



รายงานวิจัยฉบับสมบูรณ์

โครงการ สมบัติการจับคู่ในคลาสของกราฟบางคลาส

โดย

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30 มิถุนายน 2557

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

งานวิจัยครั้งนี้สำเร็จลุล่วงได้ด้วยดี โดยการสนับสนุนของสำนักงานกองทุนสนับสนุนการวิจัย ภายใต้โครงการทุนวิจัยองค์ความรู้ใหม่ที่เป็นพื้นฐานต่อการพัฒนา (30 มิถุนายน 2554 – 29 มิถุนายน 2557) รหัส BRG4680019

บทคัดย่อ

ให้ $S \subseteq V(G)$ ในกราฟ $G = (V(G), E(G))$ เราจะเรียก S ว่า dominating set ของ G เมื่อสมาชิกแต่ละ v ใน $V(G)$ เป็นสมาชิกใน S หรือประชิดกับจุดใน S เราจะกล่าวว่า S เป็น independent dominating set ของ G เมื่อ S เป็น dominating set ของ G และ $G[S]$ เป็นกราฟที่ไม่มีเส้น และ independence domination number ของกราฟ G เขียนแทนด้วย $i(G)$ คือจำนวนสมาชิกที่น้อยที่สุดในการบรรดา independent dominating set ของ G กราฟ G จะเป็นกราฟ k - i -critical เมื่อ $i(G) = k$ แต่ $i(G + e) < k$ สำหรับแต่ละเส้น $e \notin E(G)$

สำหรับกราฟอย่างง่าย G ใดๆ complementary prism ของ G เขียนแทนด้วย $G\bar{G}$ คือกราฟที่ได้จากการเชื่อมจุดแต่ละจุดใน G ที่สมนัยกับจุดใน \bar{G} ด้วยเส้นเพียง 1 เส้น

สำหรับจำนวนเต็มบวก k และกราฟไม่ขาดตอน G ที่มีอันดับมากกว่าหรือเท่ากับ $2k + 2$ เราจะกล่าวว่า G เป็นกราฟ k -extendable เมื่อสำหรับการจับคู่ M ที่มีขนาดเท่ากับ k จะมีการจับคู่สมบูรณ์ใน G ซึ่งเส้นแต่ละเส้นใน M อยู่ในการจับคู่สมบูรณ์นั้น และจะกล่าวว่ากราฟ G ที่มีขนาดเท่ากับ p เป็นกราฟ k -factor-critical เมื่อ p และ k เป็นจำนวนเต็มบวกที่มี parity เดียวกัน และเมื่อกำจัดจุดจำนวน k จุดใดๆ ใน G แล้วกราฟที่เหลือมีการจับคู่สมบูรณ์

ในรายงานการวิจัยฉบับนี้เราได้ศึกษาสมบัติของกราฟ k - i -critical graphs โดยได้แสดงว่าถ้า G เป็นกราฟไม่ขาดตอนซึ่งเป็นกราฟ k - i -critical โดยที่ $k \geq 3$ และมี u เป็น cutvertex แล้ว จำนวนคอมโพเนนต์ของ $G - u$ ไม่เกิน $k - 1$ และจะมีคอมโพเนนต์ของ $G - u$ อย่างมากที่สุดเพียงสองคอมโพเนนต์ที่แต่ละคอมโพเนนต์มีจุดมากกว่า 1 จุด และเราได้ให้ลักษณะเฉพาะเจาะจงของกราฟในกรณีที่จำนวนคอมโพเนนต์ของ $G - u$ เท่ากับ $k - 1$ ยิ่งไปกว่านั้นเราได้ให้ลักษณะเฉพาะเจาะจงของกราฟ 3 - i -critical ที่มี S เป็นเซตตัดซึ่ง $|S| = 2$ และ $|S| = 3$ และจำนวนคอมโพเนนต์ของ $G - S$ ไม่เกิน 3 รวมทั้งเราได้ให้เงื่อนไขเพียงพอที่จะทำให้กราฟ 3 - i -critical เป็นกราฟ k -factor-critical และท้ายสุดเราได้ศึกษาสมบัติของกราฟ G ในเทอมของดิกกรีที่เท่าๆกันของ G และคุณสมบัติของ extendability ของ G ที่ทำให้ $G\bar{G}$ เป็นกราฟ k -extendable.

คำสำคัญ : domination, critical, cutvertex, cutset, extendable, factor-critical, complementary prism

Abstract

A set $S \subseteq V(G)$ is a (vertex) dominating set for $G = (V(G), E(G))$ if every vertex of G either belongs to S or is adjacent to a vertex of S . A dominating set S for G is an independent dominating set if $G[S]$ has no edges. Let $i(G)$ denote the minimum cardinality of an independent dominating set for G . A graph G is said to be k - i -critical if $i(G) = k$ but $i(G + e) < k$ for each $e \notin E(G)$.

For a simple graph G , the complementary prism of G , denoted by $G\bar{G}$, is the graph obtained by taking a copy of G and a copy of \bar{G} and then joining corresponding vertices by an edge.

For a positive integer k , a connected graph G of order at least $2k + 2$ is k -extendable if for each matching M of size k in G there is a perfect matching in G containing all edges of M . A graph G of order p is k -factor-critical, where p and k are positive integers with the same parity, if the deletion of any set of k vertices results in a graph with a perfect matching.

In this report, we investigate some properties of k - i -critical graphs. We show that if G is a connected k - i -critical, for $k \geq 3$, with a cutvertex u , then the number of components of $G - u$, $c(G - u)$, is at most $k - 1$ and there are at most two non-singleton components. Further, if $c(G - u) = k - 1$, then a characterization of such graphs is given. We also give a characterization of connected 3 - i -critical graphs with a minimum cutset S of size 2 and 3 with the number of components of $G - S$ is at most 3. A sufficient condition for connected 3 - i -critical graphs to be k -factor-critical is also provided. Finally, we investigate properties of G in terms of the regularity of G and the extendability of G so that $G\bar{G}$ is k -extendable.

Keywords : domination, critical, cutvertex, cutset, extendable, factor-critical, complementary prism

หน้าสรุปโครงการ (Executive Summary)

ชื่อโครงการวิจัย

สมบัติการจับคู่ในคลาสของกราฟบางคลาส
Matching properties in some classes of graphs

ผู้วิจัย

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ระยะเวลาดำเนินการ 3 ปี (30 มิถุนายน 2554 – 29 มิถุนายน 2557)

1. Significance and problem of the project:

Graph theory is one branch of mathematics which grows rapidly in both theory and applications. It is applied to many other fields such as engineering, natural and social sciences because of simple structure of a graph. Indeed, any system which can be viewed as a collection of objects some of which are related in some way or other can be modeled by a graph in which the vertices represent the objects and the edges (or arcs) represent the relationship between the objects.

For our purposes, graphs are finite simple and connected. A set $S \subseteq V(G)$ is a dominating set for G if every vertex of G either belongs to S or is adjacent to a vertex of S . A dominating set S for G is an independent dominating set if $G[S]$ has no edges. The minimum cardinality of an independent dominating set for G is called independence domination number of G and is denoted by $i(G)$. A graph G is said to be k -independent-domination-critical (or simply k - i -critical) if $i(G) = k$, but $i(G + e) < k$ for each edge $e \in E(\overline{G})$. Clearly, if G is k - i -critical, then $i(G + e) = k - 1$ for every edge $e \in E(\overline{G})$. The concept of independent domination critical was first introduced by Ao [A] in 1994. Ao established some basic properties of 3- i -critical graphs. The only result concerning matching properties is an existence of a perfect matching in 3- i -critical graphs of even order.

The complementary prism $G\overline{G}$ of G is the graph obtained from the disjoint union of G and \overline{G} by adding the edges of a perfect matching between the corresponding vertices of G and \overline{G} . Several well-known graphs are complementary prism. For example, the Petersen graph is the complementary prism $C_5\overline{C}_5$ and the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$. Haynes et al. [HHSM] were the first who introduced the complementary prism

in 2007. Since then domination and varieties of domination were studied in complementary prisms see [H, HHM].

A set $M \subseteq E(G)$ is a matching in $G = (V(G), E(G))$ if no two edges of M have an end common vertex. We say that a vertex u of G is M -saturated if u is an end vertex of some edge in a matching M . A matching F is called a perfect matching if each vertex of G is F -saturated. A graph G is said to be k -factor-critical if for every $S \subseteq V(G)$, $G - S$ has a perfect matching. k -factor-critical graphs are called factor-critical if $k = 1$ and bicritical if $k = 2$. Factor-critical and bicritical graphs play important roles in canonical decomposition theory. The interested reader is referred to [LP] for much more on this subject. The closed related concept to k -factor-criticality is k -extendability. A graph G is said to be k -extendable if for every matching M of size k in G there is a perfect matching in G containing all edges of M . It is easy to see that $2k$ -factor critical graphs are k -extendable. In what follows, we will say that a graph G has **matching properties** if either G has a perfect matching or being k -factor-critical or being k -extendable.

A problem that arises is that of investigating matching properties in k -critical graphs and in the complementary prism \overline{GG} of G .

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2. Objectives:

- 2.1 To investigate matching properties in independent domination critical graphs.
- 2.2 To investigate matching properties of the complementary prism \overline{GG} for selected graph G .
- 2.3 To investigate graphs G with matching properties in which \overline{G} also has the same matching properties.
- 2.4 To extend the known results on matching properties of graphs.
- 2.5 To extend the known results on complementary prisms.

3. Research methodology

- 3.1 Study and investigate existence research papers on relevant topics.
- 3.2 Establish a method in order to solve the problems.

4. Outputs:

- 4.1 N. Ananchuen and W. Ananchuen, A characterization of independent domination critical graphs with a cutvertex, (submitted).
- 4.2 N. Ananchuen and W. Ananchuen, Some toughness results in independent domination critical graphs (submitted).
- 4.3 N. Ananchuen, W. Ananchuen and L. Caccetta, A characterization of connected 3-*i*-critical graphs of connectivity two (submitted).
- 4.4 N. Ananchuen, W. Ruksasakchai and W. Ananchuen, On independent domination critical graphs and k -factor-critical, *Ars Combinatoria* (accepted).
- 4.5 P. Janseana, S. Rueangthampisan and N. Ananchuen, Extendability of the complementary prism of 2-regular graphs (submitted).
- 4.6 P. Janseana and N. Ananchuen, Matching extension in complementary prism of regular graphs (submitted).
- 4.7 P. Janseana and N. Ananchuen, Extendability of complementary prism of extendable graphs (submitted).

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1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by \overline{G} . For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. S is independent if no two vertices of S are adjacent. The maximum cardinality of an independent set of G is denoted by $\alpha(G)$. The number of components of G and the number of odd components of G are denoted by $c(G)$ and $c_o(G)$, respectively. A vertex v of G is a cutvertex if $c(G - v) > c(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G denoted by $N_G(v)$ is the set of all vertices of $V(G) - \{v\}$ which are adjacent to v . $N_G[v]$ is denoted the closed neighborhood of v , i.e., $N_G[v] = N_G(v) \cup \{v\}$. The non-neighborhood of v in G denoted by $\overline{N}_G(v)$ is $V(G) - N_G[v]$. For $S \subseteq V(G)$, $N_G(v) \cap S$ and $N_G[v] \cap S$ are denoted by $N_S(v)$ and $N_S[v]$, respectively. Further, $N_G(S)$ denotes $\bigcup_{x \in S} N_G(x)$ and $N_G[S]$ denotes $\bigcup_{x \in S} N_G[x]$. For simplicity, if S is a subgraph of G , then we also denote $N_G(v) \cap V(S)$ and $N_G[v] \cap V(S)$ by $N_S(v)$ and $N_S[v]$, respectively.

The degree of a vertex u in G is denoted by $deg_G(u) = |N_G(u)|$. The minimum degree and maximum degree in a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is called r -regular graph if $deg_G(u) = r$ for all $u \in V(G)$. The distance between two vertices x and y in a graph G is the number of edges in a shortest path connecting them and denoted by $d_G(x, y)$. The number of odd components of G is denoted by $c_o(G)$. A complete graph of order r is denoted by K_r . For graphs H and G , G is called H -free if G does not contain H as an induced subgraph. A subgraph H is called a clique if $H \cong K_r$, for some r .

For graphs H_1 and H_2 , the join of H_1 and H_2 , denoted by $H_1 + H_2$ is the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2) \cup \{uv | u \in V(H_1) \text{ and } v \in V(H_2)\}$. The cartesian product $G \times H$ of two graphs G and H has the vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1 u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1 v_2 \in E(H)$. The lexicographic product $G \circ H$ of two graphs G and H has the vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent either $u_1 u_2 \in E(G)$, or $u_1 = u_2$ and $v_1 v_2 \in E(H)$.

For subsets S and T of $V(G)$, we say that S dominates T , denoted by $S \succ T$, if $T \subseteq N_G[S]$. If $S \succ T$ where $S = \{s\}$, then we write $s \succ T$ instead of $\{s\} \succ T$. Further, if $T = V(H)$ where H is a subgraph of G , then we also write $S \succ H$ instead of $S \succ V(H)$ and we say that S is a *dominating set* for H . Thus S is a dominating set for G if each vertex of $V(G)$ is either in S or adjacent to some vertex of S . The minimum cardinality of a dominating set for G is called the domination number of G and denoted by $\gamma(G)$.

For a subgraph H of G , if $S \succ H$ and S is independent, then we say that S is an *independent dominating set* for H and denoted by $S \succ_i H$. Thus S is an independent dominating set for G if $S \succ_i G$. The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and denoted by $i(G)$. Observe that for any graph G , $\gamma(G) \leq i(G)$ and if $\gamma(G) = 1$, then $i(G) = 1$.

In 1994, Ao [5] introduced the concept so called ‘‘independent domination critical’’. A graph G is k - i -critical if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G . It is easy to see that the only 1- i -critical graphs are K_n for some positive

integer n . Ao [5] proved that G is 2- i -critical if and only if $\overline{G} \cong \bigcup_{i=1}^n K_{1,r_i}$ for some positive integers r_i and n . The problem that arises is that of characterizing connected k - i -critical graphs for $k \geq 3$. But this problem seems to be very difficult.

We point out here that there are several concepts on domination critical such as (ordinary) domination critical, total domination critical, connected domination critical. Some papers on these topics are in [1, 2, 3, 9, 10, 12, 15, 22, 23, 26, 31].

A set $M \subseteq E(G)$ is called a matching if no two edges of M have common end vertex. A vertex u is saturated by M if there is an edge in M incident with u . For simplicity, the set of all vertices saturated by M is denoted by $V(M)$. M is called a maximum matching in G if there is no matching N in G of size greater than $|M|$. A perfect matching in G is a matching that saturates all vertices of G . If M_1, M_2 are matching in a graph G , then a symmetric different of M_1 and M_2 , denoted by $M_1 \Delta M_2$, is an induced subgraph $G[(M_1 - M_2) \cup (M_2 - M_1)]$.

For a positive integer k , a connected graph G of order at least $2k + 2$ is k -extendable if for every matching M of size k in G , there is a perfect matching in G containing all edges of M . A graph G is k -factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For $k = 1$ and $k = 2$, k -factor-critical graph is also called factor-critical and bicritical, respectively. For simplicity, a graph with a perfect matching is called 0-extendable and 0-factor-critical. Observe that if G is k -extendable, then $|V(G)|$ is even and if G is k -factor-critical, then $|V(G)| \equiv k \pmod{2}$.

The concept of k -extendable graphs was introduced, in 1980, by Plummer [29]. He gave a sufficient condition for a graph to be k -extendable in terms of minimum degree. A fundamental theorem (see Theorem 2.8) that mainly used in studying matching extension was established. He also proved that 2-extendable non-bipartite graphs are bicritical. Since 1980, the concept of extendable graphs have been recieved attention from many researchers, see Yu and Liu [37].

The concept of k -factor-critical graphs was introduced, in 1996, by Favaron [14]. She gave a necessary and sufficient condition for a graph to be k -factor-critical and also provided a relationship between n -extendable graphs and k -factor - critical graphs.

A *complementary prism* of G , denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of its complement \overline{G} and then joining corresponding vertices by an edge. A complementary prism is a specific case of complementary product of graphs introduced by Haynes et al.[19] in 2007. Haynes et al. ([19, 20, 21]) studied some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number and the domination number.

In this report, we investigate some properties of connected k - i -critical graphs and properties of simple graphs to be k -extendable, for some positive integer k , in complementary prism of such graphs. The results about properties of connected k - i -critical graphs are in Sections 3 - 9. Sections 10 - 14 contain results concerning properties of simple graphs to be k -extendable in complementary prism. Section 2 concerns some preliminary results that we make use of in establishing our results.

2 Preliminary results

In this section we state some results that we make use of in establishing our main results. We begin with some terminology. For a pair of non-adjacent vertices u and v of G , I_{uv} denotes a minimum independent dominating set for $G + uv$. Our first result follows immediately from the definition of k - i -critical graphs.

Lemma 2.1. *Let G be a connected k - i -critical graph and let u and v be non-adjacent vertices of G . Then $|I_{uv}| = k - 1$, $|I_{uv} \cap \{u, v\}| = 1$ and $I_{uv} \cap (N_G(u) \cup N_G(v)) = \emptyset$. \square*

Lemma 2.2. *Let G be a connected 3- i -critical graph and let u and v be non-adjacent vertices of G . Then*

1. $I_{uv} = \{u, w\}$ or $I_{uv} = \{v, w\}$ for some $w \in V(G) - \{u, v\}$.
2. If $\{w\} = I_{uv} - \{u, v\}$, then $\{u, v, w\}$ is independent. \square

The next result provides an upper bound of the diameter of connected 3- i -critical graphs.

Lemma 2.3. [5]

The diameter of a connected 3- i -critical graph is at most 3. \square

Before we state the next result, we need one more definition. A graph G is k - i -vertex-critical if $i(G) = k$ and for each $u \in V(G)$, $i(G - u) < k$. It is easy to see that if G is k - i -vertex-critical, then $i(G - u) = k - 1$.

Lemma 2.4. [5]

1. A graph G is 2- i -critical if and only if $\overline{G} \cong \bigcup_{i=1}^n K_{1,r_i}$ for some positive integers r_i and n .
2. A graph G is 2- i -vertex-critical if and only if G is isomorphic to a complete graph without a perfect matching. \square

Our next result is a useful tool for establishing our results about k - i -critical graphs.

Lemma 2.5. *Let G be a connected 3- i -critical graph and let u and v be non-adjacent vertices of G . If $I_{uv} = \{u, w\}$ for some $w \in V(G) - \{u, v\}$, then $I_{uw} - \{u, w\} = \{v\}$, i.e., $I_{uw} = \{u, v\}$ or $I_{uw} = \{w, v\}$.*

Proof. By Lemma 2.2(2), $\{u, v, w\}$ is independent. Let $x \in V(G) - \{u, v, w\}$. Then $xu \in E(G)$ or $xw \in E(G)$ since $I_{uv} = \{u, w\}$. Then $x \notin I_{uw}$ by Lemma 2.2(2). Thus $I_{uw} - \{u, w\} = \{v\}$ as required. This proves our lemma. \square

Before stating a next result, we need to introduce another notation introduced by Ao. If x, y and z are vertices of G where $\{x, y\} \succ_i G - z$, then we write $[x, y] \xrightarrow{i} z$.

Lemma 2.6. [5] *Let G be a connected 3- i -critical graph and let S be an independent set of G with size at least $n \geq 4$. Then the vertices in S may be ordered as u_1, u_2, \dots, u_n in such a way that there exists a path $x_1 x_2 \dots x_{n-1}$ in $G - S$ with $[u_i, x_i] \xrightarrow{i} u_{i+1}$ for $1 \leq i \leq n-1$. \square*

We now turn our attention to some preliminaries results concerning perfect matchings, k -extendable and k -factor-critical graphs. These results are useful tools in establishing our results about extendability of complementary prism of graphs.

Theorem 2.7. [7] *(Tutte's Theorem) A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G - S) \leq |S|$. \square*

In 1980, Plummer [29] established a fundamental theorem on k -extendable graphs as following.

Theorem 2.8. [29] *Let G be a graph of order $p \geq 2k + 2$ and $k \geq 1$. If G is k -extendable, then*

- (a) G is $(k - 1)$ -extendable, and
- (b) G is $(k + 1)$ -connected. \square

He also gave a sufficient condition for a graph to be k -extendable in terms of minimum degree and claw-free condition, respectively.

Theorem 2.9. [29] *Let G be a graph of order $2p$. If $\delta(G) \geq p + k$, for a non-negative integer k , then G is k -extendable. \square*

Theorem 2.10. [30] *Let $k \geq 1$ be an integer and let G be a $(2k+1)$ -connected claw free graph with an even number of vertices. Then G is k -extendable. \square*

Ananchuen and Caccetta [4] gave a necessary condition for a neighbor set of a vertex having minimum degree in extendable graphs. They showed that:

Theorem 2.11. [4] *If G is a k -extendable graph on $p \geq 2k + 2$ vertices with $\delta(G) = k + t$, $1 \leq t \leq k \leq p$. If $d_G(u) = \delta(G)$, then the induced subgraph $G[N_G(u)]$ has at most $t - 1$ independent edges. \square*

A necessary and sufficient condition for a graph to be k -extendable and to be k -factor-critical were provided by Yu [35] and Favaron [14], respectively.

Theorem 2.12. [35] *A graph G is k -extendable ($k \geq 1$) if and only if for any $S \subseteq V(G)$,*

- (a) $c_o(G - S) \leq |S|$ and
- (b) $c_o(G - S) = |S| - 2t$, ($0 \leq t \leq k - 1$) implies that $F(S) \leq t$, where $F(S)$ is the size of a maximum matching in $G[S]$. \square

Theorem 2.13. [14] *A graph G is k -factor-critical if and only if $|V(G)| \equiv k \pmod{2}$ and for $S \subseteq V(G)$ with $|S| \geq k$, $c_o(G - S) \leq |S| - k$. \square*

Some following properties of k -factor-critical graphs were proved in [14].

Theorem 2.14. [14] *Let G be a k -factor-critical graph. Then G is $(k - 2)$ -factor-critical. \square*

Theorem 2.15. [14] *If G is a $2k$ -extendable non-bipartite graph for $2k \geq 2$, then G is a $2k$ -factor-critical graph. \square*

Maschlanka and Volkmann [28] gave a relationship between k -extendable non-bipartite graph and the independence number.

Theorem 2.16. [28] *Let G be a k -extendable non-bipartite graph of order p . Then $\alpha(G) \leq \frac{1}{2}p - k$. \square*

In Phd. Thesis of Yu [36], he gave the following observation.

Observation 2.17. *A graph G is k -extendable if and only if for any matching M of size i ($1 \leq i \leq k$), $G - V(M)$ is a $(k - i)$ -extendable graph. \square*

An observation on k -factor-critical graphs which is similar to Observation 2.17 can be stated as following.

Observation 2.18. *Let G be a k -factor-critical graph and $S \subseteq V(G)$ where $|S| < k$. Then $G - S$ is $(k - |S|)$ -factor-critical. \square*

A following lemma follows from Theorem 2.16.

Lemma 2.19. *Let G be a k -extendable non-bipartite graph and $S \subseteq V(G)$ where $|S| \leq 2k - 2$. Then $G - S$ is a non-bipartite graph.*

Proof. Suppose to the contrary that $G - S$ is a bipartite graph. Then $\alpha(G) \geq \alpha(G - V(S)) \geq \frac{1}{2}(|V(G)| - (2k - 2)) = \frac{1}{2}|V(G)| - k + 1$. But this contradicts Theorem 2.16 and completes the proof of our lemma. \square

Our next corollary follows immediately by Observation 2.17 and Lemma 2.19

Corollary 2.20. *Let G be a k -extendable non-bipartite graph and let $M \subseteq E(G)$ where $|M| = l \leq k - 1$. Then $G - V(M)$ is $(k - l)$ -extendable non-bipartite. \square*

Note that the upper bound on $|M|$ in Corollary 2.20 is best possible. Let $G = K_{2k} + K_{t,t}$ for some positive integers $k, t \geq 2$. It is easy to see that G is k -extendable. Clearly, there is a matching M of size k in $G[K_{2k}]$ such that $G - V(M)$ is a bipartite graph.

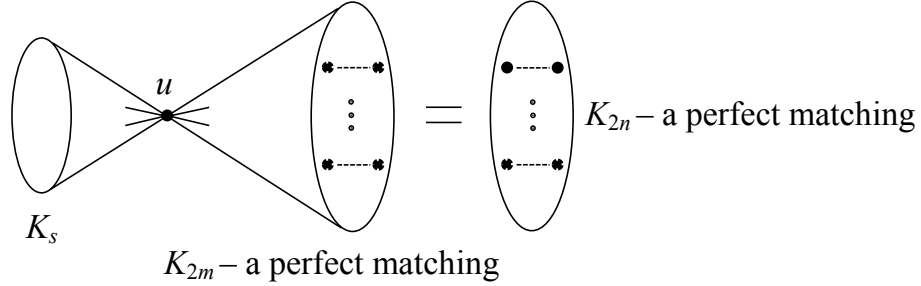


Figure 1: The structure of a graph in the class \mathcal{G}_1

3 Classes of connected k - i -critical graphs with a cutvertex

In this section, we provide seven classes of connected k - i -critical graphs with a cutvertex, four of them for $k = 3$, one for $k = 4$ and the another two for $k \geq 4$. We begin with four classes of connected 3- i -critical graphs.

I. The class \mathcal{G}_1

For positive integers n , m and $s \geq 2$, define a graph $G \in \mathcal{G}_1$ of order $2n + 2m + s + 1$ as follows. Set $V(G) = \{u\} \cup X \cup Y \cup Z$ where $|X| = 2n$, $|Y| = 2m$ and $|Z| = s$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $G[Y] = K_{2m}$ - a perfect matching and $G[Z] = K_s$. Further, join u to each vertex of $Y \cup Z$ and finally join each vertex of Y to every vertex of X . This defines the class \mathcal{G}_1 . Figure 1 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_1$ is 3- i -critical. Note that in our diagram a “double line” denotes the join.

II. The class \mathcal{G}_2

For positive integers n and $m \geq 2$, define a graph $G \in \mathcal{G}_2$ of order $2n + m + 2$ as follows. Set $V(G) = \{u, c\} \cup X \cup Y$ where $|X| = 2n$, $|Y| = m$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $G[Y] = \bigcup_{i=1}^l K_{1,r_i}$ such that $m = l + \sum_{i=1}^l r_i$, $m \geq 2l$ and $l \geq 1$. Further, join u to each vertex of $\{c\} \cup Y$ and finally join each vertex of Y to every vertex of X . This defines the class \mathcal{G}_2 . Figure 2 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_2$ is 3- i -critical.

III. The class \mathcal{G}_3

For positive integers n_1 and $s \geq 2n_1$ and non-negative integer n_2 , define a graph $G \in \mathcal{G}_3$ of order $2n_1 + 2n_2 + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X_1 \cup X_2 \cup Z$ where $|X_1| = 2n_1$, $|X_2| = 2n_2$ and $|Z| = s$. The edges of G are defined as follows. $G[X_1 \cup X_2] = K_{2n_1+2n_2} - (F_1 \cup F_2)$ where F_i is a perfect matching in $G[X_i]$ for $1 \leq i \leq 2$ and $G[Z] = K_s$. Further, join u to each vertex of $\{c\} \cup Z$ and join each vertex of Z to every vertex of X_2 (if $X_2 \neq \emptyset$). Finally, we add the set of edges E between Z and X_1 . Each vertex of Z is joined to $2n_1 - 1$ vertices of X_1 in such a way that each vertex of X_1 is a non-neighbor of some vertex of

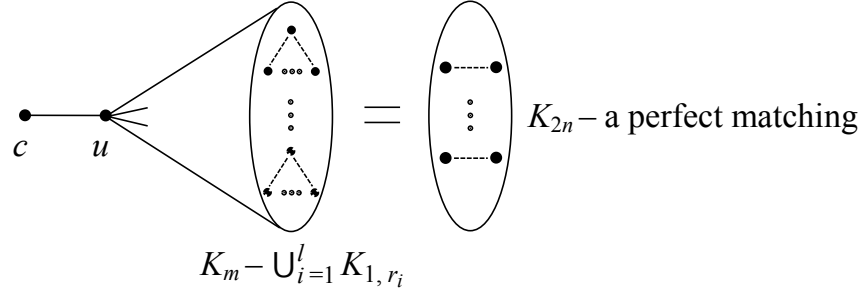


Figure 2: The structure of a graph in the class \mathcal{G}_2

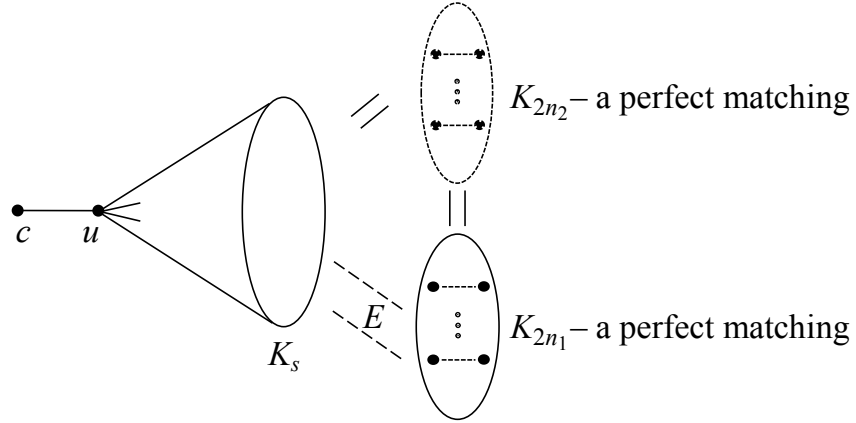


Figure 3: The structure of a graph in the class \mathcal{G}_3

Z . This defines the class \mathcal{G}_3 . Figure 3 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_3$ is 3- i -critical.

IV. The class \mathcal{G}_4

For positive integers $n_1, s \geq 2n_1$ and $m \geq 2$ and non-negative integer n_2 , define a graph $G \in \mathcal{G}_4$ of order $2n_1 + 2n_2 + m + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X_1 \cup X_2 \cup Y \cup Z$ where $|X_1| = 2n_1, |X_2| = 2n_2, |Y| = m$ and $|Z| = s$. The edges of G are defined as follows. $G[X_1 \cup X_2] = K_{2n_1+2n_2} - (F_1 \cup F_2)$ where F_i is a perfect matching in $G[X_i]$ for $1 \leq i \leq 2$, $G[Y] = \bigcup_{i=1}^l K_{1, r_i}$ such that $m = l + \sum_{i=1}^l r_i, m \geq 2l$ and $l \geq 1$ and $G[Z] = K_s$. Further, join u to each vertex of $\{c\} \cup Y \cup Z$, join each vertex of Y to every vertex of $Z \cup X_1 \cup X_2$ and join each vertex of Z to every vertex of X_2 (if $X_2 \neq \emptyset$). Finally, we add the set of edges E between Z and X_1 as defined in the class \mathcal{G}_3 . This defines the class \mathcal{G}_4 . Figure 4 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_4$ is 3- i -critical.

Our next class is a class of connected 4- i -critical graphs with a cutvertex.

V. The class \mathcal{G}_5

For positive integers n, m and $s \geq 2$, define a graph $G \in \mathcal{G}_5$ of order $2n + 2m + s + 2$ as

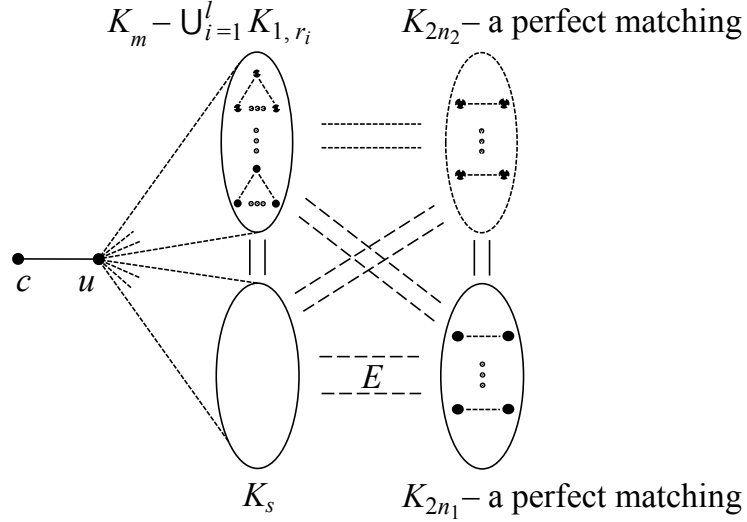


Figure 4: The structure of a graph in the class \mathcal{G}_4

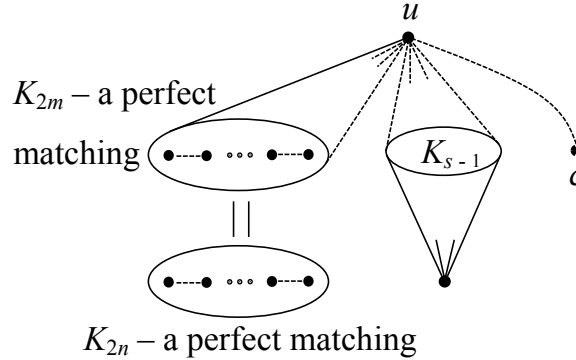


Figure 5: The structure of a graph in the class \mathcal{G}_5

follows. Set $V(G) = \{u, c\} \cup X \cup Y \cup Z$ where $|X| = 2n$, $|Y| = 2m$ and $|Z| = s$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $G[Y] = K_{2m}$ - a perfect matching and $G[Z] = K_s$. Further, join u to each vertex of $Y \cup \{c\}$ and $s - 1$ vertices of Z and finally join each vertex of Y to every vertex of X . This defines the class \mathcal{G}_5 . Figure 5 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_5$ is 4- i -critical.

We conclude this section by constructing two classes of connected k - i -critical graphs with a cutvertex for $k \geq 4$.

VI. The class \mathcal{G}_6

For positive integers s , n , and k where $s \geq n \geq k - 1 \geq 3$, define a graph $G \in \mathcal{G}_6$ of order $s + n + k - 1$ as follows. Set $V(G) = \{u\} \cup X \cup Z \cup W$ where $|X| = n$, $|Z| = s$ and $|W| = k - 2$. The edges of G are defined as follows. $G[X]$ is both $(k - 1)$ - i -critical and $(k - 1)$ - i -vertex-critical and $G[W] = \overline{K}_{k-2}$. Further, join u to each vertex of $W \cup Z$, and

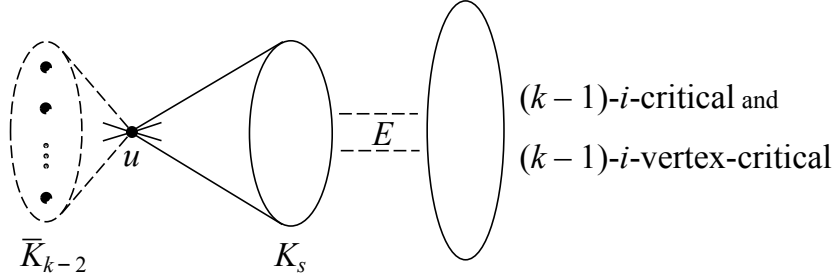


Figure 6: The structure of a graph in the class \mathcal{G}_6

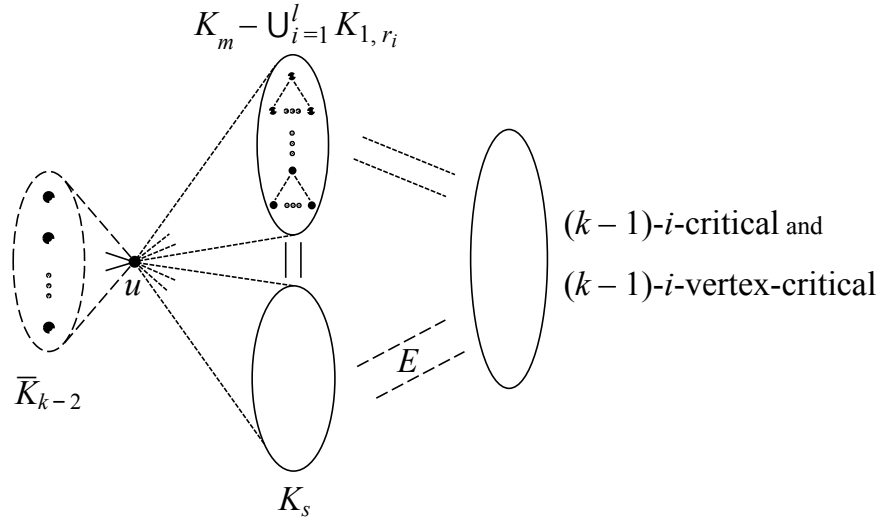


Figure 7: The structure of a graph in the class \mathcal{G}_7

then add the set of edges E between Z and X as similar as defined in the class \mathcal{G}_3 . That is, each vertex of Z is joined to $n - 1$ vertices of X in such a way that each vertex of X is a non-neighbor of some vertex of Z . This defines the class \mathcal{G}_6 . Figure 6 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_6$ is k - i -critical for $k \geq 4$.

VI. The class \mathcal{G}_7

For positive integers s, n, k and m where $s \geq n \geq k - 1 \geq 3$ and $m \geq 2$, define a graph $G \in \mathcal{G}_7$ of order $s + n + k + m - 1$ as follows. Set $V(G) = \{u\} \cup X \cup Y \cup Z \cup W$ where $|X| = n, |Y| = m, |Z| = s$ and $|W| = k - 2$. The edges of G are defined as follows. $G[X]$ is both $(k - 1)$ - i -critical and $(k - 1)$ - i -vertex-critical, $\overline{G[Y]} = \bigcup_{i=1}^l K_{1,r_i}$ such that $m = l + \sum_{i=1}^l r_i, m \geq 2l$ and $l \geq 1$, $G[Z] = K_s$ and $G[W] = \overline{K}_{k-2}$. Further, join u to each vertex of $W \cup Y \cup Z$, join each vertex of Y to every vertex of $Z \cup X$. Finally, we add the set of edges E between Z and X as similar as defined in the class \mathcal{G}_6 . This defines the class \mathcal{G}_7 . Figure 7 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_7$ is k - i -critical for $k \geq 4$.

Note that for a positive integer $k \geq 2$, \overline{K}_{k-1} is both $(k-1)$ - i -critical and $(k-1)$ - i -vertex-critical. Hence, $\mathcal{G}_6 \neq \emptyset$ and $\mathcal{G}_7 \neq \emptyset$.

4 Connected k - i -critical graphs with a cutvertex

Lemma 4.1. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Then*

1. *If $c(G-u) \geq k-1$, then $G-u$ contains at least one non-singleton and at most two non-singleton components.*
2. *$c(G-u) \leq k-1$.*

Proof. For $1 \leq i \leq t$, let C_1, C_2, \dots, C_t be the components of $G-u$. It is easy to see that $G-u$ must contain at least one non-singleton component since G is connected and $i(G) = k \geq 3$.

(1) Suppose to the contrary that $G-u$ contains $C_j, C_{j'}$ and $C_{j''}$ as non-singleton components where $\{j, j', j''\} \subseteq \{1, 2, \dots, t\}$. Choose $x \in N_{C_j}(u)$ and $y \in N_{C_{j'}}(u)$. Consider $G+xy$. By Lemma 2.1, $|I_{xy}| = k-1$, $|I_{xy} \cap \{x, y\}| = 1$ and $u \notin I_{xy}$. We may suppose without loss of generality that $x \in I_{xy}$. Since $u \notin I_{xy}$, $I_{xy} \cap V(C_i) \neq \emptyset$ for $1 \leq i \leq t$ and $i \notin \{j, j'\}$. Further, $I_{xy} \cap V(C_{j'}) \neq \emptyset$ since $V(C_{j'}) - \{y\} \neq \emptyset$. Consequently, $t = k-1$ and $|I_{xy} \cap V(C_i)| = 1$ for $1 \leq i \leq t$. Thus $x \succ C_j$. Put $\{w\} = I_{xy} \cap V(C_{j''})$. Then $w \succ C_{j''}$. Now consider $G+xw$. By Lemma 2.1, $|I_{xw}| = k-1$, $|I_{xw} \cap \{x, w\}| = 1$ and $u \notin I_{xw}$. If $x \notin I_{xw}$, then $I_{xw} \cap V(C_j) = \emptyset$ since $I_{xw} \cap (N_G(x) \cup N_G(w)) = \emptyset$ and $x \succ C_j$. But then no vertex of I_{xw} dominates $V(C_j) - \{x\}$, a contradiction. Hence, $x \in I_{xw}$. By similar arguments, $w \in I_{xw}$. But this contradicts the fact that $|I_{xw} \cap \{x, w\}| = 1$. Hence, $G-u$ contains at most two non-singleton components. This proves (1).

(2) Suppose to the contrary that $t \geq k$. By (1), we may assume that $|V(C_1)| \geq 1$ and $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq t$. Since G is connected, u is adjacent to every vertex of $\bigcup_{i=3}^t V(C_i)$. Now choose $a \in N_{C_1}(u)$ and $b \in N_{C_2}(u)$. Consider $G+ab$. By Lemma 2.1, $|I_{ab}| = k-1$, $|I_{ab} \cap \{a, b\}| = 1$ and $u \notin I_{ab}$. We may assume without loss of generality that $a \in I_{ab}$. Since $u \notin I_{ab}$, $I_{ab} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq t$. Then $t = k$ and $V(C_2) = \{b\}$ since $|I_{ab}| = k-1$. Clearly, $bu \in E(G)$ since G is connected. Further, $\overline{N}_{C_1}(u) \neq \emptyset$ otherwise $u \succ G$. Let $c \in \overline{N}_{C_1}(u)$. We now consider $G+bc$. By Lemma 2.1, $|I_{bc}| = k-1$, $|I_{bc} \cap \{b, c\}| = 1$ and $u \notin I_{bc}$. Then $I_{bc} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq t$ and $V(C_1) \cup V(C_2)$ must be dominated, in $G+bc$, by one of element in $\{b, c\}$ since $|I_{bc}| = k-1$ and $t = k$. It is easy to see that $I_{bc} \cap \{b, c\} = \{c\}$ since $a \in V(C_1) - \{c\}$. Thus $c \succ C_1$. But then $\{c, u\} \succ_i G$, a contradiction since $i(G) = k \geq 3$. This proves (2) and completes the proof of our lemma. \square

Lemma 4.2. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Suppose C_1, C_2, \dots, C_{k-1} are the components of $G - u$ where $|V(C_1)| \geq 2$, $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq k-1$. If $x \in V(C_1)$ and $y \in V(C_2)$ where $\{x, y\} \cap N_G(u) \neq \emptyset$, then $u \notin I_{xy}$, $|I_{xy} \cap V(C_1)| \geq 1$ and $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$. Further, if $|V(C_2)| \geq 2$, then $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$.*

Proof. Consider $G + xy$. By Lemma 2.1, $|I_{xy}| = k - 1$, $|I_{xy} \cap \{x, y\}| = 1$ and $u \notin I_{xy}$. Then $I_{xy} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq k - 1$. Because $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$, $|I_{xy} \cap V(C_i)| = 1$ for $3 \leq i \leq k - 1$. Since $|I_{xy}| = k - 1$ and $c(G - u) = k - 1$, it follows that $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$. Further, $|I_{xy} \cap V(C_1)| \geq 1$ because $|V(C_1)| \geq 2$. We now suppose that $|V(C_2)| \geq 2$. If $|I_{xy} \cap V(C_1)| = 2$, then no vertex of I_{xy} dominates $C_2 - y$, a contradiction. Hence, $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$. This proves our lemma. \square

Lemma 4.3. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Suppose C_1, C_2, \dots, C_{k-1} are the components of $G - u$ where $|V(C_1)| \geq 2$, $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$. Then either $i(C_1) = 1$ and $i(C_2) = 2$ or $i(C_1) = 2$ and $i(C_2) = 1$. Further, if $i(C_1) = 2$ and $i(C_2) = 1$, then C_2 is complete.*

Proof. We first suppose that $|V(C_2)| = 1$. Clearly, $i(C_1) \geq 2$ otherwise $i(G) \leq k - 1$. We need only show that $i(C_1) = 2$. Put $\{y\} = V(C_2)$. Then $yu \in E(G)$. Since $i(G) = k \geq 3$, $\overline{N}_{C_1}(u) \neq \emptyset$. Let $x \in \overline{N}_{C_1}(u)$. Consider $G + xy$. By Lemma 4.2, $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$ and $|I_{xy} \cap V(C_1)| \geq 1$. By Lemma 2.1, $|I_{xy} \cap \{x, y\}| = 1$. If $x \in I_{xy}$, then $y \notin I_{xy}$ and thus $|I_{xy} \cap V(C_1)| = 2$ since $V(C_2) = \{y\}$. If $y \in I_{xy}$, then $x \notin I_{xy}$ and thus the only vertex of $I_{xy} \cap V(C_1)$, w say, dominates $V(C_1) - \{x\}$ since $|V(C_1)| \geq 2$ and $wx \notin E(G)$. In either case, $i(C_1) = 2$.

We next suppose that $|V(C_2)| \geq 2$. Choose $a \in N_{C_1}(u)$ and $b \in N_{C_2}(u)$. Consider $G + ab$. By Lemma 4.2, $|I_{ab} \cap V(C_1)| = 1$ and $|I_{ab} \cap V(C_2)| = 1$. Put $\{a'\} = I_{ab} \cap V(C_1)$ and $\{b'\} = I_{ab} \cap V(C_2)$. If $a' = a$, then $b' \neq b$ by Lemma 2.1. Thus $a \succ C_1$ and $b' \succ C_2 - b$. Consequently, $i(C_1) = 1$ and $\{b, b'\} \succ_i C_2$. If $i(C_2) = 1$, then $i(G) \leq k - 1$, a contradiction. Hence, $i(C_2) = 2$. Similarly, if $a' \neq a$, then $b' = b$ and thus $i(C_1) = 2$ and $i(C_2) = 1$ as required.

We now suppose that $i(C_1) = 2$ and $i(C_2) = 1$. In order to finish the proof of our lemma, we need the following claim.

Claim : *If $y \in N_{C_2}(u)$, then $y \succ C_2$.*

If $|V(C_2)| = 1$, then our claim follows immediately. So we may assume that $|V(C_2)| \geq 2$. Choose $x \in V(C_1)$. Consider $G + xy$. By Lemma 4.2, $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$. Further, $u \notin I_{xy}$. If $I_{xy} \cap V(C_1) = \{x\}$, then $x \succ C_1$, contradicting the fact that $i(C_1) = 2$. Hence, $I_{xy} \cap V(C_1) \neq \{x\}$. Consequently, $I_{xy} \cap V(C_2) = \{y\}$ by Lemma 2.1 and thus $y \succ C_2$. This settles our claim.

We now ready to finish our proof. Suppose to the contrary that C_2 is not complete. Then there exist non-adjacent vertices v and w of $V(C_2)$. By the above claim, $\{v, w\} \subseteq \overline{N}_{C_2}(u)$. Let $x \in N_{C_1}(u)$. Consider $G + xv$. By Lemma 4.2, $|I_{xv} \cap V(C_1)| = 1$ and $|I_{xv} \cap V(C_2)| = 1$. Further, $u \notin I_{xv}$. If $I_{xv} \cap V(C_2) = \{v\}$, then no vertex of I_{xv} dominates w , a contradiction. Hence, $I_{xv} \cap V(C_2) \neq \{v\}$. Therefore, $I_{xv} \cap V(C_1) = \{x\}$ by Lemma 2.1. Clearly, $x \succ C_1$. But this contradicts the fact $i(C_1) = 2$. Hence, C_2 is complete as required. This completes the proof of our lemma. \square

Theorem 4.4. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex and $c(G - u) = k - 1$. Suppose $G - u$ contains exactly two non-singleton components, C_1 and C_2 say, where $i(C_1) = 2$ and $i(C_2) = 1$. Then*

1. $G[N_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching. Further, each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$.
2. $G[\overline{N}_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching.
3. $3 \leq k \leq 4$. Further, if $k = 3$, then G is isomorphic to a graph in the class \mathcal{G}_1 and if $k = 4$, then G is isomorphic to a graph in the class \mathcal{G}_5 .

