รายงานวิจัยฉบับสมบูรณ์

โครงการ

Maximal infinite order-preserving transformation semigroups

เซมิกรุปแบบใหญ่สุดเฉพาะกลุ่มของการแปลงแบบยืนยง
อันดับบนเซตอนันต์

โดย อาจารย์ ดร.ทิวดี มุสันเทียะ

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สัญญาเลขที่ TRG5780263

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บทคัดย่อ

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ชื่อโครงการ: เซมิกรุปแบบใหญ่สุดเฉพาะกลุ่มของการแปลงแบบยืนยงอันดับบนเซตอนันต์

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ระยะเวลาโครงการ: 2 ปี

บทคัดย่อ:

ในวิจัยฉบับนี้ เราศึกษาเซมิกรุปแบบใหญ่สุดเฉพาะกลุ่มของการแปลงแบบยืนยงอันดับบนเซตของจำนวนธรรมชาติและของจำนวนเต็ม ตามลำดับ เราได้บรรยายเซมิกรุปแบบใหญ่สุดเฉพาะกลุ่มของโมนอยด์ของการแปลงแบบยืนยงอันดับที่มีสมบัติหนึ่งต่อหนึ่งทั้งหมดบนเซตของจำนวนธรรมชาติและของจำนวนเต็ม จากนั้นเราได้ให้เซมิกรุปแบบใหญ่สุดเฉพาะกลุ่มของโมนอยด์ของการแปลงแบบยืนยงอันดับที่มีสมบัติหนึ่งต่อหนึ่งและทั่วถึงทั้งหมดบนเซตของจำนวนเต็ม สำหรับโมนอยด์ของการแปลงแบบยืนยงอันดับทั้งหมดบนเซตของจำนวนธรรมชาตินั้น เราให้เซมิกรุปแบบใหญ่สุดเฉพาะกลุ่มทั้งหมดที่บรรจุเซตของการแปลงบางเซต

คำหลัก : เซมิกรุปการแปลง เซมิกรุปแบบใหญ่สุดเฉพาะกลุ่ม การส่งแบบยืนยงอันดับ

Abstract

Project Code : TRG5780263

Project Title : Maximal infinite order-preserving transformation semigroups

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Abstract:

In this research, we study the maximal subsemigroups of several semigroups of order-preserving transformations on the natural numbers and the integers, respectively. We determine all maximal subsemigroups of the monoid of all order-preserving injections on the set of natural numbers as well as on the set of integers. Further, we give all maximal subsemigroups of the monoid of all bijections on the integers. For the monoid of all order-preserving transformations on the natural numbers, we classify also all its maximal subsemigroups, containing a particular set of transformations.

Keywords: Transformation semigroups, Maximal semigroups, Order-preserving mapping

Executive Summary

Maximal infinite order-preserving transformation semigroups

1. Introduction to the research problem and its significance

The project belongs to Algebra, in particular, it belongs to structure theory and representation theory of semigroups. A semigroup is a set with an associative multiplication and appears often by the modelling of sets of functions which are closed under composition. Many subjects in the pure as well as in applied mathematics, in physics and also in other sciences use semigroup theory. If the equation $ax = b$ and $b = xa$ is solvable in each case, the semigroup is a group. The natural numbers 1, 2, 3,... together with the addition is a semigroup, but not a group. Another well known semigroup is the set of linear mappings of a vector space. Linear mappings can be represented as matrices. The multiplication of matrices corresponds with the composition of linear mappings. A mapping from a semigroup into a semigroup of matrices preserving the multiplication is called linear representation of the semigroup. Any semigroup can also be represented as a semigroup of transformations on a set. The theory of the representation of semigroups (and also of other structures) is an important subject in mathematics. In structure theory, particular properties of semigroups are investigated. It is the basis of all research in semigroup theory. The representation of semigroups bases on structure theory.

The study of transformation semigroups is an important subject in semigroup theory. A transformation on a set X is a mapping from X into X. The set of all transformations on a set X forms a semigroup under the composition of functions and is called full transformation semigroup $T(X)$. A subsemigroup T of $T(X)$ ($T \leq T(X)$) is called transformation semigroup. If X is a finite set, then a semigroup $T \leq T(X)$ is called finite transformation semigroup and is already wide studied. A survey about finite transformation semigroups can be found in [3]. But the situation is quite different in the case of infinite transformation semigroups, i.e. if the set X is at least countable infinite. It is a well known fact each semigroup can be embedded in a transformation semigroup $T \leq T(X)$ for an appropriate set X. Hence the study of transformation semigroups became very important. In particular the knowledge about the subsemigroups of $T(X)$ is of large interest. By this reason, maximal subsemigroups of $T(X)$ as well as of particular subsemigroups of $T(X)$ were studied in the case that X is finite. Let us mention two important results: P. A. Bayramov [1] has characterized in 1966 all maximal subsemigroups of $T(X)$ (X is finite). The set of all order-preserving transformations on a set X forms a subsemigroup $O(X) \leq T(X)$ of $T(X)$ [8]. A transformation α is called order-preserving if $x < y$ implies $x\alpha < y\alpha$, where $<$ is a linear order on X. The maximal subsemigroups of $O(X)$ are characterized by Yang Xiuliang in 2000. A reascend result in this topic is by Ping Zhao and Mei Yang. They characterize the locally maximal idempotent generated subsemigroups of the so-called finite orientation preserving singular partial transformation semigroup [9]. If X is at least countable infinite, then the knowledge about the subsemigroups of $T(X)$ is very small. In fact, we know the maximal transformation semigroups containing the subgroup $Sym(X)$ of all bijections on X and particular subsemigroups of $Sym(X)$, respectively. There are $2^{2^{|X|}}$ subsemigroups of $T(X)$ [8]. If X is countable then there are five maximal subsemigroups of $T(X)$ containing $Sym(X)$ [4]. For an arbitrary infinite set X, M. Pinsker has determined the number of subsemigroups containing the group $Sym(X)$ [7]. In a recent work, J. Jonussas and J. D. Mitchell characterize the subsemigroups in the interval from the intersection of all maximal subsemigroups of $T(X)$ containing $Sym(X)$ [4] to the full transformation semigroup $T(X)$ for the case that X is countable infinite. They list all of the 36 transformation semigroups. In [2], the authors determine all maximal subsemigroups of $T(X)$ containing one of the following subgroups of $Sym(X)$: the pointwise stabiliser of a set $\sum \subseteq T(X)$, the stabiliser of a partition of $T(X)$, and a non-principal ultrafilter on $T(X)$, respectively.

The mentioned results reflex the current knowledge about infinite transformation semigroups. So, the research on this topic is fare from a sufficient description of the class of all infinite transformation semigroups. This project is going to contribute with an essential part in order to improve this situation and proceeds the current research in this topic. We will investigate the order-preserving transformation semigroup $O(X)$ for any linear ordered set $(X;\leq)$, an important transformation semigroup [5]. As already mentioned, the maximal subsemigroups $O(X)$ are completely determined if X is finite by Yang Xiuliang. But, if X is an infinite set, we have not yet any information. It is clear, that it will not be any simple extension of the results of the finite case since here to facts will have importance: first, the concrete cardinality of the set X and second, the concrete linear order $\leq$ on X. In contrast to the finite case, we expect for the same set $\leq$, but different linear orders $<$, essential different results.

2. Literature review

Semigroups of order-preserving transformations have been extensively studied in the recent years. In particular, O. Ganyushkin and V. Mazorchuk have studied the structure of semigroups of all injective order-preserving transformations on a finite chain [4]. We know also all maximal subsemigroups of the semigroup of all order-preserving transformations on a finite chain [12]. However, the case of infinite linearly ordered sets has been poorly studied. In 1965, G. P. Gavrilov [6] showed that there are five maximal subsemigroups of $T(X)$ containing the symmetric group on X if X is countable and in 2005, M. Pinsker [10] extended Gavrilov's result to sets of arbitrary cardinality. J. East, J. D. Mitchell, and Y. Péresse classified all maximal subsemigroups of $T(X)$ containing one of following subgroups of the symmetric group: the pointwise stabilizer of a non-empty finite subset of X, the stabilizer of an ultrafilter on X, or the stabilizer of a partition of X into finitely many subsets of equal cardinality [3]. In particular in [2], V. Doroshenko considers semigroups of all transformations of the set of integers and natural numbers that preserve the natural order on them and their subsemigroups of cofinite transformations (i.e., transformations under which the complement of image is a finite set).

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3. Objectives

1. We consider transformation semigroups of countably infinite set X where X is $\mathbb{N}$ and $\mathbb{Z}$, respectively
2. We consider the bijective part of $O(X)$, i.e. the intersection of the semigroups $Sym(X)$ and $O(X)$. Clearly, $Sym(X) \cap O(X)$ is again a semigroup.
3. In [9], the maximal subsemigroups of $T(X)$ containing the group $Sym(X)$, are determined. Using these results, we are going to find a description of the maximal subsemigroups of $O(X)$ containing the bijective part $Sym(X) \cap O(X)$.
4. If X is countable infinite, we can expect specific results, which could be quite different to the results can be expected for sets X with a cardinality greater than $\aleph_0$ (we claim this bases on other known facts concerning infinite transformation semigroup). We can start with $(\mathbb{N};\leq)$, the natural numbers $\mathbb{N}$ (having the cardinality $\aleph_0$) under the usual order $\leq$. Further, we want to determine the maximal subsemigroups of $O(\mathbb{Z})$ (the integers $\mathbb{Z}$ under the usual order $\leq$). The main aim of this item is the characterization of all maximal subsemigroups of $O(X)$, whenever X is countable infinite. The solution of this problem would give an essential contribution in the framework of the current research on infinite transformation semigroups.
5. The general aim of this project is the presentation of a satisfying description of the maximal subsemigroups of $O(X)$ for any infinite chain $(X;\leq)$.

4. Methodology

<u>ขั้นที่ 1</u> ศึกษาความรู้พื้นฐาน รวบรวมเอกสารซึ่งเป็นผลงานที่เกี่ยวข้องกับงานวิจัยนี้ทั้งหมด จากแหล่งต่างๆทั้งในและต่างประเทศ

<u>ขั้นที่ 2</u> คิดค้นวิธีการ เพื่อสร้างและออกแบบการพิสูจน์ เพื่อให้สามารถตอบคำถามได้ตามวัตถุประสงค์

<u>ขั้นที่ 3</u> Determine maximal subsemigroups of $IO(\mathbb{N})$

<u>ขั้นที่ 4</u> Determine maximal subsemigroups of $SO(\mathbb{Z})$

<u>ขั้นที่ 5</u> Determine maximal subsemigroups of $IO(\mathbb{Z})$

<u>ขั้นที่ 6</u> Determine maximal subsemigroups of $O(\mathbb{N})$

<u>ขั้นที่ 7</u> ศึกษาเฟนซ์ (fence or zig zag order) และ TF_n ซึ่งเป็นเซมิกรุปการแปลงกับ non-linear ordered set (semigroup of all order-preserving ransformations on an n- element zig-zag poset)

<u>ขั้นที่ 8</u> Characterize transformation in TF_n

เนื้อหางานวิจัย

1. Introduction

Transformation semigroups play a role in Semigroup Theory corresponding to that of the symmetric groups in Group Theory. A main result parallel to Cayley's theorem for groups is the well known result that states that every semigroup is isomorphic to a subsemigroup of a suitable full transformation semigroup. In this paper, we will deal with particular transformations, namely first of all, with order-preserving transformations on the natural linearly ordered set of natural numbers, secondly, with order-preserving injective transformations on the natural linearly ordered set of integers.

Semigroups of order-preserving transformations have been extensively studied in the recent years. In particular, O. Ganyushkin and V. Mazorchuk have studied the structure of semigroups of all injective order-preserving transformations on a finite chain [3]. We know also all maximal subsemigroups of the semigroup of all order-preserving transformations on a finite chain [7]. However, the case of infinite linearly ordered sets has been poorly studied.

Let X be an infinite set. We denote by $T(X)$ the monoid of all full transformations on X (under composition). In 1965, G. P. Gavrilov [4] showed that there are five maximal subsemigroups of $T(X)$ containing the symmetric group on X if X is countable and in 2005, M. Pinsker [6] extended Gavrilov's result to sets of arbitrary cardinality. J. East, J. D. Mitchell, and Y. Péresse classified all maximal subsemigroups of $T(X)$ containing one of following subgroups of the symmetric group: the pointwise stabilizer of a non-empty finite subset of X, the stabilizer of an ultrafilter on X, or the stabilizer of a partition of X into finitely many subsets of equal cardinality [2].

Assume that $(X,\leq)$ is a linearly ordered set. We say a transformation $\alpha \in T(X)$ is order-preserving if for all $x, y \in X$ the inequality $x \leq y$ implies $x\alpha \leq y\alpha$. We denote by $O(X)$ the submonoid of $T(X)$ of all order-preserving transformations on X . It seems extremely unlikely that a complete description in any sense of maximal subsemigroups of $O(X)$ exists. In particular in [1], V. Doroshenko considers semigroups of all transformations of the set of integers and natural numbers that preserve the natural order on them and their subsemigroups of cofinite transformations (i.e., transformations under which the complement of image is a finite set).

In this research, we determine the maximal subsemigroups of the semigroup of all injective order-preserving transformations on the natural numbers and integers, respectively. We determine also all maximal subsemigroups of the semigroup of all order-preserving transformations on the natural numbers containing a particular set. The paper is organized as follows: In Section 2, we give main definitions. All necessary definitions from the Semigroup Theory that are not mentioned in the paper, can be found e.g. in [5]. Further, we define the monoids $IO(\mathbb{N})$, $SO(\mathbb{Z})$, and $IO(\mathbb{Z})$. The maximal subsemigroups of these three monoids are characterized in Section 3, 4, and 5. In the last section, we consider a particular set $A_{\aleph_0}^{(1)} \subset O(\mathbb{N})$ and determine all maximal subsemigroups of $O(\mathbb{N})$ containing $A_{\aleph_0}^{(1)}$.

