึ่งกรุป $DT(\mathbf{P})$ ของฟังก์ชันอันดับในตัวแบบลดเป็นทั้งกึ่งกรุปปกติและกึ่งกรุปปกติบริบูรณ์ ให้ $OT(\mathbf{P})$ เป็นกึ่งกรุปของฟังก์ชันยืนยงอันดับในตัว เราได้การจำแนกการเป็นกึ่งกรุปปกติบริบูรณ์สำหรับ $OT(\mathbf{P})$ นั่นคือ $OT(\mathbf{P})$ เป็นกึ่งกรุปปกติบริบูรณ์ ก็ต่อเมื่อ $|P| \leq 4$ สุดท้ายเราศึกษาเกี่ยวกับสมาชิกปกติของ $OT(\mathbf{P})$

คำหลัก: ออบทรัคชัน, เนีย-อันนิมิตี, อัลเตอร์ อีโก, ดูอัลลิตีแบบธรรมชาติ, ปกติ

Abstract

Project Code: MRG5680113

Project Title: Dualities for quasi-varieties generated by order-primal algebras

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Let $\mathbf{P}$ be a finite relational structure that admits a $(k+1)$-ary near-unanimity polymorphism. Then the NU Duality Theorem tells us that the algebra $\underline{\mathbf{P}} = (P; \mathrm{Clo}\,\mathbf{P})$, whose operations are the polymorphisms of $\mathbf{P}$, is dualisable with a dualising alter ego can be obtained by using obstructions, as introduced by Zadori. We show that, in the case that $\mathbf{P}$ is an ordered set (and therefore $\underline{\mathbf{P}}$ is an order-primal algebra), the duality that we obtain is strong. A fence is an ordered set that the order forms a path with alternating orientation. We show that if $\mathbf{P}$ is a finite fence, then our duality is optimal.

Consider a finite fence $\mathbf{P}$. The semigroup $DT(\mathbf{P})$ of all order-decreasing self-mappings is both regular and coregular. Let $OT(\mathbf{P})$ be the semigroup of all order-preserving self-mappings of $\mathbf{P}$. A characterization of regular semigroups $OT(\mathbf{P})$ is given, that is, $OT(\mathbf{P})$ is regular if and only if $|P| \leq 4$. Finally, we discuss the regularity of elements in $OT(\mathbf{P})$.

Keywords: obstruction, near-unanimity, alter ego, natural duality, regular

Output

1 ผลงานส่งตีพิมพ์ในวารสารวิชาการนานาชาติ

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1.3 R. Srithus, Dualizability of order-primal algebras, manuscript 2015

2 การนำผลงานวิจัยไปใช้ประโยชน์

- นำไปพัฒนาการเรียนการสอน โดยนำเนื้อหาบางส่วนไปใช้ในการเรียนการสอนสำหรับนักศึกษาระดับบัณฑิตศึกษา เพื่อให้นักศึกษาเห็นความเชื่อมโยงระหว่างองค์ความรู้ทางคณิตศาสตร์ที่ต่างกัน

- ใช้แนะแนวทางการทำวิจัย และนำบางส่วนของปัญหาให้นักศึกษาเลือกทำเป็นหัวข้อสัมมนาหรือวิทยานิพนธ์ในอนาคตได้

1 Executive Summary

Constraint satisfaction problems (CSPs) are mathematical problems defined as a set of objects which satisfy all given constraints. The CSP is widely studied in many branches of mathematics such as in graph theory (see [2], [20] and [31]) and universal algebra (see [5], [17] and [26]).

Jeavons, Cohen and Pearson [19] showed that a CSP instance can be regarded as a pair of relational structures, and the solutions to the problem are the structure preserving maps between these two relational structures. A CSP is said to be *tractable* if it is solvable in polynomial time. For a fixed relational structure **P**, the problem CSP(**P**) with input **H** is the decision problem consisting of deciding whether there exists a homomorphism from **H** to **P**. The CSP(**P**) is said to be ***P**-colouring* if a relational structure **P** is a digraph. Graph colorability is actively studied in graph theory (see [1] and [4]).

For $m \geq 3$, a function $f : P^m \to P$ is called a *near-unanimity function* if for all $x, y \in P$,

$$f(x, x, x, \ldots, x, y) = f(x, x, x, \ldots, x, y, x) = \cdots = f(y, x, x, \ldots, x) = x.$$

A 3-ary near-unanimity function is called a *majority* function. The study of relational structures admitting a near-unanimity function plays an important role to describe subclasses of CSP that are tractable. It is proved in [18] that if **P** is a relational structure admitting a near-unanimity function, then the corresponding CSP(**P**) is tractable. Later, Zhuk [38] considered the **NUF-Problem** that consists of deciding whether a given finite set R of relations admits a near-unanimity function. He proved that **NUF-Problem** is decidable, that is, there exists an effective method to determine the existence of a near-unanimity function in R. Davey proved in [8] that if an algebra has a near-unanimity term operation, then it admits a natural duality. Therefore, **NUF-Problem** is linked to the problem of deciding whether a finite algebra admits a natural duality.

A *relational structure* is a set equipped with relations. Ordered sets and graphs are examples of relational structures. In basic terminology of clone theory, for any finite algebra $\underline{P}$, there is a relational structure **P** such that an n-ary operation on P is a term functions of $\underline{P}$ if and only if it is a homomorphism from $\mathbf{P}^n$ to **P**. It follows that if a finite algebra $\underline{P}$ admits a near-unanimity term, then the corresponding relational structure also admits a near-unanimity function. Such relational structures are well studied. In 1997, Zádori [36] characterized a relational structure admitting an n-ary near-unanimity function by its obstructions that is a generalization of a remark by Tardos in [12]. His characterization use the number of colored elements in every **P**-obstruction to decide whether **P** admits a near-unanimity function.

As in the literature on duality theory, we have a little knowledge about dualising alter egos of algebras admitting a near-unanimity term. Our aim is to produce simple and useful alter egos for such algebras. In this project, we study dualizable algebras admitting a near-unanimity function by the corresponding relational structures. It will be shown that dualising alter egos for $\underline{P}$ can be produced possessing a near-unanimity term by obstructions.

Next, we give the definitions that are introduced by Zádori [36].

Let $\mathbf{Q} = (Q; (r^s_Q)_{s\in S})$ and $\mathbf{P} = (P; (r^s_P)_{s\in S})$ be relational structures. A map $f : Q \to P$ is called a *morphism* from $\mathbf{Q}$ into $\mathbf{P}$, written by $f : \mathbf{Q} \to \mathbf{P}$ if f preserves each relation of $\mathbf{Q}$, i.e., if $(a_t)_{t\in T} \in r^s_Q$, then $(f(a_t))_{t\in T} \in r^s_P$ for each $s \in S$. A relational structure $\mathbf{Q} = (Q; (r^s_Q)_{s\in S})$ is a *subrelational structure* of $\mathbf{P} = (P; (r^s_P)_{s\in S})$ if $Q \subseteq P$ and $r^s_Q = r^s_P|Q$. If $r^s_Q \subseteq r^s_P|Q$, then we say that $\mathbf{Q}$ is *contained* in $\mathbf{P}$ and written by $\mathbf{Q} \subseteq \mathbf{P}$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be relational structures of the same type. A pair $(\mathbf{Q}; f)$ is called a ***P**-colored relational structure* if f is a map from a subset of Q to P. If f can be extended to a morphism $f : \mathbf{Q} \to \mathbf{P}$, then f and $(\mathbf{Q}; f)$ is called ***P**-extendible*, otherwise, f and $(\mathbf{Q}; f)$ is called ***P**-nonextendible*.

Given $\mathbf{P}$-colored relational structures $(\mathbf{H}; f)$ and $(\mathbf{Q}; g)$, we say that $(\mathbf{H}; f)$ is *contained* in $(\mathbf{Q}; g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f \subseteq g$. If $(\mathbf{H}; f)$ is contained in $(\mathbf{Q}; g)$ we write $(\mathbf{H}; f) \subseteq (\mathbf{Q}; g)$.

A finite $\mathbf{P}$-colored relational structure $(\mathbf{H}; f)$ is called a ***P**-obstruction* if $(\mathbf{H}; f)$ is $\mathbf{P}$-nonextendible and every $\mathbf{P}$-colored relational structure $(\mathbf{K}; g)$ properly contained in $(\mathbf{H}; f)$ is $\mathbf{P}$-extendible. Roughly speaking, every $\mathbf{P}$-obstruction is a finite minimal $\mathbf{P}$-nonextendible $\mathbf{P}$-colored relational structure. If a relational structure $\mathbf{P}$ is an ordered set, then $\mathbf{P}$-obstructions are called ***P**-zigzags*, see Zadori [34].

It is known that for any finite algebra $\underline{P}$, there is a relational structure $\mathbf{P} = (P; (r^s_P)_{s\in S})$ such that the clone of term operations of $\underline{P}$ is $\mathrm{Pol}(\{r^s_P \mid s \in S\})$. A finite nontrivial algebra $\underline{P}$ is said to be *order-primal* if a corresponding relational structure $\mathbf{P}$ is an ordered set. Algebraic properties of order-primal algebras are investigated by many mathematicians. There have been many research works studying order-primal algebras.

A natural duality is a special kind of dual category equivalence between a quasi-variety of algebras generated by a finite algebra and a category of structured topological spaces. For complete explanation to natural dualities, dualisability, strong dualities and optimal dualities, see Clark and Davey [6].

In the theory of natural dualities, one start with a fixed finite algebra $\underline{P} = (P; F)$ and consider *alter egos* $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ of $\underline{P}$ where G is a set of total operations, H is a set of partial operations, R is a set of finitary relations and τ is the discrete topology on P. For any alter egos $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$, $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{P}})$ is the class of all isomorphic copies of topologically closed substructures of direct power indexed over non-empty index sets of

$\underset{\sim}{\mathbf{P}}$. The quasi-variety $\mathbb{ISP}(\underline{P})$ generated by an algebra $\underline{P}$ is the class of all isomorphic copies of subalgebras of direct powers of the algebra $\underline{P}$. The aim is to find an alter ego of the algebra $\underline{P}$ such that the category $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{P}})$ is dually equivalent to the quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{P})$ via the contravariant hom-functors

$$D(-) := \mathcal{A}(-, \underline{P}) : \mathcal{A} \to \mathcal{X} \text{ and } E(-) := \mathcal{X}(-, \underset{\sim}{\mathbf{P}}) : \mathcal{X} \to \mathcal{A}.$$

In this case, we say that $\underset{\sim}{\mathbf{P}}$ *yields a (natural) duality on* $\mathcal{A}$ or $\underline{P}$ *admits a (natural) duality* (or is *dualizable*).

We say that the set $G \cup H \cup R$ of a structure $\underset{\sim}{\mathbf{P}}$ *entails* a finitary algebraic relation s on $D(\underline{A})$, where $\underline{A} \in \mathcal{A}$, if every continuous map $\varphi : D(\underline{A}) \to \underset{\sim}{\mathbf{P}}$ which preserves the operations, partial operations and relations in $G \cup H \cup R$ also preserves s. The set $G \cup H \cup R$ is said to *entail* s if $G \cup H \cup R$ entails s on $D(\underline{A})$ for all $\underline{A} \in \mathcal{A}$. Clearly, if $G \cup H \cup R$ yields a duality on $\mathcal{A}$ and $G \cup H \cup R$ entails some $s \in R$, then the smaller set $G \cup H \cup R \setminus \{s\}$ also yields a duality on $\mathcal{A}$. If $G \cup H \cup R$ is minimal with respect to yielding a duality on $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{P}}$ yields *an optimal duality* on $\mathcal{A}$.

In [11], Davey and Werner showed that if $\underline{P}$ has a $(k+1)$-ary near-unanimity term, then $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ yields a duality on $\mathcal{A}$.

Consider the 6-element ordered set $\mathbf{T}$ shown in Figure 1. It is known

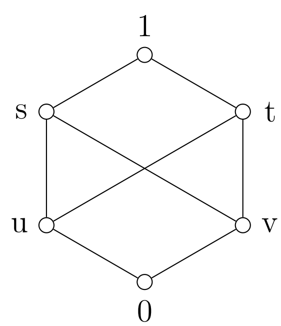

Figure 1: The 6-element ordered set $\mathbf{T}$

that $\mathbf{T}$ has a 5-ary order-preserving near-unanimity function and so the corresponding order-primal algebra $\underline{T}$ admits a duality. Davey, Quackenbush and Schweigert [13] showed that the alter ego $\underset{\sim}{\mathbf{T}}$ consisting only of the 4-ary relation

$$\rho = \{(a, b, c, d) \in T^4 \mid \exists e \in T\, (a, b \leq e \leq c, d)\}$$

yields a duality on $\underline{T}$.

In 1995, Davey, Heindorf and Mckenzie have provided a useful characterization of finite order-primal algebras admitting a duality [12]. They proved that a finite order-primal algebra $\underline{P}$ admits a duality if and only if $\underline{P}$ has a near-unanimity term. Consequently, for every dualizable order-primal algebra $\underline{P}$, the alter ego $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$, for some natural number k. The structure $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ is an important theoretical tool, but in practice we try to make the structure on $\underset{\sim}{\mathbf{P}}$ as simple as possible, for example, the structure of the alter ego $\underset{\sim}{\mathbf{T}}$ and the structure on Priestley's duality for distributive lattices is given by only a single relation and two constants.

Even though we know which order-primal algebras $\underline{P}$ admit a natural duality, we have a little knowledge about alter egos which generate the categories $\mathcal{X}$ of topological structures dual to the quasi-variety generated by $\underline{P}$. Our interest is to find such alter egos for any dualizable order-primal algebra, and in general, for any algebra with a near-unanimity term.

In [34], [35] and [36] Zádori showed that obstructions are a powerful tool in the study of relational structures, especially ordered sets, admitting a near-unanimity function. The previous results showed that near-unanimity terms play an important role to study dualizable algebras. So, we are interested in the use of obstructions to construct dualising alter egos for algebras with a near-unanimity term.

The results in Section 2.1 give the alter egos $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ to yield a duality on $\mathbb{ISP}(\underline{P})$ where $\underline{P}$ is an algebra admitting a near-unanimity term.

As we mentioned, an algebra $\underline{P}$ is order-primal if a corresponding relational structure $\mathbf{P}$ is an ordered set. In this project, we focus on an order-primal algebra $\underline{P}$ corresponding to a connected ordered set $(P; \leq)$. We shall show that, in the case of a connected ordered set, the duality from Section 2.1 is strong.

A *fence* $\mathbf{X}$ is an ordered set $(X; \leq)$ in which either

$$a_1 < a_2 > a_3, \dots, a_{2m-1} > a_{2m} < a_{2m+1}, \dots$$

or

$$a_1 > a_2 < a_3, \dots, a_{2m-1} < a_{2m} > a_{2m+1}, \dots$$

are the only comparability relations where $X = \{a_1, a_2, \dots, a_n, \dots\}$. Every element in $\mathbf{X}$ is minimal or maximal. If $a_1 < a_2$, then $\mathbf{X}$ is called an *up fence* and it is called a *down fence* if $a_1 > a_2$.

Now, we focus on the order-primal algebra $\underline{P}$ corresponding to a finite fence $\mathbf{P}$ and show that alter egos of order-primal algebras corresponding to a finite fence are optimal.

Algebraic properties of order-preserving self-mappings of fences have been long considered. Demetrovics and Rónyai [14] studied the clones of all order-preserving operations for fences. In [32], Rutkowski gave the formula for the number of order-preserving self-mappings of a fence. Later, Farley [16] computed the number of order-preserving self-mappings of a fence.

The semigroup of all order-preserving self-mappings of an ordered set $\mathbf{X}$ have been widely investigated. In [23], Gluskin showed that if $OT(\mathbf{X})$ is isomorphic to $OT(\mathbf{Y})$, then the ordered sets $\mathbf{X}$ and $\mathbf{Y}$ are isomorphic or anti-isomorphic. Higgins, Mitchell and Ruškǔc [22] found that the rank of the semigroup $T(X)$ is related to the semigroups $OT(\mathbf{X})$ for some chains $\mathbf{X} = (X; \leq)$. These results show that the order-preserving self-mappings semigroup can be used to determine the structure of the initial algebraic system. Such semigroups play an important role in the study of algebraic systems.

In Semigroup Theory the concept of regularity is one of the most-studied topics. There have been many research works studying regularity of semigroups, especially endomorphism semigroups of algebraic structures

In this project, our main purpose is to investigate the regularity of the semigroup of order-preserving self-mappings and the semigroup of order-decreasing self-mappings of a fence. Throughout we use $\operatorname{ran}\alpha$ to denote the range of a mapping α and the n^{th} composition of α is denoted by α^n.

2 Main Results

2.1 Alter egos of algebras with a near-unanimity term

The main idea of this project is to use obstructions to define relations for alter egos. Before doing it, we need to mention some basic concepts involving colored relational structures.

For a $\mathbf{P}$-colored relational structure $(\mathbf{H}; f)$, we define $C(\mathbf{H}; f) = \{x \in H \mid f(x) \text{ exists }\}$, i.e., $C(\mathbf{H}; f)$ is the domain of f and $N(\mathbf{H}; f) = N \setminus C(\mathbf{H}; f)$. The elements of $C(\mathbf{H}; f)$ and $N(\mathbf{H}; f)$ are called *colored elements* and *noncolored elements*, respectively.

For any relational structure $\mathbf{P}$, we shall define relations induced by their $\mathbf{P}$-colored relational structures on any relational structure of the same type as $\mathbf{P}$ as follows.

Definition 2.1. Let $(\mathbf{H}; f)$ be a finite $\mathbf{P}$-colored relational structure. Assume that $C(\mathbf{H}; f) = \{x_1, \dots, x_n\}$. We define the n-ary relation r_H^A on a relational structure $\mathbf{A}$ of the same type as $\mathbf{P}$ as follows:

$$r_H^A = \{\, (g(x_1), \dots, g(x_n)) \mid g : \mathbf{H} \to \mathbf{A} \text{ is a morphism}\,\}.$$

Consider an ordered set $\mathbf{P}$. We can draw a picture of a $\mathbf{P}$-colored ordered set $(\mathbf{H}; f)$. A picture consists of the covering graph of $\mathbf{H}$ and an element of $\mathbf{H}$

is drawn as a small shaded circle if f is defined on the given point. Otherwise, it is drawn as a small empty circle. Every shaded point is labelled by the value of f.

Example 2.2. Let $\mathbf{P}$ be the ordered set shown in Figure 2(a). Consider the $\mathbf{P}$-colored ordered set $(\mathbf{H}; f)$ as shown in Figure 2(b). The relation r_H^A

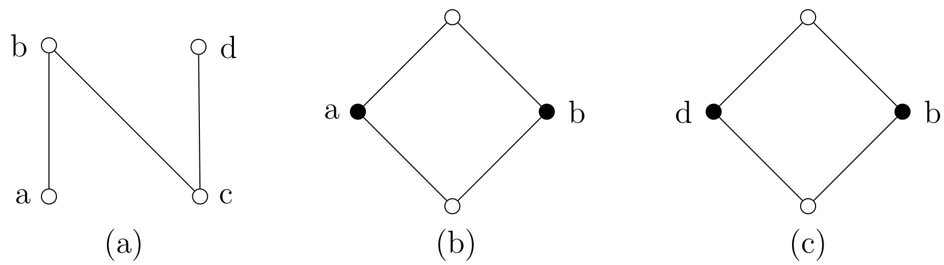

Figure 2: Ordered set $\mathbf{P}$ and $\mathbf{P}$-colored ordered sets $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$

induced by $(\mathbf{H}; f)$ is a binary relation satisfying the following condition:

$$(x, y) \in r_H^A \Leftrightarrow \text{ there exists } u, v \in A \text{ with } x \leq u \geq y \text{ and } x \geq v \leq y$$

for any ordered set $\mathbf{A}$. It is easy to see that the relation r_K^A induced by $(\mathbf{K}; g)$ as shown in Figure 2(c) is the same relation as r_H^A, although $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ are different $\mathbf{P}$-colored relational structures since $f \neq g$.

It can be proved that different $\mathbf{P}$-colored relational structures can give us the same induced relation. The relation $\sim_{\mathbf{P}}$ is defined for $\mathbf{P}$-colored relational structures producing the same induced relation.

Definition 2.3. Let $\mathbf{P}$ be a relational structure. We define the relation $\sim_{\mathbf{P}}$ on the set $\mathbf{PC}$ of all finite $\mathbf{P}$-colored relational structures as follows: $(\mathbf{H}; f) \sim_{\mathbf{P}} (\mathbf{K}; g)$ if and only if there is an isomorphism $\psi : \mathbf{H} \to \mathbf{K}$ such that $\psi|_{C(\mathbf{H};f)}$ is a bijection from $C(\mathbf{H}; f)$ onto $C(\mathbf{K}; g)$.

In 1997, Zádori characterized a relational structure admitting an n-ary near-unanimity function by its obstructions that is a generalization of a remark by Tardos in [33] as stated below.

Theorem 2.4. [36, Theorem 1.17] *Let $\boldsymbol{P}$ be a finite relational structure. Then $\boldsymbol{P}$ admits an n-ary near-unanimity function if and only if the number of colored elements in every $\boldsymbol{P}$-obstruction is at most $n - 1$.*

It follows that the number of relations r_H^A induced by its obstruction is finite. Consider a topological structure $\underset{\sim}{\mathbf{P}}$ having no partial operations and only finitely many relations. Then $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$ if $\underset{\sim}{\mathbf{P}}$ satisfies the interpolation condition (IC) with respect to $\underline{P}$ as stated below.

Theorem 2.5. [6, The Second Duality Theorem] *Assume that* $\underset{\sim}{\mathbf{P}} = (P; G, R, \tau)$ *and assume that* R *is finite. If* $\underset{\sim}{\mathbf{P}}$ *satisfies the condition:*

(IC) *for each* $n \in \mathbb{N}$ *and each substructure* X *of* P^n*, every morphism* $\alpha : X \to P$ *extends to a term function* $t : P^n \to P$ *of the algebra* $\underline{P}$*,*

then $\underset{\sim}{\mathbf{P}}$ *yields a duality on* $\mathcal{A}$ *and* $\underset{\sim}{\mathbf{P}}$ *is injective in* $\mathcal{X}$*.*

Now, we give a general theorem for constructing simple and useful alter egos of algebras admitting a near-unanimity term.

