สกว
TRF

มหาวิทยาลัยศิลปากร
SILPAKORN UNIVERSITY

รายงานวิจัยฉบับสมบูรณ์

โครงการ ตัวดำเนินการบนคอร์เนอร์แมนิโฟลด์

โดย นางสาววรรณรัตน์ รุ่งโรจน์ธีระ

เมษายน 2561

สัญญาเลขที่ MRG5980089

รายงานวิจัยฉบับสมบูรณ์

โครงการ ตัวดำเนินการบนคอร์เนอร์แมนิโฟลด์

นางสาววรรณรัตน์ รุ่งโรจน์ธีระ มหาวิทยาลัยศิลปากร

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
มหาวิทยาลัยศิลปากร

กิตติกรรมประกาศ

ดิฉันขอขอบพระคุณทางสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยศิลปากร เป็นอย่างสูง ที่ให้ทุนวิจัยแก่ดิฉันตามสัญญาเลขที่ MRG5980089 ขอบพระคุณ รศ. ดร. สืบสกุล อยู่ยืนยง นักวิจัยพี่เลี้ยง และ Prof. Dr. B.-W. Schulze เป็นอย่างสูงที่ให้ข้อมูลและคำแนะนำ อันมีค่ายิ่งในการทำงานวิจัย รวมทั้งเป็นกำลังใจ และคอยสนับสนุนดิฉันตลอด 2 ปีที่ผ่านมา ถ้าปราศจากท่านเหล่านี้ ดิฉันคงไม่สามารถทำงานวิจัยชิ้นนี้ให้สำเร็จลุล่วงไปได้ด้วยดี

นอกจากนี้ดิฉันขอบคุณอาจารย์ทุกท่านในภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากรที่เป็นกำลังใจและช่วยเหลือดิฉันในด้านต่างๆ ด้วยดีเสมอมา

วรรณรัตน์ รุ่งโรจน์ธีระ

บทคัดย่อ

รหัสโครงการ : MRG5980089

ชื่อโครงการ : ตัวดำเนินการบนคอร์เนอร์แมนิโฟลด์

ชื่อนักวิจัย : นางสาววรรณรัตน์ รุ่งโรจน์ธีระ
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร

E-mail Address : rungrottheera_w@su.ac.th, w_rungrott@outlook.co.th

ระยะเวลาโครงการ : 2 ปี (2 พฤษภาคม 2559 ถึง 1 พฤษภาคม 2561)

ให้ B เป็นแมนิโฟลด์กระชับที่มีขอบเรียบและมีมิติมากกว่า 0 เราศึกษาการมีส่วนร่วมระหว่างพีชคณิตของขอบที่ขึ้นกับตัวแปรบน B และแฟมิลีของตัวดำเนินการในแคลคูลัสคอร์เนอร์ ในกรณีคอร์เนอร์เราได้อธิบายถึงลักษณะของพาราเมทริกซ์

คำหลัก : แคลคูลัสขอบ พาราเมทริกซ์คอร์เนอร์

Abstract

Project Code : MRG5980089

Project Title : Operator on Corner Manifolds

Investigator : Wannarut Rungrottheera
Department of Mathematics, Faculty of Science, Silpakorn University

E-mail Address : rungrottheera_w@su.ac.th, w_rungrott@outlook.co.th

Project Period : 2 years (May 2, 2016 – May 1, 2018)

Let B be a compact manifold with smooth edge of dimension >0. We study the interplay between parameter-dependent edge algebra algebra on B and operator families belonging to the corner calculus, and we characterize parametrices in the corner case.

Keywords : Edge calculus, corner parametrices

EXECUTIVE SUMMARY

This article is devoted to a specific part of analysis on a manifold M with singularities, here $M := B^{\Delta}$ for

$$B^{\Delta} := (\overline{\mathbb{R}}_{+} \times B)/(\{0\} \times B) \tag{0.1}$$

where B is a manifold with edge of first order. Let us fix some terminology. The general background are stratified spaces $M \in \mathfrak{M}_k$ of singularity order $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, where the case $k = 0$ corresponds to smoothness, together with other common conditions, e.g., para-compactness, etc. We are interested in relatively simple stratifications, obtained by repeatedly forming cones $X^{\Delta}, X \in \mathfrak{M}_0$, and wedges $X^{\Delta} \times \mathbb{R}^{q_1}$ for some q_1, use those spaces as generalized charts for other spaces which are locally modeled on such cones or wedges, and then form cones, and wedges, again. In recent years there has been achieved much progress in the analysis on such spaces. The general scheme is that any $M \in \mathfrak{M}_k$ has a singular stratum $Y_k := s_k(M) \in \mathfrak{M}_0$, where $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$ and every $y_k \in Y_k$ has a neighbourhood V in M which is isomorphic (in the category of regular singularities) to a locally trivial bundle over $s_k(M)$ with fibre B^{Δ}_{k-1} for a compact $B_{k-1} \in \mathfrak{M}_{k-1}$. This definition can be iterated. In particular, we can study cones $B^{\Delta} \in \mathfrak{M}_2$ for $B \in \mathfrak{M}_1$. For $k \geq 2$ the stratification of $M \in \mathfrak{M}_k$ gives rise to a sequence

$$s(M) = (s_0(M), s_1(M), \ldots, s_k(M)) \tag{0.2}$$

of smooth subspaces $s_j(M) \in \mathfrak{M}_0, j = 0, 1, \ldots, k$, of dimension $\dim M := \dim s_0(M) > \dim s_1(M) > \cdots \geq \dim s_k(M) > 0$. Our investigations belong to the pseudo-differential calculus over such spaces M. Those contain specific corner-degenerate differential operators A with a hierarchy of principal symbols

$$\sigma(A) = (\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)). \tag{0.3}$$

The components of (0.3) are operator-valued, but $\sigma_0(A)$ is the standard homogeneous principal symbol on $T^{*}s_0(M) \setminus 0$, where 0 indicates the zero-section. Under a suitable condition on ellipticity also the parametrices of operators A belong to expected operator algebras. There are also asymptotic aspects of this program and we have structures for $B \in \mathfrak{M}_1$ (in parameter-dependent form) generating asymptotics of solutions close to edges $s_1(\cdot)$ on B and double asymptotics close to corner points $s_2(\cdot)$ of B^{Δ}, cf. [15], [8]. With $M \in \mathfrak{M}_k$ we can associate its stretched manifold $\mathbb{M}$ obtained by invariantly attaching a natural B_{k-1}- bundle (determined by the above-mentioned B^{Δ}_{k-1}-bundle) over $s_k(M)$ to $M \setminus s_k(M)$. Then, for a "negative" counterpart of $\mathbb{M}$ denoted for the moment by $\mathbb{M}_{-}$ and the former $\mathbb{M}$ by $\mathbb{M}_{+}$ we can glue together $\mathbb{M}_{-}$ and $\mathbb{M}_{+}$ along the common B_{k-1}-bundle to obtain $2\mathbb{M}$, and then $2\mathbb{M} \in \mathfrak{M}_{k-1}$.

In this article we continue and deepen the material of [2] and [3] in cases of lower orders of singularities.

Let $B \in \mathfrak{M}_1$ be a compact manifold with conical or edge singularities. In order to avoid separate comments, we assume that $\dim s_1(B) =: q > 0$. For the edge pseudo-differential calculus over B we employ suitable adapted notation. Write $Y := s_1(B)$ and $s_0(B) = B \setminus Y$. The most specific part of the edge calculus over B concerns the region close to the edge Y. The space B is locally near Y modeled on $X^\Delta \times \mathbb{R}^q$, and $B \setminus Y$ on $X^\wedge \times \mathbb{R}^q$ for a compact $X \in \mathfrak{M}_0$ for $X^\wedge := \mathbb{R}_+ \times X$. Throughout this exposition for simplicity we assume that Y has a neighbourhood W in B with the structure of trivial X^Δ-bundle over Y. The general case only needs mild extra constructions, left to the reader. Recall that any smooth manifold B with boundary belongs to $\mathfrak{M}_1$. In this case $s_0(B) = \operatorname{int} B$ and $s_1(B) = \partial B$, where W corresponds to a collar neighbourhood of ∂B, often identified with the trivial normal bundle, corresponding to a Riemannian metric. In this exposition we systematically employ the spaces $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\lambda)$ of classical parameter-dependent pseudo-differential operators of order $\mu \in \mathbb{R}$ over $X \in \mathfrak{M}_0$ of dimension n, assumed to be Riemannian, where $\lambda \in \mathbb{R}^d$ in local symbols $a(x, \xi, \lambda)$ is treated as a covariable (ξ, λ) of dimension $n+d$. Subscript "cl" means that we talk about classical symbols in Hörmander's spaces, here denoted by $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+d})$, $\Omega \subseteq \mathbb{R}^n$ open. Then $L^{-\infty}(X; \mathbb{R}^d_\lambda) = \mathcal{S}(\mathbb{R}^d_\lambda, L^{-\infty}(X))$, where $L^{-\infty}(X)$ is identified via the Riemannian metric with $C^\infty(X \times X)$, such that operators can be written $\int c(x, x')u(x')dx'$ with dx' being the measure belonging to the Riemannian metric. Then operators are locally of the form

$$\mathrm{Op}_x(a)(\lambda)u(x) = \iint e^{i(x-x')\xi} a(x, \xi, \lambda)u(x')dx' đ\xi \tag{0.4}$$

for compactly supported u, where $đ\xi = (2\pi)^{-n} d\xi$. Globally using a system of charts covering X and a subordinate partition of unity the operator families over X are determined by such local expressions modulo $L^{-\infty}(X; \mathbb{R}^d_\lambda)$. The spaces $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\lambda)$ are Fréchet in a natural way.

The main object of consideration is the space of edge operators

$$L^\mu(B, \boldsymbol{g}; \mathbb{R}^d_\lambda) \tag{0.5}$$

to be introduced in a number of steps. The properties of (0.5) will imply

$$L^\mu(B, \boldsymbol{g}; \mathbb{R}^d_\lambda) \subseteq L^\mu_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^d_\lambda)$$

Notation $\boldsymbol{g}$ indicates weight data $(\gamma_1, \gamma_1 - \mu, \Theta)$.

We also employ pseudo-differential operators based on the Mellin transform and more specific Mellin symbols that are holomorphic in $v_1 \in \mathbb{C}$.

Definition 1. Let $M^\mu_{\mathcal{O}}(X; \mathbb{R}^d)$ denote the space of all $h(v_1, \lambda) \in \mathcal{A}(\mathbb{C}_{v_1}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d))$ such that $h(\beta_1 + i\varrho_1, \lambda) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+d}_{\varrho_1, \lambda})$ for every $\beta_1 \in \mathbb{R}$, uniformly in compact β_1-intervals.

Let us prepare Definition 2 below by a notation and a list of ingredients. We fix weight data $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$, where $\gamma_1 \in \mathbb{R}$ is a weight referring to $r_1 \in \mathbb{R}_+$, the axial variable of the open stretched cone $X^\wedge := \mathbb{R}_+ \times X$, in the splitting of variables (r_1, x), and $X \in \mathfrak{M}_0$ compact, and we choose a weight interval $\Theta = (-(\theta+1), 0]$ for some $\theta \in \mathbb{N}$. We employ spaces $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ and subspaces $\mathcal{K}^{s,\gamma_1}_\Theta(X^\wedge)$ and $\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge)$ with asymptotics of type $\mathcal{P}$, cf. [14], $\mathcal{P}$ may be discrete or continuous. Below, in order to illustrate double asymptotics, we recall a few notions from this context. For the moment we freely employ these tools.

Definition 2. By

$$L^{\mu}(X^{\wedge},\boldsymbol{g};\mathbb{R}^{d}_{\lambda}\setminus\{0\}) \tag{0.6}$$

we denote the space of all operator families

$$A(\lambda):=H(\lambda)+M(\lambda)+G(\lambda) \tag{0.7}$$

where $(M+G)(\lambda)\in L^{\mu}_{\mathrm{M+G}}(X^{\wedge},\boldsymbol{g};\mathbb{R}^{d}_{\lambda}\setminus\{0\})$, see notation below, and

$$H(\lambda)=r_1^{-\mu}\mathrm{Op}_M^{\gamma_1-n/2}(h)(\lambda) \tag{0.8}$$

where

$$h(r_1,v_1,\lambda):=\tilde{h}(v_1,r_1\lambda)$$

for some $\tilde{h}(v_1,\tilde{\lambda})\in M^{\mu}_{\mathcal{O}_{v_1}}(X;\mathbb{R}^{d}_{\tilde{\lambda}})$.

In relation (0.8) we employ notation

$$\mathrm{Op}_M^{\nu}(h)(\lambda):=r_1^{\nu}\mathrm{Op}_M(T^{-\nu}h)(\lambda)r_1^{-\nu}. \tag{0.9}$$

Here $T^{-\nu}h(v_1,\lambda):=h(v_1+\nu,\lambda)$ and $\mathrm{Op}_M(f)(\lambda)u:=M^{-1}f(v_1,\lambda)Mu$ where the Mellin transform M refers to the weight line $\Gamma_{1/2}$. Here $\Gamma_{\beta}:=\{v_1\in\mathbb{C}:\mathrm{Re}\,v_1=\beta\}$.

At the beginning of the iteration from lower to larger orders we have the open stretched cone and operators have two principal symbolic components $(\sigma_0(A),\sigma_1(A))$, here depending on $\lambda\neq 0$. Operators in (0.6) are continuous in weighted Kegel spaces

$$\begin{aligned}\mathcal{K}^{s,\gamma_1}(X^{\wedge}):=\{\omega u_0+(1-\omega)u_{\infty}:\,&u_0(r_1,x)\in\mathcal{H}^{s,\gamma_1}(X^{\wedge}),\\ &u_{\infty}(r_1,x)\in H^{s}_{\mathrm{cone}}(X^{\wedge})\}\end{aligned}$$

for an arbitrary cut-off function ω, i.e., a function in $C_0^{\infty}(\overline{\mathbb{R}}_+)$ which is equal to 1 close to the origin. The spaces $\mathcal{H}^{s,\gamma_1}(X^{\wedge})$ are weighted Sobolev spaces over $X^{\wedge}=\mathbb{R}_+\times X$ of smoothness $s\in\mathbb{R}$ and weight $\gamma_1\in\mathbb{R}$. Those have the property

$$\mathcal{H}^{s,\gamma_1}(X^{\wedge})=r^{\gamma_1}\mathcal{H}^{s,0}(X^{\wedge})$$

and $\mathcal{H}^{0,0}(X^{\wedge})$ will be identified with $r_1^{-n/2}L^2(\mathbb{R}_+\times X)$ for $n=\dim X$, while $(1-\omega)H^{s}_{\mathrm{cone}}(X^{\wedge})$ can be defined first for $X=S^n$, the unit sphere in $\mathbb{R}^{1+n}_{\tilde{x}}$, where in this case

$$(1-\omega)H^{s}_{\mathrm{cone}}((S^n)^{\wedge})=(1-\omega)H^{s}(\mathbb{R}^{1+n}), \tag{0.10}$$

with (r_1,x) being polar coordinates in $\mathbb{R}^{1+n}\setminus\{0\}$. Then a simple localization argument, using a partition of unity on S^n, allows us to pass from S^n to arbitrary X by using the former definition of cone spaces for elements supported in a coordinate neighbourhood on X. This gives us relations of the kind

$$(1-\omega)\mathcal{K}^{s,\gamma_1}((S^n)^{\wedge})=(1-\omega)H^{s}_{\mathrm{cone}}((S^n)^{\wedge}), \tag{0.11}$$

$$\mathcal{K}^{0,0}(X^{\wedge})=\mathcal{H}^{0,0}(X^{\wedge}). \tag{0.12}$$

Note that the definition of $\mathcal{K}^{s,\gamma_1}(X^{\wedge})$ is independent of the choice of involved data, such as cut-off functions or specific charts. In any case if the data are fixed the arizing spaces are Hilbert in non-direct sum topology, and relation (0.12) indicates a corresponding normalization of scalar products for $s = \gamma_1 = 0$. In addition it will be essential to employ the group $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\delta : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s,\gamma_1}(X^\wedge),\ (\kappa_\delta u)(r_1, x) := \delta^{(n+1)/2} u(\delta r_1, x) \tag{0.13}$$

for any $s \in \mathbb{R}, \delta \in \mathbb{R}_+$. The spaces $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ are observed on the infinite cone up to $r_1 \to \infty$, interpreted as a conical exit of $X^\wedge$ to infinity, though close to $r_1 = 0$, at the tip of the (open stretched) cone we are interested in asymptotics of elements. In the simplest case, asymptotics are discrete. Later on we also admit continuous asymptotics.

Remark 3. The operator families (0.7) are continuous in Kegel spaces, see Theorem 5 below. In particular, $H(\lambda)$ in (0.8) satisfies the homogeneity relation

$$H(\delta\lambda) = \delta^\mu \kappa_\delta H(\lambda) \kappa_\delta^{-1} \tag{0.14}$$

for all $\delta \in \mathbb{R}_+$.

A sequence of pairs

$$\mathcal{P} = \{(p_j, m_j)\}_{j=0,\dots,N} \subset \mathbb{C} \times \mathbb{N} \tag{0.15}$$

for any $N = N(\mathcal{P}) \in \mathbb{N} \cup \{+\infty\}$ is called a discrete asymptotic type associated with the weight information $(\gamma_1, \Theta), \Theta = (-(\theta+1), 0]$, for some $\theta \in \mathbb{N}$ if

$$\pi_\mathbb{C}\mathcal{P} := \{p_j\}_{j=0,\dots,N} \subset \{v_1 \in \mathbb{C} : \frac{n+1}{2} - \gamma_1 - (\theta+1) < \operatorname{Re} v_1 < \frac{n+1}{2} - \gamma_1\} \tag{0.16}$$

and $\pi_\mathbb{C}\mathcal{P}$ is finite for finite Θ, otherwise finite or infinite, and in the latter case we assume $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. Writing for finite Θ

$$\mathcal{E}_\mathcal{P}(X^\wedge) := \left\{ \sum_{j=0}^{N} \sum_{l=0}^{m_j} \omega(r_1) c_{jl}(x) r_1^{-p_j} \log^l r_1 : c_{jl} \in C^\infty(X) \text{ for all } j, l \right\} \tag{0.17}$$

we get a Fréchet space such that $\mathcal{E}_\mathcal{P}(X^\wedge) \cap \mathcal{K}_\Theta^{s,\gamma_1}(X^\wedge) = \{0\}$ for

$$\mathcal{K}_\Theta^{s,\gamma_1}(X^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma_1-(\theta+1)-\varepsilon}(X^\wedge) \tag{0.18}$$

interpreted as the space of all flat functions relative to the weight γ_1, which is Fréchet as well. Then

$$\mathcal{K}_\mathcal{P}^{s,\gamma_1}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma_1}(X^\wedge) + \mathcal{E}_\mathcal{P}(X^\wedge) \tag{0.19}$$

is Fréchet in the topology of the direct sum. For infinite Θ we define spaces $\mathcal{K}_\mathcal{P}^{s,\gamma_1}(X^\wedge)$ in a similar manner as projective limits over spaces associated with finite flatness intervals of increasing length. Later on we employ an extension of (0.19) to continuous asymptotic types $\mathcal{P}$ which makes sense when the points of (0.16) depend on some edge variables $y_1 \in \mathbb{R}^{q_1}$. More details may be found in [13] or [14]. With asymptotic types $\mathcal{P}$ we associate the space of Green operators, depending on a parameter $\lambda \in \mathbb{R}^d$, later on in slight modification Green operator-valued symbols for the edge calculus. To this end we also control our spaces for $r_1 \to \infty$ and write

$$\mathcal{K}^{s,\gamma_1;e}(X^\wedge) = [r_1]^{-e}\mathcal{K}^{s,\gamma_1}(X^\wedge),\ \mathcal{K}_\mathcal{P}^{s,\gamma_1;e}(X^\wedge) = [r_1]^{-e}\mathcal{K}_\mathcal{P}^{s,\gamma_1}(X^\wedge). \tag{0.20}$$

for any $e \in \mathbb{R}$. Here $r_1 \to [r_1]$ is a strictly positive function which is equal to 1 close to $r_1 = 0$ and equal to r_1 for $|r_1| \geq 1$. The spaces (0.20) are Fréchet in an evident manner. In order to formulate the subspace of Green operators $L^{\mu}_{\mathrm{G}}(X^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ for weight data $\boldsymbol{g} := (\gamma_1, \gamma_1 - \mu, \Theta)$, with $\gamma_1 \in \mathbb{R}$ being the involved weight and $\mu \in \mathbb{R}$ an order we first recall some notation on Hilbert spaces with group action and associated operator-valued symbols. In order to keep the system of notation concise, we recall what we understand by operator-valued symbols. We freely employ the operator-valued generalization of Hörmander's spaces of symbols of order $\mu \in \mathbb{R}$, namely,

$$S^{\mu}(\mathbb{R}^q \times \mathbb{R}^q; H, \tilde{H}) \text{ and } S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; H, \tilde{H}) \tag{0.21}$$

where subscript "cl" indicates classical symbols. By $(H, \kappa), (\tilde{H}, \tilde{\kappa})$ we understand Hilbert spaces with corresponding group actions, and $(y, \eta) \in \mathbb{R}^q \times \mathbb{R}^q$ are variables and covariables where $y \in \mathbb{R}^q$ are local coordinates on a manifold. Charts mapping to $\mathbb{R}^q$ are taken without loss of generality. Another straightforward modification of the definition is to distinguish between dimensions of variables and covariables, see corresponding notation below. If y completely disappears we talk about symbols with constant coefficients. The well-known scalar symbolic estimates when $H = \mathbb{C}$ and $\tilde{H} = \mathbb{C}$ and $\kappa_{\delta} = \tilde{\kappa}_{\delta} = \mathrm{id}_{\mathbb{C}}$ for all $\delta \in \mathbb{R}_+$ turn to twisted symbolic estimates, namely

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^{\alpha}_y D^{\beta}_{\eta} a(y, \eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H, \tilde{H})} \leq c\, \langle\eta\rangle^{\mu - |\beta|}$$

for all $(y, \eta) \in K \times \mathbb{R}^q, K \Subset \mathbb{R}^q$, and all $\alpha \in \mathbb{N}^q, \beta \in \mathbb{N}^q$, for constants $c = c(\alpha, \beta, K) > 0$. The concept is also well-known for Fréchet spaces E and $\widetilde{E}$ with group action κ and $\tilde{\kappa}$, respectively. Classical symbols $a(y, \eta)$ are based on twisted homogeneity

$$a_{(\mu)}(y, \delta\eta) = \delta^{\nu} \tilde{\kappa}_{\delta} a_{(\mu)}(y, \eta) \kappa^{-1}_{\delta}$$

for all $\delta \in \mathbb{R}_+$. Note that in the operator-valued set-up there are non-trivial homogeneous functions defined for all η including $\eta = 0$.

Examples of Hilbert spaces with group actions are given in connection with (0.13). This scheme of notation also works when H or $\widetilde{H}$ are replaced by Fréchet spaces, see, e.g., [13] or [14]. The spaces $\mathcal{K}^{s,\gamma_1;e}(X^{\wedge})$ and $\mathcal{K}^{\infty,\gamma_1-\mu;\infty}(X^{\wedge})$ as well as subspaces with asymptotics are Fréchet where κ_{δ} is induced from (0.13) in an obvious manner, and then the above-mentioned spaces of abstract operator-valued symbols make sense with respect to those concrete spaces. Such constructions will be applied in different versions. Our definitions may be similar to each other, but in order to avoid confusion, we first look at symbols with constant coefficients, i.e., omit y at all and write λ rather than η.

Definition 4. By $L^{\mu}_{\mathrm{G}}(X^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ we denote the space of all

$$G(\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^d_{\lambda}; \mathcal{K}^{s,\gamma_1;e}(X^{\wedge}), \mathcal{K}^{\infty,\gamma_1-\mu;\infty}_{\mathcal{P}}(X^{\wedge})), \tag{0.22}$$

$$G^*(\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^d_{\lambda}; \mathcal{K}^{s,-\gamma_1+\mu;e}(X^{\wedge}), \mathcal{K}^{\infty,-\gamma_1;\infty}_{\mathcal{Q}}(X^{\wedge})) \tag{0.23}$$

(i.e., where the covariables $\lambda \in \mathbb{R}^d$ plays the role of the former η, and the symbol has constant coefficients, i.e., y is dropped at all) where (0.22), (0.23) are valid for all $s, e \in \mathbb{R}$ and fixed $\gamma_1 \in \mathbb{R}$, for asymptotics types $\mathcal{P}$ and $\mathcal{Q}$ which may depend on G.

The next issue is the class $M^{-\infty}_{\mathcal{R}}(X)$ of meromorphic operator functions, where

$$\mathcal{R} = \{(r_j, n_j)\}_{j\in\mathbb{I}} \subset \mathbb{C}\times\mathbb{N} \tag{0.24}$$

is an index set $\mathbb{I} \subseteq \{-\infty\}\cup\mathbb{Z}\cup\{+\infty\}$. We assume that $\pi_{\mathbb{C}}\mathcal{R} = \{r_j\}_{j\in\mathbb{I}}$ intersects every strip $\{v_1 \in \mathbb{C} : c \leq \operatorname{Re} v_1 \leq c'\}$ for any reals $c \leq c'$ in a finite set and $\operatorname{Re} r_j \to \pm\infty$ as $j \to \mp\infty$ once $\pi_{\mathbb{C}}\mathcal{R}$ is infinite. In addition we ask any $f(v_1) \in M^{-\infty}_{\mathcal{R}}(X) \subset \mathcal{A}(\mathbb{C}_{v_1} \setminus \pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(X))$ to be meromorphic with poles at the points $r_j \in \mathbb{C}$ of order $\leq n_j+1$ for all j, with Laurent coefficients of finite rank. Moreover, if $\chi(v_1)$ is a $\pi_{\mathbb{C}}\mathcal{R}$-excision function (i.e., $\chi(v_1) = 0$ for $\operatorname{dist}(v_1, \pi_{\mathbb{C}}\mathcal{R}) < \varepsilon_0$, $\chi(v_1) = 1$ for $\operatorname{dist}(v_1, \pi_{\mathbb{C}}\mathcal{R}) > \varepsilon_1$ for some $0 < \varepsilon_0 < \varepsilon_1 < \infty$) then $\chi(v_1)f(v_1)|_{\Gamma_\lambda} \in \mathcal{S}(\Gamma_\lambda, L^{-\infty}(X))$ for every $\lambda \in \mathbb{R}$, uniformly in finite intervals. Operator-valued symbols of order $\mu \in \mathbb{R}$, in covariables $\lambda \in \mathbb{R}^d$ connected with such meromorphic symbols are of the form

$$M(\lambda) := r_1^{-\mu}\omega_{[\lambda]}\sum_{j=0}^{N} r_1^j \sum_{|\alpha|\leq j} \operatorname{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})\lambda^\alpha \omega'_{[\lambda]} \tag{0.25}$$

for elements $f_{j\alpha}(v_1) \in M^{-\infty}_{\mathcal{R}_{j\alpha}}(X)$ and weights $\gamma_{j\alpha}$ satisfying the conditions

$$\Gamma_{\frac{n+1}{2}-\gamma_{j\alpha}} \cap \pi_{\mathbb{C}}\mathcal{R}_{j\alpha} = \emptyset \quad \text{for} \quad \gamma_1 - j \leq \gamma_{j\alpha} \leq \gamma_1 \tag{0.26}$$

for all j, α. Here $\omega_{[\lambda]}(r_1) = \omega(r_1[\lambda])$ where $\lambda \to [\lambda]$ is any smooth, strictly positive function in $\mathbb{R}^q$ such that $|\lambda| = [\lambda]$ for $|\lambda| \geq C$ for some $C > 0$ and $\Gamma_\beta = \{v_1 \in \mathbb{C} : \operatorname{Re} v_1 = \beta\}$. It is evident that then we get operator-valued symbols in λ with constant coefficients

$$M(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma_1}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu}(X^\wedge)) \tag{0.27}$$

and

$$M(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu}_{\mathcal{Q}}(X^\wedge)) \tag{0.28}$$

for every $s \in \mathbb{R}$ and any asymptotic type $\mathcal{P}$ with some resulting $\mathcal{Q}$. Also here we have a connection with weight data $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$.

The spaces $L^\mu_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ consist of all sums $M(\lambda) + G(\lambda)$ where $G(\lambda)$ are Green families as before and $M(\lambda)$ are defined as operator families of the form (0.25). In other words we completed Definition 2.

We systematically employ continuity, according to

Theorem 5. [14] *Every $A(\lambda) \in L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ induces families of continuous operators*

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge), \tag{0.29}$$

and

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}_{\mathcal{Q}}(X^\wedge), \tag{0.30}$$

for every $s \in \mathbb{R}$ and any asymptotic type $\mathcal{P}$ and some resulting $\mathcal{Q}$ which is independent of s.

In order to make the degenerate character of interior symbols more transparent we remember of the fact that our operator families are coming from operator functions

$$p(r_1, \varrho_1, \lambda) = \tilde{p}(r_1\varrho_1, r_1\lambda) \tag{0.31}$$

for

$$\tilde{p}(\tilde{\varrho}_1, \tilde{\lambda}) \in L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+d}_{\tilde{\varrho}_1, \tilde{\lambda}}) \tag{0.32}$$

where the operators in $r_1 \in \mathbb{R}_+$ act via the Fourier transform and are combined with the factor $r_1^{-\mu}$. In other words, for reasons which are explained in other papers, see [17] etc., they have the form

$$r_1^{-\mu}\mathrm{Op}_{r_1}(p)(\lambda)\,|_{r_1>0} : C_0^{\infty}(\mathbb{R}_+ \times X) \to C^{\infty}(\mathbb{R}_+ \times X). \tag{0.33}$$

Then, in order to reflect weight effects we fix a weight $\gamma_1 \in \mathbb{R}$ and rephrase (0.33) in term of the Mellin transform on the r_1 half-axis. This process is also called a Mellin quantization, and the difference between (0.33) and (0.8) consists of smoothing operators over $\mathbb{R}_+ \times X$ which are ignored under Mellin quantization, but the transformation from (0.33) to (0.8) is canonical when $\tilde{p}(r_1, \tilde{\varrho}_1, \tilde{\lambda})$ is a polynomial in $\tilde{\varrho}_1, \tilde{\lambda}$, because of the exact replacement of $-r_1\frac{\partial}{\partial r_1}$ corresponding to $-ir_1\varrho_1$ in the Fourier picture by the Mellin covariable v_1. Thus, in the program of expressing parametrices of Fuchs type operators there is no loss of information in the differential case, but there are specified remainders when we realize operators in weighted spaces for different weights. Those are, in fact, Green operators, depending on the choice of weights. In any case homogeneous principal symbols in the Fourier picture do not depend on this aspect; they can be expressed both in the Fourier and the Mellin picture and are the same up to a substitution of variables and covariables, not only for differential operators, and including parameters. When we formulate information in local coordinates $x \in \mathbb{R}^n$ on X with parameter λ and look at $A(\lambda)$ which is the same as $H(\lambda)$ modulo smoothing remainders, since Mellin plus Green operators are smoothing for $r_1 > 0$ we obtain a parameter-dependent principal symbol $\sigma_0(A)(r_1, x, \varrho_1, \xi, \lambda)$ for $(\varrho_1, \xi, \lambda) \neq 0$. This contains in its r_1-dependence also the weight factor $r_1^{-\mu}$ and in addition the r_1-degenerate behaviour in ϱ_1, λ, such that it makes sense to write

$$\sigma_0(A)(r_1, x, \varrho_1, \xi, \lambda) = r_1^{-\mu}\tilde{\sigma}_0(A)(r_1, x, r_1\varrho_1, \xi, r_1\lambda) \tag{0.34}$$

for a "reduced" symbol $\tilde{\sigma}_0(A)(r_1, x, \varrho_1, \xi, \lambda)$ which is smooth in r_1 up to $r_1 = 0$ and where (ϱ_1, λ) is involved in the meaning $(\tilde{\varrho}_1, \tilde{\lambda})$ for $\tilde{\varrho}_1 = r_1\varrho_1, \tilde{\lambda} = r_1\lambda$ which just corresponds to the background information (0.31), (0.32). Concerning $\sigma_1(\cdot)$ the advantage of the Mellin formulation is that it admits a natural definition of ellipticity with control up to $r_1 = 0$ both for A as well as for $M + G$, which entails Fredholm property in Fuchs type weighted Soboley spaces where we associate with $H(\lambda)$ also so-called conormal symbols, on the level of principal symbolic information, namely,

$$\sigma_1(H)(v_1, 0) := \tilde{h}(v_1, 0) \tag{0.35}$$

with $\tilde{h}$ being defined in Definition 2. Here there is no dependence on λ, and v_1 is varying in the complex Mellin plane. The operator families (0.35) pointwise act as continuous operators

$$\sigma_1(H)(v_1, 0) : H^s(X) \to H^{s-\mu}(X) \tag{0.36}$$

between Sobolev spaces over X. In ellipticity the operators (0.36) are parameter - dependent elliptic in the variable $\operatorname{Im} v_1$ for $v_1 \in \Gamma_\lambda$ for any fixed real λ. and hence (0.36) becomes a family of isomorphisms as soon as $\operatorname{Im} v_1$ is sufficiently large, uniformly in compact λ-intervals. Although this information is well-known, later on we refer to similar facts for singularity order $k > 1$. In order to establish conormal symbols of operator families $M(\lambda)$ of the form (0.25) we form the operator function

$$\sigma_1(M)(v_1) := f_{00}(v_1) : H^s(X) \to H^\infty(X) \tag{0.37}$$

where (0.37) is the subordinate conormal symbol of $M(\lambda)$ from the cone theory close to $r_1 = 0$. The parameter $\operatorname{Im} v_1$ is varying on Γ_λ indicated in the weight conditions of (0.25), here for $\lambda = \frac{n+1}{2} - \gamma_1$. The operators $M(\lambda) + G(\lambda)$ in Definition 2 are smoothing off $r_1 = 0$ they do not generate contributions to the interior symbolic level.