Let $(X,\leq)$ be a linearly ordered set. By $O(X)$, we denote the set of all transformations that preserve the order $\leq$, i.e., transformations $\alpha : X \to X$ such that for any $x, y \in X$, the inequality $x \leq y$ implies $x\alpha \leq y\alpha$. Then $O(X)$ is the submonoid of $T(X)$ that consists of all order-preserving transformations on X .

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of all natural numbers and integers, respectively, with natural linearly order relation on them. In what follows, X is understood as either $\mathbb{N}$ or $\mathbb{Z}$.

Let $Inj(X)$ denote the set of all injective transformations on X and let

$$IO(X) := Inj(X) \cap O(X) .$$

The set $im\,\alpha := \{x\alpha \mid x \in X\}$ is called the image of the transformation α . The cardinality of $im\,\alpha$ is called rank of α, in symbol $rank\,\alpha := |im\,\alpha|$. Note that any

$\alpha \in IO(\mathbb{N})$ is uniquely determined by $im\,\alpha$. Let $Sym(X)$ be the symmetric group on X and let

$$SO(X) := Sym(X) \cap O(X).$$

Clearly, $SO(\mathbb{N})$ contains only one element, namely the identity map $id_{\mathbb{N}}$ on $\mathbb{N}$. But $SO(\mathbb{Z})$ contains infinitely many elements, since for each $z \in \mathbb{Z}$ the mapping $l_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ defined by $xl_{\mathbb{Z}} = x + z$ is in $SO(\mathbb{Z})$.

Further, let $Sur(X)$ denote the set of all surjective transformations on X and let $SurO(X) := Sur(X) \cap O(X)$. The defect of a transformation α measures how far α is from a surjection, i.e., $d(\alpha) := |X \setminus im\,\alpha|$ is the defect of α. We denote

$$g_{\alpha}(x) := (x+1)\alpha - x\alpha$$

for any $x \in X$. The natural number $g_{\alpha}(x)$ is called the height or the jump of the transformation α at the point x. In particular, we have $g_{\alpha\beta}(x) \geq g_{\alpha}(x)$ for any $\beta \in IO(X)$. The relation $\ker\alpha := \{(x,y) | x, y \in X, x\alpha = y\alpha\}$ is an equivalence relation on X, called the kernel of α. This equivalence relation corresponds to a decomposition of X into equivalence classes (blocks). For convenience, in what follows we will write $B \in \ker\alpha$ if $B \subseteq X$ is an equivalence class of $\ker\alpha$. In particular, we have $\ker\alpha := \{x\alpha^{-1} | x \in im\,\alpha\}$, where $x\alpha^{-1} := \{y \in X \mid y\alpha = x\}$. A transversal of $\ker\alpha$ is a set $A \subseteq X$ with $|A \cap x\alpha^{-1}| = 1$ for all $x \in im\,\alpha$. More generally, a transversal of a decomposition $\mathcal{B}$ of X is a set $A \subseteq X$ with $|A \cap B| = 1$ for all $B \in \mathcal{B}$. We say that a set $A \subseteq X$ is a pseudo-transversal of the decomposition $\mathcal{B}$ of X if there is $B_1 \in \mathcal{B}$ with $B_1 \cap A = \varnothing$ and $|B \cap A| = 1$ for all $B \in \mathcal{B} \setminus \{B_1\}$. The collapse $c(\alpha)$ of α measures how far α is from an injection and it is defined by $c(\alpha) := |X \setminus A|$, where A is a transversal of $\ker\alpha$.

2. Maximal Subsemigroups of $IO(\mathbb{N})$

If we consider an order-preserving injection $\alpha $ on the natural numbers, then we can observe for any natural number x that $x\alpha \geq x$ and that $x\alpha > x$ implies $y\alpha > y$ for all natural numbers $y > x$. This motivates the consideration of the following set of order-preserving injections on $\mathbb{N}$. Let

$$IO(\mathbb{N},1) := \{\alpha \in IO(\mathbb{N}) \mid x\alpha - x = 1 \text{ for all } x \in \mathbb{N}\}.$$

In fact, if $\alpha \in IO(\mathbb{N},1)$ then there exists natural number k_α such that

$$x\alpha = \begin{cases} x & \text{for} \quad x < k_\alpha \\ x+1 & \text{for} \quad x \geq k_\alpha. \end{cases}$$

These transformations play a particular role.

Proposition 2.1 Let $\delta \in IO(\mathbb{N},1)$. Then $IO(\mathbb{N}) \setminus \{\delta\}$ is a subsemigroup of $IO(\mathbb{N})$.

Proof: We put $S := IO(\mathbb{N}) \setminus \{\delta\}$. Assume that S is not a subsemigroup of $IO(\mathbb{N})$. Then there are $\alpha, \beta \in S$ such that $\delta = \alpha\beta$. Let $x \in \mathbb{N}$. If $x < k_\delta$ then

$$x = x\delta = x\alpha\beta \geq x\alpha \geq x$$

and therefore

$$x\delta = x = x\alpha = x\beta .$$

Further, we have

$$k_\delta + 1 = k_\delta\delta = k_\delta\alpha\beta \geq k_\delta\alpha \geq k_\delta .$$

This implies $k_\delta\alpha = k_\delta + 1$ (and $k_\delta\beta = k_\delta$) or $k_\delta\alpha = k_\delta$ (and $k_\delta\beta = k_\delta + 1$).

In the first case, we have $k_\delta\alpha = k_\delta + 1$ and thus for all $x \geq k_\delta$, we obtain $x+1 = x\delta = x\alpha\beta \geq x\alpha > x$, i.e., $x\alpha = x+1$. This shows $\alpha = \delta$.

In the later case, we get $\beta = \delta$ by similar considerations. Hence, $\alpha = \delta$ or $\beta = \delta$, a contradiction. □

Proposition 2.2 Let S be a subsemigroup of $IO(\mathbb{N})$. Then S is maximal if and only if there is $\delta \in IO(\mathbb{N},1)$ such that $S = IO(\mathbb{N}) \setminus \{\delta\}$.

Proof: Suppose that S is a maximal subsemigroup of $IO(\mathbb{N})$.

Assume that $IO(\mathbb{N},1) \subseteq S$. Since S is a proper subsemigroup of $IO(\mathbb{N})$, there is $\alpha \in IO(\mathbb{N}) \setminus S$. Then there is a natural number k such that $x\alpha = x$ for $x < k$ and $x\alpha > x$ for $x \geq k$ (otherwise $x\alpha = x$ for all natural numbers x, i.e., $\alpha = id_{\mathbb{N}} \in IO(\mathbb{N},1) \subseteq S$, a contradiction). We put

$$p := k\alpha - k.$$

and define a transformation β on $\mathbb{N}$ by

$$x\beta := \begin{cases} x & \text{for} \quad x < k \\ x\alpha - p & \text{for} \quad x \geq k. \end{cases}$$

Clearly, β is an injection. Let $x, y \in \mathbb{N}$ with $x < y$.

If $x < y < k$ then $x\beta = x < y = y\beta$.

If $x < k < y$ then

$$x\beta = x < k = k\alpha - k\alpha + k = k\alpha - (k\alpha - k) = k\alpha - p \leq y\alpha - p = y\beta.$$

If $x \geq k$ then $x\beta = x\alpha - p \leq y\alpha - p = y\beta$.

This shows that β is order-preserving and thus $\beta \in IO(\mathbb{N})$. Because $\langle S, \alpha \rangle = IO(\mathbb{N})$ (since S is maximal), there are $\beta_1, \beta_2, ..., \beta_n \in S \cup \{\alpha\}$ such that $\beta = \beta_1 \beta_2 \cdots \beta_n$.

Assume that there is $i \in \{1, 2, ..., n\}$ with $\beta_i = \alpha$. Then $k\beta_1\beta_2 \cdots \beta_{i-1} \geq k$ and thus $k\beta_1\beta_2 \cdots \beta_{i-1}\beta_i \geq k\beta_i = k\alpha > k$. This implies

$$k\beta_1\beta_2 \cdots \beta_{i-1}\beta_i \cdots \beta_n > k\beta_i \cdots \beta_n \geq k,$$

i.e., $k\beta > k$, a contradiction (we have $k\beta = k$). Hence, $\beta \in S$.

We define a transformation γ on $\mathbb{N}$ by

$$x\gamma := \begin{cases} x & \text{for} \quad x < k \\ x+1 & \text{for} \quad x \geq k. \end{cases}$$

Clearly, $\gamma \in IO(\mathbb{N},1)$. Let x be a natural number.

If $x < k$ then $x\beta\gamma^p = x = x\alpha$.

If $x \geq k$ then $x\beta\gamma^p = (x\alpha - p)\gamma^p = x\alpha - p + p = x\alpha$.

This shows that $\alpha = \beta\gamma^p \in S$, a contradiction.

Hence $IO(\mathbb{N},1)$ is not a subset of S and there is $\delta \in S \setminus IO(\mathbb{N},1)$, i.e., $S \subseteq IO(\mathbb{N}) \setminus \{\delta\}$. This implies $S = IO(\mathbb{N}) \setminus \{\delta\}$ by the maximality of S.

The converse direction is due to Proposition 2.1. $\square$

3. Maximal Subsemigroups of the group $SO(\mathbb{Z})$

This section deals with the maximal subsemigroups of the group $SO(\mathbb{Z})$. It will turn out that $SO(\mathbb{Z})$ has countably infinitely many maximal subsemigroups. We observe that, if $\alpha \in SO(\mathbb{Z})$, then

$$x\alpha + 1 = (x+1)\alpha \text{ and } x\alpha - 1 = (x-1)\alpha$$

for all integers x. Indeed, we have $x\alpha + 1 \le (x+1)\alpha$. Assume that $x\alpha + 1 < (x+1)\alpha$. Then there is an integer y, different from both x and $x+1$, with $y\alpha = x\alpha + 1$, i.e., $x\alpha < y\alpha < (x+1)\alpha$. This provides $x < y < (x+1)$, i.e., $x = y$ or $x+1 = y$, a contradiction. In the same way, we check $x\alpha - 1 = (x-1)\alpha$. A successive application of the appropriate equation, starting with $x = 0$, provides $z\alpha = 0\alpha + z$ for all integers z. In this section, we will use this fact frequently. For each prime number p, let $p\mathbb{Z} := \{pz \mid z \in \mathbb{Z}\}$ and we define

$$S_p := \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha = p\mathbb{Z}\}.$$

Moreover, let

$$A_+ := \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \ge 0\} \text{ and } A_- := \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \le 0\}.$$

First of all, we observe that the just defined sets are closed under composition.

Lemma 3.1 S_p is maximal subsemigroup of $SO(\mathbb{Z})$ for each prime number p.

Proof: Let p be a prime number and let $\alpha, \beta \in S_p$. Then there are integers m and n such that $0\alpha = pm$ and $0\beta = pn$. We have

$$0\alpha\beta = (pm)\beta = pm + pn = p(m+n),$$

i.e., $\alpha\beta \in S_p$. This shows that S_p is semigroup.

Let $\alpha \in SO(\mathbb{Z}) \setminus S_p$. We will show that $\langle S_p, \alpha \rangle = SO(\mathbb{Z})$. Let $\beta \in SO(\mathbb{Z}) \setminus S_p$. Then there are integers $m_\alpha, m_\beta, n_\alpha$ and n_β with $0 < n_\alpha, n_\beta < p$ such that

$$0\alpha = m_\alpha p + n_\alpha \text{ and } 0\beta = m_\beta p + n_\beta.$$

Since p is a prime number and $0 < n_\alpha, n_\beta < p$, there is $k \in \mathbb{N}$ such that kn_α is congruent to n_β modulo p, i.e., there is integer q such that $kn_\alpha - n_\beta = qp$ and

$$0\alpha^k = k(m_\alpha p + n_\alpha) = m_\alpha kp + kn_\alpha = m_\alpha kp + qp + n_\beta = (m_\alpha k + q)p + n_\beta = rp + n_\beta$$

with $r := m_\alpha k + q$. Let $\gamma \in S_p$ with $0\gamma = (-r + m_\beta)p$. Then

$$0\alpha^k\gamma = rp + n_\beta + (-rp + m_\beta)p = m_\beta p + n_\beta = 0\beta.$$

This shows that $\beta \in \langle S_p, \alpha \rangle$. Consequently, S_p is maximal subsemigroup of $SO(\mathbb{Z})$.

□

Lemma 3.2 Both A_+ and A_- are maximal subsemigroups of $SO(\mathbb{Z})$.

Proof: First, we show that A_- is maximal subsemigroup of $SO(\mathbb{Z})$. Let $\alpha, \beta \in A_-$. Then there are non-positive integers m and l such that $0\alpha = m$ and $0\beta = l$. We have $0\alpha\beta = m\beta = m + l$, where $m + l$ is non-positive integer, i.e., $\alpha\beta \in A_-$. This shows that A_- is semigroup.

Let $\alpha \in SO(\mathbb{Z}) \setminus A_-$. We will show that $\langle A_-, \alpha \rangle = SO(\mathbb{Z})$. Let $\beta \in SO(\mathbb{Z}) \setminus A_-$. We observe that $0\alpha > 0$ as well as $0\beta > 0$ and there is natural number k such that $0\alpha^k = k(0\alpha) > 0\beta$ (if already $0\alpha > 0\beta$ then we choose $k = 1$), i.e., $0\beta - 0\alpha^k < 0$. Further, there is $\gamma \in A_-$ with $0\gamma = 0\beta - 0\alpha^k$ and we obtain $0\alpha^k\gamma = 0\alpha^k + 0\beta - 0\alpha^k = 0\beta$. This shows that $\beta = \alpha^k\gamma \in \langle A_-, \alpha \rangle$. Consequently, A_- is maximal subsemigroup of $SO(\mathbb{Z})$.