Proof. Let W be a set of all vertices in the singleton components of $G - u$. Clearly, $|W| = k - 3 \geq 0$ since $c(G - u) = k - 1$. Further, $W \subseteq N_G(u)$ since G is connected. By our hypothesis that $i(C_1) = 2$ and $i(C_2) = 1$ together with Lemma 4.3, C_2 is complete. It then follows that $\overline{N}_{C_1}(u) \neq \emptyset$ since $i(G) = k \geq 3$.

Claim 1: *If $x \in N_{C_1}(u)$, $y \in V(C_2)$, then $I_{xy} \cap V(C_1) = \{x'\}$ where $x' \in N_{C_1}(u) - \{x\}$ and $I_{xy} \cap V(C_2) = \{y\}$.*

Consider $G + xy$. By Lemma 4.2, $u \notin I_{xy}$ and $|I_{xy} \cap V(C_1)| = |I_{xy} \cap V(C_2)| = 1$. Suppose to the contrary that $x \in I_{xy}$. By Lemma 2.1, $I_{xy} \cap V(C_2) = \{y'\}$ where $y' \neq y$. Then $I_{xy} \cap N_G(y) \neq \emptyset$ since C_2 is complete. But this contradicts Lemma 2.1. Hence, $x \notin I_{xy}$. Therefore, $y \in I_{xy}$ and thus $I_{xy} \cap V(C_2) = \{y\}$. Put $\{x'\} = I_{xy} \cap V(C_1)$. Clearly, $x' \neq x$ and $x' \succ C_1 - x$. Suppose to the contrary that $x' \in \overline{N}_{C_1}(u)$. Then $\{x', u\} \succ_i G - \overline{N}_{C_2}(u)$. If $\overline{N}_{C_2}(u) = \emptyset$, then $i(G) = 2 < k$, a contradiction. Hence, $\overline{N}_{C_2}(u) \neq \emptyset$. But then $3 \leq i(G) = k \leq 3$ since C_2 is complete. Thus $k = 3$. Clearly, $d(x', y') > 3$ where $y' \in \overline{N}_{C_2}(u)$. But this contradicts Lemma 2.3. Hence, $x' \in N_{C_1}(u) - \{x\}$ as required. This settles our claim.

(1) Let $x \in N_{C_1}(u)$ and $y \in V(C_2)$. Consider $G + xy$. By Claim 1, $I_{xy} \cap V(C_1) = \{x'\}$ for some $x' \in N_{C_1}(u) - \{x\}$. Clearly, $x' \succ C_1 - x$. Now consider $G + x'y$. Again, by Claim 1, $I_{x'y} \cap V(C_1) = \{x''\}$ for some $x'' \in N_{C_1}(u) - \{x'\}$. Since $x' \succ C_1 - x$, $x'' = x$ by Lemma 2.1. Then $x'' = x \succ C_1 - x'$. If $|N_{C_1}(u)| = 2$, we are done. So suppose $|N_{C_1}(u)| \geq 3$. Let $w \in N_{C_1}(u) - \{x, x'\}$. Consider $G + wy$. By Claim 1, $I_{wy} \cap V(C_1) = \{w'\}$ for some $w' \in N_{C_1}(u) - \{w\}$. By Lemma 2.1, $w' \notin \{x, x'\}$ since $w \in N_G(x) \cap N_G(x')$. Clearly, $w' \succ C_1 - w$. Now consider $G + w'y$. By similar arguments, $I_{w'y} \cap V(C_1) = \{w\}$ and

$w \succ C_1 - w'$. Continuing in this fashion, $G[N_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching. Further, in our argument, it is clear that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$. This proves (1).

Claim 2: For $x \in \overline{N}_{C_1}(u)$ and $y \in N_{C_2}(u)$, $I_{xy} \cap V(C_1) = \{x'\}$ where $x' \in \overline{N}_{C_1}(u) - \{x\}$ and $I_{xy} \cap V(C_2) = \{y\}$.

By similar arguments as in the proof of Claim 1 and the fact that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$, by (1), our claim follows.

(2) By applying similar arguments as in the proof of (1) together with Claim 2 and the fact that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$, by (1), (2) follows.

(3) Clearly, $u \succ N_{C_1}(u) \cup N_{C_2}(u) \cup W$. We first suppose that $\overline{N}_{C_2}(u) = \emptyset$. Then $V(C_2) \subseteq N_G(u)$ and thus $i(G) = k = 3$ since $i(G[\overline{N}_{C_1}(u)]) = 2$ by (2). Hence, G is isomorphic to the graph in the class \mathcal{G}_1 .

We now suppose that $\overline{N}_{C_2}(u) \neq \emptyset$. Let $y \in \overline{N}_{C_2}(u)$. Then $\{u, y\} \succ_i N_{C_1}(u) \cup V(C_2) \cup W$. Thus, by (2), $3 \leq i(G) = k \leq 4$. By Lemma 2.3, $k \neq 3$. Hence, $k = 4$ and $|W| = 1$. We next show that $|\overline{N}_{C_2}(u)| = 1$. Suppose this is not the case. Let $y' \in \overline{N}_{C_2}(u) - \{y\}$. Consider $G + uy'$. By Lemma 2.1, $|I_{uy'}| = 3$, $|I_{uy'} \cap \{u, y'\}| = 1$. We first suppose that $I_{uy'} \cap \{u, y'\} = \{u\}$. Then $I_{uy'} \cap V(C_2) = \emptyset$ since $I_{uy'} \cap (N_G(u) \cup N_G(y')) = \emptyset$ and C_2 is complete. But then no vertex of $I_{uy'}$ dominates y , a contradiction. Hence, $I_{uy'} \cap \{u, y'\} = \{y'\}$. Then $I_{uy'} \cap W = \emptyset$ by Lemma 2.1 since the only vertex of W is adjacent to u . But then no vertex of $I_{uy'}$ dominates W , again a contradiction. Hence, $|\overline{N}_{C_2}(u)| = 1$. It follows by (1) and (2) that G is isomorphic to the graph in the class \mathcal{G}_5 . This completes the proof of our theorem. \square

Theorem 4.5. For a positive integer $k \geq 3$, let G be a connected graph with $i(G) = k$ and G contains u as a cutvertex where $c(G - u) = k - 1$. Suppose $G - u$ contains exactly one non-singleton component, C say. Then G is k - i -critical if and only if

1. $G[\overline{N}_C(u)]$ is $(k - 1)$ - i -critical and $(k - 1)$ - i -vertex-critical.
2. $N_C(u)$ consists of two disjoint sets, Y and Z say, where each vertex of Y dominates $\overline{N}_C(u)$ and $Z = N_C(u) - Y$. Further, if $Y \neq \emptyset$, then $G[Y]$ is 2- i -critical and if $Z \neq \emptyset$, then $G[Z]$ is complete and for each vertex $z \in Z$, there is exactly one vertex of $\overline{N}_C(u)$, z' say, such that $z \succ C - z'$. Moreover, if $k \geq 4$, each vertex of $\overline{N}_C(u)$ is not adjacent to at least one vertex of Z and thus $Z \neq \emptyset$ and $|Z| \geq |\overline{N}_C(u)|$.

Proof. Let W be a set of all vertices in the singleton components of $G - u$. Clearly, $|W| = k - 2 \geq 1$ since $c(G - u) = k - 1$. Further, $W \subseteq N_G(u)$ since G is connected. Note that $N_G(u) = W \cup N_C(u)$. It then follows that $\overline{N}_C(u) \neq \emptyset$ since $i(G) = k \geq 3$. Put $X = \overline{N}_C(u)$. It is easy to see that $i(G[X]) \geq k - 1 \geq 2$. For the sufficiency, it is not difficult to show that G is k - i -critical.

We now prove the necessity. Suppose that G is k - i -critical. Then the following claims follow by Lemma 2.1 and the fact that $N_G(u) = W \cup N_C(u)$.

Claim 1 : For $v \in X$, $u \in I_{uv}$ and $I_{uv} - \{u\} \subseteq X - \{v\}$.

Claim 2 : If $u \in I_{vw}$ for some $v, w \in X$, then $I_{vw} - \{u\} \subseteq X$.

Claim 3 : If $u \notin I_{vw}$ for some $v, w \in V(C)$, then either $I_{vw} = \{v\} \cup W$ or $I_{vw} = \{w\} \cup W$.

(1) Since $i(G[X]) \geq k - 1 \geq 2$, there exist non-adjacent vertices $x_1, x_2 \in X$. We first show that $G[X]$ is $(k-1)$ - i -critical. Consider $G + x_1x_2$. By Lemma 2.1, $|I_{x_1x_2}| = k - 1$ and $|I_{x_1x_2} \cap \{x_1, x_2\}| = 1$. We may assume without loss of generality that $I_{x_1x_2} \cap \{x_1, x_2\} = \{x_1\}$. We first suppose that $u \notin I_{x_1x_2}$. Then, by Claim 3, $I_{x_1x_2} = \{x_1\} \cup W$. This implies that $x_1 \succ C - x_2$ and thus $\{x_1, x_2\} \succ_i G[X]$. Consequently, $k = 3$ and $G[X]$ is 2- i -critical.

We now suppose that $u \in I_{x_1x_2}$. By our Claim 2, $I_{x_1x_2} - \{u\} \subseteq X$ and then $I_{x_1x_2} - \{u\} \succ_i X - \{x_2\}$. Hence, $(I_{x_1x_2} - \{u\}) \cup \{x_2\} \succ_i G[X]$ and thus $i(G[X]) = k - 1$. Consequently, G is $(k - 1)$ - i -critical.

By the above argument, $i(G[X]) = k - 1$. We next show that $G[X]$ is $(k - 1)$ - i -vertex-critical. Let $x \in X$. Consider $G + ux$. By Lemma 2.1, $|I_{ux}| = k - 1$ and by Claim 1, $u \in I_{ux}$ and $I_{ux} - \{u\} \subseteq X - \{x\}$. Then $I_{ux} - \{u\} \succ_i X - \{x\}$. Thus $i(G[X - \{x\}]) \leq k - 2$. If $i(G[X - \{x\}]) < k - 2$, then $i(G[X]) < k - 1$, a contradiction. Hence, $G[X]$ is $(k - 1)$ - i -vertex-critical as required. This proves (1).

(2) We first suppose that $Z \neq \emptyset$. Let $z \in Z$. Then there exists $x \in X$ such that $zx \notin E(G)$. Consider $G + zx$. By Lemma 2.1, $u \notin I_{zx}$. By Claim 3, either $I_{zx} = \{x\} \cup W$ or $I_{zx} = \{z\} \cup W$. If $I_{zx} = \{x\} \cup W$, then $x \succ C - z$ and thus $x \succ X$. But this contradicts (1) since $k \geq 3$. Hence, $I_{zx} = \{z\} \cup W$. Then $z \succ C - x$. Consequently, $G[Z]$ is complete.

We next suppose $Y \neq \emptyset$. Note that each vertex of Y dominates $Z \cup X$. If there is a vertex $y \in Y$ such that $y \succ Y$, then $\{y\} \cup W \succ_i G$. But this contradicts the fact that $i(G) = k$. Hence, for each vertex $y \in Y$, there exists a vertex $y' \in Y - \{y\}$ such that $yy' \notin E(G)$. Consider $G + yy'$. By Lemma 2.1, $u \notin I_{yy'}$. We may assume by Claim 3 that $I_{yy'} = \{y\} \cup W$. Then $y \succ C - y'$ and thus $y \succ Y - y'$. Hence, $\{y, y'\} \succ_i G[Y]$ and $y \succ G[Y] + yy'$. Therefore, $G[Y]$ is 2- i -critical.

We now suppose that $k \geq 4$. By the definition of Y , each vertex of X is adjacent to every vertex of Y . Suppose to the contrary that there exists a vertex $x \in X$ such that x is adjacent to every vertex of Z . Then $x \succ Y \cup Z$. Let $w \in W$. Consider $G + xw$. By Lemma 2.1, $|I_{xw}| = k - 1$, $|I_{xw} \cap \{x, w\}| = 1$ and $I_{xw} \cap (\{u\} \cup Y \cup Z) = \emptyset$. Since $W \subseteq N_G(u)$, $W - \{w\} \subseteq I_{xw} - \{x, w\}$. Because $|W| = k - 2$ and $|I_{xw} \cap \{x, w\}| = 1$, it follows that $|I_{xw} - (\{x, w\} \cup W)| = 1$. Put $\{a\} = I_{xw} - (\{x, w\} \cup W)$. Clearly, $a \in X - \{x\}$. Consequently, $\{a, x\} \succ X$. But this contradicts the fact that $i(G[X]) = k - 1 \geq 3$. Hence, each vertex of X is not adjacent to at least one vertex of Z . It then follows that $|Z| \geq |X|$. This proves (2) and completes the proof of our theorem.

Lemma 4.1 together with Theorems 4.4(3) and 4.5 implies the following theorem.

Theorem 4.6. *For a positive integer $k \geq 4$, if G is a connected k - i -critical graph with a cutvertex u where $c(G - u) = k - 1$, then $G \in \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$. \square*

Let Z and X be defined as in Theorem 4.5. Note that if $k = 3$, a graph in the class \mathcal{G}_2 shows that Z is empty. Further, a graph in the class $\mathcal{G}_3 \cup \mathcal{G}_4$ shows that there might exist a vertex of X which is adjacent to every vertex of Z . Our next theorem provides a characterization of connected 3- i -critical graphs containing a cutvertex.

Theorem 4.7. *If G is a connected 3- i -critical graph containing a cutvertex, then $G \in \mathcal{G}_i$, defined in Section 3, for some $1 \leq i \leq 4$.*

Proof. Let u be a cutvertex of G . By Lemma 4.1, $G - u$ contains exactly two components, at least one of them is non-singleton component. If $G - u$ contains exactly two non-singleton components, then G is isomorphic to a graph in the Class \mathcal{G}_1 by Theorem 4.4(3). We may now suppose that $G - u$ contains exactly one non-singleton component. Let C_1 and C_2 be the components of $G - u$ where $|V(C_1)| = 1$ and $|V(C_2)| \geq 2$. Clearly, $\overline{N}_{C_2}(u) \neq \emptyset$. Put $V(C_1) = \{c\}$ and $X = \overline{N}_{C_2}(u)$. Then, by Theorem 4.5(1), $G[X]$ is 2- i -vertex-critical. Thus $G[X]$ is isomorphic to a complete graph without a perfect matching by Lemma 2.4(2).

By Theorem 4.5(2), $N_{C_2}(u)$ consists of two disjoint sets, Y and Z say, where each vertex of Y dominates X and $Z = N_{C_2}(u) - Y$. Further if $Y \neq \emptyset$, then $G[Y]$ is 2- i -critical and if $Z \neq \emptyset$, then $G[Z]$ is complete and for each vertex $z \in Z$, there is exactly one vertex of X , z' say, such that $z \succ C_2 - z'$. Consequently, $N_G[z] = V(G) - \{c, z'\}$.

Claim 1: *Let $x_1, x_2 \in X$ where $x_1x_2 \notin E(G)$. If $x_1 \succ Z$, then $x_2 \succ Z$.*

By the definition of Y , $x_1 \succ Z \cup Y = N_{C_2}(u)$. Then $N_G[x_1] = V(C_2) - \{x_2\}$ since $G[X]$ is isomorphic to a complete graph without a perfect matching. Consider $G + cx_1$. Let $\{z\} = I_{cx_1} - \{c, x_1\}$. Then $\{c, x_1, z\}$ is independent by Lemma 2.1. Thus $z = x_2$ since $N_G[c] = \{c, u\}$ and $N_G[x_1] = V(C_2) - \{x_2\}$. If $I_{cx_1} = \{x_1, x_2\}$, then no vertex of I_{cx_1} is adjacent to u , a contradiction. Hence, $I_{cx_1} = \{c, x_2\}$. Since c is not adjacent to any vertex of C_2 , $x_2 \succ C_2 - x_1$. Hence, $x_2 \succ Z$. This settles our claim.

Claim 2: *If $z \in Z$ and $zx \notin E(G)$ for some $x \in X$, then the only non-neighbor of x in X is not adjacent to some vertex of $Z - \{z\}$.*

This claim follows by Claim 1 and the fact that for each vertex $z \in Z$, there is exactly one vertex of X , z' say, $N_G[z] = V(G) - \{c, z'\}$.

Let $X_1 = \{x \in X | xz \notin E(G) \text{ for some } z \in Z\}$. It is easy to see that, by the definition of Z , if $Z \neq \emptyset$, then $X_1 \neq \emptyset$. Now put $X_2 = X - X_1$. Then the following claims follow by

Claims 1 and 2.

Claim 3: For $x_1, x_2 \in X$ where $x_1x_2 \notin E(G)$, if $x_1 \in X_i$ for $1 \leq i \leq 2$, then $x_2 \in X_i$. Consequently, $|X_1|$ and $|X_2|$ are both even.

Claim 4: If $Z \neq \emptyset$, then $2 \leq |X_1| \leq |Z|$.

By Lemma 2.4, $G[Y]$ is isomorphic to $K_m - \bigcup_{i=1}^l K_{1,r_i}$ for some positive integers m, l and r_i . It is now easy to see that if $Y \neq \emptyset$ but $Z = \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_2 and if $Z \neq \emptyset$ but $Y = \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_3 . Finally, if $Y \neq \emptyset$ and $Z \neq \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_4 . This completes the proof of our theorem. \square

We conclude this section by pointing out that Ao, in [5], gave a theorem described 3- i -critical graphs with a cutvertex but his/her theorem does not depict how such graphs exactly look like while our Theorem 4.7 provides the explicit structure. Moreover, our proof is much more easier and shorter.

5 Classes of connected 3- i -critical graph G with a minimum cutset S where $2 \leq |S| \leq 3$ and $c(G - S) = 3$

In this section, we provide six classes of 3- i -critical graphs having a minimum cutset of size 2 and 3. We shall see in the next section that if G is 3- i -critical graph having a minimum cutset of size 2 and 3 with $c(G - S) = 3$, then G must belong to one of these six classes.

I. The class \mathcal{H}_1

For positive integers m and $n \geq 2$, define a graph $G \in \mathcal{H}_1$ of order $n + m + 4$ as follows. Set $V(G) = \{a, b, u, v\} \cup X \cup Y$ where $|X| = n$ and $|Y| = m$. The edges of G are defined as follows. $G[X] = K_n$ and $G[Y] = K_m$. Further, join u and v to every vertex of $\{a, b\} \cup X$. Finally, join each vertex of X to every vertex of Y . This defines the class \mathcal{H}_1 . Figure 8 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{H}_1$ is 3- i -critical containing $\{u, v\}$ as a minimum cutset.

II. The class \mathcal{H}_2

For positive integer $n \geq 3$, define a graph $G \in \mathcal{H}_2$ of order $n + 6$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3, v\} \cup X$ where $|X| = n$. The edges of G are defined as follows. $G[X] = K_n$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Further, join v to every vertex of X . Finally, add the edge u_1u_2 . This defines the class \mathcal{H}_2 . Figure 9 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{H}_2$ is 3- i -critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

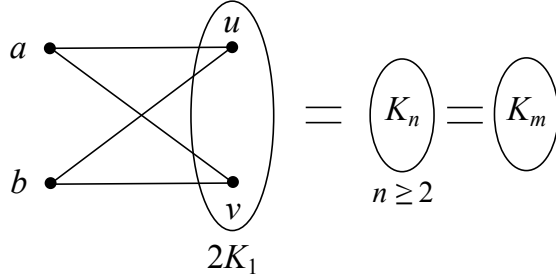


Figure 8: The structure of a graph in the class \mathcal{H}_1

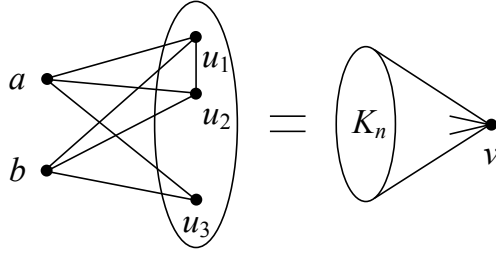


Figure 9: The structure of a graph in the class \mathcal{H}_2

III. The class \mathcal{H}_3

For positive integers n , m_1 and m_2 , define a graph $G \in \mathcal{H}_3$ of order $n + m_1 + m_2 + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y_1 \cup Y_2$ where $|X| = n$, $|Y_1| = m_1$ and $|Y_2| = m_2$. The edges of G are defined as follows. $G[X \cup Y_1 \cup Y_2] = K_{n+m_1+m_2}$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Further, for $1 \leq i \leq 2$, join u_i to every vertex of Y_i . Finally, add the edge $u_1 u_2$. This defines the class \mathcal{H}_3 . Figure 10 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{H}_3$ is 3- i -critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

IV. The class \mathcal{H}_4

For positive integer $n \geq 1$, define a graph $G \in \mathcal{H}_4$ of order $n + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X$ where $|X| = n$. The edges of G are defined as follows.

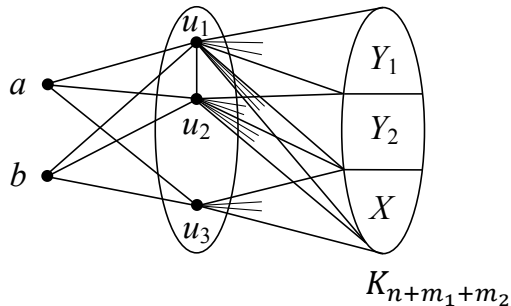


Figure 10: The structure of a graph in the class \mathcal{H}_3

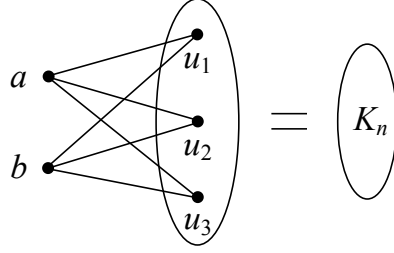


Figure 11: The structure of a graph in the class \mathcal{H}_4

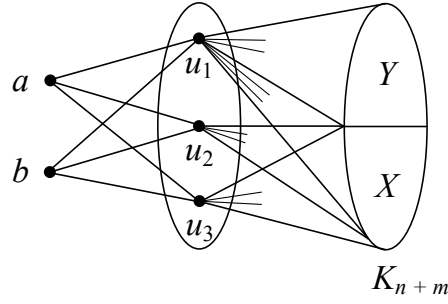


Figure 12: The structure of a graph in the class \mathcal{H}_5

$G[X] = K_n$ and join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. This defines the class \mathcal{H}_4 . Figure 11 illustrates our construction. It is easy to see that a graph $G \in \mathcal{H}_4$ is 3- i -critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

V. The class \mathcal{H}_5

For positive integers $n \geq 2$ and m , define a graph $G \in \mathcal{H}_5$ of order $n+m+5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y$ where $|X| = n$ and $|Y| = m$. The edges of G are defined as follows. $G[X \cup Y] = K_{n+m}$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Finally, join u_1 to every vertex of Y . This defines the class \mathcal{H}_5 . Figure 12 illustrates our construction. It is not difficult to see that a graph $G \in \mathcal{H}_5$ is 3- i -critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

VI. The class \mathcal{H}_6

For non-negative integer n and positive integers m_1, m_2 and m_3 , define a graph $G \in \mathcal{H}_6$ of order $n + m_1 + m_2 + m_3 + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y_1 \cup Y_2 \cup Y_3$ where $|X| = n$ and $|Y_i| = m_i$ for $1 \leq i \leq 3$. The edges of G are defined as follows. $G[X \cup Y_1 \cup Y_2 \cup Y_3] = K_{n+m_1+m_2+m_3}$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Finally, for $1 \leq i \leq 3$, join u_i to every vertex of Y_i . This defines the class \mathcal{H}_6 . Figure 13 illustrates our construction. It is easy to see that a graph $G \in \mathcal{H}_6$ is 3- i -critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

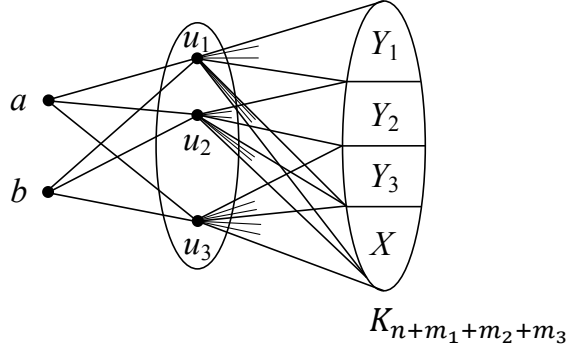


Figure 13: The structure of a graph in the class \mathcal{H}_6

6 Some toughness results of connected 3 - i - critical graphs

In this section, we establish a characterization of connected 3- i -critical graph G having a minimum cutset S of size 2 or 3 with $c(G - S) = 3$. We begin with the following theorem.

Theorem 6.1. *Let G be a connected 3- i -critical graph and S a vertex cutset. Then*

$$c(G - S) \leq \begin{cases} |S| - 2, & \text{for } |S| \geq 6, \\ 3, & \text{for } |S| \leq 5. \end{cases}$$

Proof. Suppose $G - S$ contains n components. If $n \leq 3$, then we are done. So we may suppose that $n \geq 4$. Let C_1, C_2, \dots, C_n be components of $G - S$. For $1 \leq i \leq n$, choose $a_i \in V(C_i)$. Then $A = \{a_1, a_2, \dots, a_n\}$ is an independent set of size at least 4. By Lemma 2.6, the vertices in A may be ordered as u_1, u_2, \dots, u_n in such a way that there exist a path $x_1 x_2 \dots x_{n-1}$ such that $[u_i, x_i] \xrightarrow{i} u_{i+1}$ for $1 \leq i \leq n - 1$. Without loss of generality, we may renumber the components of $G - S$ in such a way that $u_i \in V(C_i)$ for $1 \leq i \leq n$. Since $n \geq 4$, $x_i \in S$. Thus $\{x_1, x_2, \dots, x_{n-1}\} \subseteq S$. Note also that x_i is adjacent to every vertex of $\bigcup_{j=1}^n V(C_j) - (V(C_i) \cup \{u_{i+1}\})$ and $\{u_i, x_i, u_{i+1}\}$ is independent by Lemma 2.2(2). Consequently, x_i does not dominate either C_i or C_{i+1} .

We now consider $G + u_1 u_{n-k}$ where $k \in \{0, 1\}$. Let $\{z_k\} = I_{u_1 u_{n-k}} - \{u_1, u_{n-k}\}$. Then $\{u_1, u_{n-k}, z_k\}$ is independent by Lemma 2.2(2) and $z_k \succ \bigcup_{j=2}^n V(C_j) - V(C_{n-k})$. Thus, for $k \in \{0, 1\}$, $z_k \in S$ since $n \geq 4$. Further, $z_0 \neq z_1$. But $z_k \notin \{x_1, x_2, \dots, x_{n-1}\}$ since, for $1 \leq i \leq n - 1$, x_i does not dominate either C_i or C_{i+1} . Hence, $\{x_1, x_2, \dots, x_{n-1}, z_0, z_1\} \subseteq S$. Therefore, $|S| \geq n + 1$.

Suppose to the contrary that $|S| = n + 1$. Then $S = \{x_1, x_2, \dots, x_{n-1}, z_0, z_1\}$. We next consider $G + u_2 u_n$. Let $\{w\} = I_{u_2 u_n} - \{u_2, u_n\}$. Then $w \in S$ since $n \geq 4$ and $w \succ \bigcup_{j=1}^n V(C_j) - (V(C_2) \cup V(C_n))$. Clearly, $w \notin \{x_1, x_2, \dots, x_{n-1}\}$ since x_i does not dominate either C_i or C_{i+1} for $1 \leq i \leq n - 1$. Further, $w \notin \{z_0, z_1\}$ since $z_k u_1 \notin E(G)$ for $0 \leq k \leq 1$. Hence, $w \notin S$, a contradiction. Therefore, $|S| \geq n + 2$. This proves our theorem. \square

Ao [5] proved that $c(G - S) \leq |S| + 1$ where G and S are defined as in Theorem 6.1. So our result provides a better bound.

Lemma 6.2. *Let G be a connected 3- i -critical graph and S a minimum cutset where $2 \leq |S| \leq 3$. Suppose $c(G - S) = 3$. Then*

1. *There is a vertex of $G - S$, say x , such that $x \succ S$.*
2. *At least two components of $G - S$ are singleton.*
3. *Each component of $G - S$ is complete.*

Proof. Let C_1, C_2 and C_3 be the components of $G - S$.

(1) Suppose to the contrary that no vertex of $G - S$ dominates S . Since S is a minimum cutset, $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$. Further, for each $s \in S$, $N_{C_i}(s) \neq \emptyset$. Let $u_1, u_2 \in S$. Choose $x_1 \in N_{C_1}(u_1)$ and $x_2 \in N_{C_2}(u_2)$. Consider $G + x_1x_2$. By Lemma 2.2(1), we may assume that $I_{x_1x_2} = \{x_1, z\}$ for some $z \in V(G) - \{x_1, x_2\}$. Thus $z \notin \{u_1, u_2\}$ by Lemma 2.2(2). Because $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$, $z \succ (V(C_2) - \{x_2\}) \cup V(C_3)$. Thus $z \in S - \{u_1, u_2\}$. Consequently, $S = \{u_1, u_2, z\}$ since $|S| \leq 3$. Now choose $x_3 \in N_{C_3}(u_1)$. Clearly, $x_3u_1 \in E(G)$ and $x_3z \in E(G)$. Consider $G + x_2x_3$. Let $\{z_1\} = I_{x_2x_3} - \{x_2, x_3\}$. By Lemma 2.2(2), $z_1 \notin S$. If $I_{x_2x_3} = \{x_2, z_1\}$, then $z_1 \succ V(C_1) \cup (V(C_3) - \{x_3\})$ and thus $z_1 \in S$ since $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$, a contradiction. Hence, $I_{x_2x_3} = \{x_3, z_1\}$. By similar arguments, $z_1 \in S$, again a contradiction. This proves (1).

(2) By (1), we may assume that $x \in V(C_1)$. Choose $y \in V(C_2)$. Consider $G + xy$. Let $\{z\} = I_{xy} - \{x, y\}$. By Lemma 2.2(2), $z \notin S$. Then $z \in V(C_3)$. We first suppose that $I_{xy} = \{x, z\}$. Then $V(C_2) = \{y\}$. Now consider $G + xz$. Then, by Lemma 2.5, $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. If $I_{xz} = \{x, y\}$, then $|V(C_3)| = 1$. If $I_{xz} = \{z, y\}$, then $|V(C_1)| = 1$. Hence, G contains at least two singleton components as required.

We now suppose that $I_{xy} = \{y, z\}$. By similar arguments, G contains at least two singleton components. This proves (2).

(3) By (2), we may assume without loss of generality that $|V(C_1)| = |V(C_2)| = 1$. Put $\{a\} = V(C_1)$ and $\{b\} = V(C_2)$. Since S is a minimum cutset, $a \succ S$ and $b \succ S$. Suppose to the contrary that C_3 is not complete. Then there exist $x, y \in V(C_3)$ where $xy \notin E(G)$. Consider $G + ax$. Let $\{z\} = I_{ax} - \{a, x\}$. Because no vertex of $\{a, x\}$ is adjacent to a vertex of $\{b, y\}$, $z \succ \{b, y\}$. Then $z \in S$ since $b \in V(C_2)$ and $y \in V(C_3)$. But this contradicts Lemma 2.2(2) since $a \succ S$. This proves (3) and completes the proof of our lemma. \square

Theorem 6.3. *Let G be a connected 3- i -critical graph and S a minimum cutset of size 2. If $G - S$ contains exactly three components, then $G \in \mathcal{H}_1$, defined in Section 5.*

Proof. Let C_1, C_2 and C_3 be the components of $G - S$. By Lemma 6.2(2) we may assume that $|V(C_1)| = 1$ and $|V(C_2)| = 1$. Put $\{a\} = V(C_1)$, $\{b\} = V(C_2)$ and $S = \{u, v\}$. Since

S is a minimum cutset, $N_G(a) = N_G(b) = \{u, v\}$. Further, if $|V(C_3)| = 1$, then it is easy to see that $i(G) \leq 2$, a contradiction. Hence, $|V(C_3)| \geq 2$. By Lemma 6.2(3), C_3 is complete. Note that $N_{C_3}(u) \neq \emptyset$ and $N_{C_3}(v) \neq \emptyset$ since S is a minimum cutset. If there is a vertex $y \in N_{C_3}(u) - N_{C_3}(v)$, then $\{y, v\} \succ_i G$, a contradiction. Similarly, if there is a vertex $y \in N_{C_3}(v) - N_{C_3}(u)$, then $\{y, u\} \succ_i G$, again a contradiction. Hence, $N_{C_3}(u) \cup N_{C_3}(v) = N_{C_3}(u) \cap N_{C_3}(v)$ as required. It is easy to see that $|N_{C_3}(u) \cup N_{C_3}(v)| = |N_{C_3}(u) \cap N_{C_3}(v)| \geq 2$ since S is a minimum cutset of size 2. Because $i(G) = 3$, $|\overline{N}_{C_3}(u) \cap \overline{N}_{C_3}(v)| \geq 1$. Further, $uv \notin E(G)$. Therefore, $G \in \mathcal{H}_1$ as required. This completes the proof of our theorem. \square

Theorem 6.4. *Let G be a connected 3- i -critical graph and S a minimum cutset of size 3. If $G - S$ contains exactly three components, then $G \in \mathcal{H}_i$ for $2 \leq i \leq 6$, defined in Section 5.*

Proof. Let C_1, C_2 and C_3 be the components of $G - S$. By Lemma 6.2(2) we may assume that $|V(C_1)| = 1$ and $|V(C_2)| = 1$. Put $\{a\} = V(C_1)$, $\{b\} = V(C_2)$ and $S = \{u_1, u_2, u_3\}$. Since S is a minimum cutset, $N_G(a) = N_G(b) = \{u_1, u_2, u_3\}$. Further, if $|V(C_3)| = 1$, then S must be independent since $i(G) = 3$ and thus $G \in \mathcal{H}_4$. So we may suppose that $|V(C_3)| \geq 2$. Note that C_3 is complete by Lemma 6.2(3). Further, $N_{C_3}(u_i) \neq \emptyset$ for $1 \leq i \leq 3$ since S is a minimum cutset.

Claim 1: $G[S]$ contains at most one edge.

Suppose to the contrary that $G[S]$ contains at least two edges. Then there exists $i \in \{1, 2, 3\}$ such that $u_i u_j \in E(G)$ for $1 \leq i \neq j \leq 3$. Clearly, $u_i \succ S \cup \{a, b\}$. It is easy to see that $i(G) \leq 2$ since C_3 is complete. But this contradicts the fact that $i(G) = 3$. This settles our claim.

We now distinguish two cases according to the number of edges in $G[S]$.

Case 1: $G[S]$ contains exactly one edge.

We may assume without loss of generality that $u_1 u_2 \in E(G)$. Then $u_1 u_3 \notin E(G)$ and $u_2 u_3 \notin E(G)$.

Claim 2: $N_{C_3}(u_3) = N_{C_3}(u_1) \cap N_{C_3}(u_2)$.

It is easy to see that if $x \in N_{C_3}(u_3)$ but $x \notin N_{C_3}(u_1) \cap N_{C_3}(u_2)$, then $\{x, u_j\} \succ_i G$ for $1 \leq j \leq 2$, a contradiction. Hence, $N_{C_3}(u_3) \subseteq N_{C_3}(u_1) \cap N_{C_3}(u_2)$. On the other hand, if $x \in N_{C_3}(u_1) \cap N_{C_3}(u_2)$ but $x \notin N_{C_3}(u_3)$, then $\{x, u_3\} \succ_i G$, again a contradiction. Hence, $N_{C_3}(u_1) \cap N_{C_3}(u_2) \subseteq N_{C_3}(u_3)$. This settles our claim.

Now put $Y = V(C_3) - N_{C_3}(u_3)$. It is easy to see that $Y \neq \emptyset$ since $i(G) = 3$.

Claim 3: Each vertex of Y is adjacent to at most one vertex of $\{u_1, u_2\}$.

Clearly, if there exists $y \in Y$ such that $yu_1 \in E(G)$ and $yu_2 \in E(G)$, then $\{y, u_3\} \succ_i G$, a contradiction. This settles our claim.

Case 1.1: $|Y| = 1$.

Put $\{v\} = Y$. If $vu_j \in E(G)$ for some $1 \leq j \leq 2$, then $\{u_j, u_3\} \succ_i G$, a contradiction. Hence, $vu_j \notin E(G)$ for $1 \leq j \leq 2$. Consequently, $N_G(v) = V(C_3) - \{v\}$. Because S is a minimum cutset, $|V(C_3) - \{v\}| \geq 3$. Therefore, $G \in \mathcal{H}_2$.

Case 1.2: $|Y| \geq 2$.

Claim 4: Each vertex of Y is adjacent to exactly one vertex of $\{u_1, u_2\}$. Further, $|N_Y(u_1)| \geq 1$ and $|N_Y(u_2)| \geq 1$.

We first suppose to the contrary that there is a vertex $y \in Y$ such that $yu_1 \notin E(G)$ and $yu_2 \notin E(G)$. Thus $N_S(y) = \emptyset$. Since $|Y| \geq 2$, there is $y_1 \in Y - \{y\}$. Consider $G + u_3y_1$. Let $\{z_1\} = I_{u_3y_1} - \{u_3, y_1\}$. By Lemma 2.2(2), $z_1 \notin V(C_3) \cup \{a, b\}$ since C_3 is complete and $\{a, b\} \subseteq N_G(u_3)$. Then $z_1 \in \{u_1, u_2\}$. If $I_{u_3y_1} = \{u_3, z_1\}$, then no vertex of $I_{u_3y_1}$ is adjacent to y , a contradiction. Hence, $I_{u_3y_1} = \{y_1, z_1\}$. We may assume that $z_1 = u_1$. Then $u_1y_1 \notin E(G)$. Now consider $G + u_1y$. Let $\{z_2\} = I_{u_1y} - \{u_1, y\}$. By Lemma 2.2(2), $z_2 \notin V(C_3) \cup \{a, b, u_2\}$ since C_3 is complete and $\{a, b, u_2\} \subseteq N_G(u_1)$. Thus $z_2 = u_3$. If $I_{u_1y} = \{u_1, u_3\}$, then no vertex of I_{u_1y} is adjacent to y_1 , a contradiction. Hence, $I_{u_1y} = \{y, u_3\}$. But then no vertex of I_{u_1y} is adjacent to u_2 , again a contradiction. This proves that each vertex of Y is adjacent to at least one vertex of $\{u_1, u_2\}$. It then follows by Claim 3 that each vertex of Y is adjacent to exactly one vertex of $\{u_1, u_2\}$.

We next show that $N_Y(u_1) \neq \emptyset$. Suppose this is not the case. Then u_1 is not adjacent to any vertex of Y . The each vertex of Y is adjacent to u_2 by above argument. Consequently, $\{u_2, u_3\} \succ_i G$, a contradiction. Hence, $N_Y(u_1) \neq \emptyset$. Similarly, $N_Y(u_2) \neq \emptyset$. This settles our claim.

Put $Y_1 = N_Y(u_1)$ and $Y_2 = N_Y(u_2)$. It follows by Claim 4 that $G \in \mathcal{H}_3$.

Case 2: $G[S]$ is independent.

Claim 5: For each $x \in V(C_3)$, either x is adjacent to every vertex of $\{u_1, u_2, u_3\}$ or x is adjacent to exactly one vertex of $\{u_1, u_2, u_3\}$.

We first suppose to the contrary that there is a vertex $x_1 \in V(C_3)$ such that x_1 is not adjacent to any vertex of $\{u_1, u_2, u_3\}$. Consider $G + x_1u_1$. Let $\{z\} = I_{x_1u_1} - \{x_1, u_1\}$. By Lemma 2.2(2) and the fact that C_3 is complete and $\{a, b\} \subseteq N_G(u_1)$, it follows that $z \in \{u_2, u_3\}$. Since $\{x_1, u_1, u_2, u_3\}$ is independent, $z \succ \{u_2, u_3\}$. But this is not possible. Hence, each vertex of $V(C_3)$ is adjacent to at least one vertex of $\{u_1, u_2, u_3\}$.

We now suppose that there is a vertex $x_2 \in V(C_3)$ such that x_2 is adjacent to u_i and u_j where $1 \leq i \neq j \leq 3$. If $x_2u_k \notin E(G)$ where $k \in \{1, 2, 3\} - \{i, j\}$, then $\{x_2, u_k\} \succ_i G$, a contradiction. This settles our claim.

For $1 \leq i \leq 3$, let $Y_i = \{y \in V(C_3) \mid yu_i \in E(G) \text{ but } yu_j \notin E(G) \text{ for } 1 \leq i \neq j \leq 3\}$. By Claim 5, $Y_i \cap Y_j = \emptyset$ for $1 \leq i \neq j \leq 3$. Put $X = V(C_3) - (Y_1 \cup Y_2 \cup Y_3)$. Then each vertex of X is adjacent to every vertex of $\{u_1, u_2, u_3\}$.

Claim 6: For $1 \leq i \leq 3$, if $Y_i \neq \emptyset$, then either $Y_j \neq \emptyset$ for all $1 \leq i \neq j \leq 3$ or $Y_j = \emptyset$ for all $1 \leq i \neq j \leq 3$.

Assume that $Y_i \neq \emptyset$ for some $1 \leq i \leq 3$. Suppose to the contrary that $Y_j \neq \emptyset$ but $Y_k = \emptyset$ where $\{j, k\} = \{1, 2, 3\} - \{i\}$. Then $V(C_3) = Y_i \cup Y_j \cup X$. Consider $G + u_i u_j$. Let $\{z\} = I_{u_i u_j} - \{u_i, u_j\}$. By Lemma 2.2(2), $z \notin V(C_3) \cup \{a, b\}$. Then $z = u_k$. If $I_{u_i u_j} = \{u_i, u_k\}$, then no vertex of $I_{u_i u_j}$ is adjacent to a vertex of Y_j , a contradiction. Hence, $I_{u_i u_j} = \{u_j, u_k\}$. But then no vertex of $I_{u_i u_j}$ is adjacent to a vertex of Y_i , again a contradiction. This settles our claim.

We now distinguish two subcases.

Case 2.1: For $1 \leq i \leq 3$, $Y_i = \emptyset$.

It is easy to see that $V(C_3) = X$ and thus $G \in \mathcal{H}_4$.

Case 2.2: $Y_i \neq \emptyset$ for some $1 \leq i \leq 3$.

We first suppose that $Y_j = \emptyset$ for all $1 \leq i \neq j \leq 3$. If $X = \emptyset$, then u_i becomes a cutvertex of G , contradicting the fact that $S = \{u_1, u_2, u_3\}$ is a minimum cutset. Hence, $X \neq \emptyset$. Note that $\{u_i\} \cup X$ is a vertex cutset of G . Since the minimum cardinality of a vertex cutset in G is 3, $|X| \geq 2$. It is easy to see that $G \in \mathcal{H}_5$.

We now suppose that $Y_j \neq \emptyset$ for all $1 \leq i \neq j \leq 3$. Then $G \in \mathcal{H}_6$. This completes the proof of our theorem. \square

Our last result follows immediately from Theorems 6.3 and 6.4.

Corollary 6.5. *Let G be a connected 3- i -critical graph and S a minimum cutset where $2 \leq |S| \leq 3$. If $c(G - S) = 3$, then the minimum degree of G is $|S|$.*

\square

7 Classes of connected 3- i -critical graphs with a minimum cutset of size 2

In this section, we provide classes of connected 3- i -critical graphs with a minimum cutset of size 2 where if G is a graph in such classes and S is a minimum cutset of size 2, then $G - S$ contains exactly two components.

I. The class \mathcal{A}_1

For $1 \leq i \leq 4$, let n_i be a positive integer where if $n_1 \geq 2$, then $n_2 = n_3 = 1$. Define a graph $G \in \mathcal{A}_1$ of order $n_1 + n_2 + n_3 + 2n_4 + 2$ as follows. Set $V(G) = \{u, v\} \cup \bigcup_{i=1}^4 X_i$ where

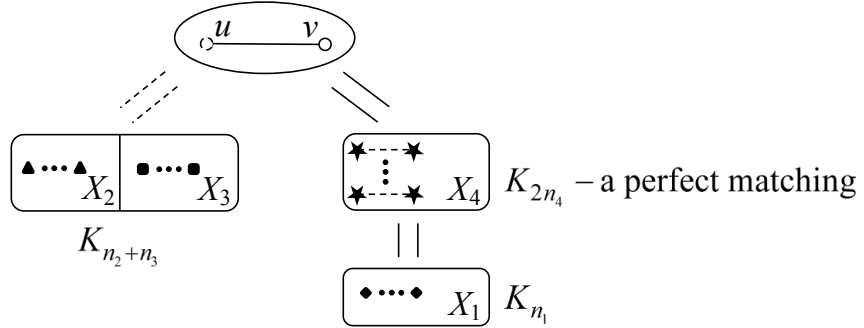


Figure 14: A structure of a graph in \mathcal{A}_1

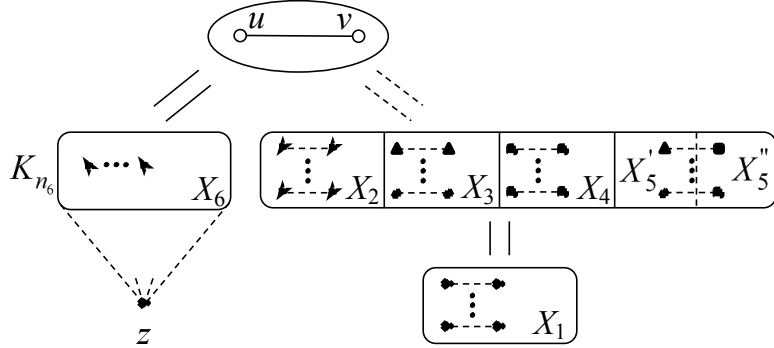


Figure 15: A structure of the graph H_1

$|X_i| = n_i$ for $1 \leq i \leq 3$ and $|X_4| = 2n_4$. Then, set $G[X_1] = K_{n_1}$, $G[X_2 \cup X_3] = K_{n_2+n_3}$ and $G[X_4]$ is a complete graph without a perfect matching. Further, join u to every vertex of $\{v\} \cup X_2 \cup X_4$ and join v to every vertex of $X_3 \cup X_4$. Finally, join each vertex of X_1 to every vertex of X_4 . Figure 14 illustrates our construction. Note that in our diagrams a “double line” denotes the join and for simplicity, a “double dash line” denotes the adjacency of $\{u, v\}$ and vertices in our diagram where the star vertices are joined to both u and v , the triangle vertices are only joined to u , the square vertices are only joined to v and the diamond vertices are neither joined to u nor v .

II. The graph H_1 and the class \mathcal{A}_2

For $1 \leq i \leq 6$, let n_i be a positive integer where $n_6 \geq 2$, define the graph H_1 of order $\sum_{i=1}^5 2n_i + n_6 + 3$ as follows. Set $V(H_1) = \{u, v, z\} \cup \bigcup_{i=1}^6 X_i$ where for $1 \leq i \leq 5$, $|X_i| = 2n_i$ and $|X_6| = n_6 \geq 2$. Put $X_5 = X_5' \cup X_5''$ where $H_1[X_5'] = H_1[X_5''] = K_{n_5}$. Now for $1 \leq i \leq 5$, set $H_1[X_i]$ to be a complete graph without a perfect matching and $H_1[X_6] = K_{n_6}$. Then for each $x \in X_5'$, there is a unique vertex $\bar{x} \in X_5''$ such that $x\bar{x} \notin E(H_1)$. Further, join u to every vertex of $\{v\} \cup X_2 \cup X_3 \cup X_5' \cup X_6$, join v to every vertex of $X_2 \cup X_4 \cup X_5'' \cup X_6$ and join z to every vertex of X_6 . Finally, join each vertex of X_i to every vertex of X_j where $1 \leq i \neq j \leq 5$.