The operators $A(\lambda) \in L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ are called elliptic if $\tilde{h}(v_1, \tilde{\lambda})$ is parameter-dependent elliptic in $M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$ and the conormal symbol

$$\sigma_1(A)(v_1) := \tilde{h}(v_1, 0) + f_{00}(v_1) : H^s(X) \to H^{s-\mu}(X)$$

is a family of isomorphisms for all $v_1 \in \Gamma_{\frac{n+1}{2} - \gamma_1}$ with a prescribed weight $\gamma_1 \in \mathbb{R}$ which belongs to the ellipticity condition. Note that in case of ellipticity the restriction of elements in $M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$ to $\Gamma_{\frac{n+1}{2} - \gamma_1} \times \mathbb{R}^d_{\tilde{\lambda}}$ are parameter-dependent elliptic in $L^\mu_{\mathrm{cl}}(X; \Gamma_{\frac{n+1}{2} - \gamma_1} \times \mathbb{R}^d_{\tilde{\lambda}})$. This holds for all weights γ_1.

We also recall the following result:

Theorem 6. *Let $A(\lambda) \in L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ be parameter-dependent elliptic with respect to the weight γ_1. Then there is a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}(X^\wedge, \boldsymbol{g}^{-1}; \mathbb{R}^d \setminus \{0\}), \boldsymbol{g}^{-1} = (\gamma_1 - \mu, \gamma_1, \Theta)$ where*

$$P(\lambda)A(\lambda) = 1 - G_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - G_{\mathrm{R}}(\lambda)$$

for some $G_{\mathrm{L}}(\lambda) \in L^{-1}_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}), G_{\mathrm{R}}(\lambda) \in L^{-1}_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d \setminus \{0\})$ for $\boldsymbol{g}_{\mathrm{L}} := (\gamma_1, \gamma_1, \Theta), \boldsymbol{g}_{\mathrm{R}} := (\gamma_1 - \mu, \gamma_1 - \mu, \Theta)$.

Corollary 7. *If $A(\lambda)$ is parameter-dependent elliptic then*

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)$$

is a family of Fredholm operators, and kernels belong to subspaces of $\mathcal{K}^{\infty,\gamma_1}(X^\wedge)$ with asymptotics. Cokernels can be represented by finite-dimensional subspaces of $\mathcal{K}^{\infty,\gamma_1-\mu}(X^\wedge)$ with asymptotics.

Theorem 8. ([11]) *For every $\gamma_1 \in \mathbb{R}$ and $s \in \mathbb{R}$ there exists a $\tilde{h}(v_1, \tilde{\lambda}) \in M^{-s}_{\mathcal{O}_{v_1}}(X, \boldsymbol{g}; \mathbb{R}^d_{\tilde{\lambda}})$ such that for $h(r_1, v_1, \lambda) = \tilde{h}(v_1, r_1\lambda)$ and sufficiently large $|\tilde{\lambda}|$ the operator*

$$R^{-s}(\lambda) := r_1^s \mathrm{Op}_M^{\gamma_1 - s - n/2}(h)(\lambda) : \mathcal{K}^{0,\gamma_1 - s}(X^\wedge) \to \mathcal{K}^{s,\gamma_1}(X^\wedge)$$

is an isomorphism.

Theorem 6 is known, but we have similar results below for base spaces of higher singular order. Therefore, we sketch the arguments. The main issue is that the ellipticity condition on $\tilde{h}(v_1, \tilde{\lambda})$ means that there is an $\tilde{h}^{(-1)}(v_1, \tilde{\lambda}) \in M^{-\mu}_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$ such that the product

$$r_1^{\mu} \mathrm{Op}_M(\tilde{h}^{(-1)})(\tilde{\lambda}) r_1^{-\mu} \mathrm{Op}_M(\tilde{h})(\tilde{\lambda}) \tag{0.38}$$

is equal to $\mathrm{Op}_M(\tilde{f})(\tilde{\lambda})$ for some $\tilde{f}(v_1, \tilde{\lambda}) \in M^{0}_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$. In this conclusion the r_1-powers cancel out modulo some translations in v_1 of the involved Mellin symbols. These translations do not influence the parameter-dependent homogeneous symbols in the sense of pseudo-differential operators on $\Gamma_{\beta_1} \times \mathbb{R}^d_{\tilde{\lambda}}$. Therefore, the product (0.38) produces identity plus a Mellin operator with parameter-dependent symbol of order 1. When we insert symbols $h^{(-1)}(r_1, v_1, \lambda) = \tilde{h}^{(-1)}(v_1, r_1\lambda)$ and $h(r_1, v_1, \lambda) = \tilde{h}(v_1, r_1\lambda)$, then there is a Leibnitz product effect in (r_1, v_1) under which only lower order terms are changed, and those again produce lower order terms. Then a formal Neumann series argument gives us, say for computing a left parametrix gives us remainders of the form $1 - \widetilde{C}_L$ for $\widetilde{C}_L \in L^{-1}_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\})$. Together with the inversion of the conormal symbol we finally get remainders in

$$L^{-\infty}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}) = L^{-1}_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}).$$

For the right parametrix we proceed in a similar manner, i.e., it can be identified with the left parametrix.

As a consequence we see that

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \tag{0.39}$$

is a family of Fredholm operators. The index is independent of $s \in \mathbb{R}$. Later on we shall interpret elements of $L^{\mu}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ with $q_1 + d$ rather than d and parameters $(\eta_1, \lambda) \neq 0$ as space of occurring edge symbols for edge operators, and the ellipticity condition in the edge case will be that the respective cone operators are families of isomorphisms. This is not automatically the case, but similarly as in boundary value problems the situation is required by adding trace and potential entries including right lower corners in corresponding 2 × 2 block matrix operators to achieve families of isomorphisms. The extra entries remind of Green operators and can be chosen as symbols with asymptotics. Later on operator families in $L^{\mu}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ occur as homogeneous principal edge symbols. In contrast to Remark 3 we did not require the elements $M(\lambda) + G(\lambda) \in L^{\mu}_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ to be homogeneous in such a sense. Of course, there is the subclass

$$L^{(\mu)}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\}) \tag{0.40}$$

of those $A(\lambda) \in L^{\mu}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ such that

$$A(\delta\lambda) = \delta^{\mu} \kappa_{\delta} A(\lambda) \kappa_{\delta}^{-1}$$

for all $\delta \in \mathbb{R}_+$. The construction of trace and potential etc. operators makes sense for the space (0.40) which gives rise to an extension of $L^{(\mu)}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ to an algebra of 2×2 block matrices

$$\boldsymbol{A}(\lambda) = \begin{pmatrix} A(\lambda)+G(\lambda) & K(\lambda) \\ T(\lambda) & Q(\lambda) \end{pmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma_1}(X^\wedge) \\ \oplus \\ \mathbb{C}^e \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \\ \oplus \\ \mathbb{C}^f \end{matrix} \tag{0.41}$$

for some $e, f \in \mathbb{N}$ where $f - e$ just equals the Fredholm index of (0.39). Here $G(\lambda) \in L^\mu_G(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ and $T(\lambda)$ are trace operators, $K(\lambda)$ potential operators, and $Q(\lambda)$ are families of classical symbols in λ with values in $f \times e$-matrices. The nature of $T(\lambda)$ and $K(\lambda)$ is determined by kernels and cokernels of solutions to elliptic equations, and they may be represented by families of functions with asymptotics. The properties are similar to those in the boundary symbolic calculus in pseudo-differential boundary vales problems. Compositions $K(\lambda) \circ T(\lambda)$ are examples of parameter-dependent Green operators, cf. Definition 4.

In order to see a connection between Definition 2 and the more complex situation of operators over (stretched) wedges

$$X^\wedge \times \mathbb{R}^q \ni (r_1, x, y_1) \tag{0.42}$$

we have a look at the shape of operators in the calculus over a compact space $B \in \mathfrak{M}_1$ with edge $s_1(B) =: Y_1$, locally close to Y_1 in local coordinates $y_1 \in \mathbb{R}^{q_1}$ modeled on wedges (0.42) with local amplitude functions for the corresponding parameter-dependent edge calculus. Those have the form

$$a(y_1, \eta_1, \lambda) := h(y_1, \eta_1, \lambda) + m(y_1, \eta_1, \lambda) + g(y_1, \eta_1, \lambda) \tag{0.43}$$

where

$$h(y_1, \eta_1, \lambda) = \omega r_1^{-\mu} \mathrm{Op}_M^{\gamma_1 - n/2}(\boldsymbol{h})(y_1, \eta_1, \lambda)\omega' \tag{0.44}$$

and Mellin amplitude functions

$$\boldsymbol{h}(r_1, y_1, v_1, \eta_1, \lambda) = \tilde{\boldsymbol{h}}(r_1, y_1, v_1, r_1\eta_1, r_1\lambda) \tag{0.45}$$

for

$$\tilde{\boldsymbol{h}}(r_1, y_1, v_1, \tilde{\eta}_1, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_1}, M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^{q_1+d}_{\tilde{\eta}_1, \tilde{\lambda}})). \tag{0.46}$$

Definition 9. By $L^\mu(B, \boldsymbol{g}; \mathbb{R}^d)$ for $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$ we denote the space of operators

$$A(\lambda) := H(\lambda) + M(\lambda) + G(\lambda) + A_{\mathrm{int}}(\lambda) + C(\lambda) \tag{0.47}$$

where $H(\lambda)$ is locally close to Y_1 expressed as a finite sum of operators

$$\varphi \mathrm{Op}_{y_1}\{\omega r_1^{-\mu} \mathrm{Op}_M^{\gamma_1 - n/2}(\boldsymbol{h})(y_1, \eta_1, \lambda)\omega'\}\varphi' \tag{0.48}$$

with $\varphi \prec \varphi'$ in $C_0^\infty(\mathbb{R}^{q_1})$ coming from a partition of unity over Y_1 and where $\boldsymbol{h}$ satisfies relations (0.45) and (0.46). Moreover, we ask that

$$A_{\mathrm{int}}(\lambda) \in L^\mu_{\mathrm{cl}}(B \setminus s_1(B); \mathbb{R}^d_\lambda),$$

such that the λ-dependent distributional kernel which is contained in $(B \setminus s_1(B)) \times (B \setminus s_1(B))$ has proper support in the respective open set. In addition $C(\lambda) \in$

$L^{-\infty}(B, \boldsymbol{g}; \mathbb{R}^d)$ is parameter-depending smoothing, i.e., characterized by the property

$$C(\lambda) \in \mathcal{S}(\mathbb{R}^d_\lambda, L^{-\infty}(B, \boldsymbol{g})) \tag{0.49}$$

where $L^{-\infty}(B, \boldsymbol{g})$ is the space of all

$$C : H^{s,\gamma_1}(B) \to H^{\infty,\gamma_1-\mu}_{\mathcal{P}}(B) \tag{0.50}$$

with

$$C^* : H^{s,-\gamma_1+\mu}(B) \to H^{\infty,-\gamma_1}_{\mathcal{Q}}(B) \tag{0.51}$$

for any $s \in \mathbb{R}$ and asymptotic types $\mathcal{P}, \mathcal{Q}$, which may depend on C. The spaces involved in (0.50) and (0.51) are defined in Subsection 1.2. We finally require

$$M(\lambda) + G(\lambda) \in L^\mu_{\mathrm{M+G}}(B, \boldsymbol{g}; \mathbb{R}^d_\lambda) \tag{0.52}$$

i.e., $M(\lambda) = \mathrm{Op}_{y_1}(m)$ is a Mellin operator family locally close to Y_1 associated with Mellin amplitude functions $m(y_1, \eta_1, \lambda)$ of similar structure as (0.25), namely, as in formula (0.56) below, and $G(\lambda) = \mathrm{Op}_{y_1}(g), g(y_1, \eta_1, \lambda)$ similarly as in Definition 4.

Let us now return to operators over a compact space $B \in \mathfrak{M}_1$ with edge $s_1(B) = Y_1$ as explained before in the context of Definition 9. So far the spaces (0.50), (0.51) and (0.52) are not yet defined. Because of y_1-dependence of symbols we prefer to employ the setting of continuous asymptotics. This requires weighted edge spaces locally along $\mathbb{R}^{q_1} \ni y_1$ and subspaces with such asymptotics. Similarly as symbol spaces (0.21) we have abstract edge Sobolev spaces $\mathcal{W}^s(\mathbb{R}^{q_1}, H)$ for any Hilbert or Fréchet space H with group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. Those already played a role in (0.21). In our context we have Hilbert spaces

$$H := \mathcal{K}^{s,\gamma_1}(X^\wedge) \quad \text{or} \quad \mathcal{K}^{s,\gamma_1;e}(X^\wedge)$$

or Fréchet subspaces with asymptotics, indicated by subscript $\mathcal{P}$. Those spaces admit the group action

$$\kappa_\delta : u(r_1, x) \to \delta^{\frac{n+1}{2}} u(\delta r_1, x),$$

$\delta \in \mathbb{R}_+$, and hence we have associated weighted edge spaces defined in terms of completion of $\mathcal{S}(\mathbb{R}^{q_1}, H)$, e.g., in the Hilbert space case with respect to

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^{q_1},H)} := \left\{ \int \langle\eta_1\rangle^{2s} \, \|\kappa^{-1}_{\langle\eta_1\rangle} \hat{u}(\eta_1)\|^2_H đ\eta_1 \right\}^{1/2}$$

(otherwise for semi-norm systems in the corresponding Fréchet space, where $\|\cdot\|_H$ runs over a countable system of semi-norms in H).

Weighted edge spaces $H^{s,\gamma_1}(B)$ over a compact $B \in \mathfrak{M}_1$ with edge are defined in terms of charts

$$\chi_i : Y_i \to \mathbb{R}^{q_1} \tag{0.53}$$

for a system of coordinate neighbourhoods $Y_i \subset Y$, $i = 1, \ldots, N$, satisfying some simple admissibility condition concerning transition maps, used in several expositions on edge spaces, see, e.g., [14] or the article [7]. Then $H^{s,\gamma_1}(B)$ is defined as the set of those $u \in H^s_{\mathrm{loc}}(B \setminus Y_1)$ such that $\varphi_i u \circ \chi_i^{-1} \in \mathcal{W}^s(\mathbb{R}^{q_1}, \mathcal{K}^{s,\gamma_1}(X^\wedge))$ for all i, with $(\varphi_1, \dots, \varphi_N)$ being a subordinate partition of unity. $H^{s,\gamma_1}(B)$ can be equipped with a Hilbert space structure in a natural way.

In an analogous manner we obtain subspaces $H^{s,\gamma_1}_{\mathcal{P}}(B)$ with asymptotics of type $\mathcal{P}$ by replacing $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ in the latter definition by $\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge)$. Those are again Fréchet spaces. Thus the spaces in relations (0.50), (0.51) are defined, where the formal adjoints refer to spaces $H^{0,0}(B)$ which may be identified with $r^{-n/2}L^2(B\backslash Y_1)$.

Analogously as $L^\mu_{\mathrm{G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ in Definition 4 by $\mathcal{R}^\mu_{\mathrm{G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$ we denote the space of

$$g(y_1, \eta_1, \lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}; \mathcal{K}^{s,\gamma_1;e}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu;\infty}_{\mathcal{P}}(X^\wedge)) \tag{0.54}$$

the pointwise formal adjoints of which have the property

$$g^*(y_1, \eta_1, \lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}; \mathcal{K}^{s,-\gamma_1+\mu;e}(X^\wedge), \mathcal{K}^{\infty,-\gamma_1;\infty}_{\mathcal{Q}}(X^\wedge)), \tag{0.55}$$

for some g-dependent asymptotic types $\mathcal{P}, \mathcal{Q}$, associated with the weight information in these relations. Those are required for all $s, e \in \mathbb{R}$. In the case of discrete asymptotics we assume $\mathcal{P}, \mathcal{Q}$ to be constant with respect to y_1; in the continuous case we assume the same, but then we may admit carriers which cover variable discrete asymptotics. Such operator-valued symbols are connected with weight data $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$ which are given in connection with the weights in the spaces and also determine the position of $\pi_{\mathbb{C}}\mathcal{P}$ and $\pi_{\mathbb{C}}\mathcal{Q}$, respectively.

Other ingredients of parameter-dependent edge amplitude functions will consist of Mellin contributions, namely,

$$m(y_1, \eta_1, \lambda) := r_1^{-\mu} \omega_{\eta_1,\lambda} \sum_{j=0}^{\theta} r_1^j \sum_{|\alpha| \le j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y_1)(\eta_1, \lambda)^\alpha \omega'_{\eta_1,\lambda} \tag{0.56}$$

where $\omega_{\eta_1,\lambda}(r_1) = \omega(r_1 \langle \eta_1, \lambda \rangle)$ and, similarly, $\omega_{\eta_1,\lambda}$, and

$$f_{j\alpha}(y_1, v_1) \in C^\infty(\mathbb{R}^{q_1}, M^{-\infty}_{\mathcal{R}_{j\alpha}}(X))$$

satisfying analogous weight conditions as (0.26). In (0.26) we assumed (constant in y_1) discrete Mellin asymptotic types. This works with the observation that we always find weights $\gamma_{j\alpha}$ in the required interval. In the continuous case this is not automatically the case, since the carriers of continuous Mellin asymptotic types do not necessary leave such gaps. However, corresponding y_1-dependent smoothing Mellin symbols can always decomposed into sums where every summand is of the required form. Corresponding decomposition identities may be found, e.g., in [10]. In that way (0.56) is asked to be represented in the form $m = m_1 + m_2$ where the summands are of the form (0.56) but with Mellin symbols $f_{j\alpha}$ who are different for the corresponding summands. For brevity we simply only argue in terms of Mellin amplitude like (0.56), keeping in mind that for an arbitrary choice of $f_{j\alpha}$ we always have decomposition like $f_{j\alpha} = f^1_{j\alpha} + f^2_{j\alpha}$ where the associated continuous Mellin asymptotic types leave gaps of the carriers sets to make the Mellin actions possible.

Let $\mathcal{R}^\mu_{\mathrm{M+G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$ be the space of all $(m+g)(y_1, \eta_1, \lambda)$ such that $g(y_1, \eta_1, \lambda) \in \mathcal{R}^\mu_{\mathrm{G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$ and $m(y_1, \eta_1, \lambda)$ a finite sum of operator functions of the kind (0.56), where it can be proved that two summands generate the whole space of such elements. Similarly as (0.54) we have

$$m(y_1,\eta_1,\lambda)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q_1}\times\mathbb{R}^{q_1+d};\mathcal{K}^{s,\gamma_1}(X^{\wedge}),\mathcal{K}^{\infty,\gamma_1-\mu}(X^{\wedge}))$$

and

$$m(y_1,\eta_1,\lambda)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q_1}\times\mathbb{R}^{q_1+d};\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^{\wedge}),\mathcal{K}^{\infty,\gamma_1-\mu}_{\mathcal{Q}}(X^{\wedge}))$$

for all s, for arbitrary asymptotic types $\mathcal{P}$ and some resulting $\mathcal{Q}$. When we evaluate homogeneous principal symbols in the operator-valued set-up it suffices to ignore the exit order e in Green symbols and to express symbols with smoothness $s-\mu$ in the image spaces. For Green symbols g we can take as $\sigma_1(g)(y_1,\eta_1,\lambda)$ the twisted homogeneous principal symbol

$$\sigma_1(g)(y_1,\eta_1,\lambda)\in S^{(\mu)}(\mathbb{R}^{q_1}\times(\mathbb{R}^{q_1+d}\setminus\{0\});\mathcal{K}^{s,\gamma_1}(X^{\wedge}),\mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge}))$$

and for the twisted homogeneous principal symbol of $m(y_1,\eta_1,\lambda)$ we have

$$\sigma_1(m)(y_1,\eta_1,\lambda)\in S^{(\mu)}(\mathbb{R}^{q_1}\times(\mathbb{R}^{q_1+d}\setminus\{0\});\mathcal{K}^{s,\gamma_1}(X^{\wedge}),\mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge}))$$

which has the form

$$\sigma_1(m)(y_1,\eta_1,\lambda)=r_1^{-\mu}\omega_{|\eta_1,\lambda|}\sum_{j=0}^{\theta}r_1^{j}\sum_{|\alpha|=j}\mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y_1)(\eta_1,\lambda)^{\alpha}\omega'_{|\eta_1,\lambda|}$$

for $\omega_{|\eta_1,\lambda|}(r_1):=\omega(r_1|\eta_1,\lambda|)$, and the same for $\omega'_{|\eta_1,\lambda|}$. Observe that the general reasons these are parameter-dependent homogeneous edge symbols of Mellin plus Green type and defined for $(\eta_1,\lambda)\neq 0$.

Note that when we change the cut-off functions or the weights in (0.56) we leave remainders of Green type. Also other expected rules hold, such as that the symbol spaces $\mathcal{R}^{\mu}_{\mathrm{M+G}}$ form algebras and that the homogeneous principal symbols behave multiplicatively. Moreover, formal adjoints are Mellin plus Green again, with compatibility of formal adjoints of twisted homogeneous symbols.

The following consideration concerns the edge calculus with parameters, and as noted before, is of analogous structure as the operator class from Definition 2, now for $B\in\mathfrak{M}_1$ rather than $X^{\wedge}$, and since B is not discussed in terms of some conical exit we admit parameters $\lambda\in\mathbb{R}^d$.

Let $B\in\mathfrak{M}_1$, not necessary compact, and $q_1=\dim Y_1>0$. By

$$L^{\mu}(B,\boldsymbol{g};\mathbb{R}^d_{\lambda}) \tag{0.57}$$

for weight data $\boldsymbol{g}=(\gamma_1,\gamma_1-\mu,\Theta_1)$, we denote the set of all families of operators (0.47), where it remains to formulate the ingredients close to Y_1 in terms of edge amplitude functions in

$$\mathcal{R}^{\mu}(\mathbb{R}^{q_1}\times\mathbb{R}^{q_1+d},\boldsymbol{g}) \tag{0.58}$$

consisting of all operator families (0.43) for $(m+g)(y_1,\eta_1,\lambda)\in\mathcal{R}^{\mu}_{\mathrm{M+G}}(\mathbb{R}^{q_1}\times\mathbb{R}^{q_1+d},\boldsymbol{g})$, cf. notation in Subsection 1.2, and $h(y_1,\eta_1,\lambda)$ as in (0.44). Then the part of operators in $L^{\mu}(B,\boldsymbol{g};\mathbb{R}^d_{\lambda})$ close to Y_1 can be described by

$$H(\lambda)=\sum_{\varphi\prec\varphi'}\varphi\mathrm{Op}_{y_1}\{r_1^{-\mu}\omega_1\mathrm{Op}_{M_{r_1}}^{\gamma_1-n/2}(\boldsymbol{h})(y_1,\eta_1,\lambda)\omega_1'\}\varphi' \tag{0.59}$$

with summation over φ from the partition of unity, $\varphi \prec \varphi'$ also smooth and of compact support in $\mathbb{R}^{q_1}$ and cut-off functions $\omega_1 \prec \omega_1'$ on the r_1 half-axis and

$$(M+G)(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}\{(m+g)(y_1, \eta_1, \lambda)\}\varphi'$$

for $(m+g)(y_1, \eta_1, \lambda) \in \mathcal{R}^{\mu}_{\mathrm{M+G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$. In more concise manner we also could express the elements of (0.47) as

$$A(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}(a(y_1, \eta_1, \lambda))\varphi' + A_{\mathrm{int}}(\lambda) + C(\lambda) \tag{0.60}$$

for arbitrary $a(y_1, \eta_1, \lambda) \in \mathcal{R}^{\mu}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$.

Note that in the edge situation we have locally over a wedge in the variables (r_1, y_1) and covariables $(\varrho_1, \eta_1, \lambda)$ analogously as (0.31), (0.32) edge-degenerate pseudo-differential families

$$p(r_1, y_1, \varrho_1, \eta_1, \lambda) = \tilde{p}(r_1, y_1, r_1\varrho_1, r_1\eta_1, r_1\lambda) \tag{0.61}$$

for

$$\tilde{p}(r_1, y_1, \tilde{\varrho}_1, \tilde{\eta}_1, \tilde{\lambda}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_1}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q_1+d}_{\tilde{\varrho}_1, \tilde{\eta}_1, \tilde{\lambda}})). \tag{0.62}$$

Then the Mellin symbols $\boldsymbol{h}$ indicated in (0.44), (0.45), (0.46) are just by applying a Mellin quantization process which turns $\tilde{\varrho}_1$ to $v_1 \in \mathbb{C}$, and which is combined with a kernel cut-off procedure. The edge symbolic hierarchy, consisting of

$$\sigma(A(\lambda)) = (\sigma_0(A(\lambda)), \sigma_1(A(\lambda)))$$

partly refers to the interior Fourier based background. From (0.62) we have by dissolving the variables into $(r_1, x, y_1, \tilde{\varrho}_1, \xi, \tilde{\eta}_1, \tilde{\lambda})$ the parameter-dependent "scalar" interior symbol

$$\sigma_0(A)(r_1, x, y_1, \varrho_1, \xi, \eta_1, \lambda) = r_1^{-\mu}\tilde{\sigma}_0(A)(r_1, x, y_1, r_1\varrho_1, \xi, r_1\eta_1, r_1\lambda) \tag{0.63}$$

for a "reduced" symbol $\tilde{\sigma}_0(A)(r_1, x, y_1, \varrho_1, \xi, \eta_1, \lambda)$ homogeneous of order μ in $(\varrho_1, \xi, \eta_1, \lambda) \neq 0$ and smooth up to $r_1 = 0$. This explains, analogously as (0.34) the symbolic level $\sigma_0(A(\lambda))$. Moreover, in this framework we have

$$\sigma_1(A(\lambda)) = \sigma_1(H(\lambda)) + \sigma_1(M(\lambda)) + \sigma_1(G(\lambda))$$

where the second summands are known from the edge calculus in [14] and

$$\sigma_1(H(\lambda))(y_1, \eta_1, \lambda) = r_1^{-\mu}\mathrm{Op}^{\gamma_1 - n/2}_{M_{r_1}}\tilde{h}_0(y_1, r_1\eta_1, r_1\lambda)$$

for $\tilde{h}_0(y_1, \tilde{\eta_1}, \tilde{\lambda}) = \tilde{h}(0, y_1, r_1\eta_1, r_1\lambda)$, which is a function with values in continuous operators

$$\sigma_1(H(\cdot))(y_1, \eta_1, \lambda) : \mathcal{K}^{s,\gamma_1}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge})$$

for $(\eta_1, \lambda) \in \mathbb{R}^{q_1+d} \setminus \{0\}$ belonging to

$$L^{\mu}(X^{\wedge}, \boldsymbol{g}; \mathbb{R}^{q_1+d} \setminus \{0\}) \tag{0.64}$$

smoothly depending on y_1. In other words (0.64) furnishes the parameter-dependent edge symbolic calculus of (0.57)

Let us make some remarks on how we can obtain special examples of parameter-dependent elliptic elements $A(\lambda)$ in (0.57). Ellipticity is defined as bijectivity of the involved symbols. We first have the interior ellipticity which means non-vanishing of $\sigma_0(A)$ for all $(\varrho_1, \xi, \eta_1, \lambda) \neq 0$ and of $\tilde{\sigma}_0(A)$ up to $r_1 = 0$. In addition the edge symbol

$$\sigma_1(A(\cdot))(y_1, \eta_1, \lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \tag{0.65}$$

which is a family of operators in $L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^{q_1+d}_{\eta_1,\lambda} \setminus \{0\})$ has to be Fredholm for every $(y_1, \eta_1, \lambda) \in \mathbb{R}^{q_1} \times (\mathbb{R}^{q_1+d} \setminus \{0\})$. It is of interest to investigate on how we can pass to a 2×2 block matrix family of isomorphisms. For this some topological obstruction on the K-theoretic index bundle of (0.65) has to vanish. If we focus on the case that (0.65) is a family of isomorphisms then the obstruction vanishes. In the articles [17] or [18] we studied special elliptic cases where the operators (0.65) (in local form) are essentially derived from parameter-dependent Laplacians and the edge symbols are then of the form

$$(|\eta_1|^2 + |\lambda|^2)^{\mu/2}$$

taking values in operators between spaces in (0.65). The desired property depends to a large extent on the behaviour of subordinate conormal symbols which have to attain isomorphisms between Sobolev spaces over X, namely

$$H^s(X) \to H^{s-\mu}(X)$$

for any $s \in \mathbb{R}$. Those are operator functions depending on $v_1 \in \mathbb{C}$ and they have to be bijective on the prescribed weight line $\Gamma_{\frac{n+1}{2}-\gamma_1}$. In applications below we need bijectivity in a particularly large weight strip. In order to achieve this it is helpful to have bijectivity first in any (particularly narrow) strip, and then to enlarge this by a dilation in the z-plane. In this manipulation we do not loose the other ellipticity conditions, and this weight strip can be stretched so wide that several manipulations, e.g., translations of reference weights, remain possible without leaving that strip. Examples have been discussed in [17], and we use them here in connection with the observation that enlarging weight strips is always possible. Let us summarize weight manipulations of that kind as

Parameter-dependent ellipticity of an $A(\lambda) \in L^\mu(B, \boldsymbol{g}; \mathbb{R}^d_\lambda)$, cf. formula (0.57), means that $A \in L^\mu(B \setminus Y; \mathbb{R}^d_\lambda) \subseteq L^\mu_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^d_\lambda)$ is parameter-dependent elliptic in the standard sense and close to the edge Y the reduced symbol $\tilde{\sigma}_0(A)$ does not vanish for $(\varrho_1, \xi, \eta_1, \lambda) \neq 0$ up to $r_1 = 0$. Concerning the principal edge symbol (0.65) we have different options. In the "best possible" case (0.65) is a family of isomorphisms, otherwise we ask (0.65) to be a family of Fredholm operators which comes extra edge conditions. For brevity we consider the first case.

Theorem 10. *Let $A(\lambda) \in L^\mu(B, \boldsymbol{g}; \mathbb{R}^d)$ be parameter-dependent elliptic. Then there is a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}(B, \boldsymbol{g}^{-1}; \mathbb{R}^d)$ such that*

$$P(\lambda)A(\lambda) = 1 - C_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - C_{\mathrm{R}}(\lambda)$$

for operators $C_{\mathrm{L}}(\lambda) \in L^{-\infty}(B, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d), C_{\mathrm{R}}(\lambda) \in L^{-\infty}(B, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d)$. Moreover, if B is compact

$$A(\lambda) : H^{s,\gamma_1}(B) \to H^{s-\mu,\gamma_1-\mu}(B)$$

is a family of Fredholm operators. It becomes a family of isomorphism for sufficiently large $|\lambda|$. This holds for all $s \in \mathbb{R}$.