Next, we show that A_+ is maximal subsemigroup of $SO(\mathbb{Z})$. Let $\alpha, \beta \in A_+$. Then there are non-negative integers n and k such that $0\alpha = n$ and $0\beta = k$. We have $0\alpha\beta = n\beta = n + k$, where $n + k$ is non-negative integer, i.e., $\alpha\beta \in A_+$. This shows that A_+ is semigroup.

Let $\alpha \in SO(\mathbb{Z}) \setminus A_+$. We will show that $\langle A_+, \alpha \rangle = SO(\mathbb{Z})$. Let $\beta \in SO(\mathbb{Z}) \setminus A_+$. We observe that $0\alpha < 0$ as well as $0\beta < 0$ and there is natural number q such that $0\alpha^q = q(0\alpha) < 0\beta$ (if already $0\alpha > 0\beta$ then we choose $k = 1$), i.e., $0\beta - 0\alpha^k > 0$. Further, there is $\gamma \in A_+$ with $0\gamma = 0\beta - 0\alpha^k$ and we obtain $0\alpha^k\gamma = 0\alpha^k + 0\beta - 0\alpha^k = 0\beta$. This shows that $\beta = \alpha^k\gamma \in \langle A_+, \alpha \rangle$. Consequently, A_+ is maximal subsemigroup of $SO(\mathbb{Z})$. □

The following theorem states that we have actually found all maximal subsemigroups of $SO(\mathbb{Z})$.

Theorem 3.1 Let S be a subsemigroup of $SO(\mathbb{Z})$. Then the following statements are equivalent:

(1) S is a maximal subsemigroup of $SO(\mathbb{Z})$;

(2) $S = A_+$ or $S = A_-$ or there is a prime number p such that $S = S_p$.

Proof: (1)→(2): Suppose that S is different from both A_+ and A_-. First, we note that there is a least natural number q_+ such that there is $\alpha_+ \in S$ with $0\alpha_+ = q_+$. Otherwise, $0\alpha < 0$ for all $\alpha \in S$, i.e., $S \subseteq A_-$. By the maximality of S, we obtain $S = A_-$, a contradiction. By a dual argument, there is a least natural number q_- such that there is $\alpha_- \in S$ with $0\alpha_- = -q_-$.

Assume that $q_- < q_+$. Then $0\alpha_+\alpha_- = q_+\alpha_- = q_+ - q_-$. But $q_+ - q_-$ is a natural number less than q_+. This contradicts the minimality of q_+. Hence, $q_- \geq q_+$. Assume that $q_- > q_+$. Then $0\alpha_+\alpha_- = q_+\alpha_- = q_+ - q_- = -(q_- - q_+)$. But $q_- - q_+$ is a natural number less than q_-. This contradicts the minimality of q_-. Hence, $q_- \leq q_+$, i.e., $q_- = q_+ (=: q)$.

Let n be a natural number. Then we observe that $0\alpha_+^n = qn$ and $0\alpha_-^n = -qn = q(-n)$ as well as $0\alpha_+\alpha_- = q\alpha_- = q - q = 0$. This shows that

$$\{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \in q\mathbb{Z}\} \subseteq \langle \alpha_+, \alpha_- \rangle \subseteq S.$$

Assume that there is $\gamma \in S \setminus \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \in q\mathbb{Z}\}$. Let m be the greatest common divisor of q and 0γ. Then there is $\beta \in S$ with $0\beta = m < q$, which contradicts the minimality of q. This shows that $S = \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \in q\mathbb{Z}\}$. This excludes $q = 1$ (since $q = 1$ would mean that $S = \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \in q\mathbb{Z}\} = SO(\mathbb{Z})$.

Finally, assume that q is not a prime number. Then there are a prime number p and a natural number k such that $q = pk$. This provides that $S = \{\alpha \in SO(\mathbb{Z}) \mid 0\alpha \in q\mathbb{Z}\}$ is a proper subsemigroup of S_p, a contradiction to the maximality of S. Hence, $S = S_q$ and q is prime number.

(2) $\rightarrow$ (1): It is obvious by Lemma 3.1 and Lemma 3.2. $\square$

4. Maximal Subsemigroups of $IO(\mathbb{Z})$

In this section, we characterize the maximal subsemigroups of $IO(\mathbb{Z})$. It can be expected that several maximal semigroups of $IO(\mathbb{Z})$ are related to the maximal subsemigroups of the group $SO(\mathbb{Z})$. We will obtain only one maximal subsemigroup of $IO(\mathbb{Z})$ containing the group $SO(\mathbb{Z})$. For this, we consider the set of all proper order-preserving injections on $\mathbb{Z}$. Let

$$IO_1(\mathbb{Z}) := IO(\mathbb{Z}) \setminus SO(\mathbb{Z})$$

be the set of all proper order-preserving injections on $\mathbb{Z}$. It is well known that proper injections on a given set form an ideal of the semigroup of all injections on this set. Restricting our attention to the monoid of all order-preserving transformations on $\mathbb{Z}$, we conclude that $IO_1(\mathbb{Z})$ is an ideal of $IO(\mathbb{Z})$.

The subsemigroups $IO_1(\mathbb{Z}) \setminus S_l$ and $IO_1(\mathbb{Z}) \setminus S_r$ of $IO_1(\mathbb{Z})$ where

$S_l := \{\alpha \in IO_1(\mathbb{Z}) \mid$ there is integer z with $g_\alpha(x) = 1$ for all $x \leq z\}$ and

$S_r := \{\alpha \in IO_1(\mathbb{Z}) \mid$ there is integer z with $g_\alpha(x) = 1$ for all $x \geq z\}$

are of particular interest.

Lemma 4.1 $IO_1(\mathbb{Z}) \setminus S_l$ and $IO_1(\mathbb{Z}) \setminus S_r$ are ideals of $IO_1(\mathbb{Z})$.

Proof: We will show that $IO_1(\mathbb{Z}) \setminus S_l$ is an ideal. The second statement can be proved similarly. Note that $\alpha \in IO_1(\mathbb{Z}) \setminus S_l$ if and only if for all integers z there is an integer $x < z$ with $g_\alpha(x) \neq 1$. Let $\alpha \in IO_1(\mathbb{Z}) \setminus S_l$ and let $\beta \in IO_1(\mathbb{Z})$. Further let z be an integer. Then there is $x < z$ with $(x+1)\alpha - x\alpha \geq 2$ and we have

$$(x+1)\alpha\beta - x\alpha\beta \geq (x\alpha + 2)\beta - x\alpha\beta \geq 2.$$

This shows that $\alpha\beta \in IO_1(\mathbb{Z}) \setminus S_l$.

Let $q := z\beta$. Then there is $y < q$ such that $(y+1)\alpha - y\alpha \geq 2$. Further, there is an integer k with $k\beta \leq y$ and $(k+1)\beta \geq y+1$, i.e.,

$$(k+1)\beta\alpha - k\beta\alpha \geq (y+1)\alpha - y\alpha \geq 2.$$

We notice that $k < z$ since $k\beta \le y < q = z\beta$ and β is injective. This shows that $\beta\alpha \in IO_1(\mathbb{Z}) \setminus S_l$.

The assertion follows. □

Let

$$S_1 := \{\alpha \in IO_1(\mathbb{Z}) \mid d(\alpha) = 1\}.$$

We notice that any $\alpha \in IO(\mathbb{Z})$ belongs to the set S_1 if and only if there exists an integer z with $X \setminus \{z\alpha + 1\} = im\,\alpha$, i.e.,

$$g_\alpha(z) = (z+1)\alpha - z\alpha = z\alpha + 2 - z\alpha = 2$$

and $$g_\alpha(x) = (x+1)\alpha - x\alpha = x\alpha + 1 - x\alpha = 1$$

for all $x < z$ as well as for all $x > z$. This shows that $S_1 \subseteq S_l \cap S_r$ and, in particular, that S_1 is the set of all $\alpha \in S_l \cap S_r$ such that there is an integer z with $g_\alpha(z) = 2$ and $g_\alpha(x) = 1$ for all integers x different from z. Let $\alpha \in S_1$. Then it is easy to verify that $\theta_1\alpha\theta_2 \in S_1$ for all $\theta_1, \theta_2 \in SO(\mathbb{Z})$. Since $\alpha \in S_1$, there is an integer a with $g_\alpha(a) = 2$. We put $a_1 = a\alpha$. Let $\beta \in S_1$. Then there is an integer b with $g_\alpha(b) = 2$ and we define order-preserving bijections θ_1, θ_2 on $\mathbb{Z}$ by

$$\theta_1 := l_{(a-b)} \text{ and } \theta_2 := l_{(b\beta - a_1)}.$$

Note that both θ_1 and θ_2 are well defined and belong to $SO(\mathbb{Z})$. Then we have

$$(b+1)\theta_1\alpha\theta_2 = (a+1)\alpha\theta_2 = (a+2)\theta_2 = b\beta + 2$$

and $$b\theta_1\alpha\theta_2 = a\alpha\theta_2 = a_1\theta_2 = b\beta,$$

i.e., $g_{\theta_1\alpha\theta_2}(b) = 2$, where $b\theta_1\alpha\theta_2 = b\beta$. This shows that $\beta^a\, \theta_1\alpha\theta_2$. The foregoing argument proves the following lemma:

Lemma 4.2 For all $\alpha \in S_1$, we have $S_1 = \{\theta_1\alpha\theta_2 \mid \theta_1, \theta_2 \in SO(\mathbb{Z})\}$.

Moreover, we observe the following technical fact:

Lemma 4.3 $S_l \cap S_r \subseteq \langle S_1 \rangle$.

Proof: Let $\gamma \in S_l \cap S_r$ and let us put $A(\gamma) := \{a \in \mathbb{Z} \mid g_\gamma(a) \neq 1\}$. Since $\gamma \in S_l \cap S_r$, the set $A(\gamma)$ is finite and

$$\Gamma(\gamma) := \sum_{a \in A(\gamma)} g_\gamma(a)$$

is a natural number.

If $\Gamma(\gamma) = 2$ then $\gamma \in S_1$, and so $\gamma \in \langle S_1 \rangle$.

Suppose that $\gamma \in \langle S_1 \rangle$ for all $\gamma \in S_l \cap S_r$ with $\Gamma(\gamma) = k$ for some natural number $k \geq 2$.

Let $\gamma \in S_l \cap S_r$ with $\Gamma(\gamma) = k+1$. Then there is integer a with $g_\gamma(a) > 1$. We define a mapping $\beta : \mathbb{Z} \to \mathbb{Z}$ by

$$x\beta := \begin{cases} x\gamma & \text{for} \quad x \leq a \\ x\gamma - 1 & \text{for} \quad x > a. \end{cases}$$

Clearly, $\beta \in S_l \cap S_r$. Then

$$(x+1)\beta - x\beta = (x+1)\gamma - x\gamma = g_\gamma(x) \text{ for all } x < a,$$

$$(a+1)\beta - a\beta = (a+1)\gamma - 1 - a\gamma = g_\gamma(a) - 1, \text{ and}$$

$$(x+1)\beta - x\beta = (x+1)\gamma - 1 - (x\gamma - 1) = (x+1)\gamma - x\gamma = g_\gamma(x) \text{ for } x > a.$$

This shows that $\Gamma(\beta) = \Gamma(\gamma) - 1 = k$, i.e., $\beta \in \langle S_1 \rangle$. Now we define a transformation α on $\mathbb{Z}$ by

$$x\alpha := \begin{cases} x & \text{for} \quad x \leq (a+1)\beta - 1 \\ x+1 & \text{for} \quad x > (a+1)\beta - 1. \end{cases}$$

Clearly, $\alpha \in S_1$. Then we have

$$x\beta\alpha = x\gamma\alpha = x\gamma \text{ for } x \leq a$$

and $$x\beta\alpha = x\beta + 1 = x\gamma - 1 + 1 = x\gamma \text{ for } x > a.$$

This shows that $\gamma = \beta\alpha \in \langle S_1 \rangle$. Altogether, we have proved that $S_l \cap S_r \subseteq \langle S_1 \rangle$.

□

We will show that $IO(\mathbb{Z}) \setminus S_1$ is a maximal subsemigroup of $IO(\mathbb{Z})$.

Lemma 4.4 $IO(\mathbb{Z}) \setminus S_1$ is a maximal subsemigroup of $IO(\mathbb{Z})$.

Proof: Let $\alpha, \beta \in IO(\mathbb{Z}) \setminus S_1$ and assume that $\alpha\beta \in S_1$. It is well known that $d(\alpha\beta) \geq d(\beta)$, i.e., $1 \geq d(\beta)$, which implies $\beta \in SO(\mathbb{Z})$. Note that the identity map $id_{\mathbb{Z}}$ on $\mathbb{Z}$ belongs to $SO(\mathbb{Z})$. Then $\alpha = id_{\mathbb{Z}}\alpha\beta\beta^{-1} = id_{\mathbb{Z}}(\alpha\beta)\beta^{-1} \in S_1$ by Lemma 4.2, a contradiction. This shows that $IO(\mathbb{Z}) \setminus S_1$ is closed under composition, i.e., $IO(\mathbb{Z}) \setminus S_1$ is a semigroup. Now we show the maximality.