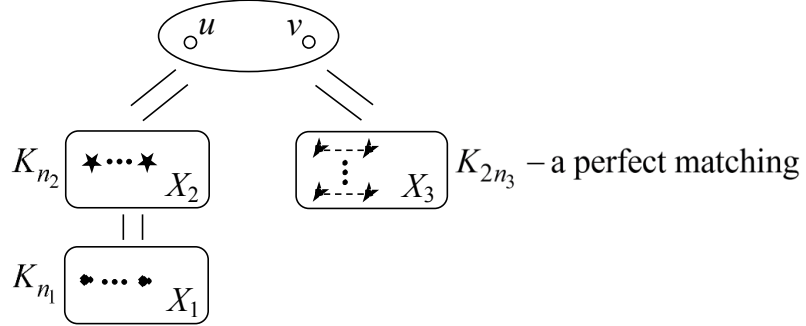


Figure 16: A structure of a graph in \mathcal{A}_3

We now define a new class of connected 3- i -critical graphs obtained from an induced subgraph of H_1 . The class \mathcal{A}_2 consists of G_1, G_2, \dots, G_7 where, for $1 \leq i \leq 7$, G_i is obtained from an induced subgraph of H_1 as follows.

Put $G_1 = H_1[\{u, v\} \cup X_3 \cup X_4 \cup X_6]$, $G_2 = H_1[\{u, v\} \cup X_2 \cup X_3 \cup X_4 \cup X_6]$, $G_3 = H_1[\{u, v, z\} \cup X_2 \cup X_5 \cup X_6]$ if $|X_5| = 2$, $G_4 = H_1[\{u, v\} \cup X_1 \cup X_2 \cup X_6]$, $G_5 = H_1[\{u, v\} \cup X_1 \cup X_2 \cup X_3 \cup X_6]$, $G_6 = H_1[\{u, v\} \cup X_1 \cup X_3 \cup X_4 \cup X_6]$ and $G_7 = H_1[\{u, v\} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_6]$.

III. The class \mathcal{A}_3

For $1 \leq i \leq 3$, let n_i be a positive integer where $n_2 \geq 2$, define a graph $G \in \mathcal{A}_3$ as follows. Set $V(G) = \{u, v\} \cup X_1 \cup X_2 \cup X_3$ where for $1 \leq i \leq 2$, $|X_i| = n_i$ and $|X_3| = 2n_3$. Now set $G[X_1] = K_{n_1}$, $G[X_2] = K_{n_2}$ and $G[X_3]$ to be a complete graph without a perfect matching. Finally, join u and v to every vertex of $X_2 \cup X_3$ and join each vertex of X_1 to every of vertex X_2 . This defines a graph $G \in \mathcal{A}_3$. Figure 16 illustrates our construction.

IV. The graph H_2 and the class \mathcal{A}_4

For $1 \leq i \leq 6$, let n_i be a positive integer where $n_1 = n_5$ and $n_6 \geq 2$, define the graph H_2 of order $n_1 + n_5 + n_6 + 2(n_2 + n_3 + n_4) + 2$ as follows. Set $V(H_2) = \{u, v\} \cup \bigcup_{i=1}^6 X_i$ where $|X_i| = n_i$ for $i \in \{1, 5, 6\}$ and $|X_i| = 2n_i$ for $i \in \{2, 3, 4\}$. Now, for $2 \leq i \leq 4$, set $H_2[X_i]$ and $H_2[X_1 \cup X_5]$ to be complete graphs without a perfect matching where $H_2[X_1] = H_2[X_5] = K_{n_1} = K_{n_5}$ and $H_2[X_6] = K_{n_6}$. Further, join u to every vertex of $X_2 \cup X_3 \cup X_5 \cup X_6$, join v to every vertex of $X_2 \cup X_4 \cup X_5 \cup X_6$. Finally, join each vertex of X_i to every vertex of X_j for $i \in \{2, 3, 4\}$ and $j \in \{1, 2, 3, 4, 5\}$ where $i \neq j$. Figure 17 illustrates our construction.

We now define a new class of connected 3- i -critical graphs obtained from the graph H_2 and an induced subgraph H_2 . The class \mathcal{A}_4 consists of $H_2, H_2[\{u, v\} \cup X_1 \cup X_5 \cup X_6]$ if $|X_5| = |X_1| \geq 2$, $H_2[\{u, v\} \cup X_1 \cup X_i \cup X_5 \cup X_6]$ where $i \in \{2, 3, 4\}$, $H_2[\{u, v\} \cup X_1 \cup X_i \cup X_j \cup X_5 \cup X_6]$ where $i, j \in \{2, 3, 4\}$ and $i \neq j$.

Before we define new classes, we need to set up some terminology. For a graph H

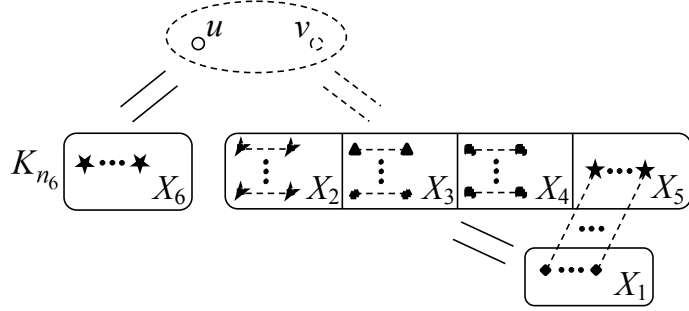


Figure 17: A structure of the graph H_2

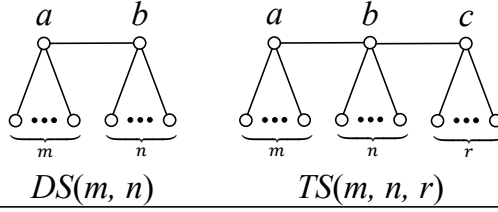


Figure 18: Graphs $DS(m, n)$ and $TS(m, n, r)$

and a positive integer n , let X_i a subgraph of H where $1 \leq i \leq n$, we denote the induced subgraph of H by $H_{X_1 \dots X_n}$ where $V(H_{X_1 \dots X_n}) = \bigcup_{i=1}^n V(X_i)$ and $V(X_i) \cap V(X_j) = \emptyset$ for $1 \leq i \neq j \leq n$. Further, if X and Y are subsets of vertices of H where $X \cap Y = \emptyset$, then $E(X, Y)$ denotes set of all edges xy where $x \in X$ and $y \in Y$.

For positive integers m, n and r , let a, b and c be a center vertex of $K_{1,m}$, $K_{1,n}$ and $K_{1,r}$, respectively. A *double star* $DS(m, n)$ is a graph obtained from $K_{1,m}$ and $K_{1,n}$ by adding the edge ab and a *triple star* $TS(m, n, r)$ is a graph obtained from $K_{1,m}$, $K_{1,n}$ and $K_{1,r}$ by adding the edges ab and bc . In what follows, a and b in $DS(m, n)$ are called the *center vertices* while the vertices b and a, c in $TS(m, n, r)$ are called the *middle center vertex* and *end center vertices*, respectively. Figure 18 illustrates $DS(m, n)$ and $TS(m, n, r)$.

For $1 \leq i \leq 3$, let $P_i \cong K_{2p_i}$ a perfect matching and $D_i = K_{d'_i} - \bigcup_{j=1}^{d_i} DS(m_j^i, n_j^i)$ where p_i, m_j^i, n_j^i are positive integers and $d'_i = \sum_{j=1}^{d_i} (m_j^i + n_j^i + 2)$. Further, let $S_i = K_{s'_i} - \bigcup_{j=1}^{s_i} K_{1, q_j^i}$ where q_j^i, s_i are positive integers and $s'_i = \sum_{j=1}^{s_i} (q_j^i + 1)$. Finally, let $T = K_{t'} - \bigcup_{j=1}^t TS(m_j, n_j, r_j)$ where $t' = \sum_{j=1}^t (m_j + n_j + r_j + 3)$.

V. The graph H_3 and the class \mathcal{A}_5

Let H_3 be a graph obtained from $P_1, P_2, D_1, D_2, S_1, S_2, S_3$ where each pair of such graphs is disjoint and $V(H_3) = \bigcup_{i=1}^2 V(P_i) \cup \bigcup_{i=1}^2 V(D_i) \cup \bigcup_{i=1}^3 V(S_i)$, $E(H_3) = \bigcup_{i=1}^2 E(P_i) \cup \bigcup_{i=1}^2 E(D_i) \cup \bigcup_{i=1}^3 E(S_i) \cup E(V(X), V(Y))$ where $X \neq Y$ and $\{X, Y\} \subseteq \{P_1, P_2, D_1, D_2, S_1, S_2, S_3\}$. For $1 \leq j \leq s_2$, let $\emptyset \neq R_j$ be a subset of $\{x | x \text{ is an end vertex of } K_{1, r_j^2} \text{ in } \overline{S_2}\}$. Put $R = \bigcup_{j=1}^{s_2} R_j$. Clearly, $|R| \geq s_2$. We now define a graph $G \in \mathcal{A}_5$ where G is obtained from H_3 by adding three new vertices, say u, v, w , and $E(H_3) \subseteq E(G)$ as follows. Set $V(G) = V(H_3) \cup \{u, v, w\}$ and $E(G) = E(H_3) \cup \{uw, vw\} \cup E(\{u, v\}, V(S_1) \cup$

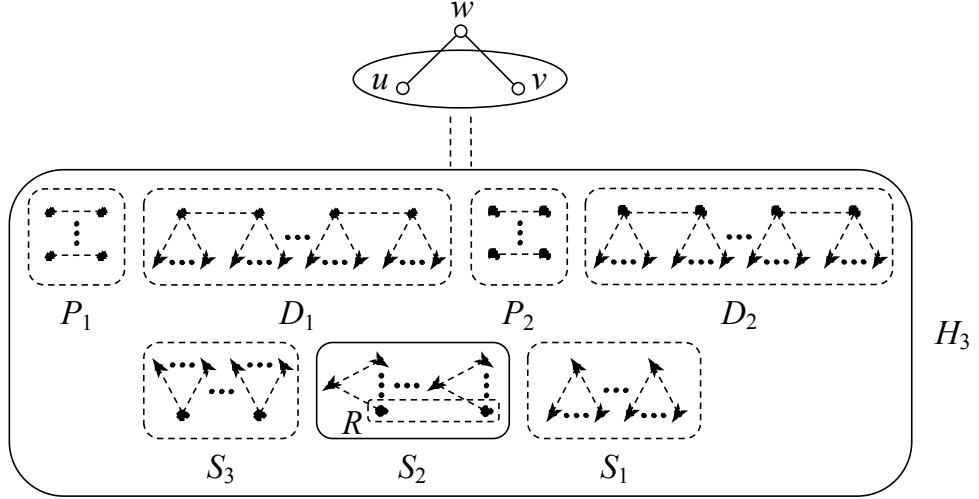


Figure 19: The structure of a graph in the class \mathcal{A}_5

$(V(S_2) - R) \cup \{x | x \text{ is an end vertex of a double star in } \overline{D_1} \cup \overline{D_2} \text{ or an end vertex of a star in } \overline{S_3}\} \cup E(\{u\}, V(P_1) \cup \{x | x \text{ is a center vertex of a double star in } \overline{D_1}\}) \cup E(\{v\}, V(P_2) \cup \{x | x \text{ is a center vertex of a double star in } \overline{D_2}\})$. Figure 19 illustrates our construction. Recall that, in our diagrams, a “double dash line” denotes the adjacency of $\{u, v\}$ and vertices in our diagram where the star vertices are joined to both u and v , the triangle vertices are only joined to u , the square vertices are only joined to v and the diamond vertices are neither joined to u nor v .

VI. The class \mathcal{A}_6

For $G \in \mathcal{A}_5$, let $G' = G[\{u, v, w\} \cup V(S_2) \cup V(S_3)]$, $G'' = G[\{u, v, w\} \cup V(S_2)]$ if $\overline{S_2}$ contains exactly one star and there are at least two end vertices of the star which are not in R and $G''' = G[\{u, v, w\} \cup V(S_3)]$ if $\overline{S_3}$ contains at least two stars. For $1 \leq i \leq n \leq 5$, let X_i be a graph in $\{P_1, P_2, D_1, D_2, S_1\}$. We now let $G_{X_1 \dots X_n}^1$ be an induced subgraph of G where $V(G_{X_1 \dots X_n}^1) = \{u, v, w\} \cup V(S_2) \cup V(S_3) \cup \bigcup_{i=1}^n V(X_i)$. and for $2 \leq j \leq 3$, let $G_{X_1 \dots X_n}^j$ be an induced subgraph of G where $V(G_{X_1 \dots X_n}^j) = \{u, v, w\} \cup V(S_j) \cup \bigcup_{i=1}^n V(X_i)$. Figure 20 shows a structure of G'' , G''' , $G_{p_1}^3$ and $G_{P_1 D_2}^2$. For $G \in \mathcal{A}_5$, let $\mathcal{A}_6 = \{G', G'', G''', G_{X_1 \dots X_n}^j | 1 \leq j \leq 3 \text{ and } 1 \leq n \leq 5\}$.

VII. The graph H_4 and the class \mathcal{A}_7

Let H_4 be a graph obtained from $P_1, P_2, P_3, D_1, D_2, D_3, S_1$ and T where each pair of such graphs is disjoint and $V(H_4) = \bigcup_{i=1}^3 V(P_i) \cup \bigcup_{i=1}^3 V(D_i) \cup V(S_1) \cup V(T)$, $E(H_4) = \bigcup_{i=1}^3 E(P_i) \cup \bigcup_{i=1}^3 E(D_i) \cup E(S_1) \cup E(T) \cup E(V(X), V(Y))$ where $X \neq Y$ and $\{X, Y\} \subseteq \{P_1, P_2, P_3, D_1, D_2, D_3, S_1, T\}$. We now define a graph $G \in \mathcal{A}_7$ where G is obtained from H_4 by adding three new vertices, say u, v, w and $E(H_4) \subseteq E(G)$ as follows. Set $V(G) = V(H_4) \cup \{u, v, w\}$ and $E(G) = E(H_4) \cup \{uv, uw, vw\} \cup E(\{u, v\}, V(S_1) \cup \{x | x \text{ is an end vertex of a double star in } \bigcup_{i=1}^3 \overline{D_i} \text{ or a triple star in } \overline{T}\}) \cup E(\{u\}, V(P_1) \cup \{x | x$

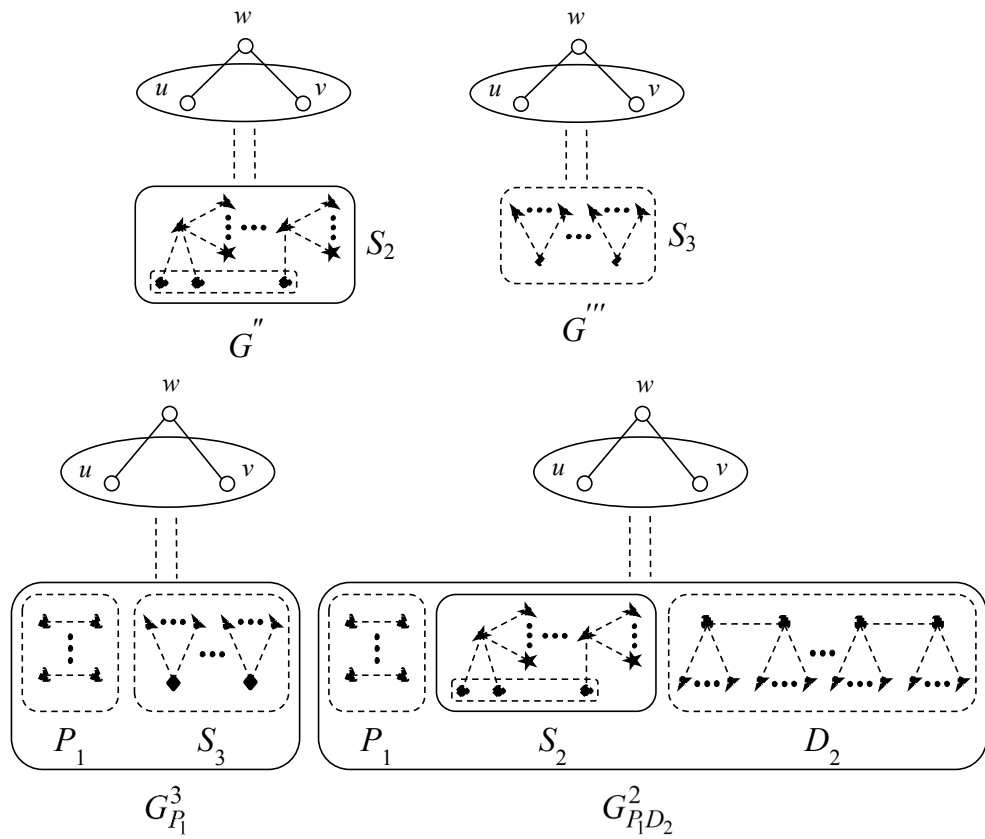


Figure 20: Some graphs in the class \mathcal{A}_6

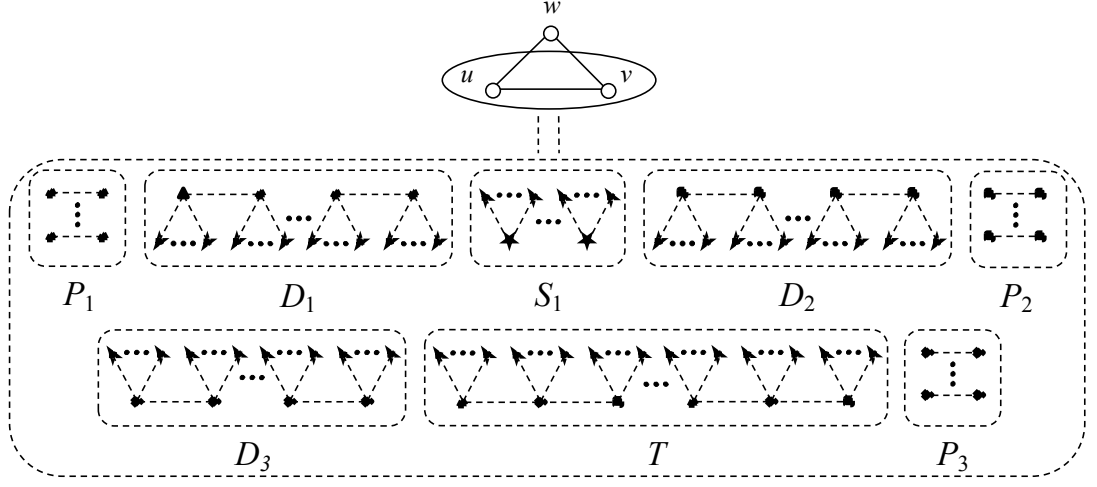


Figure 21: The structure of a graph in the class \mathcal{A}_7

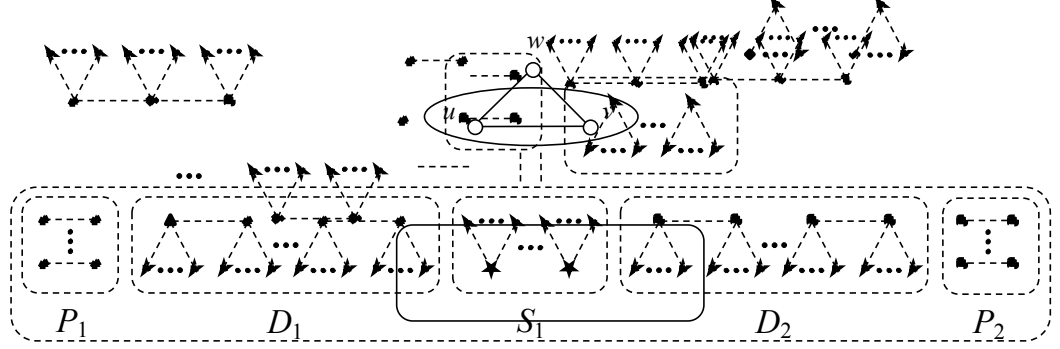


Figure 22: The structure of $G_{R_1D_1D_2S_1}$ in the class \mathcal{A}_8

is a center vertex of a double star in $\overline{D_1}$) \cup $\{x \mid x \text{ is one of end center vertices of a triple star in } \overline{T}\} \cup \{x \mid x \text{ is a center of a double star in } \overline{D_2}\} \cup \{x \mid x \text{ is the end center vertex of a triple star in } \overline{T}, \text{ that is not adjacent to } u\}$. Figure 21 illustrates our construction.

VIII. The class \mathcal{A}_8

For $G \in \mathcal{A}_7$, let $R_1 = G[\{u, v, w\} \cup V(P_1) \cup V(P_2)]$, $R_2 = G[\{u, v, w\} \cup V(P_1) \cup V(D_2)]$, $R_3 = G[\{u, v, w\} \cup V(D_1) \cup V(D_2)]$. For $1 \leq i \leq n$, let X_i be a graph in $\{R_1, R_2, R_3, S_1\}$ and denote the induced subgraph of $G \in \mathcal{A}_7$ by $G_{X_1 \dots X_n}$. We now define $\mathcal{A}_8 = \{G_{R_1}, G_{R_2}, G_{R_3}, G_{R_1D_1}, G_{R_1S_1}, G_{R_2D_1}, G_{R_2S_1}, G_{R_3S_1}, G_{R_1D_1D_2}, G_{R_1D_1S_1}, G_{R_2D_1S_1}, G_{R_1D_1D_2S_1}\}$. Figure 22 illustrates our construction of $G_{R_1D_1D_2S_1}$.

IX. The class \mathcal{A}_9

For $G \in \mathcal{A}_7$, let $W_1 = G[\{u, v, w\} \cup V(P_3)]$, $W_2 = G[\{u, v, w\} \cup V(D_3)]$, $W_3 = G[\{u, v, w\} \cup V(T)]$, $W_4 = G[\{u, v, w\} \cup V(P_3) \cup V(D_3)]$, $W_5 = G[\{u, v, w\} \cup V(P_3) \cup V(T)]$, $W_6 = G[\{u, v, w\} \cup V(D_3) \cup V(T)]$ and $W_7 = G[\{u, v, w\} \cup V(P_3) \cup V(D_3) \cup V(T)]$. Fur-

ther. let $Q_1 = G[\{u, v, w\} \cup V(D_3)]$ if \overline{D}_3 contains either at least two double stars or exactly one double star $DS(m, n)$ where $m \geq 2, n \geq 2$. For $1 \leq i \leq 5$, let F_i be a graph in $\{P_1, P_2, D_1, D_2, S_1\}$ and assume that if $1 \leq i \neq j \leq 5$, $F_i \neq F_j$. For $3 \leq k \leq 7$, $2 \leq k' \leq 7$, $1 \leq n \leq 5$ and $l \geq 2$, let $\mathcal{A}_9 = \{G_{Q_1}, G_{W_1 D_1}, G_{W_1 S_1}, G_{W_1 F_1 \dots F_l}, G_{W_k}, G_{W_{k'} F_1 \dots F_n}\}$. Note that $G_{W_7 P_1 P_2 D_1 D_2 S_1}$ is a graph in \mathcal{A}_7 .

It is not difficult to show that for $1 \leq i \leq 9$, if $G \in \mathcal{A}_i$, then G is a connected 3- i -critical graph where $\{u, v\}$ is a minimum cutset of G and $c(G - \{u, v\}) = 2$.

8 Connected 3- i -critical graphs with minimum cutset of size 2

In this section, we establish a characterization of connected 3- i -critical graph G with minimum cutset of size 2. Recall that for a pair of non-adjacent vertices x and y of G , I_{xy} denotes a minimum independent dominating set for $G + xy$. We begin with the following lemma.

Lemma 8.1. *Let G be a connected 3- i -critical graph and S a minimum cutset of size 2 where $S = \{u, v\}$ and $c(G - S) = 2$. Further, let C_1 and C_2 be the components of $G - S$. Then*

1. *Exactly one of C_1 or C_2 is complete.*
2. *Suppose C_1 is complete. Then no vertex of $N_{C_2}(u) \cup N_{C_2}(v)$ dominates C_2 . Further, $i(C_2) \leq 2$.*

Proof. Since S is a minimum cutset, for $1 \leq i \leq 2$, $N_{C_i}(u) \neq \emptyset$ and $N_{C_i}(v) \neq \emptyset$.

Claim 1: If $x \in V(C_1)$ and $y \in V(C_2)$ where x does not dominate C_1 and y does not dominate C_2 , then $I_{xy} \cap S \neq \emptyset$.

It is easy to see that if $I_{xy} \cap S = \emptyset$, then either $x \succ C_1$ or $y \succ C_2$ since $|I_{xy}| = 2$ and $|I_{xy} \cap \{x, y\}| = 1$.

(1) Suppose to the contrary that neither C_1 nor C_2 is complete. Then there exist $x_1, y_1 \in V(C_1)$ and $x_2, y_2 \in V(C_2)$ such that $x_1 y_1 \notin E(G)$ and $x_2 y_2 \notin E(G)$. By Claim 1, we may suppose without loss of generality that $I_{x_1 x_2} = \{x_1, u\}$. Then $u y_1, u y_2 \in E(G)$. Thus $u \notin I_{y_1 y_2}$ and $u \notin I_{x_1 y_2}$ by Lemma 2.2(2). It follows by Claim 1 that $v \in I_{y_1 y_2}$ and $v \in I_{x_1 y_2}$. Since $v \in I_{y_1 y_2}$, $v x_1 \in E(G)$ and since $v \in I_{x_1 y_2}$, $v x_1 \notin E(G)$ by Lemma 2.2(2), a contradiction. Hence, C_1 or C_2 is complete. Since S is a minimum cutset, $N_{C_1}(u) \neq \emptyset$ and $N_{C_2}(v) \neq \emptyset$. If C_1 and C_2 are complete, then $\{u_1, v_1\} \succ_i G$ where $u_1 \in N_{C_1}(u)$ and $v_1 \in N_{C_2}(v)$, a contradiction. Hence, exactly one of C_1 or C_2 is complete. This proves (1).

(2) By (1), C_2 is not complete.

Claim 2: No vertex of $N_{C_2}(u) \cup N_{C_2}(v)$ dominates C_2 .

Suppose to the contrary that there is a vertex $x \in N_{C_2}(u)$ such that $x \succ C_2$. Let $y \in N_{C_1}(v)$. Then $\{x, y\} \succ_i G$ since C_1 is complete, a contradiction. This settles our claim.

By Claim 2, it is easy to see that $|V(C_2)| \geq 2$. Let $x \in N_{C_1}(u)$ and $y \in N_{C_2}(v)$. Now consider $G + xy$. By Lemma 2.2(2). $I_{xy} \cap \{u, v\} = \emptyset$. Let $\{z\} = I_{xy} - \{x, y\}$. Then either $I_{xy} = \{x, z\}$ or $I_{xy} = \{y, z\}$. We first suppose that $I_{xy} = \{x, z\}$. Then $z \in V(C_2)$ since $|V(C_2)| \geq 2$. Clearly, $z \succ V(C_2) - \{y\}$ and thus $\{z, y\} \succ_i C_2$. Hence, $i(C_2) \leq 2$. We now suppose that $I_{xy} = \{y, z\}$. By Lemma 2.2(2) and the fact that C_1 is complete, $z \notin V(C_1)$. Thus $z \in V(C_2)$. Hence, $\{y, z\} \succ_i C_2$. This proves that $i(C_2) \leq 2$ and completes the proof of our lemma. \square

Theorem 8.2. *Let G and S be defined as in Lemma 8.1. Further, let C_1 and C_2 be the components of $G - S$ where C_1 is complete. If $i(C_2) = 1$, then $G \in \mathcal{A}_1$.*

Proof. Let $A = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. Since $i(C_2) = 1$, there exists $a \in V(C_2)$ where $a \succ C_2$. By Lemma 8.1(2), $a \in A$.

Claim 1: $N_{C_1}(u) \cap N_{C_1}(v) = \emptyset$. Further, $V(C_1) = N_{C_1}(u) \cup N_{C_1}(v)$.

If there is a vertex $x \in N_{C_1}(u) \cap N_{C_1}(v)$, then $\{x, a\} \succ_i G$, a contradiction. Hence, $N_{C_1}(u) \cap N_{C_1}(v) = \emptyset$. If there is a vertex $y \in V(C_1) - (N_{C_1}(u) \cup N_{C_1}(v))$, then $d(y, a) > 3$. But this contradicts Lemma 2.3. This settles our claim.

Since S is a minimum cutset, $N_{C_1}(u) \neq \emptyset$ and $N_{C_1}(v) \neq \emptyset$. Thus, by Claim 1, $|V(C_1)| = |N_{C_1}(u)| + |N_{C_1}(v)| \geq 2$.

Claim 2: $uv \in E(G)$.

Suppose this is not the case. Consider $G + uv$. We may assume without loss of generality that $I_{uv} = \{u, z\}$ for some $z \in V(G) - \{u, v\}$. Thus $z \in V(C_1) \cup V(C_2)$ and $z \succ N_{C_1}(v) \cup (V(C_2) - N_G(u))$. But this is not possible. Hence, $uv \in E(G)$ as required.

Claim 3: $N_{C_2}(u) \cap N_{C_2}(v) = N_{C_2}(u) \cup N_{C_2}(v)$.

Suppose there is a vertex $y \in N_{C_2}(u) - N_{C_2}(v)$. Consider $G + yv$. By Lemma 2.2(2) and the fact that $a \succ C_2$, $\{u, a\} \cap I_{yv} = \emptyset$. Put $\{z\} = I_{yv} - \{y, v\}$. Clearly, $z \in V(C_1) \cup V(C_2)$. If $I_{yv} = \{v, z\}$, then $z \succ N_{C_1}(u) \cup A$ which is not possible. Hence, $I_{yv} = \{y, z\}$. Since $y \in V(C_2)$, $z \in V(C_1)$. Then $y \succ C_2$. But this contradicts Lemma 8.1(2) since $y \in N_{C_2}(u)$. Hence, $N_{C_2}(u) - N_{C_2}(v) = \emptyset$. Similarly, $N_{C_2}(v) - N_{C_2}(u) = \emptyset$. This settles our claim.

Claim 4: $G' = G[N_{C_2}(u) \cup N_{C_2}(v)] \cong K_{2m}$ - a perfect matching.

Let $x \in N_{C_2}(u) \cup N_{C_2}(v)$. Choose $w \in N_{C_1}(u)$. Consider $G + xw$. By Claim 3 and Lemma 2.2(2), $I_{xw} \cap \{u, v\} = \emptyset$. Because C_1 is complete and $|V(C_1)| \geq 2$, it follows that $w \in I_{xw}$. Put $\{z\} = I_{xw} - \{w\}$. Then $z \in V(C_2)$ and thus $z \succ V(C_2) - \{x\}$. Since $wv \notin E(G)$ by Claim 1, $z \in N_{C_2}(v)$. Clearly, $\{x, z\} \succ_i G'$. Hence, $i(G') = 2$ by Lemma 8.1(2). It also follows that $z \succ G' + xz$. Hence, G' is 2- i -critical. By Lemma 2.4(1), $G' \cong K_m - \bigcup_{i=1}^s K_{1,r_i}$ for some positive integers m, r_i and s .

We next show that $r_i = 1$ for $1 \leq i \leq s$. Suppose this is not the case. Then $\overline{G'}$ contains K_{1,r_j} as an induced subgraph where $r_j \geq 2$ for some $1 \leq j \leq s$. Let x and y_1 and y_2 be the center and the end vertices of K_{1,r_j} in $\overline{G'}$, respectively. Choose $w_1 \in N_{C_1}(u)$. By Claim 1, $w_1v \notin E(G)$. Consider $G + w_1y_1$. Put $\{z_1\} = I_{w_1y_1} - \{w_1, y_1\}$. Lemma 2.2(2) together with Claim 3 and the fact that C_1 is complete implies that $z_1 \notin \{u, v\} \cup V(C_1)$. If $I_{w_1y_1} = \{w_1, z_1\}$, then $z_1 = x$ in order to dominate v . But then no vertex of $I_{w_1y_1}$ is adjacent to y_2 , a contradiction. Hence, $I_{w_1y_1} = \{y_1, z_1\}$. Then $z_1 \in V(C_2)$ and thus no vertex of $I_{w_1y_1}$ is adjacent to a vertex of $N_{C_1}(v)$, again a contradiction. This proves that $r_j = 1$ for $1 \leq j \leq s$.

Claim 5: For each $x \in A, x \succ C_2$.

Suppose to the contrary that there are $b \in A$ and $c \in V(C_2) - \{b\}$ such that $bc \notin E(G)$. Clearly, $b \neq a$ since $a \succ C_2$. Consider $G + ub$. Let $\{z\} = I_{ub} - \{u, b\}$. Then $z \notin \{a, v\}$ by Lemma 2.2(2), Claim 2 and the fact that $a \succ C_2$. If $I_{ub} = \{u, z\}$, then $z \succ N_{C_1}(v) \cup \{a\}$ which is not possible since $a \in A = V(C_2) - (N_G(u) \cup N_G(v))$. Hence, $I_{ub} = \{b, z\}$. Then $z \succ \{c\} \cup V(C_1)$ and thus $z = v$ since $c \in V(C_2)$. But this contradicts the fact that $z \notin \{a, v\}$. This settles our claim.

We now distinguish into two cases.

Case 1: $|A| = 1$.

Clearly, $A = \{a\}$. By Claims 1-5, $G \in \mathcal{A}_1$.

Case 2: $|A| \geq 2$.

By Claims 1-5, we need only to show that $|N_{C_1}(u)| = |N_{C_1}(v)| = 1$. Let $x \in A$ and $y \in N_{C_1}(u)$. Consider $G + xy$. By Lemma 2.2(2), $(I_{xy} - \{x, y\}) \cap (V(C_1) \cup V(C_2) \cup \{u\}) = \emptyset$ since $x \succ V(C_2)$ and $y \succ V(C_1) \cup \{u\}$. Then $I_{xy} - \{x, y\} = \{v\}$. If $I_{xy} = \{y, v\}$, then no vertex of I_{xy} is adjacent to vertices of $A - \{x\}$, a contradiction. Hence, $I_{xy} = \{x, v\}$. It follows by Claim 1 that $N_{C_1}(u) = \{y\}$. Thus $|N_{C_1}(u)| = 1$. By similar arguments, $|N_{C_1}(v)| = 1$. Hence, $G \in \mathcal{A}_1$ where $n_1 \geq 2$ and $n_2 = n_3 = 1$. This completes the proof of our theorem. \square

Theorem 8.3. *Let G, S, C_1 and C_2 be defined as in Theorem 8.2. If $i(C_2) = 2$ and $|V(C_1)| \geq 2$, then $G \in \bigcup_{i=2}^4 \mathcal{A}_i$.*

Proof. We begin with the following claims.

Claim 1: If $N_{C_1}(u) - N_{C_1}(v) \neq \emptyset$, then, for each $y \in N_{C_2}(v)$, there exists a unique vertex $y' \in N_{C_2}(v) - \{y\}$ such that $y \succ C_2 - y'$ and $y' \succ C_2 - y$.

Let $x \in N_{C_1}(u) - N_{C_1}(v)$. Choose $y \in N_{C_2}(v)$. Consider $G + xy$. By Lemma 2.2(2) and the fact that C_1 is complete, $(I_{xy} - \{x, y\}) \cap (\{u, v\} \cup V(C_1)) = \emptyset$. Put $\{z\} = I_{xy} - \{x, y\}$. Then $z \in V(C_2)$. If $I_{xy} = \{y, z\}$, then no vertex of I_{xy} is adjacent to vertices of $V(C_1) - \{x\}$, a contradiction. Hence, $I_{xy} = \{x, z\}$. Thus $z \succ C_2 - \{y\}$ and $zv \in E(G)$ since $xv \notin E(G)$. Now consider $G + xz$. By Lemma 2.5, either $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. By similar arguments as above, $I_{xz} = \{x, y\}$ and thus $y \succ C_2 - z$ and $yv \in E(G)$. Continuing in this fashion, if there is $w \in V(C_2) - \{y, z\}$, then there exists $w' \in V(C_2) - \{y, z, w\}$ such that $w' \succ C_2 - w$, $w'v \in E(G)$ and $w \succ C_2 - w'$, $wv \in E(G)$. This settles our claim.

By similar arguments as in the proof of Claim 1, Claim 2 follows.

Claim 2: If $N_{C_1}(v) - N_{C_1}(u) \neq \emptyset$, then for each $y \in N_{C_2}(u)$, there exists a unique vertex $y' \in N_{C_2}(u) - \{y\}$ such that $y \succ C_2 - y'$ and $y' \succ C_2 - y$.

Claim 3: $N_{C_1}(u) \cup N_{C_1}(v) = N_{C_1}(u) \cap N_{C_1}(v)$.

Suppose to the contrary that there exists $x \in N_{C_1}(u) - N_{C_1}(v)$. If $v \succ C_2$, then $\{x, v\} \succ_i G$ since C_1 is complete, a contradiction. Choose $y \in V(C_2) - N_{C_2}(v)$. Consider $G + xy$. Let $\{z\} = I_{xy} - \{x, y\}$. By Lemma 2.2(2), $z \notin V(C_1) \cup \{u\}$. Further, $z \succ \{v\}$ since x and y are not adjacent to v . We first suppose that $I_{xy} = \{x, z\}$. If $z \in V(C_2) - \{y\}$, then $z \in N_{C_2}(v)$ and thus $zy \in E(G)$ by Claim 1. But this contradicts Lemma 2.2(2). Hence, $z = v$ and thus $V(C_2) - N_{C_2}(v) = \{y\}$. It then follows by Claim 1 that $y \succ C_2$. But this contradicts our hypothesis that $i(C_2) = 2$. Hence, $I_{xy} = \{y, z\}$. Then $z = v$ since $|V(C_1)| \geq 2$ and $z \notin V(C_1) \cup \{u\}$. Thus $y \succ V(C_2) - N_{C_2}(v)$. By Claim 1, $y \succ C_2$, again a contradiction. Hence, $N_{C_1}(u) - N_{C_1}(v) = \emptyset$. By Claim 2 and similar arguments, $N_{C_1}(v) - N_{C_1}(u) = \emptyset$. This settles our claim.

Claim 4: $G[C_2] \cong K_{2m}$ - a perfect matching for some positive integer m .

Let $x \in N_{C_1}(u)$ and $y \in V(C_2)$. Thus $x \in N_{C_1}(v)$ by Claim 1. Consider $G + xy$. By Lemma 2.2(2) together with the fact that C_1 is complete and $|V(C_1)| \geq 2$, $(I_{xy} - \{x, y\}) \cap (\{u, v\} \cup V(C_1)) = \emptyset$. By applying similar arguments as in the proof of Claim 1, our claim follows.

Now put $A = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. We now distinguish two cases.

Case 1: $A = \emptyset$.

Then each vertex of $V(C_2)$ is adjacent to u or v .

Claim 5: Suppose x and y are vertices of C_2 where $xy \notin E(G)$. If $x \in N_{C_2}(u) \cap N_{C_2}(v)$, then $y \in N_{C_2}(u) \cap N_{C_2}(v)$.

Note that, by Claim 4, $N_G[x] = (V(C_2) - \{y\}) \cup \{u, v\}$ and $(V(C_2) - \{x\}) \subseteq N_G[y]$. Suppose to the contrary that $yu \notin E(G)$. Since $A = \emptyset$, $N_G[y] = (V(C_2) - \{x\}) \cup \{v\}$. Consider $G + uy$. Put $\{z\} = I_{uy} - \{u, y\}$. Then, by Lemma 2.2(2), $z \in V(C_1) - N_{C_1}(u)$. Thus $zu \notin E(G)$. By Claim 3, $zv \notin E(G)$. Now consider $G + zx$. Lemma 2.2(2) together with the fact that C_1 is complete implies that $I_{zx} - \{z, x\} = \{y\}$. If $I_{zx} = \{z, y\}$, then no vertex of I_{zx} is adjacent to u , contradiction. Hence, $I_{zx} = \{x, y\}$. But then no vertex of I_{zx} dominates $C_1 - z$, contradicting the fact that $|V(C_1)| \geq 2$. Hence, $yu \in E(G)$. By similar arguments, $yv \in E(G)$. This settles our claim.

By Claim 4, it is easy to see that for each $x \in V(C_2)$ there exists only one vertex of C_2 , say y , such that $xy \notin E(G)$. Let

$X = \{x \in N_{C_2}(u) \cap N_{C_2}(v) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(u) \cap N_{C_2}(v)\}$,

$Y = \{x \in N_{C_2}(u) - N_{C_2}(v) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(u) - N_{C_2}(v)\}$,

$Z = \{x \in N_{C_2}(v) - N_{C_2}(u) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } (N_{C_2}(v) - N_{C_2}(u))\}$

and

$W = \{x \in N_{C_2}(u) - N_{C_2}(v) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(v) - N_{C_2}(u)\} \cup \{x \in N_{C_2}(v) - N_{C_2}(u) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(u) - N_{C_2}(v)\}$.

It follows by Claim 5 that X, Y, Z and W form a partition set of $N_{C_2}(u) \cup N_{C_2}(v)$. Further, $G[X], G[Y], G[Z]$ and $G[W]$ are complete without a perfect matching.

Claim 6: If $Y \neq \emptyset$ ($Z \neq \emptyset$), then $N_{C_1}(u) \cap N_{C_1}(v) = V(C_1)$, $uv \in E(G)$ and $Z \neq \emptyset$ ($Y \neq \emptyset$).

Let $x_1, x_2 \in Y$ where $x_1x_2 \notin E(G)$. Then $vx_1, vx_2 \notin E(G)$. Consider $G + vx_1$. Put $\{z\} = I_{vx_1} - \{v, x_1\}$. If $I_{vx_1} = \{x_1, z\}$, then $z \succ \{x_2\} \cup V(C_1)$ and thus $z = u$. But this contradicts Lemma 2.2(2). Hence, $I_{vx_1} = \{v, z\}$. Since x_2 is the only non-neighbour of x_1 in C_2 and $vx_2 \notin E(G)$, it follows that $z = x_2$. Thus $v \succ C_1$. By Claim 3, $u \succ C_1$. Hence, $N_{C_1}(u) \cap N_{C_1}(v) = V(C_1)$ as required.

We now suppose to the contrary that $Z = \emptyset$. It then follows that $uv \in E(G)$ otherwise $\{u, v\} \succ_i G$ since $A = \emptyset$. If $W = \emptyset$, then $u \succ G$, a contradiction. Hence, $W \neq \emptyset$. Let $w_1, w_2 \in W$ where $w_1w_2 \notin E(G)$, $w_1 \in N_{C_2}(u) - N_{C_2}(v)$ and $w_2 \in N_{C_2}(v) - N_{C_2}(u)$. But then $\{u, w_2\} \succ_i G$, again a contradiction. Hence, $Z \neq \emptyset$. This settles our claim.

It follows by Claim 6 that $Y = \emptyset$ if and only if $Z = \emptyset$.

Claim 7: If $W \neq \emptyset$, then $Y = \emptyset$, $|W| = 2$, $|V(C_1) - (N_{C_2}(u) \cap N_{C_2}(v))| = 1$ and $uv \in E(G)$.

Let $w_1, w_2 \in W$ where $w_1w_2 \notin E(G)$, $w_1 \in N_{C_2}(u) - N_{C_2}(v)$ and $w_2 \in N_{C_2}(v) - N_{C_2}(u)$. Consider $G + vw_1$. By similar arguments as in the proof of Claim 6, $I_{vw_1} = \{v, z\}$ for some

$z \in V(G) - \{v, w_1\}$. Since w_2 is the only non-neighbour of w_1 in C_2 and $vw_2 \in E(G)$, it follows that $z \in V(C_1) - N_{C_1}(v)$. Clearly, $v \succ C_2 - w_1$. Thus $Y = \emptyset$ and $|W| = 2$, as required.

By Claim 3, $z \in V(C_1) - (N_{C_1}(u) \cap N_{C_1}(v))$. Consider $G + vz$. By Lemma 2.5, either $I_{vz} = \{v, w_1\}$ or $I_{vz} = \{z, w_1\}$. If $I_{vz} = \{z, w_1\}$, then no vertex of I_{vz} is adjacent to w_2 , a contradiction. Thus $I_{vz} = \{v, w_1\}$. Hence, $V(C_1) - (N_{C_1}(u) \cap N_{C_1}(v)) = \{z\}$ as required.

We now show that $uv \in E(G)$. Suppose this is not the case. Consider $G + uv$. Let $\{z_1\} = I_{uv} - \{u, v\}$ for some $z_1 \in V(G) - \{u, v\}$. If $I_{uv} = \{u, z_1\}$, then $z_1 \succ \{z, w_2\}$. But this is not possible since $z \in V(C_1) - (N_{C_1}(u) \cap N_{C_1}(v))$ and $w_2 \in N_{C_2}(v) - N_{C_2}(u)$. Hence, $I_{uv} = \{v, z_1\}$. But then $z_1 \succ \{z, w_1\}$ which is not possible either. Therefore, $uv \in E(G)$. This settles our claim.

We now distinguish three subcases.

Subcase 1.1: $Y \neq \emptyset$.

By Claim 7, $W = \emptyset$ and by Claim 6, $N_{C_1}(u) \cap N_{C_1}(v) = V(C_1)$, $uv \in E(G)$ and $Z \neq \emptyset$. Then $N_{C_2}(u) \cup N_{C_2}(v) = Y \cup X \cup Z$. Then G is isomorphic to the graph $G_1 \in \mathcal{A}_2$ if $X = \emptyset$ and G is isomorphic to the graph $G_2 \in \mathcal{A}_2$ if $X \neq \emptyset$.

Subcase 1.2: $W \neq \emptyset$.

By Claim 7, $Y = \emptyset$, $|W| = 2$, $|V(C_1) - (N_{C_1}(u) \cup N_{C_1}(v))| = 1$ and $uv \in E(G)$. Then $Z = \emptyset$ by Claim 6 and thus $V(C_2) = W \cup X$. It is easy to see that $X \neq \emptyset$ otherwise u or v becomes a cutvertex contradicting our hypothesis that $\{u, v\}$ is a minimum cutset. Further, $|N_{C_1}(u) \cap N_{C_1}(v)| \geq 2$. Hence, G is isomorphic to the graph $G_3 \in \mathcal{A}_2$.

Subcase 1.3: $Y = \emptyset$ and $W = \emptyset$.

By Claim 6, $Z = \emptyset$ and thus $N_{C_2}(u) \cup N_{C_2}(v) = X$. Then $u \succ C_2$ and $v \succ C_2$. It is easy to see that if $uv \in E(G)$, $i(G) \leq 2$ since C_1 is complete. Hence, $uv \notin E(G)$. Further, $N_{C_1}(u) \cup N_{C_1}(v) \neq V(C_1)$. Since $\{u, v\}$ is a minimum cutset, $|N_{C_1}(u) \cup N_{C_1}(v)| \geq 2$. It is easy to see that G is isomorphic to a graph in \mathcal{A}_3 .

Case 2: $A \neq \emptyset$.

By Lemma 2.3 and Claim 3, $V(C_1) = N_{C_1}(u) \cup N_{C_1}(v) = N_{C_1}(u) \cap N_{C_1}(v)$.

Claim 8: Suppose there exists a pair of non-adjacent vertices w_1, w_2 of C_2 where $\{w_1, w_2\} \subseteq A$. Then $uv \in E(G)$. Further, for a pair of non-adjacent vertices x and y of $V(C_2) - \{w_1, w_2\}$, if $x \in A$, then $y \in A$.

Consider $G + uw_1$. Let $\{z\} = I_{uw_1} - \{u, w_1\}$. By Lemma 2.2(2) and Claim 4, $z \in \{v, w_2\}$. If $I_{uw_1} = \{w_1, z\}$, then $z \succ \{w_2\} \cup V(C_1)$ which is not possible since $d(w_2, c) > 2$ for each $c \in V(C_1)$. Hence, $I_{uw_1} = \{u, z\}$. It follows that $z = w_2$ since $z \succ \{w_2\}$. Hence, $uv \in E(G)$ as required.

We now suppose to the contrary that $y \notin A$. Assume that $y \in N_G(u)$. Consider $G + ux$. Since $u \succ V(C_1) \cup \{v, y\}$ and $x \succ V(C_2) - \{y\}$ by Claim 4, it follows that

$I_{ux} - \{u, x\} \subseteq N_G(u) \cup N_G(x)$. But this contradicts Lemma 2.2(2). Hence, $y \in A$. This settles our claim.

Claim 9: Let x and y be a pair of non-adjacent vertices of C_2 . If $x \in N_{C_2}(u) \cap N_{C_2}(v)$, then $y \in N_{C_2}(u) \cap N_{C_2}(v)$ or $y \in A$.