Theorem 11. *Consider a sequence of operator functions*

$$f_{\mu-j}(\tilde{\tau},\tilde{\lambda}) \in L^{\mu-j}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}), \tag{0.66}$$

$j \in \mathbb{N}$, $\boldsymbol{g} = (\gamma,\gamma-\mu,\Theta)$, and let the asymptotic types contained in the Green terms of $f_{\mu-j}$ be independent of j. Then there is an $f(\tilde{\tau},\tilde{\lambda}) \in L^{\mu}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})$ such that

$$f(\tilde{\tau},\tilde{\lambda}) - \sum_{j=0}^{N} f_{\mu-j}(\tilde{\tau},\tilde{\lambda}) \in L^{\mu-(N+1)}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}) \tag{0.67}$$

for any $N \in \mathbb{N}$. Let us write

$$f(\tilde{\tau},\tilde{\lambda}) \sim \sum_{j=0}^{\infty} f_{\mu-j}(\tilde{\tau},\tilde{\lambda}),$$

called an asymptotic sum of the $f_{\mu-j}$.

Operator functions in $L^{\mu}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})$ satisfy rules in terms of differentiations with respect to parameters. In particular, we have

$$D^k_{\tilde{\tau}} D^{\alpha}_{\tilde{\lambda}} L^{\mu}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}) \subseteq L^{\mu-(k+|\alpha|)}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}) \tag{0.68}$$

for any $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$. Our next objective is to compose corner-degenerate families of operators

$$t^{-\mu}\mathrm{Op}_t(a)(\lambda)t^{-\nu}\mathrm{Op}_t(b)(\lambda) \tag{0.69}$$

for $a(t\tau,t\lambda), b(t\tau,t\lambda)$ given by

$$a(\tilde{\tau},\tilde{\lambda}) \in L^{\mu}(B,\boldsymbol{g}_2;\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}), \quad b(\tilde{\tau},\tilde{\lambda}) \in L^{\nu}(B,\boldsymbol{g}_1;\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}),$$

respectively, where

$$\boldsymbol{g}_1 := (\gamma,\gamma-\nu,\Theta),\ \boldsymbol{g}_2 := (\gamma-\nu,\gamma-(\nu+\mu),\Theta),$$

in terms of Leibniz compositions with respect to variables and covariables (t,τ).

Proposition 12. *We have*

$$t^{-\mu}\mathrm{Op}_t(a)(\lambda)t^{-\nu}\mathrm{Op}_t(b)(\lambda) = t^{-(\mu+\nu)}\mathrm{Op}_t(a\#_t b)(\lambda) + G(\lambda) \tag{0.70}$$

for $G(\lambda)$ determined by $\mathrm{Op}_t(g)(\lambda)$ for some $g(t\tau,t\lambda)$ with

$$g(\tilde{\tau},\tilde{\lambda}) \in L^{-\infty}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}).$$

Definition 13. The space $M^{\mu}_{\mathcal{O}_{v_2}}(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\tilde{\eta}_2,\lambda})$ for $\boldsymbol{g}_1 := (\gamma_1,\gamma_1-\mu,\Theta)$, is defined as the set of all $h(y_2,\eta_2,\lambda) \in \mathcal{A}(\mathbb{C}_{v_2},L^{\mu}(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\eta_2,\lambda}))$ such that

$$h(\beta_2+i\varrho_2,\lambda) \in L^{\mu}(B_1,\boldsymbol{g}_1;\mathbb{R}^{1+q_2+d}_{\varrho_2,\eta_2,\lambda})$$

for every $\beta_2 \in \mathbb{R}$, uniformly in compact β_2-intervals.

Here we employ the operator classes in Definition 9 together with notation (0.59) for the non-smoothing Mellin part close to Y_1 and the former B is denoted by B_1. The spaces

$$L^{\mu}(B_1, \boldsymbol{g}_1; \mathbb{R}^d) \tag{0.71}$$

are locally convex, and they may be interpreted as unions of Fréchet spaces. Every element of (0.71) belongs to such a subspace and we can talk about spaces of holomorphic functions

$$\mathcal{A}(\mathbb{C}_{v_2}, L^{\mu}(B_1, \boldsymbol{g}_1; \mathbb{R}^d)) \tag{0.72}$$

by taking unions coming from those Fréchet subspaces of (0.71). Note that standard elements of complex function theory also work for holomorphy of functions with values in a Fréchet space, see, e.g., the textbook of Jarchow, [9]. In such constructions the weight data $\boldsymbol{g}_1 = (\gamma_1, \gamma_1 - \mu, \Theta_1)$ for weight intervals $\Theta_1 := (-(\theta_1 + 1), 0], \theta_1 \in \mathbb{N}$, are chosen and fixed, and then

$$M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d) \tag{0.73}$$

is defined as in Definition 13, but for $q_2 = 0$. In the sequel we use notation

$$h(r_2, v_2, \lambda) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{r_2\lambda}) \tag{0.74}$$

if $h(r_2, v_2, \lambda) = \tilde{h}(v_2, r_2\lambda)$ and $\tilde{h}(v_2, \tilde{\lambda}) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$. If H is a Hilbert space with group action we define weighted "Mellin-Fourier" Sobolev spaces $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^{q_1}, H)$ as the completion of $C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^{q_1}, H)$ with respect to the norm

$$\begin{aligned} \|u\|_{\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^{q_1}, H)} := \bigg\{ \int_{\mathbb{R}^{q_1}} \int_{\Gamma_{\frac{b_1+1}{2} - \gamma_2}} &\langle v_2, \eta_1 \rangle^{2s} \\ &\times \|\kappa^{-1}_{\langle v_2, \eta_1 \rangle} (M_{r_2 \to v_2} F_{y_2 \to \eta_1} u)(v_2, \eta_1)\|^2_H \, đv_2 đ\eta_1 \bigg\}^{1/2}. \end{aligned} \tag{0.75}$$

This expression can be generalized to Y_1 rather than $\mathbb{R}^{q_1}$ by using charts like (0.53) which gives us spaces $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times Y_1, H)$ for any $s, \gamma_2 \in \mathbb{R}$. We employ this definition first to the case $s = 0$ and $H := \mathcal{K}^{0,\gamma_1}(X^{\wedge})$. In the next step we consider $\mathcal{K}$-spaces of smoothness zero and arbitrary pairs of weights $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$, namely, for $B_1 \in \mathfrak{M}_1, Y_1 = s_1(B_1)$ we set

$$\begin{aligned} \mathcal{K}^{0,\gamma}(B_1^{\wedge}) = &\omega_2 \omega_1 \mathcal{H}^{0,\gamma_2}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{0,\gamma_1}(X^{\wedge})) \\ &+ (1 - \omega_2)\omega_1 \mathcal{H}^{0,0}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{0,\gamma_1}(X^{\wedge})) \\ &+ (1 - \omega_2)(1 - \omega_1)\mathcal{K}^{0,0}((2\mathbb{B}_1)^{\wedge}) \\ &+ \omega_2(1 - \omega_1)\mathcal{K}^{0,\gamma_2}((2\mathbb{B}_1)^{\wedge}) \end{aligned} \tag{0.76}$$

for cut-off functions $\omega_2 = \omega_2(r_2), \omega_1 = \omega_1(r_1)$.

In order to give an explanation of spaces $\mathcal{K}^{s,\gamma}(B_1^{\wedge})$ for arbitrary $s \in \mathbb{R}$ we first consider the Hilbert spaces $H^{s,\gamma_1}(B_1)$ which are defined locally close to $s_1(B_1) =: Y_1$ by

$$H^{s,\gamma_1}(B_1) = \omega_1 \mathcal{W}^s(Y_1, \mathcal{K}^{s,\gamma_1}(X^{\wedge})) + (1 - \omega_1) H^s_{\mathrm{loc}}(B_1 \setminus Y_1).$$

This gives us spaces $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1))$ and also

$$\mathcal{H}^{s,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) := [r_2]^{-e}\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1))$$

and we form

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) := \varprojlim_{s_0, e\in\mathbb{R}} \mathcal{H}^{s_0,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) \tag{0.77}$$

for sufficiently large s_0, e. In particular, γ_2 remains a multiplicative weight and we have

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) = r_2^{\gamma_2}\mathcal{S}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)).$$

The spaces $\mathcal{H}^{s,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B_1))$ and also (0.77) are embedded in $\mathcal{K}^{0,\gamma}(B_1^\wedge)$. Moreover, we employ the non-degenerate sesquilinear paring

$$(\cdot,\cdot)_{\mathcal{K}^{0,0}(B_1^\wedge)} : \mathcal{K}^{0,\gamma}(B_1^\wedge) \times \mathcal{K}^{0,-\gamma}(B_1^\wedge) \to \mathbb{C} \tag{0.78}$$

for any $\gamma = (\gamma_1, \gamma_2)$.

Now with Mellin symbols

$$f(r_2, v_2, \lambda) \in M^0_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{r_2\lambda})$$

for $\boldsymbol{g}_1 = (\gamma_1, \gamma_1, \Theta)$ we have continuity

$$\mathrm{Op}^{\gamma_2 - b_1/2}_{M_{r_2}}(f)(\lambda) : \mathcal{K}^{0,\gamma}(B_1^\wedge) \to \mathcal{K}^{0,\gamma}(B_1^\wedge).$$

If (0.74) is a Mellin symbol, we have continuity of

$$r_2^{-\mu}\mathrm{Op}^{\gamma_2 - b_1/2}_{M_{r_2}}(h)(\lambda) : \mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)) \to \mathcal{S}^{\gamma_2-\mu}(\mathbb{R}_+, H^{\infty,\gamma_1-\mu}(B_1)). \tag{0.79}$$

By duality via (0.78) we also get continuity between the respective dual spaces. For

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)) = r_2^{\gamma_2}\mathcal{S}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1))$$

we have a sesquilinear pairing

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)) \times (\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)))' \to \mathbb{C}.$$

Instead of $(\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)))'$ we also write $(\mathcal{S}')^{-\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B_1))$. Then, (0.79) allows us to pass to formal adjoint operators

$$\begin{aligned}(r_2^{-\mu}\mathrm{Op}^{\gamma_2 - b_1/2}_{M_{r_2}}(h)(\lambda))^* : (&\mathcal{S}^{\gamma_2-\mu}(\mathbb{R}_+, H^{\infty,\gamma_1-\mu}(B_1)))' \\ &\to (\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)))'\end{aligned} \tag{0.80}$$

which is a map between the respective distribution spaces. The operator on the left-hand side of (0.80) is of analogous structure as that on the left of (0.79), i.e., of the form

$$r_2^{-\mu}\mathrm{Op}^{\gamma_2 - b_1/2}_{M_{r_2}}(h^*)(\lambda)$$

for some element $h^* \in M^\mu_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_\mu; \mathbb{R}^d_{r_2\lambda})$ for $\boldsymbol{g}_\mu = (-\gamma_1 + \mu, -\gamma_1, \Theta)$ obtained together with a translation in the complex v_2-plane. Because of the assumptions on holomorphy of Mellin symbols in sufficiently large v_2-strips, the map (0.80) may be realized between spaces of distributions relatively to the original weights, namely

$$\begin{aligned}r_2^{-\mu}\mathrm{Op}^{\gamma_2 - b_1/2}_{M_{r_2}}(h^*)(\lambda) : (&\mathcal{S}')^{\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B_1)) \\ &\to (\mathcal{S}')^{\gamma_2-\mu}(\mathbb{R}_+, H^{-\infty,\gamma_1-\mu}(B_1)).\end{aligned}$$

By restriction this induces an operator family

$$r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2-b_1/2}(h^*)(\lambda):\mathcal{K}^{0,\gamma}(B_1^\wedge)\to(\mathcal{S}')^{\gamma_2-\mu}(\mathbb{R}_+,H^{-\infty,\gamma_1-\mu}(B_1)) \tag{0.81}$$

which makes sense because of $\mathcal{K}^{0,\gamma}(B_1^\wedge)\subset(\mathcal{S}')^{\gamma_2}(\mathbb{R}_+,H^{-\infty,-\gamma_1}(B_1))$, but (0.81) is by no means surjective. This consideration may be applied to $s\in\mathbb{R}$ rather than μ, and a pair of weights $\gamma-s=(\gamma_1-s,\gamma_2-s)$. Then the respective operator will be called R^{-s}, and we define in new notation

$$R^{-s}(\lambda):=r_2^s\mathrm{Op}_{M_{r_2}}^{\gamma_2-s-b_1/2}(f)(\lambda):\mathcal{K}^{0,\gamma-s}(B_1^\wedge)\to\mathcal{K}^{s,\gamma}(B_1^\wedge)$$

where $f(r_2,v_2,\lambda)=\tilde f(v_2,r_2\lambda)$ for a resulting $\tilde f(v_2,\tilde\lambda)\in M_{\mathcal{O}_{v_2}}^{-s}(B_1,\boldsymbol{g}_s;\mathbb{R}^d_{\tilde\lambda})$ for weight data $\boldsymbol{g}_s:=(\gamma_1-s,\gamma_1,\Theta)$. The resulting space $\mathcal{K}^{s,\gamma}(B_1^\wedge)$ is the image of $\mathcal{K}^{0,\gamma-s}(B_1^\wedge)$ under $R^{-s}(\lambda)$ for sufficiently large λ. We altogether obtain

Theorem 14. *For every pair of weights $\gamma=(\gamma_1,\gamma_2)\in\mathbb{R}^2$ and $s\in\mathbb{R}$ there exists an $f(v_2,\tilde\lambda)\in M_{\mathcal{O}_{v_2}}^{-s}(B_1,\boldsymbol{g}_s;\mathbb{R}^d_{\tilde\lambda})$ such that for sufficiently large $|\lambda|$ the operator*

$$R^{-s}(\lambda):=r_2^s\mathrm{Op}_{M_{r_2}}^{\gamma_2-s-b_1/2}(f)(\lambda):\mathcal{K}^{0,\gamma-s}(B_1^\wedge)\to\mathcal{K}^{s,\gamma}(B_1^\wedge) \tag{0.82}$$

defines an isomorphism.

Note that the operator (0.82) is defined in terms of a parameter - dependent elliptic element $f(v_2,\tilde\lambda)\in M_{\mathcal{O}_{v_2}}^{-\infty}(B_1,\boldsymbol{g}_1;\mathbb{R}^d_{\tilde\lambda})$ and for large $|\lambda|$ we have injectivity on $\mathcal{K}^{0,\gamma-s}(B_1^\wedge)$, since there is a parameter-dependent left parametrix which leaves small remainders with increasing $|\lambda|$. In any case we set

$$\mathcal{K}^{s,\gamma}(B_1^\wedge):=\mathrm{im}\,R^{-s}(\lambda)$$

and another consideration shows that this space is independent of the specific choice of f with the indicated property.

A simple consideration shows that $\mathcal{K}^{s,\gamma}(B_1^\wedge)$ admits the group action

$$\kappa_\delta:\mathcal{K}^{s,\gamma}(B_1^\wedge)\to\mathcal{K}^{s,\gamma}(B_1^\wedge),$$

$$(\kappa_\delta u)(r_1,x):=\delta^{(b_1+1)/2}u(\delta r_1,x),\delta\in\mathbb{R}_+.$$

In addition the particular form of the Mellin symbol $f(v_2,r_2\lambda)$ gives us twisted homogeneity

$$R^{-s}(\delta\lambda)=\delta^{-s}\kappa_\delta R^{-s}(\lambda)\kappa_\delta^{-1}$$

for all $\delta\in\mathbb{R}_+$. The operator calculus on the infinite stretched cone $B^\wedge$ requires Green and smoothing Mellin operators which play a similar role as those on Subsection 1.2 for a smooth compact X. Similarly as in (0.16) we employ pairs of discrete asymptotic types $\mathcal{P}_1,\mathcal{P}_2$ and form spaces of singular functions

$$\mathcal{E}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge):=\Big\{\sum_{\nu=0}^{M}\sum_{\lambda=0}^{l_\nu}\omega_2(r_2)d_{\nu\lambda}(b)r_2^{-s_\nu}\log^\lambda r_2$$

$$:d_{\nu\lambda}\in H_{\mathcal{P}_1}^{\infty,\gamma_1}(B_1)\text{ for all }\nu,\lambda\Big\}.$$

Moreover, we have the spaces (0.75) where the involved space H may also be a Fréchet space with group action. This allows us to form spaces

$$\mathcal{H}^{s,\gamma_1,\gamma_2}_{\mathcal{P}_1,\Theta_2}(\mathbb{R}_+ \times \mathbb{R}^{q_2}, \,\cdot\,) := \varprojlim_{\varepsilon>0} \mathcal{H}^{s,\gamma_2-(\theta_2+1)-\varepsilon}(\mathbb{R}_+ \times \mathbb{R}^{q_2}, \mathcal{K}^{s,\gamma_1}_{\mathcal{P}_1}(X^\wedge))$$

for $\Theta_2 = (-(\theta_2{+}1), 0]$ which is an expression in terms of known data. A globalization gives us the spaces

$$\omega_2\omega_{1,\mathrm{glob}}\mathcal{H}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \omega_2\omega_{1,\mathrm{glob}}\mathcal{H}^{s,\gamma}_{\mathcal{P}_1,\Theta_2}(B_1^\wedge) + \mathcal{E}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$$

for cut-off functions $\omega_2 = \omega_2(r_2)$ on the half-axis and functions $\omega_{1,\mathrm{glob}}$ in the distance variable r_1 to Y_1, the edge of B_1. We also pass to spaces globally on $B_1^\wedge$ by forming

$$\omega_2\mathcal{H}^{s;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \omega_2\omega_{1,\mathrm{glob}}\mathcal{H}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) + \omega_2(1-\omega_{1,\mathrm{glob}})\mathcal{H}^{s,\gamma_2}_{\mathcal{P}_2}((2\mathbb{B}_1)^\wedge)|_{s_0(B_1^\wedge)}.$$

This gives us

$$\omega_2\mathcal{H}^{\infty;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \varprojlim_{s\in\mathbb{R}} \omega_2\mathcal{H}^{s;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge).$$

Moreover, consider

$$(1-\omega_2)\mathcal{S}(\overline{\mathbb{R}}_{+,r_2}, H^{\infty,\gamma_1}_{\mathcal{P}_1}(B_1))$$

which can be identified closed to $\mathbb{R}_+ \times Y_1$ with the space

$$(1-\omega_2) \varprojlim_{s,\delta\in\mathbb{R}} \mathcal{W}^{s,\delta}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{\infty,\gamma_1}_{\mathcal{P}_1}(X^\wedge))$$

Let us now form

$$\mathcal{K}^{\infty;\gamma_1,\gamma_2;\infty}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \omega_2\mathcal{H}^{\infty;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) + (1-\omega_2)\mathcal{S}(\overline{\mathbb{R}}_{+,r_2}, H^{\infty,\gamma_1}_{\mathcal{P}_1}(B_1)).$$

Definition 15. By $L^\mu_{\mathrm{G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ for weight data $\boldsymbol{g} := (\boldsymbol{g}_i)_{i=1,2}$, where $\boldsymbol{g}_i = (\gamma_i, \gamma_i - \mu, \Theta_i)$, $\Theta_i = (-(\theta_i+1), 0]$ for certain $\theta_i \in \mathbb{N}$, $i = 1, 2$ we denote the space of all

$$G(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d_\lambda; \mathcal{K}^{s;\gamma_1,\gamma_2;e}(B_1^\wedge), \mathcal{K}^{\infty;\gamma_1-\mu,\gamma_2-\mu;\infty}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge))$$

such that

$$G^*(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d_\lambda; \mathcal{K}^{s;-\gamma_1+\mu,-\gamma_2+\mu;e}(B_1^\wedge), \mathcal{K}^{\infty,-\gamma_1,-\gamma_2;\infty}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_1^\wedge))$$

for arbitrary $s, e \in \mathbb{R}$ and asymptotic types $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2$ depending.

Concerning a definition of spaces

$$\mathcal{K}^{s;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$$

for $s \in \mathbb{R}$ we apply an analogue of Theorem 14. The definition of $\mathcal{K}^{0,\gamma}(B_1^\wedge)$ in (0.76) allows us to pass to subspaces

$$\begin{aligned}\mathcal{K}^{0,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) =&\omega_2\omega_1\mathcal{H}^{0,\gamma_2}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{0,\gamma_1}_{\mathcal{P}_1}(X^\wedge)) \\ &+ (1-\omega_2)\omega_1\mathcal{H}^{0,0}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{0,\gamma_1}_{\mathcal{P}_1}(X^\wedge)) \\ &+ (1-\omega_2)(1-\omega_1)\mathcal{K}^{0,0}((2\mathbb{B}_1)^\wedge) + \omega_2(1-\omega_1)\mathcal{K}^{0,\gamma_2}((2\mathbb{B}_1)^\wedge).\end{aligned}$$

We now form asymptotic types $\mathcal{P}_1(s), \mathcal{P}_2(s)$ which are translations of $\mathcal{P}_1, \mathcal{P}_2$ according to the weight shift in formula (0.82) and generate $\mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$ as the image of $\mathcal{K}^{0,\gamma-s}_{\mathcal{P}_1(s),\mathcal{P}_2(s)}(B_1^\wedge)$ under the map (0.82). Then

$$R^{-s}(\lambda) : \mathcal{K}^{0,\gamma-s}_{\mathcal{P}_1(s),\mathcal{P}_2(s)}(B_1^\wedge) \to \mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$$

just generates Kegel spaces with asymptotics for arbitrary s.

Analogously as the operator families we now formulate the smoothing Mellin contribution to the class $L^\mu_{\mathrm{M+G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ for $\boldsymbol{g} = (\boldsymbol{g}_1, \boldsymbol{g}_2)$. To this and we first formulate the symbol classes $M^{-\infty}_{\mathcal{R}}(B_1, \boldsymbol{g}_1)$ for some Mellin asymptotic type $\mathcal{R}$ belonging to the singular order 1. Similarly as (0.28) in the discrete case $\mathcal{R}$ is a sequence in $\mathbb{C}_{v_2} \times \mathbb{N}$ such that $\pi_{\mathbb{C}}\mathcal{R}$ satisfies an analogous condition as before what concerns the position of points in $\mathbb{C}_{v_2}$. Now for such an $\mathcal{R}$ the space

$$M^{-\infty}_{\mathcal{R}}(B_1, \boldsymbol{g}_1)$$

is defined as the set of all $f(v_2) \in \mathcal{A}(\mathbb{C}_{v_2} \setminus \pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(B_1, \boldsymbol{g}_1))$ such that $f(v_2)$ is meromorphic with poles at the points in $\pi_{\mathbb{C}}\mathcal{R}$ of multiplicity $n_j + 1$ and finite rank Laurent coefficients in $L^{-\infty}(B_1, \boldsymbol{g}_1)$. In addition we ask

$$\chi(v_2)f(v_2)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(B_1, \boldsymbol{g}_1))$$

for any $\pi_{\mathbb{C}}\mathcal{R}$-excision function χ and every $\beta \in \mathbb{R}$, uniformly in finite intervals. Now the smoothing Mellin cone families for the calculus over $B_1^\wedge$ are assumed to be of the form

$$M(\lambda) := r_2^{-\mu}\omega_{2,[\lambda]} \sum_{j=0}^{\theta_2} r_2^j \sum_{|\alpha| \le j} \mathrm{Op}_M^{\gamma_{j\alpha}-b_1/2}(f_{j\alpha})\lambda^\alpha \omega'_{2,[\lambda]} \tag{0.83}$$

for elements $f_{j\alpha}(v_2) \in M^{-\infty}_{\mathcal{R}_{j\alpha}}(B_1, \boldsymbol{g}_1)$ and weights $\gamma_{j\alpha}$ satisfying

$$\Gamma_{\frac{b_1+1}{2}-\gamma_{j\alpha}} \cap \pi_{\mathbb{C}}\mathcal{R}_{j\alpha} = \emptyset \quad \text{for} \quad \gamma_2 - j \le \gamma_{j\alpha} \le \gamma_2$$

for all j, α. Recall that $\omega_{2,[\lambda]}(r_2) = \omega_2(r_2[\lambda])$, etc. Similarly as (0.27), (0.28) the operator families $M(\lambda)$ constitute operator-valued symbols with constant coefficients

$$M(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma_2}(B_1^\wedge), \mathcal{K}^{\infty,\gamma_2-\mu}(B_1^\wedge))$$

and similarly, between subspaces with double asymptotic types.
Note that both Green and Mellin operator families over $B_1^\wedge$ can also be formulated for continuous asymptotics, both with respect to r_1 and r_2.

As announced before the space $L^\mu_{\mathrm{M+G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ is defined as the set of sums $M(\lambda) + G(\lambda)$ for $M(\lambda)$ as it has been just defined and $G(\lambda) \in L^\mu_{\mathrm{G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$.

Definition 16. The space $L^\mu(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ is defined as the space of all operator families

$$A(\lambda) := H(\lambda) + M(\lambda) + G(\lambda)$$

where $(M + G)(\lambda) \in L^\mu_{\mathrm{M+G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$,

$$H(\lambda) = r_2^{-\mu}\mathrm{Op}_M^{\gamma_2-b/2}(h)(\lambda)$$

where $h(r_2, v_2, \lambda) := \tilde{h}(v_2, r_2\lambda)$ for some $\tilde{h}(v_2, \tilde{\lambda}) \in M^\mu_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$.

Remark 17. We have

$$\begin{aligned} L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\}) \subset & L^{\mu}((s_0(B_1))^{\wedge}, \boldsymbol{g}_1; \mathbb{R}^d_{\lambda} \setminus \{0\}) \\ & := L^{\mu}((2\mathbb{B}_1)^{\wedge}; \mathbb{R}^d_{\lambda} \setminus \{0\})|_{\mathrm{int}\,\mathbb{B}_{1_+}}, \end{aligned} \tag{0.84}$$

see notation in the introduction.

Analogously as in the symbolic characterization of $H(\lambda)$ in Definition 2 and expressions (0.31), (0.32) we have degenerate families

$$p(r_2, \varrho_2, \lambda) = \tilde{p}(r_2 \varrho_2, r_2 \lambda)$$

for

$$\tilde{p}(\tilde{\varrho}_2, \tilde{\lambda}) \in L^{\mu}(B_1, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\varrho}_2, \tilde{\lambda}}).$$

Then the non-smoothing Mellin symbols $\tilde{h}(v_2, \tilde{\lambda}) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$ in Definition 16 are obtained from $\tilde{p}$ via Mellin quantization from $\tilde{p}(\tilde{\varrho}_2, \tilde{\lambda})$ which turns $\tilde{\varrho}_2$ to v_2. Another principal symbolic level of operators $A(\lambda)$ in Definition 16 is the highest conormal symbol belonging to the corner singularity

$$\sigma_2(A)(v_2) := h(0, v_2, 0) + f_0(v_2) : H^{s,\gamma_1}(B_1) \to H^{s-\mu, \gamma_1 - \mu}(B_1) \tag{0.85}$$

with h as in Definition 16, frozen at $r_2 = 0$ and the principal conormal symbol of $M(\lambda)$, cf. relation (0.83) which is just the smoothing Mellin symbol belonging to the summand for $j = 0$, now denoted by $f_0(v_2)$. The conormal symbol (0.85) is regarded as an operator function parametrized by $v_2 \in \Gamma_{\frac{b_1+1}{2} - \gamma_2}$ for $b_1 = \dim B_1$.

Theorem 18. *Any* $A(\lambda) \in L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ *induces families of continuous operators*

$$A(\lambda) : \mathcal{K}^{s,\gamma}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu, \gamma-\mu}(B_1^{\wedge}) \tag{0.86}$$

and

$$A(\lambda) : \mathcal{K}^{s,\gamma}_{\mathcal{P}_1, \mathcal{P}_2}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu, \gamma-\mu}_{\mathcal{Q}_1, \mathcal{Q}_2}(B_1^{\wedge}) \tag{0.87}$$

for every pair of asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ *and some resulting* $\mathcal{Q}_1, \mathcal{Q}_2$.

It can be easily verified that operator families $A(\lambda) \in L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ form symbols

$$A(\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma}(B_1^{\wedge}), \mathcal{K}^{s-\mu, \gamma-\mu}(B_1^{\wedge}))$$

for any real s. The twisted homogeneity in the respective symbolic estimates refer to group actions in the involved spaces

$$(\kappa_{\delta} u)(r_2, x) = \delta^{(b_1+1)/2} u(\delta r_2, x)$$

for all $\delta \in \mathbb{R}_+$, with x denoting the variable on B_1. Let $S^{(\mu)}(\mathbb{R}^d \setminus \{0\}; \mathcal{K}^{s,\gamma}(B_1^{\wedge}), \mathcal{K}^{s-\mu, \gamma-\mu}(B_1^{\wedge}))$ be the space of principal homogeneous components. Those are also operator families in $L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ and play the role of principal edge symbols for operators of the edge calculus of second singular order in Subsection 2.1 below, with λ being replaced by (η_2, λ).

Let us call a family $A(\lambda) \in L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ parameter-dependent elliptic of order μ, with respect to the fixed weights $\gamma = (\gamma_1, \gamma_2)$ if $\sigma_0(A)$ is parameter-dependent elliptic over $B_1^{\wedge} \setminus s_0(B_1^{\wedge})$ in the standard sense and if also the reduced symbols

$$r_2^{-\mu} \tilde{\sigma}_0(A)(r_1, x, y_1, r_2, \tilde{\varrho}_1, \xi, \tilde{\varrho}_2, \tilde{\eta}_1, \tilde{\lambda})$$

for $r_2 > 0$ do not vanish for $(\tilde{\varrho}_1, \xi, \tilde{\varrho}_2, \tilde{\eta}_1, \tilde{\lambda}) \neq 0$ up to $r_1 = 0$, where $\tilde{\varrho}_1 := r_1 \varrho_1$, etc., and for r_1, r_2 close to zero

$$\hat{\tilde{\sigma}}_0(A)(r_1, x, y_1, r_2, \tilde{\varrho}_1, \xi, \hat{\tilde{\varrho}}_2, \tilde{\eta}_1, \hat{\tilde{\lambda}})$$

is non-vanishing for $(\tilde{\varrho}_1, \xi, \hat{\tilde{\varrho}}_2, \tilde{\eta}_1, \hat{\tilde{\lambda}}) \neq 0$ up to $r_2 = 0$, for $\hat{\tilde{\varrho}}_2 := r_1 r_2 \varrho_2$, $\hat{\tilde{\lambda}} := r_1 r_2 \lambda$. In addition we ask that $\tilde{h}(v_2, \tilde{\lambda}) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$, cf. Definition 16 to be parameter-dependent elliptic of order μ (this just encodes the suitable exit ellipticity when $r_2 \to \infty$), and that the principal conormal symbol (0.85) is a family of isomorphism for all $v_2 \in \Gamma_{\frac{b_1+1}{2} - \gamma_2}$.

Theorem 19. *Let $A(\lambda) \in L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ be parameter-dependent elliptic of order μ and relative to the weight $\gamma = (\gamma_1, \gamma_2)$. Then there is a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}(B_1^{\wedge}, \boldsymbol{g}^{-1}; \mathbb{R}^d \setminus \{0\}), \boldsymbol{g}^{-1} = (\boldsymbol{g}_1^{-1}, \boldsymbol{g}_2^{-1})$ such that*

$$P(\lambda)A(\lambda) = 1 - G_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - G_{\mathrm{R}}(\lambda)$$

for some $G_{\mathrm{L}}(\lambda) \in L^{-1}_{\mathrm{G}}(B_1^{\wedge}, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}), G_{\mathrm{R}}(\lambda) \in L^{-1}_{\mathrm{G}}(B_1^{\wedge}, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d \setminus \{0\})$ for $\boldsymbol{g}_{\mathrm{L}} := (\boldsymbol{g}_{1,\mathrm{L}}, \boldsymbol{g}_{2,\mathrm{L}}), \boldsymbol{g}_{\mathrm{R}} := (\boldsymbol{g}_{1,\mathrm{R}}, \boldsymbol{g}_{2,\mathrm{R}})$.

Corollary 20. *If $A(\lambda)$ is parameter-dependent elliptic then*

$$A(\lambda) : \mathcal{K}^{s,\gamma}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^{\wedge})$$

is a family of Fredholm operators, and kernels belong to subspaces of $\mathcal{K}^{\infty,\gamma}(B_1^{\wedge})$ with asymptotics. Cokernels can be represented by finite-dimensional subspaces of $\mathcal{K}^{\infty,\gamma-\mu}(X^{\wedge})$ with asymptotics.