Theorem 2.6. *Let* $\underline{P}$ *be an algebra corresponding to a relational structure* $\boldsymbol{P}$ *and* Q *be a transversal of the* $\sim_{\boldsymbol{P}}$*-blocks that contain a* $\boldsymbol{P}$*-obstruction. Assume that* $\underline{P}$ *admits an* n*-ary near-unanimity term and* $R = \{r_H^P | (H; f) \in Q\}$*. Then the following conditions hold:*

(i) every relation $r \in R$ *is a* k*-ary relation for some* $k \leq n - 1$*,*

(ii) the structure $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ *yields a duality on* $\mathcal{A} = \mathbb{ISP}(\underline{P})$*.*

Since we need the structure on $\underset{\sim}{\mathbf{P}}$ that is as simple as possible, one can ask whether our structure is optimal, that is, if any relation in the structure were deleted, the duality would be destroyed. Unfortunately, it is not optimal. Our aim is to find a method for checking whether a duality of any finite algebra admitting a near-unanimity term which is constructed from our theorem is optimal. Initially, we need the following definition.

Definition 2.7. Let $\mathbf{P}$ be a relational structure. We define the relation $\leq_{\mathbf{PC}}$ on $\mathbf{PC}$ as follows:

$(\mathbf{H}; f) \leq_{\mathbf{PC}} (\mathbf{K}; g)$ if and only if $(\mathbf{H}; f)$ is a color-preserving retract of $(\mathbf{K}; g)$, that is, there are morphisms $\psi : \mathbf{K} \to \mathbf{H}$ and $\varphi : \mathbf{H} \to \mathbf{K}$, both of which map colored elements to colored elements, such that $\psi \circ \varphi = \mathrm{id}_H$.

It is easy to see that $\leq_{\mathbf{PC}}$ is reflexive and transitive, but not antisymmetric. The following proposition shows a relation between the relations induced by $\mathbf{P}$-colored relational structures which are related under $\leq_{\mathbf{PC}}$.

Proposition 2.8. *Let* $(\boldsymbol{H}; f)$ *and* $(\boldsymbol{K}; g)$ *be finite* $\boldsymbol{P}$*-colored relational structures and let* $\boldsymbol{A}$ *be a relational structure of the same type of* $\boldsymbol{P}$*. Assume that* $C(\boldsymbol{H}; f) = \{x_1, \ldots, x_m\}$ *and* $C(\boldsymbol{K}; g) = \{y_1, \ldots, y_n\}$*. If* $(\boldsymbol{H}; f) \leq_{\boldsymbol{PC}} (\boldsymbol{K}; g)$ *vis a color-preserving retraction* ψ*, then*

$$(a_1, \ldots, a_m) \in r_H^A \text{ if and only if } (a_{\overline{\psi}(1)}, \ldots, a_{\overline{\psi}(n)}) \in r_K^A$$

where $\overline{\psi} : \{1, \ldots, n\} \to \{1, \ldots, m\}$ *defined by* $\overline{\psi}(i) = j$ *if* $\psi(y_i) = x_j$*, and hence* r_H^A *is definable from* r_K^A*.*

Proposition 2.8 implies that every relation in the alter ego $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ in Theorem 2.6 which is $\leq_{\mathbf{PC}}$-related to some relations in R can be deleted from the structure of $\underset{\sim}{\mathbf{P}}$ without destroying the duality. This gives us a sufficient condition for a structure $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ to be non-optimal as stated in the theorem below.

Theorem 2.9. *Let $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ be an alter ego in Theorem 2.6. If R contains two different relations induced from $\boldsymbol{P}$-colored relational structures which are related under $\leq_{\boldsymbol{PC}}$, then $\underset{\sim}{\mathbf{P}}$ is not optimal.*

2.2 Dualizable order-primal algebras and their properties

The results in Section 2.1 give the alter egos $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ to yield a duality on $\mathbb{ISP}(\underline{P})$ where $\underline{P}$ is an algebra admitting a near-unanimity term.

As we mentioned in Section 1, an algebra $\underline{P}$ is order-primal if a corresponding relational structure $\mathbf{P}$ is an ordered set. In this section, we focus on an order-primal algebra $\underline{P}$ corresponding to a connected ordered set $(P; \leq)$. We shall show that, in the case of a connected ordered set, the duality from Section 2.1 is strong.

To do so we need a basic result concerning upgrading dualities to strong dualities. Define

$$\mathrm{Irr}(\underline{P}) = \max\{\mathrm{irr}(\underline{Q}) \mid \underline{Q} \text{ is a subalgebra of } \underline{P}\},$$

where $\mathrm{irr}(\underline{Q})$ is the least n such that the zero congruence $\mathbf{0}^{\underline{Q}}$ on $\underline{Q}$ is a meet of n meet-irreducible congruence. Note that $\underline{Q}$ is subdirectly irreducible if and only if $\mathrm{irr}(\underline{Q}) = 1$.

Theorem 2.10. [6, Theorem 3.3.7] *Assume that $\underline{P}$ generates a congruence-distributive variety and that $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ yields a duality on $\mathcal{A}$. If $\underset{\sim}{\mathbf{P}}'$ is obtained from $\underset{\sim}{\mathbf{P}}$ by adding to $G \cup H$ all n-ary algebraic partial operations where $0 \leq n \leq \mathrm{Irr}(\underline{P})$, then $\underset{\sim}{\mathbf{P}}'$ yields a strong duality on $\mathcal{A}$.*

Davey, Quackenbush and Schweigert proved in [13] that every order-primal algebra $\underline{P}$ corresponding to the ordered set $(P; \leq)$ has no proper subalgebra and if $(P; \leq)$ is connected, then $\underline{P}$ is simple and hence, $\underline{P}$ is subdirectly irreducible. It follows that $\mathrm{Irr}(\underline{P}) = 1$. Using this result, we can prove the following theorem.

Theorem 2.11. *Let $k \geq 2$ and $\underline{P}$ be an order-primal algebra corresponding to a connected ordered set $(P; \leq)$. If $\underline{P}$ has a $(k+1)$-ary near-unanimity term, then $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ in Theorem 2.6 yields a strong duality.*

As we mentioned in Section 2.1, the set R in the structure $\underset{\sim}{\mathbf{P}}$ is produced by obstructions or zigzags if $\underline{P}$ is order-primal. To construct $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$, we need to know all zigzags of the corresponding ordered set. Next, we give concrete examples of alter egos which are produced by our methods for special order-primal algebras.

Consider an ordered set $(P; \leq)$, a $\mathbf{P}$-colored ordered set $(\mathbf{H}; f)$ is called *monotone* if f is a monotone map on its domain, otherwise $(\mathbf{H}; f)$ is *nonmonotone*. Zádori showed [35] that the $\mathbf{P}$-colored 2-element chain in which the top is colored by a and the bottom is colored by b where $b \not\leq a$, is a nonmonotone $\mathbf{P}$-zigzag and every nonmonotone $\mathbf{P}$-zigzag is of this form. Moreover, he proved that if $(P; \leq)$ is a complete lattice, then $\mathbf{P}$ has no monotone $\mathbf{P}$-zigzags. In particular, no finite lattice possesses monotone zigzags.

Example 2.12. Let $\underline{P}$ be an order-primal algebra corresponding to a finite lattice $(L; \leq)$. Then by Zádori's result, $(L; \leq)$ has no monotone $\mathbf{L}$-zigzags and hence every $\mathbf{L}$-zigzag $(\mathbf{H}; f)$ is of the following form. The corresponding

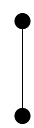

Figure 3: The form of $\mathbf{L}$-zigzags

relation r_H^P of $(\mathbf{H}; f)$ is the set $\{(a, b) \in L^2 \mid a \leq b\}$, that is, $r_H^L = \leq$. It follows that $R = \{\leq\}$ and by Theorem 2.6, the structure $\underset{\sim}{\mathbf{P}} = (L; \leq, \tau)$ yields a duality on $\mathcal{A}$. See Davey and Rival [10, Theorem 1.1] where this result is proved directly.

Now, we focus on the order-primal algebra $\underline{P}$ corresponding to a finite fence $\mathbf{P}$. Consider a transversal Q of the $\sim_{\mathbf{P}}$-blocks that contain a $\mathbf{P}$-obstruction of a finite fence $\mathbf{P}$, we easily find that

$$Q = \Big\{ (\mathbf{H}; f) \mid C(\mathbf{H}; f) = \{a, b\} \text{ and } \mathbf{H} \text{ is a subfence of } \mathbf{P} \text{ or } \mathbf{H} = \mathbf{P}^{\partial} \Big\},$$

where a and b are the endpoints of $\mathbf{H}$. So, $R = \{r_H^P \subseteq P^2 | (\mathbf{H}; f) \in Q\}$. By the use of the Test Algebra Lemma (see [6, Theorem 8.1.3]), we can prove that the structure $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ yields an optimal duality on $\mathcal{A}$ as follows.

Lemma 2.13. *Let $(\boldsymbol{H}; f)$ and $(\boldsymbol{K}; g)$ be $\boldsymbol{F}_n$-colored ordered sets such that $\boldsymbol{H}$ and $\boldsymbol{K}$ are subfences of $\boldsymbol{F}_n$ and the colored elements of both are exactly the endpoints of $\boldsymbol{H}$ and $\boldsymbol{K}$, respectively. Assume that $r = r_K^{\boldsymbol{F}_n}$ and $s = s_H^{\boldsymbol{F}_n}$.*

(i) If $\boldsymbol{H}$ is (up to isomorphism) a subfence of $\boldsymbol{K}$, then s and s^{-1} are subsets of r.

(ii) If $\boldsymbol{H}$ is (up to isomorphism) the dual $\boldsymbol{K}^\partial$ with $|H|$ is even, then $(\boldsymbol{H}; f) \sim_{\boldsymbol{P}} (\boldsymbol{K}; g)$, and hence s is definable from r and vise versa.

The results of Lemma 2.13 also hold true for $\mathbf{G}_n$.

Theorem 2.14. *Let $\underline{P}$ be an order-primal algebra corresponding to an n-element fence $\boldsymbol{P}$. Then $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ in Theorem 2.6 yields an optimal duality on $\mathcal{A}$.*

2.3 Regular subsemigroup $DT(\mathbf{X})$ of $T(X)$

For any ordered set $\mathbf{X}$ having X as the base set, the semigroup $DT(\mathbf{X})$ of all order-decresing transformations of $\mathbf{X}$ is a subsemigroup of $T(X)$. Since in general subsemigroup of $T(X)$ need not be regular, our aim is to describe fences $\mathbf{X}$ having a regular semigroup $DT(\mathbf{X})$.

In 2012, Namnak and Laysirikul characterized an ordered set $\mathbf{X}$ having a regular semigroup $DT(\mathbf{X})$ as stated below.

Theorem 2.15. [29, Theorem 2.2] *Let $\mathbf{X}$ be an ordered set. Then $DT(\mathbf{X})$ is a regular semigroup if and only if for every subchain of $\mathbf{X}$ has at most two elements.*

Since every maximal subchain of a fence is a 2-element chain, we can immediately deduce from Theorem 2.15 the following proposition.

Proposition 2.16. *For every finite fence $\mathbf{X}$, the semigroup $DT(\mathbf{X})$ is regular.*

In Section 1, we already mentioned that a coregular element is one of important cases of a regular element. Since a regular element need not be coregular, it is interested to describe fences $\mathbf{X}$ having a coregular semigroup $DT(\mathbf{X})$.

Proposition 2.17. *For every finite fence $\mathbf{X}$, the semigroup $DT(\mathbf{X})$ is always coregular.*

2.4 Regular subsemigroups $OT(\mathbf{X})$ of $T(X)$

We now investigate the regularlity for a subsemigroup $OT(\mathbf{X})$ of $T(X)$ when $\mathbf{X} = (X; \leq)$ is a finite fence. Before doing so we need a result that an order-preserving self-mapping of a fence $\mathbf{X}$ preserves subfences. An ordered set $\mathbf{P}$ is called *connected* if for all $a, b \in P$ there is a fence $\mathbf{F} \subseteq \mathbf{P}$ with endpoints a and b. It is well known that if $\mathbf{P}$ is connected and $\alpha : P \to Q$ is order-preserving, then $\alpha(\mathbf{P})$ is connected. Consequently, every order-preserving mapping maps connected sets to connected sets. Because connected subsets of a fence $\mathbf{X}$ are precisely the subfences, an order-preserving mapping $\alpha : X \to X$ maps subfences to subfences.

As we mentioned in Section 1, α is regular in $OT(\mathbf{X})$ if α is constant, that is, $|\operatorname{ran}\alpha| = 1$. Consider $|\operatorname{ran}\alpha| = 2$. Our question is that whether α is regular.

Proposition 2.18. *Let $\mathbf{X}$ be a fence with $|X| \leq \aleph_0$ and $\alpha \in OT(\mathbf{X})$. If $|\operatorname{ran}\alpha| = 2$, then α is regular.*

As a consequence of Proposition 2.18, we obtain the following corollary.

Corollary 2.19. *Let $\mathbf{X}$ be a 2-element fence. Then $OT(\mathbf{X})$ is regular.*

Next, we focus on the semigroup $OT(\mathbf{X})$ of a fence $\mathbf{X}$ having 3 or 4 elements.

Proposition 2.20. *Let $\mathbf{X}$ be a 3-element fence. Then $OT(\mathbf{X})$ is regular.*

Proposition 2.21. *Let $\mathbf{X}$ be a 4-element fence. Then $OT(\mathbf{X})$ is regular.*

In what follows, we restrict our study to the case of a mapping α in $OT(\mathbf{X})$ where $\mathbf{X}$ having 5 elements.

Example 2.22. Consider the 5-element fence $(X; \leq)$ as shown in Figure 4.

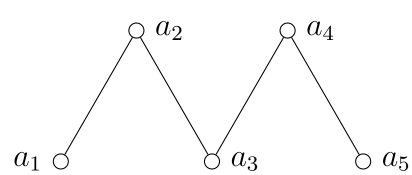

Figure 4: The 5-element fence $\mathbf{F}$

Define $\alpha : X \to X$ by

$$\alpha(x) = \begin{cases} a_3, & x = a_1, \\ a_5, & x = a_5, \\ a_4, & \text{otherwise}. \end{cases}$$

It is easy to see that α is order-preserving. Next we show that α is not regular. Suppose that α is regular. Then there is a $\beta \in OT(\mathbf{X})$ with $\alpha\beta\alpha = \alpha$, that is, $\alpha\beta\alpha(x) = \alpha(x)$ for all $x \in X$. If $x = a_1$, then $a_3 = \alpha(a_1) = \alpha\beta\alpha(a_1) = \alpha\beta(a_3)$. By the definition of α, we get that $\beta(a_3) = a_1$. Similarly, we get $\beta(a_5) = a_5$.

Consider $x = a_3$. We have $a_4 = \alpha(a_3) = \alpha\beta\alpha(a_3) = \alpha\beta(a_4)$ and so, $\beta(a_4) \notin \{a_1, a_5\}$. Since $a_3 < a_4$, so $a_1 = \beta(a_3) \leq \beta(a_4)$ implies that $\beta(a_4) \in \{a_1, a_2\}$. But $\beta(a_4) \neq a_1$, we get that $\beta(a_4) = a_2$. From $a_5 < a_4$, we have $a_5 = \beta(a_5) < \beta(a_4) = a_2$, a contradiction. Therefore α is not regular.

Example 2.22 shows that $OT(\mathbf{X})$ is not regular if $\mathbf{X}$ is a 5-element fence. It is natural to ask whether the semigroup $OT(\mathbf{X})$ of a fence $\mathbf{X}$ is not regular if $|X| \geq 5$. The answer is shown in the following proposition.

Proposition 2.23. *Let $\mathbf{X}$ be a fence with $|X| \geq 5$. Then $OT(\mathbf{X})$ is not regular.*

Theorem 2.24. *Let $\mathbf{X}$ be a finite fence. Then $OT(\mathbf{X})$ is regular if and only if $|X| \leq 4$.*

As we proved in Theorem 2.24, $OT(\mathbf{X})$ is not regular if $|X| > 4$. In this section, our goal is to investigate properties of regular elements of $OT(\mathbf{X})$.

Proposition 2.25. *Let $\alpha \in OT(\mathbf{X})$ with $|\operatorname{ran}\alpha| \geq 3$. If $|\operatorname{ran}\alpha| \geq |X| - 1$, then α is regular.*

To finish this section, we give a sufficient conditions for a mapping in $OT(\mathbf{X})$ to be not regular. It is known that the mapping α defined in Example 2.22 is not regular. The last proposition shows that every mapping in $OT(\mathbf{X})$ satisfying the same conditions as the mapping α is also not regular.

Proposition 2.26. *Let $\alpha \in OT(\mathbf{X})$ with $|\operatorname{ran}\alpha| \geq 3$ and $|\operatorname{ran}\alpha| = |X| - 2$. Assume that a and b are the initial point and endpoint of $\mathbf{X}$, respectively. If there are $u, v, w \in X \setminus \{a, b\}$ with $u < v > w$ (or $u > v < w$) and $\alpha(\{u, v, w\}) = x$ where x is neither the initial point nor endpoint of $\operatorname{ran}\alpha$, then α is not regular.*

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DUALITIES AND ALGEBRAS WITH A NEAR-UNANIMITY TERM

R. SRITHUS AND U. CHOTWATTAKAWANIT

ABSTRACT. Let $\underline{P}$ be an algebra admitting a near-unanimity term. In this paper, we produce dualising alter egos for $\underline{P}$ possessing a near-unanimity term. We give concrete alter egos for dualizable order-primal algebras and show that they yield a strong duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$. Finally, we show that alter egos of order-primal algebras corresponding to a finite fence are optimal.

1. Introduction

Constraint satisfaction problems (CSPs) are mathematical problems defined as a set of objects which satisfy all given constraints. The CSP is widely studied in many branches of mathematics such as in graph theory (see [2], [17] and [21]) and universal algebra (see [4], [14] and [18]).

Jeavons, Cohen and Pearson [16] showed that a CSP instance can be regarded as a pair of relational structures, and the solutions to the problem are the structure preserving maps between these two relational structures. A CSP is said to be *tractable* if it is solvable in polynomial time. For a fixed relational structure $\mathbf{P}$, the problem CSP($\mathbf{P}$) with input $\mathbf{H}$ is the decision problem consisting of deciding whether there exists a homomorphism from $\mathbf{H}$ to $\mathbf{P}$. The CSP($\mathbf{P}$) is said to be ***P**-colouring* if a relational structure $\mathbf{P}$ is a digraph. Graph colorability is actively studied in graph theory (see [1] and [3]).

For $m \geq 3$, a function $f : P^m \to P$ is called a *near-unanimity function* if for all $x, y \in P$,

$$f(x, x, x, \dots, x, y) = f(x, x, x, \dots, x, y, x) = \cdots = f(y, x, x, \dots, x) = x.$$

A 3-ary near-unanimity function is called a *majority* function. The study of relational structures admitting a near-unanimity function plays an important role to describe subclasses of CSP that are tractable. It is proved in [15] that if $\mathbf{P}$ is a relational structure admitting a near-unanimity function, then the corresponding CSP($\mathbf{P}$) is tractable. Later, Zhuk [26] considered the **NUF-Problem** that consists of deciding whether a given finite set R of relations admits a near-unanimity function. He proved that **NUF-Problem** is decidable, that is, there exists an effective method to determine the existence of a near-unanimity function in R. Davey proved in [6] that if an algebra has a near-unanimity term operation, then it admits a natural duality. Therefore, **NUF-Problem** is linked to the problem of deciding whether a finite algebra admits a natural duality.

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In basic terminology of clone theory, for any finite algebra $\underline{P}$, there is a relational structure $\mathbf{P}$ such that an n-ary operation on P is a term functions of $\underline{P}$ if and only if it is a homomorphism from $\mathbf{P}^n$ to $\mathbf{P}$. It follows that if a finite algebra $\underline{P}$ admits a near-unanimity term, then the corresponding relational structure also admits a near-unanimity function. Such relational structures are well studied. In 1997, Zádori [25] characterized a relational structure admitting an n-ary near-unanimity function by its obstructions that is a generalization of a remark by Tardos in [12]. His characterization use the number of colored elements in every $\mathbf{P}$-obstruction to decide whether $\mathbf{P}$ admits a near-unanimity function.

As in the literature on duality theory, we have a little knowledge about dualising alter egos of algebras admitting a near-unanimity term. Our aim is to produce simple and useful alter egos for such algebras. In this paper, we study dualizable algebras admitting a near-unanimity function by the corresponding relational structures. It will be shown that dualising alter egos for $\underline{P}$ can be produced possessing a near-unanimity term by obstructions.

2. Preliminaries

A *relational structure* is a set equipped with relations. Ordered sets and graphs are examples of relational structures. Next, we give the definitions that are introduced by Zádori [25].

Let $\mathbf{Q} = (Q; (r^s_Q)_{s\in S})$ and $\mathbf{P} = (P; (r^s_P)_{s\in S})$ be relational structures. A map $f : Q \to P$ is called a *morphism* from $\mathbf{Q}$ into $\mathbf{P}$, written by $f : \mathbf{Q} \to \mathbf{P}$ if f preserves each relation of $\mathbf{Q}$, i.e., if $(a_t)_{t\in T} \in r^s_Q$, then $(f(a_t))_{t\in T} \in r^s_P$ for each $s \in S$. A relational structure $\mathbf{Q} = (Q; (r^s_Q)_{s\in S})$ is a *subrelational structure* of $\mathbf{P} = (P; (r^s_P)_{s\in S})$ if $Q \subseteq P$ and $r^s_Q = r^s_P|Q$. If $r^s_Q \subseteq r^s_P|Q$, then we say that $\mathbf{Q}$ is *contained* in $\mathbf{P}$ and written by $\mathbf{Q} \subseteq \mathbf{P}$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be relational structures of the same type. A pair $(\mathbf{Q}; f)$ is called a ***P**-colored relational structure* if f is a map from a subset of Q to P. If f can be extended to a morphism $f : \mathbf{Q} \to \mathbf{P}$, then f and $(\mathbf{Q}; f)$ is called ***P**-extendible*, otherwise, f and $(\mathbf{Q}; f)$ is called ***P**-nonextendible*.

Given $\mathbf{P}$-colored relational structures $(\mathbf{H}; f)$ and $(\mathbf{Q}; g)$, we say that $(\mathbf{H}; f)$ is *contained* in $(\mathbf{Q}; g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f \subseteq g$. If $(\mathbf{H}; f)$ is contained in $(\mathbf{Q}; g)$ we write $(\mathbf{H}; f) \subseteq (\mathbf{Q}; g)$.