Let $\alpha \in S_1$. Then $S_1 \subseteq \langle SO(\mathbb{Z}), \alpha \rangle$ by Lemma 4.2 and since $SO(\mathbb{Z}) \subseteq IO(\mathbb{Z}) \setminus S_1$, we have $S_1 \subseteq \langle IO(\mathbb{Z}) \setminus S_1, \alpha \rangle$. So $IO(\mathbb{Z}) = \langle IO(\mathbb{Z}) \setminus S_1, \alpha \rangle$. This shows that $IO(\mathbb{Z}) \setminus S_1$ is a maximal subsemigroup of $IO(\mathbb{Z})$. □

Lemma 4.5 Let S be a maximal subsemigroup of $IO(\mathbb{Z})$ with $IO(\mathbb{Z}) \setminus S_l \subseteq S$ or $IO(\mathbb{Z}) \setminus S_r \subseteq S$. Then $S = IO(\mathbb{Z}) \setminus S_1$.

Proof: Suppose that $IO(\mathbb{Z}) \setminus S_l \subseteq S$. Let $\gamma \in S_l \setminus S$.

Assume that $\gamma \notin S_r$. Then there is an infinite chain

$$\{a_0 < a_1 < \cdots < a_n < \cdots\} = \{a \in \mathbb{Z} : g_\gamma(a) \neq 1\}$$

and we define mappings $\alpha, \beta \in IO(\mathbb{Z})$ inductively by $a_0\alpha := a_0\gamma$, $(a_0\gamma)\beta := a_0\gamma$,

$$g_\alpha(a) = \begin{cases} g_\gamma(a_i) & \text{for } a = a_i \quad \text{with } i \in 2\mathbb{N} \cup \{0\} \\ 1 & \text{otherwise} \end{cases}$$

and

$$g_\beta(a) = \begin{cases} g_\gamma(a_i) & \text{for } a = a_i\alpha \quad \text{with } i \in 2\mathbb{N} - 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then we have $a_0\alpha\beta = a_0\gamma\beta = a_0\gamma$. Further for $i \in 2\mathbb{N} \cup \{0\}$, we obtain

$$(a_i + 1)\alpha\beta = (a_i\alpha + g_\gamma(a_i))\beta = a_i\alpha\beta + g_\gamma(a_i) \text{ and thus}$$

$$g_{\alpha\beta}(a_i)=(a_i+1)\alpha\beta-a_i\alpha\beta=a_i\alpha\beta+g_\gamma(a_i)-a_i\alpha\beta=g_\gamma(a_i).$$

For $i\in 2\mathbb{N}-1$, we calculate

$$(a_i+1)\alpha\beta=(a_i\alpha+1)\beta=a_i\alpha\beta+g_\gamma(a_i) \text{ and hence}$$

$$g_{\alpha\beta}(a_i)=a_i\alpha\beta+g_\gamma(a_i)-a_i\alpha\beta=g_\gamma(a_i).$$

Moreover, for an integer a with $g_\gamma(a)=1$, we get $(a+1)\alpha\beta=(a\alpha+1)\beta=a\alpha\beta+1$ and thus $g_{\alpha\beta}(a)=a\alpha\beta+1-a\alpha\beta=1=g_\gamma(a)$. Altogether, this shows that $\gamma=\alpha\beta$.

Assume that there are $\rho_1,\rho_2\in IO(\mathbb{Z})$ such that $\alpha=\rho_1\gamma\rho_2$. We put $z:=a_0\rho_1$. Then $z\le a_0$. Otherwise, there is $z_0<a_0$ with $z_0\rho_1\le a_0$ and $(z_0+1)\rho_1>a_0$. Since $g_\alpha(a_0)\ge 2$, we get $g_{\rho_1\gamma}(z_0)\ge 2$ and thus $1=g_\alpha(z_0)=g_{\rho_1\gamma\rho_2}(z_0)\ge 2$, a contradiction. Let $c:=a_0-z$. By the recursive construction of α, we can easy verify that

$$\sum_{x=a_0}^{c+a_{2c+c}} g_\alpha(x)<\sum_{x=z}^{a_{2c+c}} g_\gamma(x).$$

From $a_0\rho_1=z$, i.e., $a_0\rho_1\gamma=z\gamma$, it follows that $g_{\rho_1\gamma}(a_0+r)\ge g_\gamma(z+r)$ for all integers $r\ge 0$. This provides $\sum_{x=a_0}^{c+a_{2c+2}} g_{\rho_1\gamma}(x)\ge\sum_{x=z}^{a_{2c+2}} g_\gamma(x)$. Because of $\alpha=\rho_1\gamma\rho_2$, we have $\sum_{x=a_0}^{c+a_{2c+2}} g_\alpha(x)\ge\sum_{x=a_0}^{c+a_{2c+2}} g_{\rho_1\gamma}(x)$. Altogether, we obtain

$$\sum_{x=a_0}^{c+a_{2c+2}} g_\alpha(x)\ge\sum_{x=a_0}^{c+a_{2c+2}} g_{\rho_1\gamma}(x)\ge\sum_{x=z}^{a_{2c+2}} g_\gamma(x)>\sum_{x=a_0}^{c+a_{2c+2}} g_\alpha(x),$$

a contradiction.

Hence $\alpha\notin\langle S,\gamma\rangle$ and thus $\alpha\in S$ since S is maximal and $\gamma\notin S$. With similar arguments, one can show that $\beta\in S$. Thus $\gamma=\alpha\beta\in S$, a contradiction.

Hence $\gamma\in S_l\cap S_r$. Then $\gamma\in\langle S_1\rangle$ by Lemma 4.3. Thus, since $S_1\subseteq\langle SO(\mathbb{Z}),\alpha\rangle$ for all $\alpha\in S_1$ (see Lemma 4.2), $SO(\mathbb{Z})\subseteq S$, and $\gamma\notin S$, we have $S_1\cap S=\varnothing$. Using the maximality of S, from $IO(\mathbb{Z})\setminus S_1\supseteq S$ it follows $IO(\mathbb{Z})\setminus S_1=S$.

If $IO(\mathbb{Z})\setminus S_r \subseteq S$ then $IO(\mathbb{Z})\setminus S_1 = S$ can be proved by dual arguments.

□

Now we are ready to prove the main result of this section, the characterization of all maximal subsemigroups of $IO(\mathbb{Z})$.

Theorem 4.1 Let $S \subseteq IO(\mathbb{Z})$. Then the following statements are equivalent:

(1) S is a maximal subsemigroup of $IO(\mathbb{Z})$;

(2) $S = IO(\mathbb{Z})\setminus S_1$ or $S = IO_1(\mathbb{Z})\cup A_+$ or $S = IO_1(\mathbb{Z})\cup A_-$ or there is a prime number p such that $S = IO_1(\mathbb{Z})\cup S_p$.

Proof: (1) $\rightarrow$ (2): Let S be a maximal subsemigroup of $IO(\mathbb{Z})$.

Suppose that $SO(\mathbb{Z}) \nsubseteq S$. Since $IO_1(\mathbb{Z})$ is an ideal of $IO(\mathbb{Z})$ and since S is maximal, we conclude that $IO_1(\mathbb{Z})\subseteq S$ and $S\cap SO(\mathbb{Z})$ has to be a maximal subsemigroup of $SO(\mathbb{Z})$. Thus, by Theorem 3.1, we have $S = IO_1(\mathbb{Z})\cup A_+$ or $S = IO_1(\mathbb{Z})\cup A_-$ or there is a prime number p such that $S = IO_1(\mathbb{Z})\cup S_p$.

Suppose that $SO(\mathbb{Z})\subseteq S$.

Assume that $S_l \subseteq S$ and $S_r \subseteq S$. Let $\alpha \in IO_1(\mathbb{Z})$. Then the map $\beta_r : \mathbb{Z}\rightarrow\mathbb{Z}$ defined by

$$x\beta_r := \begin{cases} x\alpha & \text{for} \quad x \le 0 \\ x + 0\alpha & \text{for} \quad x > 0 \end{cases}$$

belongs to $S_r \cup SO(\mathbb{Z})$. Further, we define a map $\beta_l : \mathbb{Z}\rightarrow\mathbb{Z}$ by

$$x\beta_l := \begin{cases} x & \text{for} \quad x \le 0\alpha \\ (x - 0\alpha)\alpha & \text{for} \quad x > 0\alpha. \end{cases}$$

Clearly, $\beta_l \in S_l \cup SO(\mathbb{Z})$. For $x \le 0$, we can calculate $x\beta_r\beta_l = x\alpha\beta_l = x\alpha$ (since $x\alpha \le 0\alpha$) and for $x > 0$, we have

$$x\beta_r\beta_l = (x + 0\alpha)\beta_l = (x + 0\alpha - 0\alpha)\alpha = x\alpha .$$

Hence $\alpha = \beta_r \beta_l \in S_l \cup S_r \cup SO(\mathbb{Z}) \subseteq S$. This shows that $IO_1(\mathbb{Z}) \subseteq S$, i.e., $IO_1(\mathbb{Z}) \cup SO(\mathbb{Z}) \subseteq S$ and thus $IO(\mathbb{Z}) = S$, a contradiction.

Hence $S_l \not\subseteq S$ or $S_r \not\subseteq S$. Since both $IO_1(\mathbb{Z}) \setminus S_l$ and $IO_1(\mathbb{Z}) \setminus S_r$ are ideals of $IO_1(\mathbb{Z})$ and $SO(\mathbb{Z}) \subseteq S$, the maximality of S implies $IO(\mathbb{Z}) \setminus S_l \subseteq S$ or $IO(\mathbb{Z}) \setminus S_r \subseteq S$. Then by Lemma 4.5, we obtain $S = IO(\mathbb{Z}) \setminus S_1$.

(2) $\rightarrow$ (1): $IO(\mathbb{Z})$ is the disjoint union of $IO_1(\mathbb{Z})$ and $SO(\mathbb{Z})$. Since $IO_1(\mathbb{Z})$ is an ideal of $IO(\mathbb{Z})$, we can conclude that $S = IO_1(\mathbb{Z}) \cup A_+$, $S = IO_1(\mathbb{Z}) \cup A_-$, and $S = IO_1(\mathbb{Z}) \cup S_p$ (for any prime number p) are maximal subsemigroups of $IO(\mathbb{Z})$ due to Theorem 4.1. Finally, $IO(\mathbb{Z}) \setminus S_1$ is a maximal subsemigroup of $IO(\mathbb{Z})$ by Lemma 4.4. $\square$

5. Maximal Subsemigroups of $O(\mathbb{N})$

The characterization of all maximal subsemigroups of $O(\mathbb{N})$ is still an open problem. In this section, we determine all maximal subsemigroups of $O(\mathbb{N})$ containing a certain set. For $\alpha \in O(\mathbb{N})$ with $\operatorname{rank} \alpha = \aleph_0$, let us fix the following sets:

$$G_\alpha := \{g_\alpha(n) : n \in \mathbb{N}\} \text{ and}$$

$$K_\alpha := \{|B| : B \in \ker \alpha\}.$$

For $A, B \subseteq \mathbb{N}$, we will write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. Let $\alpha \in O(\mathbb{N})$ have infinite rank. We will identify $\ker \alpha$ with the partition that it induces and write

$$\ker \alpha = \{A_1 < A_2 < \cdots < A_i < \cdots\},$$

$$\operatorname{im} \alpha = \{a_1 < a_2 < \cdots < a_i < \cdots\},$$

where $A_i = a_i \alpha^{-1}$ for every $i \in \mathbb{N}$. For any set $N \subseteq \mathbb{N}$, we write $\max N = \infty$, if for all $n_0 \in N$ there is an element $n \in N$ with $n > n_0$; otherwise $\max N$ is an element of N, i.e., $\max N \neq \infty$. Then

$$\max G_\alpha = \max\{a_{i+1} - a_i : i \in \mathbb{N}\},$$

$$\max K_\alpha = \max\{|A_i| : i \in \mathbb{N}\}.$$

Lemma 5.1 Let $\alpha, \beta \in O(\mathbb{N})$ have infinite rank. Then the following statements are true:

(1) $\max G_{\alpha\beta} \neq \infty$ implies $\max G_\beta \neq \infty.$

(2) $\max G_{\alpha\beta} \neq \infty$ and $\max K_\beta \neq \infty$ implies $\max G_\alpha \neq \infty.$

(3) $\max K_\alpha \neq \infty$ and $\max K_\beta \neq \infty$ implies $\max K_{\alpha\beta} \neq \infty.$

(4) $\max K_{\alpha\beta} = \infty$ and $\max K_\beta \neq \infty$ implies $\max K_\alpha = \infty$.

Proof: (1) Suppose that $\max G_{\alpha\beta} \neq \infty$, i.e., there is $a \in \mathbb{N}$ with $a = \max G_{\alpha\beta}$. Assume that $\max G_\beta = \infty$. Then there is $x \in \mathbb{N}$ with $g_\beta(x+1) \geq a+1$ and we have

$$y\alpha \le x \text{ and } (y+1)\alpha \ge x+1$$

for some $y \in \mathbb{N}$. Thus

$$(y+1)\alpha\beta - y\alpha\beta \ge (x+1)\beta - x\beta = g_\beta(x) \ge a+1,$$

a contradiction with $a = \max G_{\alpha\beta}$.

(2) Suppose that $\max G_{\alpha\beta} \ne \infty$ and $\max K_\beta \ne \infty$, i.e., there are $a, b \in \mathbb{N}$ with $a = \max G_{\alpha\beta}$ and $b = \max K_\beta$. Assume that $\max G_\alpha = \infty$. Thenthere exists $x \in \mathbb{N}$ with $g_\alpha(x) \ge a(b+1)+1$ and we obtain

$$(x+1)\alpha\beta - x\alpha\beta \ge (x\alpha + a(b+1)+1)\beta - x\alpha\beta \ge a+1$$

since $\left|\{z\beta : x\alpha \le z \le x\alpha + a(b+1)+1)\}\right| \ge a+1$.