Suppose this is not the case. Then we may assume that $y \in N_{C_2}(u) - N_{C_2}(v)$. Consider $G + vy$. Since $v \succ V(C_1) \cup \{x\}$ and $y \succ (V(C_2) - \{x\}) \cup \{u\}$, it then follows that $I_{vy} - \{v, y\} \subseteq N_G(v) \cup N_G(y)$. But this contradicts Lemma 2.2(2). This settles our claim.

Claim 10: Let x and y be a pair non-adjacent vertices of C_2 . If $x \in N_{C_2}(u) - N_{C_2}(v)$ ($N_{C_2}(v) - N_{C_2}(u)$), then $y \in N_{C_2}(u) - N_{C_2}(v)$ ($N_{C_2}(v) - N_{C_2}(u)$).

Suppose to the contrary that $y \in A$. Consider $G + vx$. Since $v \succ V(C_1)$ and $x \succ (V(C_2) - \{y\}) \cup \{u\}$ by Claim 4, it follows by Lemma 2.2(2) that either $I_{vx} = \{v, y\}$ or $I_{vx} = \{x, y\}$. If $I_{vx} = \{x, y\}$, then no vertex of I_{vx} dominates C_1 , a contradiction. Hence, $I_{vx} = \{v, y\}$. Thus $vu \in E(G)$ since $yu \notin E(G)$. Now consider $G + uy$. Because $u \succ V(C_1) \cup \{x, v\}$ and $y \succ V(C_2) - \{x\}$, $I_{uy} - \{u, y\} \subseteq N_G(u) \cup N_G(y)$. But this contradicts Lemma 2.2(2). Hence, $y \notin A$. By Claim 9, $y \notin N_{C_2}(u) \cap N_{C_2}(v)$. Then $y \in N_{C_2}(u) - N_{C_2}(v)$ or $y \in N_{C_2}(v) - N_{C_2}(u)$.

Suppose $y \notin N_{C_2}(u) - N_{C_2}(v)$. Then $y \in N_{C_2}(v) - N_{C_2}(u)$. Consider $G + xv$. Since $x \succ (V(C_2) - \{y\}) \cup \{u\}$ and $v \succ V(C_1) \cup \{y\}$, it follows that $I_{xv} \subseteq N_G(x) \cup N_G(v)$. But this contradicts Lemma 2.2(2). Hence, $y \in N_{C_2}(u) - N_{C_2}(v)$. This settles our claim.

Claim 11: Let x and y be a pair non-adjacent vertices of C_2 . If $x \in A$ and $y \notin A$, then $y \in N_{C_2}(u) \cap N_{C_2}(v)$ and $uv \notin E(G)$.

It follows by Claim 8 that $y \notin (N_{C_2}(u) - N_{C_2}(v)) \cup (N_{C_2}(v) - N_{C_2}(u))$. Thus $y \in N_{C_2}(u) \cap N_{C_2}(v)$. Consider $G + ux$. Since $u \succ V(C_1) \cup \{y\}$ and $x \succ V(C_2) - \{y\}$, it follows by Lemma 2.2(2) that $I_{ux} - \{u, x\} = \{v\}$. Thus $uv \notin E(G)$ by Lemma 2.2(2) as required.

Recall that $V(C_1) = N_{C_1}(u) \cap N_{C_1}(v)$ and $A \neq \emptyset$. Further, $N_{C_2}(u) \neq \emptyset$ and $N_{C_2}(v) \neq \emptyset$ since $\{u, v\}$ is a minimum cutset. We now distinguish two subcases.

Subcase 2.1: There exists a pair of non-adjacent vertices w_1, w_2 of C_2 where $\{w_1, w_2\} \subseteq A$.

By Claim 8, $uv \in E(G)$ and for each pair of non-adjacent vertices x and y of $V(C_2) - \{w_1, w_2\}$, if $x \in A$, then $y \in A$. It then follows by Claims 3, 8, 9 and 10 that G is isomorphic to a graph $G_i \in \mathcal{A}_2$ where $i \in \{4, 5, 6, 7\}$.

Subcase 2.2: For each pair of non-adjacent vertices of C_2 , say x and y , $x \notin A$ or $y \notin A$.

Since $A \neq \emptyset$, there exists $t_1 \in A$. Let t_2 be the non-neighbour of t_1 in C_2 . By our hypothesis and Claim 11, $t_2 \in N_{C_2}(u) \cap N_{C_2}(v)$. Further, $uv \notin E(G)$. It is easy to see that if $|V(C_2) - A| = 1$, then $|A| = 1$ and $V(C_2) - A = N_{C_2}(u) \cap N_{C_2}(v)$ and thus

G is disconnected. Consequently, $|V(C_2) - A| \geq 2$. It follows by Claims 9 and 10 that G is isomorphic to a graph in \mathcal{A}_4 . This completes the proof of Case 2 and our theorem. \square

In what follows, for subsets W and L of $V(G)$ where $W \cap L = \emptyset$, $W(\sim L)$ denotes the set of all vertices of W which are not adjacent to some vertex of L .

Theorem 8.4. *Let G, S, C_1 and C_2 be defined as in Theorem 8.2. If $i(C_2) = 2$ and $|V(C_1)| = 1$, then $G \in \bigcup_{i=5}^9 \mathcal{A}_i$.*

Proof. Let $\{w\} = V(C_1)$. Then $N_G(w) = \{u, v\}$ by the fact that S is a minimum cutset. Put $B = N_{C_2}(u)$, $C = N_{C_2}(v)$ and $A = V(C_2) - (B \cup C)$. By our hypothesis that $i(C_2) = 2$, we have following observation.

Observation 1: No vertex of $V(C_2)$ dominates C_2 .

Claim 1: Suppose $xy \notin E(G)$ where either $x \in B - C$ and $y \in C - B$ or x or y belongs to $B \cap C$. Then $I_{xy} - \{x, y\} = \{w\}$ and thus either $x \succ C_2 - y$ or $y \succ C_2 - x$. Further, if $x \in B \cap C$ and $y \in (B - C) \cup (C - B)$, then $I_{xy} = \{x, w\}$ and thus $x \succ C_2 - y$.

By Lemma 2.2(2), $I_{xy} \cap \{u, v\} = \emptyset$. Since $N_G(w) = \{u, v\}$, $I_{xy} - \{x, y\} = \{w\}$. If $I_{xy} = \{x, w\}$, then $x \succ C_2 - y$. Similarly, if $I_{xy} = \{y, w\}$, then $y \succ C_2 - x$. We now suppose that $x \in B \cap C$ and $y \in (B - C) \cup (C - B)$. If $I_{xy} = \{y, w\}$, then $y \succ C_2 - x$ and thus either $\{v, y\} \succ_i G$ or $\{u, y\} \succ_i G$, a contradiction. This settles our claim.

By similar arguments as in the proof of Claim 1, Claim 2 follows.

Claim 2: If $x \in B \cap C$ and $y \in A \cup (B \cap C)$ where $xy \notin E(G)$, then either $x \succ C_2 - y$ or $y \succ C_2 - x$ and thus $\{x, y\} \succ_i G - w$.

Claim 3: For $x \in B - C$, there exists a unique vertex $\bar{x} \in A \cup (B - C)$ where $x \succ (A \cup (B - C)) - \{\bar{x}\}$ and $\bar{x} \succ (A \cup (B - C)) - \{x\}$. Further, if $uv \notin E(G)$, then $\bar{x} \in B - C$, each vertex of $B - C$ dominates A and $G[B - C] \cong K_{2n}$ - a perfect matching for some positive integer n .

Consider $G + xv$. By Lemma 2.2(2), $I_{xv} \cap (\{u, w\} \cup (B \cap C) \cup (C - B)) = \emptyset$ since $xu, xv \in E(G)$ and $C \subseteq N_G(v)$. Because $N_G(w) = \{u, v\}$, $v \in I_{xv}$. Thus $I_{xv} = \{v, \bar{x}\}$ for some $\bar{x} \in (A \cup (B - C)) - \{x\}$. Clearly, $\bar{x} \succ (A \cup (B - C)) - \{x\}$. Now consider $G + \bar{x}v$. By similar arguments, $I_{\bar{x}v} = \{v, x\}$ since $\bar{x} \succ (A \cup (B - C)) - \{x\}$. Thus $x \succ (A \cup (B - C)) - \{\bar{x}\}$ as required.

We now suppose that $uv \notin E(G)$. Since $I_{xv} = \{v, \bar{x}\}$, \bar{x} dominates u and thus $\bar{x} \in (B - C) - \{x\}$. By above argument, $x \succ (A \cup (B - C)) - \{\bar{x}\}$ and $\bar{x} \succ (A \cup (B - C)) - \{x\}$. If $|B - C| = 2$, we are done. So suppose there exists $x_1 \in (B - C) - \{x, \bar{x}\}$. Consider $G + x_1v$. By the above arguments, there exists $\bar{x}_1 \in (B - C) - \{x_1\}$ such that $x_1 \succ (A \cup (B - C)) - \{\bar{x}_1\}$ and $\bar{x}_1 \succ (A \cup (B - C)) - \{x_1\}$. By Lemma 2.2(2), $\{x_1, \bar{x}_1\} \cap \{x, \bar{x}\} = \emptyset$. Continuing in this fashion, $G[B - C]$ is isomorphic to a complete graph without a perfect matching. Further, each vertex of $B - C$ dominates A .

This settles our claim.

By the same arguments, Claim 4 follows.

Claim 4: For $x \in C - B$, there exists a unique vertex $\bar{x} \in A \cup (C - B)$ where $x \succ (A \cup (C - B)) - \{\bar{x}\}$ and $\bar{x} \succ (A \cup (C - B)) - \{x\}$. Further, if $uv \notin E(G)$, then $\bar{x} \in C - B$, each vertex of $C - B$ dominates A and $G[C - B] \cong K_{2m}$ - a perfect matching for some positive integer m .

For the rest of our proof, if $x \in B - C(C - B)$, then we shall denote the only non-neighbour of x in $A \cup (B - C)(A \cup (C - B))$ by \bar{x} .

Claim 5: If $x \in B - C$ and $y \in C - B$, then $xy \in E(G)$. Thus each vertex of $B - C(C - B)$ dominates $C - B(B - C)$.

By Claims 3 and 4, there exist $\bar{x} \in A \cup (B - C)$ and $\bar{y} \in A \cup (C - B)$ such that $x\bar{x}, y\bar{y} \notin E(G)$. Suppose to the contrary that $xy \notin E(G)$. Consider $G + xy$. By Claim 1, either $x \succ C_2 - y$ or $y \succ C_2 - x$. But this is not possible since $x\bar{x}, y\bar{y} \notin E(G)$. Hence, $xy \in E(G)$. This settles our claim.

Now put $A_1 = \{x \in A | x \succ A\}$ and $A_2 = A - A_1$

Claim 6: $G[A_2]$ is isomorphic to a complete graph without a perfect matching and for each pair of non-adjacent vertices x_1 and x_2 of A_2 , $x_i \succ (B - C) \cup (C - B) \cup (A - \{x_j\})$ where $1 \leq i \neq j \leq 2$. Further, if $uv \notin E(G)$, then $A_2 = \emptyset$ and thus $G[A]$ is complete.

Let $x \in A_2$. Then there exists $y \in A_2$ such that $xy \notin E(G)$. Consider $G + xy$. Put $\{z\} = I_{xy} - \{x, y\}$. Then $z \in \{u, v, w\}$ since $N_G(w) = \{u, v\}$. We may assume without loss of generality that $I_{xy} = \{x, z\}$. Clearly, $x \succ A - \{y\}$ since z is not adjacent to any vertex of A . Now consider $G + xz$. By Lemma 2.5, $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. If $I_{xz} = \{x, y\}$, then no vertex of I_{xz} is adjacent to vertices of $\{u, v, w\} - \{z\}$, a contradiction. Hence, $I_{xz} = \{z, y\}$ and thus $y \succ A - \{x\}$. If $|A_2| = 2$, then we are done. So suppose that $|A_2| \geq 3$. By applying similar arguments as above and as in the proof of Claim 3, $G[A_2]$ is isomorphic to a complete graph without a perfect matching.

Now let $x_1, x_2 \in A_2$ where $x_1x_2 \notin E(G)$. It follows by Claims 3 and 4 that, $x_i \succ (B - C) \cup (C - B)$ for $1 \leq i \leq 2$. Since $G[A_2]$ is a complete graph without a perfect matching, $x_i \succ (B - C) \cup (C - B) \cup (A - \{x_j\})$. where $1 \leq i \neq j \leq 2$ as required.

We now suppose that $uv \notin E(G)$ and $A_2 \neq \emptyset$. Since $G[A_2]$ is a complete graph without a perfect matching, there exist $y_1, y_2 \in A_2$ where $y_1y_2 \notin E(G)$. Consider $G + uy_1$. Put $\{z_1\} = I_{uy_1} - \{u, y_1\}$. We first suppose that $I_{uy_1} = \{u, z_1\}$. Since $uv, uy_2 \notin E(G)$, z_1 dominates $\{v, y_2\}$. It follows that $z_1 \in C - B$ by Lemma 2.2(2) and $z_1 \succ C - B$. But this contradicts Claim 3. Hence, $I_{uy_1} = \{y_1, z_1\}$. Since $y_1y_2, y_1w \notin E(G)$, z_1 dominates $\{y_2, w\}$. But this is not possible since $d(y_2, w) = 3$. Hence, $A_2 = \emptyset$ as required. This settles our claim.

Let

$$\begin{aligned} X_1 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in B - C\} \\ X_2 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in C - B\} \\ X_3 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in B \cap C\} \\ X_4 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in A\}. \end{aligned}$$

By Observation 1, $B \cap C = X_1 \cup X_2 \cup X_3 \cup X_4$. The following claim follows immediately by Claims 1, 3 and 4.

Claim 7: If $x \in X_1(X_2)$, then there exists a unique vertex $y \in B - C(C - B)$ such that $N_G[x] = V(G) - \{y, w\}$.

Claim 8: For $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$, $X_i \cap X_j = \emptyset$. Further, if $uv \in E(G)$, $X_3 \cap X_4 = \emptyset$.

It follows by Claim 7 that $X_i \cap X_j = \emptyset$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$. We now assume that $uv \in E(G)$. Suppose to the contrary that there exists $x \in X_3 \cap X_4$. Let $y_1 \in B \cap C$ and $y_2 \in A$ where $xy_1, xy_2 \notin E(G)$. By Claim 1, either $x \succ C_2 - y_2$ or $y_2 \succ C_2 - x$. Since $xy_1 \notin E(G)$, $y_2 \succ C_2 - x$. But then $\{y_2, u\} \succ_i G$, a contradiction. Hence, $X_3 \cap X_4 = \emptyset$ as required.

By the definition of X_3 and X_4 together with Claims 7 and 8, we have the following claim.

Claim 9: If $x \in X_3 \cup X_4$, then $x \succ (B - C) \cup (C - B) \cup X_1 \cup X_2$. Further, suppose $uv \in E(G)$. Then if $x \in X_3$, then $x \succ A \cup (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup X_4$ and if $x \in X_4$, then $x \succ (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup X_3$.

For $x \in B - C(C - B)$, by Claim 3 (Claim 4) there is a unique vertex \bar{x} of $A \cup (B - C)(A \cup (C - B))$ which is not adjacent to x . Let

$$\begin{aligned} Y_1 &= \{x \in (B - C)(\sim X_1) \mid \bar{x} \in B - C\} \\ Y_2 &= \{x \in (B - C)(\sim X_1) \mid \bar{x} \in A\} \\ Z_1 &= \{x \in (C - B)(\sim X_2) \mid \bar{x} \in C - B\} \\ Z_2 &= \{x \in (C - B)(\sim X_2) \mid \bar{x} \in A\}. \end{aligned}$$

Observation 2: $X_1 \neq \emptyset$ if and only if $Y_1 \cup Y_2 \neq \emptyset$ and $X_2 \neq \emptyset$ if and only if $Z_1 \cup Z_2 \neq \emptyset$.

Observation 3: If $x \in Y_2 \cup Z_2$, then $\bar{x} \in A_1$ and $uv \in E(G)$.

Claim 10: If $y \in Y_1$, then $\bar{y} \in Y_1$.

Since $y \in Y_1$, there is a vertex $x \in X_1$ such that $xy \notin E(G)$. Suppose to the contrary that $\bar{y} \succ X_1$. By Claims 7 and 9, $\bar{y} \succ B \cap C$ and thus $\bar{y} \succ C_2 - y$ by Claims 3 and 5. Consider $G + w\bar{y}$. By Lemma 2.2(2) and the fact that $N_G(w) = \{u, v\}$, it follows that $I_{w\bar{y}} - \{w, \bar{y}\} = \{y\}$. If $I_{w\bar{y}} = \{w, y\}$, then no vertex of $I_{w\bar{y}}$ dominates x , a contradiction.

Hence, $I_{w\bar{y}} = \{\bar{y}, y\}$. But then no vertex of $I_{w\bar{y}}$ dominates v , again a contradiction. Hence, there exists a vertex $x_1 \in X_1$ such that $\bar{y}x_1 \notin E(G)$. This settles our claim.

By similar arguments as in the proof of Claim 10, Claim 11 follows.

Claim 11: If $z \in Z_1$, then $\bar{z} \in Z_1$.

Claim 12: If $Y_1 \neq \emptyset$, then each component of $\overline{G}[Y_1 \cup (X_1(\sim Y_1))]$ is a double star $DS(m, n)$, in \overline{C}_2 , for some positive integers m and n where the center and the end vertices of $DS(m, n)$ are in Y_1 and $X_1(\sim Y_1)$, respectively.

Let $y \in Y_1$. Then $yx_1 \notin E(G)$ for some $x_1 \in X_1$. By Claim 10, $\bar{y} \in Y_1$ and thus there exists $x_2 \in X_1$ such that $\bar{y}x_2 \notin E(G)$. By Claim 7, $x_1 \neq x_2$ and $N_G[x_1] = V(G) - \{y, w\}$ and $N_G[x_2] = V(G) - \{\bar{y}, w\}$. Further, by Claims 3, 5, 7 and 9, $V(C_2) - (\{\bar{y}\} \cup X_1) \subseteq N_G[y]$ and $V(C_2) - (\{y\} \cup X_1) \subseteq N_G[\bar{y}]$. It then follows by Claim 7 that $\overline{G}[\{y, \bar{y}, x_1, x_2\} \cup \overline{N}_{X_1}(y) \cup \overline{N}_{X_1}(\bar{y})]$ is a component in \overline{C}_2 and forms a double star $DS(m, n)$ where $m = |\overline{N}_{X_1}(y)| \geq 1$ and $n = |\overline{N}_{X_1}(\bar{y})| \geq 1$. Further, y and \bar{y} are center vertices and vertices of $\overline{N}_{X_1}(y) \cup \overline{N}_{X_1}(\bar{y})$ are end vertices of $DS(m, n)$ in \overline{C}_2 . This settles our claim.

By similar arguments as in the proof of Claim 12, Claim 13 follows.

Claim 13: If $Z_1 \neq \emptyset$, then each component of $\overline{G}[Z_1 \cup X_2(\sim Z_1)]$ is a double star $DS(m, n)$, in \overline{C}_2 , for some positive integers m and n where the center and the end vertices of $DS(m, n)$ are in Z_1 and $X_2(\sim Z_1)$, respectively.

We now distinguish two cases according to uv .

Case 1: $uv \notin E(G)$.

Then $A \neq \emptyset$ otherwise $i(G) = 2$ and thus $G[A]$ is complete by Claim 6. Further, by Claims 3 and 4, $G[B - C]$ and $G[C - B]$ are complete without a perfect matching and thus $Y_2 = Z_2 = \emptyset$. It follows that $X_1(\sim Y_1) = X_1$ and $X_2(\sim Z_1) = X_2$. Since a vertex of A dominates $A \cup (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup (X_3 - X_4)$ by the definitions of X_3 and X_4 and Claims 3, 4, 6 and 7, $X_4 \neq \emptyset$ otherwise $i(G) = 2$. Consequently, $i(G[A \cup X_3 \cup X_4]) \geq 2$.

Claim 14: Each component of $\overline{G}[A \cup X_3 \cup X_4]$ is a star $K_{1,r}$, in \overline{C}_2 , for some positive integer r . Further, for each pair of non-adjacent vertices x and y , $|\{x, y\} \cap A| \leq 1$, $\{x, y\} \succ_i G - w$ and if x is the center and y is an end vertex of $K_{1,r}$ in $\overline{G}[A \cup X_3 \cup X_4]$, then $\{y, w\} \succ_i G - x$.

Claims 1, 3, 4, 7, 8 and 9 together with the definitions of X_3 and X_4 imply that if $z \in X_3 \cup X_4$, then $\overline{N}_{C_2}(z) \subseteq X_3 \cup X_4 \cup A$. Further, if $z \in A$, then $\overline{N}_{C_2}(z) \subseteq X_4$. Now let $x, y \in A \cup X_3 \cup X_4$ where $xy \notin E(G)$. By Claim 6, $|\{x, y\} \cap A| \leq 1$. We may suppose without loss of generality that $x \notin A$. Then $x \in X_3 \cup X_4 \subseteq B \cap C$. Consider $G + xy$. By Claim 1, either $x \succ C_2 - y$ or $y \succ C_2 - x$. Hence, $i(G[A \cup X_3 \cup X_4] + xy) = 1$. Therefore, $i(G[A \cup X_3 \cup X_4]) = 2$ and $G[A \cup X_3 \cup X_4]$ is 2- i -critical. Thus, by Lemma 2.4, each

component of $\overline{G}[A \cup X_3 \cup X_4]$ is a star $K_{1,r}$, in \overline{C}_2 .

Since $x \in B \cap C$ and either $x \succ C_2 - y$ or $y \succ C_2 - x$, it follows that $\{x, y\} \succ_i G - w$ and thus $\{x, w\} \succ_i G - y$ or $\{y, w\} \succ_i G - x$ by Lemma 2.5. It is easy to see that if x is a center and y is an end vertex of $K_{1,r}$, in \overline{C}_2 , then $\{y, w\} \succ_i G - x$. This settles our claim.

Now let $S_1 = \{K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4] \mid V(K_{1,r}) \cap A = \emptyset\}$, $S_2 = \{K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4] \mid \text{some end vertices of } K_{1,r} \text{ belongs to } A\}$ and $S_3 = \{K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4] \mid \text{the center of } K_{1,r} \text{ belongs to } A\}$. Clearly, $\overline{G}[A \cup X_3 \cup X_4] = S_1 \cup S_2 \cup S_3$ and $X_3 \cap X_4 = \emptyset$ if and only if all end vertices of stars in S_2 belong to A .

Claim 15: For $1 \leq i \neq j \leq 3$, $S_i \cap S_j = \emptyset$

Let xy be an edge of $K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4]$ where x is the center and y is an end vertex of $K_{1,r}$. By Claim 6, $|\{x, y\} \cap A| \leq 1$. It is easy to see that if $|\{x, y\} \cap A| = 0$, then $\{x, y\} \subseteq X_3$. If there exists an end $y_1 \neq y$ of $K_{1,r}$ where $y_1 \in A$, then $K_{1,r} \in S_2$ otherwise $K_{1,r} \in S_1$. We now suppose that $|\{x, y\} \cap A| = 1$. If $\{x, y\} \cap A = \{x\}$, then $K_{1,r} \in S_3$ and if $\{x, y\} \cap A = \{y\}$, $K_{1,r} \in S_2$. This settles our claim.

Claim 16: Suppose $\overline{G}[A \cup X_3 \cup X_4] = K_{1,r}$ for some positive integer r . Then $K_{1,r} \in S_2$ or $K_{1,r} \in S_3$. Further, if $K_{1,r} \in S_3$, then $|B - C| \geq 2$ or $|C - B| \geq 2$ and if $K_{1,r} \in S_2$, then either at least two end vertices of $K_{1,r}$ belong to X_3 or $|B - C| \geq 2$ or $|C - B| \geq 2$.

Since $A \neq \emptyset$, and $X_4 \neq \emptyset$, it follows that $K_{1,r} \notin S_1$. Thus $K_{1,r} \in S_2$ or $K_{1,r} \in S_3$. Let x, y_1, y_2, \dots, y_r be the center and end vertices of $K_{1,r}$, respectively. We first suppose that $K_{1,r} \in S_3$. Then $x \in A$ and thus $\{y_1, y_2, \dots, y_r\} = X_4$. Because G is connected and $\{u, v\}$ is a minimum cutset of G , it follows that there exist at least two vertices of $V(C_2) - (A \cup X_4)$, joining to x . Thus $B - C$ or $C - B$ is not empty by Observation 2 and the fact that $G[B - C]$ and $G[C - B]$ are complete graphs without a perfect matching. Clearly, $|B - C| \geq 2$ or $|C - B| \geq 2$.

We now suppose that $K_{1,r} \in S_2$. Assume that $y_1 \in A$. Thus $x \in X_4$. By similar arguments as above, if $|\{y_1, y_2, \dots, y_r\} - A| \leq 1$, then $B - C$ or $C - B$ is not empty. So we suppose that $|\{y_1, y_2, \dots, y_r\} - A| \geq 2$. Thus $x \in X_3 \cup X_4$ and thus $\{y_1, y_2, \dots, y_r\} - A \subseteq X_3$ as required. This settles our claim.

We are now ready to characterize graphs in this case. Recall that $Y_2 = Z_2 = \emptyset$. Let F be a component of \overline{C}_2 and $x \in V(F)$. We first suppose that $x \in (B - C) - Y_1 ((C - B) - Z_1)$. By Claims 3(4), 5, 7 and 9, there exists a unique vertex $\bar{x} \in (B - C) - Y_1 ((C - B) - Z_1)$ where $N_{C_2}[x] = V(C_2) - \{\bar{x}\}$ and $N_{C_2}[\bar{x}] = V(C_2) - \{x\}$. Thus $V(F) = \{x, \bar{x}\}$ and F is a K_2 , in \overline{C}_2 , by Claim 3(4).

We next suppose that $x \in Y_1 \cup X_1 (\sim Y_1) = Y_1 \cup X_1$. Clearly, $V(F) \cap X_1 \neq \emptyset$ and $V(F) \cap Y_1 \neq \emptyset$. We may assume that $x \in V(F) \cap X_1$. Let $y \in V(F) \cap Y_1$. By Claim 7, $N_G[x] = V(G) - \{y, w\}$ and thus $N_{C_2}[x] = V(C_2) - \{y\}$. By Claim 10, there exists a unique vertex $\bar{y} \in Y_1$ where $V(C_2) - (\{\bar{y}\} \cup X_1) \subseteq N_{C_2}[y]$ and $V(C_2) - (\{y\} \cup X_1) \subseteq N_{C_2}[\bar{y}]$ by Claim 3, 5, 7 and 9. It then follows that $V(F) \subseteq Y_1 \cup X_1$. By Claim 12, F is a double star, in \overline{C}_2 , where y and \bar{y} are center vertices and vertices of $\overline{N}_{X_1}(y) \cup \overline{N}_{X_1}(\bar{y})$ are end

vertices.

By similar arguments, if $x \in Z_1 \cup (X_2(\sim Z_1)) = Z_1 \cup X_2$, then $V(F) \subseteq Z_1 \cup X_2$ and F is a double star in \overline{C}_2 .

Finally, we may suppose that $x \in A \cup X_3 \cup X_4$. It is easy to see that if $V(F) \cap A = \emptyset$, then $V(F) \subseteq X_3$ by Claim 9 and thus F is a star by Claim 14. So we may now suppose that $V(F) \cap A \neq \emptyset$ and $x \in V(F) \cap A$. Then there exists $y \in V(F) \cap X_4$ such that $xy \notin E(G)$. By Claim 9, $V(F) \subseteq A \cup X_3 \cup X_4$ and thus F is a star by Claim 14.

We may conclude here that a component F of \overline{C}_2 is a K_2 or a star or a double star.

Recall that $A \neq \emptyset$ and $X_4 \neq \emptyset$. We first suppose that $B - C = C - B = \emptyset$. By Observation 2, $X_1 = X_2 = \emptyset$ and thus $V(G) = \{u, v, w\} \cup A \cup X_3 \cup X_4$. It follows by Claim 16 that $\overline{G}[A \cup X_3 \cup X_4]$ contains either at least two stars or exactly one star with the center of the star belongs to X_4 , A is a subset of the set of end vertices of the star and at least two end vertices of the star are in X_3 . Therefore, G is isomorphic to a graph in $\{G', G'', G''', G_{S_1}^1, G_{S_1}^2, G_{S_1}^3\} \subseteq \mathcal{A}_6$.

We next suppose that $B - C \neq \emptyset$ but $C - B = \emptyset$. Then, by Observation 2, $X_2 = \emptyset$ and thus $V(G) = \{u, v, w\} \cup (B - C) \cup A \cup X_1 \cup X_3 \cup X_4$. Then G is isomorphic to a graph in $\{G_{P_1}^j, G_{D_1}^j, G_{P_1 D_1}^j, G_{P_1 S_1}^j, G_{D_1 S_1}^j, G_{P_1 D_1 S_1}^j\} \subseteq \mathcal{A}_6$ where $1 \leq j \leq 3$. By similar arguments, if $C - B \neq \emptyset$ but $B - C = \emptyset$, then G is isomorphic to a graph in $\{G_{P_2}^j, G_{D_2}^j, G_{P_2 D_2}^j, G_{P_2 S_1}^j, G_{D_1 S_1}^j, G_{P_1 D_1 S_1}^j\} \subseteq \mathcal{A}_6$ where $1 \leq j \leq 3$. Finally, we suppose that $B - C \neq \emptyset$ and $C - B \neq \emptyset$. Then G is isomorphic to a graph in $\{G_{P_1 P_2}^j, G_{P_1 D_2}^j, G_{P_1 P_2 D_2}^j, G_{P_1 P_2 S_1}^j, G_{P_1 D_2 S_1}^j, G_{P_1 P_2 D_2 S_1}^j, G_{D_1 P_2}^j, G_{D_1 D_2}^j, G_{D_1 P_2 D_2}^j, G_{D_1 P_2 S_1}^j, G_{D_1 D_2 S_1}^j, G_{D_1 P_1 P_2 S_1}^j, G_{P_1 D_1 P_2}^j, G_{P_1 D_1 D_2}^j, G_{P_1 D_1 P_2 D_2}^j, G_{D_1 D_2 P_2 S_1}^j, G_{P_1 D_1 D_2 S_1}^j, G_{P_1 D_1 P_2 D_2 S_1}^j\} \subseteq \mathcal{A}_6$ where $1 \leq j \leq 3$. This proves Case 1.

Case 2: $uv \in E(G)$.

Before we characterize this case, we need to establish some more claims. By Claim 8, $X_3 \cap X_4 = \emptyset$. By the definition of X_3 , $i(G[X_3]) \geq 2$. By applying similar arguments as in the proof of Claim 14 together with the fact that for each $x \in X_3$, $x \succ A \cup (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup X_4$ by Claim 9, Claim 17 follows.

Claim 17: Each component of $\overline{G}[X_3]$ is a star $K_{1,r}$ for some positive integer r . Further, if $x, y \in X_3$ where $xy \notin E(G)$, then $\{x, y\} \succ_i G - \{w\}$.

Claim 18: Suppose $x, y \in A_2$ where $xy \notin E(G)$. If x is not adjacent to some vertex $x_1 \in X_4$, then y is not adjacent to some vertex $y_1 \in X_4 - \{x_1\}$.

By applying similar arguments as in the proof of Claim 10 together with the definition of X_3 and Claims 1, 6, 7 and 9, our claim follows.

Claim 19: If $A_2(\sim X_4) \neq \emptyset$, then each component of $\overline{G}[(X_4(\sim A_2)) \cup (A_2(\sim X_4))]$ is a double star $DS(m, n)$ for some positive integers m and n where the center and the end vertices of $D(m, n)$ are in $A_2(\sim X_4)$ and $X_4(\sim A_2)$, respectively.

By applying similar arguments as in the proof of Claim 12 together with Claims 2, 6,

7 and 18, our claim follows.

Claim 20: For $x \in A_1$, there exist $y_1 \in B - C$ and $y_2 \in C - B$ such that $xy_1, xy_2 \notin E(G)$ and $x \succ (A \cup (B - C) \cup (C - B)) - \{y_1, y_2\}$ and $y_i \succ (A \cup (B - C) \cup (C - B)) - \{x\}$ for $1 \leq i \leq 2$. Further, there exist three distinct vertices of $B \cap C$, say a_1, a_2, b such that $y_1a_1, y_2a_2, xb \notin E(G)$.

It is easy to see that if $x \succ B - C$, then $\{x, v\} \succ_i G$ since $uv \in E(G)$, a contradiction. Hence, there exists $y_1 \in B - C$ such that $xy_1 \notin E(G)$. Similarly, there exists $y_2 \in C - B$ such that $xy_2 \notin E(G)$. By the definition of A_1 together with Claims 3 and 4, $x \succ (A \cup (B - C) \cup (C - B)) - \{y_1, y_2\}$. Further, by Claims 3, 4 and 5, $y_i \succ (A \cup (B - C) \cup (C - B)) - \{x\}$ for $1 \leq i \leq 2$.

Suppose to the contrary that $y_1 \succ B \cap C$. Then $y_1 \succ C_2 - x$. Consider $G + wy_1$. Put $\{z\} = I_{wy_1} - \{w, y_1\}$. Then $z = x$ by Lemma 2.2(2) since $y_1 \succ C_2 - x$ and $N_G(w) = \{u, v\}$. If $I_{wy_1} = \{w, x\}$, then no vertex of I_{wy_1} is adjacent to y_2 , a contradiction. Hence, $I_{wy_1} = \{y_1, x\}$. But then no vertex of I_{wy_1} is adjacent to v , again a contradiction. Hence, there exists $a_1 \in B \cap C$ such that $y_1a_1 \notin E(G)$. Similarly, there exists $a_2 \in B \cap C$ such that $y_2a_2 \notin E(G)$.

We next show that x does not dominate $B \cap C$. Suppose this is not the case. Then $x \succ B \cap C$ and thus $x \succ C_2 - \{y_1, y_2\}$. Consider $G + wx$. Put $\{z_1\} = I_{wx} - \{w, x\}$. By Lemma 2.2(2), $z_1 \in \{y_1, y_2\}$ since $x \succ C_2 - \{y_1, y_2\}$ and $N_G(w) = \{u, v\}$. If $I_{wx} = \{w, z_1\}$, then no vertex of I_{wx} is adjacent to either a_1 or a_2 , a contradiction. Hence, $I_{wx} = \{x, z_1\}$. Then no vertex of I_{wx} is adjacent to either v or u , again a contradiction. Hence, there exists $b \in B \cap C$ such that $xb \notin E(G)$. By Claims 1 and 2, a_1, a_2 and b are all distinct. This settles our claim.

Claim 21: Suppose $A_1 \neq \emptyset$. Then each component of $\overline{G}[A_1 \cup Y_2 \cup Z_2 \cup (X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup (X_4(\sim A_1))]$ is a triple star $TD(m, n, r)$ for some positive integers m, n and r where the middle center vertex is in A_1 one of the end center vertices is in Y_2 and the another end center vertices is in Z_2 while all the end vertices are in $(X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup (X_4(\sim A_1))$.

Let $x \in A_1$. By Claim 20, there exist $y_1 \in B - C$ and $y_2 \in C - B$ such that $xy_1, xy_2 \notin E(G)$ and $x \succ (A \cup (B - C) \cup (C - B)) - \{y_1, y_2\}$ and for $1 \leq i \leq 2$, $y_i \succ (A \cup (B - C) \cup (C - B)) - \{x\}$. Further, there exist $a_1, a_2, b \in B \cap C$ where $y_1a_1, y_2a_2, xb \notin E(G)$. Clearly, $a_1 \in X_1(\sim Y_2)$, $a_2 \in X_2(\sim Z_2)$ and $b \in X_4(\sim A_1)$. By Claims 1, 2, 5, 7, 9 and 20, $\overline{N}_{C_2}(x) \subseteq \{y_1, y_2\} \cup (X_4(\sim A_1))$, $\overline{N}_{C_2}(y_1) \subseteq \{x\} \cup (X_1(\sim Y_2))$ and $\overline{N}_{C_2}(y_2) \subseteq \{x\} \cup (X_2(\sim Z_2))$. Further, for $c \in \{y_1, y_2, x\}$, if $c_1 \in \overline{N}_{C_2}(c) - \{y_1, y_2, x\}$, then $c_1 \succ C_2 - c$. Hence, $\overline{G}[\{x, y_1, y_2\} \cup \overline{N}_{C_2}(x) \cup \overline{N}_{C_2}(y_1) \cup \overline{N}_{C_2}(y_2)]$ is a component in \overline{C}_2 and forms a triple star $TS(m, n, r)$ where $m = |\overline{N}_{C_2}(y_1)|$, $n = |\overline{N}_{C_2}(x)|$ and $r = |\overline{N}_{C_2}(y_2)|$. Clearly, m, n and r are positive integers. Further, x is the middle center vertex, y_1 and y_2 are the end center vertices and each vertex of $\overline{N}_{C_2}(x) \cup \overline{N}_{C_2}(y_1) \cup \overline{N}_{C_2}(y_2)$ is an end vertex of $TS(m, n, r)$ as required. This settles our claim.

Let F be a component of \overline{C}_2 and $x \in V(F)$. By the same arguments as in the proof

of Case 1, if $x \in (B - C) - (Y_1 \cup Y_2)((C - B) - (Z_1 \cup Z_2))$, F is K_2 . If $x \in Y_1 \cup (X_1(\sim Y_1))(Z_1 \cup (X_2(\sim Z_1)))$, F is a double star. We now suppose that $x \in X_3$. By Claim 9, $V(F) \subseteq X_3$ and thus F is a star by Claim 17.

We next suppose that $x \in X_4(\sim A_1)$. Then there exists $x_1 \in A_1$ such that $xx_1 \notin E(G)$. By Claim 20, there exist $a_1, a_2, b \in B \cap C, y_1 \in B - C, y_2 \in C - B$ where $x_1y_1, x_1y_2, y_1a_1, y_2a_2, x_1b \notin E(G)$. (Note that b and x might not be different). Then $\{x, x_1, y_1, y_2, a_1, a_2, b\} \subseteq V(F)$. By similar arguments as in the proof of Claim 21, $V(F) \subseteq (X_4(\sim A_1)) \cup (X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup \{x_1, y_1, y_2\}$. By Claim 21, F is a triple star.

Finally, we assume that $x \in X_4(\sim A_2)$. Then there exists $w_1 \in A_2$ such that $xw_1 \notin E(G)$. By Claim 6, there exists a unique vertex $w_2 \in A_2 - \{w_1\}$ such that $w_i \succ (B - C) \cup (C - B) \cup (A - \{w_j\})$ for $1 \leq i \neq j \leq 2$. By Claim 18, there exists $c \in B \cap C$ such that $w_2c \notin E(G)$. By Claim 2, $x \succ C_2 - w_1$ and $c \succ C_2 - w_2$. Further, by Claims 7 and 9, $w_i \succ X_1 \cup X_2 \cup X_3$ for $1 \leq i \leq 2$. Thus $V(F) \subseteq (X_4(\sim A_2)) \cup \{w_1, w_2\}$. Hence, F is a double star by Claim 19.

We are now ready to characterize this case by distinguishing three cases.

Case 2.1: $A_1 = A_2 = \emptyset$.

Since $uv \in E(G)$, $u \succ V(G) - (C - B)$ and $v \succ V(G) - (B - C)$. Then $B - C \neq \emptyset$ and $C - B \neq \emptyset$ otherwise $u \succ G$ or $v \succ G$. Note that $X_4 = \emptyset$ and X_1, X_2, X_3 are partition sets of $B \cap C$ by Claim 8. It then follows that $Y_2 = Z_2 = \emptyset$ and thus $X_1(\sim Y_1) = X_1$ and $X_2(\sim Z_1) = X_2$. If $X_1 = X_2 = X_3 = \emptyset$, then G is isomorphic to $G_{R_1} \in \mathcal{A}_8$. We now suppose that $X_3 \neq \emptyset$. If exactly one of $\{X_1, X_2\}$ is not empty, then G is isomorphic to a graph in $\{G_{R_1D_1}, G_{R_2}\} \subseteq \mathcal{A}_8$ and if both of X_1 and X_2 are not empty, then G is isomorphic to a graph in $\{G_{R_3}, G_{R_2D_1}, G_{R_1D_1D_2}\} \subseteq \mathcal{A}_8$. We next suppose that $X_3 \neq \emptyset$. If $X_1 = X_2 = \emptyset$, then G is isomorphic to $G_{R_1S_1} \in \mathcal{A}_8$ and if at least one of $\{X_1, X_2\}$ is not empty, then G is isomorphic to a graph in $\{G_{R_1D_1S_1}, G_{R_2S_1}, G_{R_3S_1}, G_{R_2D_1S_1}, G_{R_1D_1D_2S_1}\} \subseteq \mathcal{A}_8$. This proves Case 2.1.

Case 2.2: $A_1 = \emptyset$ but $A_2 \neq \emptyset$.

Then, by Claim 6, $G[A] = G[A_2]$ is a complete graph without a perfect matching and each vertex of $A_2 = A$ dominates $(B - C) \cup (C - B)$. It then follows by Claims 3 and 4 that $G[B - C]$ and $G[C - B]$ are complete graphs without a perfect matching. Further, $Y_2 = Z_2 = \emptyset$. By Claim 5, each vertex of $B - C$ ($C - B$) dominates $C - B(B - C)$. We now distinguish three subcases.

Case 2.2.1: $B \cap C = \emptyset$.

Then $B - C \neq \emptyset$ and $C - B \neq \emptyset$ otherwise v or u becomes a cutvertex, contradicting the fact that $\{u, v\}$ is a minimum cutset. It is easy to see that $G \cong G_{W_1P_1P_2} \in \mathcal{A}_9$.

Case 2.2.2: $X_4 = \emptyset$ but $X_1 \cup X_2 \cup X_3 \neq \emptyset$.

Then $A_2(\sim X_4) = \emptyset$. By Claims 12, 13 and 17, G is isomorphic to a graph in $\{G_{W_1D_1}, G_{W_1D_1P_1}, G_{W_1D_1P_1P_2}, G_{W_1S_1}, G_{W_1S_1P_1}, G_{W_1S_1P_1P_2}\} \subseteq \mathcal{A}_9$ if exactly one of X_1, X_2 and X_3

is not empty. If exactly two of X_1, X_2 and X_3 are not empty, then G is isomorphic to a graph in $\{G_{W_1D_1D_2}, G_{W_1D_1D_2P_1}, G_{W_1D_1D_2P_1P_2}, G_{W_1D_1S_1}, G_{W_1D_1S_1P_1}, G_{W_1D_1S_1P_1P_2}\} \subseteq \mathcal{A}_9$. Finally, if none of X_1, X_2 and X_3 is empty, then G is isomorphic to a graph in $\{G_{W_1D_1S_1D_2}, G_{W_1D_1S_1D_2P_1}, G_{W_1D_1S_1P_1P_2}\} \subseteq \mathcal{A}_9$.

Case 2.2.3: $X_4 \neq \emptyset$.

By Claim 19, $\overline{G}[(X_4(\sim A_2)) \cup (A_2(\sim X_4))] \cong \bigcup_{j=1}^d DS(m_j, n_j)$. Note that if $d = 1$, then either $m \geq 2$ and $n \geq 2$, or $A_2 - (A_2(\sim X_4)) \neq \emptyset$ or $(B - C) \cup (C - B) \neq \emptyset$ or $X_3 \neq \emptyset$ otherwise an end vertex of the double star becomes a cutvertex, contradicting the fact that $\{u, v\}$ is a minimum cutset of G . For this case, G is isomorphic to a graph in $\{G_{Q_1}, G_{w_4}, G_{W_2F_1, \dots, F_n}, G_{W_4F_1 \dots F_n}\} \subseteq \mathcal{A}_9$ where $F_i \in \{P_1, D_1, P_2, D_2, S_1\}$ and $F_i \neq F_j$ for $1 \leq i \neq j \leq n \leq 5$.

Case 2.3: $A_1 \neq \emptyset$ (A_2 may be empty).

By Claim 21, each component of $\overline{G}[A_1 \cup Y_2 \cup Z_2 \cup (X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup (X_4(\sim A_1))]$ is a triple star $TD(m, n, r)$ for some positive integers m, n and r where the middle center vertex is in A_1 , one of the end center vertices is in Y_2 and the another end center is in Z_2 while all the end vertices are in $X_1(\sim Y_2) \cup X_2(\sim Z_2) \cup X_4(\sim A_1)$. By Claims 3, 4, 5, 6, 12, 13 and 17, G is isomorphic to a graph in $\{G_{W_i}, G_{W_iF_1 \dots F_n}\} \subseteq \mathcal{A}_9$ where $i \in \{3, 5, 6, 7\}$, $F_i \in \{P_1, D_1, P_2, D_2, S\}$ and $F_i \neq F_j$ for $1 \leq i \neq j \leq n \leq 5$. This completes the proof of Case 2 and our theorem. \square

As a consequence of Lemma 8.1 and Theorems 8.2 - 8.4, we have:

Theorem 8.5. *Let G be a connected 3- i -critical graph and S a minimum cutset of size 2 where $c(G - S) = 2$. Then $G \in \bigcup_{i=1}^9 \mathcal{A}_i$.* \square

9 Independent domination critical graphs and k -factor-critical

Our first result in this section concerns an existence of a perfect matching in connected 3- i -critical graphs.

Lemma 9.1. *If G is a connected 3- i -critical graph of even order, then G has a perfect matching.*

Proof. Suppose that G does not contain a perfect matching. By Theorem 2.7, there is a set $S \subseteq V(G)$ such that $c_0(G - S) > |S|$. Since $|V(G)|$ is even, $c_0(G - S) \geq |S| + 2$ which contradicts either Lemma 4.1 or Theorem 6.1. Hence, G has a perfect matching. \square

Lemma 9.2. *Let G be a 3-connected 3-edge- i -critical graph of even order. If $\delta(G) \geq 4$, then G is bicritical.*

Proof. Suppose to the contrary that G is not bicritical. By Theorem 2.13, there exists $S \subseteq V(G)$ where $|S| \geq 2$ and $c_0(G - S) > |S| - 2$. Lemma 9.1 together with the fact that G is of even order, $c_0(G - S) = |S|$. This implies by Theorem 6.1 that $|S| \leq 3$. Then, by the hypothesis that G is 3-connected, $|S| = 3$ and thus $c_0(G - S) = c_0(G - S) = |S| = 3$. By Lemma 6.5, $\delta(G) = 3$, contradicting the minimum requirement of G . This completes the proof of our lemma. □

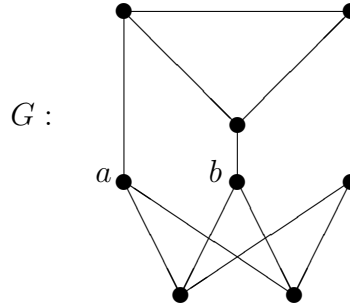


Figure 23: A 3-edge- i -critical graph of even order with $\delta(G) = 3$ which is not bicritical.

Note that the bound on the minimum degree stated in the hypotheses of Theorem 9.2 is best possible since there is a 3-edge- i -critical graph with minimum degree 3 having even order but is not bicritical. The graph G shown in Figure 23 is one of such graphs. Observe that $G - \{a, b\}$ has no perfect matching.

Lemma 9.3. *Let G be a connected 3- i -critical graph and S a cutset of G . Let x_1, x_2, y_1, y_2 be four distinct vertices of $G - S$ where $x_i y_i \notin E(G)$ for $1 \leq i \leq 2$ and these four vertices belong to at least two different components of $G - S$. Further, let $\{z_i\} = I_{x_i y_i} - \{x_i, y_i\}$, $1 \leq i \leq 2$. If either $c_0(G - S) \geq 4$ or $c_0(G - S) = 3$ but no components of $G - S$ is singleton, then $\{z_1, z_2\} \subseteq S$ and $z_1 \neq z_2$.*

Proof. By our hypothesis, it is easy to see that for $1 \leq i \leq 2$, z_i dominates at least two components. Then $z_i \in S$ and thus $\{z_1, z_2\} \subseteq S$ as required. Let C be a component of $G - S$ containing x_1 . We first suppose that $y_1 \in V(C) - \{x_1\}$. Then $x_2 \notin V(C)$ or $y_2 \notin V(C)$ and thus $z_1 x_2 \in E(G)$ or $z_1 y_2 \in E(G)$ since $I_{x_1 y_1} = \{x_1, z_1\}$ or $I_{x_1 y_1} = \{y_1, z_1\}$. Hence, $z_1 \neq z_2$ by Lemma 2.2(2).