We established the calculus of edge operators over a $B_1 \in \mathfrak{M}_1$ with parameters, including discrete or continuous asymptotics with respect to an edge Z_1. Here we slightly modify notation, since first order and second order edges for $B_2 \in \mathfrak{M}_2$ are denoted by Y_1, Y_2, according to the way of successively forming wedges, e.g., when

$$B_2 = B_1^{\Delta} \times Y_2 \tag{0.88}$$

we have $Y_2 = s_2(B_2)$ but $Y_1 = s_1(B_2 \setminus Y_2)$, and then $B_1 \in \mathfrak{M}_1$ which is the base of the new model cone has another edge, now denoted by Z_1 of some dimension p_1 which is also admitted to be zero. Thus analogously as (0.57) we have to introduce local (along Y_2 in variables $(y_2, \eta_2, \lambda), y_2 \in \mathbb{R}^{q_2}$) spaces of amplitude functions

$$\mathcal{R}^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g}) \tag{0.89}$$

for weight data $\boldsymbol{g} := (\boldsymbol{g}_i)_{i=1,2}, \boldsymbol{g}_i = (\gamma_i, \gamma_i - \mu, \Theta_i), i = 1, 2$, consisting of operator functions

$$a_2(y_2, \eta_2, \lambda) = h_2(y_2, \eta_2, \lambda) + (m_2 + g_2)(y_2, \eta_2, \lambda) \tag{0.90}$$

for the asymptotic part

$$(m_2+g_2)(y_2,\eta_2,\lambda)\in\mathcal{R}^{\mu}_{\mathrm{M+G}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d},\boldsymbol{g}). \tag{0.91}$$

Let $\mathcal{R}^{\mu}_{\mathrm{G}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d},\boldsymbol{g})$ be the space of Green symbols, i.e.,

$$g(y_2,\eta_2,\lambda)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d};\mathcal{K}^{s;\gamma;e}(B_1^{\wedge}),\mathcal{K}^{\infty;\gamma-\mu;\infty}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^{\wedge}))$$

such that the pointwise formal adjoint with respect to the $\mathcal{K}^{0,0}(B_1^{\wedge})$-scalar product has the property

$$g^{*}(y_2,\eta_2,\lambda)\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d};\mathcal{K}^{s;-\gamma+\mu}(B_1^{\wedge}),\mathcal{K}^{\infty,-\gamma;\infty}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_1^{\wedge}))$$

for g-dependent asymptotic types $\mathcal{P}_1,\mathcal{P}_2$ and $\mathcal{Q}_1,\mathcal{Q}_2$, associated with the weight information involved in the respective spaces. Let us admit here from the very beginning continuous asymptotic types for $r_1\to 0$ or $r_2\to 0$. Clearly we also have discrete asymptotics, and y_1- or y_2-independence as a special case. Analogously as (0.56) there are also Mellin amplitude functions

$$m(y_2,\eta_2,\lambda):=r_2^{-\mu}\omega_{\eta_2,\lambda}\sum_{j=0}^{\theta_2}r_2^{j}\sum_{|\alpha|\le j}\mathrm{Op}_M^{\gamma_{j\alpha}-b_1/2}(f_{j\alpha})(y_2)(\eta_2,\lambda)^{\alpha}\omega'_{\eta_2,\lambda}$$

for Mellin symbols

$$f_{j\alpha}(y_2,v_2)\in C^{\infty}(\mathbb{R}^{q_2},M^{-\infty}_{\mathcal{R}_{j\alpha}}(B_1,\boldsymbol{g}_1))$$

satisfying the conditions

$$\Gamma_{\frac{b_1+1}{2}-\gamma_{j\alpha}}\cap\pi_{\mathbb{C}}\mathcal{R}_{j\alpha}=\emptyset\quad\text{for}\quad\gamma_2-j\le\gamma_{j\alpha}\le\gamma_2$$

for all j,α. Clearly $f_{j\alpha},\mathcal{R}_{j\alpha}$ and weights $\gamma_{j\alpha}$ are different from the corresponding objects in (0.56); for convenience we employ here the same notation. The Mellin asymptotic types $\mathcal{R}_{j\alpha}$ are admitted to be continuous. As for first singularity order 1 by

$$\mathcal{R}^{\mu}_{\mathrm{M+G}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d},\boldsymbol{g})$$

we denote the space of all $(m+g)(y_2,\eta_2,\lambda)$ of the indicated structure (where for the Mellin part we also take into account more summands of the same conormal order when the asymptotic types are continuous) and we write $\mathcal{R}^{\mu}_{\mathrm{G}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d},\boldsymbol{g})$ for the respective space of Green amplitude functions. The non-smoothing holomorphic Mellin part is defined as

$$h_2(y_2,\eta_2,\lambda)=\omega_2 r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2-b_1/2}(\boldsymbol{h}_2)(y_2,\eta_2,\lambda)\omega'_2 \tag{0.92}$$

for some

$$\boldsymbol{h}_2(r_2,y_2,v_2,\eta_2,\lambda)=\tilde{\boldsymbol{h}}_2(r_2,y_2,v_2,r_2\eta_2,r_2\lambda) \tag{0.93}$$

for

$$\tilde{\boldsymbol{h}}_2(r_2,y_2,v_2,\tilde{\eta}_2,\tilde{\lambda})\in C^{\infty}(\overline{\mathbb{R}}_+\times\mathbb{R}^{q_2},M^{\mu}_{\mathcal{O}_{v_2}}(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\tilde{\eta}_2,\tilde{\lambda}})). \tag{0.94}$$

On $B_2\in\mathfrak{M}_2$ we have weighted spaces $H^{s,\gamma}(B_2)$, defined as the set of those $u\in H^{s,\gamma}_{\mathrm{loc}}(B_2\setminus Y_2)$ such that $\varphi_i u\circ\chi_i^{-1}\in\mathcal{W}^{s}(\mathbb{R}^{q_2},\mathcal{K}^{s,\gamma}(B_1^{\wedge}))$ for all i. Here $(\varphi_1,\dots,\varphi_N)$ is a partition of unity on Y_2 subordinate to an open covering of Y_2 by coordinate neighbourhoods $Y_{2,i}$, and $\chi_i:Y_{2,i}\to\mathbb{R}^{q_2}$ charts, $i=1,\dots,N$. Moreover, we have subspaces with asymptotics $H^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_2)$ consisting of elements $u \in H^{s,\gamma_1}_{\mathrm{loc},\mathcal{P}_1}(B_2 \setminus Y_2)$ such that $\varphi_i u \circ \chi_i^{-1} \in \mathcal{W}^s(\mathbb{R}^{q_2}, \mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge))$. There is then the space

$$L^{-\infty}(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$$

of all $C(\lambda) \in \mathcal{S}(\mathbb{R}^d_\lambda, L^{-\infty}(B_2, \boldsymbol{g}))$ where $L^{-\infty}(B_2, \boldsymbol{g})$ is the space of all operators

$$C(\lambda) : H^{s,\gamma}(B_2) \to H^{\infty,\gamma-\mu}_{\mathcal{P}_1,\mathcal{P}_2}(B_2) \tag{0.95}$$

such that the formal adjoint with respect to the scalar product of $\mathcal{H}^{0,0}(B_2)$ induces continuous maps

$$C^*(\lambda) : H^{s,-\gamma+\mu}(B_2) \to H^{\infty,-\gamma}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_2) \tag{0.96}$$

for C-dependent asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{Q}_1, \mathcal{Q}_2$, respectively.

Definition 21. By $L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ for $B_2 \in \mathfrak{M}_2$ and $\boldsymbol{g} = (\boldsymbol{g}_1, \boldsymbol{g}_2)$ we denote the space of operator families

$$A(\lambda) := H(\lambda) + (M+G)(\lambda) + A_{\mathrm{int}}(\lambda) + C(\lambda)$$

where

$$H(\lambda) + (M+G)(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}(a_2)(\lambda)\varphi'$$

for $a_2(y_2, \eta_2, \lambda) \in \mathcal{R}^\mu(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g})$, where the sum runs over a partition of unity of Y_2 by localizing functions φ and $\varphi' \succ \varphi$ are compactly supported smooth functions in the respective coordinate neighbourhoods. Moreover, $A_{\mathrm{int}}(\lambda)$ belongs to $L^\mu(B_2 \setminus Y_2, \boldsymbol{g}_1; \mathbb{R}^d_\lambda)$ and its kernel is supported off a small neighbourhood of Y_2. The operator family $C(\lambda)$ belongs to $L^{-\infty}(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$.

Operators in $L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ have the property, that the restrict to elements in $L^\mu(B_2 \setminus Y_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$. Applying a decomposition

$$\begin{aligned} A(\lambda) =& \omega_2 A(\lambda)\omega_2' + \omega_2 A(\lambda)(1-\omega_2') + (1-\omega_2)A(\lambda)\omega_2'' \\ &+ (1-\omega_2)A(\lambda)(\omega_2' - \omega_2'') + (1-\omega_2)A(\lambda)(1-\omega_2') \end{aligned}$$

for global cut-off functions over B_2 which are $\equiv 1$ close to Y_2 and $\omega_2'' \prec \omega_2 \prec \omega_2'$. Then $\omega_2 A(\lambda)\omega_2'$ is located close to Y_2 and can be described by amplitude function in (0.89). Those belong to $S^\mu(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s,\gamma}(B_1^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge))$ and hence induce (after a localization in $y_2 \in \mathbb{R}^{q_2}$ from both sides) continuous operators

$$\mathcal{W}^s(\mathbb{R}^{q_2}, \mathcal{K}^{s,\gamma}(B_1^\wedge)) \to \mathcal{W}^{s-\mu}(\mathbb{R}^{q_2}, \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge)).$$

Globally, assuming B_2 to be compact, there operators are continuous in the sense

$$\omega_2 A(\lambda)\omega_2' : H^{s,\gamma}(B_2) \to H^{s-\mu,\gamma-\mu}(B_2).$$

The operators $\omega_2 A(\lambda)(1-\omega_2') + (1-\omega_2)A(\lambda)\omega_2''$ belong to $L^{-\infty}(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ and induce continuous operators (0.95), (0.96). The operators

$$(1-\omega_2)A(\lambda)(\omega_2' - \omega_2'') \quad \text{and} \quad (1-\omega_2)A(\lambda)(1-\omega_2')$$

belong to $L^\mu(B_2 \setminus Y_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ and generate continuous operators

$$H^{s,\gamma_1,\tilde{\gamma}_2}(B_2) \to H^{s-\mu,\gamma_1-\mu,\tilde{\tilde{\gamma}}_2-\mu}(B_2)$$

for any $\tilde{\gamma}_2, \tilde{\tilde{\gamma}}_2 \in \mathbb{R}$. Similar considerations may be applied to subspaces with asymptotics, using that any element a of (0.89) belongs to a space $S^\mu(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_1^\wedge))$. Thus we obtain altogether the following result:

Proposition 22. *Operators $A(\lambda) \in L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ represent families of continuous operators*

$$A(\lambda) : H^{s,\gamma}(B_2) \to H^{s-\mu,\gamma-\mu}(B_2)$$

and

$$A(\lambda) : H^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_2) \to H^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_2)$$

for any pairs of asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ and some resulting $\mathcal{Q}_1, \mathcal{Q}_2$, for all $s \in \mathbb{R}$.

Let us now pass to ellipticity and Fredholm property of operators in the calculus over $B_2 \in \mathfrak{M}_2$. An operator $A(\lambda) \in L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ in notation of Definition 21 is called elliptic if $A(\lambda)|_{s_0(B_2)} \in L^\mu_{\mathrm{cl}}(s_0(B_2); \mathbb{R}^d_\lambda)$ is parameter-dependent elliptic over the smooth manifold $s_0(B_2)$ in the standard sense, moreover, if

$$\tilde{\sigma}_0(A)(r_1, x, y_1, r_2, y_2, \tilde{\varrho}_1, \xi, \tilde{\eta}_1, \tilde{\varrho}_2, \tilde{\eta}_2, \tilde{\lambda})$$

which is the reduced symbol of $A(\lambda)|_{B_1 \setminus Y_2}$ does not vanish for

$$(\tilde{\varrho}_1, \xi, \tilde{\eta}_1, \tilde{\varrho}_2, \tilde{\eta}_2, \tilde{\lambda}) \neq 0$$

up to $r_1 = 0$, where tilde indicates multiplication by r_1, and

$$\hat{\tilde{\sigma}}_0(A)(r_1, x, y_1, r_2, y_2, \tilde{\varrho}_1, \xi, \tilde{\eta}_1, \hat{\tilde{\varrho}}_2, \hat{\tilde{\eta}}_2, \hat{\tilde{\lambda}})$$

which is the reduced symbol of $A(\lambda)|_{B_1 \setminus (Y_1 \cup Y_2)}$ (computed close to $r_1 = 0$ and $r_2 = 0$) does not vanish for $(\tilde{\varrho}_1, \xi, \tilde{\eta}_1, \hat{\tilde{\varrho}}_2, \hat{\tilde{\eta}}_2, \hat{\tilde{\lambda}}) \neq 0$ up to $r_2 = 0$, where $\hat{\tilde{\ }}$ indicates multiplication by $r_1 r_2$. In addition over $B_2 \setminus Y_2$ we assume that the parameter-dependent edge symbol close to Y_1

$$\sigma_1(A)(y_1, r_2, y_2, \eta_1, \varrho_2, \eta_2, \lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)$$

is bijective for $(\eta_1, \varrho_2, \eta_2, \lambda) \neq 0$ and also its reduced symbol

$$\tilde{\sigma}_1(A)(y_1, r_2, y_2, \tilde{\eta}_1, \varrho_2, \eta_2, \tilde{\lambda}) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)$$

up to $r_1 = 0$ for $(\tilde{\eta}_1, \varrho_2, \eta_2, \tilde{\lambda}) \neq 0$. In addition close to Y_2 we ask bijectivity of the second singular edge symbol

$$\sigma_2(A)(y_2, \eta_2, \lambda) : \mathcal{K}^{s,\gamma}(B_1^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge) \tag{0.97}$$

to be bijective for $(\eta_2, \lambda) \neq 0$. Note that this entails the bijectivity of the subordinate conormal symbol (0.85). This is an operator family in $L^\mu(B_1^\wedge, \boldsymbol{g}; (\mathbb{R}^{q_2} \times \mathbb{R}^d) \setminus \{0\})$ which also depends on $y_2 \in Y_2$.

Theorem 23. *Let $A(\lambda) \in L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d)$ be parameter-dependent elliptic. Then there is a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}(B_2, \boldsymbol{g}^{-1}; \mathbb{R}^d)$, $\boldsymbol{g}^{-1} = (\boldsymbol{g}_1^{-1}, \boldsymbol{g}_2^{-1})$ such that*

$$P(\lambda)A(\lambda) = 1 - C_{\mathrm{L}}(\lambda), \quad A(\lambda)P(\lambda) = 1 - C_{\mathrm{R}}(\lambda) \tag{0.98}$$

for $C_{\mathrm{L}}(\lambda) \in L^{-\infty}(B_2, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d)$, $C_{\mathrm{R}}(\lambda) \in L^{-\infty}(B_2, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d)$ *for* $\boldsymbol{g}_{\mathrm{L}} := (\boldsymbol{g}_{1,\mathrm{L}}, \boldsymbol{g}_{2,\mathrm{L}})$, $\boldsymbol{g}_{\mathrm{R}} := (\boldsymbol{g}_{1,\mathrm{R}}, \boldsymbol{g}_{2,\mathrm{R}})$. *Moreover, if* B_2 *is compact*

$$A(\lambda) : H^{s,\gamma}(B_2) \to H^{s-\mu,\gamma-\mu}(B_2) \tag{0.99}$$

is a family of Fredholm operators. It becomes a family of isomorphism for sufficiently large $|\lambda|$. *This holds for all* $s \in \mathbb{R}$. *In addition, solutions* u *of* $A(\lambda)u = f$ *for* $f \in H^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_2)$ *belong to* $H^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_2)$ *for pairs of continuous asymptotic types* $\mathcal{Q}_1, \mathcal{Q}_2$ *with resulting* $\mathcal{P}_1, \mathcal{P}_2$.

References

[1] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[2] D.-C. Chang and B.-W. Schulze, *Calculus on spaces with higher singularities*, J. Pseudo-Differ. Oper. Appl. **8** (4) (2016), 585–622.

[3] D.-C. Chang and B.-W. Schulze, *Ellipticity on spaces with higher singularities*, CHINA SCIENCE Math. **60** (11) (2017), 2053–2076.

[4] H.-J. Flad, G. Harutyunyan, and B.-W. Schulze, *Explicit Green operators for quantum mechanical Hamiltonians. II. Edge singularities of the helium atom*, manuscripta math. **135** (2011), 497–519.

[5] H.-J. Flad, G. Harutyunyan, and B.-W. Schulze, *Asymptotic parametrices on singular cones.* (in preparation, 2017)

[6] J.B. Gil, B.-W. Schulze, and J. Seiler, *Holomorphic operator-valued symbols for edge-degenerate pseudo-differential operators*, in 'Differential Equations, Asymptotic Analysis and Mathematical Physics", eds. M. Demuth et al. Mathematical Research, vol. 100, pp. 113-137. Akademie Verlag (1997).

[7] J.B. Gil, B.-W. Schulze, and J. Seiler, *Cone pseudodifferential operators in the edge symbolic calculus*, Osaka J. Math. **37** (2000), 219–258.

[8] M. Hedayat-Mahmoudi and B.-W. Schulze, *A new approach to the second order edge calculus*, J. Pseudo-Differ. Oper. Appl. (2017). doi:10.1007/s11868-017-0191-2

[9] H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, (1981).

[10] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[11] X. Lyu and B.-W. Schulze, *Mellin operators in the edge calculus*, Complex Analysis and Operator Theory **10** (2016), 965–10000.

[12] E. Schrohe and B.-W. Schulze, *Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities I*, Math. Topics **5**, Adv. in: PDE "Pseudo-Differential Calculus and Mathematical Physics", Akademie Verlag, Berlin, (1994).

[13] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[14] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[15] B.-W. Schulze, *Operators with symbol hierarchies and iterated asymptotics*, Publ. Res. Inst. Math. Sci., Kyoto **38** (2002), 735–802.

[16] B.-W. Schulze, *The iterative structure of the corner calculus*, in: Rodino, L., Wong, M.W., Zhu, H.(eds.) Operator Theory: Advances and Applications, vol. 213, pp. 79–103. Pseudo-Differential Operators: Analysis, Application and Computations. Birkhäuser Verlag, Basel, (2011).

[17] W. Rungrottheera, *Parameter-dependent corner operators*, Asian-Eur. J. Math. **6** (2013). doi:10.1142/S1793557113500022

[18] W. Rungrottheera and B.-W. Schulze, *Holomorphic operator families on a manifold with edge*, J. Pseudo-Differ. Oper. Appl. **4** (2013), 297–315.

[19] W. Rungrottheera, B.-W. Schulze and M.W. Wong, *Iterative properties of pseudo-differential operators on edge spaces*, J. Pseudo-Differ. Oper. Appl. **5** (2014), 455–479.

PARAMETER-DEPENDENT EDGE CALCULUS AND CORNER PARAMETRICES

WANNARUT RUNGROTTHEERA, XIAOJING LYU, AND BERT-WOLFGANG SCHULZE

Abstract. Let B be a compact manifold with smooth edge of dimension > 0. We study the interplay between parameter-dependent edge algebra algebra on B and operator families belonging to the corner calculus, and we characterize parametrices in the corner case.

1. Introduction

This article is devoted to a specific part of analysis on a manifold M with singularities, here $M := B^{\triangle}$ for

$$B^{\triangle} := (\overline{\mathbb{R}}_+ \times B)/(\{0\} \times B) \tag{1.1}$$

where B is a manifold with edge of first order. Let us fix some terminology. The general background are stratified spaces $M \in \mathfrak{M}_k$ of singularity order $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, where the case $k = 0$ corresponds to smoothness, together with other common conditions, e.g., para-compactness, etc. We are interested in relatively simple stratifications, obtained by repeatedly forming cones $X^{\triangle}, X \in \mathfrak{M}_0$, and wedges $X^{\triangle} \times \mathbb{R}^{q_1}$ for some q_1, use those spaces as generalized charts for other spaces which are locally modeled on such cones or wedges, and then form cones, and wedges, again. In recent years there has been achieved much progress in the analysis on such spaces. The general scheme is that any $M \in \mathfrak{M}_k$ has a singular stratum $Y_k := s_k(M) \in \mathfrak{M}_0$, where $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$ and every $y_k \in Y_k$ has a neighbourhood V in M which is isomorphic (in the category of regular singularities) to a locally trivial bundle over $s_k(M)$ with fibre $B^{\triangle}_{k-1}$ for a compact $B_{k-1} \in \mathfrak{M}_{k-1}$. This definition can be iterated. In particular, we can study cones $B^{\triangle} \in \mathfrak{M}_2$ for $B \in \mathfrak{M}_1$. For $k \geq 2$ the stratification of $M \in \mathfrak{M}_k$ gives rise to a sequence

$$s(M) = (s_0(M), s_1(M), \ldots, s_k(M)) \tag{1.2}$$

of smooth subspaces $s_j(M) \in \mathfrak{M}_0, j = 0, 1, \ldots, k$, of dimension $\dim M := \dim s_0(M) > \dim s_1(M) > \cdots \geq \dim s_k(M) > 0$. Our investigations belong to the pseudo-differential calculus over such spaces M. Those contain specific corner-degenerate differential operators A with a hierarchy of principal symbols

$$\sigma(A) = (\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)). \tag{1.3}$$

The components of (1.3) are operator-valued, but $\sigma_0(A)$ is the standard homogeneous principal symbol on $T^* s_0(M) \setminus 0$, where 0 indicates the zero-section. Under a

---

2010 *Mathematics Subject Classification.* Primary 35S35; Secondary 35J70.

Key words and phrases. Edge calculus, corner parametrices.

This research has been supported by the Thailand Research Fund (TRF) grant MRG5980089.

suitable condition on ellipticity also the parametrices of operators A belong to expected operator algebras. There are also asymptotic aspects of this program and we have structures for $B \in \mathfrak{M}_1$ (in parameter-dependent form) generating asymptotics of solutions close to edges $s_1(\cdot)$ on B and double asymptotics close to corner points $s_2(\cdot)$ of B^Δ, cf. [15], [8]. With $M \in \mathfrak{M}_k$ we can associate its stretched manifold $\mathbb{M}$ obtained by invariantly attaching a natural B_{k-1}- bundle (determined by the above-mentioned B^Δ_{k-1}-bundle) over $s_k(M)$ to $M \setminus s_k(M)$. Then, for a "negative" counterpart of $\mathbb{M}$ denoted for the moment by $\mathbb{M}_-$ and the former $\mathbb{M}$ by $\mathbb{M}_+$ we can glue together $\mathbb{M}_-$ and $\mathbb{M}_+$ along the common B_{k-1}-bundle to obtain $2\mathbb{M}$, and then $2\mathbb{M} \in \mathfrak{M}_{k-1}$.

The present paper is organized as follows. In Section 2 we establish an edge pseudo-differential calculus over B with parameters. Those are involved as extra covariables, degenerate in a similar manner as edge covariables. We then formulate the existence of parameter-dependent parametrices in this framework with controlled remainders. Section 3 is devoted to elements of a analogous approach for manifolds with edges of singularity order 2. In particular, we formulate a new level of operator-valued corner symbols and compute corner parametrices in terms of a new set of variables and covariables in the new corner axis direction.

In this article we continue and deepen the material of [2] and [3] in cases of lower orders of singularities.

2. PARAMETER-DEPENDENT EDGE CALCULUS

2.1. **Basic notation.** Let $B \in \mathfrak{M}_1$ be a compact manifold with conical or edge singularities. In order to avoid separate comments, we assume that $\dim s_1(B) =: q > 0$. For the edge pseudo-differential calculus over B we employ suitable adapted notation. Write $Y := s_1(B)$ and $s_0(B) = B \setminus Y$. The most specific part of the edge calculus over B concerns the region close to the edge Y. The space B is locally near Y modeled on $X^\Delta \times \mathbb{R}^q$, and $B \setminus Y$ on $X^\wedge \times \mathbb{R}^q$ for a compact $X \in \mathfrak{M}_0$ for $X^\wedge := \mathbb{R}_+ \times X$. Throughout this exposition for simplicity we assume that Y has a neighbourhood W in B with the structure of trivial X^Δ-bundle over Y. The general case only needs mild extra constructions, left to the reader. Recall that any smooth manifold B with boundary belongs to $\mathfrak{M}_1$. In this case $s_0(B) = \operatorname{int} B$ and $s_1(B) = \partial B$, where W corresponds to a collar neighbourhood of ∂B, often identified with the trivial normal bundle, corresponding to a Riemannian metric. In this exposition we systematically employ the spaces $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\lambda)$ of classical parameter-dependent pseudo-differential operators of order $\mu \in \mathbb{R}$ over $X \in \mathfrak{M}_0$ of dimension n, assumed to be Riemannian, where $\lambda \in \mathbb{R}^d$ in local symbols $a(x, \xi, \lambda)$ is treated as a covariable (ξ, λ) of dimension $n+d$. Subscript "cl" means that we talk about classical symbols in Hörmander's spaces, here denoted by $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+d}), \Omega \subseteq \mathbb{R}^n$ open. Then $L^{-\infty}(X; \mathbb{R}^d_\lambda) = \mathcal{S}(\mathbb{R}^d_\lambda, L^{-\infty}(X))$, where $L^{-\infty}(X)$ is identified via the Riemannian metric with $C^\infty(X \times X)$, such that operators can be written $\int c(x, x')u(x')dx'$ with dx' being the measure belonging to the Riemannian metric. Then operators are locally of the form

$$\mathrm{Op}_x(a)(\lambda)u(x) = \iint e^{i(x-x')\xi} a(x, \xi, \lambda)u(x')dx' đ\xi \tag{2.1}$$

for compactly supported u, where $đ\xi = (2\pi)^{-n}d\xi$. Globally using a system of charts covering X and a subordinate partition of unity the operator families over X are determined by such local expressions modulo $L^{-\infty}(X;\mathbb{R}^d_\lambda)$. The spaces $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^d_\lambda)$ are Fréchet in a natural way.

The main object of consideration in the present section is the space of edge operators

$$L^\mu(B,\boldsymbol{g};\mathbb{R}^d_\lambda) \tag{2.2}$$

to be introduced in a number of steps. The properties of (2.2) will imply

$$L^\mu(B,\boldsymbol{g};\mathbb{R}^d_\lambda) \subseteq L^\mu_{\mathrm{cl}}(B\setminus Y;\mathbb{R}^d_\lambda)$$

Notation $\boldsymbol{g}$ indicates weight data $(\gamma_1,\gamma_1-\mu,\Theta)$.

We also employ pseudo-differential operators based on the Mellin transform and more specific Mellin symbols that are holomorphic in $v_1 \in \mathbb{C}$.

Definition 2.1. Let $M^\mu_{\mathcal{O}}(X;\mathbb{R}^d)$ denote the space of all $h(v_1,\lambda)\in\mathcal{A}(\mathbb{C}_{v_1}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^d))$ such that $h(\beta_1+i\varrho_1,\lambda)\in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+d}_{\varrho_1,\lambda})$ for every $\beta_1\in\mathbb{R}$, uniformly in compact β_1-intervals.

Remark 2.2. The space $M^\mu_{\mathcal{O}}(X;\mathbb{R}^d)$ is Fréchet with the system of semi-norms of $\mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^d))$, namely, $\sup_{v_1\in K}\pi_j(h(v_1,\cdot))$, where $\pi_j, j\in\mathbb{N}$, is a semi-norm system for $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^d), K\Subset\mathbb{C}$, together with

$$\sup_{-l\le\beta_1\le l}\pi_j(h(\beta_1+i\varrho_1,\cdot))$$

where $\pi_j, j\in\mathbb{N}$, is a semi-norm system for $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+d}_{\varrho_1,\lambda})$, and $l\in\mathbb{N}$.

Let us prepare Definition 2.3 below by a notation and a list of ingredients. We fix weight data $\boldsymbol{g}=(\gamma_1,\gamma_1-\mu,\Theta)$, where $\gamma_1\in\mathbb{R}$ is a weight referring to $r_1\in\mathbb{R}_+$, the axial variable of the open stretched cone $X^\wedge := \mathbb{R}_+\times X$, in the splitting of variables (r_1,x), and $X\in\mathfrak{M}_0$ compact, and we choose a weight interval $\Theta=(-(\theta+1),0]$ for some $\theta\in\mathbb{N}$. We employ spaces $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ and subspaces $\mathcal{K}^{s,\gamma_1}_\Theta(X^\wedge)$ and $\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge)$ with asymptotics of type $\mathcal{P}$, cf. [14], $\mathcal{P}$ may be discrete or continuous. Below, in order to illustrate double asymptotics, we recall a few notions from this context. For the moment we freely employ these tools.

Definition 2.3. By

$$L^\mu(X^\wedge,\boldsymbol{g};\mathbb{R}^d_\lambda\setminus\{0\}) \tag{2.3}$$

we denote the space of all operator families

$$A(\lambda) := H(\lambda)+M(\lambda)+G(\lambda) \tag{2.4}$$

where $(M+G)(\lambda)\in L^\mu_{\mathrm{M+G}}(X^\wedge,\boldsymbol{g};\mathbb{R}^d_\lambda\setminus\{0\})$, see notation below, and

$$H(\lambda) = r_1^{-\mu}\mathrm{Op}_M^{\gamma_1-n/2}(h)(\lambda) \tag{2.5}$$

where

$$h(r_1,v_1,\lambda) := \tilde{h}(v_1,r_1\lambda)$$

for some $\tilde{h}(v_1,\tilde{\lambda})\in M^\mu_{\mathcal{O}_{v_1}}(X;\mathbb{R}^d_{\tilde{\lambda}})$.

In relation (2.5) we employ notation

$$\mathrm{Op}_M^{\nu}(h)(\lambda) := r_1^{\nu}\mathrm{Op}_M(T^{-\nu}h)(\lambda)r_1^{-\nu}. \tag{2.6}$$

Here $T^{-\nu}h(v_1,\lambda) := h(v_1+\nu,\lambda)$ and $\mathrm{Op}_M(f)(\lambda)u := M^{-1}f(v_1,\lambda)Mu$ where the Mellin transform M refers to the weight line $\Gamma_{1/2}$. Here $\Gamma_\beta := \{v_1 \in \mathbb{C} : \mathrm{Re}\, v_1 = \beta\}$.

At the beginning of the iteration from lower to larger orders we have the open stretched cone and operators have two principal symbolic components $(\sigma_0(A), \sigma_1(A))$, here depending on $\lambda \neq 0$. Operators in (2.3) are continuous in weighted Kegel spaces

$$\begin{aligned}\mathcal{K}^{s,\gamma_1}(X^\wedge) := \{\omega u_0 + (1-\omega)u_\infty : & u_0(r_1,x) \in \mathcal{H}^{s,\gamma_1}(X^\wedge),\\ & u_\infty(r_1,x) \in H^s_{\mathrm{cone}}(X^\wedge)\}\end{aligned}$$

for an arbitrary cut-off function ω, i.e., a function in $C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 close to the origin. The spaces $\mathcal{H}^{s,\gamma_1}(X^\wedge)$ are weighted Sobolev spaces over $X^\wedge = \mathbb{R}_+ \times X$ of smoothness $s \in \mathbb{R}$ and weight $\gamma_1 \in \mathbb{R}$. Those have the property

$$\mathcal{H}^{s,\gamma_1}(X^\wedge) = r^{\gamma_1}\mathcal{H}^{s,0}(X^\wedge)$$

and $\mathcal{H}^{0,0}(X^\wedge)$ will be identified with $r_1^{-n/2}L^2(\mathbb{R}_+ \times X)$ for $n = \dim X$, while $(1-\omega)H^s_{\mathrm{cone}}(X^\wedge)$ can be defined first for $X = S^n$, the unit sphere in $\mathbb{R}^{1+n}_{\tilde{x}}$, where in this case

$$(1-\omega)H^s_{\mathrm{cone}}((S^n)^\wedge) = (1-\omega)H^s(\mathbb{R}^{1+n}), \tag{2.7}$$

with (r_1, x) being polar coordinates in $\mathbb{R}^{1+n} \setminus \{0\}$. Then a simple localization argument, using a partition of unity on S^n, allows us to pass from S^n to arbitrary X by using the former definition of cone spaces for elements supported in a coordinate neighbourhood on X. This gives us relations of the kind

$$(1-\omega)\mathcal{K}^{s,\gamma_1}((S^n)^\wedge) = (1-\omega)H^s_{\mathrm{cone}}((S^n)^\wedge), \tag{2.8}$$

$$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge). \tag{2.9}$$

Note that the definition of $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ is independent of the choice of involved data, such as cut-off functions or specific charts. In any case if the data are fixed the arizing spaces are Hilbert in non-direct sum topology, and relation (2.9) indicates a corresponding normalization of scalar products for $s = \gamma_1 = 0$. In addition it will be essential to employ the group $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\delta : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s,\gamma_1}(X^\wedge),\ (\kappa_\delta u)(r_1,x) := \delta^{(n+1)/2}u(\delta r_1, x) \tag{2.10}$$

for any $s \in \mathbb{R}, \delta \in \mathbb{R}_+$. The spaces $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ are observed on the infinite cone up to $r_1 \to \infty$, interpreted as a conical exit of $X^\wedge$ to infinity, though close to $r_1 = 0$, at the tip of the (open stretched) cone we are interested in asymptotics of elements. In the simplest case, asymptotics are discrete. Later on we also admit continuous asymptotics.