(3) Suppose that $\max K_\alpha \ne \infty$ and $\max K_\beta \ne \infty$. Then there are $b, c \in \mathbb{N}$ with $b = \max K_\beta$ and $c = \max K_\alpha$. For any $x \in \operatorname{im} \alpha\beta$, we have $\left|x\beta^{-1}\right| \le b$ and $\left|y\alpha^{-1}\right| \le c$ for all $y \in x\beta^{-1}$, i.e.,$\left|x(\alpha\beta)^{-1}\right| \le b \cdot c$. Thus $\max K_{\alpha\beta} \le b \cdot c$, i.e., $K_{\alpha\beta} \ne \infty$.

(4) Assume that $\max K_\alpha \ne \infty$. Then $\max K_{\alpha\beta} \ne \infty$ by (3), a contradiction.

□

Then we put

$$S_{\aleph_0} := \{\alpha \in O(\mathbb{N}) : \operatorname{rank} \alpha = \aleph_0, \max K_\alpha = \infty, \text{ and } \max G_\alpha \ne \infty\}.$$

Lemma 5.2 $O(\mathbb{N}) \setminus S_{\aleph_0}$ is a semigroup.

Proof: We will use all the items (1), (2), and (3) of Lemma 5.1. Let $\alpha, \beta \in O(\mathbb{N}) \setminus S_{\aleph_0}$. Assume that $\alpha\beta \in S_{\aleph_0}$. Since $\operatorname{rank} \alpha\beta \le \operatorname{rank} \alpha, \operatorname{rank} \beta$, so $\operatorname{rank} \alpha = \operatorname{rank} \beta = \aleph_0$. Further, $\max G_\beta \ne \infty$ since $\max G_{\alpha\beta} \ne \infty$. Then $\max K_\beta \ne \infty$ because $\beta \notin S_{\aleph_0}$. Since $\max G_{\alpha\beta} \ne \infty$ and $\max K_\beta \ne \infty$, we conclude $\max G_\alpha \ne \infty$. But because $\alpha \notin S_{\aleph_0}$, we get $\max K_\alpha \ne \infty$. But $\max K_\alpha \ne \infty$ and $\max K_\beta \ne \infty$ implies $\max K_{\alpha\beta} \ne \infty$ (see Lemma 5.1 (3)), i.e., $\alpha\beta \notin S_{\aleph_0}$, a contradiction. □

Lemma 5.3 $O(\mathbb{N})\setminus S_{\aleph_0}$ is a maximal subsemigroup of $O(\mathbb{N})$.

Proof: Let $\alpha,\beta \in S_{\aleph_0}$. Suppose that $\operatorname{im}\alpha = \{a_1 < a_2 < \cdots < a_i < \cdots\}$. Then there is a set $A := \{\tilde{a}_1 < \tilde{a}_2 < \cdots < \tilde{a}_i < \cdots\}$ of natural numbers with $\tilde{a}_i\alpha = a_i$ for $i \in \mathbb{N}$, i.e., A is transversal of $\ker\alpha$. Further, there is a decomposition $A_1 < A_2 < \cdots < A_i < \cdots$ of $\mathbb{N}$ which is the kernel of β. Then we define a transformation γ on $\mathbb{N}$ by

$$x\gamma := \tilde{a}_i \text{ whenever } x \in A_i, i \in \mathbb{N}.$$

Clearly, $\gamma \in O(\mathbb{N})$ and $\max G_\gamma = \infty$ (since $\max K_\alpha = \infty$). Thus $\gamma \notin S_{\aleph_0}$. Further, there is decomposition $B_1 < B_2 < \cdots < B_i < \cdots$ of $\mathbb{N}$ such that $a_i \in B_i$ for $i \in \mathbb{N}$. Then we define a transformation δ on $\mathbb{N}$ by

$$x\delta := b_i \text{ whenever } x \in B_i$$

where $b_i\beta^{-1} = A_i$ for $i \in \mathbb{N}$. Clearly, $\delta \in O(\mathbb{N})$ and $\max K_\delta \neq \infty$ since $\max G_\alpha \neq \infty$. Thus $\delta \notin S_{\aleph_0}$.

Let $x \in \mathbb{N}$. Then there is $i \in \mathbb{N}$ such that $x \in A_i$ and $x\gamma\alpha\delta = \tilde{a}_i\alpha\delta = a_i\delta = b_i = x\beta$. This shows that $\beta = \gamma\alpha\delta \in \langle O(\mathbb{N})\setminus S_{\aleph_0}, \alpha\rangle$ and consequently, we have proved that $O(\mathbb{N})\setminus S_{\aleph_0}$ is maximal. □

Since the product of two transformations is injective only if the first one is an injection and since the product of two transformations is surjective only if the second factor is surjective, we conclude that both $O(\mathbb{N})\setminus IO(\mathbb{N})$ and $O(\mathbb{N})\setminus SurO(\mathbb{N})$ are semigroups. Let us mention that the identity map $id_{\mathbb{N}}$ is the only bijection in $O(\mathbb{N})$. In particular, both $(O(\mathbb{N})\setminus IO(\mathbb{N}))\cup\{id_{\mathbb{N}}\}$ and $(O(\mathbb{N}\setminus SurO(\mathbb{N}))\cup\{id_{\mathbb{N}}\}$ are subsemigroups of $O(\mathbb{N})$.

Lemma 5.4 $(O(\mathbb{N})\setminus IO(\mathbb{N}))\cup\{id_{\mathbb{N}}\}$ and $(O(\mathbb{N})\setminus SurO(\mathbb{N}))\cup\{id_{\mathbb{N}}\}$ are maximal subsemigroups of $O(\mathbb{N})$.

Proof: Let $\alpha,\beta \in IO(\mathbb{N})\setminus\{id_{\mathbb{N}}\}$. Suppose that $\operatorname{im}\alpha = \{a_1 < a_2 < \cdots < a_i < \cdots\}$ and let

$$\mathcal{B} := \{B_1 < B_2 < \cdots < B_i < \cdots\}$$

be a decomposition of $\mathbb{N}$ such that $a_i \in B_i$ for $i \in \mathbb{N}$. Then $\text{im } \alpha$ is transversal of $\mathcal{B}$. Since $\alpha \in IO(\mathbb{N}) \setminus \{id_{\mathbb{N}}\}$, we know that $\alpha \notin SurO(\mathbb{N})$ and thus $\mathbb{N} \setminus \text{im } \alpha \neq \emptyset$. Hence there is $j \in \mathbb{N}$ with $|B_j| \geq 2$. Let δ be the transformation on $\mathbb{N}$ defined by $x\delta := b_i$ whenever $x \in B_i$ for $i \in \mathbb{N}$ where $\text{im } \beta = \{b_1 < b_2 < \cdots < b_i < \cdots\}$. Then it is easy to see that $\delta \in O(\mathbb{N}) \setminus IO(\mathbb{N})$. For $x \in \mathbb{N}$, we have $x\alpha = a_x$ and $x\beta = b_x$. Hence $x\alpha\delta = a_x\delta = b_x = x\beta$, i.e., $\beta = \alpha\delta \in \langle O(\mathbb{N}) \setminus IO(\mathbb{N}), \alpha \rangle$. This shows that $(O(\mathbb{N}) \setminus IO(\mathbb{N})) \cup \{id_{\mathbb{N}}\}$ is maximal.

Let $\alpha, \beta \in SurO(\mathbb{N}) \setminus \{id_{\mathbb{N}}\}$ and let $A := \{a_1 < a_2 < \cdots < a_i < \cdots\}$ such that $a_i \in A_i$ for $i \in \mathbb{N}$ with $\{A_1 < A_2 < \cdots < A_i < \cdots\}$ being the kernel of α. Since $\alpha \in SurO(\mathbb{N}) \setminus \{id_{\mathbb{N}}\}$, we know that $\alpha \notin IO(\mathbb{N})$ and there is $j \in \mathbb{N}$ such that $|A_j| \geq 2$. Let γ be the transformation on $\mathbb{N}$ defined by $x\gamma := a_i$, whenever $x \in C_i$ for $i \in \mathbb{N}$, where $\{C_1 < C_2 < \cdots < C_i < \cdots\}$ is the kernel of β. Then it is easy to see that $\gamma \in O(\mathbb{N}) \setminus SurO(\mathbb{N})$. For $x \in \mathbb{N}$, there is $i \in \mathbb{N}$ such that $x \in C_i$ and we have $i\alpha^{-1} = A_i$, $i\beta^{-1} = C_i$ and thus

$$x\gamma\alpha = a_i\alpha = i = x\beta, \text{ i.e., } \beta = \gamma\alpha \in \langle O(\mathbb{N}) \setminus SurO(\mathbb{N}), \alpha \rangle.$$

This shows that $(O(\mathbb{N}) \setminus SurO(\mathbb{N})) \cup \{id_{\mathbb{N}}\}$ is maximal. □

Now we investigate the subsemigroups of $O(\mathbb{N})$ containing the set

$$K(\aleph_0) := \{\alpha \in O(\mathbb{N}) : \text{rank } \alpha = \aleph_0\}.$$

Lemma 5.5 Any subsemigroup S of $O(\mathbb{N})$ with $K(\aleph_0) \subseteq S$ is not maximal.

Proof: Let S be a subsemigroup of $O(\mathbb{N})$ with $K(\aleph_0) \subseteq S$ and assume that S is maximal. Then there is $\alpha \in O(\mathbb{N}) \setminus S$ such that $\text{rank } \alpha = n$ for some $n \in \mathbb{N}$. Let $\beta \in O(\mathbb{N})$ with $\text{rank } \beta = n+1$. Suppose that $\text{im } \alpha = \{a_1 < a_2 < \cdots < a_n\}$ and $\text{im } \beta = \{b_1 < b_2 < \cdots < b_n < b_{n+1}\}$. We choose $\tilde{b}_i \in b_i\beta^{-1}$ for $1 \leq i \leq n+1$. Note that $b \in b_{n+1}\beta^{-1}$ for $b > \tilde{b}_{n+1}$. Let γ be the transformation on $\mathbb{N}$ defined by

$$x\gamma := \begin{cases} \tilde{b}_i & \text{if} \quad x \in a_i\alpha^{-1} \text{ for } 1 \le i < n \\ x + \tilde{b}_{n+1} & \text{if} \quad x \in a_n\alpha^{-1}\% \end{cases}$$

and let δ be the transformation on $\mathbb{N}$ defined by

$$x\delta := \begin{cases} a_1 & \text{if} \quad x < b_1 \\ a_i & \text{if} \quad b_i \le x < b_{i+1} \qquad \text{for } 1 \le i < n-1 \\ a_{n-1} & \text{if} \quad b_{n-1} \le x < b_{n+1} \\ a_n & \text{if} \quad x = b_{n+1} \\ x + a_n & \text{if} \quad x > b_{n+1}. \end{cases}$$

It is easy to verify that $\gamma, \delta \in K(\aleph_0)$. Let $x \in \mathbb{N}$. Then there is $1 \le i \le n$ such that $x \in a_i\alpha^{-1}$. If $i < n$ then we have

$$x\gamma\beta\delta = \tilde{b}_i\beta\delta = b_i\delta = a_i = x\alpha$$

and in the case $i = n$ we can calculate

$$x\gamma\beta\delta = (x + \tilde{b}_{n+1})\beta\delta = b_{n+1}\delta = a_n = x\alpha .$$

This shows that $\alpha = \gamma\beta\delta$ and we conclude that $\beta \notin S$ (since $\alpha \notin S$). Since $J(n) := \{\varphi \in O(\mathbb{N}) : \text{rank } \varphi \le n\}$ is an ideal of $O(\mathbb{N})$, the maximality of S implies $S \cup J(n) = O(\mathbb{N})$. This is a contradiction since $\beta \notin S$ and $\beta \notin J(n)$. □

Corollary 5.1 If S is a maximal subsemigroup of $O(\mathbb{N})$ then

$$J(\aleph_0) := \{\beta \in O(\mathbb{N}) : \text{rank } \beta < \aleph_0\} \subseteq S .$$

Proof: Suppose that S is maximal and $J(\aleph_0) \not\subseteq S$. Then, since $J(\aleph_0)$ is an ideal of $O(\mathbb{N})$, $S \cup J(\aleph_0) = O(\mathbb{N})$. Since $K(\aleph_0) \cap J(\aleph_0) = \varnothing$ we have $K(\aleph_0) \subseteq S$, which contradicts Lemma 5.5. □

Let us consider the injection α on $\mathbb{N}$ defined by $x\alpha := x+1$ for all $x \in \mathbb{N}$ and the surjection β on $\mathbb{N}$ defined by $1\beta := 1$ and $x\beta := x-1$ for each natural number $x \ge 2$. Then we see that $\alpha\beta$ is the identity map $id_{\mathbb{N}}$. Hence $id_{\mathbb{N}}$ belongs to each maximal subsemigroup of $O(\mathbb{N})$.

Now we are going to show that the following sets form maximal subsemigroups of $O(\mathbb{N})$:

$$S^{(1)} := \{\alpha \in O(\mathbb{N}) : 1\alpha = 1\} \cup J(\aleph_0)$$

and
$$S_{(1)} := \{\alpha \in O(\mathbb{N}) : 2\alpha \geq 2\} \cup J(\aleph_0).$$

Lemma 5.6 $S^{(1)}$ is a maximal subsemigroup of $O(\mathbb{N})$.