A finite $\mathbf{P}$-colored relational structure $(\mathbf{H}; f)$ is called a ***P**-obstruction* if $(\mathbf{H}; f)$ is $\mathbf{P}$-nonextendible and every $\mathbf{P}$-colored relational structure $(\mathbf{K}; g)$ properly contained in $(\mathbf{H}; f)$ is $\mathbf{P}$-extendible. Roughly speaking, every $\mathbf{P}$-obstruction is a finite minimal $\mathbf{P}$-nonextendible $\mathbf{P}$-colored relational structure. If a relational structure $\mathbf{P}$ is an ordered set, then $\mathbf{P}$-obstructions are called ***P**-zigzags*, see Zadori [23].

It is known that for any finite algebra $\underline{P}$, there is a relational structure $\mathbf{P} = (P; (r^s_P)_{s\in S})$ such that the clone of term operations of $\underline{P}$ is $\mathrm{Pol}(\{r^s_P \mid s \in S\})$. A finite nontrivial algebra $\underline{P}$ is said to be *order-primal* if a corresponding relational structure $\mathbf{P}$ is an ordered set. Algebraic properties of order-primal algebras are investigated by many mathematicians. There have been many research works studying order-primal algebras (see [11], [13] and [19] the references therein).

A natural duality is a special kind of dual category equivalence between a quasivariety of algebras generated by a finite algebra and a category of structured topological spaces. For complete explanation to natural dualities, dualisability, strong dualities and optimal dualities, see Clark and Davey [5].

In the theory of natural dualities, one start with a fixed finite algebra $\underline{P} = (P; F)$ and consider *alter egos* $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ of $\underline{P}$ where G is a set of total operations, H is a set of partial operations, R is a set of finitary relations and τ is the discrete topology on P. For any alter egos $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$, $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{P}})$ is the class of all isomorphic copies of topologically closed substructures of direct power indexed over non-empty index sets of $\underset{\sim}{\mathbf{P}}$. The quasi-variety $\mathbb{ISP}(\underline{P})$ generated by an algebra $\underline{P}$ is the class of all isomorphic copies of subalgebras of direct powers of the algebra $\underline{P}$. The aim is to find an alter ego of the algebra $\underline{P}$ such that the category $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{P}})$ is dually equivalent to the quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{P})$ via the contravariant hom-functors

$$D(-) := \mathcal{A}(-, \underline{P}) : \mathcal{A} \to \mathcal{X} \text{ and } E(-) := \mathcal{X}(-, \underset{\sim}{\mathbf{P}}) : \mathcal{X} \to \mathcal{A}.$$

In this case, we say that $\underset{\sim}{\mathbf{P}}$ *yields a* (*natural*) *duality on* $\mathcal{A}$ or $\underline{P}$ *admits a* (*natural*) *duality* (or is *dualizable*).

We say that the set $G \cup H \cup R$ of a structure $\underset{\sim}{\mathbf{P}}$ *entails* a finitary algebraic relation s on $D(\underline{A})$, where $\underline{A} \in \mathcal{A}$, if every continuous map $\varphi : D(\underline{A}) \to \underset{\sim}{\mathbf{P}}$ which preserves the operations, partial operations and relations in $G \cup H \cup R$ also preserves s. The set $G \cup H \cup R$ is said to *entail* s if $G \cup H \cup R$ entails s on $D(\underline{A})$ for all $\underline{A} \in \mathcal{A}$. Clearly, if $G \cup H \cup R$ yields a duality on $\mathcal{A}$ and $G \cup H \cup R$ entails some $s \in R$, then the smaller set $G \cup H \cup R \setminus \{s\}$ also yields a duality on $\mathcal{A}$. If $G \cup H \cup R$ is minimal with respect to yielding a duality on $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{P}}$ yields *an optimal duality* on $\mathcal{A}$.

In [9], Davey and Werner showed that if $\underline{P}$ has a $(k+1)$-ary near-unanimity term, then $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ yields a duality on $\mathcal{A}$.

Consider the 6-element ordered set $\mathbf{T}$ shown in Figure 1. It is known that $\mathbf{T}$ has

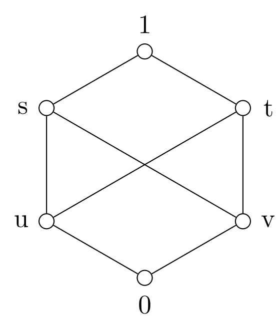


FIGURE 1. The 6-element ordered set $\mathbf{T}$

a 5-ary order-preserving near-unanimity function and so the corresponding order-primal algebra $\underline{T}$ admits a duality. Davey, Quackenbush and Schweigert [11] showed that the alter ego $\underset{\sim}{\mathbf{T}}$ consisting only of the 4-ary relation

$$\rho = \{(a, b, c, d) \in T^4 \mid \exists e \in T\, (a, b \le e \le c, d)\}$$

yields a duality on $\underline{T}$.

In 1995, Davey, Heindorf and Mckenzie have provided a useful characterization of finite order-primal algebras admitting a duality [10]. They proved that a finite order-primal algebra $\underline{P}$ admits a duality if and only if $\underline{P}$ has a near-unanimity term. Consequently, for every dualizable order-primal algebra $\underline{P}$, the alter ego

$\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$, for some natural number k. The structure $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ is an important theoretical tool, but in practice we try to make the structure on $\underset{\sim}{\mathbf{P}}$ as simple as possible, for example, the structure of the alter ego $\underset{\sim}{\mathbf{T}}$ and the structure on Priestley's duality for distributive lattices is given by only a single relation and two constants.

Even though we know which order-primal algebras $\underline{P}$ admit a natural duality, we have a little knowledge about alter egos which generate the categories $\mathcal{X}$ of topological structures dual to the quasi-variety generated by $\underline{P}$. Our interest is to find such alter egos for any dualizable order-primal algebra, and in general, for any algebra with a near-unanimity term.

In [23], [24] and [25] Zádori showed that obstructions are a powerful tool in the study of relational structures, especially ordered sets, admitting a near-unanimity function. The previous results showed that near-unanimity terms play an important role to study dualizable algebras. So, we are interested in the use of obstructions to construct dualising alter egos for algebras with a near-unanimity term.

3. Alter egos of algebras with a near-unanimity term

The main idea of this paper is to use obstructions to define relations for alter egos. Before doing it, we need to mention some basic concepts involving colored relational structures.

For a $\mathbf{P}$-colored relational structure $(\mathbf{H}; f)$, we define $C(\mathbf{H}; f) = \{x \in H \mid f(x)$ exists $\}$, i.e., $C(\mathbf{H}; f)$ is the domain of f and $N(\mathbf{H}; f) = N \setminus C(\mathbf{H}; f)$. The elements of $C(\mathbf{H}; f)$ and $N(\mathbf{H}; f)$ are called *colored elements* and *noncolored elements*, respectively.

For any relational structure $\mathbf{P}$, we shall define relations induced by their $\mathbf{P}$-colored relational structures on any relational structure of the same type as $\mathbf{P}$ as follows.

Definition 3.1. Let $(\mathbf{H}; f)$ be a finite $\mathbf{P}$-colored relational structure. Assume that $C(\mathbf{H}; f) = \{x_1, \ldots, x_n\}$. We define the n-ary relation r_H^A on a relational structure $\mathbf{A}$ of the same type as $\mathbf{P}$ as follows:

$$r_H^A = \{\, (g(x_1), \ldots, g(x_n)) \mid g : \mathbf{H} \to \mathbf{A} \text{ is a morphism} \,\}.$$

Consider an ordered set $\mathbf{P}$. We can draw a picture of a $\mathbf{P}$-colored ordered set $(\mathbf{H}; f)$. A picture consists of the covering graph of $\mathbf{H}$ and an element of $\mathbf{H}$ is drawn as a small shaded circle if f is defined on the given point. Otherwise, it is drawn as a small empty circle. Every shaded point is labelled by the value of f.

Example 3.2. Let $\mathbf{P}$ be the ordered set shown in Figure 2(a). Consider the $\mathbf{P}$-colored ordered set $(\mathbf{H}; f)$ as shown in Figure 2(b). The relation r_H^A induced by $(\mathbf{H}; f)$ is a binary relation satisfying the following condition:

$$(x, y) \in r_H^A \Leftrightarrow \text{ there exists } u, v \in A \text{ with } x \leq u \geq y \text{ and } x \geq v \leq y$$

for any ordered set $\mathbf{A}$. It is easy to see that the relation r_K^A induced by $(\mathbf{K}; g)$ as shown in Figure 2(c) is the same relation as r_H^A, although $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ are different $\mathbf{P}$-colored relational structures since $f \neq g$.

Example 3.2 shows that different $\mathbf{P}$-colored relational structures can give us the same induced relation. Consider, $\mathbf{P}$-colored relational structures $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ in Example 3.2, we see that they have the same set of colored elements and

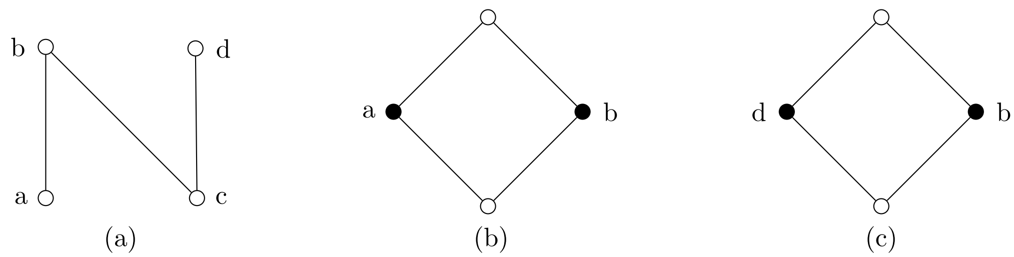


FIGURE 2. Ordered set $\mathbf{P}$ and $\mathbf{P}$-colored ordered sets $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$

the covering graphs of $\mathbf{H}$ and $\mathbf{K}$ have the same shape. More generally, it can be said that $\mathbf{H}$ and $\mathbf{K}$ are isomorphic under an isomorphism such that its restriction function on $C(\mathbf{H}; f)$ is a bijection from $C(\mathbf{H}; f)$ onto $C(\mathbf{K}; g)$. The relation $\sim_{\mathbf{P}}$ is defined for $\mathbf{P}$-colored relational structures satisfying this property.

Definition 3.3. Let $\mathbf{P}$ be a relational structure. We define the relation $\sim_{\mathbf{P}}$ on the set $\mathbf{PC}$ of all finite $\mathbf{P}$-colored relational structures as follows: $(\mathbf{H}; f) \sim_{\mathbf{P}} (\mathbf{K}; g)$ if and only if there is an isomorphism $\psi : \mathbf{H} \to \mathbf{K}$ such that $\psi|_{C(\mathbf{H};f)}$ is a bijection from $C(\mathbf{H}; f)$ onto $C(\mathbf{K}; g)$.

Clearly, $\sim_{\mathbf{P}}$ is an equivalence relation. It is natural to ask whether $\mathbf{P}$-colored relational structures which are $\sim_{\mathbf{P}}$-related to a $\mathbf{P}$-obstruction are $\mathbf{P}$-obstructions. The answer is shown in the following example.

Example 3.4. Let $\mathbf{P}$ be the ordered set shown in Figure 2(a). Consider the $\mathbf{P}$-colored relational structures $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ as shown in Figure 3. It is shown in

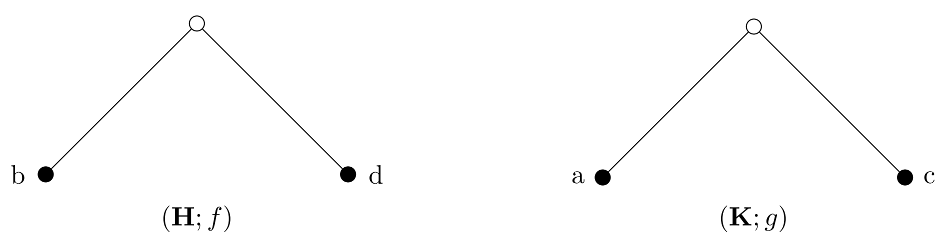


FIGURE 3. Two $\mathbf{P}$-colored ordered sets $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$

[23] that $(\mathbf{H}; f)$ is a $\mathbf{P}$-zigzag. Since g can be extended to a full defined monotone map on K, $(\mathbf{K}; g)$ is not a $\mathbf{P}$-zigzag. Clearly, $(\mathbf{H}; f) \sim_{\mathbf{P}} (\mathbf{K}; g)$ and therefore, there is a $\mathbf{P}$-colored relational structure which is $\sim_{\mathbf{P}}$-related to a $\mathbf{P}$-obstruction, but it is not a $\mathbf{P}$-obstruction.

The following lemma shows that $\mathbf{P}$-colored relational structures which are related under $\sim_{\mathbf{P}}$ produce relations r_H^A which are definable from each other.

Lemma 3.5. *Let $(\boldsymbol{H}; f)$ and $(\boldsymbol{K}; g)$ be finite $\boldsymbol{P}$-colored relational structures and let $\boldsymbol{A}$ be a relational structure of the same type of $\boldsymbol{P}$. Then following conditions hold:*

(i) $r_H^{A^k} = (r_H^A)^k$ *for all* $k \in \mathbb{N}$.

(ii) *Let* $(\boldsymbol{H};f) \sim_{\boldsymbol{P}} (\boldsymbol{K};g)$ *under an isomorphism* ψ. *Assume that* $C(\boldsymbol{H};f) = \{x_1,\dots,x_n\}$ *and* $C(\boldsymbol{K};g) = \{y_1,\dots,y_n\}$. *Then*

$$(a_1,\dots,a_n) \in r^A_K \text{ if and only if } (a_{\sigma(1)},\dots,a_{\sigma(n)}) \in r^A_H,$$

where $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ *defined by* $\sigma(i) = j$ *if* $\psi(x_i) = y_j$. *Hence* r^A_H *is definable from* r^A_K *and vise versa.*

Proof. (i) This follows since $\lambda : \mathbf{H} \to \mathbf{A}^k$ is a homomorphism if and only if $\pi_j \circ \lambda : \mathbf{H} \to \mathbf{A}$ is a homomorphism for all $j \in \{1,\dots,k\}$.

(ii) Let $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ be defined by $\sigma(i) = j$ if $\psi(x_i) = y_j$. Since $\psi|_{C(\mathbf{H};f)}$ is bijective, the map σ is bijective. Assume that $(a_1,\dots,a_n) \in r^A_K$. Then there is a morphism $h : \mathbf{K} \to \mathbf{A}$ such that $h(y_i) = a_i$ for all $i \in \{1,\dots,n\}$ implying that $h \circ \psi : \mathbf{H} \to \mathbf{A}$ is a morphism. Since $h \circ \psi(x_i) = h(\psi(x_i)) = h(y_j) = a_j = a_{\sigma(i)}$ where $\psi(x_i) = y_j$, so $(a_{\sigma(1)},\dots,a_{\sigma(n)}) \in r^A_H$. Conversely, let $(a_{\sigma(1)},\dots,a_{\sigma(n)}) \in r^A_H$. Then there is a morphism $h : \mathbf{H} \to \mathbf{A}$ such that $h(x_i) = a_{\sigma(i)}$ for all $i \in \{1,\dots,n\}$. Assume that $\psi(x_i) = y_i$. Then $\sigma(i) = j$ implies $h \circ \psi^{-1}(y_j) = h(x_i) = a_{\sigma(i)} = a_j$ and hence, $(a_1,\dots,a_n) \in r^A_K$. □

Lemma 3.6. *Let* $(\boldsymbol{H};f)$ *be a finite* $\boldsymbol{P}$*-obstruction with* $C(\boldsymbol{H};f) = \{x_1,\dots,x_n\}$. *Then* $(f(x_1),\dots,f(x_n)) \notin r^P_H$.

Proof. Suppose that $(f(x_1),\dots,f(x_n)) \in r^P_H$. Then there is a morphism $g : \mathbf{H} \to \mathbf{P}$ such that $(g(x_1),\dots,g(x_n)) = (f(x_1),\dots,f(x_n))$ implying that $g(x_i) = f(x_i)$ for all $x_i \in C(\mathbf{H};f)$. So, g is an extension morphism of f. It follows that $(\mathbf{H};f)$ is $\mathbf{P}$-extendible, a contradiction. □

In 1997, Zádori characterized a relational structure admitting an n-ary near-unanimity function by its obstructions that is a generalization of a remark by Tardos in [22] as stated below.

Theorem 3.7. [25, Theorem 1.17] *Let* $\boldsymbol{P}$ *be a finite relational structure. Then* $\boldsymbol{P}$ *admits an* n*-ary near-unanimity function if and only if the number of colored elements in every* $\boldsymbol{P}$*-obstruction is at most* $n-1$.

It follows that the number of relations r^A_H induced by its obstruction is finite. Lemma 3.5 implies that it is enough to use only the relations r^A_H induced by representations of $\sim_{\mathbf{P}}$-blocks containing a $\mathbf{P}$-obstruction to construct an alter ego of the corresponding algebra $\underline{P}$. Indeed, the structure of such alter ego includes only relations.

Consider a topological structure $\underset{\sim}{\mathbf{P}}$ having no partial operations and only finitely many relations. Then $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$ if $\underset{\sim}{\mathbf{P}}$ satisfies the interpolation condition (IC) with respect to $\underline{P}$ as stated below.

Theorem 3.8. [5, The Second Duality Theorem] *Assume that* $\underset{\sim}{\mathbf{P}} = (P;G,R,\tau)$ *and assume that* R *is finite. If* $\underset{\sim}{\mathbf{P}}$ *satisfies the condition:*

(IC) *for each* $n \in \mathbb{N}$ *and each substructure* X *of* P^n, *every morphism* $\alpha : X \to P$ *extends to a term function* $t : P^n \to P$ *of the algebra* $\underline{P}$,

then $\underset{\sim}{\mathbf{P}}$ *yields a duality on* $\mathcal{A}$ *and* $\underset{\sim}{\mathbf{P}}$ *is injective in* $\mathcal{X}$.

Now, we give a general theorem for constructing simple and useful alter egos of algebras admitting a near-unanimity term.

Theorem 3.9. *Let $\underline{P}$ be an algebra corresponding to a relational structure $\boldsymbol{P}$ and Q be a transversal of the $\sim_{\mathbf{P}}$-blocks that contain a $\boldsymbol{P}$-obstruction. Assume that $\underline{P}$ admits an n-ary near-unanimity term and $R = \{r_H^P | (H; f) \in Q\}$. Then the following conditions hold:*

(i) *every relation $r \in R$ is a k-ary relation for some $k \leq n-1$,*
(ii) *the structure $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$.*

Proof. By Theorem 3.7, we know that the number of colored elements in every $\mathbf{P}$-obstruction of any finite relational structure $\mathbf{P}$ admitting an n-ary near-unanimity function is at most $n-1$. Since the arity of any relation r_H^P in R is equal to a number of colour elements of a $\mathbf{P}$-obstruction, r_H^P is a k-ary relation for some $k \leq n-1$. So, condition (i) holds.

(ii) Since the number of relations r_H^A induced by $\mathbf{P}$-obstructions is finite, R is finite. We shall show that $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$ by proving that $\underset{\sim}{\mathbf{P}}$ satisfies (IC).

Let $\varphi : X \subseteq P^m \to P$ be a R-preserving. To show that (IC) holds, it is enough to show that $(\mathbf{P}^m; \varphi)$ is $\mathbf{P}$-extendible. Suppose that $(\mathbf{P}^m; \varphi)$ is $\mathbf{P}$-nonextendible. Then as P^m is finite, $(\mathbf{P}^m; \varphi)$ contains a minimal $\mathbf{P}$-nonextendible, i.e., $(\mathbf{P}^m; \varphi)$ contains a $\mathbf{P}$-obstruction, say $(\mathbf{K}; \varphi|K)$ where $\mathbf{K}$ is a relational structure contained in the relational structure $\mathbf{P}^m$. By the assumption, there is a $\mathbf{P}$-colored relational structure $(\mathbf{H}; f)$ in Q which is $\sim_{\mathbf{P}}$-related to $(\mathbf{K}; \varphi|K)$. Assume that $(\mathbf{H}; f) \sim_{\mathbf{P}} (\mathbf{K}; \varphi|K)$ under an isomorphism ψ. Since id_K is a morphism from $\mathbf{K}$ into itself, $(x_1, \ldots, x_k) \in r_K^{P^m}$ where $C(\mathbf{K}; \varphi|K) = \{x_1, \ldots, x_k\}$. By Lemma 3.5(ii), $(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \in r_H^{P^m}$ where $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ defined by $\sigma(i) = j$ if $\psi(x_i) = y_i$. It follows that $(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \in (r_H^P)^m$. From φ is R-preserving and $r_H^P \in R$, we have $(\varphi(x_{\sigma(1)}), \ldots, \varphi(x_{\sigma(k)})) \in r_H^P$ and then $(\varphi(x_1), \ldots, \varphi(x_k)) \in r_K^P$ which is contradict to Lemma 3.6. Therefore, $(\mathbf{P}^m; \varphi)$ is $\mathbf{P}$-extendible. □

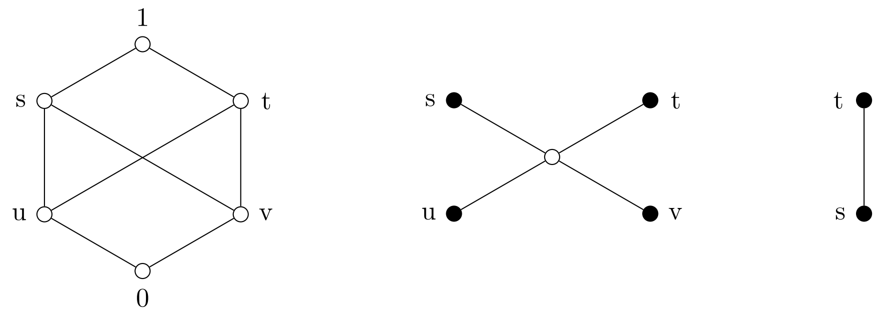
Since we need the structure on $\underset{\sim}{\mathbf{P}}$ that is as simple as possible, one can ask whether our structure is optimal, that is, if any relation in the structure were deleted, the duality would be destroyed. Unfortunately, it is not optimal as shown by the following example.