Remark 2.4. The operator families (2.4) are continuous in Kegel spaces, see Theorem 2.6 below. In particular, $H(\lambda)$ in (2.5) satisfies the homogeneity relation

$$H(\delta\lambda) = \delta^\mu \kappa_\delta H(\lambda)\kappa_\delta^{-1} \tag{2.11}$$

for all $\delta \in \mathbb{R}_+$.

A sequence of pairs

$$\mathcal{P} = \{(p_j, m_j)\}_{j=0,\dots,N} \subset \mathbb{C} \times \mathbb{N} \tag{2.12}$$

for any $N = N(\mathcal{P}) \in \mathbb{N} \cup \{+\infty\}$ is called a discrete asymptotic type associated with the weight information $(\gamma_1, \Theta), \Theta = (-(\theta+1), 0]$, for some $\theta \in \mathbb{N}$ if

$$\pi_{\mathbb{C}}\mathcal{P} := \{p_j\}_{j=0,\dots,N} \subset \{v_1 \in \mathbb{C} : \frac{n+1}{2} - \gamma_1 - (\theta+1) < \operatorname{Re} v_1 < \frac{n+1}{2} - \gamma_1\} \tag{2.13}$$

and $\pi_{\mathbb{C}}\mathcal{P}$ is finite for finite Θ, otherwise finite or infinite, and in the latter case we assume $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. Writing for finite Θ

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) := \left\{ \sum_{j=0}^{N} \sum_{l=0}^{m_j} \omega(r_1) c_{jl}(x) r_1^{-p_j} \log^l r_1 : c_{jl} \in C^\infty(X) \text{ for all } j, l \right\} \tag{2.14}$$

we get a Fréchet space such that $\mathcal{E}_{\mathcal{P}}(X^\wedge) \cap \mathcal{K}^{s,\gamma_1}_{\Theta}(X^\wedge) = \{0\}$ for

$$\mathcal{K}^{s,\gamma_1}_{\Theta}(X^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma_1-(\theta+1)-\varepsilon}(X^\wedge) \tag{2.15}$$

interpreted as the space of all flat functions relative to the weight γ_1, which is Fréchet as well. Then

$$\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge) := \mathcal{K}^{s,\gamma_1}_{\Theta}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge) \tag{2.16}$$

is Fréchet in the topology of the direct sum. For infinite Θ we define spaces $\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge)$ in a similar manner as projective limits over spaces associated with finite flatness intervals of increasing length. Later on we employ an extension of (2.16) to continuous asymptotic types $\mathcal{P}$ which makes sense when the points of (2.13) depend on some edge variables $y_1 \in \mathbb{R}^{q_1}$. More details may be found in [13] or [14]. With asymptotic types $\mathcal{P}$ we associate the space of Green operators, depending on a parameter $\lambda \in \mathbb{R}^d$, later on in slight modification Green operator-valued symbols for the edge calculus. To this end we also control our spaces for $r_1 \to \infty$ and write

$$\mathcal{K}^{s,\gamma_1;e}(X^\wedge) = [r_1]^{-e}\mathcal{K}^{s,\gamma_1}(X^\wedge),\ \mathcal{K}^{s,\gamma_1;e}_{\mathcal{P}}(X^\wedge) = [r_1]^{-e}\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge). \tag{2.17}$$

for any $e \in \mathbb{R}$. Here $r_1 \to [r_1]$ is a strictly positive function which is equal to 1 close to $r_1 = 0$ and equal to r_1 for $|r_1| \geq 1$. The spaces (2.17) are Fréchet in an evident manner. In order to formulate the subspace of Green operators $L^\mu_{\mathrm{G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ for weight data $\boldsymbol{g} := (\gamma_1, \gamma_1 - \mu, \Theta)$, with $\gamma_1 \in \mathbb{R}$ being the involved weight and $\mu \in \mathbb{R}$ an order we first recall some notation on Hilbert spaces with group action and associated operator-valued symbols. In order to keep the system of notation concise, we recall what we understand by operator-valued symbols. We freely employ the operator-valued generalization of Hörmander's spaces of symbols of order $\mu \in \mathbb{R}$, namely,

$$S^\mu(\mathbb{R}^q \times \mathbb{R}^q; H, \tilde{H}) \text{ and } S^\mu_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; H, \tilde{H}) \tag{2.18}$$

where subscript "cl" indicates classical symbols. By $(H, \kappa), (\tilde{H}, \tilde{\kappa})$ we understand Hilbert spaces with corresponding group actions, and $(y, \eta) \in \mathbb{R}^q \times \mathbb{R}^q$ are variables and covariables where $y \in \mathbb{R}^q$ are local coordinates on a manifold. Charts mapping to $\mathbb{R}^q$ are taken without loss of generality. Another straightforward modification of the definition is to distinguish between dimensions of variables and covariables, see corresponding notation below. If y completely disappears we talk about symbols with constant coefficients. The well-known scalar symbolic estimates when $H = \mathbb{C}$ and $\tilde{H} = \mathbb{C}$ and $\kappa_\delta = \tilde{\kappa}_\delta = \mathrm{id}_{\mathbb{C}}$ for all $\delta \in \mathbb{R}_+$ turn to twisted symbolic estimates, namely

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})} \leq c\,\langle\eta\rangle^{\mu-|\beta|}$$

for all $(y,\eta) \in K \times \mathbb{R}^q, K \Subset \mathbb{R}^q$, and all $\alpha \in \mathbb{N}^q, \beta \in \mathbb{N}^q$, for constants $c = c(\alpha,\beta,K) > 0$. The concept is also well-known for Fréchet spaces E and $\widetilde{E}$ with group action κ and $\tilde{\kappa}$, respectively. Classical symbols $a(y,\eta)$ are based on twisted homogeneity

$$a_{(\mu)}(y,\delta\eta) = \delta^\nu \tilde{\kappa}_\delta a_{(\mu)}(y,\eta)\kappa^{-1}_\delta$$

for all $\delta \in \mathbb{R}_+$. Note that in the operator-valued set-up there are non-trivial homogeneous functions defined for all η including $\eta = 0$.

Examples of Hilbert spaces with group actions are given in connection with (2.10). This scheme of notation also works when H or $\widetilde{H}$ are replaced by Fréchet spaces, see, e.g., [13] or [14]. The spaces $\mathcal{K}^{s,\gamma_1;e}(X^\wedge)$ and $\mathcal{K}^{\infty,\gamma_1-\mu;\infty}(X^\wedge)$ as well as subspaces with asymptotics are Fréchet where κ_δ is induced from (2.10) in an obvious manner, and then the above-mentioned spaces of abstract operator-valued symbols make sense with respect to those concrete spaces. Such constructions will be applied in different versions. Our definitions may be similar to each other, but in order to avoid confusion, we first look at symbols with constant coefficients, i.e., omit y at all and write λ rather than η.

Definition 2.5. By $L^\mu_{\mathrm{G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ we denote the space of all

$$G(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d_\lambda; \mathcal{K}^{s,\gamma_1;e}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu;\infty}_{\mathcal{P}}(X^\wedge)), \tag{2.19}$$

$$G^*(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d_\lambda; \mathcal{K}^{s,-\gamma_1+\mu;e}(X^\wedge), \mathcal{K}^{\infty,-\gamma_1;\infty}_{\mathcal{Q}}(X^\wedge)) \tag{2.20}$$

(i.e., where the covariables $\lambda \in \mathbb{R}^d$ plays the role of the former η, and the symbol has constant coefficients, i.e., y is dropped at all) where (2.19), (2.20) are valid for all $s, e \in \mathbb{R}$ and fixed $\gamma_1 \in \mathbb{R}$, for asymptotics types $\mathcal{P}$ and $\mathcal{Q}$ which may depend on G.

The next issue is the class $M^{-\infty}_{\mathcal{R}}(X)$ of meromorphic operator functions, where

$$\mathcal{R} = \{(r_j, n_j)\}_{j\in\mathbb{I}} \subset \mathbb{C} \times \mathbb{N} \tag{2.21}$$

is an index set $\mathbb{I} \subseteq \{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$. We assume that $\pi_{\mathbb{C}}\mathcal{R} = \{r_j\}_{j\in\mathbb{I}}$ intersects every strip $\{v_1 \in \mathbb{C} : c \leq \mathrm{Re}\, v_1 \leq c'\}$ for any reals $c \leq c'$ in a finite set and $\mathrm{Re}\, r_j \to \pm\infty$ as $j \to \mp\infty$ once $\pi_{\mathbb{C}}\mathcal{R}$ is infinite. In addition we ask any $f(v_1) \in M^{-\infty}_{\mathcal{R}}(X) \subset \mathcal{A}(\mathbb{C}_{v_1} \setminus \pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(X))$ to be meromorphic with poles at the points $r_j \in \mathbb{C}$ of order $\leq n_j+1$ for all j, with Laurent coefficients of finite rank. Moreover, if $\chi(v_1)$ is a $\pi_{\mathbb{C}}\mathcal{R}$-excision function (i.e., $\chi(v_1) = 0$ for $\mathrm{dist}(v_1, \pi_{\mathbb{C}}\mathcal{R}) < \varepsilon_0$, $\chi(v_1) = 1$ for $\mathrm{dist}(v_1, \pi_{\mathbb{C}}\mathcal{R}) > \varepsilon_1$ for some $0 < \varepsilon_0 < \varepsilon_1 < \infty$) then $\chi(v_1)f(v_1)|_{\Gamma_\lambda} \in \mathcal{S}(\Gamma_\lambda, L^{-\infty}(X))$ for every $\lambda \in \mathbb{R}$, uniformly in finite intervals. Operator-valued symbols of order $\mu \in \mathbb{R}$, in covariables $\lambda \in \mathbb{R}^d$ connected with such meromorphic symbols are of the form

$$M(\lambda) := r_1^{-\mu}\omega_{[\lambda]} \sum_{j=0}^{N} r_1^j \sum_{|\alpha|\leq j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})\lambda^\alpha \omega'_{[\lambda]} \tag{2.22}$$

for elements $f_{j\alpha}(v_1) \in M^{-\infty}_{\mathcal{R}_{j\alpha}}(X)$ and weights $\gamma_{j\alpha}$ satisfying the conditions

$$\Gamma_{\frac{n+1}{2}-\gamma_{j\alpha}} \cap \pi_{\mathbb{C}}\mathcal{R}_{j\alpha} = \emptyset \quad \text{for} \quad \gamma_1 - j \leq \gamma_{j\alpha} \leq \gamma_1 \tag{2.23}$$

for all j, α. Here $\omega_{[\lambda]}(r_1) = \omega(r_1[\lambda])$ where $\lambda \to [\lambda]$ is any smooth, strictly positive function in $\mathbb{R}^q$ such that $|\lambda| = [\lambda]$ for $|\lambda| \geq C$ for some $C > 0$ and $\Gamma_\beta = \{v_1 \in \mathbb{C} : \operatorname{Re} v_1 = \beta\}$. It is evident that then we get operator-valued symbols in λ with constant coefficients

$$M(\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma_1}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu}(X^\wedge)) \tag{2.24}$$

and

$$M(\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu}_{\mathcal{Q}}(X^\wedge)) \tag{2.25}$$

for every $s \in \mathbb{R}$ and any asymptotic type $\mathcal{P}$ with some resulting $\mathcal{Q}$. Also here we have a connection with weight data $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$.

The spaces $L^{\mu}_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ consist of all sums $M(\lambda) + G(\lambda)$ where $G(\lambda)$ are Green families as before and $M(\lambda)$ are defined as operator families of the form (2.22). In other words we completed Definition 2.3.

We systematically employ continuity, according to

Theorem 2.6. [14] *Every $A(\lambda) \in L^{\mu}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ induces families of continuous operators*

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge), \tag{2.26}$$

and

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}_{\mathcal{Q}}(X^\wedge), \tag{2.27}$$

for every $s \in \mathbb{R}$ and any asymptotic type $\mathcal{P}$ and some resulting $\mathcal{Q}$ which is independent of s.

Remark 2.7. Theorem 2.6 is well-known in the cone calculus over the infinite stretched cone $X^\wedge$ for smooth compact X, though specific cone aspects are suppressed for $r_1 \to \infty$ since at $r_1 = \infty$ the Kegel spaces coincide with "cone"-spaces from (2.7), and those are not affected by the weight γ_1 which is responsible for $r_1 \to 0$. In other words the structure of Mellin operators

$$(1-\omega) r_1^{-\mu} \mathrm{Op}_M^{\gamma_1 - n/2}(h)(\lambda) : (1-\omega') H^s_{\mathrm{cone}}(X^\wedge) \to H^{s-\mu}_{\mathrm{cone}}(X^\wedge)$$

which are involved in (2.26) ignores the weight γ_1 completely and caused by the holomorphic covariable of h in v_1 including the other elements of the operator-valued Mellin symbol classes in (2.5). This effect comes from [6], [7] which contains a proof of the equivalence of traditional quantizations with (in this case) λ-depending cut-offs together with Fourier quantization at $r_1 \to \infty$ and Mellin quantization.

Remark 2.8. Notation (2.3), especially $\lambda \neq 0$ is motivated by the edge symbolic structure of operator families $L^{\mu}(B, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ for some $B \in \mathfrak{M}_1$ with edge $s_1(B)$ of dimension $q_1 > 0$. In this moment the amplitude functions depend on covariables (η_1, λ) with η_1 being the covariables on the edge $s_1(B)$ and, if necessary, on parameters $\lambda \in \mathbb{R}^d$. Then edge symbols will be twisted homogeneous functions in $(\eta_1, \lambda) \neq 0$. We come later on the explicit shape of such amplitude functions. In any case, it also makes sense to admit the case $d = 0$, and then only $\eta \neq 0$ remains can be rephrased to $\lambda \neq 0$ as we did so far. In other words, up to notation change, the class (2.3) just contains twisted homogeneous subclasses which come from the parameter-dependent symbolic structure of (2.5). Since those conventions are essential we briefly stretch the definition.

In order to make the degenerate character of interior symbols more transparent we remember of the fact that our operator families are coming from operator functions

$$p(r_1, \varrho_1, \lambda) = \tilde{p}(r_1\varrho_1, r_1\lambda) \tag{2.28}$$

for

$$\tilde{p}(\tilde{\varrho}_1, \tilde{\lambda}) \in L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+d}_{\tilde{\varrho}_1, \tilde{\lambda}}) \tag{2.29}$$

where the operators in $r_1 \in \mathbb{R}_+$ act via the Fourier transform and are combined with the factor $r_1^{-\mu}$. In other words, for reasons which are explained in other papers, see [17] etc., they have the form

$$r_1^{-\mu}\mathrm{Op}_{r_1}(p)(\lambda)|_{r_1>0} : C_0^{\infty}(\mathbb{R}_+ \times X) \to C^{\infty}(\mathbb{R}_+ \times X). \tag{2.30}$$

Then, in order to reflect weight effects we fix a weight $\gamma_1 \in \mathbb{R}$ and rephrase (2.30) in term of the Mellin transform on the r_1 half-axis. This process is also called a Mellin quantization, and the difference between (2.30) and (2.5) consists of smoothing operators over $\mathbb{R}_+ \times X$ which are ignored under Mellin quantization, but the transformation from (2.30) to (2.5) is canonical when $\tilde{p}(r_1, \tilde{\varrho}_1, \tilde{\lambda})$ is a polynomial in $\tilde{\varrho}_1, \tilde{\lambda}$, because of the exact replacement of $-r_1\frac{\partial}{\partial r_1}$ corresponding to $-ir_1\varrho_1$ in the Fourier picture by the Mellin covariable v_1. Thus, in the program of expressing parametrices of Fuchs type operators there is no loss of information in the differential case, but there are specified remainders when we realize operators in weighted spaces for different weights. Those are, in fact, Green operators, depending on the choice of weights. In any case homogeneous principal symbols in the Fourier picture do not depend on this aspect; they can be expressed both in the Fourier and the Mellin picture and are the same up to a substitution of variables and covariables, not only for differential operators, and including parameters. When we formulate information in local coordinates $x \in \mathbb{R}^n$ on X with parameter λ and look at $A(\lambda)$ which is the same as $H(\lambda)$ modulo smoothing remainders, since Mellin plus Green operators are smoothing for $r_1 > 0$ we obtain a parameter-dependent principal symbol $\sigma_0(A)(r_1, x, \varrho_1, \xi, \lambda)$ for $(\varrho_1, \xi, \lambda) \neq 0$. This contains in its r_1-dependence also the weight factor $r_1^{-\mu}$ and in addition the r_1-degenerate behaviour in ϱ_1, λ, such that it makes sense to write

$$\sigma_0(A)(r_1, x, \varrho_1, \xi, \lambda) = r_1^{-\mu}\tilde{\sigma}_0(A)(r_1, x, r_1\varrho_1, \xi, r_1\lambda) \tag{2.31}$$

for a "reduced" symbol $\tilde{\sigma}_0(A)(r_1, x, \varrho_1, \xi, \lambda)$ which is smooth in r_1 up to $r_1 = 0$ and where (ϱ_1, λ) is involved in the meaning $(\tilde{\varrho}_1, \tilde{\lambda})$ for $\tilde{\varrho}_1 = r_1\varrho_1, \tilde{\lambda} = r_1\lambda$ which just corresponds to the background information (2.28), (2.29). Concerning $\sigma_1(\cdot)$ the advantage of the Mellin formulation is that it admits a natural definition of ellipticity with control up to $r_1 = 0$ both for A as well as for $M + G$, which entails Fredholm property in Fuchs type weighted Soboley spaces where we associate with $H(\lambda)$ also so-called conormal symbols, on the level of principal symbolic information, namely,

$$\sigma_1(H)(v_1, 0) := \tilde{h}(v_1, 0) \tag{2.32}$$

with $\tilde{h}$ being defined in Definition 2.3. Here there is no dependence on λ, and v_1 is varying in the complex Mellin plane. The operator families (2.32) pointwise act as continuous operators

$$\sigma_1(H)(v_1, 0) : H^s(X) \to H^{s-\mu}(X) \tag{2.33}$$

between Sobolev spaces over X. In ellipticity the operators (2.33) are parameter - dependent elliptic in the variable $\operatorname{Im} v_1$ for $v_1 \in \Gamma_\lambda$ for any fixed real λ. and hence (2.33) becomes a family of isomorphisms as soon as $\operatorname{Im} v_1$ is sufficiently large, uniformly in compact λ-intervals. Although this information is well-known, later on we refer to similar facts for singularity order $k > 1$. In order to establish conormal symbols of operator families $M(\lambda)$ of the form (2.22) we form the operator function

$$\sigma_1(M)(v_1) := f_{00}(v_1) : H^s(X) \to H^\infty(X) \tag{2.34}$$

where (2.34) is the subordinate conormal symbol of $M(\lambda)$ from the cone theory close to $r_1 = 0$. The parameter $\operatorname{Im} v_1$ is varying on Γ_λ indicated in the weight conditions of (2.22), here for $\lambda = \frac{n+1}{2} - \gamma_1$. The operators $M(\lambda) + G(\lambda)$ in Definition 2.3 are smoothing off $r_1 = 0$ they do not generate contributions to the interior symbolic level.

The operators $A(\lambda) \in L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ are called elliptic if $\tilde{h}(v_1, \tilde{\lambda})$ is parameter-dependent elliptic in $M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$ and the conormal symbol

$$\sigma_1(A)(v_1) := \tilde{h}(v_1, 0) + f_{00}(v_1) : H^s(X) \to H^{s-\mu}(X)$$

is a family of isomorphisms for all $v_1 \in \Gamma_{\frac{n+1}{2}-\gamma_1}$ with a prescribed weight $\gamma_1 \in \mathbb{R}$ which belongs to the ellipticity condition. Note that in case of ellipticity the restriction of elements in $M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$ to $\Gamma_{\frac{n+1}{2}-\gamma_1} \times \mathbb{R}^d_{\tilde{\lambda}}$ are parameter-dependent elliptic in $L^\mu_{\mathrm{cl}}(X; \Gamma_{\frac{n+1}{2}-\gamma_1} \times \mathbb{R}^d_{\tilde{\lambda}})$. This holds for all weights γ_1.

We also recall the following result:

Theorem 2.9. *Let $A(\lambda) \in L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ be parameter-dependent elliptic with respect to the weight γ_1. Then there is a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}(X^\wedge, \boldsymbol{g}^{-1}; \mathbb{R}^d \setminus \{0\}), \boldsymbol{g}^{-1} = (\gamma_1 - \mu, \gamma_1, \Theta)$ where*

$$P(\lambda)A(\lambda) = 1 - G_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - G_{\mathrm{R}}(\lambda)$$

for some $G_{\mathrm{L}}(\lambda) \in L^{-1}_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}), G_{\mathrm{R}}(\lambda) \in L^{-1}_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d \setminus \{0\})$ for $\boldsymbol{g}_{\mathrm{L}} := (\gamma_1, \gamma_1, \Theta), \boldsymbol{g}_{\mathrm{R}} := (\gamma_1 - \mu, \gamma_1 - \mu, \Theta)$.

Corollary 2.10. *If $A(\lambda)$ is parameter-dependent elliptic then*

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)$$

is a family of Fredholm operators, and kernels belong to subspaces of $\mathcal{K}^{\infty,\gamma_1}(X^\wedge)$ with asymptotics. Cokernels can be represented by finite-dimensional subspaces of $\mathcal{K}^{\infty,\gamma_1-\mu}(X^\wedge)$ with asymptotics.

Theorem 2.11. ([11]) *For every* $\gamma_1 \in \mathbb{R}$ *and* $s \in \mathbb{R}$ *there exists a* $\tilde{h}(v_1, \tilde{\lambda}) \in M^{-s}_{\mathcal{O}_{v_1}}(X, \boldsymbol{g}; \mathbb{R}^d_{\tilde{\lambda}})$ *such that for* $h(r_1, v_1, \lambda) = \tilde{h}(v_1, r_1\lambda)$ *and sufficiently large* $|\tilde{\lambda}|$ *the operator*

$$R^{-s}(\lambda) := r_1^s \mathrm{Op}_M^{\gamma_1 - s - n/2}(h)(\lambda) : \mathcal{K}^{0,\gamma_1 - s}(X^\wedge) \to \mathcal{K}^{s,\gamma_1}(X^\wedge)$$

is an isomorphism.

Theorem 2.9 is known, but we have similar results below for base spaces of higher singular order. Therefore, we sketch the arguments. The main issue is that the ellipticity condition on $\tilde{h}(v_1, \tilde{\lambda})$ means that there is an $\tilde{h}^{(-1)}(v_1, \tilde{\lambda}) \in M^{-\mu}_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$ such that the product

$$r_1^{\mu} \mathrm{Op}_M(\tilde{h}^{(-1)})(\tilde{\lambda}) r_1^{-\mu} \mathrm{Op}_M(\tilde{h})(\tilde{\lambda}) \tag{2.35}$$

is equal to $\mathrm{Op}_M(\tilde{f})(\tilde{\lambda})$ for some $\tilde{f}(v_1, \tilde{\lambda}) \in M^0_{\mathcal{O}_{v_1}}(X; \mathbb{R}^d_{\tilde{\lambda}})$. In this conclusion the r_1-powers cancel out modulo some translations in v_1 of the involved Mellin symbols. These translations do not influence the parameter-dependent homogeneous symbols in the sense of pseudo-differential operators on $\Gamma_{\beta_1} \times \mathbb{R}^d_{\tilde{\lambda}}$. Therefore, the product (2.35) produces identity plus a Mellin operator with parameter-dependent symbol of order 1. When we insert symbols $h^{(-1)}(r_1, v_1, \lambda) = \tilde{h}^{(-1)}(v_1, r_1\lambda)$ and $h(r_1, v_1, \lambda) = \tilde{h}(v_1, r_1\lambda)$, then there is a Leibnitz product effect in (r_1, v_1) under which only lower order terms are changed, and those again produce lower order terms. Then a formal Neumann series argument gives us, say for computing a left parametrix gives us remainders of the form $1 - \widetilde{C}_L$ for $\widetilde{C}_L \in L^{-1}_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\})$. Together with the inversion of the conormal symbol we finally get remainders in

$$L^{-\infty}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}) = L^{-1}_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}).$$

For the right parametrix we proceed in a similar manner, i.e., it can be identified with the left parametrix.

As a consequence we see that

$$A(\lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \tag{2.36}$$

is a family of Fredholm operators. The index is independent of $s \in \mathbb{R}$. Later on we shall interpret elements of $L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ with $q_1 + d$ rather than d and parameters $(\eta_1, \lambda) \neq 0$ as space of occurring edge symbols for edge operators, and the ellipticity condition in the edge case will be that the respective cone operators are families of isomorphisms. This is not automatically the case, but similarly as in boundary value problems the situation is required by adding trace and potential entries including right lower corners in corresponding 2 × 2 block matrix operators to achieve families of isomorphisms. The extra entries remind of Green operators and can be chosen as symbols with asymptotics. Later on operator families in $L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ occur as homogeneous principal edge symbols. In contrast to Remark 2.4 we did not require the elements $M(\lambda) + G(\lambda) \in L^\mu_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ to be homogeneous in such a sense. Of course, there is the subclass

$$L^{(\mu)}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\}) \tag{2.37}$$

of those $A(\lambda) \in L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ such that

$$A(\delta\lambda) = \delta^\mu \kappa_\delta A(\lambda) \kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$. The construction of trace and potential etc. operators makes sense for the space (2.37) which gives rise to an extension of $L^{(\mu)}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ to an algebra of 2×2 block matrices

$$\boldsymbol{A}(\lambda) = \begin{pmatrix} A(\lambda) + G(\lambda) & K(\lambda) \\ T(\lambda) & Q(\lambda) \end{pmatrix} : \begin{array}{c} \mathcal{K}^{s,\gamma_1}(X^\wedge) \\ \oplus \\ \mathbb{C}^e \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \\ \oplus \\ \mathbb{C}^f \end{array} \tag{2.38}$$

for some $e, f \in \mathbb{N}$ where $f - e$ just equals the Fredholm index of (2.36). Here $G(\lambda) \in L^\mu_G(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ and $T(\lambda)$ are trace operators, $K(\lambda)$ potential operators, and $Q(\lambda)$ are families of classical symbols in λ with values in $f \times e$-matrices. The nature of $T(\lambda)$ and $K(\lambda)$ is determined by kernels and cokernels of solutions to elliptic equations, and they may be represented by families of functions with asymptotics. The properties are similar to those in the boundary symbolic calculus in pseudo-differential boundary vales problems. Compositions $K(\lambda) \circ T(\lambda)$ are examples of parameter-dependent Green operators, cf. Definition 2.5.

In order to see a connection between Definition 2.3 and the more complex situation of operators over (stretched) wedges

$$X^\wedge \times \mathbb{R}^q \ni (r_1, x, y_1) \tag{2.39}$$

we have a look at the shape of operators in the calculus over a compact space $B \in \mathfrak{M}_1$ with edge $s_1(B) =: Y_1$, locally close to Y_1 in local coordinates $y_1 \in \mathbb{R}^{q_1}$ modeled on wedges (2.39) with local amplitude functions for the corresponding parameter-dependent edge calculus. Those have the form

$$a(y_1, \eta_1, \lambda) := h(y_1, \eta_1, \lambda) + m(y_1, \eta_1, \lambda) + g(y_1, \eta_1, \lambda) \tag{2.40}$$

where

$$h(y_1, \eta_1, \lambda) = \omega r_1^{-\mu} \mathrm{Op}_M^{\gamma_1 - n/2}(\boldsymbol{h})(y_1, \eta_1, \lambda)\omega' \tag{2.41}$$

and Mellin amplitude functions

$$\boldsymbol{h}(r_1, y_1, v_1, \eta_1, \lambda) = \tilde{\boldsymbol{h}}(r_1, y_1, v_1, r_1\eta_1, r_1\lambda) \tag{2.42}$$

for

$$\tilde{\boldsymbol{h}}(r_1, y_1, v_1, \tilde{\eta}_1, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_1}, M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^{q_1+d}_{\tilde{\eta}_1, \tilde{\lambda}})). \tag{2.43}$$

Definition 2.12. By $L^\mu(B, \boldsymbol{g}; \mathbb{R}^d)$ for $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$ we denote the space of operators

$$A(\lambda) := H(\lambda) + M(\lambda) + G(\lambda) + A_{\mathrm{int}}(\lambda) + C(\lambda) \tag{2.44}$$

where $H(\lambda)$ is locally close to Y_1 expressed as a finite sum of operators

$$\varphi \mathrm{Op}_{y_1}\{\omega r_1^{-\mu} \mathrm{Op}_M^{\gamma_1 - n/2}(\boldsymbol{h})(y_1, \eta_1, \lambda)\omega'\}\varphi' \tag{2.45}$$

with $\varphi \prec \varphi'$ in $C_0^\infty(\mathbb{R}^{q_1})$ coming from a partition of unity over Y_1 and where $\boldsymbol{h}$ satisfies relations (2.42) and (2.43). Moreover, we ask that

$$A_{\mathrm{int}}(\lambda) \in L^\mu_{\mathrm{cl}}(B \setminus s_1(B); \mathbb{R}^d_\lambda),$$

such that the λ-dependent distributional kernel which is contained in $(B \setminus s_1(B)) \times (B \setminus s_1(B))$ has proper support in the respective open set. In addition $C(\lambda) \in$

$L^{-\infty}(B, \boldsymbol{g}; \mathbb{R}^d)$ is parameter-depending smoothing, i.e., characterized by the property

$$C(\lambda) \in \mathcal{S}(\mathbb{R}^d_\lambda, L^{-\infty}(B, \boldsymbol{g})) \tag{2.46}$$

where $L^{-\infty}(B, \boldsymbol{g})$ is the space of all

$$C : H^{s,\gamma_1}(B) \to H^{\infty,\gamma_1-\mu}_{\mathcal{P}}(B) \tag{2.47}$$

with

$$C^* : H^{s,-\gamma_1+\mu}(B) \to H^{\infty,-\gamma_1}_{\mathcal{Q}}(B) \tag{2.48}$$

for any $s \in \mathbb{R}$ and asymptotic types $\mathcal{P}, \mathcal{Q}$, which may depend on C. The spaces involved in (2.47) and (2.48) are defined in Subsection 2.2. We finally require

$$M(\lambda) + G(\lambda) \in L^\mu_{\mathrm{M+G}}(B, \boldsymbol{g}; \mathbb{R}^d_\lambda) \tag{2.49}$$

i.e., $M(\lambda) = \mathrm{Op}_{y_1}(m)$ is a Mellin operator family locally close to Y_1 associated with Mellin amplitude functions $m(y_1, \eta_1, \lambda)$ of similar structure as (2.22), namely, as in formula (2.53) below, and $G(\lambda) = \mathrm{Op}_{y_1}(g), g(y_1, \eta_1, \lambda)$ similarly as in Definition 2.5.

2.2. **Asymptotics on a manifold with edge.** Let us now return to operators over a compact space $B \in \mathfrak{M}_1$ with edge $s_1(B) = Y_1$ as explained before in the context of Definition 2.12. So far the spaces (2.47), (2.48) and (2.49) are not yet defined. Because of y_1-dependence of symbols we prefer to employ the setting of continuous asymptotics. This requires weighted edge spaces locally along $\mathbb{R}^{q_1} \ni y_1$ and subspaces with such asymptotics. Similarly as symbol spaces (2.18) we have abstract edge Sobolev spaces $\mathcal{W}^s(\mathbb{R}^{q_1}, H)$ for any Hilbert or Fréchet space H with group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. Those already played a role in (2.18). In our context we have Hilbert spaces

$$H := \mathcal{K}^{s,\gamma_1}(X^\wedge) \quad \text{or} \quad \mathcal{K}^{s,\gamma_1;e}(X^\wedge)$$

or Fréchet subspaces with asymptotics, indicated by subscript $\mathcal{P}$. Those spaces admit the group action

$$\kappa_\delta : u(r_1, x) \to \delta^{\frac{n+1}{2}} u(\delta r_1, x),$$

$\delta \in \mathbb{R}_+$, and hence we have associated weighted edge spaces defined in terms of completion of $\mathcal{S}(\mathbb{R}^{q_1}, H)$, e.g., in the Hilbert space case with respect to

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^{q_1},H)} := \left\{ \int \langle\eta_1\rangle^{2s} \, \|\kappa^{-1}_{\langle\eta_1\rangle} \hat{u}(\eta_1)\|^2_H \, đ\eta_1 \right\}^{1/2}$$

(otherwise for semi-norm systems in the corresponding Fréchet space, where $\|\cdot\|_H$ runs over a countable system of semi-norms in H).