Proof: Clearly, $S^{(1)}$ is a semigroup. It remains to show that $S^{(1)}$ is maximal. For this let $\alpha, \beta \in O(\mathbb{N}) \setminus S^{(1)}$. Suppose that

$$\operatorname{im} \alpha = \{a_1 < a_2 < \cdots < a_i < \cdots\},$$

$$\operatorname{im} \beta = \{b_1 < b_2 < \cdots < b_i < \cdots\}$$

for $2 \leq i \in \mathbb{N}$, we choose $\tilde{a}_i \in a_i\alpha^{-1}$ and set $\tilde{a}_1 := 1$. Then let γ, δ be transformations on $\mathbb{N}$ defined by

$$x\gamma := \tilde{a}_i$$

whenever $x \in b_i\beta^{-1}$ for $\mathrm{i} \in \mathbb{N}$ and

$$x\delta := \begin{cases} 1 & \text{if} \quad x = 1 \\ b_i & \text{if} \quad a_{i-1} < x \leq a_i \text{ for } i \in \mathbb{N} \end{cases}$$

where $a_0 := 1$. Since $\alpha \notin S^{(1)}$, we know that $1\alpha > 1$ and thus $a_1 > 1$. This guarantees that δ is well defined. Because of $1\gamma = 1\delta = 1$, we conclude $\gamma, \delta \in S^{(1)}$. Let $x \in \mathbb{N}$. If $x = 1$ then we get $x\gamma\alpha\delta = b_1 = 1\beta = x\beta$. Suppose that $x > 1$. Then there is $i \in \mathbb{N}$ with $x \in b_i\beta^{-1}$ and we have

$$x\gamma\alpha\delta = \tilde{a}_i\alpha\delta = a_i\delta = b_i = x\beta.$$

Thus $\beta = \gamma\alpha\delta \in \langle S^{(1)}, \alpha \rangle$. This shows that $S^{(1)}$ is maximal. □

Lemma 5.7 $S_{(1)}$ is a maximal subsemigroup of $O(\mathbb{N})$.

Proof: It is easy to verify that $S_{(1)}$ is semigroup. We are going to show that $S_{(1)}$ is maximal. For this let $\alpha, \beta \in O(\mathbb{N}) \setminus S_{(1)}$. Again we suppose that

$$\operatorname{im} \alpha = \{a_1 < a_2 < \cdots < a_i < \cdots\},$$

$$\operatorname{im} \beta = \{b_1 < b_2 < \cdots < b_i < \cdots\},$$

and for $2 \le i \in \mathbb{N}$ we choose $\tilde{a}_i \in a_i\alpha^{-1}$ and set $\tilde{a}_1 := 2$ (note that $\alpha \notin S_{(1)}$ implies $2\alpha = 1 = a_1$, i.e., $1, 2 \in 1\alpha^{-1}$, so we can choose $\tilde{a}_1 = 2$). Then we define transformations γ, δ by

$$x\gamma := \tilde{a}_i$$

whenever $x \in b_i\beta^{-1}$ for $i \in \mathbb{N}$ and

$$x\delta := b_i \text{ if } a_{i-1} < x \le a_i \text{ for } i \in \mathbb{N},$$

where $a_0 := 0$. It is easy to verify that both maps γ and δ are well defined. Clearly, $\gamma, \delta \in O(\mathbb{N})$. Since $1 \notin \operatorname{im} \gamma$, we have $2\gamma \ge 2$ and $a_1 < 2 \le a_2$ implies $2\delta = b_2 \ge 2$. Hence $\gamma, \delta \in S_{(1)}$.

Let $x \in \mathbb{N}$. Then there is $i \in \mathbb{N}$ with $x \in b_i\beta^{-1}$ and we have

$$x\gamma\alpha\delta = \tilde{a}_i\alpha\delta = a_i\delta = b_i = x\beta.$$

Thus $\beta = \gamma\alpha\delta \in \langle S_{(1)}, \alpha \rangle$. This shows that $S_{(1)}$ is maximal. □

In the next step, we characterize the maximal subsemigroups of $O(\mathbb{N})$ containing the set

$$\tilde{S}^{(1)} := \{\alpha \in S^{(1)} : 1\alpha^{-1} = \{1\}\} \cup J(\aleph_0).$$

Proposition 5.1 Let S be a subsemigroup of $O(\mathbb{N})$ with $\tilde{S}^{(1)} \subseteq S$. Then S is maximal if and only if $S = S_{(1)}$ or $S = S^{(1)}$.

Proof: One direction is clear by Lemmas 6.6 and 6.7. Conversely, suppose that S is maximal. Assume that $S \ne S_{(1)}$ as well as $S \ne S^{(1)}$. Since S, $S^{(1)}$, and $S_{(1)}$ are maximal, it follows that $S \nsubseteq S^{(1)}$ and $S \nsubseteq S_{(1)}$. From $S \nsubseteq S^{(1)}$, it follows that there is $\alpha_1 \in S \setminus J(\aleph_0)$ with $1\alpha_1 > 1$. From $S \nsubseteq S_{(1)}$, it follows that there is $\alpha_2 \in S \setminus J(\aleph_0)$ with $2\alpha_2 = 1$.

We suppose that $\operatorname{im}\alpha_2 = \{a_1 < a_2 < \cdots < a_i < \cdots\}$ and choose $\tilde{a}_i \in a_i\alpha_2^{-1}$ for $i \in \mathbb{N}$, where $\tilde{a}_1 := 2$ (it is possible since $1,2 \in a_1\alpha_2^{-1}$). Let $\beta \in O(\mathbb{N}) \setminus S \neq \varnothing$. Then $\beta \in K(\aleph_0)$ and we can suppose that $\operatorname{im}\beta = \{b_1 < b_2 < \cdots < b_i < \cdots\}$. Then $2\beta = 1$ or $1\beta > 1$.

If $1 = 2\beta$ then we define transformations $\gamma,\ \delta$ on $\mathbb{N}$ by

$$x\gamma := \begin{cases} 1 & \text{if } x = 1 \\ \tilde{a}_i & \text{whenever } x \neq 1 \text{ and } x \in b_i\beta^{-1} \text{ for } i \in \mathbb{N}; \end{cases}$$

$$x\delta := \begin{cases} 1 & \text{if } \quad x = 1 \\ b_i & \text{if } \quad a_{i-1} < x \le a_i \text{ for } 2 \le i \in \mathbb{N}. \end{cases}$$

It is easy to verify that $\gamma, \delta \in \tilde{S}^{(1)}$. Let $x \in \mathbb{N}$. If $x \neq 1$ then there is $i \in \mathbb{N}$ such that $x \in b_i\beta^{-1}$ and we have

$$x\gamma\alpha_2\delta = \tilde{a}_i\alpha_2\delta = a_i\delta = b_i = x\beta .$$

Moreover, $1\gamma\alpha_2\delta = 1\alpha_2\delta = 1\delta = 1 = 1\beta$. Hence, $\beta = \gamma\alpha_2\delta \in \left\langle \tilde{S}^{(1)}, \alpha_1, \alpha_2 \right\rangle \subseteq S$.

If $1\beta > 1$ then by the foregoing argument applied to $\alpha_2\alpha_1$, there exist $\eta, \theta \in S^{(1)}$ such that $\beta = \eta\alpha_2\alpha_1\theta$. By the same matter, we replace α_2 by $\alpha_2\alpha_1$, we obtain that $\beta \in \left\langle \tilde{S}^{(1)}, \alpha_1, \alpha_2 \right\rangle$. Suppose that $\operatorname{im}\alpha_2\alpha_1 = \{c_1 < c_2 < \cdots < c_i < \cdots\}$ and let us choose $\tilde{c}_i \in c_i(\alpha_2\alpha_1)^{-1}$ for $i \in \mathbb{N}$ where $\tilde{c}_1 \ge 2$ (possible since $1,2 \in c_1(\alpha_2\alpha_1)^{-1}$). Then we define transformations η, θ on $\mathbb{N}$ by

$$x\eta := \begin{cases} 1 & \text{if } x = 1 \\ \tilde{c}_i & \text{whenever } x \neq 1 \text{ and } x \in b_i\beta^{-1} \text{ for } i \in \mathbb{N}; \end{cases}$$

$$x\theta := \begin{cases} 1 & \text{if } \quad x = 1 \\ b_i & \text{if } \quad c_{i-1} < x \le c_i \text{ for } i \in \mathbb{N}, \end{cases}$$

where $c_0 := 1$ (it is possible since $c_1 \ge 2$). Clearly, $\eta, \theta \in \tilde{S}^{(1)}$. Let $x \in \mathbb{N}$. If $x \neq 1$ then there is $i \in \mathbb{N}$ such that $x \in b_i\beta^{-1}$ and we have $x\eta\alpha_2\alpha_1\theta = \tilde{c}_i\alpha_2\alpha_1\theta = c_i\theta = b_i = x\beta$. Moreover, $1\eta\alpha_2\alpha_1\theta = 1\alpha_2\alpha_1\theta = c_1\theta = b_1 = 1\beta$. Hence

$$\beta = \eta\alpha_2\alpha_1\theta \in \left\langle \tilde{S}^{(1)}, \alpha_1, \alpha_2 \right\rangle \subseteq S .$$

Altogether, this shows that $\beta \in S$, a contradiction. □

Let

$$S_f^1 := \{\alpha \in O(\mathbb{N}) : 2\alpha = 1 \text{ and } \max G_\alpha \neq \infty\} \cup \{\alpha \in \tilde{S}^{(1)} : \max G_\alpha \neq \infty \text{ and } \max K_\alpha = \infty\}.$$

Lemma 5.8 $O(\mathbb{N}) \setminus S_f^1$ is a maximal subsemigroup of $O(\mathbb{N})$.

Proof: We will use all the items (1), (2), and (4) of Lemma 5.1. Let $\alpha, \beta \in O(\mathbb{N}) \setminus S_f^1$. Assume that $\alpha\beta \in S_f^1$. Then α and β have infinite rank. Further, we can conclude $1 \in \operatorname{im}\beta$ and $\max G_\beta \neq \infty$. This implies $\beta \in \tilde{S}^{(1)}$ and thus $\max K_\beta \neq \infty$. Since $\max G_{\alpha\beta} \neq \infty$, this implies that $\max G_\alpha \neq \infty$. Since $\alpha\beta \notin S_f^1$, we have either $2(\alpha\beta) = 1$ or $\max K_{\alpha\beta} = \infty$ and $1(\alpha\beta)^{-1} = 1$. In the latter case, we obtain

$$\max K_\alpha = \infty \text{ and } \{1\} = 1(\alpha\beta)^{-1} = 1\beta^{-1}\alpha^{-1} = 1\alpha^{-1},$$

i.e., $\alpha \in S_f^1$, a contradiction. In the former case, we get $2\alpha = 1$ because $2\beta \neq 1$, i.e., $\alpha \in S_f^1$, a contradiction too. This shows that $O(\mathbb{N}) \setminus S_f^1$ is a subsemigroup of $O(\mathbb{N})$.

Now we show that $O(\mathbb{N}) \setminus S_f^1$ is maximal. Let $\alpha, \beta \in S_f^1$ and we are going to show that $\beta \in \langle O(\mathbb{N}) \setminus S_f^1, \alpha \rangle$. Suppose that $\ker\alpha = \{A_1 < A_2 < \ldots < A_n < \ldots\}$ and $\operatorname{im}\alpha = \{a_1 < a_2 < \ldots < a_n < \ldots\}$ with $a_i\alpha^{-1} = A_i$ for $i \in \mathbb{N}$, as well as $\ker\beta = \{B_1 < B_2 < \ldots < B_n < \ldots\}$ and $\operatorname{im}\beta = \{b_1 < b_2 < \ldots < b_n < \ldots\}$ with $b_i\alpha^{-1} = B_i$ for $i \in \mathbb{N}$. Let $\{c_1 < c_2 < \ldots < c_n < \ldots\}$ be a set of natural numbers such that $c_i \in A_i$ for $i \in \mathbb{N}$. In the case $\max K_\alpha \neq \infty$, we know that $2\alpha = 1$ and can require that $c_1 = 2$. Further, let $\gamma : \mathbb{N} \to \mathbb{N}$ with $\operatorname{im}\gamma = \{c_1, c_2, \ldots, c_n, \ldots\}$ defined by $c_i\gamma^{-1} = B_i$ for $i \in \mathbb{N}$. Clearly, $\gamma \in O(\mathbb{N})$. Since $1 \notin \operatorname{im}\gamma$ or $\max G_\gamma = \infty$ (it is the case if $\max K_\alpha = \infty$), we can conclude that $\gamma \notin S_f^1$.

Let $\{D_1 < D_2 < \ldots < D_n < \ldots\}$ with $D_1 := \{1\}$ be a decomposition of $\mathbb{N}$ such that $a_i \in D_i$ for $i \in \mathbb{N}$. Then we define a map $\delta : \mathbb{N} \to \mathbb{N}$ with $\operatorname{im}\delta = \operatorname{im}\beta$ by $b_i\delta^{-1} = D_i$ for $i \in \mathbb{N}$. Clearly, $\delta \in O(\mathbb{N})$ and $\max G_\alpha \neq \infty$ implies $\max K_\delta \neq \infty$. But since $1\delta^{-1} = \{1\}$, we obtain $\delta \notin S_f^1$.

Let $x \in \mathbb{N}$. Then there is $i \in \mathbb{N}$ with $x \in B_i$ and $x\gamma\alpha\delta = c_i\alpha\delta = a_i\delta = b_i = x\beta$. Therefore, $\beta = \gamma\alpha\delta \in \left\langle O(\mathbb{N}) \setminus S_f^1, \alpha \right\rangle$. □

Now we state the main result of this section.

Definition 5.1 Let $A_{\aleph_0}^{(1)}$ be the set of all $\alpha \in \tilde{S}^{(1)} \setminus J(\aleph_0)$ with one or both of the following properties:

(1) $2 \le \max K_\alpha \ne \infty$ and $\alpha \notin SurO(\mathbb{N})$.

(2) $\max G_\alpha = \infty$ and $\alpha \notin IO(\mathbb{N})$.

It is easy to verify that

$$A_{\aleph_0}^{(1)} = (\tilde{S}^{(1)} \cap \bar{S}_{\aleph_0}) \setminus J(\aleph_0)$$

with $\bar{S}_{\aleph_0} := O(\mathbb{N}) \setminus (SurO(\mathbb{N}) \cup IO(\mathbb{N}) \cup S_{\aleph_0})$. The main result of this section characterizes all maximal subsemigroups of $O(\mathbb{N})$ containing $A_{\aleph_0}^{(1)}$.