Example 3.10. Consider the order-primal algebra $\underline{T}$ corresponding to the 6-element ordered set $(T; \leq)$ as shown in Figure 4(a). It is known that $\underline{T}$ has a 5-ary near-unanimity term. Zádori showed in [23] that $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ in Figure 4(b) and 4(c) are $\mathbf{T}$-zigzags. The relations r_H^T and r_K^T induced by $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$, respectively, are the 4-ary relation

$$\rho = \{(a, b, c, d) \in T^4 \mid \exists e \in T\, (a, b \leq e \leq c, d)\}$$

and the order of $(T; \leq)$, respectively. We can see that $(\mathbf{H}; f) \not\sim_{\mathbf{P}} (\mathbf{K}; g)$ and then the relations r_H^T and r_H^T are contained in the structure of $\underset{\sim}{\mathbf{T}} = (T; R, \tau)$ in Theorem 3.9. As we mentioned in Section 2, Davey, Quackenbush and Schweigert proved that the topological structure $(T; \rho, \tau)$ where τ is the discrete topology yields a natural duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$. Therefore, the structure $\underset{\sim}{\mathbf{T}} = (T; R, \tau)$ is not optimal since we can delete the order $\leq$ from R and the result structure still yields a duality on $\mathcal{A}$.

Our aim is to find a method for checking whether a duality of any finite algebra admitting a near-unanimity term which is constructed from our theorem is optimal. Initially, we need the following definition.



(a) (b) (c)

FIGURE 4. The 6-element ordered set and its zigzags

Definition 3.11. Let $\mathbf{P}$ be a relational structure. We define the relation $\leq_{\mathbf{PC}}$ on $\mathbf{PC}$ as follows:

$(\mathbf{H}; f) \leq_{\mathbf{PC}} (\mathbf{K}; g)$ if and only if $(\mathbf{H}; f)$ is a color-preserving retract of $(\mathbf{K}; g)$, that is, there are morphisms $\psi : \mathbf{K} \to \mathbf{H}$ and $\varphi : \mathbf{H} \to \mathbf{K}$, both of which map colored elements to colored elements, such that $\psi \circ \varphi = \mathrm{id}_H$.

It is easy to see that $\leq_{\mathbf{PC}}$ is reflexive and transitive, but not antisymmetric. The following proposition shows a relation between the relations induced by $\mathbf{P}$-colored relational structures which are related under $\leq_{\mathbf{PC}}$.

Proposition 3.12. *Let $(\boldsymbol{H}; f)$ and $(\boldsymbol{K}; g)$ be finite $\boldsymbol{P}$-colored relational structures and let $\boldsymbol{A}$ be a relational structure of the same type of $\boldsymbol{P}$. Assume that $C(\boldsymbol{H}; f) = \{x_1, \dots, x_m\}$ and $C(\boldsymbol{K}; g) = \{y_1, \dots, y_n\}$. If $(\boldsymbol{H}; f) \leq_{\boldsymbol{PC}} (\boldsymbol{K}; g)$ vis a color-preserving retraction ψ, then*

$$(a_1, \dots, a_m) \in r_H^A \text{ if and only if } (a_{\overline{\psi}(1)}, \dots, a_{\overline{\psi}(n)}) \in r_K^A$$

where $\overline{\psi} : \{1, \dots, n\} \to \{1, \dots, m\}$ defined by $\overline{\psi}(i) = j$ if $\psi(y_i) = x_j$, and hence r_H^A is definable from r_K^A.

Proof. Assume that $(\mathbf{H}; f) \leq_{\mathbf{PC}} (\mathbf{K}; g)$ under a morphism ψ and let $(a_1, \dots, a_m) \in r_H^A$. Then there is a morphism $h : \mathbf{H} \to \mathbf{A}$ such that $h(x_i) = a_i$ for all $i \in \{1, \dots, m\}$. By definition, for all $i \in \{1, \dots, n\}$, we have $\psi(y_i) = x_j$, for some $j \in \{1, \dots, m\}$. Then $\overline{\psi}(i) = j$ and we get $h \circ \psi(y_i) = h(x_j) = a_j = a_{\overline{\psi}(i)}$. Since $h \circ \psi : \mathbf{K} \to \mathbf{A}$ is a morphism, $(a_{\overline{\psi}(1)}, \dots, a_{\overline{\psi}(n)}) \in r_K^A$.

Conversely, let $(a_{\overline{\psi}(1)}, \dots, a_{\overline{\psi}(n)}) \in r_K^A$. Then there is a morphism $k : \mathbf{K} \to \mathbf{A}$ with $k(y_i) = a_{\overline{\psi}(i)}$ for all $i \in \{1, \dots, n\}$. By definition, there is a morphism $\varphi : \mathbf{H} \to \mathbf{K}$ that preserves colored elements and satisfies $\psi \circ \varphi = \mathrm{id}_H$. Assume that $\varphi(x_j) = y_i$. Then $\psi(y_i) = x_j$ implies $\overline{\psi}(i) = j$. It follows that $k \circ \varphi(x_j) = k(y_i) = a_{\overline{\psi}(i)} = a_j$ and therefore, $(a_1, \dots, a_m) \in r_H^A$. □

Proposition 3.12 implies that every relation in the alter ego $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ in Theorem 3.9 which is $\leq_{\mathbf{PC}}$-related to some relations in R can be deleted from the structure of $\underset{\sim}{\mathbf{P}}$ without destroying the duality. This gives us a sufficient condition for a structure $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ to be non-optimal as stated in the theorem below.

Theorem 3.13. *Let $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ be an alter ego in Theorem 3.9. If R contains two different relations induced from $\boldsymbol{P}$-colored relational structures which are related under $\leq_{\boldsymbol{PC}}$, then $\underset{\sim}{\mathbf{P}}$ is not optimal.*

4. Dualizable order-primal algebras and their properties

The results in Section 3 give the alter egos $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ to yield a duality on $\mathbb{ISP}(\underline{P})$ where $\underline{P}$ is an algebra admitting a near-unanimity term.

As we mentioned in Section 2, an algebra $\underline{P}$ is order-primal if a corresponding relational structure $\mathbf{P}$ is an ordered set. In this section, we focus on an order-primal algebra $\underline{P}$ corresponding to a connected ordered set $(P; \leq)$. We shall show that, in the case of a connected ordered set, the duality from Section 3 is strong.

To do so we need a basic result concerning upgrading dualities to strong dualities. Define

$$\mathrm{Irr}(\underline{P}) = \max\{\mathrm{irr}(\underline{Q}) \mid \underline{Q} \text{ is a subalgebra of } \underline{P}\},$$

where $\mathrm{irr}(\underline{Q})$ is the least n such that the zero congruence $\mathbf{0}^{\underline{Q}}$ on $\underline{Q}$ is a meet of n meet-irreducible congruence. Note that $\underline{Q}$ is subdirectly irreducible if and only if $\mathrm{irr}(\underline{Q}) = 1$.

Theorem 4.1. [5, Theorem 3.3.7] *Assume that $\underline{P}$ generates a congruence-distributive variety and that $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ yields a duality on $\mathcal{A}$. If $\underset{\sim}{\mathbf{P}}'$ is obtained from $\underset{\sim}{\mathbf{P}}$ by adding to $G \cup H$ all n-ary algebraic partial operations where $0 \leq n \leq \mathrm{Irr}(\underline{P})$, then $\underset{\sim}{\mathbf{P}}'$ yields a strong duality on $\mathcal{A}$.*

Davey, Quackenbush and Schweigert proved in [11] that every order-primal algebra $\underline{P}$ corresponding to the ordered set $(P; \leq)$ has no proper subalgebra and if $(P; \leq)$ is connected, then $\underline{P}$ is simple and hence, $\underline{P}$ is subdirectly irreducible. It follows that $\mathrm{Irr}(\underline{P}) = 1$. Using this result, we can prove the following theorem.

Theorem 4.2. *Let $k \geq 2$ and $\underline{P}$ be an order-primal algebra corresponding to a connected ordered set $(P; \leq)$. If $\underline{P}$ has a $(k+1)$-ary near-unanimity term, then $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ in Theorem 3.9 yields a strong duality.*

Proof. We shall apply Theorem 4.1, which applies since $\underline{P}$ generates a congruence-distributive variety since it has a near-unanimity term. Since $\underline{P}$ has no proper subalgebra, and $\underline{P}$ is simple, we have $\mathrm{Irr}(\underline{P}) = 1$. Thus, by Theorem 4.1, to upgrade the duality to a strong duality given by Theorem 3.9 it suffices to add to the alter ego all endomorphism of $\underline{P}$. Since $\underline{P}$ is order-primal and every constant map is order-preserving, every element of P is the value of a constant unary term function of $\underline{P}$. Consequently, the only endomorphism of $\underline{P}$ is id_P, which can always be removed from an alter ego without destroying a strong duality. Therefore, $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ yields a strong duality on $\mathcal{A}$. □

As we mentioned in Section 3, the set R in the structure $\underset{\sim}{\mathbf{P}}$ is produced by obstructions or zigzags if $\underline{P}$ is order-primal. To construct $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$, we need to know all zigzags of the corresponding ordered set. Next, we give concrete examples of alter egos which are produced by our methods for special order-primal algebras.

Consider an ordered set $(P; \leq)$, a $\mathbf{P}$-colored ordered set $(\mathbf{H}; f)$ is called *monotone* if f is a monotone map on its domain, otherwise $(\mathbf{H}; f)$ is *nonmonotone*. Zádori showed [24] that the $\mathbf{P}$-colored 2-element chain in which the top is colored by a and the bottom is colored by b where $b \not\leq a$, is a nonmonotone $\mathbf{P}$-zigzag and every nonmonotone **P**-zigzag is of this form. Moreover, he proved that if $(P;\leq)$ is a complete lattice, then **P** has no monotone **P**-zigzags. In particular, no finite lattice possesses monotone zigzags.

Example 4.3. Let $\underline{P}$ be an order-primal algebra corresponding to a finite lattice $(L;\leq)$. Then by Zádori's result, $(L;\leq)$ has no monotone **L**-zigzags and hence every **L**-zigzag $(\mathbf{H};f)$ is of the following form. The corresponding relation r_H^P of $(\mathbf{H};f)$

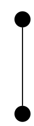


FIGURE 5. The form of **L**-zigzags

is the set $\{(a,b)\in L^2 \mid a\leq b\}$, that is, $r_H^L = \leq$. It follows that $R=\{\leq\}$ and by Theorem 3.9, the structure $\underset{\sim}{\mathbf{P}} = (L;\leq,\tau)$ yields a duality on $\mathcal{A}$. See Davey and Rival [8, Theorem 1.1] where this result is proved directly.

In a connected ordered set $(P;\leq)$, we define the *up distance* from a to b to be the least positive integer n such that there is a subset $\{a_0,\dots,a_n\}\subseteq P$ with

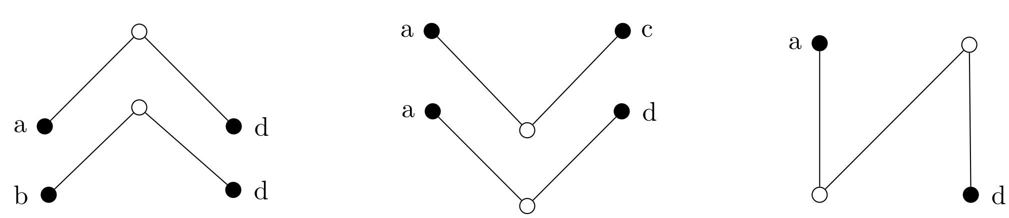
$$a=a_0, b=a_n \text{ and } a_0\leq a_1\geq a_2\leq\dots.$$

The *down distance* from a to b is defined dually. Let $\uparrow(a,b)$ and $\downarrow(a,b)$ denote the up and down distance from a to b, respectively. The *distance* between a and b is $d(a,b)=\min\{\uparrow(a,b),\downarrow(a,b)\}$, that is, the length of the shortest fence from a to b. We denote by $\mathbf{F}_n$ and $\mathbf{G}_n$ an n-element up and down fence, respectively.

In 1993, Zádori described all zigzags of a finite connected ordered set admitting a majority function as shown below.

Theorem 4.4. [23, Theorem 3.1] *Let $\boldsymbol{P}$ be a finite connected ordered set admitting a majority function. Then every $\boldsymbol{P}$-zigzag $(\boldsymbol{H};f)$ is a $\boldsymbol{P}$-colored fence satisfying the following properties. If a and b denote the endpoints of the colored fence, then a and b are only colored points and at least one of the inequalities $\uparrow(a,b)<\uparrow(f(a),f(b))$ and $\downarrow(a,b)<\downarrow(f(a),f(b))$ holds.*

He noted [23] that we easily get all the monotone zigzags of fences by Theorem 4.4. For example, the 4-element fence $\mathbf{F}_4$ has exactly 5 monotone zigzags shown in Figure 6.



$(\mathbf{H}_1;f_1)$ and $(\mathbf{H}_3;f_3)$ $(\mathbf{H}_2;f_2)$ and $(\mathbf{H}_4;f_4)$ $(\mathbf{H}_5;f_5)$

FIGURE 6. The monotone $\mathbf{F}_4$-zigzags

Example 4.5. Consider the order-primal algebra $\underline{P}$ corresponding to $\mathbf{F}_4$, we shall construct an alter ego which yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$. As in Example 4.3, a nonmonotone $\mathbf{F}_4$-zigzag processes the order of $\mathbf{F}_4$. It is easy to see that

$$r_{H_1}^{F_4} = \{(x, y) \in (F_4)^2 \mid \exists z \in F_4\,(x \leq z \geq y)\} = r_{H_3}^{F_4} \text{ and}$$
$$r_{H_2}^{F_4} = \{(x, y) \in (F_4)^2 \mid \exists z \in F_4\,(x \geq z \leq y)\} = r_{H_4}^{F_4}.$$

Moreover,

$$r_{H_1}^{F_4} = \{(x, y) \in (F_4)^2 \mid \exists u, v \in F_4\,(x \geq u \leq v \geq y)\}.$$

It follows from Theorem 3.9 that the structure

$$\underset{\sim}{\mathbf{P}} = (F_4; R, \tau), \text{ where } R = \{\leq, r_{H_1}^{F_4}, r_{H_2}^{F_4}, r_{H_5}^{F_4}\}$$

yields a strong duality on $\mathcal{A}$.

Now, we focus on the order-primal algebra $\underline{P}$ corresponding to a finite fence $\mathbf{P}$. It is well-known [12] that $\mathbf{P}$ admits a majority function and then every $\mathbf{P}$-zigzag $(\mathbf{H}; f)$ is a $\mathbf{P}$-colored fence satisfying the properties in Theorem 4.4. Since $\mathbf{H}$ is a fence with one of the inequalities $\uparrow(a, b) < \uparrow(f(a), f(b))$ and $\downarrow(a, b) < \downarrow(f(a), f(b))$ holds where a and b are the endpoints of $\mathbf{H}$, it is easy to see that $\mathbf{H}$ is (up to isomorphism) a subfence of $\mathbf{P}$ or the dual $\mathbf{P}^{\partial}$ and the endpoints a and b are the exactly two colored elements of $(\mathbf{H}; f)$.

Consider a transversal Q of the $\sim_{\mathbf{P}}$-blocks that contain a $\mathbf{P}$-obstruction of a finite fence $\mathbf{P}$, we easily find that

$$Q = \left\{ (\mathbf{H}; f) \mid C(\mathbf{H}; f) = \{a, b\} \text{ and } \mathbf{H} \text{ is a subfence of } \mathbf{P} \text{ or } \mathbf{H} = \mathbf{P}^{\partial} \right\},$$

where a and b are the endpoints of $\mathbf{H}$. So, $R = \{r_H^P \subseteq P^2 | (\mathbf{H}; f) \in Q\}$. By the use of the Test Algebra Lemma (see [5, Theorem 8.1.3]), we can prove that the structure $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ yields an optimal duality on $\mathcal{A}$ as follows.

Lemma 4.6. *Let $(\boldsymbol{H}; f)$ and $(\boldsymbol{K}; g)$ be $\boldsymbol{F}_n$-colored ordered sets such that $\boldsymbol{H}$ and $\boldsymbol{K}$ are subfences of $\boldsymbol{F}_n$ and the colored elements of both are exactly the endpoints of $\boldsymbol{H}$ and $\boldsymbol{K}$, respectively. Assume that $r = r_K^{\boldsymbol{F}_n}$ and $s = s_H^{\boldsymbol{F}_n}$.*

(i) *If $\boldsymbol{H}$ is (up to isomorphism) a subfence of $\boldsymbol{K}$, then s and s^{-1} are subsets of r.*

(ii) *If $\boldsymbol{H}$ is (up to isomorphism) the dual $\boldsymbol{K}^{\partial}$ with $|H|$ is even, then $(\boldsymbol{H}; f) \sim_{\boldsymbol{P}} (\boldsymbol{K}; g)$, and hence s is definable from r and vise versa.*

Proof. First, we assume that $F_n = \{1, 2, \ldots, n\}$.

(i) Assume that $\mathbf{H}$ is a subfence of $\mathbf{K}$ with

$$\mathbf{H} = \{j, j + 1, \ldots, k\} \text{ and } \mathbf{K} = \{i, i + 1, \ldots, l\}.$$

Then $i \leq j$ and $k \leq l$ in $\mathbb{N}$. Let $(x, y) \in s = r_H^{F_n}$. Then there is an order-preserving map $\psi : \mathbf{H} \to \mathbf{F}_n$ with $\psi(j) = x$ and $\psi(k) = y$.

Define $\bar{\psi} : \mathbf{K} \to \mathbf{F}_n$ by

$$\bar{\psi}(t) = \begin{cases} x, \ t \in \{i, i + 1, \ldots, j - 1\}, \\ \psi(t), \ t \in H, \\ y, \ t \in \{k + 1, k + 2, \ldots, l\}. \end{cases}$$

Because ψ is order-preserving, $\bar{\psi}$ is also order-preserving with $\bar{\psi}(i) = x$ and $\bar{\psi}(l) = y$ and hence, $(x, y) \in r_K^{F_n} = r$. Therefore, $s \subseteq r$

Next, we show that $s^{-1} \subseteq r$ by considering $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ in the following cases.

Case 1: $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ are up fences. Without loss of generality, we assume that $i = 1 = j$. Then $\mathbf{H} = \{1, 2, \ldots, k\}$ and $\mathbf{K} = \{1, 2 \ldots, l\}$. We consider the endpoint k of $\mathbf{H}$ in the following cases.

Case 1.1: k is maximal. Define $\bar{\psi} : \mathbf{K} \to \mathbf{F}_n$ by

$$\bar{\psi}(t) = \begin{cases} y, \ t = 1, \\ \psi(k - t + 2), \ t \in \{2, 3, \ldots, k, k + 1\}, \\ x, \ t \in \{k + 2, k + 3, \ldots, l\}. \end{cases}$$

Case 1.2: k is minimal. Define $\bar{\psi} : \mathbf{K} \to \mathbf{F}_n$ by

$$\bar{\psi}(t) = \begin{cases} \psi(k - t + 1), \ t \in \{1, 2, \ldots, k\}, \\ x, \ t \in \{k + 1, k + 2, \ldots, l\}. \end{cases}$$

Case 2: $(\mathbf{H}; f)$ and $(\mathbf{K}; g)$ are down fences. The $\bar{\psi}$ is defined dually to Case 1.

Case 3: $(\mathbf{H}; f)$ is an up fence and $(\mathbf{K}; g)$ is a down fence. Without loss of generality, we assume that $i = 2$ and $j = 3$. Then $\mathbf{H} = \{3, 4, \ldots, k\}$ and $\mathbf{K} = \{2, 3 \ldots, l\}$. We consider k again in 2 cases.

Case 3.1: k is maximal. Define $\psi : \mathbf{H} \to \mathbf{F}_n$ by

$$\bar{\psi}(t) = \begin{cases} \psi(k - t + 2), \ t \in \{2, 3, \ldots, k - 1\}, \\ x, \ t \in \{k, k + 1, k + 2, \ldots, l\}. \end{cases}$$

Case 3.2: k is minimal. Define $\bar{\psi} : \mathbf{K} \to \mathbf{F}_n$ by

$$\bar{\psi}(t) = \begin{cases} y, \ t = 2, \\ \psi(k - t + 3), \ t \in \{3, 4, \ldots, k\}, \\ x, \ t \in \{k + 1, k + 2, \ldots, l\}. \end{cases}$$

Case 4: $(\mathbf{H}; f)$ is a down fence and $(\mathbf{K}; g)$ is an up fence. The $\bar{\psi}$ is defined dually to Case 3.

It is easy to see that $\bar{\psi}$ defined in Case 1-4 is order-preserving with $\bar{\psi}(i) = y$ and $\bar{\psi}(l) = x$. So, $(y, x) \in r_K^{F_n} = r$ and therefore, $s^{-1} \subseteq r$.

(ii) Let $\mathbf{H} = \{j, j+1, \ldots, j+m = k\}$ and $\mathbf{K} = \{i, i+1, \ldots, i+m = l\}$. As $|H|$ is even, we may assume that j is minimal and k is maximal. Then i is maximal and l is minimal since $\mathbf{H}$ is the dual $\mathbf{K}^{\partial}$. We define $\psi : H \to K$ by $\psi(j + t) = i + (m - t)$ for all $t \in \{0, 1, \ldots, m\}$. Clearly, ψ is an isomorphism with $\psi(j) = l$ and $\psi(k) = i$. It follows that $(\mathbf{H}; f) \sim_{\mathbf{P}} (\mathbf{K}; g)$, and hence s is definable from r and vise versa. □

The results of Lemma 4.6 also hold true for $\mathbf{G}_n$.

Theorem 4.7. *Let $\underline{P}$ be an order-primal algebra corresponding to an n-element fence $\boldsymbol{P}$. Then $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ in Theorem 3.9 yields an optimal duality on $\mathcal{A}$.*

Proof. We may assume that $\mathbf{P} = \mathbf{F}_n$ with $P = \{1, 2, \ldots, n\}$. Since any fence is a connected ordered set admitting a majority term, R contains only binary relations and $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ yields a strong duality on $\mathcal{A}$ by Theorem 4.2. It follows that for each subalgebra $\underline{A}$ of $\underline{P}^n$, the structure $D(\underline{A})$ is generated by $\{\rho_1, \ldots, \rho_n\}$, where ρ_i is the i^{th} n-ary projection: see [5, Exercise 9.8(iii)]. As $\underset{\sim}{\mathbf{P}}$ is purely relational, we have $D(\underline{A}) = \{\rho_1, \ldots, \rho_n\}$. Clearly, all relations in R are reflexive since every constant map between two ordered sets is always order-preserving.