Weighted edge spaces $H^{s,\gamma_1}(B)$ over a compact $B \in \mathfrak{M}_1$ with edge are defined in terms of charts

$$\chi_i : Y_i \to \mathbb{R}^{q_1} \tag{2.50}$$

for a system of coordinate neighbourhoods $Y_i \subset Y$, $i = 1, \ldots, N$, satisfying some simple admissibility condition concerning transition maps, used in several expositions on edge spaces, see, e.g., [14] or the article [7]. Then $H^{s,\gamma_1}(B)$ is defined as the set of those $u \in H^s_{\mathrm{loc}}(B \setminus Y_1)$ such that $\varphi_i u \circ \chi_i^{-1} \in \mathcal{W}^s(\mathbb{R}^{q_1}, \mathcal{K}^{s,\gamma_1}(X^\wedge))$ for all i, with $(\varphi_1, \dots, \varphi_N)$ being a subordinate partition of unity. $H^{s,\gamma_1}(B)$ can be equipped with a Hilbert space structure in a natural way.

In an analogous manner we obtain subspaces $H^{s,\gamma_1}_{\mathcal{P}}(B)$ with asymptotics of type $\mathcal{P}$ by replacing $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ in the latter definition by $\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge)$. Those are again Fréchet spaces. Thus the spaces in relations (2.47), (2.48) are defined, where the formal adjoints refer to spaces $H^{0,0}(B)$ which may be identified with $r^{-n/2}L^2(B\backslash Y_1)$.

Analogously as $L^\mu_{\mathrm{G}}(X^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ in Definition 2.5 by $\mathcal{R}^\mu_{\mathrm{G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$ we denote the space of

$$g(y_1, \eta_1, \lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}; \mathcal{K}^{s,\gamma_1;e}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu;\infty}_{\mathcal{P}}(X^\wedge)) \tag{2.51}$$

the pointwise formal adjoints of which have the property

$$g^*(y_1, \eta_1, \lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}; \mathcal{K}^{s,-\gamma_1+\mu;e}(X^\wedge), \mathcal{K}^{\infty,-\gamma_1;\infty}_{\mathcal{Q}}(X^\wedge)), \tag{2.52}$$

for some g-dependent asymptotic types $\mathcal{P}, \mathcal{Q}$, associated with the weight information in these relations. Those are required for all $s, e \in \mathbb{R}$. In the case of discrete asymptotics we assume $\mathcal{P}, \mathcal{Q}$ to be constant with respect to y_1; in the continuous case we assume the same, but then we may admit carriers which cover variable discrete asymptotics. Such operator-valued symbols are connected with weight data $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta)$ which are given in connection with the weights in the spaces and also determine the position of $\pi_{\mathbb{C}}\mathcal{P}$ and $\pi_{\mathbb{C}}\mathcal{Q}$, respectively.

Other ingredients of parameter-dependent edge amplitude functions will consist of Mellin contributions, namely,

$$m(y_1, \eta_1, \lambda) := r_1^{-\mu}\omega_{\eta_1,\lambda} \sum_{j=0}^{\theta} r_1^j \sum_{|\alpha| \le j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y_1)(\eta_1, \lambda)^\alpha \omega'_{\eta_1,\lambda} \tag{2.53}$$

where $\omega_{\eta_1,\lambda}(r_1) = \omega(r_1 \langle \eta_1, \lambda \rangle)$ and, similarly, $\omega_{\eta_1,\lambda}$, and

$$f_{j\alpha}(y_1, v_1) \in C^\infty(\mathbb{R}^{q_1}, M^{-\infty}_{\mathcal{R}_{j\alpha}}(X))$$

satisfying analogous weight conditions as (2.23). In (2.23) we assumed (constant in y_1) discrete Mellin asymptotic types. This works with the observation that we always find weights $\gamma_{j\alpha}$ in the required interval. In the continuous case this is not automatically the case, since the carriers of continuous Mellin asymptotic types do not necessary leave such gaps. However, corresponding y_1-dependent smoothing Mellin symbols can always decomposed into sums where every summand is of the required form. Corresponding decomposition identities may be found, e.g., in [10]. In that way (2.53) is asked to be represented in the form $m = m_1 + m_2$ where the summands are of the form (2.53) but with Mellin symbols $f_{j\alpha}$ who are different for the corresponding summands. For brevity we simply only argue in terms of Mellin amplitude like (2.53), keeping in mind that for an arbitrary choice of $f_{j\alpha}$ we always have decomposition like $f_{j\alpha} = f^1_{j\alpha} + f^2_{j\alpha}$ where the associated continuous Mellin asymptotic types leave gaps of the carriers sets to make the Mellin actions possible.

Let $\mathcal{R}^\mu_{\mathrm{M+G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$ be the space of all $(m+g)(y_1, \eta_1, \lambda)$ such that $g(y_1, \eta_1, \lambda) \in \mathcal{R}^\mu_{\mathrm{G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$ and $m(y_1, \eta_1, \lambda)$ a finite sum of operator functions of the kind (2.53), where it can be proved that two summands generate the whole space of such elements. Similarly as (2.51) we have

$$m(y_1,\eta_1,\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}; \mathcal{K}^{s,\gamma_1}(X^{\wedge}), \mathcal{K}^{\infty,\gamma_1-\mu}(X^{\wedge}))$$

and

$$m(y_1,\eta_1,\lambda) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}; \mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^{\wedge}), \mathcal{K}^{\infty,\gamma_1-\mu}_{\mathcal{Q}}(X^{\wedge}))$$

for all s, for arbitrary asymptotic types $\mathcal{P}$ and some resulting $\mathcal{Q}$. When we evaluate homogeneous principal symbols in the operator-valued set-up it suffices to ignore the exit order e in Green symbols and to express symbols with smoothness $s-\mu$ in the image spaces. For Green symbols g we can take as $\sigma_1(g)(y_1,\eta_1,\lambda)$ the twisted homogeneous principal symbol

$$\sigma_1(g)(y_1,\eta_1,\lambda) \in S^{(\mu)}(\mathbb{R}^{q_1} \times (\mathbb{R}^{q_1+d} \setminus \{0\}); \mathcal{K}^{s,\gamma_1}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge}))$$

and for the twisted homogeneous principal symbol of $m(y_1,\eta_1,\lambda)$ we have

$$\sigma_1(m)(y_1,\eta_1,\lambda) \in S^{(\mu)}(\mathbb{R}^{q_1} \times (\mathbb{R}^{q_1+d} \setminus \{0\}); \mathcal{K}^{s,\gamma_1}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge}))$$

which has the form

$$\sigma_1(m)(y_1,\eta_1,\lambda) = r_1^{-\mu}\omega_{|\eta_1,\lambda|}\sum_{j=0}^{\theta} r_1^j \sum_{|\alpha|=j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y_1)(\eta_1,\lambda)^{\alpha}\omega'_{|\eta_1,\lambda|}$$

for $\omega_{|\eta_1,\lambda|}(r_1) := \omega(r_1|\eta_1,\lambda|)$, and the same for $\omega'_{|\eta_1,\lambda|}$. Observe that the general reasons these are parameter-dependent homogeneous edge symbols of Mellin plus Green type and defined for $(\eta_1,\lambda) \neq 0$.

Note that when we change the cut-off functions or the weights in (2.53) we leave remainders of Green type. Also other expected rules hold, such as that the symbol spaces $\mathcal{R}^{\mu}_{\mathrm{M+G}}$ form algebras and that the homogeneous principal symbols behave multiplicatively. Moreover, formal adjoints are Mellin plus Green again, with compatibility of formal adjoints of twisted homogeneous symbols.

2.3. **Edge calculus of first singularity order.** The following consideration concerns the edge calculus with parameters, and as noted before, is of analogous structure as the operator class from Definition 2.3, now for $B \in \mathfrak{M}_1$ rather than $X^{\wedge}$, and since B is not discussed in terms of some conical exit we admit parameters $\lambda \in \mathbb{R}^d$.

Let $B \in \mathfrak{M}_1$, not necessary compact, and $q_1 = \dim Y_1 > 0$. By

$$L^{\mu}(B,\boldsymbol{g};\mathbb{R}^d_{\lambda}) \tag{2.54}$$

for weight data $\boldsymbol{g} = (\gamma_1, \gamma_1-\mu, \Theta_1)$, we denote the set of all families of operators (2.44), where it remains to formulate the ingredients close to Y_1 in terms of edge amplitude functions in

$$\mathcal{R}^{\mu}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g}) \tag{2.55}$$

consisting of all operator families (2.40) for $(m+g)(y_1,\eta_1,\lambda) \in \mathcal{R}^{\mu}_{\mathrm{M+G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$, cf. notation in Subsection 2.2, and $h(y_1,\eta_1,\lambda)$ as in (2.41). Then the part of operators in $L^{\mu}(B,\boldsymbol{g};\mathbb{R}^d_{\lambda})$ close to Y_1 can be described by

$$H(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}\{r_1^{-\mu}\omega_1 \mathrm{Op}_{M_{r_1}}^{\gamma_1-n/2}(\boldsymbol{h})(y_1,\eta_1,\lambda)\omega'_1\}\varphi' \tag{2.56}$$

with summation over φ from the partition of unity, $\varphi \prec \varphi'$ also smooth and of compact support in $\mathbb{R}^{q_1}$ and cut-off functions $\omega_1 \prec \omega_1'$ on the r_1 half-axis and

$$(M+G)(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}\{(m+g)(y_1,\eta_1,\lambda)\}\varphi'$$

for $(m+g)(y_1,\eta_1,\lambda) \in \mathcal{R}^{\mu}_{\mathrm{M+G}}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$. In more concise manner we also could express the elements of (2.44) as

$$A(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}(a(y_1,\eta_1,\lambda))\varphi' + A_{\mathrm{int}}(\lambda) + C(\lambda) \tag{2.57}$$

for arbitrary $a(y_1,\eta_1,\lambda) \in \mathcal{R}^{\mu}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g})$.

Note that in the edge situation we have locally over a wedge in the variables (r_1, y_1) and covariables $(\varrho_1, \eta_1, \lambda)$ analogously as (2.28), (2.29) edge-degenerate pseudo-differential families

$$p(r_1, y_1, \varrho_1, \eta_1, \lambda) = \tilde{p}(r_1, y_1, r_1\varrho_1, r_1\eta_1, r_1\lambda) \tag{2.58}$$

for

$$\tilde{p}(r_1, y_1, \tilde{\varrho}_1, \tilde{\eta}_1, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_1}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q_1+d}_{\tilde{\varrho}_1,\tilde{\eta}_1,\tilde{\lambda}})). \tag{2.59}$$

Then the Mellin symbols $\boldsymbol{h}$ indicated in (2.41), (2.42), (2.43) are just by applying a Mellin quantization process which turns $\tilde{\varrho}_1$ to $v_1 \in \mathbb{C}$, and which is combined with a kernel cut-off procedure. The edge symbolic hierarchy, consisting of

$$\sigma(A(\lambda)) = (\sigma_0(A(\lambda)), \sigma_1(A(\lambda)))$$

partly refers to the interior Fourier based background. From (2.59) we have by dissolving the variables into $(r_1, x, y_1, \tilde{\varrho}_1, \xi, \tilde{\eta}_1, \tilde{\lambda})$ the parameter-dependent "scalar" interior symbol

$$\sigma_0(A)(r_1, x, y_1, \varrho_1, \xi, \eta_1, \lambda) = r_1^{-\mu}\tilde{\sigma}_0(A)(r_1, x, y_1, r_1\varrho_1, \xi, r_1\eta_1, r_1\lambda) \tag{2.60}$$

for a "reduced" symbol $\tilde{\sigma}_0(A)(r_1, x, y_1, \varrho_1, \xi, \eta_1, \lambda)$ homogeneous of order μ in $(\varrho_1, \xi, \eta_1, \lambda) \neq 0$ and smooth up to $r_1 = 0$. This explains, analogously as (2.31) the symbolic level $\sigma_0(A(\lambda))$. Moreover, in this framework we have

$$\sigma_1(A(\lambda)) = \sigma_1(H(\lambda)) + \sigma_1(M(\lambda)) + \sigma_1(G(\lambda))$$

where the second summands are known from the edge calculus in [14] and

$$\sigma_1(H(\lambda))(y_1, \eta_1, \lambda) = r_1^{-\mu}\mathrm{Op}^{\gamma_1 - n/2}_{M_{r_1}}\tilde{h}_0(y_1, r_1\eta_1, r_1\lambda)$$

for $\tilde{h}_0(y_1, \tilde{\eta}_1, \tilde{\lambda}) = \tilde{h}(0, y_1, r_1\eta_1, r_1\lambda)$, which is a function with values in continuous operators

$$\sigma_1(H(\cdot))(y_1, \eta_1, \lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)$$

for $(\eta_1, \lambda) \in \mathbb{R}^{q_1+d} \setminus \{0\}$ belonging to

$$L^{\mu}(X^\wedge, \boldsymbol{g}; \mathbb{R}^{q_1+d} \setminus \{0\}) \tag{2.61}$$

smoothly depending on y_1. In other words (2.61) furnishes the parameter-dependent edge symbolic calculus of (2.54)

Let us make some remarks on how we can obtain special examples of parameter-dependent elliptic elements $A(\lambda)$ in (2.54). Ellipticity is defined as bijectivity of the involved symbols. We first have the interior ellipticity which means non-vanishing of $\sigma_0(A)$ for all $(\varrho_1, \xi, \eta_1, \lambda) \neq 0$ and of $\tilde{\sigma}_0(A)$ up to $r_1 = 0$. In addition the edge symbol

$$\sigma_1(A(\cdot))(y_1, \eta_1, \lambda) : \mathcal{K}^{s,\gamma_1}(X^\wedge) \rightarrow \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge) \tag{2.62}$$

which is a family of operators in $L^\mu(X^\wedge, \boldsymbol{g}; \mathbb{R}^{q_1+d}_{\eta_1,\lambda} \setminus \{0\})$ has to be Fredholm for every $(y_1, \eta_1, \lambda) \in \mathbb{R}^{q_1} \times (\mathbb{R}^{q_1+d} \setminus \{0\})$. It is of interest to investigate on how we can pass to a 2×2 block matrix family of isomorphisms. For this some topological obstruction on the K-theoretic index bundle of (2.62) has to vanish. If we focus on the case that (2.62) is a family of isomorphisms then the obstruction vanishes. In the articles [17] or [18] we studied special elliptic cases where the operators (2.62) (in local form) are essentially derived from parameter-dependent Laplacians and the edge symbols are then of the form

$$(|\eta_1|^2 + |\lambda|^2)^{\mu/2}$$

taking values in operators between spaces in (2.62). The desired property depends to a large extent on the behaviour of subordinate conormal symbols which have to attain isomorphisms between Sobolev spaces over X, namely

$$H^s(X) \rightarrow H^{s-\mu}(X)$$

for any $s \in \mathbb{R}$. Those are operator functions depending on $v_1 \in \mathbb{C}$ and they have to be bijective on the prescribed weight line $\Gamma_{\frac{n+1}{2}-\gamma_1}$. In applications below we need bijectivity in a particularly large weight strip. In order to achieve this it is helpful to have bijectivity first in any (particularly narrow) strip, and then to enlarge this by a dilation in the z-plane. In this manipulation we do not loose the other ellipticity conditions, and this weight strip can be stretched so wide that several manipulations, e.g., translations of reference weights, remain possible without leaving that strip. Examples have been discussed in [17], and we use them here in connection with the observation that enlarging weight strips is always possible. Let us summarize weight manipulations of that kind as

Remark 2.13. Prescribed weights in suitable examples may be variable in large intervals without violating the conormal ellipticity condition in v_1.

The effect of making such a choice of special cases is that the contribution from Mellin symbols to asymptotics only consists of composing singular functions by a holomorphic (operator-valued) factor, up to translations in the complex plane.

The present material on the edge calculus of first singularity order is a background for the calculus of higher singularity order, especially, ellipticity and parametrices. Let us add here for completeness a theorem which states Fredholm property and parametrices within the edge algebra, cf. [8].

Parameter-dependent ellipticity of an $A(\lambda) \in L^\mu(B, \boldsymbol{g}; \mathbb{R}^d_\lambda)$, cf. formula (2.54), means that $A \in L^\mu(B \setminus Y; \mathbb{R}^d_\lambda) \subseteq L^\mu_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^d_\lambda)$ is parameter-dependent elliptic in the standard sense and close to the edge Y the reduced symbol $\tilde{\sigma}_0(A)$ does not vanish for $(\varrho_1, \xi, \eta_1, \lambda) \neq 0$ up to $r_1 = 0$. Concerning the principal edge symbol (2.62) we have different options. In the "best possible" case (2.62) is a family of isomorphisms, otherwise we ask (2.62) to be a family of Fredholm operators which comes extra edge conditions. For brevity we consider the first case.

Theorem 2.14. *Let* $A(\lambda) \in L^{\mu}(B, \boldsymbol{g}; \mathbb{R}^d)$ *be parameter-dependent elliptic. Then there is a parameter-dependent parametrix* $P(\lambda) \in L^{-\mu}(B, \boldsymbol{g}^{-1}; \mathbb{R}^d)$ *such that*

$$P(\lambda)A(\lambda) = 1 - C_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - C_{\mathrm{R}}(\lambda)$$

for operators $C_{\mathrm{L}}(\lambda) \in L^{-\infty}(B, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d), C_{\mathrm{R}}(\lambda) \in L^{-\infty}(B, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d)$*. Moreover, if* B *is compact*

$$A(\lambda) : H^{s,\gamma_1}(B) \to H^{s-\mu,\gamma_1-\mu}(B)$$

is a family of Fredholm operators. It becomes a family of isomorphism for sufficiently large $|\lambda|$*. This holds for all* $s \in \mathbb{R}$*.*

Proof. It suffices to consider operators close to Y, since the interior part is elliptic in $L^{\mu}_{\mathrm{cl}}(B \setminus s_0(B); \mathbb{R}^d)$. The construction of $P(\lambda)$ can be carried out first in terms of amplitude functions (2.55) by first inserting the κ_δ-homogeneous principal non-smoothing part and then to compose the corresponding test parametrix to $A(\lambda)$. Then we can invert the resulting $(1 + m_1 + g_1)(y_1, \eta_1, \lambda)$ by using the theorem on inverse of 1+ smoothing Mellin symbol within this class of operator functions. The resulting conormal symbol then vanishes, and a subsequent formal Neumann series argument leaves us a Green remainder. □

3. Corner operators

3.1. **Calculus of corner symbols.**

Theorem 3.1. *Consider a sequence of operator functions*

$$f_{\mu-j}(\tilde{\tau}, \tilde{\lambda}) \in L^{\mu-j}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}), \tag{3.1}$$

$j \in \mathbb{N}$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$*, and let the asymptotic types contained in the Green terms of* $f_{\mu-j}$ *be independent of* j*. Then there is an* $f(\tilde{\tau}, \tilde{\lambda}) \in L^{\mu}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})$ *such that*

$$f(\tilde{\tau}, \tilde{\lambda}) - \sum_{j=0}^{N} f_{\mu-j}(\tilde{\tau}, \tilde{\lambda}) \in L^{\mu-(N+1)}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}) \tag{3.2}$$

for any $N \in \mathbb{N}$*. Let us write*

$$f(\tilde{\tau}, \tilde{\lambda}) \sim \sum_{j=0}^{\infty} f_{\mu-j}(\tilde{\tau}, \tilde{\lambda}),$$

called an asymptotic sum of the $f_{\mu-j}$*.*

Proof. Elements A of $L^{\mu}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})$ can be representd modulo

$$L^{-\infty}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}}) = L^{-\infty}_{\mathrm{G}}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})$$

in the form $A = A_0 + A_1$ for some

$$A_0 \in \varphi L^{\mu}_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})\varphi',\ A_1 \in \omega_{\mathrm{glob}} L^{\mu}(B, \boldsymbol{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\lambda}})\omega'_{\mathrm{glob}}, \tag{3.3}$$

for certain $\varphi, \varphi' \in C_0^{\infty}(B \setminus Y)$, $\varphi \prec \varphi'$, and global cut-off functions $\omega_{\mathrm{glob}} \prec \omega'_{\mathrm{glob}}$ with respect to Y. The construction of $f := f_0 + f_1$ can be carried out over $B \setminus Y$ and close to Y separately in a straightforward way. □

Operator functions in $L^\mu(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde\tau,\tilde\lambda})$ satisfy rules in terms of differentiations with respect to parameters. In particular, we have

$$D^k_{\tilde\tau}D^\alpha_{\tilde\lambda}L^\mu(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde\tau,\tilde\lambda})\subseteq L^{\mu-(k+|\alpha|)}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde\tau,\tilde\lambda}) \tag{3.4}$$

for any $k\in\mathbb{N}$, $\alpha\in\mathbb{N}^d$. Our next objective is to compose corner-degenerate families of operators

$$t^{-\mu}\mathrm{Op}_t(a)(\lambda)t^{-\nu}\mathrm{Op}_t(b)(\lambda) \tag{3.5}$$

for $a(t\tau,t\lambda),b(t\tau,t\lambda)$ given by

$$a(\tilde\tau,\tilde\lambda)\in L^\mu(B,\boldsymbol{g}_2;\mathbb{R}^{1+d}_{\tilde\tau,\tilde\lambda}),\quad b(\tilde\tau,\tilde\lambda)\in L^\nu(B,\boldsymbol{g}_1;\mathbb{R}^{1+d}_{\tilde\tau,\tilde\lambda}),$$

respectively, where

$$\boldsymbol{g}_1:=(\gamma,\gamma-\nu,\Theta),\ \boldsymbol{g}_2:=(\gamma-\nu,\gamma-(\nu+\mu),\Theta),$$

in terms of Leibniz compositions with respect to variables and covariables (t,τ).

Proposition 3.2. *We have*

$$t^{-\mu}\mathrm{Op}_t(a)(\lambda)t^{-\nu}\mathrm{Op}_t(b)(\lambda)=t^{-(\mu+\nu)}\mathrm{Op}_t(a\#_t b)(\lambda)+G(\lambda) \tag{3.6}$$

for $G(\lambda)$ determined by $\mathrm{Op}_t(g)(\lambda)$ *for some $g(t\tau,t\lambda)$ with*

$$g(\tilde\tau,\tilde\lambda)\in L^{-\infty}(B,\boldsymbol{g};\mathbb{R}^{1+d}_{\tilde\tau,\tilde\lambda}).$$

For our purposes Proposition 3.2 only plays the role of giving an idea on what happens with symbols under compositions and might be studied in more detail. We simply want to illustrate in which way corner operators, obtained by replacing parameters $(\tilde\tau,\tilde\lambda)$ in operator-valued symbols by $(t\tau,t\lambda)$ reflect some Leibniz composition behaviour. Compositions in other form will be formulated in combination with Mellin quantizations applied to the respective symbols.

Proof of Poposition 3.2. The proof only employs repeatedly applying the chain rule and other elementary conclusions concerning compositions of operator functions, where

$$a\#_t b\sim\sum_{k=0}^{\infty}\frac{1}{k!}\partial^k_\tau a D^k_t b.$$

$\square$

Definition 3.3. The space $M^\mu_{\mathcal{O}_{v_2}}(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\tilde\eta_2,\lambda})$ for $\boldsymbol{g}_1:=(\gamma_1,\gamma_1-\mu,\Theta)$, is defined as the set of all $h(y_2,\eta_2,\lambda)\in\mathcal{A}(\mathbb{C}_{v_2},L^\mu(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\eta_2,\lambda}))$ such that

$$h(\beta_2+i\varrho_2,\lambda)\in L^\mu(B_1,\boldsymbol{g}_1;\mathbb{R}^{1+q_2+d}_{\varrho_2,\eta_2,\lambda})$$

for every $\beta_2\in\mathbb{R}$, uniformly in compact β_2-intervals.

Here we employ the operator classes in Definition 2.12 together with notation (2.56) for the non-smoothing Mellin part close to Y_1 and the former B is denoted by B_1. The spaces

$$L^\mu(B_1,\boldsymbol{g}_1;\mathbb{R}^d) \tag{3.7}$$

are locally convex, and they may be interpreted as unions of Fréchet spaces. Every element of (3.7) belongs to such a subspace and we can talk about spaces of holomorphic functions

$$\mathcal{A}(\mathbb{C}_{v_2}, L^{\mu}(B_1, \boldsymbol{g}_1; \mathbb{R}^d)) \tag{3.8}$$

by taking unions coming from those Fréchet subspaces of (3.7). Note that standard elements of complex function theory also work for holomorphy of functions with values in a Fréchet space, see, e.g., the textbook of Jarchow, [9]. In such constructions the weight data $\boldsymbol{g}_1 = (\gamma_1, \gamma_1 - \mu, \Theta_1)$ for weight intervals $\Theta_1 := (-(\theta_1 + 1), 0], \theta_1 \in \mathbb{N}$, are chosen and fixed, and then

$$M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d) \tag{3.9}$$

is defined as in Definition 3.3, but for $q_2 = 0$. In the sequel we use notation

$$h(r_2, v_2, \lambda) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{r_2\lambda}) \tag{3.10}$$

if $h(r_2, v_2, \lambda) = \tilde{h}(v_2, r_2\lambda)$ and $\tilde{h}(v_2, \tilde{\lambda}) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$. If H is a Hilbert space with group action we define weighted "Mellin-Fourier" Sobolev spaces $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^{q_1}, H)$ as the completion of $C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^{q_1}, H)$ with respect to the norm

$$\begin{aligned} \|u\|_{\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^{q_1},H)} := \bigg\{ \int_{\mathbb{R}^{q_1}} \int_{\Gamma_{\frac{b_1+1}{2}-\gamma_2}} & \langle v_2, \eta_1 \rangle^{2s} \\ & \times \|\kappa^{-1}_{\langle v_2,\eta_1\rangle}(M_{r_2\to v_2}F_{y_2\to\eta_1}u)(v_2,\eta_1)\|^2_H \, đv_2 đ\eta_1 \bigg\}^{1/2}. \end{aligned} \tag{3.11}$$

This expression can be generalized to Y_1 rather than $\mathbb{R}^{q_1}$ by using charts like (2.50) which gives us spaces $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+ \times Y_1, H)$ for any $s, \gamma_2 \in \mathbb{R}$. We employ this definition first to the case $s = 0$ and $H := \mathcal{K}^{0,\gamma_1}(X^{\wedge})$. In the next step we consider $\mathcal{K}$-spaces of smoothness zero and arbitrary pairs of weights $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$, namely, for $B_1 \in \mathfrak{M}_1, Y_1 = s_1(B_1)$ we set

$$\begin{aligned} \mathcal{K}^{0,\gamma}(B_1^{\wedge}) = & \omega_2\omega_1\mathcal{H}^{0,\gamma_2}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{0,\gamma_1}(X^{\wedge})) \\ & + (1-\omega_2)\omega_1\mathcal{H}^{0,0}(\mathbb{R}_+ \times Y_1, \mathcal{K}^{0,\gamma_1}(X^{\wedge})) \\ & + (1-\omega_2)(1-\omega_1)\mathcal{K}^{0,0}((2\mathbb{B}_1)^{\wedge}) \\ & + \omega_2(1-\omega_1)\mathcal{K}^{0,\gamma_2}((2\mathbb{B}_1)^{\wedge}) \end{aligned} \tag{3.12}$$

for cut-off functions $\omega_2 = \omega_2(r_2), \omega_1 = \omega_1(r_1)$.

In order to give an explanation of spaces $\mathcal{K}^{s,\gamma}(B_1^{\wedge})$ for arbitrary $s \in \mathbb{R}$ we first consider the Hilbert spaces $H^{s,\gamma_1}(B_1)$ which are defined locally close to $s_1(B_1) =: Y_1$ by

$$H^{s,\gamma_1}(B_1) = \omega_1\mathcal{W}^s(Y_1, \mathcal{K}^{s,\gamma_1}(X^{\wedge})) + (1-\omega_1)H^s_{\mathrm{loc}}(B_1 \setminus Y_1).$$

This gives us spaces $\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1))$ and also

$$\mathcal{H}^{s,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) := [r_2]^{-e}\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1))$$

and we form

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) := \varprojlim_{s_0,e\in\mathbb{R}} \mathcal{H}^{s_0,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) \tag{3.13}$$

for sufficiently large s_0, e. In particular, γ_2 remains a multiplicative weight and we have

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)) = r_2^{\gamma_2}\mathcal{S}(\mathbb{R}_+, H^{s,\gamma_1}(B_1)).$$

The spaces $\mathcal{H}^{s,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B_1))$ and also (3.13) are embedded in $\mathcal{K}^{0,\gamma}(B_1^\wedge)$. Moreover, we employ the non-degenerate sesquilinear paring

$$(\cdot,\cdot)_{\mathcal{K}^{0,0}(B_1^\wedge)} : \mathcal{K}^{0,\gamma}(B_1^\wedge) \times \mathcal{K}^{0,-\gamma}(B_1^\wedge) \to \mathbb{C} \tag{3.14}$$

for any $\gamma = (\gamma_1, \gamma_2)$.

Now with Mellin symbols

$$f(r_2, v_2, \lambda) \in M^0_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{r_2\lambda})$$

for $\boldsymbol{g}_1 = (\gamma_1, \gamma_1, \Theta)$ we have continuity

$$\mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_1/2}(f)(\lambda) : \mathcal{K}^{0,\gamma}(B_1^\wedge) \to \mathcal{K}^{0,\gamma}(B_1^\wedge).$$

If (3.10) is a Mellin symbol, we have continuity of

$$r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_1/2}(h)(\lambda) : \mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)) \to \mathcal{S}^{\gamma_2-\mu}(\mathbb{R}_+, H^{\infty,\gamma_1-\mu}(B_1)). \tag{3.15}$$

By duality via (3.14) we also get continuity between the respective dual spaces. For

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)) = r_2^{\gamma_2}\mathcal{S}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1))$$

we have a sesquilinear pairing

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)) \times (\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)))' \to \mathbb{C}.$$

Instead of $(\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)))'$ we also write $(\mathcal{S}')^{-\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B_1))$. Then, (3.15) allows us to pass to formal adjoint operators

$$\begin{aligned}(r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_1/2}(h)(\lambda))^* : (\mathcal{S}^{\gamma_2-\mu}(\mathbb{R}_+, H^{\infty,\gamma_1-\mu}(B_1)))' \\ \to (\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B_1)))'\end{aligned} \tag{3.16}$$

which is a map between the respective distribution spaces. The operator on the left-hand side of (3.16) is of analogous structure as that on the left of (3.15), i.e., of the form

$$r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_1/2}(h^*)(\lambda)$$

for some element $h^* \in M^\mu_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_\mu; \mathbb{R}^d_{r_2\lambda})$ for $\boldsymbol{g}_\mu = (-\gamma_1 + \mu, -\gamma_1, \Theta)$ obtained together with a translation in the complex v_2-plane. Because of the assumptions on holomorphy of Mellin symbols in sufficiently large v_2-strips, the map (3.16) may be realized between spaces of distributions relatively to the original weights, namely

$$\begin{aligned}r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_1/2}(h^*)(\lambda) : (\mathcal{S}')^{\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B_1)) \\ \to (\mathcal{S}')^{\gamma_2-\mu}(\mathbb{R}_+, H^{-\infty,\gamma_1-\mu}(B_1)).\end{aligned}$$

By restriction this induces an operator family

$$r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_1/2}(h^*)(\lambda) : \mathcal{K}^{0,\gamma}(B_1^\wedge) \to (\mathcal{S}')^{\gamma_2-\mu}(\mathbb{R}_+, H^{-\infty,\gamma_1-\mu}(B_1)) \tag{3.17}$$

which makes sense because of $\mathcal{K}^{0,\gamma}(B_1^\wedge) \subset (\mathcal{S}')^{\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B_1))$, but (3.17) is by no means surjective. This consideration may be applied to $s \in \mathbb{R}$ rather than μ, and a pair of weights $\gamma - s = (\gamma_1 - s, \gamma_2 - s)$. Then the respective operator will be called R^{-s}, and we define in new notation

$$R^{-s}(\lambda) := r_2^s \mathrm{Op}_{M_{r_2}}^{\gamma_2 - s - b_1/2}(f)(\lambda) : \mathcal{K}^{0,\gamma-s}(B_1^\wedge) \to \mathcal{K}^{s,\gamma}(B_1^\wedge)$$

where $f(r_2, v_2, \lambda) = \tilde{f}(v_2, r_2\lambda)$ for a resulting $\tilde{f}(v_2, \tilde{\lambda}) \in M_{\mathcal{O}_{v_2}}^{-s}(B_1, \boldsymbol{g}_s; \mathbb{R}_{\tilde{\lambda}}^d)$ for weight data $\boldsymbol{g}_s := (\gamma_1 - s, \gamma_1, \Theta)$. The resulting space $\mathcal{K}^{s,\gamma}(B_1^\wedge)$ is the image of $\mathcal{K}^{0,\gamma-s}(B_1^\wedge)$ under $R^{-s}(\lambda)$ for sufficiently large λ. We altogether obtain

Theorem 3.4. *For every pair of weights $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ and $s \in \mathbb{R}$ there exists an $f(v_2, \tilde{\lambda}) \in M_{\mathcal{O}_{v_2}}^{-s}(B_1, \boldsymbol{g}_s; \mathbb{R}_{\tilde{\lambda}}^d)$ such that for sufficiently large $|\lambda|$ the operator*

$$R^{-s}(\lambda) := r_2^s \mathrm{Op}_{M_{r_2}}^{\gamma_2 - s - b_1/2}(f)(\lambda) : \mathcal{K}^{0,\gamma-s}(B_1^\wedge) \to \mathcal{K}^{s,\gamma}(B_1^\wedge) \tag{3.18}$$

defines an isomorphism.