We now assume that $y_1 \notin V(C)$. Since $\{x_2, y_2\} \subseteq V(G) - S$, it follows that $|N_G(x_2) \cap \{x_1, y_1\}| \leq 1$ and $|N_G(y_2) \cap \{x_1, y_1\}| \leq 1$. Thus $x_1 \in N_G(z_2)$ or $y_1 \in N_G(z_2)$ since $\{z_2\} = I_{x_2 y_2} - \{x_2, y_2\}$. Consequently, $z_1 \neq z_2$ because $\{x_1, y_1, z_1\}$ is independent by Lemma 2.2(2). This completes the proof of our lemma □

Lemma 9.4. *Let G be a connected 3- i -critical graph and S a cutset of G . Let x_1, x_2 and x_3 be three distinct vertices of $G - S$ and no any two vertices of $\{x_1, x_2, x_3\}$ belong to the same component of $G - S$. Further, let $\{z_1\} = I_{x_1x_2} - \{x_1, x_2\}$ and $\{z_2\} = I_{x_1x_3} - \{x_1, x_3\}$. If either $c_0(G - S) \geq 4$ or $c_0(G - S) = 3$ but no components of $G - S$ is singleton, then $\{z_1, z_2\} \subseteq S$ and $z_1 \neq z_2$.*

Proof. By similar arguments as in the proof of Lemma 9.3, $\{z_1, z_2\} \subseteq S$. By Lemma 2.2(2), $\{x_1, x_2, z_1\}$ and $\{x_1, x_3, z_2\}$ are independent. Since x_3 does not belong to the same component containing x_1 or x_2 , $z_1x_3 \in E(G)$ because $\{z_1\} = I_{x_1x_2} - \{x_1, x_2\}$. By Lemma 2.2(2), $z_1 \neq z_2$. This completes the proof of our lemma \square

Theorem 9.5. *For a positive integer k , let G be a t -connected 3- i -critical graph of even order where*

$$t = \begin{cases} 2k + 1, & \text{for } 1 \leq k \leq 36, \\ \lfloor \frac{9k}{4} - 7 \rfloor, & \text{for } k \geq 37. \end{cases}$$

Further, for $1 \leq k \leq 36$, $\delta(G) \geq 2k + 2$. Then G is $2k$ -factor-critical.

Proof. We shall prove by mathematical induction. Clearly, our result holds for $k = 1$ by Theorem 9.2. We now suppose that our result holds for $k - 1$ where $k \geq 2$. Suppose to the contrary that G satisfies our hypothesis but G is not $2k$ -factor-critical. By Theorem 2.13, there is a vertex cutset S where $|S| \geq 2k$ and $c_0(G - S) > |S| - 2k$. By our induction hypothesis and Theorem 2.13, $c_0(G - S) \leq |S| - 2(k - 1)$. Since G is of even order, $c_0(G - S) = |S| - 2k + 2 \geq 2$. Then $|S| \geq t \geq 2k + 1$ and thus $c_0(G - S) \geq 3$. Because $\delta(G) \geq 2k + 2$, if $c_0(G - S) = 3$ (that is $|S| = 2k + 1$), then no components of $G - S$ is singleton.

For $1 \leq i \leq |S| - 2k + 2$, let C_i be an odd component of $G - S$. Choose $x_i \in V(C_i)$. Then, for $1 \leq i \neq j \leq |S| - 2k + 2$, the only vertex of $I_{x_ix_j} - \{x_i, x_j\}$ must be in S by Lemma 9.3. Thus $|S| \geq \binom{|S| - 2k + 2}{2}$. It also follows by Lemmas 2.2(2), 9.3 and 9.4 that x_i is not adjacent to at least $|S| - 2k + 1$ vertices of S . Then $|N_S(x_i)| \leq |S| - (|S| - 2k + 1) = 2k - 1$. Since $\delta(G) \geq 2k + 2$, $2k + 2 \leq \deg_G(x_i) \leq (|V(C_i)| - 1) + |N_S(x_i)| \leq |V(C_i)| + 2k - 2$. Hence, for $1 \leq i \leq |S| - 2k + 2$, $|V(C_i)| \geq 5$ because C_i is odd.

Since $c_0(G - S) = |S| - 2k + 2$ and $|V(C_i)| \geq 5$ for $1 \leq i \leq |S| - 2k + 2$, it follows that there are $\binom{|S| - 2k + 2}{2} \binom{5}{1} \binom{5}{1}$ choices for choosing vertices u and v where $u \in V(C_i)$ and $v \in V(C_j)$ and $i \neq j$. By Lemmas 9.3 and 9.4, $I_{uv} - \{u, v\} \subseteq S$. Thus, in fact, $|S| \geq \binom{|S| - 2k + 2}{2} \binom{5}{1} \binom{5}{1} = \frac{25}{2}(|S| - 2k + 2)(|S| - 2k + 1)$. Hence, $25|S|^2 + (73 - 100k)|S| + 25(4k^2 - 6k + 2) \leq 0$. Consequently, $|S| \leq \frac{100k - 73 + \sqrt{(73 - 100k)^2 - 4(25)(25)(4k^2 - 6k + 2)}}{50} = \frac{100k - 73 + \sqrt{400k + 329}}{50}$.

We next show that $\frac{100k - 73 + \sqrt{400k + 329}}{50} \leq 2k$ for $1 \leq k \leq 36$ and $\frac{100k - 73 + \sqrt{400k + 329}}{50} \leq \frac{9k}{4} - 8$ for $k \geq 37$. We first assume that $1 \leq k \leq 36$. Suppose to the contrary that $\frac{100k - 73 + \sqrt{400k + 329}}{50} \geq 2k + 1$. Then $\sqrt{400k + 329} \geq 123$ and thus $k \geq 37$, a contradiction.

Hence, $\frac{100k-73+\sqrt{400k+329}}{50} \leq 2k$ for $1 \leq k \leq 36$ as required. We now assume that $k \geq 37$. Again, suppose to the contrary that $\frac{100k-73+\sqrt{400k+329}}{50} \geq \frac{9k}{4} - 7$. Then $\sqrt{400k+329} \geq \frac{50k-58}{4} > 0$ and thus $0 \geq 2500k^2 - 12200k - 1900 = (50k - 122)^2 - 16784 \geq (50(37) - 122)^2 - 16784 = 2,962,200$ since $k \geq 37$, a contradiction. Hence, $\frac{100k-73+\sqrt{400k+329}}{50} \leq \frac{9k}{4} - 8$ for $k \geq 37$ as required. Therefore, $|S| < t$, contradicting the connectivity of G . Hence, G is $2k$ -factor-critical as required. This completes the proof of our theorem. \square

We now turn our attention to 3-i-critical graphs of odd order.

Lemma 9.6. *Let G be a 2-connected 3-i-critical graph of odd order. If $\delta(G) \geq 3$, then G is factor-critical.*

Proof. Suppose to the contrary that G is not factor-critical. By Theorem 2.13, there exists $S \subseteq V(G)$ where $|S| \geq 1$ and $c_0(G - S) > |S| - 1$. Since G is of odd order, $c_0(G - S) \geq |S| + 1$. This implies by Theorem 6.1 that $1 \leq |S| \leq 2$. Then, by the hypothesis that G is 2-connected, $|S| = 2$ and thus $c(G - S) = c_0(G - S) = |S| = 3$. By Lemma 6.5, $\delta(G) = 2$, contradicting the minimum requirement of G . This completes the proof of our lemma. \square

By applying similar arguments as in the proof of Theorem 9.5, we have the following results.

Theorem 9.7. *For a positive integer k , let G be a t -connected 3-i-critical graph of odd order where*

$$t = \begin{cases} 2k, & \text{for } 1 \leq k \leq 36, \\ \lfloor \frac{9k}{4} - 8 \rfloor, & \text{for } k \geq 37. \end{cases}$$

Further, for $1 \leq k \leq 36$, $\delta(G) \geq 2k + 1$. Then G is $(2k - 1)$ -factor-critical.

\square

10 Some basic results for matching extension of the complementary prism of graphs

Lemma 10.1. *Let G be a k -extendable graph for some integer $k \geq 2$ and let $S \subseteq V(G)$ be a cutset of G . If $G[S]$ contains $t \leq k - 1$ independent edges, then $|S| \geq k + t + 1$.*

Proof. Let $S' = S - V(F)$ where F is a matching of size t in $G[S]$. By Observation 2.17, $G' = G - V(F)$ is $(k - t)$ -extendable. Observe that $k - t \geq 1$. By Theorem 2.8(b), G' is $(k - t + 1)$ -connected. Since S' is a cutset of G' , $|S'| \geq k - t + 1$ and thus $|S| \geq 2t + k - t + 1 = k + t + 1$ as required. This proves our lemma. \square

Let x be a real number, $\lfloor x \rfloor_e$ and $\lfloor x \rfloor_o$ denote the even integer and the odd integer less than or equal to x , respectively. Clearly, $\lfloor x \rfloor_e = 2\lfloor \frac{x}{2} \rfloor$ and $\lfloor x \rfloor_o = 2\lfloor (x - 1)/2 \rfloor + 1$. Note that for an even integer k , if x is an integer and $\lfloor x \rfloor_e = k$ then $x = k$ or $x = k + 1$.

Lemma 10.2. *Let G be a k -extendable non-bipartite graph for $k \geq 2$. Further, let $M \subseteq E(G)$ be a matching of size m and let $S = \emptyset$ or $S \subseteq V(G) - V(M)$ be an independent set such that $k - m - |S| = t \geq 0$ for some integer t . Then*

(a) *If $|S|$ is even, then $G - (V(M) \cup S)$ is t -extendable. Further $G - (V(M) \cup S)$ is $\lfloor t \rfloor_e$ -factor-critical. Consequently, there is a perfect matching in $G - (V(M) \cup S)$.*

(b) *If $|S|$ is odd and $t \geq 1$, then $G - (V(M) \cup S)$ is $\lfloor t \rfloor_o$ -factor-critical. Thus $G - (V(M) \cup S)$ is 1-factor-critical.*

(c) *If $|S|$ is odd, $t = 0$ and there is a vertex $v \in V(G) - (V(M) \cup S)$ such that $vs \in E(G)$ for some $s \in S$, then $G - (V(M) \cup S \cup \{v\})$ contains a perfect matching.*

Proof. We first suppose $m = k$. So $S = \emptyset$ and thus $G - (V(M) \cup S) = G - V(M)$ contains a perfect matching by Theorem 2.8(a) and it is 0-factor-critical as required. We now suppose that $m \leq k - 1$. By Corollary 2.20, $G - V(M)$ is $(k - m)$ -extendable non-bipartite. Since $k - m = |S| + t$, $G - V(M)$ is $(|S| + t)$ -extendable non-bipartite.

(a) $|S|$ is even. By Theorem 2.8(a), $G - V(M)$ is $(|S| + \lfloor t \rfloor_e)$ -extendable and thus it is $(|S| + \lfloor t \rfloor_e)$ -factor-critical by Theorem 2.15. Hence, by Observation 2.18, $G - (V(M) \cup S)$ is $\lfloor t \rfloor_e$ -factor-critical as required. It then follows by Theorem 2.8(a) that $G - (V(M) \cup S)$ contains a perfect matching. This proves (a).

(b) $|S|$ is odd and $t \geq 1$. By Theorem 2.8(a), $G - V(M)$ is $(|S| + \lfloor t \rfloor_o)$ -extendable and thus it is $(|S| + \lfloor t \rfloor_o)$ -factor-critical by Theorem 2.15. By Observation 2.18, $G - (V(M) \cup S)$ is $\lfloor t \rfloor_o$ -factor-critical. Since $t \geq 1$, $\lfloor t \rfloor_o \geq 1$. Further, by Theorem 2.14, $G - (V(M) \cup S)$ is 1-factor-critical as required. This proves (b).

(c) Let $M' = M \cup \{vs\}$ and $S' = S - \{s\}$. Hence, our result follows from (a). This completes the proof of our lemma. \square

Lemma 10.3. *Let G be a k -extendable graph for some integer k and let $S \subseteq V(G)$ be a cutset of G . If $G[S]$ contains t independent edges for $t \leq k$, then $c_o(G - S) \leq |S| - 2t$. Further, if $1 \leq t \leq k - 1$ and $c_o(G - S) = |S| - 2t$ then $G - S$ contains no even components.*

Proof. Let F be a matching of size t in $G[S]$. Since G is a k -extendable graph, $G - V(F)$ contains a perfect matching by Theorem 2.8(a). By Theorem 2.7, $c_o(G - S) = c_o((G - V(F)) - (S - V(F))) \leq |S - V(F)| = |S| - 2t$, as required. We now suppose that $1 \leq t \leq k - 1$ and $c_o(G - S) = |S| - 2t$. Let D be an even component of $G - S$. By Lemma 10.1 and the fact that $t \leq k - 1 < k + 1$, $V(F)$ is not a cutset of G . Then there is an edge $e = sd$ joining a vertex s in $S - V(F)$ and a vertex d in D . Since G is k -extendable and $F \cup \{e\}$ is a matching of size $t + 1 \leq k$, it follows that there is a perfect matching in $G' = G - (V(F) \cup \{s, d\})$. Let $S' = S - (V(F) \cup \{s\})$. Clearly, $c_o(G' - S') = c_o(G - S) + 1 = |S| - 2t + 1$. Since G' contains a perfect matching, by Theorem 2.7, $|S| - 2t + 1 \leq c_o(G' - S') \leq |S| - (|V(F)| + 1) = |S| - 2t - 1$, a contradiction. Hence, there is no even component in $G - S$. This proves our lemma. \square

Lemma 10.4. *Let G be a l -extendable graph and let M be a matching of size $l + t$ where $t \geq 1$. Then there is a maximum matching in $G - V(M)$ saturates all except at most $2t$ non-adjacent vertices in $G - V(M)$.*

Proof. Let $T \subseteq M$ where $|T| = t$. Thus $M - T$ is a matching of size l in G . So there is a perfect matching F in $G - V(M - T)$. Clearly, $|V(F) \cap V(T)| = 2t$. Let $F_1 = \{xy \in F \mid \{x, y\} \cap V(M) = \emptyset\}$ and $F_2 = \{xy \in F \mid x \in V(M) \text{ and } y \notin V(M)\}$. Further, let F'_2 be a maximum matching in $G[V(F_2) - V(M)]$. Then, $F_1 \cup F'_2$ is a matching in $G - V(M)$ saturates all except at most $2t$ non-adjacent vertices as required. \square

By similar arguments as in the proof of Lemma 10.4, the next lemma follows.

Lemma 10.5. *Let G be a k -factor-critical graph and let $T \subseteq V(G)$ where $|T| = k + t$. Then there is a maximum matching in $G - V(T)$ saturates all except at most t non-adjacent vertices.* \square

Lemma 10.6. *Let G be an 1-extendable graph of order $p \geq 6$ and let v be a vertex of degree 2 in G . Then there are perfect matchings M_1, M_2 in G such that v is a vertex of C_{2n} in $M_1 \Delta M_2$ where $n \geq 3$. Further, there is a vertex $x \in V(C_{2n})$ where C_{2n} is a subgraph of $M_1 \Delta M_2$ such that $vx \notin E(G)$ and $G - \{v, x\}$ contains a perfect matching.*

Proof. Let $\{u_1, u_2\} = N_G(v)$. We first suppose $N_G(u_1) \cap N_G(u_2) = \{v\}$. Let M_1 be a perfect matching in G containing vu_1 and M_2 a perfect matching in G containing vu_2 . Clearly, $\{vu_1, u_2u_3\} \subseteq M_1$ and $\{vu_2, u_1u_4\} \subseteq M_2$ for some $u_3, u_4 \in V(G)$. Since $\{v\} = N_G(u_1) \cap N_G(u_2)$, $u_3 \neq u_4$. Hence, $u_3u_2vu_1u_4$ is a path of length 4 containing v . It must be contained in an even cycle of order at least 6 in $M_1 \Delta M_2$ as required.

So we now suppose that $N_G(u_1) \cap N_G(u_2) \neq \{v\}$. Then there is a vertex $v \neq u_3 \in N_G(u_1) \cap N_G(u_2)$. Since G is 2-connected by Theorem 2.8(b) and G is of order at least 6, it follows that u_3 is not a cut vertex. Then there is a vertex $u_4 \in N_G(u_1) \cup N_G(u_2)$ where $u_4 \neq u_3$. Without loss of generality, suppose $u_4 \in N_G(u_1)$. Let M_1 be a perfect matching in G containing u_1u_4 and M_2 be a perfect matching in G containing u_2u_3 . It is easy to see that $\{u_1u_4, vu_2\} \subseteq M_1$ and $\{u_2u_3, vu_1\} \subseteq M_2$. Hence, $u_4u_1vu_2u_3$ is a path of length 4 containing v . It must be contained in an even cycle of order at least 6 in $M_1 \Delta M_2$ as required. Further, let $x \in V(C_{2n})$ such that the distance between v and x along the cycle C_{2n} is 3. Clearly, $xv \notin E(G)$ and it is easy to see that $G - \{v, x\}$ contains a perfect matching. This completes the proof of our lemma. \square

11 The extendability of complementary prism of 2-regular graphs

For a matching M , we simply denote the set of end vertices of edges in M by $V(M)$.

Lemma 11.1. *Let $G\bar{G}$ be a k -extendable graph for some positive integer k . Suppose M is a matching in $G\bar{G}$ and $S \subseteq V(\bar{G})$ where $V(M) \cap S = \emptyset$ and $|M| + |S| \leq k$. Then*

1. *If $|S|$ is even, then there is a perfect matching in $G\bar{G} - (V(M) \cup S)$.*

2. If $|S|$ is odd, then there is a vertex $\bar{y} \in V(\overline{G}) - (V(M) \cup S)$ such that $G\overline{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching.

Proof. Observe that $G\overline{G}$ is non-bipartite.

(1) It is easy to see that if $S = \emptyset$, then, by Theorem 2.8(1), $G\overline{G} - (V(M) \cup S) = G\overline{G} - V(M)$ contains a perfect matching since $G\overline{G}$ is k -extendable. So we may now assume that $S \neq \emptyset$. Then $2 \leq |S| \leq |M| + |S| \leq k$. Thus $|M| \leq k - 2$. By Corollary 2.20 and the fact that $G\overline{G}$ is non-bipartite, $G\overline{G} - V(M)$ is $(k - |M|)$ -extendable non-bipartite. Since $|S| \leq k - |M|$, $G\overline{G} - V(M)$ is $|S|$ -extendable non-bipartite by Theorem 2.8(1). Hence, $G\overline{G} - V(M)$ is $|S|$ -factor-critical by Theorem 2.15 and the fact that $|S|$ is even. Therefore, $G\overline{G} - (V(M) \cup S)$ contains a perfect matching. This proves (1).

(2) Since $|S|$ is odd, $|S| \geq 1$ and thus $|M| \leq k - |S| \leq k - 1$. We first show that there are a vertex $\bar{u} \in S$ and a vertex $\bar{v} \in V(\overline{G}) - (V(M) \cup S)$ such that $\bar{u}\bar{v} \in E(\overline{G})$. Suppose this is not the case. Let $\bar{u}_0 \in S$. Then $N_{G\overline{G}}[\bar{u}_0] \subseteq S \cup V(M) \cup \{u_0\}$ where u_0 is the only vertex in G which is adjacent to \bar{u}_0 . Put $S' = (S - \{\bar{u}_0\}) \cup \{u_0\}$. Clearly, \bar{u}_0 becomes an isolated vertex in $G\overline{G} - (V(M) \cup S')$ and $|V(M) \cup S'| = 2|M| + |S'| = 2|M| + |S| \leq k + |M|$. So $V(M) \cup S'$ is a cutset of $G\overline{G}$. But this contradicts Lemma 10.1 since $G\overline{G}[V(M) \cup S']$ contains a matching of size at least $|M|$ and at most $|M| + \frac{1}{2}|S'| < |M| + |S| \leq k$ and $|V(M) \cup S'| \leq k + |M|$. Hence, there are a vertex $\bar{u} \in S$ and a vertex $\bar{v} \in V(\overline{G}) - (V(M) \cup S)$ such that $\bar{u}\bar{v} \in E(\overline{G})$ as required.

Now let $\bar{x} \in S$ and a vertex $\bar{y} \in V(\overline{G}) - (V(M) \cup S)$ such that $\bar{x}\bar{y} \in E(\overline{G})$. Consider $M \cup \{\bar{x}\bar{y}\}$. Clearly, $|M \cup \{\bar{x}\bar{y}\}| \leq k$. We first suppose that $|M \cup \{\bar{x}\bar{y}\}| = k$. Because $|M| \leq k - |S|$, $|S| = 1$ and thus $S = \{\bar{x}\}$. Since $G\overline{G}$ is k -extendable, $G\overline{G} - (V(M) \cup \{\bar{x}\bar{y}\}) = G\overline{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching as required. So we now suppose that $|M \cup \{\bar{x}\bar{y}\}| \leq k - 1$. By Corollary 2.20 and the fact that $G\overline{G}$ is non-bipartite, $G\overline{G} - (V(M) \cup \{\bar{x}\bar{y}\})$ is $(k - (|M| + 1))$ -extendable non-bipartite. Since $k - |M| - 1 \geq |S| - 1$ and $|S| - 1$ is even, it then follows by Theorems 2.8(1) and 2.15 that $G\overline{G} - (V(M) \cup \{\bar{x}\bar{y}\})$ is $(|S| - 1)$ -factor-critical. Hence, $G\overline{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching as required. This proves (2) and completes the proof of our lemma. \square

Theorem 11.2. For positive integers i and l where $1 \leq i \leq l$, let G_1, \dots, G_l be components of G . If $G_i\overline{G}_i$ is k -extendable of order $p_i \geq 2k + 2$ for some positive integer k , then $G\overline{G}$ is k -extendable. \square

Proof. Clearly, our result holds for $l = 1$. So we now suppose $l \geq 2$. For simplicity, the induced subgraphs $G\overline{G}[V(G_i)]$, $G\overline{G}[V(\overline{G}_i)]$ and $G\overline{G}[V(G_i\overline{G}_i)]$ are denoted by G_i , \overline{G}_i and $G_i\overline{G}_i$, respectively.

Let M be a matching of size k in $G\overline{G}$. For $1 \leq i \leq l$, let $M_i = M \cap E(G_i\overline{G}_i)$ and $S_i = \{x \in V(G_i\overline{G}_i) | xy \in M \text{ and } y \notin V(G_i\overline{G}_i)\}$. Observe that $S_i \subseteq V(\overline{G}_i)$ and $E(G\overline{G}[\bigcup_{i=1}^l S_i]) = M - \bigcup_{i=1}^l M_i$. We first suppose that $|S_i|$ is even for $1 \leq i \leq l$. Then, by Lemma 11.1(1), there is a perfect matching F_i in $G_i\overline{G}_i - (V(M_i) \cup S_i)$ for $1 \leq i \leq l$. Hence, $(\bigcup_{i=1}^l F_i) \cup M$ is a perfect matching in $G\overline{G}$ containing M as required.

We now suppose that $|S_i|$ is odd for some i . Let l_o be the number of components G_i of G in which $|S_i|$ is odd. We may now renumber the components of G in such a way that for the first l_o components of G , $|S_i|$ is odd for $1 \leq i \leq l_o$ and for the last $l - l_o$ components of G , $|S_i|$ is even. Since $\sum_{i=1}^{l_o} |S_i| = 2(|M - \bigcup_{i=1}^l M_i|) - \sum_{i>l_o} |S_i|$ is even, l_o is even. By Lemma 11.1(2), there is $\bar{y}_i \in V(\bar{G}_i) - (V(M_i) \cup S_i)$ such that $G_i \bar{G}_i - (V(M_i) \cup S_i \cup \{\bar{y}_i\})$ contains a perfect matching, say F'_i , for $1 \leq i \leq l_o$. Clearly, $G \bar{G}[\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_o}\}]$ is a complete graph of even order. So there is a perfect matching in $G \bar{G}[\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_o}\}]$, say F' . By Lemma 11.1(1), if $l_o < l$, then there is a perfect matching F'_i in $G_i \bar{G}_i - (V(M_i) \cup S_i)$ for $l_o + 1 \leq i \leq l$. Therefore, $\bigcup_{i=1}^l F'_i \cup F' \cup M$ is a perfect matching in $G \bar{G}$ containing M as required. Hence, $G \bar{G}$ is k -extendable. This completes the proof of our theorem. \square

Our next result follows immediately from Theorems 11.2 and 2.8(1).

Corollary 11.3. *For positive integers i and l where $1 \leq i \leq l$, let G_1, \dots, G_l be components of G . If $G_i \bar{G}_i$ is k_i -extendable of order $p_i \geq 2k_i + 2$ for some positive integer k_i , then $G \bar{G}$ is k_0 -extendable where $k_0 = \min\{k_1, k_2, \dots, k_l\}$. \square*

For simplicity, we now assume that G is a connected 2-regular graph and put $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_{i+1} | 1 \leq i \leq n\}$ where the subscript is read modulo n . Observe that $G \cong C_n$, $\bar{G} \cong K_n - \{v_i v_{i+1} | 1 \leq i \leq n\}$ and thus \bar{G} is $(n - 3)$ -regular of order n . In what follows, the symbols G , \bar{G} , $V(G)$, $E(G)$, n and v_k for $1 \leq k \leq n$ are referred to these set up. Further, all subscripts are read modulo n . Our first lemma follows immediately from the fact that if x is a vertex of C_p where $p \geq 3$, then $C_p - x$ is a path of order $p - 1$.

Lemma 11.4. *1. If n is even and e is an edge of G , then there is a perfect matching in G containing the edge e .*

2. If n is odd, then, for each $1 \leq k \leq n$, $G - v_k$ contains a maximum matching of size $\frac{n-1}{2}$. In fact, a maximum matching of size $\frac{n-1}{2}$ is $\{v_{k+1}v_{k+2}, v_{k+3}v_{k+4}, \dots, v_{k+n-2}v_{k+n-1}\}$ which is also a perfect matching in $G - v_k$. \square

Lemma 11.5. *For an integer $n \geq 5$, \bar{G} is $(n - 3)$ -connected.*

Proof. Let S be a minimum cutset of \bar{G} . For a positive integer $k \geq 2$, let H_1, \dots, H_k be components of $\bar{G} - S$. Since \bar{G} is $(n - 3)$ -regular, $|V(H_i)| \geq n - 2 - |S|$. Then $n = |V(\bar{G})| = \sum_{i=1}^k |V(H_i)| + |S| \geq 2(n - 2 - |S|) + |S| = 2n - 4 - |S|$ and thus $|S| \geq n - 4$. Suppose $|S| = n - 4$. It is easy to see that $|V(H_i)| = 2$ and $k = 2$. Thus $n \geq 7$ since \bar{G} is $(n - 3)$ -regular. It follows that $\bar{G} \cong 2K_2 \vee H$ where H is $(n - 7)$ -regular of order $n - 4$. Thus G contains C_4 as an induced subgraph. But this contradicts the fact that $G \cong C_n$ where $n \geq 5$. Hence, $|S| \geq n - 3$ and then \bar{G} is $(n - 3)$ -connected. This completes the proof of our lemma. \square

Lemma 11.6. *For an integer $n \geq 6$, if n is even, then \overline{G} is $(\frac{n-4}{2})$ -extendable and if n is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is $(\frac{n-5}{2})$ -extendable.*

Proof. Observe that \overline{G} is $K_{1,3}$ -free otherwise G contains C_3 as an induced subgraph which contradicts the fact that $G \cong C_n$ and $n \geq 6$. By Theorem 2.10 and Lemma 11.5, \overline{G} is $(\frac{n-4}{2})$ -extendable if n is even. We now suppose that n is odd. Then $n \geq 7$ and $\overline{G} - v_k$ is $(n-4)$ -connected by Lemma 11.5. Hence, by Theorem 2.10, $\overline{G} - v_k$ is $(\frac{n-5}{2})$ -extendable. This proves our lemma. \square

As a consequence of Theorem 2.8(1) and Lemma 11.6, we have the following corollaries.

Corollary 11.7. *For an integer n , if $n \geq 8$ is even, then \overline{G} is 2-extendable and if $n \geq 9$ is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is 2-extendable.* \square

Corollary 11.8. *For an integer n , if $n \geq 6$ is even, then \overline{G} is 1-extendable and if $n \geq 7$ is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is 1-extendable.* \square

Corollary 11.9. *For an integer $n \geq 6$, let v_i, v_j, v_k be three distinct vertices of \overline{G} where $1 \leq i, j, k \leq n$, then $\overline{G} - \{v_i, v_j\}$ has a perfect matching if n is even and $\overline{G} - \{v_i, v_j, v_k\}$ has a perfect matching if n is odd.*

Proof. Our result follows from Theorems 2.8(1) and 2.15 together with Lemma 11.6 if $n \geq 8$. For $6 \leq n \leq 7$, our result follows from Theorem 2.7, Lemma 11.5 and the fact that \overline{G} is $K_{1,3}$ -free. \square

For simplicity, put $V(\overline{G}) = \{u_1, \dots, u_n\}$ where $u_i \in V(\overline{G})$ corresponds to $v_i \in V(G)$. Then $V(G\overline{G}) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v_i u_i | 1 \leq i \leq n\}$.

Theorem 11.10. *Let G be a connected 2-regular graph of order $n \geq 6$. Then $G\overline{G}$ is 2-extendable.*

Proof. Let $T = \{e_1, e_2\}$ be a matching of size 2 in $G\overline{G}$. It is easy to see that if $\{e_1, e_2\} \subseteq \{v_i u_i | 1 \leq i \leq n\}$, then $\{v_i u_i | 1 \leq i \leq n\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . So we may now assume without loss of generality that $e_1 \notin \{v_i u_i | 1 \leq i \leq n\}$. For simplicity, the set of end vertices of the edge e_i is denoted by $V(e_i)$ for $1 \leq i \leq 2$. To show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 , we distinguish five cases according to the edges e_1 and e_2 .

Case 1: $\{e_1, e_2\} \subseteq E(\overline{G})$.

By Corollary 11.7 and the fact that $G \cong C_n$, it is easy to see that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 if $n \geq 8$ is even. For $n = 6$, it is not

difficult to show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 as well.

So we now suppose that $n \geq 9$ is odd. Choose a vertex $u_j \in V(\overline{G}) - (V(e_1) \cup V(e_2))$. Then, by Corollary 11.7, there is a perfect matching \overline{M}_1 , in $\overline{G} - u_j$, containing the edges e_1 and e_2 . By Lemma 11.4(2), there is a perfect matching M_1 in $G - v_j$. Hence, $M_1 \cup \overline{M}_1 \cup \{v_j u_j\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now consider $n = 7$. Observe that $V(\overline{G}) - (V(e_1) \cup V(e_2))$ contains an edge, say e_3 , otherwise G contains C_3 as an induced subgraph. Put $\{u_{j'}\} = V(\overline{G}) - \bigcup_{i=1}^3 V(e_i)$. By Lemma 11.4(2), there is a perfect matching M_2 in $G - v_{j'}$. Thus $M_2 \cup \{e_1, e_2, e_3, v_{j'} u_{j'}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 1.

Case 2: $e_1 \in E(G), e_2 \in E(\overline{G})$.

Suppose $e_1 = v_j v_{j+1}$ and $e_2 = u_k u_{k'}$ where $1 \leq j, k, k' \leq n$ and $k \neq k'$. By Lemma 11.4(1) and Corollary 11.8, it is easy to see that there is a perfect matching containing the edges e_1 and e_2 if n is even. So we now suppose that n is odd.

We first suppose that $j + 2 \notin \{k, k'\}$. Then a maximum matching M_1 , in $G - v_{j+2}$, containing the edge $e_1 = v_j v_{j+1}$ is a matching of size $\frac{n-1}{2}$. Thus M_1 is a perfect matching in $G - v_{j+2}$ by Lemma 11.4(2). By Corollary 11.8, $\overline{G} - u_{j+2}$ has a perfect matching \overline{M}_1 containing the edge e_2 . Then $M_1 \cup \overline{M}_1 \cup \{v_{j+2} u_{j+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

By similar arguments, if $j - 1 \notin \{k, k'\}$, then there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . We may now assume that $\{j - 1, j + 2\} = \{k, k'\}$. Then $e_2 = u_k u_{k'} = u_{j-1} u_{j+2}$. Now consider $G - v_{j+4}$. Since $n \geq 7$, $j + 4 \notin \{j - 1, j + 2\}$. Then a maximum matching M_2 , in $G - v_{j+4}$, of size $\frac{n-1}{2}$ must contain the edge $e_1 = v_j v_{j+1}$. By Lemma 11.4(2), M_2 is a perfect matching in $G - v_{j+4}$. By Corollary 11.8, $\overline{G} - u_{j+4}$ has a perfect matching \overline{M}_2 containing the edge e_2 . Then $M_2 \cup \overline{M}_2 \cup \{v_{j+4} u_{j+4}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

Case 3: $e_1 \in E(G), e_2 \in \{v_i u_i | 1 \leq i \leq n\}$.

Let $e_2 = v_k u_k$ for some $1 \leq k \leq n$. Consider $G - v_k$. Observe that $G - v_k$ is a path of order $n - 1$. Let M_1 and M_2 be matchings in $G - v_k$ where $E(G - v_k) = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. We may assume that $|M_1| \geq |M_2|$. We first suppose that n is odd. Then $|M_1| = \frac{n-1}{2}$ and $|M_2| = \frac{n-3}{2}$. Further, v_{k-1} and v_{k+1} are M_2 -unsaturated. By Lemma 11.4(2), M_1 is a perfect matching in $G - v_k$. If $e_1 \in M_1$, then $M_1 \cup \overline{M}_1 \cup \{v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_1 is a perfect matching, in $\overline{G} - u_k$. Note that \overline{M}_1 exists by Corollary 11.8. We now suppose that $e_1 \in M_2$. By Corollary 11.9, there is a perfect matching \overline{M}_2 , in $\overline{G} - \{u_{k-1}, u_k, u_{k+1}\}$. Hence, $M_2 \cup \overline{M}_2 \cup \{v_{k-1} u_{k-1}, v_k u_k, v_{k+1} u_{k+1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now suppose that n is even. Then $|M_1| = |M_2| = \frac{n-2}{2}$. Then either v_{k-1} or v_{k+1} is M' -unsaturated where $M' \in \{M_1, M_2\}$. Suppose without loss of generality that $e_1 \in M_1$ and v_{k-1} is M_1 -unsaturated. By Corollary 11.9, there is a perfect matching \overline{M}_3 , in $\overline{G} - \{u_{k-1}, u_k\}$. Hence, $M_1 \cup \overline{M}_3 \cup \{v_{k-1} u_{k-1}, v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 3.

Case 4: $e_1 \in E(\overline{G}), e_2 \in \{v_i u_i | 1 \leq i \leq n\}$.

Let $e_1 = u_j u_{j'}$ and $e_2 = v_k u_k$ for some $1 \leq j, j', k \leq n$. Clearly, $k \notin \{j, j'\}$. We first suppose that n is odd. By Lemma 11.4(2), $G - v_k$ contains M_1 as a perfect matching. By Corollary 11.8, $\overline{G} - u_k$ has a perfect matching containing the edge e_1 , say \overline{M}_1 . Thus $M_1 \cup \overline{M}_1 \cup \{v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now suppose that $n \geq 8$ is even. Let M_2 and M_3 be perfect matchings in G containing the $v_k v_{k+1}$ and $v_{k-1} v_k$, respectively. Observe that if $S \subseteq V(\overline{G})$ with $|S| = 4$, then $\overline{G}[S]$ contains a matching of size two since \overline{G} is $(n-3)$ -regular and G does not contain C_3 as an induced subgraph. We first suppose that $k+1 \notin \{j, j'\}$. By Corollary 11.7, $\overline{G} - \{u_j, u_{j'}, u_k, u_{k+1}\}$ contains \overline{M}_2 as a perfect matching. Then $(M_2 - \{v_k v_{k+1}\}) \cup \overline{M}_2 \cup \{u_j u_{j'}, v_k u_k, v_{k+1} u_{k+1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . By similar arguments, if $k-1 \notin \{j, j'\}$, then $(M_3 - \{v_{k-1} v_k\}) \cup \overline{M}_3 \cup \{u_j u_{j'}, v_{k-1} u_{k-1}, v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_3 is a perfect matching in $\overline{G} - \{u_j, u_{j'}, u_{k-1}, u_k\}$. Finally, we suppose that $\{j, j'\} = \{k-1, k+1\}$. By Corollary 11.7 and the observation that $\overline{G}[S]$ contains a matching of size two if $S \subseteq V(\overline{G})$ with $|S| = 4$, $\overline{G} - \{u_{k-1}, u_k, u_{k+1}, u_{k+3}\}$ contains \overline{M}_4 as a perfect matching. Then $(M_2 - \{v_k v_{k+1}, v_{k+2} v_{k+3}\}) \cup \overline{M}_4 \cup \{u_j u_{j'}, v_k u_k, v_{k+1} v_{k+2}, v_{k+3} u_{k+3}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . For $n = 6$, it is routine to show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 4.

Case 5: $\{e_1, e_2\} \subseteq E(G)$.

Let M be a maximum matching in G containing the edge e_1 . Clearly, M is a perfect matching if n is even and if n is odd, then there is exactly one M -unsaturated vertex, say v_j , for some $1 \leq j \leq n$. We first suppose that $e_2 \in M$. Then there is a perfect matching F containing the edges e_1 and e_2 where $F = M \cup \overline{M}$ if n is even and $F = M \cup \overline{M}_1 \cup \{v_j u_j\}$ if n is odd where \overline{M} and \overline{M}_1 are perfect matchings in \overline{G} and $\overline{G} - u_j$, respectively. Such \overline{M} and \overline{M}_1 exist by Lemma 11.6.

We now suppose that $e_2 \notin M$. Put $e_2 = v_k v_{k+1}$ where $1 \leq k \leq n$. We first assume that n is even. Then $\{v_{k-1} v_k, v_{k+1} v_{k+2}\} \subseteq M - \{e_1\}$ since $\{e_1, e_2\}$ is a matching, M is a perfect matching and $G \cong C_n$. Clearly, $\{v_{k-1}, v_{k+2}\} \cap V(e_1) = \emptyset$. By Corollary 11.9, there exists a perfect matching in \overline{M}_2 in $\overline{G} - \{u_{k-1}, u_{k+2}\}$. Then $(M - \{v_{k-1} v_k, v_{k+1} v_{k+2}\}) \cup \overline{M}_2 \cup \{v_k v_{k+1}, v_{k-1} u_{k-1}, v_{k+2} u_{k+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now suppose that n is odd. Recall that v_j is the only M -unsaturated of G . If $\{v_k, v_{k+1}\} \cap \{v_j\} = \{v_k\}$, then $\{v_{k+1} v_{k+2}\} \subseteq M - \{e_1\}$ and thus $(M - \{v_{k+1} v_{k+2}\}) \cup \overline{M}_3 \cup \{v_k v_{k+1}, v_{k+2} u_{k+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_3 is a perfect matching in $\overline{G} - u_{k+2}$. Note that \overline{M}_3 exists by Corollary 11.8. Similarly, if $\{v_k, v_{k+1}\} \cap \{v_j\} = \{v_{k+1}\}$, then $M - \{v_{k-1} v_k\} \cup \overline{M}_4 \cup \{v_k v_{k+1}, v_{k-1} u_{k-1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_4 is a perfect matching in $\overline{G} - u_{k-1}$. We now consider the case that $\{v_k, v_{k+1}\} \cap \{v_j\} = \emptyset$. Observe that $j \notin \{k-1, k+2\}$ since $e_2 \notin M$ and v_j is M -unsaturated. Then $\{v_{k-1} v_k, v_{k+1} v_{k+2}\} \subseteq M - \{e_1\}$. By Corollary 11.9, there exists a perfect matching \overline{M}_5 in $\overline{G} - \{u_j, u_{k-1}, u_{k+2}\}$. Then $(M - \{v_{k-1} v_k, v_{k+1} v_{k+2}\})$

$\cup \overline{M}_5 \cup \{v_k v_{k+1}, v_{k-1} u_{k-1}, v_{k+2} u_{k+2}, v_j u_j\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 5 and completes the proof of our theorem \square

Note that the bound on n in Theorem 11.10 is sharp since the graph $C_5\overline{C}_5$ in Figure 23 is not 2-extendable because there is no perfect matching containing the edges $v_1 u_1$ and $v_3 v_4$. Further, the condition of H -free and the extendability of $G\overline{G}$ stated in Theorem 11.12 are all best possible. For positive integers $n \geq 8$ and $3 \leq i \leq 5$, let $H_i = C_i \cup C_{n-i}$. Then the graph $H_i\overline{H}_i$, shown in Figure 24, is not 2-extendable since there is no perfect matching containing the edge $x_1 x_2$ and $y_1 y_2$. Note that “a double line” in our diagram denotes the join between corresponding graphs. Hence, the hypothesis H -free where $H \in \{C_3, C_4, C_5\}$ in Theorem 11.12 cannot be dropped. Finally, the extendability of $G\overline{G}$ in Theorem 11.12 is best possible by Theorem 2.8(2) and the fact that the minimum degree of $G\overline{G}$ is 3.

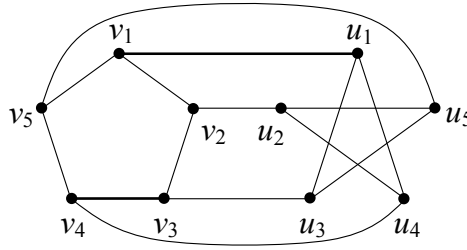


Figure 23: The graph $C_5\overline{C}_5$

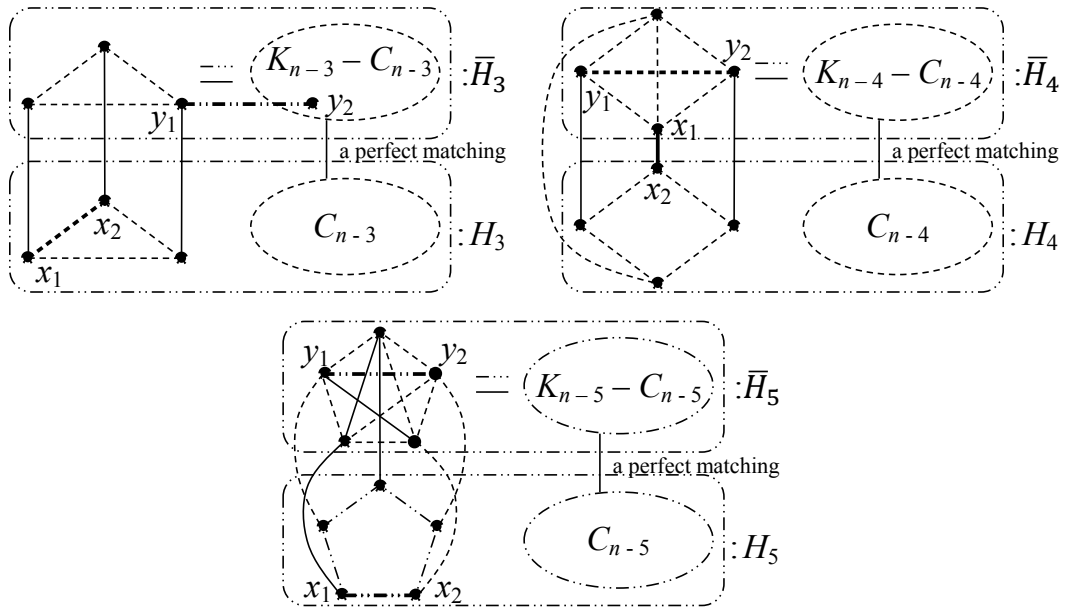


Figure 24: The graph $H_i\overline{H}_i$, $i \in \{3, 4, 5\}$

Corollary 11.11. *Let G be a connected 2-regular graph of order $n \geq 4$. Then $G\overline{G}$ is 1-extendable.*

Proof. Our result follows from Theorems 2.8(1) and 11.10 if $n \geq 6$. It is not difficult to show that the result is true for $4 \leq n \leq 5$. \square

It is easy to see that the next result follows from Theorems 11.2 and 11.10.

Theorem 11.12. *Let G be a 2-regular H -free graph where $H \in \{C_3, C_4, C_5\}$. Then $G\overline{G}$ is 2-extendable.* \square

12 The extendability of complementary prism of r -regular graphs for $r \geq 3$

We begin this section by establishing some lemmas concerning complementary prism of graphs and of regular graphs. These results are essential for establishing Theorem 12.10, a main result of this section. To simplify our discussion of complementary prisms, G and \overline{G} are referred to subgraph copies of G and \overline{G} , respectively, in $G\overline{G}$. For a vertex v of G , there is exactly one vertex of \overline{G} which is adjacent to v in $G\overline{G}$. This vertex is denoted by \overline{v} . That is $\{\overline{v}\} = N_{\overline{G}}(v)$. Conversely, v is the only vertex of G which is adjacent to \overline{v} . Similarly, for $\phi \neq X = \{x_1, x_2, \dots, x_k\} \subseteq V(G)$, $\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_k\} \subseteq V(\overline{G})$ is denoted by \overline{X} and vice versa. Clearly, $|X| = |\overline{X}|$.

Lemma 12.1. *Let G be a graph. Then $G\overline{G}$ is even and connected.*

Proof. Clearly, $G\overline{G}$ is even. Let $u, v \in V(G\overline{G})$. It is easy to see that if $u, v \in V(G)(V(\overline{G}))$, then either $uv \in E(G)$ or $u\overline{u}\overline{v}v$ is a $u - v$ path. We may now assume that $u \in V(G)$ and $v \in V(\overline{G})$. Clearly, $uv \in E(G\overline{G})$ if $v = \overline{u}$. So suppose that $v = \overline{w}$ for some $w \in V(G) - \{u\}$. Then either $u\overline{w}\overline{w}$ or $uw\overline{w}$ is a $u - v$ path. This proves that $G\overline{G}$ is connected and completes the proof of our lemma. \square

For a graph G , it is easy to see that $G\overline{G}$ has a perfect matching. It then follows by Theorem 2.7 that for a cutset $S \subseteq V(G\overline{G})$, $c_o(G\overline{G} - S) \leq |S|$. The next lemma provides a relationship of a cutset and the number of odd components in a complementary prism.

Lemma 12.2. *Let $G\overline{G}$ be a complementary prism and let $S = A \cup \overline{B}$ be a cutset of $G\overline{G}$, where $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Then*

$$a) c_o(G\overline{G} - S) = |S| - 2t, \text{ for some } t \geq 0.$$

$$b) c_o(G\overline{G} - S) = |A| + |B| - 2t \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |A| + |B| - 2|A \cap B|.$$

Consequently, $|A \cap B| \leq t$.

c) If $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |A| + |B| - 2|A \cap B|$, then each component of $G[B - A]$ and $\overline{G}[\overline{A} - \overline{B}]$ is singleton and hence $G[A - B]$ is a clique.

Proof. a) Since $G\bar{G}$ contains a perfect matching and is of even order, it follows by Theorem 2.7 that there is a non-negative integer t such that $c_o(G\bar{G} - S) = |S| - 2t$, for any cutset $S \subseteq V(G\bar{G})$.