Proposition 5.2 Let S be a subsemigroup of $O(\mathbb{N})$ with $A_{\aleph_0}^{(1)} \subseteq S$. Then S is maximal if and only if $S = S_{(1)}$ or $S = S^{(1)}$ or $S = O(\mathbb{N}) \setminus S_f^1$ or $S = (O(\mathbb{N}) \setminus SurO(\mathbb{N})) \cup \{id_{\mathbb{N}}\}$ or $S = (O(\mathbb{N}) \setminus IO(\mathbb{N})) \cup \{id_{\mathbb{N}}\}$ or $S = O(\mathbb{N}) \setminus S_{\aleph_0}$.

Proof: The six listed semigroups are maximal (see the previous lemmas) and it is easy to verify that each of them contains $A_{\aleph_0}^{(1)}$.

Conversely, let S be maximal. Assume that S is different from all of the six listed semigroups. We are going to show that then $\tilde{S}^{(1)} \subseteq S$. In fact, we have to show that any $\beta \in \tilde{S}^{(1)} \setminus \bar{S}_{\aleph_0} = \tilde{S}^{(1)} \cap (SurO(\mathbb{N}) \cup IO(\mathbb{N}) \cup S_{\aleph_0})$ belongs to S. There are $\alpha_1 \in SurO(\mathbb{N}) \setminus \{id_{\mathbb{N}}\}, \alpha_2 \in IO(\mathbb{N}) \setminus \{id_{\mathbb{N}}\}, \alpha_3 \in S_{\aleph_0}, \alpha_4 \in S_f^1, \alpha_5 \in O(\mathbb{N}) \setminus S_{(1)}$, and $\alpha_6 \in O(\mathbb{N}) \setminus S^{(1)}$ with $\alpha_i \in S$, $\operatorname{im} \alpha_i = \{a_{1,i} < a_{2,i} < \ldots < a_{n,i} < \ldots\}$, and $\ker \alpha_i = \{A_{1,i} < A_{2,i} < \ldots < A_{n,i} < \ldots\}$ for $i \in \{1,2,3,4,5,6\}$.

First of all, we show that there is $\alpha \in S \cap S_{\aleph_0}$ with $1 \in \operatorname{im} \alpha$. If $1 \in \operatorname{im} \alpha_3$, we have nothing to show. Suppose now that $1 \notin \operatorname{im} \alpha_3$. We consider α_4. We have $1 \in \operatorname{im} \alpha_4$. If $\alpha_4 \in S_{\aleph_0}$, we have nothing to show. Suppose that $\alpha_4 \notin S_{\aleph_0}$. Then

$2\alpha_4 = 1$ and $\max G_{\alpha_4} \neq \infty$. Let $\{D_1 < D_2 < \ldots < D_n < \ldots\}$ be a decomposition of $\mathbb{N} \setminus \{1\}$ with $a_{1,3} \in D_1$ and $a_{i+1,3} \in D_i$ for $i \in \mathbb{N}$. Let $\{d_1, d_2, \ldots, d_n, \ldots\}$ be a set of natural numbers such that $d_i \in A_{i,4}$ for $i \in \mathbb{N}$ where $d_1 > 1$ (this is possible since $|A_{1,4}| > 1$). Then we define a transformation δ on $\mathbb{N}$ with $\mathrm{im}\ \delta = \{1 < d_1 < d_2 < \ldots < d_n < \ldots\}$ by

$$1\delta^{-1} = \{1\} \text{ and } d_i\delta^{-1} = D_i$$

for $i \in \mathbb{N}$. It is easy to verify that $\delta \in O(\mathbb{N})$ but $\delta \notin SurO(\mathbb{N}) \cup IO(\mathbb{N})$. From $\max G_{\alpha_3} \neq \infty$, it follows $\max K_\delta \neq \infty$. Moreover, $1\delta^{-1} = \{1\}$. Hence, $\delta \in A_{\aleph_0}^{(1)} \subseteq S$. Then $1\alpha_3\delta\alpha_4 = a_{1,3}\delta\alpha_4 = d_1\alpha_4 = 1$. Note, $\mathrm{im}\ \alpha_3$ is a transversal of $\ker\delta$ and $\mathrm{im}\ \delta$ is a transversal of $\ker\alpha_4$. Hence, $\ker\alpha_3 = \ker\alpha_3\delta\alpha_4$ and $\mathrm{im}\ \alpha_4 = \mathrm{im}\ \alpha_3\delta\alpha_4$. Thus $\max G_{\alpha_3\delta\alpha_4} \neq \infty$ (since $\max G_{\alpha_4} \neq \infty$) and $\max K_{\alpha_3\delta\alpha_4} = \infty$ (since $\max K_{\alpha_3} = \infty$). This shows that $\alpha_3\delta\alpha_4 \in S \cap S_{\aleph_0}$ with $1 \in \mathrm{im}\ \alpha_3\delta\alpha_4$. Hence, there is $\alpha \in S \cap S_{\aleph_0}$ with $1 \in \mathrm{im}\ \alpha$ and suppose that $\ker\alpha = \{A_1 < A_2 < \ldots < A_n < \ldots\}$ and $\mathrm{im}\ \alpha = \{a_1 < a_2 < \ldots < a_n < \ldots\}$, i.e., $a_i\alpha^{-1} = A_i$ for $i \in \mathbb{N}$.

Let

$$\beta \in \tilde{S}^{(1)} \cap (SurO(\mathbb{N}) \cup IO(\mathbb{N}) \cup S_{\aleph_0})$$

where $\ker\beta = \{B_1 < B_2 < \ldots < B_n < \ldots\}$ and $\mathrm{im}\ \beta = \{b_1 < b_2 < \ldots < b_n < \ldots\}$ such that

$$b_i\beta^{-1} = B_i$$

for $i \in \mathbb{N}$. Let $\{c_1 < c_2 < \ldots < c_n < \ldots\}$ be a set natural numbers with $c_1 = 1$ such that $c_1 \in A_1$ and $c_i \in A_{i+1}$ for all $2 \leq i \in \mathbb{N}$. Next, we show that there is a transformation β_1 in $S \setminus SurO(\mathbb{N})$ with $\max G_{\beta_1} \neq \infty$, $1 \in \mathrm{im}\ \beta_1$, and $\ker\beta = \ker\beta_1$. We consider two possible cases $\beta \in IO(\mathbb{N})$ and $\beta \notin IO(\mathbb{N})$.

Suppose that $\beta \notin IO(\mathbb{N})$. Let γ be a transformation on $\mathbb{N}$ with $\mathrm{im}\ \gamma = \{c_1 < c_2 < \ldots < c_n < \ldots\}$ defined by

$$c_i\gamma^{-1} = B_i \text{ for } i \in \mathbb{N}.$$

Clearly, $\ker\gamma = \ker\beta$ and $\gamma \in O(\mathbb{N})$. Since $\operatorname{im}\gamma$ is a pseudo-transversal of $\ker\alpha$, from $\max K_\alpha = \infty$, it follows $\max G_\gamma = \infty$. Moreover, $\gamma \notin IO(\mathbb{N})$ (since $\ker\gamma = \ker\beta$), and so $\gamma \in A^{(1)}_{\aleph_0} \subseteq S$. We also have $1\gamma^{-1} = c_1\gamma^{-1} = B_1 = 1\beta^{-1} = \{1\}$. Since $\operatorname{im}\gamma$ is a pseudo-transversal of $\ker\alpha$, we have $\ker\beta = \ker\gamma\alpha$ (since $\ker\beta = \ker\gamma$), $\max G_{\gamma\alpha} \neq \infty$ (since $\max G_\alpha \neq \infty$), and $1 \in \operatorname{im}\gamma\alpha$ (since $1 \in \operatorname{im}\alpha$). Since $a_2 \notin \operatorname{im}\gamma\alpha$, we have $\gamma\alpha \notin SurO(\mathbb{N})$. So, we can take $\beta_1 := \gamma\alpha \in S \setminus SurO(\mathbb{N})$.

Suppose that $\beta \in IO(\mathbb{N})$. If $\max G_{\alpha_2} = \infty$ then we consider a decomposition $\{E_1 < E_2 < \ldots < E_n < \ldots\}$ of $\mathbb{N}$ with $E_1 = \{1\}$ such that $a_{i,2} \in E_i$ for $i \in \mathbb{N}$ (if $a_{1,2} = 1$) and $a_{i,2} \in E_{i+1}$ for $i \in \mathbb{N}$ (if $a_{1,2} \neq 1$), respectively. Further, let $\{e_1 < e_2 < \ldots < e_n < \ldots\}$ be a set of natural numbers with $e_1 = 1$ such that $e_i \in A_{i,3}$ for $i \in \mathbb{N}$. Let ε be the transformation on $\mathbb{N}$ with $\operatorname{im}\varepsilon = \{e_1 < e_2 < \ldots < e_n < \ldots\}$ and

$$e_i\varepsilon^{-1} = E_i \text{ for } i \in \mathbb{N}.$$

Clearly, $\varepsilon \in O(\mathbb{N})$ but $\varepsilon \notin IO(\mathbb{N})$. We can calculate $1\varepsilon^{-1} = e_1\varepsilon^{-1} = E_1 = \{1\}$. Since $\max K_{\alpha_3} = \infty$, where $\operatorname{im}\varepsilon$ is a transversal of $\ker\alpha_3$, we conclude that $\max G_\varepsilon = \infty$ and $\varepsilon \in A^{(1)}_{\aleph_0} \subseteq S$. It is easy to verify that $\alpha_2\varepsilon\alpha_3 \in IO(\mathbb{N})$. Because $\operatorname{im}\alpha_2$ is a transversal of $\ker\varepsilon$ (of $\ker\varepsilon \setminus \{\{1\}\}$, respectively) as well as since $\operatorname{im}\varepsilon$ is a transversal of $\ker\alpha_3$, we have $\max G_{\alpha_2\varepsilon\alpha_3} \neq \infty$ (since $\max G_{\alpha_3} \neq \infty$). By this reason, we can assume that $\max G_{\alpha_2} \neq \infty$.

If $\max G_{\alpha_5} = \infty$ then let $\{F_1 < F_2 < \ldots < F_n < \ldots\}$ be a decomposition of $\mathbb{N}$ with $F_1 = \{1\}$ such that $a_{i,5} \in F_i$ for $i \in \mathbb{N}$. Let ϵ be the transformation on $\mathbb{N}$ with $\operatorname{im}\epsilon = \{c_1 < c_2 < \ldots < c_n < \ldots\}$ such that

$$c_i\epsilon^{-1} = F_i \text{ for } i \in \mathbb{N}.$$

Clearly, $\epsilon \in O(\mathbb{N})$. Note that $\operatorname{im}\epsilon$ and $\operatorname{im}\alpha_5$ are transversals of $\ker\alpha$ and $\ker\epsilon$, respectively. This implies that $\epsilon \notin IO(\mathbb{N})$ (since $\max G_{\alpha_5} = \infty$) and $\max G_\epsilon = \infty$ (since $\max K_\alpha = \infty$). Hence $\epsilon \in A^{(1)}_{\aleph_0} \subseteq S$. It is easy to verify that $2\alpha_5\epsilon\alpha = 1\epsilon\alpha = 1\alpha = 1$ and that $\max G_{\alpha_5\epsilon\alpha} \neq \infty$. By this reason, we can assume that

$\max G_{\alpha_5} \neq \infty$. Let $\{x_1 < x_2 < \ldots < x_n < \ldots\}$ be a set of natural numbers with $x_1 = 1$ (if $1 \in \text{im } \alpha_2$) and $x_1 = 2$ (if $1 \notin \text{im } \alpha_2$), such that $x_i \in A_{i+1,5}$ for $2 \leq i \in \mathbb{N}$. Let $\{G_1 < G_2 < \ldots < G_n < \ldots\}$ be a decomposition of $\mathbb{N}$ with $G_1 = \{1\}$ (if $1 \in \text{im } \alpha_2$) and of $\mathbb{N} \setminus \{1\}$ (if $1 \notin \text{im } \alpha_2$), such that $a_{i,2} \in G_i$ for $i \in \mathbb{N}$. Let ν be the transformation on $\mathbb{N}$ with

$$\text{im } \nu = \{x_1 < x_2 < \ldots < x_n < \ldots\} \text{ (if } 1 \in \text{im } \alpha_2\text{)}$$

and
$$\text{im } \nu = \{1 < x_1 < x_2 < \ldots < x_n < \ldots\} \text{ (if } 1 \notin \text{im } \alpha_2 \text{),}$$

such that

$$x_i \nu^{-1} = G_i \text{ for } i \in \mathbb{N}$$

(and $1\nu^{-1} = \{1\}$ if $1 \notin \text{im } \alpha_2$). Clearly, $\nu \in O(\mathbb{N})$. Since $\alpha_2 \neq id_{\mathbb{N}}$, we get $\nu \notin IO(\mathbb{N})$. Because $\text{im } \alpha_2$ is a transversal of $\ker \nu$ (a pseudo-transversal of $\ker \nu$ if $1 \notin \text{im } \alpha_2$), we obtain $\max K_\nu \neq \infty$ (since $\max G_{\alpha_2} \neq \infty$) and $1\nu^{-1} = \{1\}$ (note that $1\nu^{-1} = x_1\nu^{-1} = G_1 = \{1\}$ if $1 \notin \text{im } \alpha_2$). This shows that $\nu \in A_{\aleph_0}^{(1)} \subseteq S$. It is easy to verify that $\alpha_2\nu\alpha_5 \in IO(\mathbb{N})$ and $1 \in \text{im } \alpha_2\nu\alpha_5$. Further, we observe that $\text{im } \alpha_2$ is a transversal of $\ker \nu$ (a pseudo-transversal of $\ker \nu$ if $1 \notin \text{im } \alpha_2$) as well as $\text{im } \nu$ is a pseudo-transversal of $\ker \alpha_5$. Thus, $\max G_{\alpha_2\nu\alpha_5} \neq \infty$ (since $\max G_{\alpha_5} \neq \infty$) and, in particular, $\ker \beta = \ker \alpha_2\nu\alpha_5$ (since both α_2 and β are injective). Moreover, $\alpha_2\nu\alpha_5$ is not surjective since $a_{2,5} \notin \text{im } \alpha_2\nu\alpha_5$. So, we can choose $\beta_1 := \alpha_2\nu\alpha_5 \in S \setminus SurO(\mathbb{N})$.