Remark 3.5. In the construction of $f(v_2, \tilde{\lambda}) \in M_{\mathcal{O}_{v_2}}^{-s}(B_1, \boldsymbol{g}_s; \mathbb{R}_{\tilde{\lambda}}^d)$ we make a special choice in connection with Definition 3.3. It is not necessary to admit an arbitrary $L^\mu(B_1, \boldsymbol{g}_s; \mathbb{R}^{1+q_2+d})$-valued holomorphic function but only take a function with values in operators $H(\cdot) + A_{\mathrm{int}}(\cdot)$ as indicated in Definition 2.12. This allows us to take into account that also asymptotics for $r_1 \to 0$ are respected (up to a translation) under the map with Mellin symbol, holomorphic in v_1.

Note that the operator (3.18) is defined in terms of a parameter - dependent elliptic element $f(v_2, \tilde{\lambda}) \in M_{\mathcal{O}_{v_2}}^{-\infty}(B_1, \boldsymbol{g}_1; \mathbb{R}_{\tilde{\lambda}}^d)$ and for large $|\lambda|$ we have injectivity on $\mathcal{K}^{0,\gamma-s}(B_1^\wedge)$, since there is a parameter-dependent left parametrix which leaves small remainders with increasing $|\lambda|$. In any case we set

$$\mathcal{K}^{s,\gamma}(B_1^\wedge) := \operatorname{im} R^{-s}(\lambda)$$

and another consideration shows that this space is independent of the specific choice of f with the indicated property.
A simple consideration shows that $\mathcal{K}^{s,\gamma}(B_1^\wedge)$ admits the group action

$$\kappa_\delta : \mathcal{K}^{s,\gamma}(B_1^\wedge) \to \mathcal{K}^{s,\gamma}(B_1^\wedge),$$

$$(\kappa_\delta u)(r_1, x) := \delta^{(b_1+1)/2} u(\delta r_1, x), \delta \in \mathbb{R}_+.$$

In addition the particular form of the Mellin symbol $f(v_2, r_2\lambda)$ gives us twisted homogeneity

$$R^{-s}(\delta\lambda) = \delta^{-s} \kappa_\delta R^{-s}(\lambda) \kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$. The operator calculus on the infinite stretched cone $B^\wedge$ requires Green and smoothing Mellin operators which play a similar role as those on Subsection 2.2 for a smooth compact X. Similarly as in (2.13) we employ pairs of discrete asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ and form spaces of singular functions

$$\mathcal{E}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \Big\{ \sum_{\nu=0}^{M} \sum_{\lambda=0}^{l_\nu} \omega_2(r_2) d_{\nu\lambda}(b) r_2^{-s_\nu} \log^\lambda r_2 \\ : d_{\nu\lambda} \in H_{\mathcal{P}_1}^{\infty,\gamma_1}(B_1) \text{ for all } \nu, \lambda \Big\}.$$

Moreover, we have the spaces (3.11) where the involved space H may also be a Fréchet space with group action. This allows us to form spaces

$$\mathcal{H}^{s,\gamma_1,\gamma_2}_{\mathcal{P}_1,\Theta_2}(\mathbb{R}_+\times\mathbb{R}^{q_2},\,\cdot\,) := \varprojlim_{\varepsilon>0} \mathcal{H}^{s,\gamma_2-(\theta_2+1)-\varepsilon}(\mathbb{R}_+\times\mathbb{R}^{q_2}, \mathcal{K}^{s,\gamma_1}_{\mathcal{P}_1}(X^\wedge))$$

for $\Theta_2 = (-(\theta_2{+}1), 0]$ which is an expression in terms of known data. A globalization gives us the spaces

$$\omega_2\omega_{1,\mathrm{glob}}\mathcal{H}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \omega_2\omega_{1,\mathrm{glob}}\mathcal{H}^{s,\gamma}_{\mathcal{P}_1,\Theta_2}(B_1^\wedge) + \mathcal{E}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$$

for cut-off functions $\omega_2 = \omega_2(r_2)$ on the half-axis and functions $\omega_{1,\mathrm{glob}}$ in the distance variable r_1 to Y_1, the edge of B_1. We also pass to spaces globally on $B_1^\wedge$ by forming

$$\omega_2\mathcal{H}^{s;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \omega_2\omega_{1,\mathrm{glob}}\mathcal{H}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) + \omega_2(1-\omega_{1,\mathrm{glob}})\mathcal{H}^{s,\gamma_2}_{\mathcal{P}_2}((2\mathbb{B}_1)^\wedge)|_{s_0(B_1^\wedge)}.$$

This gives us

$$\omega_2\mathcal{H}^{\infty;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \varprojlim_{s\in\mathbb{R}} \omega_2\mathcal{H}^{s;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge).$$

Moreover, consider

$$(1-\omega_2)\mathcal{S}(\overline{\mathbb{R}}_{+,r_2}, H^{\infty,\gamma_1}_{\mathcal{P}_1}(B_1))$$

which can be identified closed to $\mathbb{R}_+\times Y_1$ with the space

$$(1-\omega_2)\varprojlim_{s,\delta\in\mathbb{R}} \mathcal{W}^{s,\delta}(\mathbb{R}_+\times Y_1, \mathcal{K}^{\infty,\gamma_1}_{\mathcal{P}_1}(X^\wedge))$$

Let us now form

$$\mathcal{K}^{\infty;\gamma_1,\gamma_2;\infty}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) := \omega_2\mathcal{H}^{\infty;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) + (1-\omega_2)\mathcal{S}(\overline{\mathbb{R}}_{+,r_2}, H^{\infty,\gamma_1}_{\mathcal{P}_1}(B_1)).$$

Definition 3.6. By $L^\mu_{\mathrm{G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d\setminus\{0\})$ for weight data $\boldsymbol{g} := (\boldsymbol{g}_i)_{i=1,2}$, where $\boldsymbol{g}_i = (\gamma_i, \gamma_i-\mu, \Theta_i)$, $\Theta_i = (-(\theta_i+1), 0]$ for certain $\theta_i\in\mathbb{N}$, $i=1,2$ we denote the space of all

$$G(\lambda)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^d_\lambda; \mathcal{K}^{s;\gamma_1,\gamma_2;e}(B_1^\wedge), \mathcal{K}^{\infty;\gamma_1-\mu,\gamma_2-\mu;\infty}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge))$$

such that

$$G^*(\lambda)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^d_\lambda; \mathcal{K}^{s;-\gamma_1+\mu,-\gamma_2+\mu;e}(B_1^\wedge), \mathcal{K}^{\infty,-\gamma_1,-\gamma_2;\infty}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_1^\wedge))$$

for arbitrary $s, e\in\mathbb{R}$ and asymptotic types $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2$ depending.

Concerning a definition of spaces

$$\mathcal{K}^{s;\gamma_1,\gamma_2}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$$

for $s\in\mathbb{R}$ we apply an analogue of Theorem 3.4. The definition of $\mathcal{K}^{0,\gamma}(B_1^\wedge)$ in (3.12) allows us to pass to subspaces

$$\begin{aligned}\mathcal{K}^{0,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge) = {}&\omega_2\omega_1\mathcal{H}^{0,\gamma_2}(\mathbb{R}_+\times Y_1, \mathcal{K}^{0,\gamma_1}_{\mathcal{P}_1}(X^\wedge))\\ &+ (1-\omega_2)\omega_1\mathcal{H}^{0,0}(\mathbb{R}_+\times Y_1, \mathcal{K}^{0,\gamma_1}_{\mathcal{P}_1}(X^\wedge))\\ &+ (1-\omega_2)(1-\omega_1)\mathcal{K}^{0,0}((2\mathbb{B}_1)^\wedge) + \omega_2(1-\omega_1)\mathcal{K}^{0,\gamma_2}((2\mathbb{B}_1)^\wedge).\end{aligned}$$

We now form asymptotic types $\mathcal{P}_1(s), \mathcal{P}_2(s)$ which are translations of $\mathcal{P}_1, \mathcal{P}_2$ according to the weight shift in formula (3.18) and generate $\mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$ as the image of $\mathcal{K}^{0,\gamma-s}_{\mathcal{P}_1(s),\mathcal{P}_2(s)}(B_1^\wedge)$ under the map (3.18). Then

$$R^{-s}(\lambda) : \mathcal{K}^{0,\gamma-s}_{\mathcal{P}_1(s),\mathcal{P}_2(s)}(B_1^\wedge) \to \mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge)$$

just generates Kegel spaces with asymptotics for arbitrary s.

Remark 3.7. All those constructions makes sense for continuous asymptotic types in r_2.

Analogously as the operator families we now formulate the smoothing Mellin contribution to the class $L^\mu_{\mathrm{M+G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ for $\boldsymbol{g} = (\boldsymbol{g}_1, \boldsymbol{g}_2)$. To this and we first formulate the symbol classes $M^{-\infty}_{\mathcal{R}}(B_1, \boldsymbol{g}_1)$ for some Mellin asymptotic type $\mathcal{R}$ belonging to the singular order 1. Similarly as (2.25) in the discrete case $\mathcal{R}$ is a sequence in $\mathbb{C}_{v_2} \times \mathbb{N}$ such that $\pi_{\mathbb{C}}\mathcal{R}$ satisfies an analogous condition as before what concerns the position of points in $\mathbb{C}_{v_2}$. Now for such an $\mathcal{R}$ the space

$$M^{-\infty}_{\mathcal{R}}(B_1, \boldsymbol{g}_1)$$

is defined as the set of all $f(v_2) \in \mathcal{A}(\mathbb{C}_{v_2} \setminus \pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(B_1, \boldsymbol{g}_1))$ such that $f(v_2)$ is meromorphic with poles at the points in $\pi_{\mathbb{C}}\mathcal{R}$ of multiplicity $n_j + 1$ and finite rank Laurent coefficients in $L^{-\infty}(B_1, \boldsymbol{g}_1)$. In addition we ask

$$\chi(v_2) f(v_2)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(B_1, \boldsymbol{g}_1))$$

for any $\pi_{\mathbb{C}}\mathcal{R}$-excision function χ and every $\beta \in \mathbb{R}$, uniformly in finite intervals. Now the smoothing Mellin cone families for the calculus over $B_1^\wedge$ are assumed to be of the form

$$M(\lambda) := r_2^{-\mu} \omega_{2,[\lambda]} \sum_{j=0}^{\theta_2} r_2^j \sum_{|\alpha| \le j} \mathrm{Op}_M^{\gamma_{j\alpha} - b_1/2}(f_{j\alpha}) \lambda^\alpha \omega'_{2,[\lambda]} \tag{3.19}$$

for elements $f_{j\alpha}(v_2) \in M^{-\infty}_{\mathcal{R}_{j\alpha}}(B_1, \boldsymbol{g}_1)$ and weights $\gamma_{j\alpha}$ satisfying

$$\Gamma_{\frac{b_1+1}{2} - \gamma_{j\alpha}} \cap \pi_{\mathbb{C}}\mathcal{R}_{j\alpha} = \emptyset \quad \text{for} \quad \gamma_2 - j \le \gamma_{j\alpha} \le \gamma_2$$

for all j, α. Recall that $\omega_{2,[\lambda]}(r_2) = \omega_2(r_2[\lambda])$, etc. Similarly as (2.24), (2.25) the operator families $M(\lambda)$ constitute operator-valued symbols with constant coefficients

$$M(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma_2}(B_1^\wedge), \mathcal{K}^{\infty,\gamma_2-\mu}(B_1^\wedge))$$

and similarly, between subspaces with double asymptotic types.
Note that both Green and Mellin operator families over $B_1^\wedge$ can also be formulated for continuous asymptotics, both with respect to r_1 and r_2.

As announced before the space $L^\mu_{\mathrm{M+G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ is defined as the set of sums $M(\lambda) + G(\lambda)$ for $M(\lambda)$ as it has been just defined and $G(\lambda) \in L^\mu_{\mathrm{G}}(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$.

Definition 3.8. The space $L^\mu(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ is defined as the space of all operator families

$$A(\lambda) := H(\lambda) + M(\lambda) + G(\lambda)$$

where $(M+G)(\lambda) \in L^{\mu}_{\mathrm{M+G}}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$,

$$H(\lambda) = r_2^{-\mu} \mathrm{Op}_M^{\gamma_2 - b/2}(h)(\lambda)$$

where $h(r_2, v_2, \lambda) := \tilde{h}(v_2, r_2\lambda)$ for some $\tilde{h}(v_2, \tilde{\lambda}) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$.

Remark 3.9. We have

$$\begin{aligned} L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\}) \subset & L^{\mu}((s_0(B_1))^{\wedge}, \boldsymbol{g}_1; \mathbb{R}^d_{\lambda} \setminus \{0\}) \\ & := L^{\mu}((2\mathbb{B}_1)^{\wedge}; \mathbb{R}^d_{\lambda} \setminus \{0\})|_{\mathrm{int}\, \mathbb{B}_{1_+}}, \end{aligned} \tag{3.20}$$

see notation in the introduction.

Analogously as in the symbolic characterization of $H(\lambda)$ in Definition 2.3 and expressions (2.28), (2.29) we have degenerate families

$$p(r_2, \varrho_2, \lambda) = \tilde{p}(r_2\varrho_2, r_2\lambda)$$

for

$$\tilde{p}(\tilde{\varrho}_2, \tilde{\lambda}) \in L^{\mu}(B_1, \boldsymbol{g}_1; \mathbb{R}^{1+d}_{\tilde{\varrho}_2, \tilde{\lambda}}).$$

Then the non-smoothing Mellin symbols $\tilde{h}(v_2, \tilde{\lambda}) \in M^{\mu}_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$ in Definition 3.8 are obtained from $\tilde{p}$ via Mellin quantization from $\tilde{p}(\tilde{\varrho}_2, \tilde{\lambda})$ which turns $\tilde{\varrho}_2$ to v_2. Another principal symbolic level of operators $A(\lambda)$ in Definition 3.8 is the highest conormal symbol belonging to the corner singularity

$$\sigma_2(A)(v_2) := h(0, v_2, 0) + f_0(v_2) : H^{s,\gamma_1}(B_1) \to H^{s-\mu, \gamma_1 - \mu}(B_1) \tag{3.21}$$

with h as in Definition 3.8, frozen at $r_2 = 0$ and the principal conormal symbol of $M(\lambda)$, cf. relation (3.19) which is just the smoothing Mellin symbol belonging to the summand for $j = 0$, now denoted by $f_0(v_2)$. The conormal symbol (3.21) is regarded as an operator function parametrized by $v_2 \in \Gamma_{\frac{b_1+1}{2} - \gamma_2}$ for $b_1 = \dim B_1$.

Theorem 3.10. *Any $A(\lambda) \in L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d_{\lambda} \setminus \{0\})$ induces families of continuous operators*

$$A(\lambda) : \mathcal{K}^{s,\gamma}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu, \gamma-\mu}(B_1^{\wedge}) \tag{3.22}$$

and

$$A(\lambda) : \mathcal{K}^{s,\gamma}_{\mathcal{P}_1, \mathcal{P}_2}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu, \gamma-\mu}_{\mathcal{Q}_1, \mathcal{Q}_2}(B_1^{\wedge}) \tag{3.23}$$

for every pair of asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ and some resulting $\mathcal{Q}_1, \mathcal{Q}_2$.

Proof. The continuity (3.22) is proved in [2]. For (3.23) the arguments are similar to those of [12, Theorem 3.2.8, page 155]. The crucial information comes from the mapping properties of Mellin operators close to zero. Let us illustrate phenomena first for the case of $\mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^{\wedge})$ for an asymptotic type $\mathcal{P}$. The mapping of asymptotic contributions under Mellin pseudo-differential operators is first determined by the weighted Mellin transform $M_{\gamma_1 - n/2}$ which induces a map

$$M_{\gamma_1 - n/2} : \omega \mathcal{K}^{s,\gamma_1}_{\mathcal{P}_1}(X^{\wedge}) \to \mathcal{A}^{s,\gamma_1}_{\mathcal{P}_1}(X^{\wedge})$$

where $\mathcal{A}^{s,\gamma_1}_{\mathcal{P}_1}(X^\wedge)$ is the space of holomorphic functions u in $\mathbb{C}_{v_1} \setminus \pi_{\mathbb{C}}\mathcal{P}_1$ with values in $H^s(X)$ which generate asymptotics of type $\mathcal{P}_1$ in such a way that

$$\|\chi M_{\beta-n/2} u\|_{\mathcal{H}^{s,\beta}(X^\wedge)} < \infty \tag{3.24}$$

for a $\pi_{\mathbb{C}}\mathcal{P}_1$-excision function χ and any β such that $\beta - n/2 < \gamma_1 - n/2$, uniformly in compact subintervals in the weight strip $\{\gamma_1 - n/2 - (1+\theta) < \operatorname{Re} v_1 < \gamma_1 - n/2\}$. Another observation is that Mellin operators contain operators of multiplication by symbols which contribute their singular behaviour in the respective weight strip by multiplication, and this generates a new asymptotic type $\mathcal{Q}_1$, where the smoothness s in (3.24) is shifted by the order μ. In other words the action of our operator composed from both sides with cut-off functions defines the required map

$$\tilde{\omega} M^{-1}_{\gamma_1-n/2} f M_{\gamma_1-n/2}\omega : \mathcal{K}^{s,\gamma_1}_{\mathcal{P}_1}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}_{\mathcal{Q}_1}(X^\wedge).$$

In a similar manner we can argue with double asymptotics. Then in (3.24) we have to replace $\mathcal{H}^{s,\beta}(X^\wedge)$ by $\mathcal{H}^{s,\beta}_{\mathcal{P}_1}(B_1^\wedge)$, and the arguments concerns $\mathcal{P}_2$ rather than $\mathcal{P}_1$. □

It can be easily verified that operator families $A(\lambda) \in L^\mu(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ form symbols

$$A(\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^d; \mathcal{K}^{s,\gamma}(B_1^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge))$$

for any real s. The twisted homogeneity in the respective symbolic estimates refer to group actions in the involved spaces

$$(\kappa_\delta u)(r_2, x) = \delta^{(b_1+1)/2} u(\delta r_2, x)$$

for all $\delta \in \mathbb{R}_+$, with x denoting the variable on B_1. Let $S^{(\mu)}(\mathbb{R}^d \setminus \{0\}; \mathcal{K}^{s,\gamma}(B_1^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge))$ be the space of principal homogeneous components. Those are also operator families in $L^\mu(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ and play the role of principal edge symbols for operators of the edge calculus of second singular order in Subsection 3.2 below, with λ being replaced by (η_2, λ).

Let us call a family $A(\lambda) \in L^\mu(B_1^\wedge, \boldsymbol{g}; \mathbb{R}^d_\lambda \setminus \{0\})$ parameter-dependent elliptic of order μ, with respect to the fixed weights $\gamma = (\gamma_1, \gamma_2)$ if $\sigma_0(A)$ is parameter-dependent elliptic over $B_1^\wedge \setminus s_0(B_1^\wedge)$ in the standard sense and if also the reduced symbols

$$r_2^{-\mu}\tilde{\sigma}_0(A)(r_1, x, y_1, r_2, \tilde{\varrho}_1, \xi, \tilde{\varrho}_2, \tilde{\eta}_1, \tilde{\lambda})$$

for $r_2 > 0$ do not vanish for $(\tilde{\varrho}_1, \xi, \tilde{\varrho}_2, \tilde{\eta}_1, \tilde{\lambda}) \neq 0$ up to $r_1 = 0$, where $\tilde{\varrho}_1 := r_1\varrho_1$, etc., and for r_1, r_2 close to zero

$$\hat{\tilde{\sigma}}_0(A)(r_1, x, y_1, r_2, \tilde{\varrho}_1, \xi, \hat{\tilde{\varrho}}_2, \tilde{\eta}_1, \hat{\tilde{\lambda}})$$

is non-vanishing for $(\tilde{\varrho}_1, \xi, \hat{\tilde{\varrho}}_2, \tilde{\eta}_1, \hat{\tilde{\lambda}}) \neq 0$ up to $r_2 = 0$, for $\hat{\tilde{\varrho}}_2 := r_1 r_2 \varrho_2, \hat{\tilde{\lambda}} := r_1 r_2 \lambda$. In addition we ask that $\tilde{h}(v_2, \tilde{\lambda}) \in M^\mu_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_{\tilde{\lambda}})$, cf. Definition 3.8 to be parameter-dependent elliptic of order μ (this just encodes the suitable exit ellipticity when $r_2 \to \infty$), and that the principal conormal symbol (3.21) is a family of isomorphism for all $v_2 \in \Gamma_{\frac{b_1+1}{2}-\gamma_2}$.

Theorem 3.11. *Let* $A(\lambda) \in L^{\mu}(B_1^{\wedge}, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ *be parameter-dependent elliptic of order* μ *and relative to the weight* $\gamma = (\gamma_1, \gamma_2)$*. Then there is a parameter-dependent parametrix* $P(\lambda) \in L^{-\mu}(B_1^{\wedge}, \boldsymbol{g}^{-1}; \mathbb{R}^d \setminus \{0\}), \boldsymbol{g}^{-1} = (\boldsymbol{g}_1^{-1}, \boldsymbol{g}_2^{-1})$ *such that*

$$P(\lambda)A(\lambda) = 1 - G_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - G_{\mathrm{R}}(\lambda)$$

for some $G_{\mathrm{L}}(\lambda) \in L_{\mathrm{G}}^{-1}(B_1^{\wedge}, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d \setminus \{0\}), G_{\mathrm{R}}(\lambda) \in L_{\mathrm{G}}^{-1}(B_1^{\wedge}, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d \setminus \{0\})$ *for* $\boldsymbol{g}_{\mathrm{L}} := (\boldsymbol{g}_{1,\mathrm{L}}, \boldsymbol{g}_{2,\mathrm{L}}), \boldsymbol{g}_{\mathrm{R}} := (\boldsymbol{g}_{1,\mathrm{R}}, \boldsymbol{g}_{2,\mathrm{R}})$*.*

Proof. The proof follows in a similar manner as Theorem 3.17 below. □

Corollary 3.12. *If* $A(\lambda)$ *is parameter-dependent elliptic then*

$$A(\lambda) : \mathcal{K}^{s,\gamma}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^{\wedge})$$

is a family of Fredholm operators, and kernels belong to subspaces of $\mathcal{K}^{\infty,\gamma}(B_1^{\wedge})$ *with asymptotics. Cokernels can be represented by finite-dimensional subspaces of* $\mathcal{K}^{\infty,\gamma-\mu}(X^{\wedge})$ *with asymptotics.*

3.2. **Edge calculus of second singularity order.** In Chapter 2 we established the calculus of edge operators over a $B_1 \in \mathfrak{M}_1$ with parameters, including discrete or continuous asymptotics with respect to an edge Z_1. Here we slightly modify notation, since first order and second order edges for $B_2 \in \mathfrak{M}_2$ are denoted by Y_1, Y_2, according to the way of successively forming wedges, e.g., when

$$B_2 = B_1^{\Delta} \times Y_2 \tag{3.25}$$

we have $Y_2 = s_2(B_2)$ but $Y_1 = s_1(B_2 \setminus Y_2)$, and then $B_1 \in \mathfrak{M}_1$ which is the base of the new model cone has another edge, now denoted by Z_1 of some dimension p_1 which is also admitted to be zero. Thus analogously as (2.54) we have to introduce local (along Y_2 in variables $(y_2, \eta_2, \lambda), y_2 \in \mathbb{R}^{q_2}$) spaces of amplitude functions

$$\mathcal{R}^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g}) \tag{3.26}$$

for weight data $\boldsymbol{g} := (\boldsymbol{g}_i)_{i=1,2}, \boldsymbol{g}_i = (\gamma_i, \gamma_i - \mu, \Theta_i), i = 1, 2$, consisting of operator functions

$$a_2(y_2, \eta_2, \lambda) = h_2(y_2, \eta_2, \lambda) + (m_2 + g_2)(y_2, \eta_2, \lambda) \tag{3.27}$$

for the asymptotic part

$$(m_2 + g_2)(y_2, \eta_2, \lambda) \in \mathcal{R}_{\mathrm{M+G}}^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g}). \tag{3.28}$$

Let $\mathcal{R}_{\mathrm{G}}^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g})$ be the space of Green symbols, i.e.,

$$g(y_2, \eta_2, \lambda) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s;\gamma;e}(B_1^{\wedge}), \mathcal{K}_{\mathcal{P}_1,\mathcal{P}_2}^{\infty;\gamma-\mu;\infty}(B_1^{\wedge}))$$

such that the pointwise formal adjoint with respect to the $\mathcal{K}^{0,0}(B_1^{\wedge})$-scalar product has the property

$$g^*(y_2, \eta_2, \lambda) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s;-\gamma+\mu}(B_1^{\wedge}), \mathcal{K}_{\mathcal{Q}_1,\mathcal{Q}_2}^{\infty,-\gamma;\infty}(B_1^{\wedge}))$$

for g-dependent asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{Q}_1, \mathcal{Q}_2$, associated with the weight information involved in the respective spaces. Let us admit here from the very beginning continuous asymptotic types for $r_1 \to 0$ or $r_2 \to 0$. Clearly we also have discrete asymptotics, and y_1- or y_2-independence as a special case. Analogously as (2.53) there are also Mellin amplitude functions

$$m(y_2,\eta_2,\lambda) := r_2^{-\mu}\omega_{\eta_2,\lambda}\sum_{j=0}^{\theta_2} r_2^j \sum_{|\alpha|\le j}\mathrm{Op}_M^{\gamma_{j\alpha}-b_1/2}(f_{j\alpha})(y_2)(\eta_2,\lambda)^\alpha\omega'_{\eta_2,\lambda}$$

for Mellin symbols

$$f_{j\alpha}(y_2,v_2)\in C^\infty(\mathbb{R}^{q_2},M^{-\infty}_{\mathcal{R}_{j\alpha}}(B_1,\boldsymbol{g}_1))$$

satisfying the conditions

$$\Gamma_{\frac{b_1+1}{2}-\gamma_{j\alpha}}\cap\pi_{\mathbb{C}}\mathcal{R}_{j\alpha}=\emptyset \quad\text{for}\quad \gamma_2-j\le\gamma_{j\alpha}\le\gamma_2$$

for all j,α. Clearly $f_{j\alpha},\mathcal{R}_{j\alpha}$ and weights $\gamma_{j\alpha}$ are different from the corresponding objects in (2.53); for convenience we employ here the same notation. The Mellin asymptotic types $\mathcal{R}_{j\alpha}$ are admitted to be continuous. As for first singularity order 1 by

$$\mathcal{R}^\mu_{\mathrm{M+G}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d},\boldsymbol{g})$$

we denote the space of all $(m+g)(y_2,\eta_2,\lambda)$ of the indicated structure (where for the Mellin part we also take into account more summands of the same conormal order when the asymptotic types are continuous) and we write $\mathcal{R}^\mu_{\mathrm{G}}(\mathbb{R}^{q_2}\times\mathbb{R}^{q_2+d},\boldsymbol{g})$ for the respective space of Green amplitude functions. The non-smoothing holomorphic Mellin part is defined as

$$h_2(y_2,\eta_2,\lambda)=\omega_2 r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\gamma_2-b_1/2}(\boldsymbol{h}_2)(y_2,\eta_2,\lambda)\omega'_2 \tag{3.29}$$

for some

$$\boldsymbol{h}_2(r_2,y_2,v_2,\eta_2,\lambda)=\tilde{\boldsymbol{h}}_2(r_2,y_2,v_2,r_2\eta_2,r_2\lambda) \tag{3.30}$$

for

$$\tilde{\boldsymbol{h}}_2(r_2,y_2,v_2,\tilde\eta_2,\tilde\lambda)\in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^{q_2},M^\mu_{\mathcal{O}_{v_2}}(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\tilde\eta_2,\tilde\lambda})). \tag{3.31}$$

On $B_2\in\mathfrak{M}_2$ we have weighted spaces $H^{s,\gamma}(B_2)$, defined as the set of those $u\in H^{s,\gamma}_{\mathrm{loc}}(B_2\setminus Y_2)$ such that $\varphi_i u\circ\chi_i^{-1}\in\mathcal{W}^s(\mathbb{R}^{q_2},\mathcal{K}^{s,\gamma}(B_1^\wedge))$ for all i. Here $(\varphi_1,\dots,\varphi_N)$ is a partition of unity on Y_2 subordinate to an open covering of Y_2 by coordinate neighbourhoods $Y_{2,i}$, and $\chi_i : Y_{2,i}\to\mathbb{R}^{q_2}$ charts, $i=1,\dots,N$. Moreover, we have subspaces with asymptotics $H^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_2)$ consisting of elements $u\in H^{s,\gamma_1}_{\mathrm{loc},\mathcal{P}_1}(B_2\setminus Y_2)$ such that $\varphi_i u\circ\chi_i^{-1}\in\mathcal{W}^s(\mathbb{R}^{q_2},\mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge))$. There is then the space

$$L^{-\infty}(B_2,\boldsymbol{g};\mathbb{R}^d_\lambda)$$

of all $C(\lambda)\in\mathcal{S}(\mathbb{R}^d_\lambda,L^{-\infty}(B_2,\boldsymbol{g}))$ where $L^{-\infty}(B_2,\boldsymbol{g})$ is the space of all operators

$$C(\lambda) : H^{s,\gamma}(B_2)\to H^{\infty,\gamma-\mu}_{\mathcal{P}_1,\mathcal{P}_2}(B_2) \tag{3.32}$$

such that the formal adjoint with respect to the scalar product of $\mathcal{H}^{0,0}(B_2)$ induces continuous maps

$$C^*(\lambda) : H^{s,-\gamma+\mu}(B_2)\to H^{\infty,-\gamma}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_2) \tag{3.33}$$

for C-dependent asymptotic types $\mathcal{P}_1,\mathcal{P}_2$ and $\mathcal{Q}_1,\mathcal{Q}_2$, respectively.

Definition 3.13. By $L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ for $B_2 \in \mathfrak{M}_2$ and $\boldsymbol{g} = (\boldsymbol{g}_1, \boldsymbol{g}_2)$ we denote the space of operator families

$$A(\lambda) := H(\lambda) + (M+G)(\lambda) + A_{\mathrm{int}}(\lambda) + C(\lambda)$$

where

$$H(\lambda) + (M+G)(\lambda) = \sum_{\varphi \prec \varphi'} \varphi \mathrm{Op}_{y_1}(a_2)(\lambda)\varphi'$$

for $a_2(y_2, \eta_2, \lambda) \in \mathcal{R}^\mu(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g})$, where the sum runs over a partition of unity of Y_2 by localizing functions φ and $\varphi' \succ \varphi$ are compactly supported smooth functions in the respective coordinate neighbourhoods. Moreover, $A_{\mathrm{int}}(\lambda)$ belongs to $L^\mu(B_2 \setminus Y_2, \boldsymbol{g}_1; \mathbb{R}^d_\lambda)$ and its kernel is supported off a small neighbourhood of Y_2. The operator family $C(\lambda)$ belongs to $L^{-\infty}(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$.