We first observe that $|B - A| + |\bar{A} - \bar{B}| = |B - A| + |A - B| = |A| + |B| - 2|A \cap B|$ since $|A| = |A - B| + |A \cap B|$ and $|B| = |B - A| + |A \cap B|$.

b) Let $C = V(G) - (A \cup B)$. By Lemma 12.1, if $C \neq \phi$, then $G\bar{G}[C \cup \bar{C}]$ is even and connected. Thus $c_o(G\bar{G} - S) \leq c_o(G\bar{G} - (S \cup C \cup \bar{C})) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) \leq |B - A| + |\bar{A} - \bar{B}| = |A| + |B| - 2|A \cap B|$ as required.

c) follows by the fact that $|B - A| + |A - B| = |A| + |B| - 2|A \cap B|$. \square

For an induced subgraph H of G , Com_H denotes the set of all components in H . If $X \subseteq V(G)$, then we use Com_X for $Com_{G[X]}$. For a cutset S of $G\bar{G}$, put $A = S \cap V(G)$, $\bar{B} = S \cap V(\bar{G})$ and $C = V(G) - (A \cup B)$. Thus $S = A \cup \bar{B}$. Further, let $T_{B-A} = \{F | F \text{ is an odd component of } G[B - A] \text{ and } N_G(u) - V(F) \subseteq A \text{ for all } u \in V(F)\}$. $T_{\bar{A}-\bar{B}} = \{F | F \text{ is an odd component of } \bar{G}[\bar{A} - \bar{B}] \text{ and } N_{\bar{G}}(\bar{u}) - V(F) \subseteq \bar{B} \text{ for all } \bar{u} \in V(F)\}$. Finally, let $L = L_G \cup L_{\bar{G}}$, where $L_G = \{F | F \text{ is an odd component in } G[B - A] \text{ and } N_{G\bar{G}}(V(F)) \cap C \neq \phi\}$ and $L_{\bar{G}} = \{F | F \text{ is an odd component in } \bar{G}[\bar{A} - \bar{B}] \text{ and } N_{G\bar{G}}(V(F)) \cap \bar{C} \neq \phi\}$. Note that if $C = \phi$, then $L = \phi$. Clearly, $T_{B-A} \cap L_G = \phi$ and $T_{\bar{A}-\bar{B}} \cap L_{\bar{G}} = \phi$. It is easy to see that, if G is connected and $G[B - A]$ contains only odd components, then $Com_{B-A} = T_{B-A} \cup L_G$. Similarly, if \bar{G} is connected and $\bar{G}[\bar{A} - \bar{B}]$ contains only odd components, then $Com_{\bar{A}-\bar{B}} = T_{\bar{A}-\bar{B}} \cup L_{\bar{G}}$. In what follows, the symbols Com_H , S , A , \bar{B} , C , T_{B-A} , $T_{\bar{A}-\bar{B}}$, L , L_G and $L_{\bar{G}}$ are referred to these set up.

The next lemma follows from our set up.

Lemma 12.3. *Let G be an r -regular connected graph of order $p \geq 2r + 1$ and $G\bar{G}$ a complementary prism. If $|A| < r$, then T_{B-A} contains no singleton components. Similarly, if $|\bar{B}| < p - r - 1$, then $T_{\bar{A}-\bar{B}}$ contains no singleton components.* \square

Lemma 12.4. *For $r \geq 3$, let G be a connected r -regular graph of order $p \geq 2r + 1$. Let $A, B, T_{B-A}, T_{\bar{A}-\bar{B}}$ be defined as above. Then*

a) *If $G[A] = K_r$, then each component of T_{B-A} is of order at least 3.*

b) *If $|A \cap B| = 1$ and $G[A - B] \cong K_r$, then the number of singleton components in T_{B-A} is at most 1.*

c) *If $|A \cap B| = 1$ and $G[A - B] \cong K_{r-1}$, then the number of singleton components in T_{B-A} is at most 2.*

Proof. a) It follows by the fact that G is connected r -regular of order $p \geq 2r + 1$.

b) Suppose to the contrary that T_{B-A} contains two singleton components, say F_1 and F_2 where $V(F_1) = \{y_1\}$ and $V(F_2) = \{y_2\}$. Because $|A \cap B| = 1$, y_1 and y_2 are adjacent to at least $r - 1$ vertices of $A - B$. Since $G[A - B] = K_r$ and $r \geq 3$, it follows that there exists a vertex of $A - B$, say y_3 , such that $\{y_1, y_2\} \cup (A - B) \subseteq N_G(y_3)$. Thus $d_G(y_3) \geq r + 1$, a contradiction

c) By applying similar arguments as in the proof of (b), (c) follows. \square

Lemma 12.5. *Let $G\bar{G}$ be a complementary prism and $L = L_G \cup L_{\bar{G}}$ be defined as above. Then $c_o(G\bar{G} - S) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) - \lfloor |L| \rfloor_e$. Consequently $c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) - c_o(G\bar{G} - S) \leq |L| \leq c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) - c_o(G\bar{G} - S) + 1$.*

Proof. If $C = \phi$, then $|L| = 0$ and thus $c_o(G\bar{G} - S) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}])$ as required. We now suppose that $C \neq \phi$. By Lemmas 12.2(a) and (b), $c_o(G\bar{G} - S) \leq c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}])$. By Lemma 12.1, $G\bar{G}[C \cup \bar{C}]$ is even and connected. So it must be contained in some component of $G\bar{G} - S$, say F . If $x \in V(F) - (C \cup \bar{C})$, then x is in some component of $G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]$, say M . So $V(M) \subseteq V(F)$. If M is odd, then $M \in L$. Note that each odd component of L is a subgraph of F . Hence, $|V(F)|$ has the same parity with $|L|$ and $c_o(G\bar{G} - S) = c_o(G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]) - |L| + \epsilon$, where $\epsilon = 1$ if $|L|$ is odd and $\epsilon = 0$ if $|L|$ is even. So $c_o(G\bar{G} - S) = c_o(G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]) - \lfloor |L| \rfloor_e$. Thus $\lfloor |L| \rfloor_e = c_o(G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]) - c_o(G\bar{G} - S)$. By properties of $\lfloor x \rfloor_e$, our result follows. This proves our lemma. \square

Lemma 12.6. *If G is an r -regular graph of order $p \geq 2r + 1$, then \bar{G} is connected.*

Proof. Note that \bar{G} is $(p - r - 1)$ -regular graph of order p . Suppose \bar{G} is disconnected. Then each component must have order at least $p - r$. So $p \geq 2(p - r)$ and thus $p \leq 2r$, a contradiction. This proves our lemma. \square

Lemma 12.7. *Let G be a connected r -regular graph of order $p \geq 2r + 1$. Let S be a cutset of $G\bar{G}$. Then $S \cap V(G) \neq \phi$ and $S \cap V(\bar{G}) \neq \phi$.*

Proof. By Lemma 12.6, \bar{G} is connected. Hence, G and \bar{G} are connected. Suppose without loss of generality that $S \cap V(G) = \phi$. So $S \subseteq V(\bar{G})$. Since $G = G\bar{G} - V(\bar{G})$ is connected and each vertex \bar{u} of $V(\bar{G}) - S$ is adjacent to a vertex u in G , it follows that $G\bar{G} - S$ is connected, a contradiction. Hence, $S \cap V(G) \neq \phi$. By similar arguments, $S \cap V(\bar{G}) \neq \phi$. This proves our lemma. \square

Theorem 12.8. *Let G be a connected r -regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. Then $G\bar{G}$ is bicritical. Consequently, $G\bar{G}$ is 1-extendable.*

Proof. Suppose $G\bar{G}$ is not bicritical. By Theorem 2.13, there is a cutset $S \subseteq V(G\bar{G})$, where $|S| \geq 2$ such that $c_o(G\bar{G} - S) > |S| - 2$. It follows by Lemmas 12.2(a) that $c_o(G\bar{G} - S) = |S|$ for $|S| \geq 2$. Note that, by Lemma 12.7, $A = S \cap V(G)$ and $\bar{B} = S \cap V(\bar{G})$ are not empty. Thus \bar{A} and B are not empty. By Lemma 12.2 (b), $A \cap B = \phi$ and thus $c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) = c_o(G[B]) + c_o(\bar{G}[\bar{A}]) = c_o(G\bar{G} - S) = |S| = |B| + |\bar{A}|$. By Lemma 12.2(c), each component of $G[B]$ and $\bar{G}[\bar{A}]$ is singleton. Hence, $G[A] \cong K_{|A|}$. Since G is r -regular of order $p \geq 2r + 1$, $|A| \leq r + 1$. If $|A| = r + 1$, then $G[A] \cong K_{r+1}$ is a disconnected component in G , a contradiction. So $1 \leq |A| \leq r$. By Lemmas 12.3 and 12.4(a), no singleton component in $G[B]$ belongs to T_{B-A} . Since each component of $G[B]$ is singleton, $T_{B-A} = \phi$. Because $c_o(G\bar{G} - S) = c_o(\bar{G}[\bar{A}]) + c_o(G[B])$, it follows by Lemma 12.5 that $0 \leq |L| \leq 1$. Since $B \neq \phi$ and $G[B]$ contains only singleton components, it follows that $1 \leq |B| = |T_{B-A}| + |L_G| \leq 1$. Hence, $|B| = |L_G| = 1$. Therefore, $|\bar{B}| = 1 < r \leq p - r - 1$. By Lemma 12.3, $T_{\bar{A}-\bar{B}}$ contains no singleton components.

Hence, $T_{\overline{A}-\overline{B}} = \phi$. Since each component of $\overline{G}[\overline{A}]$ is singleton, it is contained in $L_{\overline{G}}$. So $|L_{\overline{G}}| = |\overline{A}| = |A| \geq 1$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| \geq 2$, a contradiction. Hence, $G\overline{G}$ is bicritical. It then follows that $G\overline{G}$ is 1-extendable. This proves our theorem. \square

The next lemma follows by Theorem 2.11.

Lemma 12.9. *Let G be a connected r -regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. If G contains a triangle, then $G\overline{G}$ is not r -extendable.* \square

By Lemma 12.9, if G is a 3-regular graph of order $p \geq 8$ containing a triangle, then $G\overline{G}$ is not 3-extendable. The next theorem provides a sufficient condition for a connected r -regular graph G which $G\overline{G}$ is 2-extendable, for $r \geq 4$. In case $r = 3$, if G contains the graph F in Figure 25 as an induced subgraph, then $\{yz, \overline{w}\overline{x}\}$ cannot be extended to a perfect matching in $G\overline{G}$. Hence, $G\overline{G}$ is not 2-extendable. We next show that the complementary prism of connected 3-regular F -free graphs and connected r -regular graphs for $r \geq 4$ are 2-extendable.

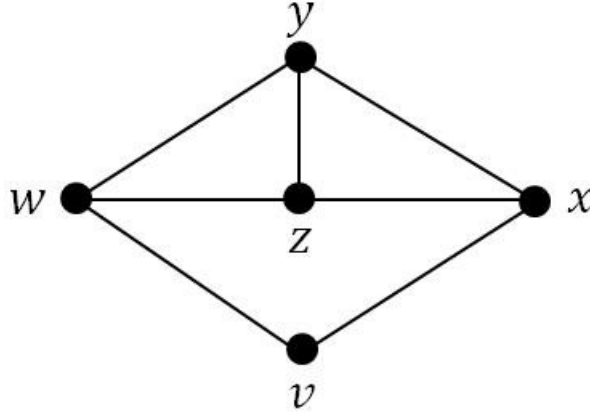


Figure 25: the graph F

Theorem 12.10. *Suppose G is a connected graph of order p . If G is either 3-regular F -free where $p \geq 8$ and F is the graph in Figure 1 or r_0 -regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

Proof. Observe that \overline{G} is $(p - r - 1)$ -regular where $r \in \{3, r_0\}$ and $p - r - 1 \geq 4$. By Theorem 12.8, $G\overline{G}$ is bicritical. Suppose to the contrary that $G\overline{G}$ is not 2-extendable. Then there is a matching $M \subseteq E(G\overline{G})$ of size two such that $G\overline{G} - V(M)$ contains no perfect matching. By Theorem 2.7, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. Let $S = T \cup V(M)$. Clearly, $|S| \geq 4$. Thus $c_o(G\overline{G} - S) > |S| - 4$. Because $G\overline{G}$ is bicritical, by Theorem 2.13, $c_o(G\overline{G} - S) \leq |S| - 2$. It follows by parity that $c_o(G\overline{G} - S) = |S| - 2$ and $G\overline{G}[S]$ contains a matching of size at least two. Let $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$. By Lemma 12.2 (b), $|A \cap \overline{B}| \leq 1$. Further, by

Lemma 12.7, $A \neq \phi$ and $\bar{B} \neq \phi$. So $\bar{A} \neq \phi$ and $B \neq \phi$. We distinguish 2 cases according to $|A \cap B|$.

Case 1: $|A \cap B| = 1$. Put $\{u\} = A \cap B$. By Lemma 12.2(b) $c_o(G\bar{G} - S) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) = |S| - 2$. By Lemma 12.5, $|L| \leq 1$. Further, by Lemma 12.2(c), each component of $\bar{G}[\bar{A} - \bar{B}] \cup G[B - A]$ is singleton. Thus, $G[A - B]$ is a clique, $|Com_{\bar{A}-\bar{B}}| = |\bar{A} - \bar{B}|$ and $|Com_{B-A}| = |B - A|$. Since G is connected, it is easy to see that if $|A - B| \geq r + 1$, then $G[A - B] \cong K_{|A-B|}$ contains a vertex of degree greater than r or $G \cong K_{r+1}$ is a graph of order less than p , a contradiction. Hence, $|A - B| \leq r$.

We first show that $|T_{B-A}| \geq 2$. Suppose to the contrary that $|T_{B-A}| \leq 1$. Since $G[B - A]$ contains only singleton components and $|L_G| \leq |L| \leq 1$, it follows that $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 2$. Thus $|\bar{B}| = |B| = |B - A| + |B \cap A| \leq 3 < 4 \leq p - r - 1$. By Lemma 12.3, $T_{\bar{A}-\bar{B}}$ contains no singleton components. Thus $T_{\bar{A}-\bar{B}} = \phi$. Consequently, $Com_{\bar{A}-\bar{B}} = T_{\bar{A}-\bar{B}} \cup L_{\bar{G}} = L_{\bar{G}}$. Therefore, $|\bar{A} - \bar{B}| = |L_{\bar{G}}| \leq 1$ since $\bar{G}[\bar{A} - \bar{B}]$ contains only singleton components. So $|A| = |\bar{A}| = |\bar{A} - \bar{B}| + |\bar{A} \cap \bar{B}| \leq 2 < r$. By Lemma 12.3, T_{B-A} contains no singleton components. So $T_{B-A} = \phi$. Since $T_{\bar{A}-\bar{B}} = \phi$ and $T_{B-A} = \phi$, it follows that every odd component of $\bar{G}[\bar{A} - \bar{B}] \cup G[B - A]$ is in L . Because $|L| \leq 1$ and $\bar{G}[\bar{A} - \bar{B}] \cup G[B - A]$ contains only singleton components, it follows that $|\bar{A} - \bar{B}| + |B - A| \leq 1$. Hence, $|S| = |A - B| + |A \cap B| + |\bar{A} \cap \bar{B}| + |\bar{B} - \bar{A}| = |A - B| + 2|A \cap B| + |\bar{B} - \bar{A}| \leq 3 < 4$, contradicting the fact that $|S| \geq 4$. Therefore, $|T_{B-A}| \geq 2$.

Let $D_1, D_2 \in T_{B-A}$. Since $G[B - A]$ contains only singleton components, $D_i \cong K_1$, for $1 \leq i \leq 2$. Put $\{v_i\} = V(D_i)$. By Lemma 12.3, $|A| \geq r$. Consequently, $|A - B| \geq r - 1$. Because $|A - B| \leq r$, $r - 1 \leq |A - B| \leq r$. Since $G[A - B]$ is clique, $|A \cap B| = 1$ and $|T_{B-A}| \geq 2$, it follows by Lemmas 12.4 (b) and (c) that $|A - B| = r - 1$ and $|T_{B-A}| = 2$. Thus $|A| = |A - B| + |A \cap B| = r$. Because $r - 1 = |A - B| = |\bar{A} - \bar{B}| = |Com_{\bar{A}-\bar{B}}| = |T_{\bar{A}-\bar{B}}| + |L_{\bar{G}}| \leq |T_{\bar{A}-\bar{B}}| + 1$, it follows that $|T_{\bar{A}-\bar{B}}| \geq r - 2 \geq 1$. Thus $T_{\bar{A}-\bar{B}}$ contains a singleton component. By Lemma 12.3, $|\bar{B}| \geq p - r - 1 \geq 4$. Therefore, $|\bar{B} - \bar{A}| = |\bar{B}| - |\bar{B} \cap \bar{A}| \geq p - r - 2 \geq 3$. On the other hand, $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 3$. Then $|B - A| = |\bar{B} - \bar{A}| = 3$. Thus $3 = |T_{B-A}| + |L_G| = 2 + |L_G|$. It follows that $L = L_G = \{K_1\}$ and consequently $L_{\bar{G}} = \phi$. Since $|A| = r$, $deg_G v_1 = deg_G v_2 = r$ and $N_G(v_1) = N_G(v_2) \subseteq A$, it follows that $N_G(v_1) = N_G(v_2) = A$.

We now put $\{\bar{w}\} = V(K_1)$ where $K_1 \in T_{\bar{A}-\bar{B}}$. Clearly, $N_{\bar{G}}(\bar{w}) \subseteq \bar{B} - \{\bar{v}_1, \bar{v}_2\}$ since v_1 and v_2 are adjacent to every vertex in A . Because $|\bar{B}| = |\bar{B} - \bar{A}| + |\bar{A} \cap \bar{B}| = 3 + 1 = 4$, $|N_{\bar{G}}(\bar{w})| \leq |\bar{B}| - |\{\bar{v}_1, \bar{v}_2\}| = 2$ thus \bar{G} is t -regular where $t \leq 2$. This contradicts the fact that \bar{G} is $(p - r - 1)$ -regular where $p - r - 1 \geq 4$. Therefore, Case 1 cannot occur.

Case 2: $|A \cap B| = 0$. By Lemmas 12.2(a) and (b), $|S| - 2 = c_o(G\bar{G} - S) \leq c_o(\bar{G}[\bar{A}]) + c_o(G[B]) \leq |\bar{A}| + |B| = |S|$. By parity, $c_o(\bar{G}[\bar{A}]) + c_o(G[B]) = |S|$ or $c_o(\bar{G}[\bar{A}]) + c_o(G[B]) = |S| - 2$. We distinguish 2 cases.

Case 2.1 : $c_o(\bar{G}[\bar{A}]) + c_o(G[B]) = |S| = |\bar{A}| + |B|$. Clearly, each component of $\bar{G}[\bar{A}] \cup G[B]$ is singleton. So $G[A] \cong K_{|A|}$. It is easy to see that if $|A| \geq r + 1$, then $G[A]$ contains a vertex of degree greater than r or $G[A]$ is a disconnected component in G , a contradiction. Hence, $|A| \leq r$. By Lemmas 12.3 and 12.4(a), T_{B-A} contains no singleton components. Therefore, $T_{B-A} = \phi$. Thus $|L_G| = |B|$. Because $c_o(G[B]) + c_o(\bar{G}[\bar{A}]) -$

$c_o(G\bar{G} - S) = |S| - (|S| - 2) = 2$, by Lemma 12.5, $2 \leq |L| \leq 3$. Since $B \neq \phi$ and $|B| = |L_G| \leq |L|$, it follows that $1 \leq |B| \leq 3$. Because $|\bar{B}| = |B| \leq 3 < 4 \leq p - r - 1$, by Lemma 12.3, $T_{\bar{A}-\bar{B}}$ contains no singleton components. Thus $T_{\bar{A}-\bar{B}} = \phi$. Hence, $|L_{\bar{G}}| = |\bar{A}| = |A|$. Therefore, $|L| = |L_G| + |L_{\bar{G}}| = |B| + |\bar{A}| = |S|$ and thus $2 \leq |S| \leq 3$ since $2 \leq |L| \leq 3$, contradicting the fact that $|S| \geq 4$. Hence, Case 2.1 cannot occur.

Case 2.2 : $c_o(\bar{G}[\bar{A}]) + c_o(G[B]) = |S| - 2 = |\bar{A}| + |B| - 2$. Put $s = |S|$. It is easy to see that $\bar{G}[\bar{A}] \cup G[B]$ contains all singleton components except exactly one non-singleton component which is of order 2 or 3. Hence, $\bar{G}[\bar{A}] \cup G[B]$ is isomorphic to a graph in $\{(s-2)K_1 \cup K_2, (s-3)K_1 \cup P_3, (s-3)K_1 \cup K_3\}$. If $|\bar{A}| \geq r+2 \geq 5$, then $\bar{G}[\bar{A}]$ must contain a singleton component, say F , where $V(F) = \{\bar{u}\}$. It follows that $\deg_{\bar{G}} u \geq r+1$, a contradiction. Hence, $|A| = |\bar{A}| \leq r+1$. Since $c_o(\bar{G}[\bar{A}]) + c_o(G[B]) - c_o(G\bar{G} - S) = (|S| - 2) - (|S| - 2) = 0$, by Lemma 12.5, $|L| \leq 1$. We distinguish 2 subcases according to the non-singleton component.

Subcase 2.2.1 : The only non-singleton component in $\bar{G}[\bar{A}] \cup G[B]$ is contained in $G[B]$. So $\bar{G}[\bar{A}] \cong |\bar{A}|K_1$ and $G[A] \cong K_{|\bar{A}|} \cong K_{|A|}$. Clearly, $|A| \leq r$ otherwise $G[A]$ is a disconnected component in G . By Lemmas 12.3 and 12.4(a), T_{B-A} contains no singleton components. So every singleton component in $G[B]$ is contained in L_G . Since $|L_G| \leq |L| \leq 1$, $G[B]$ contains at most 1 singleton component. We first show that $T_{\bar{A}-\bar{B}} = \phi$. Suppose this is not the case. Then there is $K_1 \in T_{\bar{A}-\bar{B}}$ since $\bar{G}[\bar{A}]$ contains only singleton components. By Lemma 12.3, $|B| = |\bar{B}| \geq p - r - 1 \geq 4$. Because $G[B]$ contains a non-singleton component of order either 2 or 3 and at most 1 singleton component, it follows that $G[B]$ is isomorphic to a graph in $\{K_1 \cup P_3, K_1 \cup K_3\}$. Thus $|B| = 4$ and either $T_{B-A} = \{P_3\}$ or $T_{B-A} = \{K_3\}$, and $L_G = \{K_1\}$. Thus $L_{\bar{G}} = \phi$. So $Com_{\bar{A}} = T_{\bar{A}-\bar{B}} \cup L_{\bar{G}} = T_{\bar{A}-\bar{B}}$. Therefore, each vertex of \bar{A} is adjacent to every vertex of \bar{B} since \bar{G} is $(p-r-1)$ -regular and $p-r-1 \geq 4$. It follows that there is no edge joining vertices of A and B . But this contradicts the fact that $T_{B-A} \neq \phi$. Hence, $T_{\bar{A}-\bar{B}} = \phi$ as required.

Therefore, $Com_{\bar{A}} = L_{\bar{G}}$. Since $|L_{\bar{G}}| \leq |L| \leq 1$ and $|\bar{A}| = |A| \neq 0$, it follows that $|Com_{\bar{A}}| = |L_{\bar{G}}| = 1$. Further, $L_G = \phi$ and $\bar{G}[\bar{A}] = K_1$. Thus $Com_B = T_{B-A}$. Because $|A| = |\bar{A}| = 1 < r \leq 3$, by Lemma 12.3, T_{B-A} contains no singleton components. So $G[B]$ contains no singleton components and $G[B]$ is isomorphic to a graph in $\{P_3, K_3\}$ since $|B| = |S| - |A| \geq 3$. Then $G\bar{G}[S] = G[A] \cup \bar{G}[\bar{B}]$ contains a matching of size less than two, contradicting the fact that $G\bar{G}[S]$ contains a matching of size at least two. Hence, Subcase 2.2.1 cannot occur.

Subcase 2.2.2 : The only non-singleton component in $\bar{G}[\bar{A}] \cup G[B]$ is contained in $\bar{G}[\bar{A}]$. So $G[B] \cong |B|K_1$. We first show that $T_{B-A} \neq \phi$. Suppose this is not the case. Then $T_{B-A} = \phi$ and thus $Com_B = T_{B-A} \cup L_G = L_G$. Since $B \neq \phi$ and $|L_G| + |L_{\bar{G}}| = |L| \leq 1$, it follows that $|L_G| = 1$ and $|L_{\bar{G}}| = 0$. Consequently, $|B| = 1$ since $G[B] \cong |B|K_1$. Because $|\bar{B}| = |B| = 1 < r$, $T_{\bar{A}-\bar{B}}$ contains no singleton components by Lemma 12.3. Hence, $G[\bar{A}]$ contains exactly one non-singleton component of order 2 or 3. Thus $|A| = |\bar{A}| \leq 3$. It is easy to see that $G\bar{G}[S] = G[A] \cup \bar{G}[\bar{B}]$ contains a matching of size at most one since $|\bar{B}| = 1$. This contradicts the fact that $G\bar{G}[S]$ contains a matching of size at least two. Hence, $T_{B-A} \neq \phi$. Further, $|T_{B-A}| \geq |B| - 1$ since $|L_G| \leq |L| \leq 1$ and $|T_{B-A}| + |L_G| = |B|$.

Because $G[B] \cong |B|K_1$, there exists $K_1 \in T_{B-A}$. By Lemma 12.3, $|A| \geq r$. So $r \leq |A| \leq r+1$. We first suppose that $|A| = r+1$. Let F_t be the non-singleton component of order t in $\overline{G}[\overline{A}]$ and let $\overline{A}_1 = V(F_t)$. Then $2 \leq t \leq 3$ and $\overline{G}[\overline{A}] \cong (r+1-t)K_1 \cup F_t$. It is easy to see that $G[A]$ contains $r+1-t$ vertices of degree r and each vertex of $A_1 = \overline{A}_1$ has degree, in $G[A]$, at least $r+1-t$ and at most $r-1$. Let $\{w\} = V(K_1)$ where $K_1 \in T_{B-A}$, then $N_G(w) \subseteq A_1$ and thus $3 \leq r = \deg_G(w) \leq t \leq 3$. It then follows that $N_G(w) = A_1$ and $t = r = 3$. Thus \bar{w} is not adjacent to any vertex of \overline{A}_1 and $\overline{G}[\overline{A}] \cong K_1 \cup F_3$. Further, each vertex of A_1 has degree at least $|T_{B-A}| + 1 = |B| - |L_G| + 1 \geq |B|$ since $|L_G| \leq 1$. Thus $|B| \leq 3$ since G is now 3-regular. Because \overline{G} is $(p-r-1)$ -regular where $p-r-1 \geq 4$ and each vertex of $V(F_3) = \overline{A}_1$ has degree at most 3 in $\overline{G}[\overline{A} \cup \overline{B}]$ since it must be adjacent to at most one vertex in \overline{B} , it follows that $F_3 \in L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$, the only singleton component, K_1 , of $\overline{G}[\overline{A}]$ must be in $T_{\overline{A}-\overline{B}}$. By Lemma 12.3, $|\overline{B}| \geq p-r-1 \geq 4$. But this contradicts the fact that $|\overline{B}| = |B| \leq 3$. Therefore, $|A| = r$.

Consequently, for each $w \in V(K_1)$ where $K_1 \in T_{B-A}$, $N_G(w) = A$. Now let $\bar{v} \in \overline{A}$. Then $\deg_{\overline{B}}(\bar{v}) \leq |\overline{B}| - |T_{B-A}| = |B| - |T_{B-A}| = |L_G| \leq 1$. Further, $\deg_{\overline{A}}(\bar{v}) \leq 2$ since each component of $\overline{G}[\overline{A}]$ has order at most 3. Because \overline{G} is $(p-r-1)$ -regular where $p-r-1 \geq 4$, \bar{v} is adjacent to some vertex of \overline{C} . Consequently, each odd component of $\overline{G}[\overline{A}]$ is contained in $L_{\overline{G}}$. Because $|\overline{A}| = |A| = r \geq 3$, $\overline{G}[\overline{A}]$ contains a non-singleton component of order either 2 or 3 and $|L_{\overline{G}}| \leq |L| \leq 1$, it follows that $c_o(\overline{G}[\overline{A}]) = 1$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{K_1 \cup K_2, P_3, K_3\}$. Hence, $r = |A| = 3$, $|L| = |L_{\overline{G}}| = 1$, $Com_B = T_{B-A} = \{|B|K_1\}$. Further, for $x \in B, y \in A, N_G(x) = A$ and $\deg_G(y) = r = 3 \geq |B| = |\overline{B}|$.

We first suppose that $\overline{G}[\overline{A}] \cong K_3$. Then $G[A]$ is independent and thus $\overline{G}[\overline{B}]$ must contain a matching of size at least two since $G\overline{G}[S]$ contains a matching of size at least two. So $|B| = |\overline{B}| \geq 4$. But this contradicts the fact that $|B| = |\overline{B}| \leq 3$. Hence, $\overline{G}[\overline{A}] \neq K_3$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{P_3, K_1 \cup K_2\}$. In either case, $G[A]$ contains a maximum matching of size one. Then $2 \leq |\overline{B}| \leq 3$ since $G\overline{G}[A \cup \overline{B}]$ contains a matching of size at least two.

We now suppose that $\overline{G}[\overline{A}] \cong K_1 \cup K_2$. Then $G[A] \cong P_3$ and then the vertex of degree two in P_3 has degree, in G , greater than $r = 3$, again a contradiction. Hence, $\overline{G}[\overline{A}] \neq K_1 \cup K_2$. Consequently, $\overline{G}[\overline{A}] \cong P_3$ and then $G[A] \cong K_1 \cup K_2$. Clearly, $|B| \neq 3$ otherwise $G[A]$ contains a vertex of degree greater than $r = 3$. So $|B| = 2$ and thus $G[A \cup B]$ contains the graph F in Figure 1 as an induced subgraph. But this contradicts our hypothesis that G is 3-regular F -free graph. This completes the proof of our theorem. \square

It is clear that a connected 3-regular graph containing F , in Figure 25, as an induced subgraph contains v as a cut vertex. So 2-connected 3-regular graphs are F -free. The next corollary follows by this fact and Theorem 12.10.

Corollary 12.11. *If G is a 2-connected r -regular graph of order $p \geq 2r + 1$, for $r \geq 3$, then $G\overline{G}$ is 2-extendable.* \square

According to Theorems 11.2 and 12.10, we have the following theorem.

Theorem 12.12. *If each component G_i of G is 3-regular F -free of order at least 8 where F is the graph in Figure 1 or r_0 -regular of order at least $2r_0 + 1 \geq 9$, then $G\bar{G}$ is 2-extendable.*

13 Some constructions of extendable non-bipartite graphs

In this section, we provide two constructions of extendable non-bipartite graphs in which their complement graphs are also extendable by using cartesian and lexicographic products of two extendable graphs.

The first two results in this section concern the extendability of graphs obtained from a cartesian product, established by Györi and Plummer [18] and a lexicographic product established by Bai et al. [6].

Theorem 13.1. [18] *For non-negative integers l_1 and l_2 , let G_i be a l_i -extendable graph for $1 \leq i \leq 2$. Then $G_1 \times G_2$ is $(l_1 + l_2 + 1)$ -extendable. \square*

Theorem 13.2. [6] *For non-negative integers l_1 and l_2 , let G_i be a l_i -extendable graph for $1 \leq i \leq 2$. Then $G_1 \circ G_2$ is $2(l_1 + 1)(l_2 + 1)$ -factor-critical. In particular, $G_1 \circ G_2$ is $(l_1 + 1)(l_2 + 1)$ -extendable. \square*

We are now ready for our constructions.

Theorem 13.3. *For non-negative integers $l_1, l_2, p_1 \geq 2l_1 + 2$ and $p_2 \geq 2l_2 + 2$ and $1 \leq i \leq 2$, let H_i be l_i -extendable of order p_i . Further, let $G = H_1 \times H_2$. If $\Delta(H_1) = p_1 - 1 - t_1$ and $\Delta(H_2) = p_2 - 1 - t_2$ for some non-negative integers t_1 and t_2 , then G is $(l_1 + l_2 + 1)$ -extendable and \bar{G} is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$ -extendable.*

Proof. By Theorem 13.1, $G = H_1 \times H_2$ is $(l_1 + l_2 + 1)$ -extendable as required. We need only show that \bar{G} is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$ -extendable. Clearly, G and \bar{G} are of order $p_1 p_2$. Since $N_G((u, v)) = \{(x, v) | xu \in E(H_1)\} \cup \{(u, y) | vy \in E(H_2)\}$, $\deg_G((u, v)) = \deg_{H_1}(u) + \deg_{H_2}(v)$. Thus $\Delta(G) = \Delta(H_1) + \Delta(H_2) = p_1 + p_2 - 2 - t_1 - t_2$. Therefore, $\delta(\bar{G}) = p_1 p_2 - p_1 - p_2 + 2 + t_1 + t_2 - 1 = \frac{1}{2} p_1 p_2 + \frac{1}{2} p_1 p_2 - p_1 - p_2 + t_1 + t_2 + 1 = \frac{1}{2} p_1 p_2 + \frac{1}{2} (p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1$. By Theorem 2.9, \bar{G} is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$ -extendable as required. This proves our theorem. \square

Corollary 13.4. *Let H_1, H_2 and G be graphs defined in Theorem 13.3. If either H_1 or H_2 is non-bipartite, then G and \bar{G} are also non-bipartite. \square*

Theorem 13.5. *For non-negative integers $h_1, h_2, \bar{h}_1, \bar{h}_2$, let H_i be a h_i -extendable and let \bar{H}_i be a \bar{h}_i -extendable for $1 \leq i \leq 2$. Then $G = H_1 \circ H_2$ is $(h_1 + 1)(h_2 + 1)$ -extendable graph and \bar{G} is $(\bar{h}_1 + 1)(\bar{h}_2 + 1)$ -extendable graph.*

Proof. By Theorem 13.2, $G = H_1 \circ H_2$ is $(h_1 + 1)(h_2 + 1)$ -extendable. We first show that $\bar{G} = \bar{H}_1 \circ \bar{H}_2$. Clearly, $V(\bar{G}) = V(\overline{H_1 \circ H_2}) = V(H_1) \times V(H_2) = V(\bar{H}_1) \times V(\bar{H}_2) =$

$V(\overline{H_1} \circ \overline{H_2})$. Let $(u_1, v_1), (u_2, v_2) \in V(H_1) \times V(H_2)$ and let $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1} \circ \overline{H_2})$. Thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$.

If $u_1 = u_2$, then $v_1v_2 \notin E(H_2)$ and thus $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1} \circ \overline{H_2})$. Further, if $u_1 \neq u_2$, then $u_1u_2 \notin E(H_1)$. And again $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1} \circ \overline{H_2})$. Hence, $E(\overline{H_1} \circ \overline{H_2}) \subseteq E(\overline{H_1} \circ \overline{H_2})$.

We now suppose that $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1} \circ \overline{H_2})$. If $u_1 = u_2$, then $v_1v_2 \in E(\overline{H_2})$. Thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$. Further, if $u_1 \neq u_2$, then $u_1u_2 \in E(\overline{H_1})$ and thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$. In either case $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1} \circ \overline{H_2})$. Hence, $E(\overline{H_1} \circ \overline{H_2}) \subseteq E(\overline{H_1} \circ \overline{H_2})$. Therefore, $E(\overline{H_1} \circ \overline{H_2}) = E(\overline{H_1} \circ \overline{H_2})$. Thus $\overline{H_1} \circ \overline{H_2} = \overline{H_1} \circ \overline{H_2}$. It follows by Theorem 13.2 that \overline{G} is $(\overline{h_1} + 1)(\overline{h_2} + 1)$ -extendable graph as required. This proves our theorem. \square

Corollary 13.6. *For $1 \leq i \leq 2$, let H_i, \overline{H}_i and G be graphs defined in Theorem 13.5. If H_1 is connected, $E(H_2) \neq \phi$ and $E(\overline{H}_2) \neq \phi$ then G and \overline{G} are non-bipartite.* \square

According to Theorems 13.3 and 13.5, we have shown that there exists a graph G such that G is l_1 -extendable and \overline{G} is l_2 -extendable for some integers l_1 and l_2 . Theorem 13.8 establishes that for any positive integers l_1 and l_2 , there is a graph G such that G is l_1 -extendable and \overline{G} is l_2 -extendable.

Lemma 13.7. *Let P_t be a path of order t . If $t \geq 4$ is an even integer, then P_t is 0-extendable and \overline{P}_t is $(t - 4)$ -factor-critical. Further, \overline{P}_t is $\frac{1}{2}(t - 4)$ -extendable.*

Proof. Clearly, P_t contains a perfect matching. We only show that \overline{P}_t is $(t - 4)$ -factor-critical. Let $T \subseteq V(\overline{P}_t)$ such that $|T| = t - 4$. Clearly, $\overline{P}_t - T$ is connected and contains P_4 as a subgraph. Thus $\overline{P}_t - T$ is one of a graph in $\{K_4, K_4 - e, C_4, K_4 - \{e_1, e_2\}, P_4\}$, where e_1 and e_2 have a common end vertex. In either case, $\overline{P}_t - T$ contains a perfect matching. Thus $\overline{P}_t - T$ is $(t - 4)$ -factor-critical as required. It then follows by definition of k -extendable that \overline{P}_t is $\frac{1}{2}(t - 4)$ -extendable. This proves our lemma. \square

Theorem 13.8. *For positive integers l_1 and l_2 , there is a graph G such that G is l_1 -extendable and \overline{G} is l_2 -extendable. Further, G and \overline{G} are non-bipartite.*

Proof. Let $H_1 = \overline{P}_{2l_1+2}$ and $H_2 = P_{2l_2+2}$. By Lemma 13.7, H_1 is $(l_1 - 1)$ -extendable, \overline{H}_1 is 0-extendable, H_2 is 0-extendable and \overline{H}_2 is $(l_2 - 1)$ -extendable. Let $G = H_1 \circ H_2$. By Theorem 13.5, G is l_1 -extendable and \overline{G} is l_2 -extendable as required. Further, it is clear that G and \overline{G} are non-bipartite. This proves our theorem. \square

14 Results on extendability of complementary prism of extendable graphs

In this section, we establish the extendability of the complementary prism $G\overline{G}$ of G where G and \overline{G} are l_1 -extendable and l_2 -extendable non-bipartite graphs, respectively. We begin with some lemmas.

Lemma 14.1. *For a graph G , let $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Suppose $c_o(G-A) = |A| - t_1$ and $c_o(\overline{G}-\overline{B}) = |\overline{B}| - t_2$, for some non-negative integers t_1, t_2 . Then $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2)$. Further, if $A \cup B \neq V(G)$ and $G - A$ and $\overline{G} - \overline{B}$ contain no even components, then $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2) - 2$.*

Proof. It is easy to see that $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2)$. We now suppose that $A \cup B \neq V(G)$ and $G - A, \overline{G} - \overline{B}$ contain no even components. Let $x \in V(G) - (A \cup B)$. Then x is in an odd component of $G - A$, say C . Clearly, $\overline{x} \notin \overline{A} \cup \overline{B}$ and thus \overline{x} is in an odd component of $\overline{G} - \overline{B}$, say D . Hence, $G\overline{G}[V(C) \cup V(D)]$ forms an even component in $G\overline{G} - (A \cup \overline{B})$. Therefore $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2) - 2$ as required. This proves our lemma. \square

Lemma 14.2. *Let G and \overline{G} be l_1 -extendable and l_2 -extendable graphs, respectively where l_1 and l_2 are positive integers. Further, let M be a matching of size $l + 1$ in $G\overline{G}$ where $l = \min\{l_1, l_2\}$. If either $M = \{x_i \overline{x}_i | x_i \in V(G) \text{ for } 1 \leq i \leq l + 1\}$ or $M \subseteq (E(G) \cup E(\overline{G}))$, then $G\overline{G}$ has a perfect matching containing M .*

Proof. Clearly, if $M = \{x_i \overline{x}_i | x_i \in V(G) \text{ for } 1 \leq i \leq l + 1\}$, then $\{v \overline{v} | v \in V(G)\}$ is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that $M \subseteq (E(G) \cup E(\overline{G}))$. Put $M_G = M \cap E(G)$ and $M_{\overline{G}} = M \cap E(\overline{G})$. If $1 \leq |M_G| \leq l$ and $1 \leq |M_{\overline{G}}| \leq l$, then it is easy to see that $M = M_G \cup M_{\overline{G}}$ can be extended to a perfect matching in $G\overline{G}$, by Theorem 2.8(a), since G is l_1 -extendable and \overline{G} is l_2 -extendable. Hence, we suppose without loss of generality that $|M_G| = l + 1$. Suppose there is no perfect matching in G containing M_G . By Lemma 10.4, there is a maximum matching F_1 in $G - V(M)$ saturates all except two non-adjacent vertices, say x and y . So $\overline{x} \overline{y} \in E(\overline{G})$. Since \overline{G} is l_2 -extendable where $l_2 \geq 1$ and by Theorem 2.8(a), it follows that there is a perfect matching F_2 in \overline{G} containing $\overline{x} \overline{y}$. Hence, $M \cup F_1 \cup (F_2 - \{\overline{x} \overline{y}\}) \cup \{x \overline{x}, y \overline{y}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our lemma. \square

We are now ready to prove our main result. We begin with the extendability of $G\overline{G}$ where G is l_1 -extendable and \overline{G} is l_2 -extendable for $l_1 \geq 4$ and $l_2 \geq 4$.

Theorem 14.3. *For positive integers $l_1 \geq 4, l_2 \geq 4$, let G and \overline{G} be l_1 -extendable and l_2 -extendable non-bipartite graphs of order $p \geq 2l + 2$, respectively, where $l = \min\{l_1, l_2\}$. Then $G\overline{G}$ is $(l + 1)$ -extendable.*

Proof. Let $M \subseteq E(G\overline{G})$ be a matching of size $l + 1$ in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Note that $M_{G\overline{G}} = \{x \overline{x} | \text{for some } x \in V(G)\}$. If $M_{G\overline{G}} = M$ or $M_{G\overline{G}} = \phi$, then, by Lemma 14.2, there is a perfect matching in $G\overline{G}$ containing M as required. We now suppose that $M_{G\overline{G}} \neq M$ and $M_{G\overline{G}} \neq \phi$. Without loss of generality, we may suppose that $|M_G| \geq |M_{\overline{G}}|$. Hence, $M_G \neq \phi$.

Put $S = V(G) \cap V(M_{G\overline{G}})$. Let N_S be a maximum matching in $G[S]$. Put $I_S = S - V(N_S)$. Clearly, I_S is an independent set. Similarly, let $N_{\overline{S}}$ be a maximum matching in $\overline{G}[\overline{S}]$ and put $I_{\overline{S}} = \overline{S} - V(N_{\overline{S}})$. For simplicity, we denote the cardinalities of each set by its small letter, i.e., $m_G = |M_G|$, $m_{\overline{G}} = |M_{\overline{G}}|$, $m_{G\overline{G}} = |M_{G\overline{G}}|$, $s = |S|$, $i_S = |I_S|$, etc.

Clearly, $1 \leq m_G \leq l$, $s = \bar{s}$, $n_S + i_S \geq 1$ and $n_{\bar{S}} + i_{\bar{S}} \geq 1$ since $s = \bar{s} = m_{G\bar{G}} \geq 1$. Therefore,

$$l + 1 = m_G + m_{\bar{G}} + m_{G\bar{G}} \quad (1)$$

$$l + 1 = m_G + m_{\bar{G}} + s \quad (2)$$

$$l + 1 = m_G + m_{\bar{G}} + 2n_S + i_S \quad (3)$$

$$l + 1 = m_G + m_{\bar{G}} + 2n_{\bar{S}} + i_{\bar{S}}. \quad (4)$$

Consequently, $m_G + n_S = l + 1 - (m_{\bar{G}} + n_S + i_S) \leq l$ since $n_S + i_S \geq 1$ and $m_{\bar{G}} + n_{\bar{S}} = l + 1 - (m_G + n_{\bar{S}} + i_{\bar{S}}) \leq l$ since $1 \leq m_G \leq l$. Further, $s \equiv i_S \pmod{2}$ and $\bar{s} \equiv i_{\bar{S}} \pmod{2}$ because $s = 2n_S + i_S$ and $s = \bar{s} = 2n_{\bar{S}} + i_{\bar{S}}$.

We first suppose that $i_S = 0$. Since $m_G + n_S \leq l$, by Theorem 2.8(a), there is a perfect matching in $G - (V(M_G) \cup N_S)$, say F_G . Now consider \bar{G} . By Equation 4, $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1 \geq 0$ since $m_G \geq 1$. By Lemma 10.2(a), there is a perfect matching in $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$, say $F_{\bar{G}}$. Hence, $M \cup F_G \cup F_{\bar{G}}$ is a perfect matching in $G\bar{G}$ containing M as required.

So we now suppose that $i_S \geq 1$. We distinguish 2 cases according to parity of s .

Case 1 : s is even. So $i_S \geq 2$ and $i_{\bar{S}} \geq 0$ are also even. We distinguish 2 subcases according to $m_{\bar{G}} + n_S$.

Subcase 1.1 : $m_{\bar{G}} + n_S \geq 1$. So by Equation 3, $l - (m_G + n_S + i_S) = m_{\bar{G}} + n_S - 1 \geq 0$. By Lemma 10.2(a), there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S)$, say F_G . Further, by Equation 4 and the fact that $m_G \geq 1$, $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1 \geq 0$. So, by Lemma 10.2(a), there is a perfect matching in $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$, say $F_{\bar{G}}$. Hence, $M \cup F_G \cup F_{\bar{G}}$ is a perfect matching in $G\bar{G}$ containing M as required.

Subcase 1.2 : $m_{\bar{G}} + n_S = 0$. We first show that $n_{\bar{S}} \leq \frac{l}{2}$. Since $n_S = 0$, $G[S]$ is independent and thus $\bar{G}[\bar{S}]$ is a complete graph. Because s is even, $n_{\bar{S}} = \frac{1}{2}\bar{s} = \frac{1}{2}s$. So, by Equation 2 and the fact that $m_G \geq 1$, $n_{\bar{S}} = \frac{1}{2}s = \frac{1}{2}(l + 1 - m_G - m_{\bar{G}}) = \frac{1}{2}(l + 1 - m_G) \leq \frac{l}{2}$ as required. By Equation 3, $l - m_G = m_{\bar{G}} + 2n_S + i_S - 1 = i_S - 1$ and $\lfloor i_S - 1 \rfloor_e = i_S - 2 \geq 0$ since $i_S = s$ is even. It follows by Lemma 10.2(a) that $G' = G - V(M_G)$ is $(i_S - 2)$ -factor-critical. By Lemma 10.5, there is a maximum matching F_G in $G' - I_S$ saturates all except at most 2 vertices in $G' - I_S$.

We next consider \bar{G} . By Equation 4 and the fact that $m_G \geq 1$, $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1 \geq n_{\bar{S}} \geq 0$. By Lemma 10.2(a), there is a perfect matching in $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$, say $F_{\bar{G}}$. Clearly, if F_G is a perfect matching in $G' - I_S$, then $M \cup F_G \cup F_{\bar{G}}$ is a perfect matching in $G\bar{G}$ as required.

We now suppose that F_G is not a perfect matching. Let $x, y \in V(G') - I_S$ where x and y are unsaturated by F_G . Clearly, $xy \notin E(G)$. So $\bar{x}\bar{y} \in E(\bar{G})$. Because $n_{\bar{S}} \leq \frac{l}{2}$ and $l \geq 4$, it follows that $m_{\bar{G}} + n_{\bar{S}} + 1 = n_{\bar{S}} + 1 \leq \frac{l}{2} + 1 \leq \frac{l}{2} + (\frac{l}{2} - 1) \leq l - 1$. By Theorem 2.8(a), there is a perfect matching in $\bar{G} - V(M_{\bar{G}} \cup N_{\bar{S}} \cup \{\bar{x}\bar{y}\})$, say $F'_{\bar{G}}$. Hence, $M \cup F_G \cup (F'_{\bar{G}} - \{\bar{x}\bar{y}\}) \cup \{x\bar{x}, y\bar{y}\}$ is a perfect matching in $G\bar{G}$ containing M as required. This proves Case 1.

Case 2 : s is odd. So i_S and $i_{\bar{S}}$ are also odd. We distinguish 3 subcases according to $m_{\bar{G}} + n_S$.

Subcase 2.1 : $m_{\bar{G}} = n_S = 0$. By Equation 3, $l - (m_G + (i_S - 1)) = m_{\bar{G}} + 2n_S = 0$. Let $i \in I_S$, by Lemma 10.2(a), there is a perfect matching in $G - (V(M_G) \cup (I_S - \{i\}))$, say F_G . Let $iv \in F_G$. We now consider \bar{G} . Since $n_S = 0$, $G[S]$ is independent and thus $\bar{G}[\bar{S}]$ is a complete graph of odd order s . Therefore, $n_{\bar{S}} = \frac{1}{2}(s - 1)$ and $i_{\bar{S}} = 1$. By Equation 2, $l = m_G + s - 1$. So $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = l - (n_{\bar{S}} + i_{\bar{S}}) = m_G + s - 1 - (\frac{1}{2}(s - 1) + 1) = m_G + \frac{1}{2}(s - 3)$.