Let $s \in R$. Then there is a $\mathbf{F}_n$-zigzag $(\mathbf{H}; f) \in Q$ with $s = r_H^{F_n}$. Next, we show that $R \setminus \{s\}$ does not entail s by proving that $R \setminus \{s\}$ does not entail s on $D(s)$. Since s is a binary relation, $D(s) = \{\rho_1, \rho_2\}$. To avoid confusion, we shall denote a relation r on $D(s)$ by $r_{D(s)}$.

Let $\mathbf{H} = \{j, j+1, \dots, k\}$. Without loss of generality, we assume that $d(j,k) = d(f(j), f(k))$, that is, the fence with the endpoints $f(j)$ and $f(k)$ is the dual $\mathbf{H}^\partial$. Lemma 3.6 implies that $(f(j), f(k)) \notin s$. Define $\varphi : D(S) \to P$ by $\varphi(\rho_1) = f(j)$ and $\varphi(\rho_2) = f(k)$.

Now, we show that φ preserves all relations in $R \setminus \{s\}$, but does not preserve s.

Let $r \in R \setminus \{s\}$. Then there is a $\mathbf{F}_n$-zigzag $(\mathbf{K}; g) \in Q$ with $r = r_K^{F_n}$.

Let $\mathbf{K} = \{i, i+1, \dots, l\}$. We consider $\mathbf{K}$ in the following 3 cases.

Case 1: $d(i,l) < d(j,k)$. Because $id_{F_n}|_H : \mathbf{H} \to \mathbf{F}_n$ is order-preserving, $(j,k) \in s$. From $d(i,l) < d(j,k)$, we have $(j,k),(k,j) \notin r = r_K^P$ implying that $(\rho_1,\rho_2),(\rho_2,\rho_1) \notin r_{D(s)}$ since otherwise, (j,k) or $(k,j) \in r = r_K^P$. Hence, $r_{D(s)} = \Delta_{D(s)}$ implies that φ preserves r.

Case 2: $d(i,l) > d(j,k)$. Then $\mathbf{H}$ is (up to isomorphism) a subfence of $\mathbf{K}$ and by Lemma 4.6(i), s and s^{-1} are subsets of r. So, $(\rho_1(x,y), \rho_2(x,y)) = (x,y) \in s \subseteq r$ and $(\rho_2(x,y), \rho_1(x,y)) = (y,x) \in s^{-1} \subseteq r$ for all $(x,y) \in s$, that is, $(\rho_1,\rho_2),(\rho_2,\rho_1) \in r_{D(s)}$. Hence, $r_{D(s)} = D(s) \times D(s)$. Since $d(f(j), f(k)) = d(j,k) < d(i,l)$, we have $(f(j), f(k)), (f(k), f(j)) \in r$ implying $(\varphi(\rho_1), \varphi(\rho_2)) = (f(j), f(k)) \in r$ and $(\varphi(\rho_2), \varphi(\rho_1)) = (f(k), f(j)) \in r$. Therefore, φ preserves r.

Case 3: $d(i,l) = d(j,k)$. Then $\mathbf{H}$ is (up to isomorphism) the dual $\mathbf{K}^\partial$. Since s and r are contained in R, so, $(\mathbf{H}; f)$ cannot be $\sim_{\mathbf{P}}$-related to $(\mathbf{K}; g)$. It follows from Lemma 4.6(ii) that $|H|$ is odd and then the endpoints j and k are maximal or the endpoints j and k are minimal. Hence, $(y,x) \in s$ for all $(x,y) \in s$. Because of $\mathbf{H} = \mathbf{K}^\partial$, we have $(j,k),(k,j) \notin r$. From $(\rho_1(j,k), \rho_2(j,k)) = (j,k)$ and $(\rho_2(j,k), \rho_1(j,k)) = (k,j)$, we get that $(\rho_1,\rho_2),(\rho_2,\rho_1) \notin r_{D(s)}$, i.e., $r_{D(s)} = \Delta_{D(s)}$. Therefore, φ preserves r.

It is easy to see that $(\rho_1,\rho_2) \in s_{D(s)}$. But $(\varphi(\rho_1), \varphi(\rho_2)) = (f(j), f(k)) \notin s$ and hence, φ does not preserve s. Altogether, we have proved that $\underset{\sim}{\mathbf{P}} = (P; R, \tau)$ yields an optimal duality. □

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DUALIZABILITY OF ORDER-PRIMAL ALGEBRAS

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ABSTRACT. Let $\mathbf{T} = (T; \leq)$ be a rooted tree and let $\underline{T}$ and $\underline{T}^{\partial}$ be order-primal algebras corresponding to the ordered set $\mathbf{T}$ and its dual $\mathbf{T}^{\partial}$, respectively. In this paper, we produce dualising alter egos for $\underline{T}$ and $\underline{T}^{\partial}$, respectively. We show that they yield a strong duality on quasi-varieties generated by $\underline{T}$ and $\underline{T}^{\partial}$, respectively. Finally, we prove that the alter egos of $\underline{T}$ and $\underline{T}^{\partial}$ are optimal if $\mathbf{T}$ is not a chain.

1. INTRODUCTION AND PRELIMINARIES

An *order-primal* algebra $\underline{T}$ corresponding to an ordered set $\mathbf{P} = (P; \leq)$ is a finite non-trivial algebra for which the clone of term operations of $\underline{P}$ is $Pol(\leq)$.

Algebraic properties of order-primal algebras are investigated by many mathematicians. There have been many research works studying order-primal algebras (see [8], [9] and [11]).

A natural duality is a special kind of dual category equivalence between a quasi-variety of algebras generated by a finite algebra and a category of structured topological spaces. It provides a method for translating problem in a class of algebras into a different class of mathematical structures. For example, problems about distributive lattice can be translated into problems about ordered topological structures by Priestley's duality [12]. The theory of natural dualities is a powerful tool for studying algebra. For complete explanation to natural dualities, dualizability, strong dualities and optimal dualities, see Clark and Davey [1].

For a finite algebra $\underline{P} = (P; F)$, we consider *alter egos* $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ of $\underline{P}$ where G is a set of total operations, H is a set of partial operations, R is a set of finitary relations and τ is the discrete topology on P. For any alter egos $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$, $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{P}})$ is the class of all isomorphic copies of topologically closed substructures of direct power indexed over non-empty index sets of $\underset{\sim}{\mathbf{P}}$. The quasi-variety $\mathbb{ISP}(\underline{P})$ generated by an algebra $\underline{P}$ is the class of all isomorphic copies of subalgebras of direct powers of the algebra $\underline{P}$. The aim is to find an alter ego of the algebra $\underline{P}$ such that the category $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{P}})$ is dually equivalent to the quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{P})$ via the contravariant hom-functors

$$D(-) := \mathcal{A}(-, \underline{P}) : \mathcal{A} \to \mathcal{X} \text{ and } E(-) := \mathcal{X}(-, \underset{\sim}{\mathbf{P}}) : \mathcal{X} \to \mathcal{A}.$$

In this case, we say that $\underset{\sim}{\mathbf{P}}$ *yields a* (*natural*) *duality on* $\mathcal{A}$ or $\underline{P}$ *admits a* (*natural*) *duality* (or is *dualizable*).

A subset $X \subseteq P^s$ is *term-closed* if for all $y \in P^s \setminus X$, there exists s-ary term functions $\sigma, \tau : P^s \to P$ on $\underline{P}$ that agree on X but not at y. An alter ego $\underset{\sim}{\mathbf{P}}$ yields

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a strong duality on $\mathcal{A}$ if $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A}$ and every closed substructure of a power of $\underset{\sim}{\mathbf{P}}$ is term-closed.

We say that the set $G \cup H \cup R$ of a structure $\underset{\sim}{\mathbf{P}}$ *entails* a finitary algebraic relation s on $D(\underline{A})$, where $\underline{A} \in \mathcal{A}$, if every continuous map $\varphi : D(\underline{A}) \to \underset{\sim}{\mathbf{P}}$ which preserves the operations, partial operations and relations in $G \cup H \cup R$ also preserves s. The set $G \cup H \cup R$ is said to *entail* s if $G \cup H \cup R$ entails s on $D(\underline{A})$ for all $\underline{A} \in \mathcal{A}$. Clearly, if $G \cup H \cup R$ yields a duality on $\mathcal{A}$ and $G \cup H \cup R$ entails some $s \in R$, then the smaller set $G \cup H \cup R \setminus \{s\}$ also yields a duality on $\mathcal{A}$. If $G \cup H \cup R$ is minimal with respect to yielding a duality on $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{P}}$ yields *an optimal duality* on $\mathcal{A}$.

There have known examples of dualizable algebras. Pontryagin [13] showed that every finite cyclic group is dualizable. Davey and Werner [6] proved that finite one-dimensional vector spaces are dualizable. They also showed that if $\underline{P}$ has a $(k+1)$-ary near-unanimity term, then $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ yields a duality on $\mathcal{A}$. Later, Clark, Idziak, Sabourin, Szabó and Willard [2] characterized dualizable commutative ring with identity.

Our interest is to find alter egos for dualizable order-primal algebra. Davey and Rival [5, Theorem 1.1] proved that if $\underline{P}$ is an order-primal algebra corresponding to a lattice $(L; \leq)$, then the alter ego $\underset{\sim}{\mathbf{P}} = (L; \leq, \tau)$ yields a duality on $\mathcal{A}$. Applying results of Davey, Heindorf and Mckenzie [7] it follows that an order-primal $\underline{P}$ is dualizable if and only if $\underline{P}$ has an near-unanimity term if and only if $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ yields a duality on $\mathcal{A}$ for some natural number k. Consider the 6-element ordered set $\mathbf{P}$ shown in Figure 1. Davey, Quackenbush and Schweigert [8] showed that

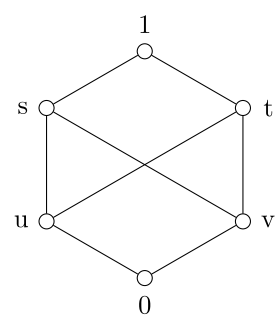

FIGURE 1. The 6-element ordered set $\mathbf{P}$

the corresponding order-primal algebra $\underline{P}$ admits a duality and the alter ego $\underset{\sim}{\mathbf{P}}$ consisting only of the 4-ary relation

$$\rho = \{(a, b, c, d) \in P^4 \mid \exists e \in P\, (a, b \leq e \leq c, d)\}$$

yields a duality on $\underline{P}$.

Graph theoretic properties of the diagrams of ordered sets is one of the most interest in Lattice Theory. For example, distributive lattices and modular lattices can be expressed as properties of their diagrams. Recently, we consider any finite ordered set whose diagram is a rooted tree. Such an ordered set is called a rooted tree. Next, we give the concrete definition. A finite ordered set $\mathbf{T}$ is a *rooted tree* if $\mathbf{T}$ has a unique maximal element and for each non-comparable elements x and y, there is no $z \in T$ with $z \leq x$ and $z \leq y$. The maximal and minimal elements are called the *root* and *leaves*, respectively.

Consider the order-primal algebras $\underline{T}$ and $\underline{T}^\partial$ corresponding to a rooted tree $\mathbf{T}$ and its dual $\mathbf{T}^\partial$, respectively. Our purpose is to prove that such algebras yield a duality. By the definition of a rooted tree, for any element $x \in T$ there is a unique path from x to the root and every two paths of x and y to the root have a minimal common point. So, the least upper bound of x and y exists and therefore, $\mathbf{T}$ is a join-semilattice. Dually, $\mathbf{T}^\partial$ is a meet-semilattice. It is shown in [8, Corollary 5.4(vi)] that every finite semilattice whose diagram is a tree has a 3-ary order-preserving near-unanimity function. Thus, $\mathbf{T}$ and $\mathbf{T}^\partial$ have a 3-ary order-preserving near-unanimity function and therefore, $\underset{\sim}{\mathbf{T}} = (T; \mathbb{S}(\underline{T}^2), \tau)$ and $\underset{\sim}{\mathbf{T}}^\partial = (T; \mathbb{S}((\underline{T}^\partial)^2), \tau)$ yield a duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$ and $\mathcal{A}^\partial = \mathbb{ISP}(\underline{T}^\partial)$, respectively. It follows that $\underline{T}$ and $\underline{T}^\partial$ are dualizable. Even though the structure $\underset{\sim}{\mathbf{P}} = (P; \mathbb{S}(\underline{P}^k), \tau)$ is an important dualising alter ego for any dualizable order-primal algebra, it take a time to find all subalgebras of $\underline{P}^k$ and its structure can be complicated, especially if k and the number of elements in P are large. In this paper, we are interested in producing alter egos $\underset{\sim}{\mathbf{P}}$ for which their structures on $\underset{\sim}{\mathbf{P}}$ are simple as possible.

2. Some properties of order-primal algebras

In this section, we focus on an order-primal algebra $\underline{P}$ corresponding to a connected ordered set $(P; \leq)$. We shall show that, in the case of a connected ordered set, the duality for such an algebra is strong.

To do so we need a basic result concerning upgrading dualities to strong dualities. Define

$$\mathrm{Irr}(\underline{P}) = \max\{\mathrm{irr}(\underline{Q}) \mid \underline{Q} \text{ is a subalgebra of } \underline{P}\},$$

where $\mathrm{irr}(\underline{Q})$ is the least n such that the zero congruence $\mathbf{0}^{\underline{Q}}$ on $\underline{Q}$ is a meet of n meet-irreducible congruence. Note that $\underline{Q}$ is subdirectly irreducible if and only if $\mathrm{irr}(\underline{Q}) = 1$.

Theorem 2.1. [1, Theorem 3.3.7] *Assume that $\underline{P}$ generates a congruence-distributive variety and that $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ yields a duality on $\mathcal{A}$. If $\underset{\sim}{\mathbf{P}}'$ is obtained from $\underset{\sim}{\mathbf{P}}$ by adding to $G \cup H$ all n-ary algebraic partial operations where $0 \leq n \leq \mathrm{Irr}(\underline{P})$, then $\underset{\sim}{\mathbf{P}}'$ yields a strong duality on $\mathcal{A}$.*

Davey, Quackenbush and Schweigert proved in [8] that every order-primal algebra $\underline{P}$ corresponding to the ordered set $(P; \leq)$ has no proper subalgebra and if $(P; \leq)$ is connected, then $\underline{P}$ is simple and hence, $\underline{P}$ is subdirectly irreducible. It follows that $\mathrm{Irr}(\underline{P}) = 1$. Using this result, we can prove the following theorem.

Proposition 2.2. *Let $k \geq 2$ and $\underline{P}$ be an order-primal algebra corresponding to a connected ordered set $(P; \leq)$ and let $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ yields a duality on $\mathcal{A}$. Then $\underset{\sim}{\mathbf{P}}$ yields a strong duality on $\mathcal{A}$.*

Proof. The algebra $\underline{P}$ has a near-unanimity term since $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A}$, So, $\underline{P}$ generates a congruence-distributive variety.

Next, we show that $\underset{\sim}{\mathbf{P}}$ yields a strong duality on $\mathcal{A}$ by Theorem 2.1. Because $\underline{P}$ has no proper subalgebra and it is simple, we have $\mathrm{Irr}(\underline{P}) = 1$. To upgrade the duality to a strong duality, it suffices to add to $\underset{\sim}{\mathbf{P}}$ all endomorphism of $\underline{P}$. Since $\underline{P}$ is order-primal and every constant map is order-preserving, every element of P is the value of a constant unary term function of $\underline{P}$. Consequently, the only endomorphism of $\underline{P}$ is id_P. It is known that id_P can be removed from an alter ego without destroying a strong duality. Therefore, $\underset{\sim}{\mathbf{P}} = (P; G, H, R, \tau)$ yields a strong duality. □

In what follows we start with a choice of $\underset{\sim}{\mathbf{T}}$ that yields a duality on $\mathcal{A}$. In order to develop a duality we need a set R of relations on T. Since $\underline{T}$ is order-primal, the order $\leq$ of the corresponding ordered set is an our first choice for a relation of a structure $\underset{\sim}{\mathbf{P}}$. As we mentioned in Section 1, if $\mathbf{P}$ is a lattice, then the structure $\underset{\sim}{\mathbf{P}} = (P; \leq, \tau)$ is a dualising alter ego for the corresponding order-primal algebra $\underline{P}$. One can ask whether the structure $\underset{\sim}{\mathbf{T}} = (T; \leq, \tau)$ yields a duality on $\mathcal{A}$. Unfortunately, it seems not to be the case as shown in the following proposition.

Proposition 2.3. *Let $\underline{T}$ be an order-primal algebra corresponding to a rooted tree $\mathbf{T} = (T; \leq)$ with n leaves where $n \geq 2$. Then $\underset{\sim}{\mathbf{T}} = (T; \leq, \tau)$ does not yield a duality on $\mathcal{A}$.*

Proof. Suppose that $\underset{\sim}{\mathbf{T}} = (T; \leq, \tau)$ yields a duality on $\mathcal{A}$. Then by Proposition 2.2, $\underset{\sim}{\mathbf{T}}$ yields a strong duality on $\mathcal{A}$ implying that $\underset{\sim}{\mathbf{T}}$ is injective in the category $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{T}})$.

Let $\hat{\mathbf{T}}$ be the Dedekind-MacNeille completion of $\mathbf{T}$. Then there exists an element $0 \in \hat{T}$ for which 0 is the minimum element of $\hat{\mathbf{T}}$ and $\hat{\mathbf{T}} \setminus \{0\} = \mathbf{T}$.

Next, we show that $\hat{\mathbf{T}} \in \mathcal{X}$. Assume that $a_1, a_2, \ldots, a_n$ and 1 are leaves and the root of $\mathbf{T}$, respectively. For an $i \in I = \{1, 2, \ldots, n\}$, we denote by C_i the path from a_i to the root.

Recall that t covers s if and only if $s < t$, but there does not exist a $z \in T$ with $s < z < t$. For any $t \in T$, we denote by t_i the element in a path C_i that is covered by t and $I_t = \{i \in I \mid t \in C_i\}$. Let $X = \{(a_1, a_2, \ldots, a_n), (1, 1, \ldots, 1)\} \cup \bigcup_{i \in I} \{(b_1, b_2, \ldots, b_{i-1}, t, b_{i+1} \ldots, b_n) \mid t \in C_i \setminus \{a_i\}\}$ where $b_j = a_j$ and $b_k = t$ for all $j \in I \setminus I_{t_i}$ and $k \in I_{t_i}$. Then X forms a closed substructure of $\underset{\sim}{\mathbf{T}}^n$. It is obvious that $(a_1, a_2, \ldots, a_n)$ and $(1, 1, \ldots, 1)$ are the minimum and the maximum elements of X, respectively. Define $\psi : X \to \hat{T}$ by

$$\psi(t) = \begin{cases} 0, \ x = (a_1, a_2, \ldots, a_n), \\ t_i, \ x = (b_1, b_2, \ldots, b_{i-1}, t, b_{i+1} \ldots, b_n), \\ 1, \ x = (1, 1, \ldots, 1). \end{cases}$$

To show that $\hat{\mathbf{T}} \in \mathcal{X}$, we need to prove that $\psi(t)$ is an order-isomorphism.

First, we show that $\psi(t)$ is an order-embedding, that is, for each $x, y \in X$, $x \leq y$ if and only if $\psi(x) \leq \psi(y)$. Let $x, y \in X$ with $x \leq y$. If $x = (a_1, a_2, \ldots, a_n)$, then $\psi(x) = 0$ and from 0 is the minimum element of $\hat{\mathbf{T}}$, we have $\psi(x) \leq \psi(y)$. If $x = (1, 1, \ldots, 1)$, then x is the maximum element of X and from $x \leq y$, we have $x = y$ and hence, $\psi(x) \leq \psi(y)$.

Let $x = (b_1, b_2, \ldots, b_{i-1}, s, b_{i+1} \ldots, b_n)$ and $y = (c_1, c_2, \ldots, c_{j-1}, t, c_{j+1} \ldots, c_n)$ where $s \in C_i \setminus \{a_i\}$ and $t \in C_j \setminus \{a_j\}$. We may assume that $i \leq j$. Then $b_1 \leq c_1, \ldots, s \leq c_i, \ldots, b_j \leq t, \ldots, b_n \leq c_n$. Since $b_k \in \{a_1, a_2, \ldots, a_n, s\}$ and $c_l \in \{a_1, a_2, \ldots, a_n, t\}$ for all $k, l \in I$ and $s \leq c_i$, so $c_i = t$. Because of $c_l = t$ if $l \in I_{t_j}$, so $i \in I_{t_j}$ implying $t_j \in C_i$. From s_i is covered by s and $s \leq t$, we have $s_i \leq t_j$. By the definition of ψ, we get $\psi(x) = s_i \leq t_j = \psi(y)$.

Conversely, assume that $\psi(x) \leq \psi(y)$. If $\psi(x) = 0$, then $x = (a_1, a_2, \ldots, a_n)$ and so, $x \leq y$. If $\psi(x) = 1$, then $\psi(x) \leq \psi(y)$, we have $\psi(y) = 1$. By the definition of

ψ, we get that $x = (1, 1 \ldots, 1) = y$. If $\psi(y) = 0$, then from $\psi(x) \leq \psi(y)$, we have $\psi(x) = 0$. By the definition of ψ, we get $x = (a_1, a_2, \ldots, a_n) = y$. If $\psi(y) = 1$, then $y = (1, 1 \ldots, 1)$ and so, $x \leq y$.

Consider $\psi(x), \psi(y) \notin \{0, 1\}$. Clearly, every pair of elements in $\mathbf{T}$ are comparable if and only if they belong to the same path and then there are $s_i, t_i \in C_i$ with $\psi(x) = s_i \leq t_i = \psi(y)$. If $s_i \in C_j$, then $t_i \in C_j$ since otherwise there is a $z \in T$ with $z > s_i < t_i$ which is contradict to that $\mathbf{T}$ is a tree. Consequently, $I_{s_i} \subseteq I_{t_i}$. Again from $\mathbf{T}$ is a tree, there exist unique elements $s, t \in T$ that are covered s_i and t_i, respectively. It follows that $x = (b_1, b_2, \ldots, b_n)$ and $y = (c_1, c_2, \ldots, c_n)$ where $b_j = a_j$, $b_k = s$, $c_l = a_l$ and $c_m = t$ for all $j \in I \setminus \{I_{s_i}\}$, $k \in \{I_{s_i}\}$, $l \in I \setminus \{I_{t_i}\}$ and $m \in \{I_{t_i}\}$. From $s_i \leq t_i$, we get $s \leq t$ implying that $b_k \leq c_k$ for all $k \in I$. Therefore, $x \leq y$.