Operators in $L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ have the property, that the restrict to elements in $L^\mu(B_2 \setminus Y_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$. Applying a decomposition

$$\begin{aligned} A(\lambda) = &\omega_2 A(\lambda)\omega_2' + \omega_2 A(\lambda)(1-\omega_2') + (1-\omega_2)A(\lambda)\omega_2'' \\ &+ (1-\omega_2)A(\lambda)(\omega_2' - \omega_2'') + (1-\omega_2)A(\lambda)(1-\omega_2') \end{aligned}$$

for global cut-off functions over B_2 which are $\equiv 1$ close to Y_2 and $\omega_2'' \prec \omega_2 \prec \omega_2'$. Then $\omega_2 A(\lambda)\omega_2'$ is located close to Y_2 and can be described by amplitude function in (3.26). Those belong to $S^\mu(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s,\gamma}(B_1^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge))$ and hence induce (after a localization in $y_2 \in \mathbb{R}^{q_2}$ from both sides) continuous operators

$$\mathcal{W}^s(\mathbb{R}^{q_2}, \mathcal{K}^{s,\gamma}(B_1^\wedge)) \to \mathcal{W}^{s-\mu}(\mathbb{R}^{q_2}, \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^\wedge)).$$

Globally, assuming B_2 to be compact, there operators are continuous in the sense

$$\omega_2 A(\lambda)\omega_2' : H^{s,\gamma}(B_2) \to H^{s-\mu,\gamma-\mu}(B_2).$$

The operators $\omega_2 A(\lambda)(1-\omega_2') + (1-\omega_2)A(\lambda)\omega_2''$ belong to $L^{-\infty}(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ and induce continuous operators (3.32), (3.33). The operators

$$(1-\omega_2)A(\lambda)(\omega_2' - \omega_2'') \quad \text{and} \quad (1-\omega_2)A(\lambda)(1-\omega_2')$$

belong to $L^\mu(B_2 \setminus Y_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ and generate continuous operators

$$H^{s,\gamma_1,\tilde{\gamma}_2}(B_2) \to H^{s-\mu,\gamma_1-\mu,\tilde{\tilde{\gamma}}_2-\mu}(B_2)$$

for any $\tilde{\gamma}_2, \tilde{\tilde{\gamma}}_2 \in \mathbb{R}$. Similar considerations may be applied to subspaces with asymptotics, using that any element a of (3.26) belongs to a space $S^\mu(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_1^\wedge))$. Thus we obtain altogether the following result:

Proposition 3.14. *Operators $A(\lambda) \in L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ represent families of continuous operators*

$$A(\lambda) : H^{s,\gamma}(B_2) \to H^{s-\mu,\gamma-\mu}(B_2)$$

and

$$A(\lambda) : H^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_2) \to H^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_2)$$

for any pairs of asymptotic types $\mathcal{P}_1, \mathcal{P}_2$ and some resulting $\mathcal{Q}_1, \mathcal{Q}_2$, for all $s \in \mathbb{R}$.

The proof employs similar tools as for a corresponding theorem of the edge calculus of first singular order. In fact it suffices to observe that the operators functions of (3.26) are contained in

$$S^{\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}; \mathcal{K}^{s,\gamma}(B_1^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^{\wedge}))$$

and in subspaces referring to $\mathcal{K}^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_1^{\wedge})$ and $\mathcal{K}^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_1^{\wedge})$.

Remark 3.15. The operators in Definition 3.13 form an algebra. For $A(\lambda) \in L^{\mu}(B_2, \boldsymbol{g}; \mathbb{R}^d_{\lambda})$ and $B(\lambda) \in L^{\nu}(B_2, \boldsymbol{k}; \mathbb{R}^d_{\lambda})$ for weight data $\boldsymbol{g} = (\boldsymbol{g}_i)_{i=1,2}$, $\boldsymbol{k} = (\boldsymbol{k}_j)_{j=1,2}$, $\boldsymbol{g}_i = (\gamma - \nu, \gamma - (\mu + \nu), \Theta), \boldsymbol{k}_j = (\gamma, \gamma - \nu, \Theta), i, j = 1, 2$, we have

$$A(\lambda)B(\lambda) \in L^{\mu+\nu}(B_2, \boldsymbol{g} \circ \boldsymbol{k}; \mathbb{R}^d_{\lambda})$$

for $\boldsymbol{g} \circ \boldsymbol{k} = (\boldsymbol{g}_i \circ \boldsymbol{k}_j)$ and $\boldsymbol{g}_i \circ \boldsymbol{k}_j = (\gamma, \gamma - (\mu + \nu), \Theta), i = 1, 2$. The principal symbols are multiplicative; we have, in particular,

$$\tilde{\sigma}_0(AB) = \tilde{\sigma}_0(A)\tilde{\sigma}_0(B),\ \hat{\tilde{\sigma}}_0(A) = \hat{\tilde{\sigma}}_0(A)\hat{\tilde{\sigma}}_0(B),$$

moreover,

$$\sigma_1(AB) = \sigma_1(A)\sigma_1(B),\ \tilde{\sigma}_1(AB) = \tilde{\sigma}_1(A)\tilde{\sigma}_1(B)$$

and

$$\sigma_2(AB) = \sigma_2(A)\sigma_2(B).$$

Let us now pass to ellipticity and Fredholm property of operators in the calculus over $B_2 \in \mathfrak{M}_2$. An operator $A(\lambda) \in L^{\mu}(B_2, \boldsymbol{g}; \mathbb{R}^d_{\lambda})$ in notation of Definition 3.13 is called elliptic if $A(\lambda)|_{s_0(B_2)} \in L^{\mu}_{\mathrm{cl}}(s_0(B_2); \mathbb{R}^d_{\lambda})$ is parameter-dependent elliptic over the smooth manifold $s_0(B_2)$ in the standard sense, moreover, if

$$\tilde{\sigma}_0(A)(r_1, x, y_1, r_2, y_2, \tilde{\varrho}_1, \xi, \tilde{\eta}_1, \tilde{\varrho}_2, \tilde{\eta}_2, \tilde{\lambda})$$

which is the reduced symbol of $A(\lambda)|_{B_1 \setminus Y_2}$ does not vanish for

$$(\tilde{\varrho}_1, \xi, \tilde{\eta}_1, \tilde{\varrho}_2, \tilde{\eta}_2, \tilde{\lambda}) \neq 0$$

up to $r_1 = 0$, where tilde indicates multiplication by r_1, and

$$\hat{\tilde{\sigma}}_0(A)(r_1, x, y_1, r_2, y_2, \tilde{\varrho}_1, \xi, \tilde{\eta}_1, \hat{\tilde{\varrho}}_2, \hat{\tilde{\eta}}_2, \hat{\tilde{\lambda}})$$

which is the reduced symbol of $A(\lambda)|_{B_1 \setminus (Y_1 \cup Y_2)}$ (computed close to $r_1 = 0$ and $r_2 = 0$) does not vanish for $(\tilde{\varrho}_1, \xi, \tilde{\eta}_1, \hat{\tilde{\varrho}}_2, \hat{\tilde{\eta}}_2, \hat{\tilde{\lambda}}) \neq 0$ up to $r_2 = 0$, where $\hat{\tilde{}}$ indicates multiplication by $r_1 r_2$. In addition over $B_2 \setminus Y_2$ we assume that the parameter-dependent edge symbol close to Y_1

$$\sigma_1(A)(y_1, r_2, y_2, \eta_1, \varrho_2, \eta_2, \lambda) : \mathcal{K}^{s,\gamma_1}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge})$$

is bijective for $(\eta_1, \varrho_2, \eta_2, \lambda) \neq 0$ and also its reduced symbol

$$\tilde{\sigma}_1(A)(y_1, r_2, y_2, \tilde{\eta}_1, \varrho_2, \eta_2, \tilde{\lambda}) : \mathcal{K}^{s,\gamma_1}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^{\wedge})$$

up to $r_1 = 0$ for $(\tilde{\eta}_1, \varrho_2, \eta_2, \tilde{\lambda}) \neq 0$. In addition close to Y_2 we ask bijectivity of the second singular edge symbol

$$\sigma_2(A)(y_2, \eta_2, \lambda) : \mathcal{K}^{s,\gamma}(B_1^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B_1^{\wedge}) \tag{3.34}$$

to be bijective for $(\eta_2, \lambda) \neq 0$. Note that this entails the bijectivity of the subordinate conormal symbol (3.21). This is an operator family in $L^\mu(B_1^\wedge, \boldsymbol{g}; (\mathbb{R}^{q_2} \times \mathbb{R}^d) \setminus \{0\})$ which also depends on $y_2 \in Y_2$.

Remark 3.16. As in first order singular calculus we could require the Fredholmness of the various edge symbols. This would require corresponding bijectivity conditions of block-matrix-valued symbols with addition trace, potential and Green entries. For brevity we consider here the case of bijectivities without such extra data.

Theorem 3.17. *Let* $A(\lambda) \in L^\mu(B_2, \boldsymbol{g}; \mathbb{R}^d)$ *be parameter-dependent elliptic. Then there is a parameter-dependent parametrix* $P(\lambda) \in L^{-\mu}(B_2, \boldsymbol{g}^{-1}; \mathbb{R}^d)$, $\boldsymbol{g}^{-1} = (\boldsymbol{g}_1^{-1}, \boldsymbol{g}_2^{-1})$ *such that*

$$P(\lambda)A(\lambda) = 1 - C_{\mathrm{L}}(\lambda),\ A(\lambda)P(\lambda) = 1 - C_{\mathrm{R}}(\lambda) \tag{3.35}$$

for $C_{\mathrm{L}}(\lambda) \in L^{-\infty}(B_2, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d)$, $C_{\mathrm{R}}(\lambda) \in L^{-\infty}(B_2, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d)$ *for* $\boldsymbol{g}_{\mathrm{L}} := (\boldsymbol{g}_{1,\mathrm{L}}, \boldsymbol{g}_{2,\mathrm{L}})$, $\boldsymbol{g}_{\mathrm{R}} := (\boldsymbol{g}_{1,\mathrm{R}}, \boldsymbol{g}_{2,\mathrm{R}})$. *Moreover, if* B_2 *is compact*

$$A(\lambda) : H^{s,\gamma}(B_2) \to H^{s-\mu,\gamma-\mu}(B_2) \tag{3.36}$$

is a family of Fredholm operators. It becomes a family of isomorphism for sufficiently large $|\lambda|$. *This holds for all* $s \in \mathbb{R}$. *In addition, solutions* u *of* $A(\lambda)u = f$ *for* $f \in H^{s-\mu,\gamma-\mu}_{\mathcal{Q}_1,\mathcal{Q}_2}(B_2)$ *belong to* $H^{s,\gamma}_{\mathcal{P}_1,\mathcal{P}_2}(B_2)$ *for pairs of continuous asymptotic types* $\mathcal{Q}_1, \mathcal{Q}_2$ *with resulting* $\mathcal{P}_1, \mathcal{P}_2$.

Proof. The assertion follows the lines of Theorem 2.14. Therefore, it remains to compute a parametrix $P(\lambda)$ in a neighbourhood of Y_2; then over B_2 we may apply a construction separately for $B_2 \setminus Y_2$ and near Y_2, and then, using a partition of unity we can glue together the respective parametrices. The main condition is the bijectivity condition (3.34). The inverse of (3.34) is obtained first by constructing a y_2-depending parametrix in $L^\mu(B_1^\wedge, \boldsymbol{g}; (\mathbb{R}^{q_2} \times \mathbb{R}^d) \setminus \{0\})$ according to Theorem 3.11, and then a Leibniz inverse $a^{(-1)}(y_2, \eta_2, \lambda)$ of the amplitude function (3.27) of the given elliptic operators which exists in $\mathcal{R}^{-\mu}(\mathbb{R}^{q_2} \times \mathbb{R}^{q_2+d}, \boldsymbol{g}^{-1})$. Then the associated operators, locally expressed by $\mathrm{Op}_{y_2}(a^{(-1)})$ are just the desired two-sided parametrix. The claimed Fredholm property of (3.36) is then a consequence, since by assumption B_2 is compact and the remainders in (3.35) are compact in the respective spaces. ☐

References

[1] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[2] D.-C. Chang and B.-W. Schulze, *Calculus on spaces with higher singularities*, J. Pseudo-Differ. Oper. Appl. **8** (4) (2016), 585–622.

[3] D.-C. Chang and B.-W. Schulze, *Ellipticity on spaces with higher singularities*, CHINA SCIENCE Math. **60** (11) (2017), 2053–2076.

[4] H.-J. Flad, G. Harutyunyan, and B.-W. Schulze, *Explicit Green operators for quantum mechanical Hamiltonians. II. Edge singularities of the helium atom*, manuscripta math. **135** (2011), 497–519.

[5] H.-J. Flad, G. Harutyunyan, and B.-W. Schulze, *Asymptotic parametrices on singular cones.* (in preparation, 2017)

[6] J.B. Gil, B.-W. Schulze, and J. Seiler, *Holomorphic operator-valued symbols for edge-degenerate pseudo-differential operators*, in 'Differential Equations, Asymptotic Analysis and Mathematical Physics", eds. M. Demuth et al. Mathematical Research, vol. 100, pp. 113-137. Akademie Verlag (1997).

[7] J.B. Gil, B.-W. Schulze, and J. Seiler, *Cone pseudodifferential operators in the edge symbolic calculus*, Osaka J. Math. **37** (2000), 219–258.

[8] M. Hedayat-Mahmoudi and B.-W. Schulze, *A new approach to the second order edge calculus*, J. Pseudo-Differ. Oper. Appl. (2017). doi:10.1007/s11868-017-0191-2

[9] H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, (1981).

[10] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[11] X. Lyu and B.-W. Schulze, *Mellin operators in the edge calculus*, Complex Analysis and Operator Theory **10** (2016), 965–10000.

[12] E. Schrohe and B.-W. Schulze, *Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities I*, Math. Topics **5**, Adv. in: PDE "Pseudo-Differential Calculus and Mathematical Physics", Akademie Verlag, Berlin, (1994).

[13] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[14] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[15] B.-W. Schulze, *Operators with symbol hierarchies and iterated asymptotics*, Publ. Res. Inst. Math. Sci., Kyoto **38** (2002), 735–802.

[16] B.-W. Schulze, *The iterative structure of the corner calculus*, in: Rodino, L., Wong, M.W., Zhu, H.(eds.) Operator Theory: Advances and Applications, vol. 213, pp. 79–103. Pseudo-Differential Operators: Analysis, Application and Computations. Birkhäuser Verlag, Basel, (2011).

[17] W. Rungrottheera, *Parameter-dependent corner operators*, Asian-Eur. J. Math. **6** (2013). doi:10.1142/S1793557113500022

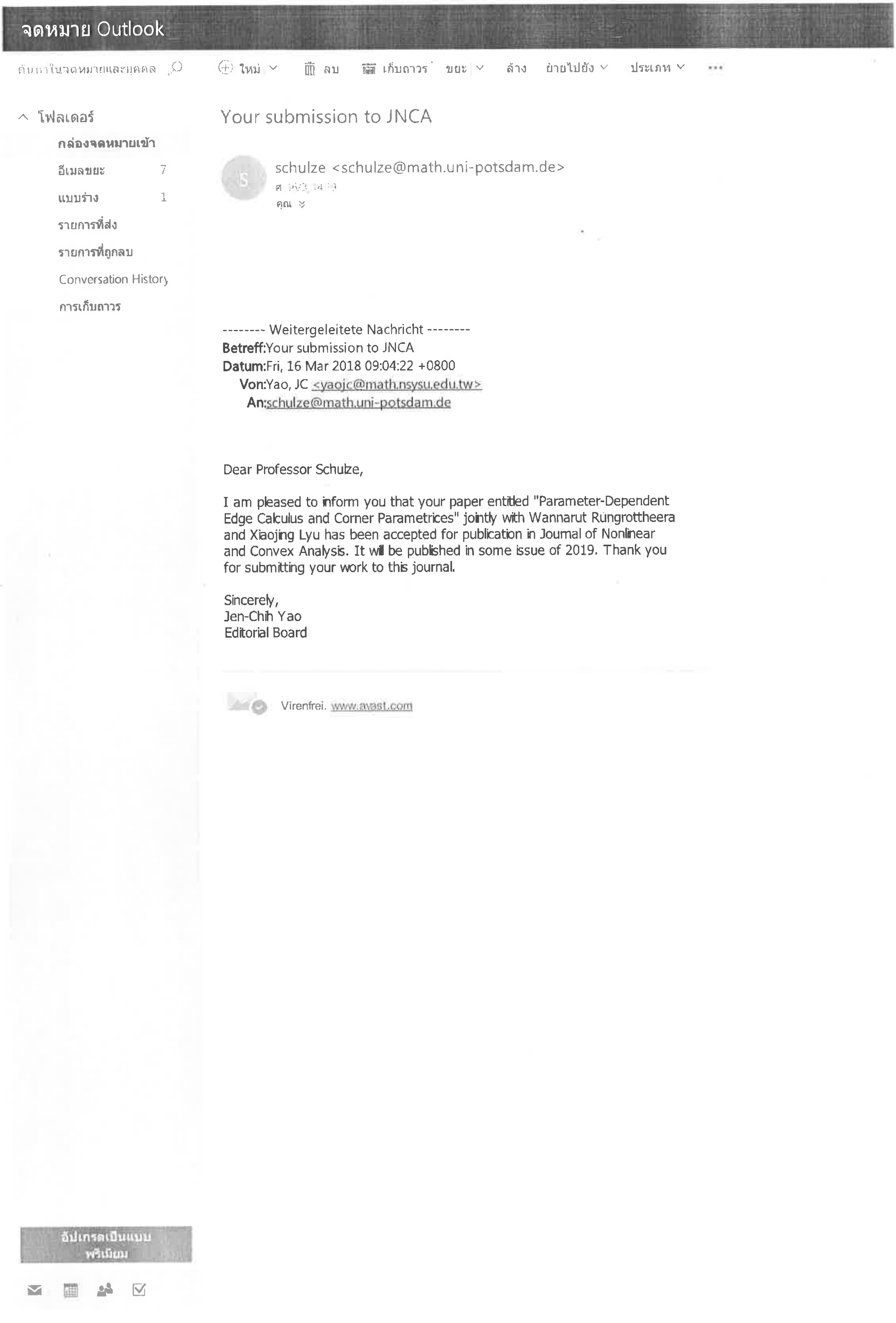
[18] W. Rungrottheera and B.-W. Schulze, *Holomorphic operator families on a manifold with edge*, J. Pseudo-Differ. Oper. Appl. **4** (2013), 297–315.

[19] W. Rungrottheera, B.-W. Schulze and M.W. Wong, *Iterative properties of pseudo-differential operators on edge spaces*, J. Pseudo-Differ. Oper. Appl. **5** (2014), 455–479.

(W. Rungrottheera) Department of Mathematics, Faculty of Science, University of Silpakorn, Nakhon Pathom 73000, Thailand
E-mail address: `w_rungrott@outlook.co.th`

(X. Lyu) College of Science, Tianjin University of Technology and Education, Tianjin 300222, People Republic of China
E-mail address: `xiaojinglv2003@tute.edu.cn`

(B.-W. Schulze) University of Potsdam, Campus Golm, Institute for Mathematics, Karl-Liebknecht-Strasse 24-25, 14476 Potsdam, Germany
E-mail address: `schulze@math.uni-potsdam.de`



Your submission to JNCA

schulze <schulze@math.uni-potsdam.de>
ศ 16/3 14:[illegible]
คุณ

-------- Weitergeleitete Nachricht --------
Betreff: Your submission to JNCA
Datum: Fri, 16 Mar 2018 09:04:22 +0800
Von: Yao, JC <yaojc@math.nsysu.edu.tw>
An:schulze@math.uni-potsdam.de

Dear Professor Schulze,

I am pleased to inform you that your paper entitled "Parameter-Dependent Edge Calculus and Corner Parametrices" jointly with Wannarut Rungrottheera and Xiaojing Lyu has been accepted for publication in Journal of Nonlinear and Convex Analysis. It will be published in some issue of 2019. Thank you for submitting your work to this journal.

Sincerely,
Jen-Chih Yao
Editorial Board

Output

1. ผลงานส่งตีพิมพ์ในวารสารวิชาการนานาชาติ

W. Rungrottheera, X. Lyu and B.-W. Schulze, Parameter-Dependent Edge Calculus and Corner Parametrices, accepted for publication in Journal of Nonlinear and Convex Analysis (JNCA), 2019

2. การนำผลงานวิจัยไปใช้ประโยชน์

- นำไปพัฒนาการเรียนการสอน โดยนำเนื้อหาบางส่วนไปใช้ในการเรียนการสอนสำหรับนักศึกษาระดับบัณฑิตศึกษา เพื่อให้นักศึกษาเห็นความเชื่อมโยงระหว่างองค์ความรู้ทางคณิตศาสตร์ที่ต่างกัน
- ใช้แนะแนวทางการทำวิจัย และนำบางส่วนของปัญหาให้นักศึกษาเลือกทำเป็นหัวข้อสัมมนาหรือวิทยานิพนธ์ในอนาคตได้

แบบสรุปปิดโครงการวิจัย

สัญญาเลขที่ MRG5980089 **ชื่อโครงการ** ตัวดำเนินการบนคอร์เนอร์แมนิโฟลด์
หัวหน้าโครงการ นางสาววรรณรัตน์ รุ่งโรจน์ธีระ **หน่วยงาน** มหาวิทยาลัยศิลปากร
โทรศัพท์ 081-4544831 **โทรสาร** 034-273042 **อีเมล์** w_rungrott@outlook.co.th
สถานะผลงาน ☐ **ปกปิด** ☑ **ไม่ปกปิด**

ความสำคัญ / ความเป็นมา

This project belong to Partial differential equations, in particular, it belongs to structure theory and representation theory of pseudo-differential operators. Partial differential equations, in particular, ellipticity, boundary value problems, asymptotics and other aspects of regularity of solutions, are relevant in many applications, e.g., mechanics, material sciences, or particle physics. Also other mathematical structures theories, such as geometry, topology, or index theory, manifolds with singularities become more and more important. These applications require the investigation of the respective algebra of pseudo-differential operators in high generality.

Pseudo-differential operators generalize the class of differential operators by elements, whose symbols are not polynomials. Let $A=\sum_{|\alpha|\le\mu} a_\alpha(x)D_x^\alpha$ be a differential operator in a domain $\Omega\subseteq\mathbb{R}^n$ with coefficients $a_\alpha(x)$ are smoothing operators on Ω. Then A can be expressed by the Fourier transform F as $A=F^{-1}a(x,\xi)F$ with $a(x,\xi)=\sum_{|\alpha|\le\mu} a_\alpha(x)\xi^\alpha$. Thus, pseudo-differential operators will be of the form $Au(x)=\iint e^{i(x-x')\xi}a(x,x',\xi)u(x')dx'\bar{d}\xi,\ \bar{d}\xi=(2\pi)^{-n}d\xi$. The homogeneous principal symbol of A of order μ defined by $\sigma_\psi(A)(x,\xi)=\sum_{|\alpha|=\mu} a_\alpha(x)\xi^\alpha$. A is called elliptic, if $\sigma_\psi(A)(x,\xi)\neq 0$ for all $(x,\xi)\in\Omega\times(\mathbb{R}^n\setminus\{0\})$. Cone manifold X^Δ is a manifold with base X, defined as $X^\Delta:=(\overline{\mathbb{R}}_+\times X)/(\{0\}\times X)$. Manifolds with conical singularities are locally modelled on cones with base spaces that are smooth manifolds. A manifold with edge is locally a Cartesian product between the model cone X^Δ and edge Y. A manifold with corner is locally modelled on a cone with base having as base space a manifold with edge. Ellipticity is described by a two-component principal symbolic hierarchy, namely $\sigma(\cdot)=(\sigma_\psi(\cdot),\sigma_M(\cdot))$ in the conical case and $\sigma(\cdot)=(\sigma_\psi(\cdot),\sigma_\wedge(\cdot))$ in the edge case. Here $\sigma_\psi(\cdot)$ has a similar meaning as before, while $\sigma_M(\cdot)$ is the (principal) conormal symbol, $\sigma_\wedge(\cdot)$ the (principal) edge symbol of the respective operator. In the conical case the conormal symbol of an operator A is an operator function $\sigma_M(A)(z):H^s(X)\to H^{s-\mu}(X)$ between the Sobolev spaces on the base X of the cone (with $\mu=\operatorname{ord}(A)$), parametrized by the covariable z to the (local) cone axis variable, varying on a weight line $\left\{\operatorname{Re} z=\frac{n+1}{2}-\gamma\right\}$ in the complex plane for a weight $\gamma\in\mathbb{R}$, $n=\dim X$. In the case of a manifold with edge Y the edge symbol of an operator A is an operator function $\sigma_\wedge(A)(y,\eta):K^{s,\gamma}(X^\wedge)\to K^{s-\mu,\gamma-\mu}(X^\wedge)$ on the infinite open stretched model cone $X^\wedge:=\mathbb{R}_+\times X$ between weighted edge Sobolev space.

Formally, pseudo-differential operators those are motivated by the problem to constructing parametrices of elliptic differential operators and the characterization of the regularity of solutions to elliptic equations. This is well-known on a smooth manifold. Among the manifolds with singularities are manifolds with boundary, moreover with conical and edge singularities, and also with corners.

A manifold M with corners of second order is a topological space containing a smooth manifold $s_2(M)$ such that $M\setminus s_2(M)$ is a manifold with corners of first order (i.e., with conical points or edges), and $s_2(M)$ has a neighbourhood in M with the structure of a locally trivial B^{Δ}-bundle over $s_2(M)$ for some B of first singularity order. The space B in turn contains a smooth manifold $s_1(B)$ such that $B \setminus s_1(B)$ is smooth and $s_1(B)$ has a neighbourhood in B with the structure of a locally trivial X^{Δ}-bundle over $s_1(B)$ for some smooth manifold X.

The analysis of operators on singular manifolds or with degenerate symbols is motivated by models of the applied sciences, especially of mechanics and elasticity theory, and also by pure mathematics, such as geometry and topology. It has been a point of interest for many researchers in these fields. See, for instance, the works of Kondratyev [3], Kondratyev and Oleynik [4], Plamenevskij [5]. Applications and more references may be found in Harutyunyan and Schulze [2]. Recall that conical and edge singularities are of order 1 in a hierarchy of corner singularities of any order. One of the main issues of the cone and edge theories is to construct parametrices of elliptic elements and to establish regularity of solutions in weighted Sobolev spaces. Rungrottheera [6] construct parameter-dependent operators on a manifold with conical singularities or edge. Such operator families play a role as operator-valued amplitude functions in operators on spaces with higher singularities. Rungrottheera and Schulze [7] apply these results to the construction of holomorphic functions with values in the edge calculus, also with an appropriate order reducing property. They construct specific holomorphic families of operators on a manifold with edge that have the order reducing property between weighted spaces for arbitrary weights varying in a prescribed compact interval and for all complex parameters in a strip parallel to the imaginary axis of arbitrary finite width. These operator functions are constructed as parameter-dependent elliptic elements of the edge calculus, here without additional conditions of trace and potential type. They play a role as operator-valued Mellin symbols in the higher corner pseudo-differential calculus.

Differential operators and principal symbolic hierarchies on higher corner spaces have been studied in Schulze [11]. The pseudo-differential machinery for singular cones up to order 2, including iterated asymptotics, is developed in Schulze [10]. Iterative Mellin quantisations for higher singularities have been constructed in Habal and Schulze [1]. Another iterative paper of Rungrotteera and Schulze [8] studies aspects of higher corner spaces and pseudo-differential operators from the point of view of kernel cut-off, weighted corner Sobolev spaces and subspaces with iterated discrete and continuous asymptotics, and their relationship with corner pseudo-differential operators. Rungrottheera, Schulze and Wong [9] study the underlying abstract concept on manifolds with singularities and establish a new iteration result, motivated by applications in the higher corner case and establish a new theorem on iterated pseudo-differential operators belonging to second order corners, i.e., $k=2$. Then by induction give k fold iterated operators for any $k \in \mathbb{N},\ k \geq 2$. These investigate belong to framework of the study of pseudo-differential operators on manifold with higher corner singularities.

In recent years the analysis of (pseudo-)differential operators on manifolds with second and higher order corners made considerable progress, and essential new structures have been developed. The study of weighted corner Sobolev spaces and subspaces with iterated asymptotics belong to the background of pseudo-differential algebras on corner manifolds. We complete and deepen in this project an iterative approach of establishing the calculus of pseudo-differential operators for the case of second and higher corner singularities

วัตถุประสงค์ของโครงการ

We will give the following results:

(1) We study elements of the pseudo-differential calculus on manifold with second order corners.

(2) We establish parameter-dependent edge calculus and Kegel spaces of second singularity order.

(3) We study weighted corner spaces, quantisation for higher corners

ผลการวิจัย

We establish an edge pseudo-differential calculus over B with parameters. Those are involved as extra covariables, degenerate in a similar manner as edge covariables. We then formulate the existence of parameter-dependent parametrices in this framework with controlled remainders. Section 2 is devoted to elements of a analogous approach for manifolds with edges of singularity order 2. In particular, we formulate a new level of operator-valued corner symbols and compute corner parametrices in terms of a new set of variables and covariables in the new corner axis direction. Moreover, we continue and deepen the material of D.-C. Chang and B.-W. Schulze in cases of lower orders of singularities.

คำสืบค้น (Keywords)
Edge calculus, corner parametrices

การนำผลงานวิจัยไปใช้ประโยชน์

☐ **ด้านนโยบาย** **โดยใคร** *(กรุณาให้ข้อมูลเจาะจง)*..
มีการนำไปใช้อย่างไร
....
....

☐ **ด้านสาธารณะ** **โดยใคร** *(กรุณาให้ข้อมูลเจาะจง)* ....
มีการนำไปใช้อย่างไร
....
....

☐ **ด้านชุมชนและพื้นที่** **โดยใคร** *(กรุณาให้ข้อมูลเจาะจง)* ....
มีการนำไปใช้อย่างไร
....
....

☐ **ด้านพาณิชย์** **โดยใคร** *(กรุณาให้ข้อมูลเจาะจง)* ....
มีการนำไปใช้อย่างไร
....
....

☑ **ด้านวิชาการ** **โดย D.-C. Chang, W. Rungrottheera and B.-W. Schulze**
มีการนำไปใช้อย่างไร
ผลที่ได้จากการศึกษาของโครงการ “ตัวดำเนินการบนคอร์เนอร์แมนิโฟลด์” ได้นำไปศึกษาตัวดำเนินการ รวมถึงฟังก์ชันเมลลินและฟังก์ชันกรีนบนแมนิโฟลด์ที่มีอันดับมากกว่า 2

☐ **ยังไม่มีการนำไปใช้** (โปรดกรอกในกรอบถัดไป)

(กรณีที่ยังไม่มีการใช้ประโยชน์) **ผลงานวิจัยมีศักยภาพในการนำไปใช้ประโยชน์**
☐ **ด้านนโยบาย** ☐ **ด้านสาธารณะ** ☐ **ด้านชุมชนและพื้นที่** ☐ **ด้านพาณิชย์** ☐ **ด้านวิชาการ**
ข้อเสนอแนะเพื่อให้ผลงานถูกนำไปใช้ประโยชน์
....
....

การเผยแพร่/ประชาสัมพันธ์ *(กรุณาให้รายละเอียด พร้อมแนบหลักฐาน)*

1. สิ่งพิมพ์ หรือสื่อทั่วไป

   ☐ **หนังสือพิมพ์** ☐ **วารสาร** ☐ **โทรทัศน์** ☐ **วิทยุ** ☐ **เว็บไซต์** ☐ **คู่มือ/แผ่นพับ** ☐ **จัดประชุม/อบรม** ☐ **อื่น ๆ**

   ………

   ………

2. สิ่งพิมพ์ทางวิชาการ *(วารสาร, การประชุม ให้ระบุรายละเอียดแบบการเขียนเอกสารอ้างอิง เพื่อการค้นหาซึ่งควรประกอบด้วย ชื่อผู้แต่ง ชื่อเรื่อง แหล่งพิมพ์ ปี พ.ศ. (ค.ศ.) ฉบับที่ หน้า)*

   W. Rungrottheera, X. Lyu and B.-W. Schulze, Parameter-Dependent Edge Calculus and Corner Parametrices, accepted for publication in Journal of Nonlinear and Convex Analysis (JNCA), 2019