Altogether, we have verified the existence of a $\beta_1 \in S \setminus SurO(\mathbb{N})$ with

$$\max G_{\beta_1} \neq \infty,\ 1 \in \text{im } \beta_1, \text{ and } \ker \beta = \ker \beta_1.$$

We write $\text{im } \beta_1 = \{m_1 < m_2 < \cdots < m_n < \cdots\}$. Finally, we show that there is $\delta_1 \in S$ with $\text{im } \delta_1 = \text{im } \beta$ such that $\text{im } \beta_1$ is a transversal of $\ker \delta_1$. We consider two possible cases $\beta \in Sur(\mathbb{N})$ and $\beta \notin Sur(\mathbb{N})$.

Suppose that $\beta \notin SurO(\mathbb{N})$. Let $\{M_1 < M_2 < \ldots < M_n < \ldots\}$ be a decomposition of $\mathbb{N}$ with $M_1 = \{1\}$ such that $m_i \in M_i$ for $i \in \mathbb{N}$ (it is possible since $1 \in \operatorname{im} \beta_1$). Let δ_1 be the transformation on $\mathbb{N}$ with $\operatorname{im} \delta_1 = \operatorname{im} \beta$ such that

$$b_i \delta_1^{-1} = M_i$$

for $i \in \mathbb{N}$. Clearly, $\delta_1 \in O(\mathbb{N})$ and $\max G_{\beta_1} \neq \infty$ implies $\max K_{\delta_1} \neq \infty$ since $\operatorname{im} \beta_1$ is a transversal of $\ker \delta_1$. From $\beta \notin Sur(\mathbb{N})$, we can conclude that $\delta_1 \notin Sur(\mathbb{N})$ (since $\operatorname{im} \beta = \operatorname{im} \delta_1$). Further, $1\delta_1^{-1} = b_1 \delta_1^{-1} = M_1 = \{1\}$ and consequently, $\delta_1 \in A_{\aleph_0}^{(1)} \subseteq S$.

Suppose that $\beta \in SurO(\mathbb{N})$. Then let $\{g_1 < g_2 < \ldots < g_n < \ldots\}$ be a set of natural numbers with $g_1 = 1$ such that $g_i \in A_{i,1}$ for $i \in \mathbb{N}$. Further, let $\{L_1 < L_2 < \ldots < L_n < \ldots\}$ be a decomposition of $\mathbb{N}$ with $L_1 = \{1\}$ such that $m_i \in L_i$ for $i \in \mathbb{N}$ (it is possible since $1 \in \operatorname{im} \beta_1$). Let η be the transformation on $\mathbb{N}$ with $\operatorname{im} \eta = \{g_1 < g_2 < \ldots < g_n < \ldots\}$ such that

$$g_i \eta^{-1} = L_i$$

for $i \in \mathbb{N}$. Clearly, $\eta \in O(\mathbb{N})$ and $\max G_{\beta_1} \neq \infty$ implies $\max K_{\eta} \neq \infty$ since $\operatorname{im} \beta_1$ is a transversal of $\ker \eta$. Further, we have $1\eta^{-1} = g_1 \eta^{-1} = L_1 = \{1\}$. Since $\operatorname{im} \eta$ is a transversal of $\ker \alpha_1$, where α_1 is not injective, we conclude that η is not surjective. Since $\beta_1 \notin SurO(\mathbb{N})$, we obtain $2 \leq \max K_{\eta} \neq \infty$. Thus $\eta \in A_{\aleph_0}^{(1)} \subseteq S$. So, we can take $\delta_1 := \eta \alpha_1 \in S$.

Altogether, we have verified the existence of a $\delta_1 \in S$ with $\operatorname{im} \delta_1 = \operatorname{im} \beta$, such that $\operatorname{im} \beta_1$ is a transversal of $\ker \delta_1$.

Let $x \in \mathbb{N}$. Note that $\operatorname{im} \delta_1 = \operatorname{im} \beta$, $\ker \beta_1 = \ker \beta$, and $\operatorname{im} \beta_1$ is a transversal of $\ker \delta_1$. This shows that

$$\beta = \beta_1 \delta_1 \in S .$$

Altogether, we have shown that $\tilde{S}^{(1)} \subseteq S$. Then by Proposition 5.1, we obtain that $S = S_{(1)}$ or $S = S^{(1)}$, a contradiction. □

6. The Idempotent Elements on TF_n

Let $n \in \mathbb{N}$ be a natural number and denote by T_n the monoid (under composition) of all full transformations on the set $\overline{n} := \{1, \ldots, n\}$ of the first n natural numbers. Let $\alpha \in T_n$. We say that α is an order-preserving transformation (with respect to a partial order $\preceq$ on $\overline{n}$) if $x\alpha \preceq y\alpha$, whenever $x \prec y$. If $\preceq$ is a linear order (also called totally order) then the monoid O_n of all order-preserving transformations on $\overline{n}$ have been the object of study by several authors and several papers.

A non-linear order "next" to the linear order is the so-called zig-zag order. The pair $(\overline{n}, \preceq)$ is called zig-zag poset or fence if

$1 \prec 2 \succ 3 \prec \cdots \prec n-1 \succ n,$ whenever n is odd, and

$1 \prec 2 \succ 3 \prec \cdots \succ n-1 \prec n,$ whenever n is even, respectively, or dually

$1 \succ 2 \prec 3 \succ \cdots \succ n-1 \prec n,$ whenever n is odd, and

$1 \succ 2 \prec 3 \succ \cdots \prec n-1 \succ n,$ whenever n is even, respectively.

The definition of the partial order $\preceq$ is self-explanatory. Every element in a fence is either minimal or maximal. Denote by TF_n the submonoid of T_n, whose elements preserve the zig-zag order $\preceq$.

We consider the generating sets and the idempotent elements in TF_n. Without loss of generality, we will assume that $\preceq$ is an up-fence, i.e. $1 \prec 2 \succ 3 \prec \ldots$. A subset Σ of $\overline{n}$ of the form $\Sigma = \{k, k+1, \ldots, k+l-1\}$ with $k, l \in \overline{n}$ and $k + l \leq n+1$ is called subfence (for short: fence).

Let $x, y \in \overline{n}$. We say that x and y are comparable (in symbols: $x \perp y$) if $x \prec y$ or $x = y$ or $y \prec x$. Otherwise, we write $x \,||\, y$. It is clear that $x \perp y$ if and only if $x \in \{y-1, y, y+1\}$. Let Σ_1 and Σ_2 be disjunct subsets of $\overline{n}$. We write $\Sigma_1 \perp \Sigma_2$, whenever $x_1 \perp x_2$ for all $x_1 \in \Sigma_1$ and all $x_2 \in \Sigma_2$. It is easy to see that $\Sigma_1 \perp \Sigma_2$ if and only if $\Sigma_1 = \{x, x+2\}$ and $\Sigma_2 = \{x+1\}$ (or conversely) for some $x \in \{1, \ldots, n-2\}$. We write $\Sigma_1 \| \Sigma_2$, whenever $x_1 \| x_2$ for all $x_1 \in \Sigma_1$ and all $x_2 \in \Sigma_2$. This is just the case if $\Sigma_1 \cap \{x-1, x, x+1 : x \in \Sigma_2\} = \varnothing$.

We write transformations to the right of their argument and compose from left to right. Let $\alpha \in TF_n$. For general background on Semigroup Theory and standard notation, we refer the reader to Howie's book.

Clearly, the identity mapping id_n on $\overline{n}$ is order-preserving. Also all the n constant mappings are order-preserving. Whenever n is odd, exactly two bijections on $\overline{n}$ are order-preserving, namely the identity mapping and the reflection

$$\gamma_n := \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

The idempotent elements in a semigroup are of particular interest. In particular, they give important information about the subgroups of the given semigroup. An idempotent in T_n is characterized by the property that its restriction to its range is the identity mapping. The number of idempotents in other subsemigroups of TF_n were determined by Fernandes, Gomes and Jesus. As usually, we denote by E_n the idempotent elements in TF_n.

The rank of a (finite) semigroup S is usually defined by

$$\operatorname{rank} S := \min\{|A| : A \subseteq S \text{ with } \langle A \rangle = S\},$$

with other words, the rank of S is the minimal size of a generating set of S. Gomes and Howie first showed that the rank of the semigroup O_n is $\frac{2+n(n-1)}{2}$. This result was generalized by Howie and McFadden, who showed that for $1 \le r < n$, the rank of the ideal $K(n,r) := \{\alpha \in S : rank\alpha \le r\}$ is equal to $S(n,r)$, the sterling number of the second kind. Ruškuc gave an alternative proof of this result. The aim of the next research project we want to determine the rank of TF_n for both cases, n is even and n is odd. In particular, we provide in both cases a minimal generating set for TF_n and provide a formula which gives the number of idempotents in TF_n. In order to be able to do this it is useful to know how a transformation in TF_n looks. Hence first, let us give a characterization of the transformations in TF_n, whenever $n \geq 3$.

Theorem 6.1 Let $3 \leq n \in \mathbb{N}$ and let $\alpha \in T_n$. Then $\alpha \in TF_n$ if and only if

(1) $|x\alpha - (x+1)\alpha| \leq 1$ for all $x \in \{1,\ldots,n-1\}$;

(2) $x\alpha$ and x have the same parity or $(x-1)\alpha = x\alpha = (x+1)\alpha$, for all $x \in \{2,\ldots,n-1\}$.

Proof: Suppose that $\alpha \in TF_n$. Assume that there is $x \in \{1,\ldots,n-1\}$ such that $|x\alpha - (x+1)\alpha| > 1$. Then $x\alpha \,||\, (x+1)\alpha$ and thus $x \,||\, x+1$ since $\alpha \in TF_n$, a contradiction. This shows (1). Let now $x \in \{2,\ldots,n-1\}$. Without loss of generality, we can assume that

$$x-1 \succ x \prec x+1 .$$

Since $\alpha \in TF_n$, we obtain $(x-1)\alpha \succeq x\alpha \preceq (x+1)\alpha$. If $(x-1)\alpha \neq x\alpha$ or $x\alpha \neq (x+1)\alpha$ then $(x-1)\alpha \succ x\alpha$ or $x\alpha \prec (x+1)\alpha$, i.e. $x\alpha$ and x have the same parity by the definition of the relation $\prec$. This shows (2).

Now we suppose that (1) and (2) are satisfied. Let $x \in \{2,\ldots,n-1\}$ with $x \prec x+1$. Suppose that $x\alpha \neq (x+1)\alpha$. Then (2) provides that $x\alpha$ and x have the same parity. On the other hand, $x\alpha \neq (x+1)\alpha$ and (1) provides $|x\alpha - (x+1)\alpha| = 1$, i.e. $(x+1)\alpha = x\alpha + 1$ or $(x+1)\alpha = x\alpha - 1$. Since $x\alpha$ and x have the same parity, we conclude that $x\alpha - 1 \succ x\alpha$ and $x\alpha \prec x\alpha + 1$, respectively, from $x \prec x+1$. Hence, $x\alpha \prec (x+1)\alpha$.

Let $x \in \{2,\ldots,n-1\}$ with $x-1 \prec x$. Then we can verify $(x-1)\alpha \preceq x\alpha$ in the same way.

Notice that $1 \prec 2$ and either $n-1 \prec n$ or $n \prec n-1$. Suppose that $1\alpha \neq 2\alpha$ or $n\alpha \neq (n-1)\alpha$. Since $n \neq 2$ and $n-1 \neq 1$, we have the previous case for $x = 2$ and $x = n-1$, respectively. Hence, we obtain $1\alpha \prec 2\alpha$ as well as $(n-1)\alpha \prec n\alpha$ and

$n\alpha \prec (n-1)\alpha$, respectively. Altogether, we have shown that α preserves the partial order $\prec$. □

Theorem 6.1 shows that the range of a transformation in TF_n is a fence.

Corollary 6.1 Let $\alpha \in TF_n$. Then there are $k, l \in \overline{n}$ with $k+l \leq n+1$ such that $\operatorname{im}\alpha = \{k, k+1, \ldots, k+l-1\}$, i.e. $\operatorname{im}\alpha$ is a fence.

Proof: Let k be the least and let m be greatest natural number in $\operatorname{im}\alpha$ with respect to the usual order on the set $\mathbb{N}$ of natural numbers, Assume that there is $p \in \{k, k+1, \ldots, m\}$ with $p \notin \operatorname{im}\alpha$. Then there is an element $x \in \overline{n}$ with $x\alpha < p$ and $(x+1)\alpha > p$ (or conversely) with respect to the usual order on $\mathbb{N}$. So we have $|x\alpha - (x+1)\alpha| > 1$. But this contradicts Theorem 6.1 (1). This shows that $\operatorname{im}\alpha = \{k, k+1, \ldots, k+l-1\}$ with $l := m+1-k$. □

OUTPUT ที่ได้จากโครงการ

T. Musunthia and J. Koppitz, Maximal Subsemigroups of some semigroups of Order-preserving Mappings on a Countably Infinite Set, Forum Mathematicum, DOI: 10.1515/forum-2015-0093.