It is clear that ψ is surjective. Altogether, we can prove that ψ is an order-isomorphism. Therefore, $\hat{\mathbf{T}} \in \mathcal{X}$.

Consider the map id_T and the embedding $\varphi : T \to \hat{T}$ defined by $\varphi(t) = t$ for all $t \in T$. Both are morphisms in the category $\mathcal{X}$. Since $\underset{\sim}{\mathbf{T}}$ is injective, there is an order-preserving map $\beta : \hat{\mathbf{T}} \to \mathbf{T}$ with $\beta\varphi = id_T$. From $a_1 > 0 < a_2$ in $\hat{\mathbf{T}}$, we have

$$a_1 = id_P(a_1) = \beta\varphi(a_1) = \beta(a_1) \geq \beta(0) \leq \beta(a_2) = \beta\varphi(a_2) = id_P(a_2) = a_2$$

in $\mathbf{T}$ which is a contradiction since a_1 and a_2 are different minimal elements in $\mathbf{T}$. Therefore, $\underset{\sim}{\mathbf{T}} = (T; \leq, \tau)$ does not yield a duality on $\mathcal{A}$. □

In Section 3 we present alter egos that yield a duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$. Before doing it, we need to mention some basic concepts which are introduced by Davey, Quackenbush and Schweigert [8]. They used them as a tool for proving that the structure $\underset{\sim}{\mathbf{P}} = (P; \rho, \tau)$ yields a duality.

Recall that for any k-ary relation r on a set P, we define the k-ary relation r^n on P^n where n is a natural number as follows:

$$(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k) \in r^n \text{ if and only if } (a_{1j}, a_{2j}, \ldots, a_{kj}) \in r$$

for all $j \in \{1, 2, \ldots, n\}$ where $\bar{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in P^n$.

Given $X \subseteq P^n$ and $f : X \to P$, we say that f *preserves* r if

$$(f(x_1), f(x_2), \ldots, f(x_n)) \in r$$

for all $x_1, x_2, \ldots, x_n \in X$ with $(x_1, x_2, \ldots, x_n) \in r^n$. For convenience, we denote $\leq_{P^n}$ by $\leq$.

Let $\mathbf{P} = (P; \leq)$ be an ordered set and $X \subseteq P^n$ for some $n \in \mathbb{N}$. For a map $f : X \to P$ and $a \in P$, we denote

$$X_a^f = \{x \in X \mid f(x) = a\}.$$

For an element $y \in P^n \setminus X$, we define $y < X_a^f$ if there is an element $x \in X_a^f$ with $y < x$ and $X_a^f < y$ is defined dually.

Denote $U_y^f = \{a \in P \mid y < X_a^f\}$ and $L_y^f = \{a \in P \mid X_a^f < y\}$. An element $a \in P$ is called an *upper associate* of y with respect to f if a is minimal in U_a^f and a *lower associate* of y with respect to f is defined dually.

Example 2.4. Consider the 6-element ordered set $\mathbf{P}$ as mentioned in Section 1. Let $X = \{(0, u), (0, v), (1, s), (1, t)\} \subseteq P^2$ and let $f : X \to P^2$ be a map defined by $f(a, b) = b$ for all $(a, b) \in X$. Let $y = (0, s)$. Then from $(0, u) < (0, s)$ and $f(0, u) = u$, we have $u \in L_y^f$. Similarly, $v \in L_y^f$ and hence $L_y^f = \{u, v\}$. If we consider $y = (1, u)$, then from $(1, u) < (1, s)$ and $f(1, s) = s$, we have $s \in U_y^f$. Similarly, $t \in U_y^f$ and hence $U_y^f = \{s, t\}$.

Observe that different upper associates (lower associates) of y in Example 2.4 are non-comparable. The following proposition shows that every ordered set satisfies such properties.

Proposition 2.5. *Let $\boldsymbol{P} = (P; \leq)$ be an ordered set and $X \subseteq P^n$ for some $n \in \mathbb{N}$. Let $f : X \to T$ be a map and $y \in P^n$. Then the following conditions hold:*

(i) *If u_1 and u_2 are upper associates of y with $u_1 \neq u_2$, then $u_1 \| u_2$.*
(ii) *If l_1 and l_2 are lower associates of y with $l_1 \neq l_2$, then $l_1 \| l_2$.*
(iii) *Assume that f preserves $\leq$. If l and u are a lower associate and an upper associate of y, respectively, then $l \leq u$.*

Proof. (i) Since u_1 and u_2 are upper associates of y, so u_1 and u_2 are minimal elements in $U_y^f = \{a \in T \mid y < X_a^f\}$. Because of $u_1 \neq u_2$, we have $u_1 \not\leq u_2$ and $u_2 \not\leq u_1$, that is, $u_1 \| u_2$.

(ii) The proof is similar to that of (i).

(iii) Since l and u are a lower associate and an upper associate of y, respectively, $X_l^f < y < X_u^f$. So, there are $l' \in X_l^f$ and $u' \in X_u^f$ such that $l' < y < u'$. Because f preserves $\leq$, we have $l = f(l') \leq f(u') = u$. □

3. Dualising alter egos for order-primal algebras

In Section 2 we showed that the structure $\underset{\sim}{\mathbf{T}} = (T; \leq, \tau)$ does not yield a duality. Our aim is to find a dualising alter ego for $\underline{T}$. To produce so we need to add some relations on T to $\underset{\sim}{\mathbf{T}}$.

Next, we give an idea to define such relations on T. Consider the Dedekind-MacNeille completion $\hat{\mathbf{P}}$ of the 6-element ordered set $\mathbf{P}$ as shown in Figure 2.

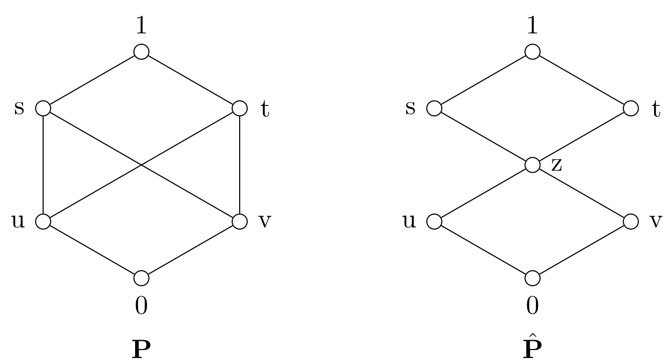

Figure 2. The Dedekind-MacNeille completion $\hat{\mathbf{P}}$

It can be seen that the diagrams of $\mathbf{P}$ and $\hat{\mathbf{P}}$ are almost the same, except that there exists an element $z \in \hat{\mathbf{P}}$ with $u, v \leq z \geq s, t$. Because the relation ρ in the dualising alter ego of $\underline{T}$ in Section 1 is defined by

$$\rho = \{(a, b, c, d) \in T^4 \mid \exists e \in T\,(a, b \leq e \leq c, d)\},$$

the relation ρ seems to be defined by the existence of z in its Dedekind-MacNeille completion. It is known that the dualising alter ego $\underset{\sim}{\mathbf{P}}$ in Section 1 consisting only the relation ρ. The order $\leq$ of $\mathbf{P}$ need not to add to the structure $\underset{\sim}{\mathbf{P}}$. since it can be proved that $x \leq y$ if and only if $(x, x, y, y) \in \rho$. So, This leads us to produce a relation on T by this method.

For a rooted tree $\mathbf{T} = (T; \leq)$ with n leaves, its Dedekind-MacNeille completion $\hat{\mathbf{T}}$ of $\mathbf{T}$ has an element $0 \in \hat{T}$ that is lesser than all leaves of $\mathbf{T}$ as shown in Figure 4.

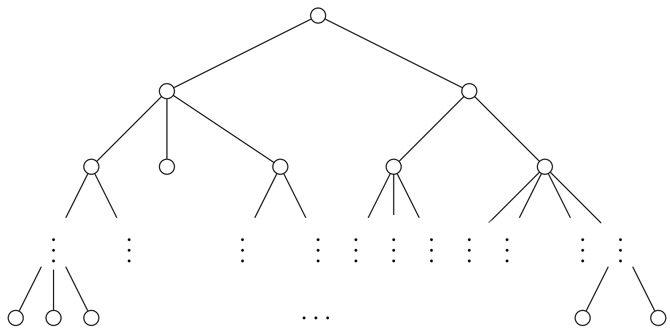

FIGURE 3. The rooted tree $\mathbf{T}$

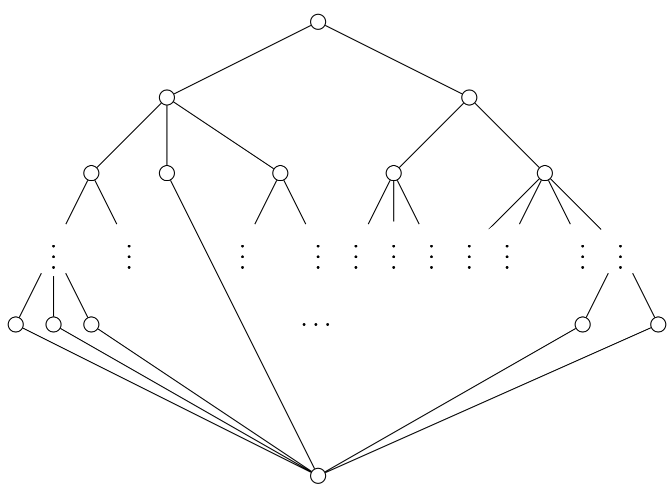

FIGURE 4. The Dedekind-MacNeille completion $\hat{\mathbf{T}}$

We define the n-ary relation β on T by

$$(b_1, b_2, \ldots, b_n) \in \beta \Leftrightarrow \exists e \in T \forall i \in \{1, 2, \ldots, n\}(e \leq b_i).$$

It is easy to see that the arity of β depend on the number of leaves of $\mathbf{T}$.

We now prove a technical proposition which will be the main too for yielding a duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$.

Proposition 3.1. *Let $\boldsymbol{T} = (T; \leq)$ be a rooted tree with n leaves and $X \subseteq T^n$. Assume that $f : X \to T$ preserves β and $y \in T^n \setminus X$. Then following conditions hold:*

(i) *y has at most 1 upper associate.*
(ii) *Consider the extension map $\bar{f} : X \cup \{y\} \to T$ defined by*

$$\bar{f}(x) = \begin{cases} f(x) \text{ if } x \in X, \\ u \text{ if } x = y \text{ and } y \text{ has an upper associate}, \\ 1 \text{ if } x = y \text{ and } y \text{ has no upper associate}. \end{cases}$$

where u and 1 are the upper associate and the root of y and $\boldsymbol{T}$, respectively. If f preserves $\leq$, then $\bar{f}$ preserves $\{\leq, \beta\}$.

Proof. Let $y \in T^n \setminus X$.

(i) Suppose that s and t are different upper associates of y. Then $y < X_s^f$ and $y < X_t^f$ and by Propsition 2.5(i), we have $s \| t$. If follows that there are $s' \in X_s^f$ and $t' \in X_t^f$ such that $y < s'$ and $y < t'$. Thus $(s', t', \ldots, t') \in \beta_{P^n}$ and from f preserves β on X, we have $(s, t, \ldots, t) = (f(s'), f(t'), \ldots, f(t')) \in \beta$. So, there is a $k \in T$ with $k \leq s$ and $k \leq t$ which is a contradiction since $\mathbf{T}$ is a tree.

(ii) Let $a, b \in X \cup \{y\}$ such that $a \leq b$. We consider a and b in the following 4 cases.

<u>Case 1</u> $a, b \in X$. Then $\bar{f}(a) = f(a) \leq f(b) = \bar{f}(b)$ since f preserves $\leq$ on X.

<u>Case 2</u> $a \in X$ and $b = y$. Then $a < y$. If y has no upper associate, then $\bar{f}(y) = 1$ implies that

$$\bar{f}(a) = f(a) \leq 1 = \bar{f}(y) = \bar{f}(b).$$

If y has an upper associate, then $\bar{f}(y) = u$ and $y < X_u^f$. So, there is an $u' \in X_u^f$ with $a < y < u'$. Thus $\bar{f}(a) = f(a) \leq f(u') = u = \bar{f}(y) = \bar{f}(b)$.

<u>Case 3</u> $a = y$ and $b \in X$. Then $y < b$ implies that $y < X_{f(b)}^f$ and hence $f(b) \in U_y^f$. Since $U_y^f \subseteq T$ and $\mathbf{T}$ is a tree with n leave, U_y^f has a minimal element. It follows that y has an upper associate and so, $\bar{f}(y) = u$. Since y has at most 1 upper associate, u is the minimum element in U_y^f and therefore, $u \leq f(b)$. Thus $\bar{f}(a) = \bar{f}(y) = u \leq f(b) = \bar{f}(b)$.

<u>Case 4</u> $a = y = b$. Then $\bar{f}(a) = \bar{f}(y) = \bar{f}(b)$.

Altogether, we can prove that $\bar{f}$ preserves $\leq$ on $X \cup \{y\}$.

Next we show that $\bar{f}$ preserves β on $X \cup \{y\}$.

Let $(b_1, b_2, \ldots, b_n) \in (\beta_{T^n}) \cap (X \cup \{y\})$. Then there is an $e \in T^n$ with $e \leq b_i$ for all $i \in \{1, 2, \ldots, n\}$. Denote $B = \{b_1, b_2, \ldots, b_n\}$. We consider $(b_1, b_2, \ldots, b_n)$ in the following 3 cases.

<u>Case 1</u> $b_1, b_2, \ldots, b_n \in X$. Then $\bar{f}(b_i) = f(b_i)$ for all $i \in \{1, 2, \ldots, n\}$ and from f preserves β on X, $(\bar{f}(b_1), \bar{f}(b_2), \ldots, \bar{f}(b_n)) = (f(b_1), f(b_2), \ldots, f(b_n)) \in \beta$.

<u>Case 2</u> $B \cap X \neq \emptyset$ and $B \cap \{y\} \neq \emptyset$. Let $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$ be all elements in $B \cap X$ and let $b_{j_1}, b_{j_2}, \ldots, b_{j_l}$ be all elements in B that are y.

<u>Case 2.1</u> y has no upper associate. Then $\bar{f}(y) = 1$ implies that

$$\bar{f}(b_{j_1}) = \bar{f}(b_{j_2}) = \cdots = \bar{f}(b_{j_l}) = 1.$$

Since $e \leq b_i$ for all $i \in \{1, 2, \ldots, n\}$, so $e \leq b_{i_p}$ for all $p \in \{1, 2, \ldots, k\}$ implies that $(b_{i_1}, b_{i_2}, \ldots, b_{i_k}, \ldots, b_{i_k}) \in \beta_{T^n}$. Because of $b_{i_1}, b_{i_2}, \ldots, b_{i_k} \in X$ and f preserves β

on X, we have $(f(b_{i_1}), f(b_{i_2}), \ldots, f(b_{i_k}), \ldots, f(b_{i_k})) \in \beta$. So, there is an $x \in T$ with $x \leq f(b_{i_p})$ for all $p \in \{1, 2, \ldots, k\}$. Since 1 is the maximum element of $\mathbf{T}$, so $x \leq \bar{f}(b_{j_q})$ for all $q \in \{1, 2, \ldots, l\}$. Thus $x \leq \bar{f}(b_i)$ for all $i \in \{1, 2, \ldots, n\}$ and therefore, $(\bar{f}(b_1), \bar{f}(b_2), \ldots, \bar{f}(b_n)) \in \beta$.

Case 2.2 y has an upper associate. Then $\bar{f}(y) = u$ and $y < X_u^f$ implies that there is an $u' \in X_u^f$ with $y < u'$. Because of $e \leq b_{j_1} = y$, we have $e \leq u'$. Since $e \leq b_{i_p}$ for all $p \in \{1, 2, \ldots, k\}$, so $(b_{i_1}, b_{i_2}, \ldots, b_{i_k}, u', \ldots, u') \in \beta_{T^n}$. From $b_{i_1}, b_{i_2}, \ldots, b_{i_k}, u' \in X$ and f preserves β on X, we have $(f(b_{i_1}), f(b_{i_2}), \ldots, f(b_{i_k}), f(u'), \ldots, f(u')) \in \beta$ implying that there is a $z \in T$ with $z \leq f(b_{i_p}) = \bar{f}(b_{i_p})$ for all $p \in \{1, 2, \ldots, k\}$ and $z \leq f(u')$. Since $f(u') = u = \bar{f}(y) = \bar{f}(b_{j_q})$ for all $q \in \{1, 2, \ldots, l\}$, so $z \leq \bar{f}(b_{j_q})$ for all $q \in \{1, 2, \ldots, l\}$. Hence, $z \leq \bar{f}(b_i)$ for all $i \in \{1, 2, \ldots, n\}$ and therefore, $(\bar{f}(b_1), \bar{f}(b_2), \ldots, \bar{f}(b_n)) \in \beta$.

Case 3 $b_i = y$ for all $i \in \{1, 2, \ldots, n\}$. Then $\bar{f}(b_i) = \bar{f}(y)$ for all $i \in \{1, 2, \ldots, n\}$. implies that $(\bar{f}(b_1), \bar{f}(b_2), \ldots, \bar{f}(b_n)) \in \beta$. □

Dually, the n-ary relation β^∂ on T is defined by

$$(b_1, b_2, \ldots, b_n) \in \beta^\partial \ \Leftrightarrow \exists e \in T \forall i \in \{1, 2, \ldots, n\}(e \geq b_i).$$

The similar results of Proposition 3.1 are obtained for β^∂ as shown in the following proposition.

Proposition 3.2. *Let $\boldsymbol{T} = (T; \leq)$ be a rooted tree with n leaves and $X \subseteq T^n$. Assume that $f : X \to T$ preserves β^∂ and $y \in T^n \setminus X$. Then following conditions hold:*

(i) *y has at most 1 lower associate.*
(ii) *Consider the extension map $\bar{f} : X \cup \{y\} \to T$ defined by*

$$\bar{f}(x) = \begin{cases} f(x) \text{ if } x \in X, \\ l \text{ if } x = y \text{ and } y \text{ has a lower associate}, \\ 0 \text{ if } x = y \text{ and } y \text{ has no lower associate}. \end{cases}$$

where l and 0 are the lower associate and the minimum of y and $\boldsymbol{T}$, respectively. If f preserves $\leq$, then $\bar{f}$ preserves $\{\leq, \beta^\partial\}$.

Proof. The proof is dual to Proposition 3.1. □

Consider a topological structure $\underset{\sim}{\mathbf{P}}$ having no partial operations and only finitely many relations. Then $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P})$ if $\underset{\sim}{\mathbf{P}}$ satisfies the interpolation condition (IC) with respect to $\underline{P}$ as stated below.

Theorem 3.3. [1, The Second Duality Theorem] *Assume that $\underset{\sim}{\mathbf{P}} = (P; G, R, \tau)$ and assume that R is finite. If $\underset{\sim}{\mathbf{P}}$ satisfies the condition:*

(IC) *for each $n \in \mathbb{N}$ and each substructure X of P^n, every morphism $\alpha : X \to P$ extends to a term function $t : P^n \to P$ of the algebra $\underline{P}$,*

then $\underset{\sim}{\mathbf{P}}$ yields a duality on $\mathcal{A}$ and $\underset{\sim}{\mathbf{P}}$ is injective in $\mathcal{X}$.

Now, we give dualising alter egos for the order-primal algebras corresponding to a rooted tree and its dual.

Theorem 3.4. *Let $\underline{T}$ be an order-primal algebra corresponding to a rooted tree $\mathbf{T} = (T; \leq)$ with n leaves and let $\underset{\sim}{\mathbf{T}} = (T; \leq, \beta, \tau)$ and $\underset{\sim}{\mathbf{T}}^\partial = (T; \leq, \beta^\partial, \tau)$ where τ is the discrete topology. Then $\underset{\sim}{\mathbf{T}}$ and $\underset{\sim}{\mathbf{T}}^\partial$ yield a duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$ and*

$\mathcal{A}^\partial = \mathbb{ISP}(\underline{T}^\partial)$, respectively. Moreover, $\underset{\sim}{\mathbf{T}}$ and $\underset{\sim}{\mathbf{T}}^\partial$ yield a strong duality on $\mathcal{A}$ and $\mathcal{A}^\partial$, respectively.

Proof. Since the number of relations of $\underset{\sim}{\mathbf{T}}$ is finite. We shall show that $\underset{\sim}{\mathbf{T}}$ yields a duality on $\mathcal{A}$ by proving that $\underset{\sim}{\mathbf{T}}$ satisfies (IC).

Let $f : X \subseteq T^n \to T$ be a $\{\leq, \beta\}$-preserving and let $y \in T^n \setminus X$. To show that (IC) holds, it suffices to extend f to $X \cup \{y\}$. Proposition 3.1(ii) shows that the mapping f can be extended to $\bar{f} : X \cup \{y\} \to T$ that preserves $\leq$ and β. Thus $\underset{\sim}{\mathbf{T}}$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$. By Proposition 2.2, $\underset{\sim}{\mathbf{T}}$ yields a strong duality on $\mathcal{A}$.

To show that $\underset{\sim}{\mathbf{T}}^\partial$ yield a duality on $\mathcal{A}^\partial$, the proof is dual to the previous one. □

Since we need the structure on $\underset{\sim}{\mathbf{P}}$ that is as simple as possible, one can ask whether our structure is optimal, that is, if any relation in the structure were deleted, the duality would be destroyed. Unfortunately, it is not optimal as shown by the following example.