We next show that $m_G + \frac{1}{2}(s - 3) \geq 1$. Suppose to the contrary that $m_G + \frac{1}{2}(s - 3) = 0$. Since $m_G \geq 1$ and s is a positive odd integer, it follows that $m_G = 1$ and $s = 1$. By Equation 2, $l + 1 = m_G + m_{\bar{G}} + s = 1 + 0 + 1 = 2$. Thus $l = 1$, contradicting the fact that $l \geq 4$. Hence, $m_G + \frac{1}{2}(s - 3) \geq 1$ as required.

Therefore, $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + \frac{1}{2}(s - 3) \geq 1$. By Lemma 10.2(b), $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$ is 1-factor-critical. Recall that $iv \in F_G$. Clearly, $\bar{v} \notin V(M_{\bar{G}})$ since $m_{\bar{G}} = 0$. So there is a perfect matching in $\bar{G} - ((V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}}) \cup \{\bar{v}\})$, say $F_{\bar{G}}$. Hence, $M \cup (F_G - \{iv\}) \cup F_{\bar{G}} \cup \{v\bar{v}\}$ is a perfect matching in $G\bar{G}$ containing M as required. This proves Subcase 2.1.

Subcase 2.2 : $m_{\bar{G}} + n_S \geq 2$. By Equation 3, $l - (m_G + n_S + i_S) = m_{\bar{G}} + n_S - 1 \geq 1$. By Lemma 10.2(b), $G - (V(M_G \cup N_S) \cup I_S)$ is 1-factor-critical.

We now consider \bar{G} . By Equation 4, $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1$.

We first suppose that $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1 \geq 1$. By Lemma 10.2(b), $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$ is 1-factor-critical. Let $x \in V(G)$ such that $x, \bar{x} \notin V(M)$. Clearly, x exists because $|V(M_G \cup M_{\bar{G}}) \cup S| \leq 2l + 1$ and G and \bar{G} are of order at least $2l + 2$. Since $G - (V(M_G \cup N_S) \cup I_S)$ and $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$ are 1-factor-critical, It follows that there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S \cup \{x\})$, say F_G , and there is a perfect matching in $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}} \cup \{\bar{x}\})$, say $F_{\bar{G}}$. Hence, $M \cup F_G \cup F_{\bar{G}} \cup \{x\bar{x}\}$ is a perfect matching in $G\bar{G}$ containing M as required.

So we next suppose that $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1 = 0$. It follows that $n_{\bar{S}} = 0$ and $m_G = 1$ since $m_G \geq 1$. Thus $i_{\bar{S}} = \bar{s}$ and $m_{\bar{G}} \leq m_G = 1$. Put $M_G = \{xy\}$. Since \bar{G} is l_2 -extendable, for $l_2 \geq 4$, by Theorem 2.8(b), \bar{G} is 5-connected. So $\{\bar{x}, \bar{y}\} \cup V(M_{\bar{G}})$ is not a cutset of \bar{G} since $|\{\bar{x}, \bar{y}\} \cup V(M_{\bar{G}})| \leq 4$. There is an edge joining a vertex in $V(\bar{G}) - (\{\bar{x}, \bar{y}\} \cup V(M_{\bar{G}}))$, say \bar{u} , and a vertex in \bar{S} , say \bar{w} . Because $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = 0$ and $i_{\bar{S}} = \bar{s}$, it follows that $l - (m_{\bar{G}} + n_{\bar{S}} + 1 + (i_{\bar{S}} - 1)) = l - (m_{\bar{G}} + n_{\bar{S}} + 1 + (\bar{s} - 1)) = 0$. By Lemma 10.2(a), there is a perfect matching in $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}} \cup \{\bar{u}\bar{w}\}) \cup (\bar{S} - \{\bar{w}\}))$, say $F_{\bar{G}}$. Since $G - (V(M_G \cup N_S) \cup I_S)$ is 1-factor-critical and $u \notin V(M_G)$, it follows that there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S \cup \{u\})$, say F_G . Hence, $M \cup F_G \cup F_{\bar{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\bar{G}$ containing M as required. This proves Subcase 2.2.

Subcase 2.3 : $m_{\bar{G}} + n_S = 1$. By Equation 3 and the fact that i_S is odd, $m_G + n_S = l + 1 - (m_{\bar{G}} + n_S + i_S) \leq l - 1$. We distinguish 2 subcases according to $m_{\bar{G}}$ and n_S .

Subcase 2.3.1 : $m_{\bar{G}} = 0$ and $n_S = 1$. Observe that $G[V(M_G \cup N_S)]$ contains $m_G + n_S \leq l - 1$ independent edges and $|V(M_G \cup N_S)| = 2(m_G + n_S) = (m_G + n_S) + m_G + n_S \leq l - 1 + (m_G + n_S)$. It follows by Lemma 10.1 that $V(M_G \cup N_S)$ is not a cutset of G . Then there are a vertex $u \in V(G) - (V(M_G) \cup S)$ and a vertex $z \in I_S$ such that $uz \in E(G)$.

Since $l - ((m_G + n_S + 1) + (i_S - 1)) = l - (m_G + n_S + i_S) = m_{\bar{G}} + n_S - 1 = 0$, by Lemma 10.2(a), there is a perfect matching in $G - (V(M_G \cup N_S \cup \{uz\}) \cup (I_S - \{z\}))$, say F_G . We now consider \bar{G} . We next show that $m_G + n_{\bar{S}} \geq 2$. Suppose to the contrary that $m_G + n_{\bar{S}} = 1$. Since $m_G \geq 1$, $n_{\bar{S}} = 0$ and $m_G = 1$. By Equation 3, $l + 1 = m_G + m_{\bar{G}} + 2n_S + i_S = 3 + i_S$. So $i_S = l - 2 \geq 4 - 2 = 2$. It follows that $\bar{G}[\bar{S}]$ contains K_2 as an induced subgraph. Thus $n_{\bar{S}} \geq 1$, contradicting the fact that $n_{\bar{S}} = 0$. Hence, $m_G + n_{\bar{S}} \geq 2$.

By Equation 3, $l - (m_{\bar{G}} + n_{\bar{S}} + i_{\bar{S}}) = m_G + n_{\bar{S}} - 1 \geq 1$. By Lemma 10.2(b), $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}})$ is 1-factor-critical. Recall that $m_{\bar{G}} = 0$. So $\bar{u} \notin V(M_{\bar{G}})$. Therefore, there is a perfect matching in $\bar{G} - (V(M_{\bar{G}} \cup N_{\bar{S}}) \cup I_{\bar{S}} \cup \{\bar{u}\})$, say $F_{\bar{G}}$. Hence, $M \cup F_G \cup F_{\bar{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\bar{G}$ containing M as required. This completes the proof of Subcase 2.3.1.

Subcase 2.3.2 : $m_{\bar{G}} = 1$ and $n_S = 0$. Put $m_{\bar{G}} = \{\bar{x}_1\bar{x}_2\}$. Note that $m_G + n_{\bar{S}} \geq 2$ since $|m| = l + 1 \geq 5$ and $n_S = 0$. If there is a vertex $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$ such that $uz \in E(G)$ for some $z \in S$, then by applying similar argument as in the proof of Subcase 2.3.1, there is a perfect matching in $G\bar{G}$ containing M as required. So we now suppose that there is no vertex $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$ such that $uz \in E(G)$ for some $z \in S$. Thus $V(M_G) \cup \{x_1, x_2\}$ is a cutset of G and $\{x_1, x_2\}$ is a cutset of $G - V(M_G)$. We next show that $s = 1$. Suppose to the contrary that $s \geq 3$. By Equation 2, $m_G = l + 1 - m_{\bar{G}} - s = l - s \leq l - 3$. By Observation 2.17, $G - V(M_G)$ is $(l - m_G)$ -extendable. Because $l - m_G \geq 3$, by Theorem 2.8(b), $G - V(M_G)$ is 4-connected, contradicting the fact that $\{x_1, x_2\}$ is a cutset of $G - V(M_G)$. Hence, $s = 1$. Put $S = \{z\}$. Therefore, $zu \notin E(G)$ for $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$. So $N_G(z) \subseteq V(M_G) \cup \{x_1, x_2\}$.

By Equation 2, $m_G = l + 1 - m_{\bar{G}} - s = l - 1$. By Observation 2.17, $G' = G - V(M_G)$ is 1-extendable. By Theorem 2.8(b), G' is 2-connected. Therefore, $N_{G'}(z) = \{x_1, x_2\}$ and $\deg_{G'}(z) = 2$. By Lemma 10.6, there is a vertex $u \in V(G')$ such that $uz \notin E(G')$ and $G' - \{u, z\}$ contains a perfect matching, say F_G . We now consider \bar{G} . Since $l \geq 4$, $m_{\bar{G}} = 1$ and $\bar{s} = s = 1$, it follows that $l - (m_{\bar{G}} + \bar{s}) = l - 2 \geq 2$. By Lemma 10.2(b), $\bar{G}' = \bar{G} - V(M_{\bar{G}} \cup \bar{S})$ is 1-factor-critical. Then there is a perfect matching in $\bar{G}' - \{\bar{u}\}$, say $F_{\bar{G}}$. Hence, $M \cup F_G \cup F_{\bar{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\bar{G}$ containing M as required. This completes the proof of Subcase 2.3.2. and thus completes the proof of our theorem. \square

We now turn our attention to the extendability of $G\bar{G}$ when G or \bar{G} is l -extendable for $1 \leq l \leq 3$.

We first provide an example of a graph G where both G and \bar{G} are 1-extendable but $G\bar{G}$ is not 2-extendable. Let G be a graph where $V(G) = \{u_i \mid \text{for } 1 \leq i \leq 4\} \cup \{v_i \mid \text{for } 1 \leq i \leq 6\}$ and $E(G) = \{u_i u_{i+1} \mid \text{for } 1 \leq i \leq 3\} \cup \{v_i v_{i+1} \mid \text{for } 1 \leq i \leq 6\}$ where the subscript is read modulo 6 $\cup \{u_1 v_1, u_4 v_2, u_4 v_3\}$. Observe that $G[\{u_i \mid \text{for } 1 \leq i \leq 4\}]$ and $G[\{v_i \mid \text{for } 1 \leq i \leq 6\}]$ are a path of order 4 and a cycle of order 6, respectively. It is routine to verify that G and \bar{G} are 1-extendable. But $G\bar{G}$ is not 2-extendable since $\{\bar{u}_2 \bar{u}_4, u_3 \bar{u}_3\}$ cannot be extended to a perfect matching in $G\bar{G}$.

We now scope our attention to extendability of $G\bar{G}$ where G is l_1 -extendable and \bar{G}

is l_2 -extendable for $l_1 \geq 2$ and $l_2 \geq 2$. We first consider the case $l_1 = 2$ and $l_2 \geq 2$. We begin with the following lemma. Recall that if $\phi \neq \{x_1, \dots, x_t\} \subseteq V(G)$, then $\{\bar{x}_1, \dots, \bar{x}_t\} \subseteq V(\bar{G})$ is denoted by \bar{X} and vice versa.

Lemma 14.4. *Let G and \bar{G} be 2-extendable non-bipartite graphs of order $p \geq 10$ and let $M = \{x_1x_2, \bar{y}_1\bar{y}_2, z\bar{z}\}$ be a matching of size 3 in $G\bar{G}$, where $\{x_1, x_2, z\} \subseteq V(G)$ and $\{\bar{y}_1, \bar{y}_2, \bar{z}\} \subseteq V(\bar{G})$. Then there is a perfect matching in $G\bar{G}$ containing M .*

Proof. Suppose to the contrary that there is no perfect matching in $G\bar{G}$ containing M . By Theorem 2.7, there is a cutset $T \subseteq V(G\bar{G}) - V(M)$ such that $c_o(G\bar{G} - (V(M) \cup T)) > |T|$. By parity, $c_o(G\bar{G} - (V(M) \cup T)) \geq |T| + 2$. Put $S = T \cup V(M)$. So $c_o(G\bar{G} - S) \geq |S| - 4$. Put $A = S \cap V(G)$, $\bar{B} = S \cap V(\bar{G})$ and $C = V(G) - (A \cup B)$. Observe that $|A| \geq 3$ and $|\bar{B}| \geq 3$.

By Theorem 2.15, G and \bar{G} are bicritical. Thus, by Theorem 2.13, $c_o(G - A) \leq |A| - 2$ and $c_o(\bar{G} - \bar{B}) \leq |\bar{B}| - 2$. We first show that $c_o(G - A) = |A| - 2$ and $c_o(\bar{G} - \bar{B}) = |\bar{B}| - 2$. Suppose to the contrary that $c_o(G - A) < |A| - 2$. By parity, $c_o(G - A) \leq |A| - 4$. It then follows by Lemma 14.1 that $c_o(G\bar{G} - S) \leq c_o(G - A) + c_o(\bar{G} - \bar{B}) \leq |A| + |\bar{B}| - 6$, contradicting the fact that $c_o(G\bar{G} - S) \geq |S| - 4$. Hence, $c_o(G - A) = |A| - 2$. Similarly, $c_o(\bar{G} - \bar{B}) = |\bar{B}| - 2$.

Since G and \bar{G} are 2-extendable, by Theorem 2.12(b), $G[A]$ and $\bar{G}[\bar{B}]$ contain at most one independent edge. Because $\{x_1, x_2, z\} \subseteq A$ and $\{\bar{y}_1, \bar{y}_2, \bar{z}\} \subseteq \bar{B}$, $G[A]$ and $\bar{G}[\bar{B}]$ contain exactly 1 independent edge. By Lemma 10.3, $G - A$ and $\bar{G} - \bar{B}$ contain no even components. If $A \cup B \neq V(G)$, then, by Lemma 14.1, $c_o(G\bar{G} - S) = c_o(G\bar{G} - (A \cup \bar{B})) \leq |A| + |\bar{B}| - 6 = |S| - 6$, again a contradiction. Hence, $A \cup B = V(G)$. Observe that if $c_o(G - A) \geq 4$, $G[B] = G - A$ contains at least 4 independence vertices and thus $\bar{G}[\bar{B}]$ contains a matching of size at least two, a contradiction. Hence, $c_o(G - A) \leq 3$. Similarly, $c_o(\bar{G} - \bar{B}) \leq 3$ and each component of $\bar{G} - \bar{B}$ is singleton otherwise $G[A] = G - B$ contains at least 2 independent edges, a contradiction. Therefore, $c_o(G[B - A]) = c_o(G - A) \leq 3$ and $\bar{G}[\bar{A} - \bar{B}] = c_o(\bar{G} - \bar{B}) \leq 3$. Since $c_o(G - A) = |A| - 2$ and $c_o(\bar{G} - \bar{B}) = |\bar{B}| - 2$, it follows that $|A| = 2 + c_o(G - A) \leq 5$ and $|\bar{B}| = 2 + c_o(\bar{G} - \bar{B}) \leq 5$. Because $z \in A \cap B$, $|A \cup B| = |A| + |\bar{B}| - |A \cap B| \leq 5 + 5 - 1 \leq 9$, contradicting the fact that $|V(G)| = p \geq 10$. This completes the proof of our lemma. \square

The next theorem shows that if G is a 2-extendable non-bipartite graph and \bar{G} is a l -extendable non-bipartite graph of order $p \geq 10$ and $l \geq 2$, then $G\bar{G}$ is 3-extendable.

Theorem 14.5. *Let G be a 2-extendable non-bipartite graph of order $p \geq 10$. If \bar{G} is l -extendable non-bipartite for some positive integer $l \geq 2$, then $G\bar{G}$ is 3-extendable.*

Proof. By Theorem 2.8(b), \bar{G} is 2-extendable non-bipartite graph. Let M be a matching of size 3 in $G\bar{G}$. Put $M_G = M \cap E(G)$, $M_{\bar{G}} = M \cap E(\bar{G})$ and $M_{G\bar{G}} = M - (M_G \cup M_{\bar{G}})$. Further, put $m_G = |M_G|$, $m_{\bar{G}} = |M_{\bar{G}}|$ and $m_{G\bar{G}} = |M_{G\bar{G}}|$. If $m_{G\bar{G}} = 0$ or $m_{G\bar{G}} = 3$, then, by Lemma 14.2, there is a perfect matching in $G\bar{G}$ containing M as required. So we now consider $1 \leq m_{G\bar{G}} \leq 2$. We distinguish 2 cases according to $m_{G\bar{G}}$.

Case 1: $m_{G\bar{G}} = 1$. If $m_G = m_{\bar{G}} = 1$, then, by Lemma 14.4, there is a perfect matching in $G\bar{G}$ containing M as required. So we suppose without loss of generality that $m_G = 2$,

$m_{\overline{G}} = 0$. By applying similar arguments as in the proof of Subcase 2.1 in Theorem 14.3, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 2: $m_{G\overline{G}} = 2$. By applying similar arguments as in the proof of Case 1 in Theorem 14.3, there is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our theorem. \square

We point out here that the bound on the order of graphs in Theorem 14.5 is best possible and the hypothesis that G and \overline{G} are non-bipartite is essential. Let G be a 3-regular bipartite graph of order 8 with bipartition (X, Y) where $X = \{x_i | 1 \leq i \leq 4\}$ and $Y = \{y_i | 1 \leq i \leq 4\}$ and $E(G) = \{x_i y_j | 1 \leq i \neq j \leq 4\}$. It is not difficult to show that $\overline{G} \cong K_4 \times K_2$ and both G and \overline{G} are 2-extendable. However, $G\overline{G}$ is not 3-extendable since $\{x_1 \bar{x}_1, x_2 y_1, \bar{y}_2 \bar{y}_3\}$ cannot be extended to a perfect matching in $G\overline{G}$.

We finally turn our attention to 3-extendable graphs.

Lemma 14.6. *Suppose G and \overline{G} are 3-extendable non-bipartite graphs of order $p \geq 8$. Let $\{x, y, z_1, z_2, z_3\} \subseteq V(G)$ and $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} \subseteq V(\overline{G})$ such that $G[\{z_1, z_2, z_3\}] \cong K_3$. Further, let $M = \{xy, z_1 \bar{z}_1, z_2 \bar{z}_2, z_3 \bar{z}_3\}$ be a matching of size 4 in $G\overline{G}$. Then there is a perfect matching in $G\overline{G}$ containing M .*

Proof. Suppose there is no perfect matching in $G\overline{G} - V(M)$. Then by Theorem 2.7, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (T \cup V(M))) > |T|$. By parity, $c_o(G\overline{G} - (T \cup V(M))) \geq |T| + 2$. Put $S = T \cup V(M)$. So $c_o(G\overline{G} - S) \geq |S| - 6$. Since $G\overline{G}$ contains a perfect matching, by Theorem 2.7, $c_o(G\overline{G} - S) \leq |S|$. Thus $|S| - 6 \leq c_o(G\overline{G} - S) \leq |S|$. Put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup \overline{B})$. Clearly, $\{z_1, z_2, z_3\} \subseteq A \cap \overline{B}$. By Lemma 12.2(b), $c_o(G\overline{G} - S) \leq |S| - 6$. So $c_o(G\overline{G} - S) = |S| - 6$.

Since $xy, z_1 z_2 \in E(G)$, by Lemma 10.3, $c_o(G - A) \leq |A| - 4$. On the other hand, since \overline{G} is 3-extendable non-bipartite graph, by Theorems 2.8(a) and 2.15, \overline{G} is bicritical. Therefore, by Theorem 2.13, $c_o(\overline{G} - \overline{B}) \leq |\overline{B}| - 2$. We first show that $c_o(G - A) = |A| - 4$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Suppose to the contrary that $c_o(G - A) \neq |A| - 4$. By parity, $c_o(G - A) \leq |A| - 6$. By Lemma 14.1(a), $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| - 6 + |\overline{B}| - 2 = |S| - 8$, a contradiction. Hence, $c_o(G - A) = |A| - 4$. By similar argument, $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. By Lemma 10.3, $G - A$ contains no even components. We next show that $\overline{G} - \overline{B}$ contains no even components. Suppose this is not the case. Then $\overline{G} - \overline{B}$ contains an even component, say \overline{D} . Let $\bar{b}\bar{d} \in E(\overline{G})$ such that $\bar{b} \in \overline{B}$ and $\bar{d} \in V(\overline{D})$. By Corollary 2.20, $\overline{G}' = \overline{G} - \{\bar{b}, \bar{d}\}$ is 2-extendable non-bipartite. By Theorem 2.15, \overline{G}' is bicritical. Since $c_o(\overline{G} - (\overline{B} \cup \{\bar{d}\})) = |\overline{B}| - 1$, $c_o(\overline{G}' - (\overline{B} - \{\bar{b}\})) = |\overline{B} - \{\bar{b}\}|$, contradicting Theorem 2.13. Hence, $\overline{G} - \overline{B}$ contains no even components.

If $A \cup \overline{B} \neq V(G)$, then by Lemma 14.1, $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq c_o(G - A) + c_o(\overline{G} - \overline{B}) - 2 = |A| + |\overline{B}| - 8 = |S| - 8$, a contradiction. So $A \cup \overline{B} = V(G)$.

Note that $G[A - \overline{B}]$ contains the edge xy . We first show that $G[A - \overline{B}]$ contains exactly one independent edge. Suppose $G[A - \overline{B}]$ contains 2 independent edges. Since $z_1 z_2 \in E(G[A \cap \overline{B}])$, there are at least 3 independent edges in $G[A]$. Therefore, by Lemma 10.3, $c_o(G - A) \leq |A| - 6$, contradicting the fact that $c_o(G - A) = |A| - 4$. Hence, $G[A - \overline{B}]$

contains exactly one independent edge. We next show that $\overline{G[\overline{B}]}$ contains no edges. Suppose to the contrary that \overline{B} contains an edge $\overline{u_1\overline{u_2}}$. By Corollary 2.20, $\overline{G} - \{\overline{u_1}, \overline{u_2}\}$ is 2-extendable non-bipartite graph. By Theorem 2.15, $\overline{G} - \{\overline{u_1}, \overline{u_2}\}$ is bicritical. Then, by Theorem 2.13, $c_o(\overline{G} - \overline{B}) = c_o((\overline{G} - \{\overline{u_1}, \overline{u_2}\}) - (\overline{B} - \{\overline{u_1}, \overline{u_2}\})) \leq |\overline{B} - \{\overline{u_1}, \overline{u_2}\}| - 2 = |\overline{B}| - 4$, contradicting the fact that $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Hence, $\overline{G[\overline{B}]}$ contains no edges and $\overline{G[\overline{B}]}$ is independent. So $G[B]$ and $G[B - A]$ are clique and thus $c_o(G[B - A]) \leq 1$.

Therefore, $|A| - 4 = c_o(G - A) = c_o(G[B - A]) \leq 1$. So $|A| \leq 5$. If $\overline{G[\overline{A} - \overline{B}]}$ contains at least 4 components, then $G[A - B]$ contains at least two independent edges. But this contradicts the fact that $G[A - B]$ contains exactly one independent edges. Hence, $\overline{G[\overline{A} - \overline{B}]}$ contains at most 3 components. Therefore, $c_o(\overline{G[\overline{A} - \overline{B}]}) = c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2 \leq 3$. Hence, $|B| = |\overline{B}| \leq 5$. It follows that $|V(G)| = |A \cup B| = |A| + |B| - |A \cap B| \leq 5 + 5 - 3 = 7$, a contradiction. This proves our lemma. \square

Lemma 14.7. *Suppose G and \overline{G} are 3-extendable non-bipartite graphs of order $p \geq 8$. Let $\{x, y, z_1, z_2, z_3\} \subseteq V(G)$ and $\{\overline{z}_1, \overline{z}_2, \overline{z}_3\} \subseteq V(\overline{G})$ such that $G[\{z_1, z_2, z_3\}] \not\cong K_3$. Further, let $M = \{xy, z_1\overline{z}_1, z_2\overline{z}_2, z_3\overline{z}_3\}$ be a matching of size 4 in $G\overline{G}$. Then there is a perfect matching in $G\overline{G}$ containing M .*

Proof. Suppose $M = \{xy, z_1\overline{z}_1, z_2\overline{z}_2, z_3\overline{z}_3\}$ where $x, y \in V(G)$. Since $G[\{z_1, z_2, z_3\}] \not\cong K_3$, we may suppose that $z_1z_2 \notin E(G)$. Since $xy \in E(G)$, by Lemma 10.2(a), there is a perfect matching in $G - \{x, y, z_1, z_2\}$, say F_G . Let $z_3w \in F_G$. Again, because $\overline{z}_1\overline{z}_2 \in E(\overline{G})$, by Lemma 10.2(a), there is a perfect matching in $\overline{G} - \{\overline{z}_1, \overline{z}_2, \overline{w}, \overline{z}_3\}$, say $F_{\overline{G}}$. Thus $M \cup (F_G - \{z_3w\}) \cup F_{\overline{G}} \cup \{w\overline{w}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our lemma. \square

Theorem 14.8. *Let G be a 3-extendable non-bipartite graph of order $p \geq 8$. If \overline{G} is l -extendable non-bipartite for some positive integer $l \geq 3$, then $G\overline{G}$ is 4-extendable.*

Proof. By Theorem 2.8(b), \overline{G} is 3-extendable non-bipartite graph. Let M be a matching of size 4 in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Without loss of generality, suppose $|M_G| \geq |M_{\overline{G}}|$. If $M_{G\overline{G}} = \phi$ or $M_{G\overline{G}} = M$, then, by Lemma 14.2, there is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that $M_{G\overline{G}} \neq \phi$ and $M_{G\overline{G}} \neq M$. Therefore, $1 \leq |M_{G\overline{G}}| \leq 3$. We distinguish 3 cases according to $|M_{G\overline{G}}|$.

Case 1: $|M_{G\overline{G}}| = 1$. By applying similar arguments as in the proof of Subcase 2.1 (if $|M_{\overline{G}}| = 0$) or Subcase 2.3 (if $|M_{\overline{G}}| = 1$) in Theorem 14.3, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 2: $|M_{G\overline{G}}| = 2$. By applying similar arguments as in the proof of Case 1 in Theorem 14.3, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 3: $|M_{G\overline{G}}| = 3$. Then, $|M_G| = 1$ and $|M_{\overline{G}}| = 0$. So, by Lemmas 14.6 and 14.7, there is a perfect matching in $G\overline{G}$ containing M as required.

This completes the proof of our theorem. \square

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A characterization of independent domination critical graphs with a cutvertex

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Abstract

Let $i(G)$ denote the minimum cardinality of an independent dominating set for G . A graph G is k - i -critical if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G . In this paper, we show that if G is a connected k - i -critical graph, for $k \geq 3$, with a cutvertex u , then the number of components of $G - u$, $\omega(G - u)$, is at most $k - 1$ and there are at most two non-singleton components. Further, if $\omega(G - u) = k - 1$, then a characterization of such graphs is given.

keyword: independent domination, critical, cutvertex
MSC2000: 05C69

1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by \overline{G} . For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. S is independent if no two vertices of S are adjacent. The number of components of G is denoted by $\omega(G)$. A vertex v of G is a cutvertex if $\omega(G - v) > \omega(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G denoted by $N_G(v)$ is the set of all vertices of $V(G) - \{v\}$ which are adjacent to v . $N_G[v]$ is denoted the closed neighborhood

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of v , i.e., $N_G[v] = N_G(v) \cup \{v\}$. The non-neighborhood of v in G denoted by $\overline{N}_G(v)$ is $V(G) - N_G[v]$. For $S \subseteq V(G)$, $N_G(v) \cap S$ and $N_G[v] \cap S$ are denoted by $N_S(v)$ and $N_S[v]$, respectively. Further, $N_G(S)$ denotes $\bigcup_{x \in S} N_G(x)$ and $N_G[S]$ denotes $\bigcup_{x \in S} N_G[x]$. For simplicity, if S is a subgraph of G , then we also denote $N_G(v) \cap V(S)$ and $N_G[v] \cap V(S)$ by $N_S(v)$ and $N_S[v]$, respectively.

For subsets S and T of $V(G)$, we say that S dominates T , denoted by $S \succ T$, if $T \subseteq N_G[S]$. If $S \succ T$ where $S = \{s\}$, then we write $s \succ T$ instead of $\{s\} \succ T$. Further, if $T = V(H)$ where H is a subgraph of G , then we also write $S \succ H$ instead of $S \succ V(H)$ and we say that S is a *dominating set* for H . Thus S is a dominating set for G if each vertex of $V(G)$ is either in S or adjacent to some vertex of S . The minimum cardinality of a dominating set for G is called the domination number of G and denoted by $\gamma(G)$.

For a subgraph H of G , if $S \succ H$ and S is independent, then we say that S is an *independent dominating set* for H and denoted by $S \succ_i H$. Thus S is an independent dominating set for G if $S \succ_i G$. The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and denoted by $i(G)$. Observe that for any graph G , $\gamma(G) \leq i(G)$ and if $\gamma(G) = 1$, then $i(G) = 1$.

In 1994, Ao [4] introduced the concept so called “independent domination critical”. A graph G is k - i -critical if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G . It is easy to see that the only 1- i -critical graphs are K_n for some positive integer n . Ao [4] proved that G is 2- i -critical if and only if $G \cong \bigcup_{i=1}^n K_{1,r_i}$ for some positive integers r_i and n . The problem that arises is that of characterizing connected k - i -critical graphs for $k \geq 3$. Up to date, there is no characterization of such graphs. Further, there are not many known results concerning properties of them especially when $k \geq 4$. This might be because the problem is getting harder and harder when k is getting bigger. So it makes sense to add some additional hypothesis in order to investigate connected k - i -critical graphs for $k \geq 3$. In this paper, we concentrate on connected k - i -critical graphs, for $k \geq 3$, with a cutvertex. We provide some necessary conditions for such graphs. In fact, we show that if G is a connected k - i -critical graphs, for $k \geq 3$, with a cutvertex u , then $\omega(G - u) \leq k - 1$ and there are at most two non-singleton components. Further, if $\omega(G - u) = k - 1$, then a characterization of such graphs is given.

We conclude this section by pointing out that domination is one area in graph theory that can be applied to different fields. Most of problems concern facility location such as police stations, hospitals. They can also be found in monitoring networks such as communication, electric. A reader is directed to [9, 10]. Further, there are several concepts on domination critical such as (ordinary) domination critical, total domination critical, connected domination critical. These critical graphs have the same situation with k - i -critical graphs in the sense that it is much more difficult to study these graphs when its (ordinary) domination (total domination or connected domination) number is getting bigger. Some papers on these topics are in [1, 2, 3, 5, 6, 7, 8, 11, 12, 13, 14].

2 Preliminaries results

In this section we state some results that we make use of in establishing our main results. We begin with some terminology. For a pair of non-adjacent vertices u and v of G , I_{uv} denotes a minimum independent dominating set for $G + uv$. Our first result follows immediately from the definition of k - i -critical graphs.

Lemma 2.1. *Let G be a connected k - i -critical graph and let u and v be non-adjacent vertices of G . Then $|I_{uv}| = k - 1$, $|I_{uv} \cap \{u, v\}| = 1$ and $I_{uv} \cap (N_G(u) \cup N_G(v)) = \emptyset$. \square*

The next result provides an upper bound of the diameter of connected 3- i -critical graphs.

Lemma 2.2. [4]

The diameter of a connected 3- i -critical graph is at most 3. \square

Before we state the last result, we need one more definition. A graph G is k - i -vertex-critical if $i(G) = k$ and for each $u \in V(G)$, $i(G - u) < k$. It is easy to see that if G is k - i -vertex-critical, then $i(G - u) = k - 1$.

Lemma 2.3. [4]

1. *A graph G is 2- i -critical if and only if $\overline{G} \cong \bigcup_{i=1}^n K_{1,r_i}$ for some positive integers r_i and n .*
2. *A graph G is 2- i -vertex-critical if and only if G is isomorphic to a complete graph without a perfect matching. \square*

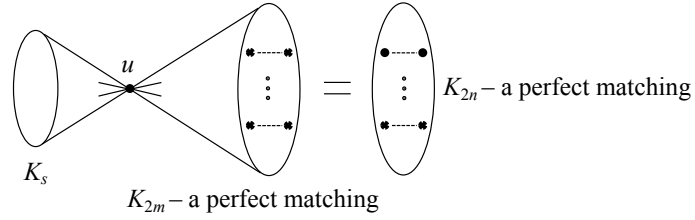
3 Classes of connected k - i -critical graphs

In this section, we provide seven classes of connected k - i -critical graphs with a cutvertex, four of them for $k = 3$, one for $k = 4$ and the another two for $k \geq 4$. We begin with four classes of connected 3- i -critical graphs.

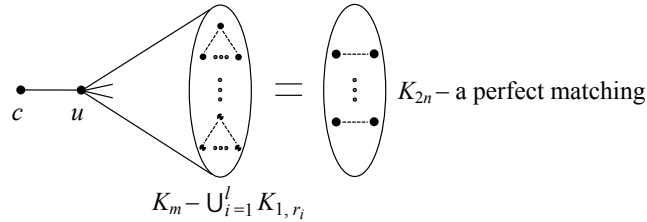
I. The class \mathcal{G}_1

For positive integers n , m and $s \geq 2$, define a graph $G \in \mathcal{G}_1$ of order $2n + 2m + s + 1$ as follows. Set $V(G) = \{u\} \cup X \cup Y \cup Z$ where $|X| = 2n$, $|Y| = 2m$ and $|Z| = s$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $G[Y] = K_{2m}$ - a perfect matching and $G[Z] = K_s$. Further, join u to each vertex of $Y \cup Z$ and finally join each vertex of Y to every vertex of X . This defines the class \mathcal{G}_1 . Figure 1 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_1$ is 3- i -critical. Note that in our diagram a “double line” denotes the join.

II. The class \mathcal{G}_2

Figure 1: The structure of a graph in the class \mathcal{G}_1

For positive integers n and $m \geq 2$, define a graph $G \in \mathcal{G}_2$ of order $2n + m + 2$ as follows. Set $V(G) = \{u, c\} \cup X \cup Y$ where $|X| = 2n$, $|Y| = m$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $\overline{G[Y]} = \bigcup_{i=1}^l K_{1, r_i}$ such that $m = l + \sum_{i=1}^l r_i$, $m \geq 2l$ and $l \geq 1$. Further, join u to each vertex of $\{c\} \cup Y$ and finally join each vertex of Y to every vertex of X . This defines the class \mathcal{G}_2 . Figure 2 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_2$ is 3- i -critical.

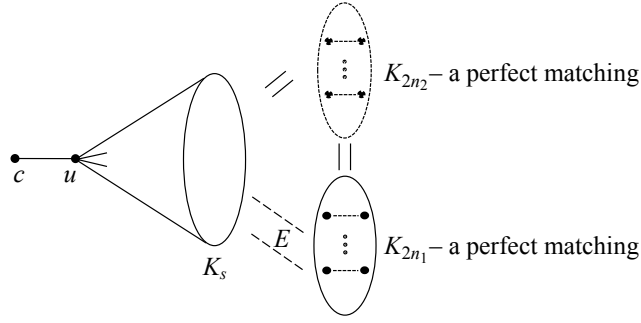
Figure 2: The structure of a graph in the class \mathcal{G}_2

III. The class \mathcal{G}_3

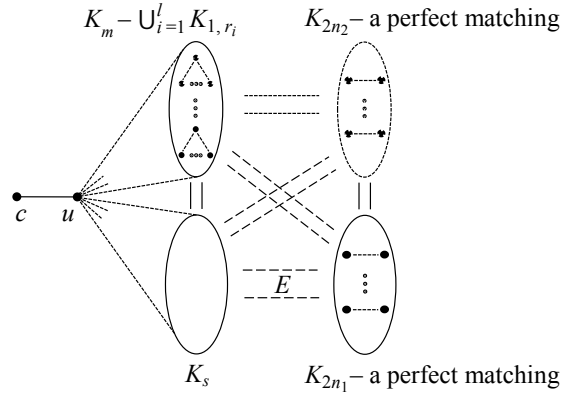
For positive integers n_1 and $s \geq 2n_1$ and non-negative integer n_2 , define a graph $G \in \mathcal{G}_3$ of order $2n_1 + 2n_2 + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X_1 \cup X_2 \cup Z$ where $|X_1| = 2n_1$, $|X_2| = 2n_2$ and $|Z| = s$. The edges of G are defined as follows. $G[X_1 \cup X_2] = K_{2n_1 + 2n_2} - (F_1 \cup F_2)$ where F_i is a perfect matching in $G[X_i]$ for $1 \leq i \leq 2$ and $G[Z] = K_s$. Further, join u to each vertex of $\{c\} \cup Z$ and join each vertex of Z to every vertex of X_2 (if $X_2 \neq \emptyset$). Finally, we add the set of edges E between Z and X_1 . Each vertex of Z is joined to $2n_1 - 1$ vertices of X_1 in such a way that each vertex of X_1 is a non-neighbor of some vertex of Z . This defines the class \mathcal{G}_3 . Figure 3 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_3$ is 3- i -critical.

IV. The class \mathcal{G}_4

For positive integers n_1 , $s \geq 2n_1$ and $m \geq 2$ and non-negative integer n_2 , define a graph $G \in \mathcal{G}_4$ of order $2n_1 + 2n_2 + m + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X_1 \cup X_2 \cup Y \cup Z$ where $|X_1| = 2n_1$, $|X_2| = 2n_2$, $|Y| = m$ and $|Z| = s$.

Figure 3: The structure of a graph in the class \mathcal{G}_3

The edges of G are defined as follows. $G[X_1 \cup X_2] = K_{2n_1+2n_2} - (F_1 \cup F_2)$ where F_i is a perfect matching in $G[X_i]$ for $1 \leq i \leq 2$, $\overline{G[Y]} = \bigcup_{i=1}^l K_{1,r_i}$ such that $m = l + \sum_{i=1}^l r_i$, $m \geq 2l$ and $l \geq 1$ and $G[Z] = K_s$. Further, join u to each vertex of $\{c\} \cup Y \cup Z$, join each vertex of Y to every vertex of $Z \cup X_1 \cup X_2$ and join each vertex of Z to every vertex of X_2 (if $X_2 \neq \emptyset$). Finally, we add the set of edges E between Z and X_1 as defined in the class \mathcal{G}_3 . This defines the class \mathcal{G}_4 . Figure 4 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_4$ is 3- i -critical.

Figure 4: The structure of a graph in the class \mathcal{G}_4

Our next class is a class of connected 4- i -critical graphs with a cutvertex.

V. The class \mathcal{G}_5

For positive integers n , m and $s \geq 2$, define a graph $G \in \mathcal{G}_5$ of order $2n + 2m + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X \cup Y \cup Z$ where $|X| = 2n$, $|Y| = 2m$ and $|Z| = s$. The edges of G are defined as follows. $G[X] = K_{2n} - a$

perfect matching, $G[Y] = K_{2m}$ - a perfect matching and $G[Z] = K_s$. Further, join u to each vertex of $Y \cup \{c\}$ and $s - 1$ vertices of Z and finally join each vertex of Y to every vertex of X . This defines the class \mathcal{G}_5 . Figure 5 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_5$ is 4- i -critical.

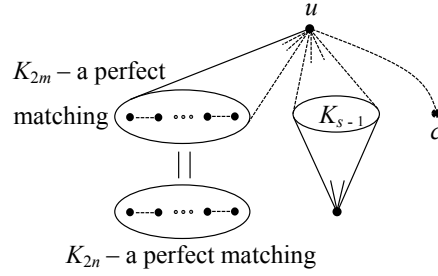


Figure 5: The structure of a graph in the class \mathcal{G}_5

We conclude this section by constructing two classes of connected k - i -critical graphs with a cutvertex for $k \geq 4$.

VI. The class \mathcal{G}_6

For positive integers s , n , and k where $s \geq n \geq k - 1 \geq 3$, define a graph $G \in \mathcal{G}_6$ of order $s + n + k - 1$ as follows. Set $V(G) = \{u\} \cup X \cup Z \cup W$ where $|X| = n$, $|Z| = s$ and $|W| = k - 2$. The edges of G are defined as follows. $G[X]$ is both $(k - 1)$ - i -critical and $(k - 1)$ - i -vertex-critical and $G[W] = \overline{K}_{k-2}$. Further, join u to each vertex of $W \cup Z$, and then add the set of edges E between Z and X as similar as defined in the class \mathcal{G}_3 . That is, each vertex of Z is joined to $n - 1$ vertices of X in such a way that each vertex of X is a non-neighbor of some vertex of Z . This defines the class \mathcal{G}_6 . Figure 6 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_6$ is k - i -critical for $k \geq 4$.

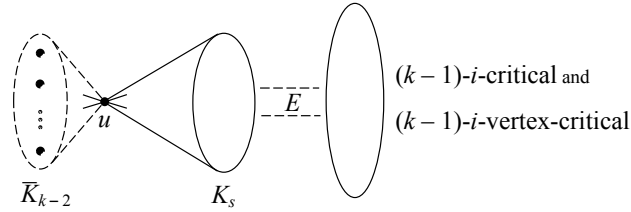


Figure 6: The structure of a graph in the class \mathcal{G}_6

VI. The class \mathcal{G}_7

For positive integers s , n , k and m where $s \geq n \geq k - 1 \geq 3$ and $m \geq 2$, define a graph $G \in \mathcal{G}_7$ of order $s + n + k + m - 1$ as follows. Set $V(G) = \{u\} \cup X \cup Y \cup Z \cup W$ where $|X| = n$, $|Y| = m$, $|Z| = s$ and $|W| = k - 2$.

The edges of G are defined as follows. $G[X]$ is both $(k-1)$ - i -critical and $(k-1)$ - i -vertex-critical, $\overline{G[Y]} = \bigcup_{i=1}^l K_{1,r_i}$ such that $m = l + \sum_{i=1}^l r_i$, $m \geq 2l$ and $l \geq 1$, $G[Z] = K_s$ and $G[W] = \overline{K}_{k-2}$. Further, join u to each vertex of $W \cup Y \cup Z$, join each vertex of Y to every vertex of $Z \cup X$. Finally, we add the set of edges E between Z and X as similar as defined in the class \mathcal{G}_6 . This defines the class \mathcal{G}_7 . Figure 7 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_7$ is k - i -critical for $k \geq 4$.

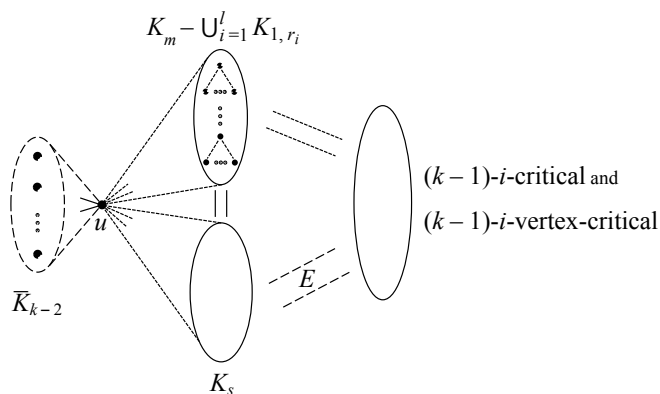


Figure 7: The structure of a graph in the class \mathcal{G}_7

Note that for a positive integer $k \geq 2$, \overline{K}_{k-1} is both $(k-1)$ - i -critical and $(k-1)$ - i -vertex-critical. Hence, $\mathcal{G}_6 \neq \emptyset$ and $\mathcal{G}_7 \neq \emptyset$.

4 The main results

Lemma 4.1. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Then*

1. *If $\omega(G - u) \geq k - 1$, then $G - u$ contains at least one non-singleton and at most two non-singleton components.*

2. $\omega(G - u) \leq k - 1$.

Proof. For $1 \leq i \leq t$, let C_1, C_2, \dots, C_t be the components of $G - u$. It is easy to see that $G - u$ must contain at least one non-singleton component since G is connected and $i(G) = k \geq 3$.

(1) Suppose to the contrary that $G - u$ contains $C_j, C_{j'}$ and $C_{j''}$ as non-singleton components where $\{j, j', j''\} \subseteq \{1, 2, \dots, t\}$. Choose $x \in N_{C_j}(u)$ and $y \in N_{C_{j'}}(u)$. Consider $G + xy$. By Lemma 2.1, $|I_{xy}| = k - 1$, $|I_{xy} \cap \{x, y\}| = 1$ and $u \notin I_{xy}$. We may suppose without loss of generality that $x \in I_{xy}$. Since $u \notin I_{xy}$, $I_{xy} \cap V(C_i) \neq \emptyset$ for $1 \leq i \leq t$ and $i \notin \{j, j'\}$. Further, $I_{xy} \cap V(C_{j'}) \neq \emptyset$ since $V(C_{j'}) - \{y\} \neq \emptyset$. Consequently, $t = k - 1$ and $|I_{xy} \cap V(C_i)| = 1$ for

$1 \leq i \leq t$. Thus $x \succ C_j$. Put $\{w\} = I_{xy} \cap V(C_j)$. Then $w \succ C_j$. Now consider $G + xw$. By Lemma 2.1, $|I_{xw}| = k - 1$, $|I_{xw} \cap \{x, w\}| = 1$ and $u \notin I_{xw}$. If $x \notin I_{xw}$, then $I_{xw} \cap V(C_j) = \emptyset$ since $I_{xw} \cap (N_G(x) \cup N_G(w)) = \emptyset$ and $x \succ C_j$. But then no vertex of I_{xw} dominates $V(C_j) - \{x\}$, a contradiction. Hence, $x \in I_{xw}$. By similar arguments, $w \in I_{xw}$. But this contradicts the fact that $|I_{xw} \cap \{x, w\}| = 1$. Hence, $G - u$ contains at most two non-singleton components. This proves (1).

(2) Suppose to the contrary that $t \geq k$. By (1), we may assume that $|V(C_1)| \geq 1$ and $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq t$. Since G is connected, u is adjacent to every vertex of $\bigcup_{i=3}^t V(C_i)$. Now choose $a \in N_{C_1}(u)$ and $b \in N_{C_2}(u)$. Consider $G + ab$. By Lemma 2.1, $|I_{ab}| = k - 1$, $|I_{ab} \cap \{a, b\}| = 1$ and $u \notin I_{ab}$. We may assume without loss of generality that $a \in I_{ab}$. Since $u \notin I_{ab}$, $I_{ab} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq t$. Then $t = k$ and $V(C_2) = \{b\}$ since $|I_{ab}| = k - 1$. Clearly, $bu \in E(G)$ since G is connected. Further, $\bar{N}_{C_1}(u) \neq \emptyset$ otherwise $u \succ G$. Let $c \in \bar{N}_{C_1}(u)$. We now consider $G + bc$. By Lemma 2.1, $|I_{bc}| = k - 1$, $|I_{bc} \cap \{b, c\}| = 1$ and $u \notin I_{bc}$. Then $I_{bc} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq t$ and $V(C_1) \cup V(C_2)$ must be dominated, in $G + bc$, by one of element in $\{b, c\}$ since $|I_{bc}| = k - 1$ and $t = k$. It is easy to see that $I_{bc} \cap \{b, c\} = \{c\}$ since $a \in V(C_1) - \{c\}$. Thus $c \succ C_1$. But then $\{c, u\} \succ_i G$, a contradiction since $i(G) = k \geq 3$. This proves (2) and completes the proof of our lemma. \square

Lemma 4.2. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Suppose C_1, C_2, \dots, C_{k-1} are the components of $G - u$ where $|V(C_1)| \geq 2$, $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$. If $x \in V(C_1)$ and $y \in V(C_2)$ where $\{x, y\} \cap N_G(u) \neq \emptyset$, then $u \notin I_{xy}$, $|I_{xy} \cap V(C_1)| \geq 1$ and $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$. Further, if $|V(C_2)| \geq 2$, then $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$.*

Proof. Consider $G + xy$. By Lemma 2.1, $|I_{xy}| = k - 1$, $|I_{xy} \cap \{x, y\}| = 1$ and $u \notin I_{xy}$. Then $I_{xy} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq k - 1$. Because $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$, $|I_{xy} \cap V(C_i)| = 1$ for $3 \leq i \leq k - 1$. Since $|I_{xy}| = k - 1$ and $\omega(G - u) = k - 1$, it follows that $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$. Further, $|I_{xy} \cap V(C_1)| \geq 1$ because $|V(C_1)| \geq 2$. We now suppose that $|V(C_2)| \geq 2$. If $|I_{xy} \cap V(C_1)| = 2$, then no vertex of I_{xy} dominates $C_2 - y$, a contradiction. Hence, $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$. This proves our lemma. \square

Lemma 4.3. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Suppose C_1, C_2, \dots, C_{k-1} are the components of $G - u$ where $|V(C_1)| \geq 2$, $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$. Then either $i(C_1) = 1$ and $i(C_2) = 2$ or $i(C_1) = 2$ and $i(C_2) = 1$. Further, if $i(C_1) = 2$ and $i(C_2) = 1$, then C_2 is complete.*

Proof. We first suppose that $|V(C_2)| = 1$. Clearly, $i(C_1) \geq 2$ otherwise $i(G) \leq k - 1$. We need only show that $i(C_1) = 2$. Put $\{y\} = V(C_2)$. Then

$yu \in E(G)$. Since $i(G) = k \geq 3$, $\overline{N}_{C_1}(u) \neq \emptyset$. Let $x \in \overline{N}_{C_1}(u)$. Consider $G + xy$. By Lemma 4.2, $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$ and $|I_{xy} \cap V(C_1)| \geq 1$. By Lemma 2.1, $|I_{xy} \cap \{x, y\}| = 1$. If $x \in I_{xy}$, then $y \notin I_{xy}$ and thus $|I_{xy} \cap V(C_1)| = 2$ since $V(C_2) = \{y\}$. If $y \in I_{xy}$, then $x \notin I_{xy}$ and thus the only vertex of $I_{xy} \cap V(C_1)$, say w , dominates $V(C_1) - \{x\}$ since $|V(C_1)| \geq 2$ and $wx \notin E(G)$. In either case, $i(C_1) = 2$.