Example 3.5. Consider the order-primal algebra $\underline{T}$ corresponding to a finite chain $\mathbf{T} = (T; \leq)$ as shown in Figure 5. It is clear that $\mathbf{T}$ is a rooted tree with 1 leaf. So,

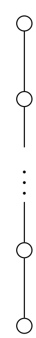

FIGURE 5. A finite chain $\mathbf{T}$

the relation β is a unary relation defined by $\{b \in T \mid \exists e \in T(e \leq b)\}$. Since $b \leq b$ for all $b \in T$, so $\beta = T$. Theorem 3.4 implies that $\underset{\sim}{\mathbf{T}} = (T; \leq, T, \tau)$ yields a duality on $\mathcal{A}$. It is known that every chain is a lattice. Applying the result of Davey and Rival [5, Theorem 1.1], it follows that the structure $\underset{\sim}{\mathbf{T}}' = (T; \leq, \tau)$ yields a duality on $\mathcal{A}$. Therefore, the structure $\underset{\sim}{\mathbf{T}}$ is not optimal since we can delete the relation β from $\underset{\sim}{\mathbf{T}}$ and the result structure still yields a duality on $\mathcal{A}$.

It is clear that chains being the only exception, every ordered set $\mathbf{T}$ whose diagram is a tree is not a lattice. Proposition 2.3 shows that $\underset{\sim}{\mathbf{T}}' = (T; \leq, \tau)$ is not a dualising alter ego for the order-primal algebra $\underline{T}$ corresponding to $\mathbf{T}$. Our question is that the alter egos in Theorem 3.4 is optimal if $\mathbf{T}$ is not a chain. The answer is shown in the following theorem.

Theorem 3.6. *Let $\underline{T}$ be an order-primal algebra corresponding to a rooted tree $\mathbf{T} = (T; \leq)$ with n leaves where $n \geq 2$. Then $\underset{\sim}{\mathbf{T}} = (T; \leq, \beta, \tau)$ and $\underset{\sim}{\mathbf{T}}^\partial = (T; \leq, \beta^\partial, \tau)$ in Theorem 3.4 yield an optimal duality on $\mathcal{A}$ and $\mathcal{A}^\partial$, respectively.*

Proof. By Proposition 2.2, $\underset{\sim}{\mathbf{T}} = (T; \leq, \beta, \tau)$ in Theorem 3.4 yields a strong duality on $\mathcal{A}$. It follows that for each subalgebra $\underline{A}$ of $\underline{P}^n$, the structure $D(\underline{A})$ is generated by $\{\rho_1, \dots, \rho_n\}$, where ρ_i is the i^{th} n-ary projection: see [1, Exercise 9.8(iii)]. As $\underset{\sim}{\mathbf{P}}$ is purely relational, we have $D(\underline{A}) = \{\rho_1, \dots, \rho_n\}$.

To show that $\leq$ does not entail β, it suffices to prove that $\leq$ does not entail β on $D(\beta)$. Since β is an n-ary relation, $D(\beta) = \{\rho_1, \dots, \rho_n\}$. To avoid confusion, we shall denote a relation r on $D(\beta)$ by $r_{D(\beta)}$.

Let $a_1, \dots, a_n$ and 1 be different leaves and the root of $\mathbf{T}$, respectively. Then from $n \geq 2$, we have $a_1 \neq a_2$ and hence there is no $e \in T$ with $e \leq a_1$ and $e \leq a_2$. So, $(a_1, \dots, a_n) \notin \beta$.

Now, we show that $\leq_{D(\beta)} = \Delta_{D(\beta)}$. Suppose that $\leq_{D(\beta)} \neq \Delta_{D(\beta)}$. Then there are different numbers $i, j \in \{1, 2, \dots, n\}$ with $(\rho_i, \rho_j) \in \leq_{D(\beta)}$. We may assume that $i < j$. From $|T| \geq 2$ and $\mathbf{T}$ is connected, there are different elements $a, b \in T$ with $a < b$. By the definition of β, we get $(b, \dots, b, \underbrace{a}_{j\text{th}}, b, \dots, b) \in \beta$ implying that

$$(\rho_i(b, \dots, b, a, b, \dots, b), \rho_j(b, \dots, b, a, b, \dots, b)) = (b, a) \in \leq$$

that is, $b \leq a$, a contradiction.

Define $\varphi : D(\beta) \to T$ by $\varphi(\rho_i) = a_i$ for all $i \in \{1, 2, \dots, n\}$.

To show that $\leq$ does not entail β on $D(\beta)$, it suffices to show that φ preserves $\leq$, but does not preserve β. Because of $\leq_{D(\beta)} = \Delta_{D(\beta)}$, we get that φ preserves $\leq$. Since $(\rho_1(b_1, b_2, \dots, b_n), \rho_2(b_1, b_2, \dots, b_n), \dots, \rho_n(b_1, b_2, \dots, b_n)) = (b_1, b_2, \dots, b_n) \in \beta$ for all $(b_1, b_2, \dots, b_n) \in \beta$, so $(\rho_1, \rho_2, \dots, \rho_n) \in \beta_{D(\beta)}$. But

$$(\varphi(\rho_1), \varphi(\rho_2), \dots, \varphi(\rho_n)) = (a_1, a_2, \dots, a_n) \notin \beta,$$

so φ does not preserve β.

The remainder of the proof is to show that β does not entail $\leq$ on $D(\leq)$. From $\leq$ is a binary relation, $D(\leq) = \{\rho_1, \rho_2\}$. Consider an element $(\rho_{1_1}, \rho_{1_2}, \dots, \rho_{1_n}) \in (D(\leq))^n$ and $a \leq b$. The projection $\rho_{1_i}(a, b) \in \{a, b\}$ for all $i \in \{1, 2, \dots, n\}$. By the definition of β, we have

$$(\rho_{1_1}(a, b), \rho_{1_2}(a, b), \dots, \rho_{1_n})(a, b) \in \beta.$$

So, $(\rho_{1_1}, \rho_{1_2}, \dots, \rho_{1_n}) \in \beta_{D(\leq)}$ and hence $\beta_{D(\leq)} = (D(\leq))^n$.

Define $\psi : D(\leq) \to T$ defined by $\psi(\rho_1) = 1$ and $\psi(\rho_2) = a_1$.

Since $(\psi(\rho_{1_1}), \psi(\rho_{1_2}), \dots, \psi(\rho_{1_n})) = (b_1, b_2, \dots, b_n)$ where $b_i \in \{a_1, 1\}$ for all $i \in \{1, 2, \dots, n\}$ and $a_1 \leq 1$, so $(\psi(\rho_{1_1}), \psi(\rho_{1_2}), \dots, \psi(\rho_{1_n})) \in \beta$. Consequently, ψ preserve β. It is obvious that $(\rho_1, \rho_2) \in \leq_{D(\leq)}$. By the definition of ψ, we have $(\psi(\rho_1), \psi(\rho_2)) = (1, a_1)$. But $(1, a_1) \notin \leq$, so ψ does not preserve $\leq$. Therefore, $\underset{\sim}{\mathbf{T}} = (T; \leq, \beta, \tau)$ yields an optimal duality. □

Although the structure of $\underset{\sim}{\mathbf{T}}$ in Theorem 3.4 is given by two relations, it may not be simple since the arity of the relation β in $\underset{\sim}{\mathbf{T}}$ can be very large depending on the number of leaves in T. Our goal is to find a new alter ego for which every relation of its structure is independent of the number of leaves in $\mathbf{T}$. It is known that the structure $\underset{\sim}{\mathbf{T}} = (T; \mathbb{S}(\underline{T}^2), \tau)$ yields a duality on $\mathcal{A}$. This leads us to consider an alter ego of $\underline{T}$ given by only binary relations.

To finish this section, it remains to find such binary relations. Reduce the arity of the relation β to be 2. We get the binary relation θ on T defined by

$$(b_1, b_2) \in \theta \ \Leftrightarrow \exists e \in T, e \leq b_1 \wedge e \leq b_2.$$

Dually, the relation θ^∂ is defined by

$$(b_1, b_2) \in \theta^\partial \ \Leftrightarrow \exists e \in T, e \geq b_1 \wedge e \geq b_2.$$

We obtain new dualsing alter egos for the order-primal algebras $\underline{T}$ and $\underline{T}^\partial$ as stated below.

Theorem 3.7. *Let $\underline{T}$ be an order-primal algebra corresponding to a rooted tree $\mathbf{T} = (T; \leq)$ and let $\underset{\sim}{\mathbf{T}} = (T; \leq, \theta, \tau)$ and $\underset{\sim}{\mathbf{T}}^\partial = (T; \leq, \theta^\partial, \tau)$ where τ is the discrete topology. Then $\underset{\sim}{\mathbf{T}}$ and $\underset{\sim}{\mathbf{T}}^\partial$ yield a duality on $\mathcal{A} = \mathbb{ISP}(\underline{T})$ and $\mathcal{A}^\partial = \mathbb{ISP}(\underline{T}^\partial)$, respectively. Moreover, $\underset{\sim}{\mathbf{T}}$ and $\underset{\sim}{\mathbf{T}}^\partial$ yield a strong duality on $\mathcal{A}$ and $\mathcal{A}^\partial$, respectively.*

Proof. The proof is similar to that of Theorem 3.4. □

The following theorem shows that the alter egos in Theorem 3.7 yield an optimal duality.

Theorem 3.8. *Let $\underline{T}$ be an order-primal algebra corresponding to a rooted tree $\mathbf{T} = (T; \leq)$ with n leaves where $n \geq 2$. Then $\underset{\sim}{\mathbf{T}} = (T; \leq, \theta, \tau)$ and $\underset{\sim}{\mathbf{T}}^\partial = (T; \leq, \theta^\partial, \tau)$ in Theorem 3.7 yield an optimal duality on $\mathcal{A}$ and $\mathcal{A}^\partial$, respectively.*

Proof. The proof is similar to that of Theorem 3.6. □

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Regular subsemigroups of the transformation semigroups

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It is well known that the transformation semigroup $T(X)$ is regular, but their subsemigroups need not be. A fence is an ordered set that the order forms a path with alternating orientation. Consider X as the base set of a fence $\mathbf{X} = (X; \leq)$. Two subsemigroups of $T(X)$ are studied. Namely, the semigroup $DT(\mathbf{X})$ of all order-decreasing self-mappings of $\mathbf{X}$ and the semigroup $OT(\mathbf{X})$ of all order-preserving self-mappings of $\mathbf{X}$. In this paper, we obtain that $DT(\mathbf{X})$ is both a regular and a coregular subsemigroup of $T(X)$. A characterization of regular subsemigroups $OT(\mathbf{X})$ of $T(X)$ is given, that is, $OT(\mathbf{X})$ is regular if and only if $|X| \leq 4$. Finally, we discuss the regularity of elements in $OT(\mathbf{X})$.

Keywords: order-decreasing; order-preserving; fence; transformation semigroup; regular; coregular.

AMS Subject Classification: 0M20, 20M05, 20M17

1. Introduction and Preliminaries

Let X be an arbitrary set and let $T(X)$ be the semigroup of transformations of X. There have been many research works studying the semigroup $T(X)$, especially over the last fifty years.

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The concept of regularity is one of the most-studied topics in Semigroup Theory. It is well known that $T(X)$ is regular (see [2], page 33). But subsemigroups of $T(X)$ need not be regular. In this paper we are concerned with question: What is a regular subsemigroup of $T(X)$? The different types of regular subsemigroups of $T(X)$ have been much studied. In 2005, Huisheng [5] considered the subsemigroup

$$T_E(X) = \{\alpha \in T(X) \mid \forall (x, y) \in E, (\alpha(x), \alpha(y)) \in E\}$$

of $T(X)$ where E is an equivalence relation on X. He discussed regularity for $T_E(X)$. Later, Sanwong and Sommanee gave a necessary and sufficient condition for the subsemigroup $T(X, Y)$ of $T(X)$ to be regular where $Y \subseteq X$ and

$$T(X, Y) = \{\alpha \in T(X) \mid \alpha(X) \subseteq Y\}.$$

A mapping $\alpha : P \to Q$ is *order-preserving* from an ordered set $\mathbf{P}$ to an ordered set $\mathbf{Q}$ if $x \leq y$ in $\mathbf{P}$, then $\alpha(x) \leq \alpha(y)$ in $\mathbf{Q}$. Consider X as the base set of a chain $\mathbf{X} = (X; \leq)$. The semigroup $OT(\mathbf{X})$ of all order-preserving self-mappings of $\mathbf{X}$ is a regular subsemigroup of $T(X)$ (see [3]). Keprasit and Changphas [7] extended this result to subchains of $\mathbb{Z}$. A characterization of regular semigroups $OT(\mathbf{X})$ where X is an interval in $\mathbb{R}$ is given in [7]. Ma, You Lua, Yang and Wang [8] investigated the regularity of a subsemigroup

$$EOP_X = \{\alpha \in T(X) \mid \forall (a, b) \in E, (x, y) \in E \text{ and } x \leq y$$

$$\Rightarrow (\alpha(x), \alpha(y)) \in E \text{ and } \alpha(x) \leq \alpha(y)\}$$

of $T(X)$ where E is an equivalence relation on X. The semigroup $OT(\mathbf{X})$ has been studied by many mathematicians. Our interest focuses on ordered sets which are "next" to chains. Such ordered sets are fences.

A *fence* $\mathbf{X}$ is an ordered set $(X; \leq)$ in which either

$$a_1 < a_2 > a_3, \ldots, a_{2m-1} > a_{2m} < a_{2m+1}, \ldots$$

or

$$a_1 > a_2 < a_3, \ldots, a_{2m-1} < a_{2m} > a_{2m+1}, \ldots$$

are the only comparability relations where $X = \{a_1, a_2, \ldots, a_n, \ldots\}$. Every element in $\mathbf{X}$ is minimal or maximal. If $a_1 < a_2$, then $\mathbf{X}$ is called an *up fence* and it is called a *down fence* if $a_1 > a_2$. A special case of a regular element is a coregular element. An element a in a semigroup S is called *coregular* if there is an element $b \in S$ with $aba = a = bab$ and S is called *coregular* if every element of S is coregular. Coregular semigroup was first introduced and studied in [1] by Bijev and Todorov. They proved that a semigroup S is coregular if and only if $a^3 = a$ for all $a \in S$. It can prove that an element a in a semigroup S is coregular if and only if $a^3 = a$.

In this paper the set X under our consideration is the base set of a fence. Two types of subsemigroups of $T(X)$ with respect to a fence $\mathbf{X} = (X; \leq)$ are considered. Namely, the semigroup $OT(\mathbf{X})$ of all order-preserving self-mappings of

$\mathbf{X}$ and and the semigroup $DT(\mathbf{X})$ of all order-decreasing self-mappings of $\mathbf{X}$. A mapping $\alpha : X \to X$ is *order-decreasing* if $\alpha(x) \leq x$ for all $x \in X$. Our main purpose is to investigate the regularity for both subsemigroups. Throughout the paper, the range of a mapping α is denoted by $\operatorname{ran}\alpha$ and $|A|$ means the cardinality of a set A. We write $x \parallel y$ when x and y are non-comparable.

2. Regular subsemigroup $DT(\mathbf{X})$ of $T(X)$

Recall that for any ordered set $\mathbf{X}$ having X as the base set, the semigroup $DT(\mathbf{X})$ of all order-decresing transformations of $\mathbf{X}$ is a subsemigroup of $T(X)$. Since in general subsemigroup of $T(X)$ need not be regular, our aim is to describe fences $\mathbf{X}$ having a regular semigroup $DT(\mathbf{X})$.

In 2012, Namnak and Laysirikul characterized an ordered set $\mathbf{X}$ having a regular semigroup $DT(\mathbf{X})$ as stated below.

Theorem 2.1. [9] *Let $\mathbf{X}$ be an ordered set. Then $DT(\mathbf{X})$ is a regular semigroup if and only if for every subchain of $\mathbf{X}$ has at most two elements.*

Since every maximal subchain of a fence is a 2-element chain, we can immediately deduce from Theorem 2.1 the following proposition.

Proposition 2.2. *For every finite fence $\mathbf{X}$, the semigroup $DT(\mathbf{X})$ is regular.*

In Section 1, we already mentioned that a coregular element is one of important cases of a regular element. Since a regular element need not be coregular, it is interested to describe fences $\mathbf{X}$ having a coregular semigroup $DT(\mathbf{X})$. We begin our investigation by characterizing order-decreasing mappings on a fence.

Lemma 2.1. *Let $\mathbf{X} = (X; \leq)$ be an n-element up fence with $X = \{a_1, a_2, \dots, a_n\}$. Then $\alpha \in DT(\mathbf{X})$ if and only if α satisfies the following conditions:*

(i) $\alpha(a_m) = a_m$ *if m is odd.*
(ii) $\alpha(a_n) \in \{a_{n-1}, a_n\}$ *if n is even.*
(iii) $\alpha(a_m) \in \{a_{m-1}, a_m, a_{m+1}\}$ *if $m \neq n$ and m is even.*

Proof. Let $\alpha \in DT(\mathbf{X})$ and $a_m \in X$. Then $\alpha(a_m) \leq a_m$. If m is odd, then a_m is minimal and so, $\alpha(a_m) = a_m$.

Consider m as even. Then a_m is maximal. Since $\mathbf{X}$ is a fence, $a_{m-1} < a_m > a_{m+1}$ and $a_m \parallel x$ for all $x \in X \setminus \{a_{m-1}, a_m, a_{m+1}\}$. It follows that $a_m \in \{a_{m-1}, a_m, a_{m+1}\}$ if $m \neq n$ and $a_m \in \{a_{m-1}, a_m\}$ if $m = n$.

The converse follows immediately from Condition (i)-(iii). □

Lemma 2.2. *Let $\mathbf{X} = (X; \leq)$ be an n-element down fence with $X = \{a_1, a_2, \dots, a_n\}$. Then $\alpha \in DT(\mathbf{X})$ if and only if α satisfies the following conditions:*

(i) $\alpha(a_m) = a_m$ *if m is even.*

(ii) $\alpha(a_1) \in \{a_1, a_2\}$.
(iii) $\alpha(a_n) \in \{a_{n-1}, a_n\}$ *if* n *is odd.*
(iv) $\alpha(a_m) \in \{a_{m-1}, a_m, a_{m+1}\}$ *if* m *is odd and* $m \notin \{1, n\}$.

Proof. The proof is similar to that of Lemma 2.1. □

In the remainder of this section, we obtain the following proposition.

Proposition 2.3. *For every finite fence* $\mathbf{X}$*, the semigroup* $DT(\mathbf{X})$ *is always coregular.*

Proof. Let $X = \{a_1, a_2, \ldots, a_n\}$ and $\alpha \in DT(\mathbf{X})$. If a_m is minimal, then by Lemma 2.1(i) and Lemma 2.2(i), $\alpha(a_m) = a_m$ and so, $\alpha^3(a_m) = \alpha(a_m)$.

Consider a_m as maximal. If $\mathbf{X}$ is an up fence, then m is even. By Lemma 2.1(ii) and (iii), $\alpha(a_m) \in \{a_{m-1}, a_{,m}, a_{m+1}\}$. If $\alpha(a_m) = a_m$, then $\alpha^3(a_m) = \alpha(a_m)$. If $\alpha(a_m) = a_{m-1}$, then from $m-1$ is even, $\alpha(a_{m-1}) = a_{m-1}$ and so, $\alpha^3(a_m) = a_{m-1} = \alpha(a_m)$. Similarly, we have $\alpha^3(a_m) = a_{m+1} = \alpha(a_m)$ if $\alpha(a_m) = a_{m+1}$. By Lemma 2.2, $\alpha^3(a_m) = \alpha(a_m)$ if $\mathbf{X}$ is a down fence. Hence, $\alpha^3 = \alpha$ and therefore, α is coregular. □

3. Regular subsemigroups $OT(\mathbf{X})$ of $T(X)$

We now investigate the regularlity for a subsemigroup $OT(\mathbf{X})$ of $T(X)$ when $\mathbf{X} = (X; \leq)$ is a finite fence. Before doing so we need a result that an order-preserving self-mapping of a fence $\mathbf{X}$ preserves subfences. An ordered set $\mathbf{P}$ is called *connected* if for all $a, b \in P$ there is a fence $\mathbf{F} \subseteq \mathbf{P}$ with endpoints a and b. It is well known that if $\mathbf{P}$ is connected and $\alpha : P \to Q$ is order-preserving, then $\alpha(\mathbf{P})$ is connected. Consequently, every order-preserving mapping maps connected sets to connected sets. Because connected subsets of a fence $\mathbf{X}$ are precisely the subfences, an order-preserving mapping $\alpha : X \to X$ maps subfences to subfences.

As we mentioned in Section 1, α is regular in $OT(\mathbf{X})$ if α is constant, that is, $|\operatorname{ran}\alpha| = 1$. Consider $|\operatorname{ran}\alpha| = 2$. Our question is that whether α is regular. To answer this question, we need the following lemma.

Lemma 3.1. *Let* $\mathbf{X}$ *be a fence with* $|X| \leq \aleph_0$ *and* $\alpha \in OT(\mathbf{X})$*. If* $\operatorname{ran}\alpha$ *is a 2-element chain, then there is a 2-element chain* C *in* $\mathbf{X}$ *with* $\alpha(C) = \operatorname{ran}\alpha$.

Proof. First, we assume that $X = \{a_i \mid i \in I \subseteq \mathbb{N}\}$. Then from $\operatorname{ran}\alpha$ is a 2-element chain, we may assume that $\operatorname{ran}\alpha = \{a_k, a_{k+1}\}$ and $a_k < a_{k+1}$ for some $k \in I$. It follows that there are $i, j \in I$ with $\alpha(a_i) = a_k$ and $\alpha(a_j) = a_{k+1}$. Let $i < j$. We denote $M := max\{l \in \{i, i+1, \ldots, j\} \mid \alpha(a_l) = a_k\}$. From $\alpha(a_j) = a_{k+1}$, we have $M < j$ and so, $M + 1 \leq j$. By the maximality of M, we get $\alpha(a_{M+1}) = a_{k+1}$. From either $a_M < a_{M+1}$ or $a_M > a_{M+1}$, the set $\{a_M, a_{M+1}\}$ is a 2-element chain C in $\mathbf{X}$. □

Proposition 3.1. *Let* $\mathbf{X}$ *be a fence with* $|X| \leq \aleph_0$ *and* $\alpha \in OT(\mathbf{X})$*. If* $|\operatorname{ran}\alpha| = 2$*, then* α *is regular.*

Proof. Let $\operatorname{ran}\alpha = \{a, b\}$ with $a < b$. By Lemma 3.1, there is a 2-element chain $C = \{c, d\}$ with $c < d$, $\alpha(c) = a$ and $\alpha(d) = b$. Define $\beta : X \to X$ by

$$\beta(x) = \begin{cases} d, & \text{if } x = b, \\ c, & \text{if } x \neq b. \end{cases}$$

It is clear that β is order-preserving. Let $x \in X$. If $x = d$, then $\alpha\beta\alpha(x) = \alpha\beta\alpha(d) = \alpha\beta(b) = \alpha(d) = \alpha(x)$. Consider $x \neq d$. From $\operatorname{ran}\alpha = \{a, b\}$, we have $\alpha(x) = a$ or $\alpha(x) = b$. If $\alpha(x) = a$, then $\alpha\beta\alpha(x) = \alpha\beta(a) = \alpha(c) = a = \alpha(x)$. If $\alpha(x) = b$, then $\alpha\beta\alpha(x) = \alpha\beta(b) = \alpha(d) = b = \alpha(x)$. □

As a consequence of Proposition 3.1, we obtain the following corollary.

Corollary 3.2. *Let* $\mathbf{X}$ *be a 2-element fence. Then* $OT(\mathbf{X})$ *is regular.*

Next, we focus on the semigroup $OT(\mathbf{X})$ of a fence $\mathbf{X}$ having 3 or 4 elements.

Proposition 3.3. *Let* $\mathbf{X}$ *be a 3-element fence. Then* $OT(\mathbf{X})$ *is regular.*

Proof. First, we assume that $\mathbf{X} = \{a, b, c\}$ is a fence with $a < b > c$. Let $\alpha \in OT(\mathbf{X})$. If $|\operatorname{ran}\alpha| = 1$, then α is a constant mapping and hence, α is regular. If $|\operatorname{ran}\alpha| = 2$, then by Proposition 3.1, α is regular.

Consider α with $|\operatorname{ran}\alpha| = 3$. Since b is maximal, $\alpha(b) = b$ otherwise, α is not bijective. It follows that $\alpha(a) \in \{a, c\}$. If $\alpha(a) = a$, then $\alpha(c) = c$ and hence, $\alpha = id_X$. So, α is regular. If $\alpha(a) = c$, then $\alpha(c) = a$ and hence, $\alpha^3 = \alpha$. So, α is coregular and then is regular. For the case $a > b < c$, we prove dually to the case $a < b > c$. □

Proposition 3.4. *Let* $\mathbf{X}$ *be a 4-element fence. Then* $OT(\mathbf{X})$ *is regular.*

Proof. First, we assume that $\mathbf{X} = \{a, b, c, d\}$ is an up fence with $a < b > c < d$. Let $\alpha \in OT(\mathbf{X})$. Then $|\operatorname{ran}\alpha| \leq 4$. As we shown in the proof of Proposition 3.3, α is regular if $|\operatorname{ran}\alpha| \leq 2$. To finish the proof, it remains to consider $|\operatorname{ran}\alpha|$ in the following 2 cases.

Case 1: $|\operatorname{ran}\alpha| = 3$. Since $\alpha(\mathbf{X})$ is a subfence of $\mathbf{X}$, so $\operatorname{ran}\alpha = \{a, b, c\}$ or $\operatorname{ran}\alpha = \{b, c, d\}$. We may assume that $\operatorname{ran}\alpha = \{a, b, c\}$. If $\alpha(b) = a$ (or c), then from $a < b > c$ and a is minimal, we have $\alpha(a) = a$ (or c) $= \alpha(c)$ and hence, $\operatorname{ran}\alpha = \{a \text{ (or } c), \alpha(d)\}$, that is, $|\operatorname{ran}\alpha| \leq 2$ which is a contradiction. So, $\alpha(b) = b$.

If $\alpha(a) = b$, then from $a < b$ and α is order-preserving, we have $\alpha(b) = b$ and hence $\alpha(c), \alpha(d) \in \{a, c\}$. Since $c < d$, so $\alpha(c) < \alpha(d)$. But $a \parallel c$, we get $\alpha(c) = \alpha(d)$ implying $|\operatorname{ran}\alpha| \leq 2$, a contradiction. So, $\alpha(a) \neq b$. Similarly, we can prove that $\alpha(c) \neq b$. It follows that $\alpha(a), \alpha(c) \in \{a, c\}$.

If $\alpha(a) = \alpha(c)$, then from $c < d$, we have $\alpha(c) \leq \alpha(d)$ and hence, $\alpha(d) \in \{\alpha(a), b\}$. Since $\alpha(b) = b$, so $\operatorname{ran}\alpha = \{\alpha(a), b\}$ implies that $|\operatorname{ran}\alpha| \leq 2$ which is a contradiction. So, $\alpha(a) \neq \alpha(c)$.

Next, we show that $\alpha^3 = \alpha$, that is, α is coregular. Let $x \in X$. If $\alpha(x) = b$, then $\alpha^2(x) = \alpha(b) = b$ and so, $\alpha^3(x) = b = \alpha(x)$. If $\alpha(x) = a$ and $\alpha(a) = a$, then $\alpha^3(x) = a = \alpha(x)$. If $\alpha(x) = a$ and $\alpha(a) = c$, then $\alpha(c) = a$ implies that $\alpha^3(x) = \alpha(\alpha(a)) = \alpha(c) = a = \alpha(x)$. Similarly, we can prove that $\alpha^3(x) = \alpha(x)$ if $\alpha(x) = c$.

Case 2: $|\operatorname{ran}\alpha| = 4$. Then α is bijective. It is clear that $\alpha(a) \notin \{b, d\}$ since otherwise $\alpha(a) = \alpha(b)$. So, $\alpha(a) = a$ or $\alpha(a) = c$. If $\alpha(a) = c$, then $\alpha(c) = a$. Since $c < b$ and $c < d$, so $a = \alpha(c) \leq \alpha(b)$ and $a = \alpha(c) \leq \alpha(d)$ implies that $\alpha(b) = b = \alpha(d)$, a contradiction. Hence, $\alpha(a) = a$. It follows that $\alpha(x) = x$ for all $x \in \{b, c, d\}$ and so, $\alpha = id_X$. Thus α is regular.

The above results also hold true for the case that **X** is a down fence. □

In what follows, we restrict our study to the case of a mapping α in $OT(\mathbf{X})$ where **X** having 5 elements.

Example 3.1. Consider the 5-element fence $(X; \leq)$ as shown in Figure 1.

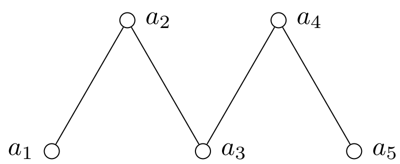

Fig. 1. The 5-element fence **F**

Define $\alpha : X \to X$ by

$$\alpha(x) = \begin{cases} a_3, & x = a_1, \\ a_5, & x = a_5, \\ a_4, & \text{otherwise.} \end{cases}$$

It is easy to see that α is order-preserving. Next we show that α is not regular. Suppose that α is regular. Then there is a $\beta \in OT(\mathbf{X})$ with $\alpha\beta\alpha = \alpha$, that is, $\alpha\beta\alpha(x) = \alpha(x)$ for all $x \in X$. If $x = a_1$, then $a_3 = \alpha(a_1) = \alpha\beta\alpha(a_1) = \alpha\beta(a_3)$. By the definition of α, we get that $\beta(a_3) = a_1$. Similarly, we get $\beta(a_5) = a_5$.

Consider $x = a_3$. We have $a_4 = \alpha(a_3) = \alpha\beta\alpha(a_3) = \alpha\beta(a_4)$ and so, $\beta(a_4) \notin \{a_1, a_5\}$. Since $a_3 < a_4$, so $a_1 = \beta(a_3) \leq \beta(a_4)$ implies that $\beta(a_4) \in \{a_1, a_2\}$. But $\beta(a_4) \neq a_1$, we get that $\beta(a_4) = a_2$. From $a_5 < a_4$, we have $a_5 = \beta(a_5) < \beta(a_4) = a_2$, a contradiction. Therefore α is not regular.

Example 3.1 shows that $OT(\mathbf{X})$ is not regular if $\mathbf{X}$ is a 5-element fence. It is natural to ask whether the semigroup $OT(\mathbf{X})$ of a fence $\mathbf{X}$ is not regular if $|X| \geq 5$. The answer is shown in the following proposition.

Proposition 3.5. *Let $\mathbf{X}$ be a fence with $|X| \geq 5$. Then $OT(\mathbf{X})$ is not regular.*

Proof. Let $|X| = n$. We consider n in the following 2 cases.

Case 1: n is odd. Define the extension $\alpha : X \to X$ of the mapping given in Example 3.1 by

$$\alpha(x) = \begin{cases} a_3, & x = a_1, \\ a_5, & x = a_n, \\ a_4, & \text{otherwise.} \end{cases}$$

Then α is order-preserving, but not regualr.

Case 2: n is even. Define $\alpha : X \to X$ by

$$\alpha(x) = \begin{cases} a_3, & x = a_1, \\ a_5, & x \in \{a_{n-1}, a_n\}, \\ a_4, & \text{otherwise.} \end{cases}$$

Clearly, α is order-preserving and so, $\alpha \in OT(\mathbf{X})$. Suppose that α is regular. Then there is a $\beta \in OT(\mathbf{X})$ with $\alpha\beta\alpha = \alpha$. By the definition of α, we get that $a_3 = \alpha(a_1) = \alpha\beta\alpha(a_1) = \alpha\beta(a_3)$. But $\alpha(x) = a_3$ when $x = a_1$, so $\beta(a_3) = a_1$. Similarly, $\beta(a_5) \in \{a_{n-1}, a_n\}$. From $a_3 < a_4$, we have $a_1 = \beta(a_3) \leq \beta(a_4)$ and so, $\beta(a_4) \in \{a_1, a_2\}$. Again from $a_4 > a_5$, we have $\beta(a_4) \geq \beta(a_5)$. Because of $n \geq 5$, we get $a_{n-1} \parallel x$ for all $x \in \{1, 2\}$ and hence, $\beta(a_4) \parallel \beta(a_5)$ which is a contradiction. Therefore, $OT(\mathbf{X})$ is not regular. □

As a consequence of Corollary 3.2, Proposition 3.3-3.4 and Proposition 3.5, a necessary and sufficient condition for $OT(\mathbf{X})$ to be regular is given.

Theorem 3.6. *Let $\mathbf{X}$ be a finite fence. Then $OT(\mathbf{X})$ is regular if and only if $|X| \leq 4$.*

4. Regular elements in $OT(\mathbf{X})$ and their properties

As we proved in Theorem 3.6, $OT(\mathbf{X})$ is not regular if $|X| > 4$. In this section, our goal is to investigate properties of regular elements of $OT(\mathbf{X})$. We start by proving a lemma.

Lemma 4.1. *Let $\mathbf{X}$ be a finite fence and $\alpha \in OT(\mathbf{X})$ with $|\operatorname{ran} \alpha| = |X| - 1$. Then there are no distinct elements $u, v \in \operatorname{ran} \alpha \setminus \{a, b\}$ with $\alpha(u) = \alpha(v)$ where a and b are the initial point and endpoint of $\mathbf{X}$, respectively.*

Proof. First, we let $X = \{a = a_1, a_2, \dots, a_n = b\}$. Suppose that there are distinct elements $a_k, a_l \in \operatorname{ran}\alpha \setminus \{a_1, a_n\}$ with $\alpha(a_k) = a_m = \alpha(a_l)$. We consider a_k in the following 2 cases.

Case 1 a_k is minimal. Then $a_{k-1} > a_k < a_{k+1}$ and so, $\alpha(a_{k-1}) \geq \alpha(a_k) \leq \alpha(a_{k+1})$. If a_m is maximal, then $\alpha(a_{k-1}) = a_m = \alpha(a_{k+1})$ implies that $|\operatorname{ran}\alpha| < |X| - 1$, a contradiction.

Consider a_m as minimal. If $a_k < a_l$, then a_l is maximal implying $a_{l-1} < a_l > a_{l+1}$. So, $\alpha(a_{l-1}) \leq \alpha(a_l) = a_m \geq \alpha(a_{l+1})$. By the minimality of a_m, we have $\alpha(a_{l-1}) = a_m = \alpha(a_{l+1})$ implying $|\operatorname{ran}\alpha| < |X| - 1$, a contradiction. If $a_k \parallel a_l$, then $|\{a_{k-1}, a_{k+1}, a_{l-1}, a_{l+1}\}| \geq 3$. From $\alpha(a_i) \in \{a_{m-1}, a_m, a_{m+1}\}$ for all $i \in \{k-1, k, k+1, l-1, l, l+1\}$, we have $|\operatorname{ran}\alpha| < |X| - 1$, a contradiction.

Case 2 a_k is maximal. The proof is similar to that of Case 1. □

It is known from Section 3 that every mapping $\alpha \in OT(\mathbf{X})$ with $|\operatorname{ran}\alpha| < 2$ is always regular, while α need not be regular if $|\operatorname{ran}\alpha| \geq 3$. The next proposition gives us sufficient conditions for such a mapping α to be regular. To do so we need the following lemma.

Lemma 4.2. [6] *Let* $\mathbf{S}$ *be a subfence of a finite fence* $\mathbf{X}$ *and let* α *be a bijection with* $\operatorname{ran}\alpha = S$. *Assume that* $S = \{a_1, a_2, \dots, a_n\}$ *and* $\alpha(a_k) = a_l$. *Let* $w \in \mathbb{N}$ *with* $w \geq 2$. *Then the following conditions hold:*

(i) Assume that $\alpha(a_{k-1}) = a_{l+1}$. *If* $a_{k\pm w} \in S$, *then* $\alpha(a_{k\pm w}) = a_{l\mp w}$.
(ii) Assume that $\alpha(a_{k-1}) = a_{l-1}$. *If* $a_{k\pm w} \in S$, *then* $\alpha(a_{k\pm w}) = a_{l\pm w}$.

Proposition 4.1. *Let* $\alpha \in OT(\mathbf{X})$ *with* $|\operatorname{ran}\alpha| \geq 3$.

(i) If $\operatorname{ran}\alpha = X$, *then* α *is regular.*
(ii) If $|\operatorname{ran}\alpha| = |X| - 1$, *then* α *is regular.*

Proof. First, we assume that $\mathbf{X}$ is an up fence having a and b as the initial point and endpoint, respectively. Then there is a unique element $z \in X$ with $a < z$.

(i) Let $X = \{a_1, a_2, \dots, a_n\}$ with $a = a_1 < a_2 > a_3 \cdots > (<) a_n = b$. Because of $\operatorname{ran}\alpha = X$, the mapping α is bijective. We consider n the following 2 cases.

Case 1 : n is even. Then a_n is maximal. If $\alpha(a_1) = a_n$, then from $a_1 < a_2$, we have $\alpha(a_2) = a_n$ which contradicts to the injectivity of α.

Suppose that $\alpha(a_1) \neq a_1$. Then there is a $k \in \{2, \dots, n-1\}$ with $\alpha(a_1) = a_k$. Because of $a_1 < a_2$, we have $\alpha(a_2) \in \{a_{k-1}, a_{k+1}\}$. Since there exists a unique a_2 that can be mapped to a_{k-1} and a_{k+1}, either $a_{k-1} \notin \operatorname{ran}\alpha$ or $a_{k+1} \notin \operatorname{ran}\alpha$. It follows that $\operatorname{ran}\alpha \neq X$, a contradiction. Therefore $\alpha(a_1) = a_1$. From $a_1 < a_2$ and α is bijective, $\alpha(a_2) = a_2$. By Lemma 4.2, $\alpha(a_k) = a_k$ for all $a_k \in X$. So, $\alpha = id_X$ and therefore α is regular.

Case 2 : n is odd. If $\alpha(a_1) = a_1$, then from $a_1 < a_2$ and α is bijective, $\alpha(a_2) = a_2$. Again by Lemma 4.2, $\alpha(a_k) = a_k$ for all $a_k \in X$ implies that $\alpha = id_X$. So, α is regular. If $\alpha(a_1) \neq a_1$, then $\alpha(a_1) = a_k$ for some $k \in \{2, \dots, n-1\}$ and there is no element in X that maps to a_{k-1}(or a_{k+1}) if $\alpha(a_2) = a_{k+1}$(or $\alpha(a_2) = a_{k-1}$). So, either $a_{k-1} \notin \operatorname{ran}\alpha$ or $a_{k+1} \notin \operatorname{ran}\alpha$ implies that $\operatorname{ran}\alpha \neq X$, a contradiction. Thus $\alpha(a_1) = a_n$. Because there exists a unique element $a_{n-1} \in X$ that is comparable to a_n and $a_1 < a_2$, it follows that $\alpha(a_2) = a_{n-1}$. By Lemma 4.2, $\alpha(a_k) = a_{n-(k-1)}$ for all $k \in \{1, \ldots, n\}$. Let $a_k \in X$. Then $a^3(a_k) = \alpha(\alpha(\alpha(a_k))) = \alpha(\alpha(a_{n-(k-1)})) = \alpha(a_{n-(n-k+1-1)}) = \alpha(a_k)$. Hence, α is regular.

(ii) Because $\operatorname{ran}\alpha$ is a subfence of $\mathbf{X}$, so $a \notin \operatorname{ran}\alpha$ or $b \notin \operatorname{ran}\alpha$. We may assume that $a \notin \operatorname{ran}\alpha$. Then $\operatorname{ran}\alpha = X \setminus \{a\}$ and by Lemma 4.1, there are $u, v \in \operatorname{ran}\alpha$ with $\alpha(a) = v = \alpha(u)$ or $\alpha(b) = v = \alpha(u)$. Without loss of generality, let $\alpha(a) = v = \alpha(u)$. Because of $|\operatorname{ran}\alpha| = |X| - 1$, the restriction $\alpha|_{X\setminus\{a\}}$ which is denoted by γ is a bijection onto $X \setminus \{a\}$.

Define $\beta : X \to X$ by

$$\beta(x) = \begin{cases} \gamma^{-1}(x), \ x \in X \setminus \{a\}, \\ \gamma^{-1}(z), \ x = a. \end{cases}$$

Let $x, y \in X$ with $x \leq y$. We consider x, y in the following cases.

Case 1: $x, y \in X \setminus \{a\}$. By the definition of β, there are $x', y' \in X \setminus \{a\}$ with $\beta(x) = \gamma^{-1}(x) = x'$ and $\beta(x) = \gamma^{-1}(y) = y'$. So, $\alpha(x') = x$ and $\alpha(y') = y$. Suppose that $x' \nleq y'$. Then $x' > y'$ or $x' \parallel y'$. If $x' > y'$, then $x = \alpha(x') > \alpha(y') = y$ which is a contradiction. Consider $x' \parallel y'$. There is a subfence $\mathbf{F}$ of $X \setminus \{a\}$ with $|F| \geq 3$ having x' and y' as the endpoints. Since $\alpha(\mathbf{F})$ is also a subfence of $\mathbf{X}$, so $\alpha(F) = \{x, y\}$. Thus there is a $c \in X \setminus \{a, x', y'\}$ with $\alpha(c) = x = \alpha(x')$ or $\alpha(c) = y = \alpha(y')$. Consequently, $|\operatorname{ran}\alpha| < |X| - 1$ which is a contradiction and therefore $\beta(x) \leq \beta(y)$.

Case 2: $x = a$ or $y = a$. If $y = a$, then by the minimality of a, we have $x = a = y$ implying $\beta(x) = \beta(y)$. Let $x = a$. Then $y = z$ and by the definition of β, we have $\beta(x) = \gamma^{-1}(z) = \beta(y)$. Altogether, we can prove that $\beta \in OT(\mathbf{X})$.

To prove that α is regular, it remains to show that $\alpha\beta\alpha = \alpha$. Consider $x \in X$. If $x = a$, then $\alpha\beta\alpha(a) = \alpha\beta(v) = \alpha\gamma^{-1}(v) = \alpha(u) = \alpha(a)$. If $x \neq a$, then $\alpha\beta\alpha(x) = \alpha\beta\gamma(x) = \alpha\gamma^{-1}(\gamma(x)) = \alpha(x)$. □

To finish this section, we give a sufficient conditions for a mapping in $OT(\mathbf{X})$ to be not regular. It is known that the mapping α defined in Example 3.1 is not regular. The last proposition shows that every mapping in $OT(\mathbf{X})$ satisfying the same conditions as the mapping α is also not regular.

Proposition 4.2. *Let $\alpha \in OT(\mathbf{X})$ with $|\operatorname{ran}\alpha| \geq 3$ and $|\operatorname{ran}\alpha| = |X| - 2$. Assume that a and b are the initial point and endpoint of $\mathbf{X}$, respectively. If there are $u, v, w \in X \setminus \{a, b\}$ with $u < v > w$ (or $u > v < w$) and $\alpha(\{u, v, w\}) = x$ where x is neither the initial point nor endpoint of* $\operatorname{ran}\alpha$*, then α is not regular.*

Proof. Suppose that α is regular. Then there is a mapping $\beta \in OT(\mathbf{X})$ with $\alpha\beta\alpha = \alpha$. Let $X = \{a_1, a_2, \ldots, a_n\}$. Then by the assumption, there are $k \in \{3, 4, \ldots, n -$

$2\}$ and $m \in \{1, 2, \ldots, n\}$ with $a_{k-1} < a_k > a_{k+1}$ (or $a_{k-1} > a_k < a_{k+1}$) and $\alpha(\{a_{k-1}, a_k, a_{k+1}\}) = a_m$ where a_m is neither the initial point nor endpoint of $\operatorname{ran}\alpha$. From $|\operatorname{ran}\alpha| = |X| - 2$, the restriction $\alpha|_{X\setminus\{a_{k-1},a_k,a_{k+1}\}}$ is a bijection onto $\operatorname{ran}\alpha \setminus \{a_m\}$. So, $\alpha(a_{k-2}), \alpha(a_{k+2}) \in \{a_{m-1}, a_{m+1}\}$. If $\alpha(a_{k-2}) = a_{m-1}$, then $\alpha(a_{k+2}) = a_{m+1}$. Thus $a_{m-1} = \alpha(a_{k-2}) = \alpha\beta\alpha(a_{k-2}) = \alpha\beta(a_{m-1})$. Since $\alpha|_{X\setminus\{a_{k-1},a_k,a_{k+1}\}}$ is bijective, $\beta(a_{m-1}) = a_{k-2}$ and $\beta(a_{m+1}) = a_{k+2}$. Because of $a_{m-1} < a_m > a_{m+1}$ (or $a_{m-1} > a_m < a_{m+1}$), we have $a_{k-2} = \beta(a_{m-1}) \leq \beta(a_m) \geq \beta(a_{m+1}) = a_{k+2}$ which is a contradiction. If $\alpha(a_{k-2}) = a_{m+1}$, then we get the same contradiction as in the previous case. Therefore α is not regular. □

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