We next suppose that $|V(C_2)| \geq 2$. Choose $a \in N_{C_1}(u)$ and $b \in N_{C_2}(u)$. Consider $G + ab$. By Lemma 4.2, $|I_{ab} \cap V(C_1)| = 1$ and $|I_{ab} \cap V(C_2)| = 1$. Put $\{a'\} = I_{ab} \cap V(C_1)$ and $\{b'\} = I_{ab} \cap V(C_2)$. If $a' = a$, then $b' \neq b$ by Lemma 2.1. Thus $a \succ C_1$ and $b' \succ C_2 - b$. Consequently, $i(C_1) = 1$ and $\{b, b'\} \succ_i C_2$. If $i(C_2) = 1$, then $i(G) \leq k - 1$, a contradiction. Hence, $i(C_2) = 2$. Similarly, if $a' \neq a$, then $b' = b$ and thus $i(C_1) = 2$ and $i(C_2) = 1$ as required.

We now suppose that $i(C_1) = 2$ and $i(C_2) = 1$. In order to finish the proof of our lemma, we need the following claim.

Claim : *If $y \in N_{C_2}(u)$, then $y \succ C_2$.*

If $|V(C_2)| = 1$, then our claim follows immediately. So we may assume that $|V(C_2)| \geq 2$. Choose $x \in V(C_1)$. Consider $G + xy$. By Lemma 4.2, $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$. Further, $u \notin I_{xy}$. If $I_{xy} \cap V(C_1) = \{x\}$, then $x \succ C_1$, contradicting the fact that $i(C_1) = 2$. Hence, $I_{xy} \cap V(C_1) \neq \{x\}$. Consequently, $I_{xy} \cap V(C_2) = \{y\}$ by Lemma 2.1 and thus $y \succ C_2$. This settles our claim.

We are now ready to finish our proof. Suppose to the contrary that C_2 is not complete. Then there exist non-adjacent vertices v and w of $V(C_2)$. By the above claim, $\{v, w\} \subseteq \overline{N}_{C_2}(u)$. Let $x \in N_{C_1}(u)$. Consider $G + xv$. By Lemma 4.2, $|I_{xv} \cap V(C_1)| = 1$ and $|I_{xv} \cap V(C_2)| = 1$. Further, $u \notin I_{xv}$. If $I_{xv} \cap V(C_2) = \{v\}$, then no vertex of I_{xv} dominates w , a contradiction. Hence, $I_{xv} \cap V(C_2) \neq \{v\}$. Therefore, $I_{xv} \cap V(C_1) = \{x\}$ by Lemma 2.1. Clearly, $x \succ C_1$. But this contradicts the fact $i(C_1) = 2$. Hence, C_2 is complete as required. This completes the proof of our lemma. \square

Theorem 4.4. *For a positive integer $k \geq 3$, let G be a connected k -critical graph containing u as a cutvertex and $\omega(G - u) = k - 1$. Suppose $G - u$ contains exactly two non-singleton components, say C_1 and C_2 , where $i(C_1) = 2$ and $i(C_2) = 1$. Then*

1. $G[N_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching. Further, each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$.
2. $G[\overline{N}_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching.
3. $3 \leq k \leq 4$. Further, if $k = 3$, then G is isomorphic to a graph in the class \mathcal{G}_1 and if $k = 4$, then G is isomorphic to a graph in the class \mathcal{G}_5 .

Proof. Let W be a set of all vertices in the singleton components of $G - u$. Clearly, $|W| = k - 3 \geq 0$ since $\omega(G - u) = k - 1$. Further, $W \subseteq N_G(u)$ since G is connected. By our hypothesis that $i(C_1) = 2$ and $i(C_2) = 1$ together with Lemma 4.3, C_2 is complete. It then follows that $\overline{N}_{C_1}(u) \neq \emptyset$ since $i(G) = k \geq 3$.

Claim 1: *If $x \in N_{C_1}(u)$, $y \in V(C_2)$, then $I_{xy} \cap V(C_1) = \{x'\}$ where $x' \in N_{C_1}(u) - \{x\}$ and $I_{xy} \cap V(C_2) = \{y\}$.*

Consider $G + xy$. By Lemma 4.2, $u \notin I_{xy}$ and $|I_{xy} \cap V(C_1)| = |I_{xy} \cap V(C_2)| = 1$. Suppose to the contrary that $x \in I_{xy}$. By Lemma 2.1, $I_{xy} \cap V(C_2) = \{y'\}$ where $y' \neq y$. Then $I_{xy} \cap N_G(y) \neq \emptyset$ since C_2 is complete. But this contradicts Lemma 2.1. Hence, $x \notin I_{xy}$. Therefore, $y \in I_{xy}$ and thus $I_{xy} \cap V(C_2) = \{y\}$. Put $\{x'\} = I_{xy} \cap V(C_1)$. Clearly, $x' \neq x$ and $x' \succ C_1 - x$. Suppose to the contrary that $x' \in \overline{N}_{C_1}(u)$. Then $\{x', u\} \succ_i G - \overline{N}_{C_2}(u)$. If $\overline{N}_{C_2}(u) = \emptyset$, then $i(G) = 2 < k$, a contradiction. Hence, $\overline{N}_{C_2}(u) \neq \emptyset$. But then $3 \leq i(G) = k \leq 3$ since C_2 is complete. Thus $k = 3$. Clearly, $d(x', y') > 3$ where $y' \in \overline{N}_{C_2}(u)$. But this contradicts Lemma 2.2. Hence, $x' \in N_{C_1}(u) - \{x\}$ as required. This settles our claim.

(1) Let $x \in N_{C_1}(u)$ and $y \in V(C_2)$. Consider $G + xy$. By Claim 1, $I_{xy} \cap V(C_1) = \{x'\}$ for some $x' \in N_{C_1}(u) - \{x\}$. Clearly, $x' \succ C_1 - x$. Now consider $G + x'y$. Again, by Claim 1, $I_{x'y} \cap V(C_1) = \{x''\}$ for some $x'' \in N_{C_1}(u) - \{x'\}$. Since $x' \succ C_1 - x$, $x'' = x$ by Lemma 2.1. Then $x'' = x \succ C_1 - x'$. If $|N_{C_1}(u)| = 2$, we are done. So suppose $|N_{C_1}(u)| \geq 3$. Let $w \in N_{C_1}(u) - \{x, x'\}$. Consider $G + wy$. By Claim 1, $I_{wy} \cap V(C_1) = \{w'\}$ for some $w' \in N_{C_1}(u) - \{w\}$. By Lemma 2.1, $w' \notin \{x, x'\}$ since $w \in N_G(x) \cap N_G(x')$. Clearly, $w' \succ C_1 - w$. Now consider $G + w'y$. By similar arguments, $I_{w'y} \cap V(C_1) = \{w\}$ and $w \succ C_1 - w'$. Continuing in this fashion, $G[N_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching. Further, in our argument, it is clear that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$. This proves (1).

Claim 2: *For $x \in \overline{N}_{C_1}(u)$ and $y \in N_{C_2}(u)$, $I_{xy} \cap V(C_1) = \{x'\}$ where $x' \in \overline{N}_{C_1}(u) - \{x\}$ and $I_{xy} \cap V(C_2) = \{y\}$.*

By similar arguments as in the proof of Claim 1 and the fact that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$, by (1), our claim follows.

(2) By applying similar arguments as in the proof of (1) together with Claim 2 and the fact that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$, by (1), (2) follows.

(3) Clearly, $u \succ N_{C_1}(u) \cup N_{C_2}(u) \cup W$. We first suppose that $\overline{N}_{C_2}(u) = \emptyset$. Then $V(C_2) \subseteq N_G(u)$ and thus $i(G) = k = 3$ since $i(G[\overline{N}_{C_1}(u)]) = 2$ by (2). Hence, G is isomorphic to the graph in the class \mathcal{G}_1 .

We now suppose that $\overline{N}_{C_2}(u) \neq \emptyset$. Let $y \in \overline{N}_{C_2}(u)$. Then $\{u, y\} \succ_i N_{C_1}(u) \cup V(C_2) \cup W$. Thus, by (2), $3 \leq i(G) = k \leq 4$. By Lemma 2.2, $k \neq 3$. Hence, $k = 4$ and $|W| = 1$. We next show that $|\overline{N}_{C_2}(u)| = 1$. Suppose this

is not the case. Let $y' \in \overline{N}_{C_2}(u) - \{y\}$. Consider $G + uy'$. By Lemma 2.1, $|I_{uy'}| = 3$, $|I_{uy'} \cap \{u, y'\}| = 1$. We first suppose that $I_{uy'} \cap \{u, y'\} = \{u\}$. Then $I_{uy'} \cap V(C_2) = \emptyset$ since $I_{uy'} \cap (N_G(u) \cup N_G(y')) = \emptyset$ and C_2 is complete. But then no vertex of $I_{uy'}$ dominates y , a contradiction. Hence, $I_{uy'} \cap \{u, y'\} = \{y'\}$. Then $I_{uy'} \cap W = \emptyset$ by Lemma 2.1 since the only vertex of W is adjacent to u . But then no vertex of $I_{uy'}$ dominates W , again a contradiction. Hence, $|\overline{N}_{C_2}(u)| = 1$. It follows by (1) and (2) that G is isomorphic to the graph in the class \mathcal{G}_5 . This completes the proof of our theorem. \square

Theorem 4.5. *For a positive integer $k \geq 3$, let G be a connected k - i -critical graph with a cutvertex u where $\omega(G - u) = k - 1$. Suppose $G - u$ contains exactly one non-singleton component, say C . Then*

1. $G[\overline{N}_C(u)]$ is $(k - 1)$ - i -critical and $(k - 1)$ - i -vertex-critical.
2. $N_C(u)$ consists of two disjoint sets, say Y and Z , where each vertex of Y dominates $\overline{N}_C(u)$ and $Z = N_C(u) - Y$. Further, if $Y \neq \emptyset$, then $G[Y]$ is 2- i -critical and if $Z \neq \emptyset$, then $G[Z]$ is complete and for each vertex $z \in Z$, there is exactly one vertex of $\overline{N}_C(u)$, say z' , such that $z \succ C - z'$. Moreover, if $k \geq 4$, each vertex of $\overline{N}_C(u)$ is not adjacent to at least one vertex of Z and thus $Z \neq \emptyset$ and $|Z| \geq |\overline{N}_C(u)|$.

Proof. Let W be a set of all vertices in the singleton components of $G - u$. Clearly, $|W| = k - 2 \geq 1$ since $\omega(G - u) = k - 1$. Further, $W \subseteq N_G(u)$ since G is connected. Note that $N_G(u) = W \cup N_C(u)$. It then follows that $\overline{N}_C(u) \neq \emptyset$ since $i(G) = k \geq 3$. Put $X = \overline{N}_C(u)$. It is easy to see that $i(G[X]) \geq k - 1 \geq 2$. Then the following claims follow by Lemma 2.1 and the fact that $N_G(u) = W \cup N_C(u)$.

Claim 1 : *For $v \in X$, $u \in I_{uv}$ and $I_{uv} - \{u\} \subseteq X - \{v\}$.*

Claim 2 : *If $u \in I_{vw}$ for some $v, w \in X$, then $I_{vw} - \{u\} \subseteq X$.*

Claim 3 : *If $u \notin I_{vw}$ for some $v, w \in V(C)$, then either $I_{vw} = \{v\} \cup W$ or $I_{vw} = \{w\} \cup W$.*

(1) Since $i(G[X]) \geq k - 1 \geq 2$, there exist non-adjacent vertices $x_1, x_2 \in X$. We first show that $G[X]$ is $(k - 1)$ - i -critical. Consider $G + x_1x_2$. By Lemma 2.1, $|I_{x_1x_2}| = k - 1$ and $|I_{x_1x_2} \cap \{x_1, x_2\}| = 1$. We may assume without loss of generality that $I_{x_1x_2} \cap \{x_1, x_2\} = \{x_1\}$. We first suppose that $u \notin I_{x_1x_2}$. Then, by Claim 3, $I_{x_1x_2} = \{x_1\} \cup W$. This implies that $x_1 \succ C - x_2$ and thus $\{x_1, x_2\} \succ_i G[X]$. Consequently, $k = 3$ and $G[X]$ is 2- i -critical.

We now suppose that $u \in I_{x_1x_2}$. By our Claim 2, $I_{x_1x_2} - \{u\} \subseteq X$ and then $I_{x_1x_2} - \{u\} \succ_i X - \{x_2\}$. Hence, $(I_{x_1x_2} - \{u\}) \cup \{x_2\} \succ_i G[X]$ and thus $i(G[X]) = k - 1$. Consequently, G is $(k - 1)$ - i -critical.

By the above argument, $i(G[X]) = k-1$. We next show that $G[X]$ is $(k-1)$ - i -vertex-critical. Let $x \in X$. Consider $G + ux$. By Lemma 2.1, $|I_{ux}| = k-1$ and by Claim 1, $u \in I_{ux}$ and $I_{ux} - \{u\} \subseteq X - \{x\}$. Then $I_{ux} - \{u\} \succ_i X - \{x\}$. Thus $i(G[X - \{x\}]) \leq k-2$. If $i(G[X - \{x\}]) < k-2$, then $i(G[X]) < k-1$, a contradiction. Hence, $G[X]$ is $(k-1)$ - i -vertex-critical as required. This proves (1).

(2) We first suppose that $Z \neq \emptyset$. Let $z \in Z$. Then there exists $x \in X$ such that $zx \notin E(G)$. Consider $G + zx$. By Lemma 2.1, $u \notin I_{zx}$. By Claim 3, either $I_{zx} = \{x\} \cup W$ or $I_{zx} = \{z\} \cup W$. If $I_{zx} = \{x\} \cup W$, then $x \succ C - z$ and thus $x \succ X$. But this contradicts (1) since $k \geq 3$. Hence, $I_{zx} = \{z\} \cup W$. Then $z \succ C - x$. Consequently, $G[Z]$ is complete.

We next suppose $Y \neq \emptyset$. Note that each vertex of Y dominates $Z \cup X$. If there is a vertex $y \in Y$ such that $y \succ Y$, then $\{y\} \cup W \succ_i G$. But this contradicts the fact that $i(G) = k$. Hence, for each vertex $y \in Y$, there exists a vertex $y' \in Y - \{y\}$ such that $yy' \notin E(G)$. Consider $G + yy'$. By Lemma 2.1, $u \notin I_{yy'}$. We may assume by Claim 3 that $I_{yy'} = \{y\} \cup W$. Then $y \succ C - y'$ and thus $y \succ Y - y'$. Hence, $\{y, y'\} \succ_i G[Y]$ and $y \succ G[Y] + yy'$. Therefore, $G[Y]$ is 2- i -critical.

We now suppose that $k \geq 4$. By the definition of Y , each vertex of X is adjacent to every vertex of Y . Suppose to the contrary that there exists a vertex $x \in X$ such that x is adjacent to every vertex of Z . Then $x \succ Y \cup Z$. Let $w \in W$. Consider $G + xw$. By Lemma 2.1, $|I_{xw}| = k-1$, $|I_{xw} \cap \{x, w\}| = 1$ and $I_{xw} \cap (\{u\} \cup Y \cup Z) = \emptyset$. Since $W \subseteq N_G(u)$, $W - \{w\} \subseteq I_{xw} - \{x, w\}$. Because $|W| = k-2$ and $|I_{xw} \cap \{x, w\}| = 1$, it follows that $|I_{xw} - (\{x, w\} \cup W)| = 1$. Put $\{a\} = I_{xw} - (\{x, w\} \cup W)$. Clearly, $a \in X - \{x\}$. Consequently, $\{a, x\} \succ X$. But this contradicts the fact that $i(G[X]) = k-1 \geq 3$. Hence, each vertex of X is not adjacent to at least one vertex of Z . It then follows that $|Z| \geq |X|$. This proves (2) and completes the proof of our theorem. \square

We are now ready to establish our characterizations.

Theorem 4.6. *For a positive integer $k \geq 4$, if G is a connected k - i -critical graph with a cutvertex u where $\omega(G - u) = k-1$, then $G \in \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$.*

Proof. By Lemma 4.1(1), $G - u$ contains at least one non-singleton and at most two non-singleton components. If $G - u$ contains exactly two non-singleton components, then $k = 4$ and G is isomorphic to a graph in the class \mathcal{G}_5 by Theorem 4.4(3). We now suppose that $G - u$ contains C as the only non-singleton component. Let Y and Z be defined as in Theorem 4.5. Then $Z \neq \emptyset$ and $G[Z]$ is complete. It is easy to see that, by Theorem 4.5, if $Y = \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_6 and if $Y \neq \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_7 . This completes the proof of our theorem \square

We now consider connected 3- i -critical graphs. Let Z and X be defined as in Theorem 4.5. Note that a graph in the class \mathcal{G}_2 shows that Z is empty.

Further, a graph in the class $\mathcal{G}_3 \cup \mathcal{G}_4$ shows that there might exist a vertex of X which is adjacent to every vertex of Z . Our next theorem provides a characterization of connected 3- i -critical graphs containing a cutvertex.

Theorem 4.7. *If G is a connected 3- i -critical graph containing a cutvertex, then $G \in \mathcal{G}_i$, defined in Section 3, for some $1 \leq i \leq 4$.*

Proof. Let u be a cutvertex of G . By Lemma 4.1, $G - u$ contains exactly two components, at least one of them is non-singleton. If both components of $G - u$ are non-singleton, then G is isomorphic to a graph in the class \mathcal{G}_1 by Theorem 4.4(3). We may now suppose that $G - u$ contains exactly one non-singleton component. Let C_1 and C_2 be the components of $G - u$ where $|V(C_1)| = 1$ and $|V(C_2)| \geq 2$. Clearly, $\overline{N}_{C_2}(u) \neq \emptyset$. Put $V(C_1) = \{c\}$ and $X = \overline{N}_{C_2}(u)$. Then, by Lemma 2.3(2) and Theorem 4.5(1), $G[X]$ is isomorphic to a complete graph without a perfect matching.

By Theorem 4.5(2), $N_{C_2}(u)$ consists of two disjoint sets, say Y and Z , where each vertex of Y dominates X and $Z = N_{C_2}(u) - Y$. Further if $Y \neq \emptyset$, then $G[Y]$ is 2- i -critical and if $Z \neq \emptyset$, then $G[Z]$ is complete and for each vertex $z \in Z$, there is exactly one vertex of X , say z' , such that $z \succ C_2 - z'$. Consequently, $N_G[z] = V(G) - \{c, z'\}$.

Claim 1: *Let $x_1, x_2 \in X$ where $x_1x_2 \notin E(G)$. If $x_1 \succ Z$, then $x_2 \succ Z$.*

By the definition of Y , $x_1 \succ Z \cup Y = N_{C_2}(u)$. Then $N_G[x_1] = V(C_2) - \{x_2\}$ since $G[X]$ is isomorphic to a complete graph without a perfect matching. Consider $G + cx_1$. Let $\{z\} = I_{cx_1} - \{c, x_1\}$. Then $\{c, x_1, z\}$ is independent by Lemma 2.1. Thus $z = x_2$ since $N_G[c] = \{c, u\}$ and $N_G[x_1] = V(C_2) - \{x_2\}$. If $I_{cx_1} = \{x_1, x_2\}$, then no vertex of I_{cx_1} is adjacent to u , a contradiction. Hence, $I_{cx_1} = \{c, x_2\}$. Since c is not adjacent to any vertex of C_2 , $x_2 \succ C_2 - x_1$. Hence, $x_2 \succ Z$. This settles our claim.

Claim 2: *If $z \in Z$ and $zx \notin E(G)$ for some $x \in X$, then the only non-neighbor of x in X is not adjacent to some vertex of $Z - \{z\}$.*

This claim follows by Claim 1 and the fact that for each vertex $z \in Z$, there is exactly one vertex of X , say z' , $N_G[z] = V(G) - \{c, z'\}$.

Let $X_1 = \{x \in X \mid xz \notin E(G) \text{ for some } z \in Z\}$. It is easy to see that, by the definition of Z , if $Z \neq \emptyset$, then $X_1 \neq \emptyset$. Now put $X_2 = X - X_1$. Then the following claims follow by Claims 1 and 2.

Claim 3: *For $x_1, x_2 \in X$ where $x_1x_2 \notin E(G)$, if $x_1 \in X_i$ for $1 \leq i \leq 2$, then $x_2 \in X_i$. Consequently, $|X_1|$ and $|X_2|$ are both even.*

Claim 4: *If $Z \neq \emptyset$, then $2 \leq |X_1| \leq |Z|$.*

By Lemma 2.3, $G[Y]$ is isomorphic to $K_m - \bigcup_{i=1}^l K_{1,r_i}$ for some positive

integers m , l and r_i . Clearly, $m = l + \sum_{i=1}^l r_i$ and $m \geq 2l \geq 2$. It is now easy to see that if $Y \neq \emptyset$ but $Z = \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_2 and if $Z \neq \emptyset$ but $Y = \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_3 . Finally, if $Y \neq \emptyset$ and $Z \neq \emptyset$, then G is isomorphic to a graph in the class \mathcal{G}_4 . This completes the proof of our theorem. \square

We conclude our paper by pointing out that Ao, in [4], gave a theorem described 3- i -critical graphs with a cutvertex but his/her theorem does not depict how such graphs exactly look like while our Theorem 4.7 provides the explicit structure. Moreover, our proof is much more easier and shorter.

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3 **SOME TOUGHNESS RESULTS IN INDEPENDENT**
4 **DOMINATION CRITICAL GRAPHS**

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16 **Abstract**

17 A subset S of $V(G)$ is an independent dominating set for G if S is
18 independent and each vertex of G is either in S or adjacent to some vertex of
19 S . Let $i(G)$ denote the minimum cardinality of an independent dominating
20 set for G . Graph G is k - i -critical if $i(G) = k$, but $i(G + uv) < k$ for any
21 pair of non-adjacent vertices u and v of G . In this paper, we establish that
22 if G is connected 3- i -critical and S is a vertex cutset of G , then $\omega(G - S) \leq$
23 $|S| - 2$ for $|S| \geq 6$ and $\omega(G - S) \leq 3$ if $|S| \leq 5$. Further, if $|S| \geq 3$,
24 then $\omega(G - S) \leq \frac{1 + \sqrt{8|S| + 1}}{2}$, improving a result proved by Ao [3]. We also
25 provide a characterization of connected 3- i -critical graph G that attains the
26 maximum number of $\omega(G - S)$ when S is a minimum cutset of size 2 or 3.

27 **Keywords:** domination critical, toughness.

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1. INTRODUCTION

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complete graph of order n is denoted by K_n . A star is a complete bipartite graph $K_{1,n}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. S is independent if no two vertices of S are adjacent. The number of components of G is denoted by $\omega(G)$. A vertex subset S of $V(G)$ is a cutset of G if $\omega(G - S) > \omega(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G , denoted by $N_G(v)$, is the set of all vertices of $V(G)$ which are adjacent to v . $N_G[v]$ denotes the closed neighborhood of v , i.e., $N_G[v] = N_G(v) \cup \{v\}$. The non-neighborhood of v in G denoted by $\overline{N}_G(v)$ is $V(G) - N_G[v]$. For $S \subseteq V(G)$, $N_G(v) \cap S$, $N_G[v] \cap S$, $\overline{N}_G(v) \cap S$ and $\overline{N}_G[v] \cap S$ are denoted by $N_S(v)$, $N_S[v]$, $\overline{N}_S(v)$ and $\overline{N}_S[v]$, respectively. Further, $N_G(S)$ denotes $\bigcup_{x \in S} N_G(x)$ and $N_G[S]$ denotes $\bigcup_{x \in S} N_G[x]$. For simplicity, if S is a subgraph of G , then we also denote $N_G(v) \cap V(S)$, $N_G[v] \cap V(S)$, $\overline{N}_G(v) \cap S$ and $\overline{N}_G[v] \cap S$ by $N_S(v)$, $N_S[v]$, $\overline{N}_S(v)$ and $\overline{N}_S[v]$, respectively.

For subsets S and T of $V(G)$, we say that S dominates T , denoted by $S \succ T$, if $T \subseteq N_G[S]$. If $S \succ T$ where $S = \{s\}$, then we write $s \succ T$ instead of $\{s\} \succ T$. Further, if $T = V(H)$ where H is a subgraph of G , then we write $S \succ H$ instead of $S \succ V(H)$ and we say that S is a *dominating set* for H . Thus S is a dominating set for G if each vertex of $V(G)$ is either in S or adjacent to some vertex of S . The minimum cardinality of a dominating set for G is called the domination number of G and denoted by $\gamma(G)$.

For a subgraph H of G , if $S \succ H$ and S is independent, then we say that S is an *independent dominating set* for H and denoted by $S \succ_i H$. Thus S is an independent dominating set for G if $S \succ_i G$. The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and denoted by $i(G)$. Observe that for any graph G , $\gamma(G) \leq i(G)$ and if $\gamma(G) = 1$, then $i(G) = 1$.

In 1994, Ao [3] introduced the concept of “independent domination critical”. A graph G is k -*i-critical* if $i(G) = k$, but $i(G+uv) < k$ for any pair of non-adjacent vertices u and v of G . In [3], Ao established some properties of 3-*i-critical* graphs including diameter and a sufficient condition for such graphs to be hamiltonian. He/she also showed that if G is a connected 3-*i-critical* graph and S is a vertex cutset of G , then $\omega(G - S) \leq |S| + 1$. We found that the upper bound of this result can be improved if $|S| \geq 2$. In fact, in this paper, we shall prove the following two main theorems in Section 2.

Theorem 1. *Let G be a connected 3-*i-critical* graph and S a vertex cutset of G . Then*

$$\omega(G - S) \leq \begin{cases} |S| - 2, & \text{for } |S| \geq 6, \\ 3, & \text{for } |S| \leq 5. \end{cases}$$

68 **Theorem 2.** *Let G be a connected 3- i -critical graph and S a vertex cutset of G .*
 69 *If $|S| \geq 3$, then $\omega(G - S) \leq \frac{1+\sqrt{8|S|+1}}{2}$. Further, the bound is sharp.*

70 In studying the concept of critical graphs, we always seek for a characteri-
 71 zation of graphs. This also applies to connected k - i -critical graphs. It is easy
 72 to see that K_n , where n is a positive integer, is the only 1- i -critical graph. Ao
 73 [3] showed that G is 2- i -critical if and only if the complement of G is a union of
 74 stars. For $k \geq 3$, it is not possible to characterize connected k - i -critical graphs
 75 unless some addition hypothesis is added. In [1], the authors proved that if G is a
 76 connected k - i -critical graph, for $k \geq 3$, with a cutvertex u , then $\omega(G - u) \leq k - 1$.
 77 Further, a characterization of such graph G with $\omega(G - u) = k - 1$ was given. So
 78 the characterization of connected 3- i -critical graphs with a cutvertex is known.
 79 According to Theorem 1, if G is a connected 3- i -critical graph with a cutset S of
 80 size at most five, then $\omega(G - S)$ is at most three. A problem that arises is that
 81 of characterizing such graphs. In Section 3, we establish that if G is connected
 82 3- i -critical with a minimum cutset S of size 2 or 3 and $\omega(G - S) = 3$, then G
 83 must be in one of six classes. In [2], a characterization of connected 3- i -critical
 84 graph G with a minimum cutset S of size 2 and $\omega(G - S) = 2$ is given. We then
 85 conclude this section by posing the following remaining cases.

- 86
- 87 1. Characterize connected 3- i -critical graph G with a minimum cutset S of
 88 size 3 and $\omega(G - S) = 2$.
 - 89
 - 90 2. Characterize connected 3- i -critical graph G with a minimum cutset S of
 91 size 4 or 5 and $\omega(G - S) = k$ where $k \in \{2, 3\}$.
 - 92

93 2. THE PROOF OF THEOREMS 1 AND 2

94 We begin this section with some terminology and preliminary results that we
 95 make use of in establishing our main results. For a pair of non-adjacent vertices
 96 u and v of G , I_{uv} denotes a minimum independent dominating set for $G + uv$.
 97 Our first two results follow immediately from the definition of k - i -critical graphs.
 98

99 **Lemma 3.** *Let G be a connected k - i -critical graph and let u and v be non-adjacent*
 100 *vertices of G . Then $|I_{uv}| = k - 1$ and $|I_{uv} \cap \{u, v\}| = 1$.*

101 **Lemma 4.** *Let G be a connected 3- i -critical graph and let u and v be non-adjacent*
 102 *vertices of G . Then*

- 103 1. $I_{uv} = \{u, w\}$ or $I_{uv} = \{v, w\}$ for some $w \in V(G) - \{u, v\}$.

104 2. If $\{w\} = I_{uv} - \{u, v\}$, then $\{u, v, w\}$ is independent.

105 Our next result is a useful tool for establishing our main results.

106 **Lemma 5.** *Let G be a connected 3- i -critical graph and let u and v be non-adjacent*
 107 *vertices of G . If $I_{uv} = \{u, w\}$ for some $w \in V(G) - \{u, v\}$, then $I_{uw} - \{u, w\} =$*
 108 *$\{v\}$, i.e., $I_{uw} = \{u, v\}$ or $I_{uw} = \{w, v\}$.*

109 **Proof.** By Lemma 4(2), $\{u, v, w\}$ is independent. Let $x \in V(G) - \{u, v, w\}$.
 110 Then $xu \in E(G)$ or $xw \in E(G)$ since $I_{uv} = \{u, w\}$. Then $x \notin I_{uw}$ by Lemma
 111 4(2). Thus $I_{uw} - \{u, w\} = \{v\}$ as required. This proves our lemma. ■

112 The next result proved by Ao [3]. Before stating this result, we need to
 113 introduce another notation introduced by Ao. If x, y and z are vertices of G
 114 where $\{x, y\} \succ_i G - z$, then we write $[x, y] \xrightarrow[i]{} z$

115 **Lemma 6.** [3] *Let G be a connected 3- i -critical graph and let S be an independent*
 116 *set of G with size at least $n \geq 4$. Then the vertices in S may be ordered as*
 117 *u_1, u_2, \dots, u_n in such a way that there exists a path $x_1x_2 \dots x_{n-1}$ in $G - S$ with*
 118 *$[u_i, x_i] \xrightarrow[i]{} u_{i+1}$ for $1 \leq i \leq n - 1$.*

119 We are now ready to prove Theorems 1 and 2.

120

121 The proof of Theorem 1

122

123 **Proof.** Suppose $G - S$ contains n components. If $n \leq 3$, then we are done. So
 124 we may suppose that $n \geq 4$. Let C_1, C_2, \dots, C_n be components of $G - S$. For
 125 $1 \leq i \leq n$, choose $a_i \in V(C_i)$. Then $A = \{a_1, a_2, \dots, a_n\}$ is an independent set of
 126 size at least 4. By Lemma 6, the vertices in A may be ordered as u_1, u_2, \dots, u_n
 127 in such a way that there exists a path $x_1x_2 \dots x_{n-1}$ such that $[u_i, x_i] \xrightarrow[i]{} u_{i+1}$ for
 128 $1 \leq i \leq n - 1$. Without loss of generality, we may renumber the components of
 129 $G - S$ in such a way that $u_i \in V(C_i)$ for $1 \leq i \leq n$. Since $n \geq 4$, $x_i \in S$. Thus
 130 $\{x_1, x_2, \dots, x_{n-1}\} \subseteq S$. Note also that $x_i \succ \bigcup_{j=1}^n V(C_j) - (V(C_i) \cup \{u_{i+1}\})$ and
 131 $\{u_i, x_i, u_{i+1}\}$ is independent by Lemma 4(2). Consequently, x_i does not dominate
 132 either C_i or C_{i+1} .

133 Now consider $G + u_1u_{n-k}$ where $k \in \{0, 1\}$. Let $\{z_k\} = I_{u_1u_{n-k}} - \{u_1, u_{n-k}\}$.
 134 Then $\{u_1, u_{n-k}, z_k\}$ is independent by Lemma 4(2) and $z_k \succ \bigcup_{j=2}^n V(C_j) -$
 135 $V(C_{n-k})$. Thus, for $k \in \{0, 1\}$, $z_k \in S$ since $n \geq 4$. Further, $z_0 \neq z_1$. But
 136 $z_k \notin \{x_1, x_2, \dots, x_{n-1}\}$ since, for $1 \leq i \leq n - 1$, x_i does not dominate either C_i
 137 or C_{i+1} . Hence, $\{x_1, x_2, \dots, x_{n-1}, z_0, z_1\} \subseteq S$. Therefore, $|S| \geq n + 1$.

138 Suppose that $|S| = n + 1$. Then $S = \{x_1, x_2, \dots, x_{n-1}, z_0, z_1\}$. We next
 139 consider $G + u_2u_n$. Let $\{w\} = I_{u_2u_n} - \{u_2, u_n\}$. Then $w \in S$ since $n \geq 4$ and
 140 $w \succ \bigcup_{j=1}^n V(C_j) - (V(C_2) \cup V(C_n))$. Clearly, $w \notin \{x_1, x_2, \dots, x_{n-1}\}$ since x_i

141 does not dominate either C_i or C_{i+1} for $1 \leq i \leq n-1$. Further, $w \notin \{z_0, z_1\}$
 142 since $z_k u_1 \notin E(G)$ for $0 \leq k \leq 1$. Hence, $w \notin S$, a contradiction. Therefore,
 143 $|S| \neq n+1$ and hence $|S| \geq n+2$. This proves our theorem. ■

144 The proof of Theorem 2

145

146 **Proof.** By Theorem 1, it is easy to see that our result holds for $3 \leq |S| \leq 6$. So
 147 we may now assume that $|S| \geq 7$. Put $t = \omega(G - S)$. Suppose to the contrary
 148 that $t \geq \frac{1+\sqrt{8|S|+1}}{2} + 1$. Then $t \geq 5$ since $|S| \geq 7$.

149 For $1 \leq i \leq t$, let C_i be a component of $G - S$. Choose $v_i \in V(C_i)$. Consider
 150 $G + v_i v_j$ where $1 \leq i \neq j \leq t$. Put $\{z_{ij}\} = I_{v_i v_j} - \{v_i, v_j\}$. Because $t \geq 5$,
 151 $z_{ij} \in S$. It is easy to see that $z_{ij} \succ \bigcup_{k=1}^t V(C_k) - (V(C_i) \cup V(C_j))$. By Lemma
 152 4(2), $\{v_i, v_j, z_{ij}\}$ is independent. Consequently, $z_{ij} \neq z_{lm}$ if $\{i, j\} \neq \{l, m\}$ for
 153 $1 \leq i, j, l, m \leq t$. Since there are $\binom{t}{2}$ subsets of size 2 of $\{1, \dots, t\}$, we can choose
 154 $\binom{t}{2}$ pair of vertices $v_i \in V(C_i)$ and $v_j \in V(C_j)$ where $1 \leq i \neq j \leq t$. It then
 155 follows that $|S| \geq \binom{t}{2} = \frac{t(t-1)}{2}$. Then $|S| \geq \frac{1}{2} \left(\frac{1+\sqrt{8|S|+1}}{2} + 1 \right) \left(\frac{1+\sqrt{8|S|+1}}{2} \right) =$
 156 $\frac{1}{8}(8|S| + 4\sqrt{8|S|+1} + 4) > |S| + 4$ since $|S| \geq 7$, a contradiction. Hence, $t \leq$
 157 $\frac{1+\sqrt{8|S|+1}}{2}$ as required.

158 We next show that our bound is sharp. Let $U = \{u_1, u_2, \dots, u_t\}$ be an
 159 independent set of vertices of size t . Put $\bar{U} = \{\bar{u}_{ij} | \text{the (unordered) pair } \{i, j\} (i \neq$
 160 $j) \text{ ranges over all } \binom{t}{2} \text{ subsets of size 2 of } \{1, \dots, t\}\}$. Now let H be a graph with
 161 $V(H) = U \cup \bar{U}$ and $E(H) = \{\bar{u}_{ij} \bar{u}_{lm} | \{i, j\} \neq \{l, m\}\} \cup \{u_i \bar{u}_{ij} | i \notin \{l, j\}\}$. It
 162 is easy to see that, in fact, $H[U] = tK_1$ and $H[\bar{U}] = K_{\binom{t}{2}}$. Further, for $u_i \in$
 163 U , $N_H(u_i) = \bar{U} - \{\bar{u}_{ij} | i \in \{l, j\}\}$ and, for $\bar{u}_{ij} \in \bar{U}$, $N_H(\bar{u}_{ij}) = (U \cup \bar{U}) - \{\bar{u}_{ij}, u_i, u_j\}$.
 164 Clearly, $i(H) = 3$, $\{u_i, \bar{u}_{ij}\} \succ_i H + u_i u_j$ and $\{u_j, \bar{u}_{ij}\} \succ_i H + u_i \bar{u}_{ij}$. Hence, H is
 165 3- i -critical where $|\bar{U}| = \binom{t}{2} = \frac{t(t-1)}{2}$ and $\omega(H - \bar{U}) = |U| = \frac{1+\sqrt{8|\bar{U}|+1}}{2} = t$. This
 166 completes the proof of our theorem. ■

167 3. THE CHARACTERIZATION OF CONNECTED 3- i -CRITICAL GRAPHS HAVING 168 MINIMUM CUTSET OF SIZE 2 AND 3

169 In this section, we establish the characterization of connected 3- i -critical graph
 170 G having a minimum cutset S of size 2 or 3 with $\omega(G - S) = 3$. We begin our
 171 section by providing six classes of connected 3- i -critical graphs having a mini-
 172 mum cutset of size 2 and 3. The last two theorems in this section show that if
 173 G is a connected 3- i -critical graph having a minimum cutset of size 2 or 3 with
 174 $\omega(G - S) = 3$, then G must belong to one of these six classes.

175

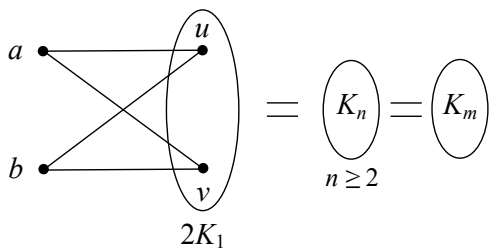


Figure 1. The structure of a graph in the class \mathcal{H}_1

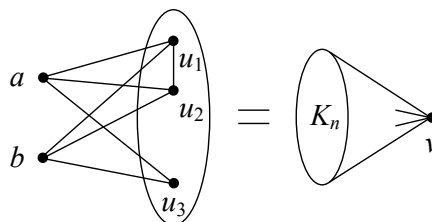


Figure 2. The structure of a graph in the class \mathcal{H}_2

176 **I. The class \mathcal{H}_1**

177 For positive integers m and $n \geq 2$, define a graph $G \in \mathcal{H}_1$ of order $n + m + 4$
 178 as follows. Set $V(G) = \{a, b, u, v\} \cup X \cup Y$ where $|X| = n$ and $|Y| = m$. The
 179 edges of G are defined as follows. $G[X] = K_n$ and $G[Y] = K_m$. Further, join
 180 u and v to every vertex of $\{a, b\} \cup X$. Finally, join each vertex of X to every
 181 vertex of Y . This defines the class \mathcal{H}_1 . Figure 1 illustrates our construction. It
 182 is not difficult to show that a graph $G \in \mathcal{H}_1$ is 3- i -critical containing $\{u, v\}$ as a
 183 minimum cutset. Note that in our diagram a “double line” denotes the join.

184

185 **II. The class \mathcal{H}_2**

186 For positive integer $n \geq 3$, define a graph $G \in \mathcal{H}_2$ of order $n + 6$ as follows.
 187 Set $V(G) = \{a, b, u_1, u_2, u_3, v\} \cup X$ where $|X| = n$. The edges of G are defined as
 188 follows. $G[X] = K_n$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$.
 189 Further, join v to every vertex of X . Finally, add the edge $u_1 u_2$. This defines the
 190 class \mathcal{H}_2 . Figure 2 illustrates our construction. It is not difficult to show that a
 191 graph $G \in \mathcal{H}_2$ is 3- i -critical containing $\{u_1, u_2, u_3\}$ as a minimum cutset.

192

193 **III. The class \mathcal{H}_3**

194 For positive integers n , m_1 and m_2 , define a graph $G \in \mathcal{H}_3$ of order $n +$
 195 $m_1 + m_2 + 5$ as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y_1 \cup Y_2$ where
 196 $|X| = n$, $|Y_1| = m_1$ and $|Y_2| = m_2$. The edges of G are defined as follows.
 197 $G[X \cup Y_1 \cup Y_2] = K_{n+m_1+m_2}$, join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of
 198 $\{a, b\} \cup X$. Further, for $1 \leq i \leq 2$, join u_i to every vertex of Y_i . Finally, add the
 199 edge $u_1 u_2$. This defines the class \mathcal{H}_3 . Figure 3 illustrates our construction. It is
 200 not difficult to show that a graph $G \in \mathcal{H}_3$ is 3- i -critical containing $\{u_1, u_2, u_3\}$
 201 as a minimum cutset.

202

203 **IV. The class \mathcal{H}_4**

204 For a positive integer n , define a graph $G \in \mathcal{H}_4$ of order $n + 5$ as follows.
 205 Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X$ where $|X| = n$. The edges of G are defined

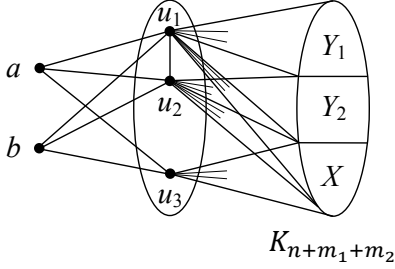


Figure 3. The structure of a graph in the class \mathcal{H}_3

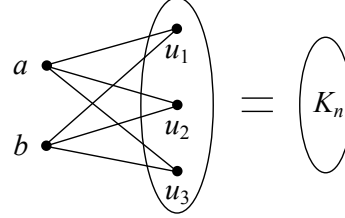


Figure 4. The structure of a graph in the class \mathcal{H}_4

206 as follows. $G[X] = K_n$ and join each vertex of $\{u_1, u_2, u_3\}$ to every vertex of
 207 $\{a, b\} \cup X$. This defines the class \mathcal{H}_4 . Figure 4 illustrates our construction. It
 208 is easy to see that a graph $G \in \mathcal{H}_4$ is 3-*i*-critical containing $\{u_1, u_2, u_3\}$ as a
 209 minimum cutset.

210

211 V. The class \mathcal{H}_5

212 For positive integers $n \geq 2$ and m , define a graph $G \in \mathcal{H}_5$ of order $n + m + 5$
 213 as follows. Set $V(G) = \{a, b, u_1, u_2, u_3\} \cup X \cup Y$ where $|X| = n$ and $|Y| = m$.
 214 The edges of G are defined as follows. $G[X \cup Y] = K_{n+m}$, join each vertex of
 215 $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Finally, join u_1 to every vertex of Y .
 216 This defines the class \mathcal{H}_5 . Figure 5 illustrates our construction. It is not difficult
 217 to see that a graph $G \in \mathcal{H}_5$ is 3-*i*-critical containing $\{u_1, u_2, u_3\}$ as a minimum
 218 cutset.

219

220 VI. The class \mathcal{H}_6

221 For non-negative integer n and positive integers m_1, m_2 and m_3 , define
 222 a graph $G \in \mathcal{H}_6$ of order $n + m_1 + m_2 + m_3 + 5$ as follows. Set $V(G) =$
 223 $\{a, b, u_1, u_2, u_3\} \cup X \cup Y_1 \cup Y_2 \cup Y_3$ where $|X| = n$ and $|Y_i| = m_i$ for $1 \leq i \leq 3$.
 224 The edges of G are defined as follows. $G[X \cup Y_1 \cup Y_2 \cup Y_3] = K_{n+m_1+m_2+m_3}$, join
 225 each vertex of $\{u_1, u_2, u_3\}$ to every vertex of $\{a, b\} \cup X$. Finally, for $1 \leq i \leq 3$,
 226 join u_i to every vertex of Y_i . This defines the class \mathcal{H}_6 . Figure 6 illustrates our
 227 construction. It is easy to see that a graph $G \in \mathcal{H}_6$ is 3-*i*-critical containing
 228 $\{u_1, u_2, u_3\}$ as a minimum cutset.

229

230 Before establishing our main results in this section, we need a following result.

231

232 **Lemma 7.** *Let G be a connected 3-*i*-critical graph and S a minimum cutset*
 233 *where $2 \leq |S| \leq 5$. Suppose $\omega(G - S) = 3$. Then*

234

1. *There is a vertex of $G - S$, say x , such that $x \succ S$.*

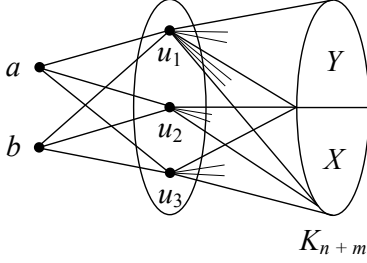


Figure 5. The structure of a graph in the class \mathcal{H}_5

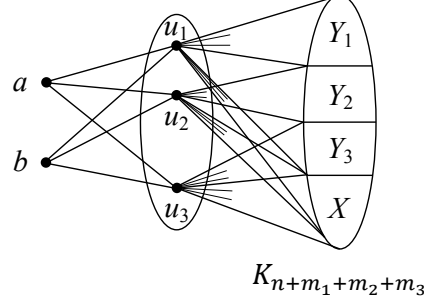


Figure 6. The structure of a graph in the class \mathcal{H}_6

235 2. At least two components of $G - S$ are singleton.

236 3. Each component of $G - S$ is complete.

237 **Proof.** Let C_1 , C_2 and C_3 be the components of $G - S$.

238 (1) Suppose to the contrary that no vertex of $G - S$ dominates S . Since
 239 S is a minimum cutset, $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$. Further, for each $s \in S$,
 240 $N_{C_i}(s) \neq \emptyset$. Let $u_1, u_2 \in S$. For $1 \leq i \leq 3$, let $x_i \in N_{C_i}(u_1)$ and $y_i \in N_{C_i}(u_2)$.
 241 Consider $G + x_i y_j$ where $1 \leq i \neq j \leq 3$. Put $\{z_{x_i y_j}\} = I_{x_i y_j} - \{x_i, y_j\}$. By
 242 Lemma 4(2), $\{x_i, y_j, z_{x_i y_j}\}$ is independent and thus $z_{x_i y_j} \notin \{u_1, u_2\}$. Since
 243 $|V(C_i)| \geq 2$ for $1 \leq i \leq 3$, $z_{x_i y_j} \in S$. Further, $z_{x_i y_j} \succ C_k$ where $1 \leq k \leq 3$
 244 and $\{i, j, k\} = \{1, 2, 3\}$. It then follows that $|\{z_{x_1 y_2}, z_{x_2 y_3}, z_{x_1 y_3}\}| = 3$. Since
 245 $|S| \leq 5$, $S = \{u_1, u_2, z_{x_1 y_2}, z_{x_2 y_3}, z_{x_1 y_3}\}$. Consequently, $z_{y_1 x_3} = z_{x_1 y_3}$ otherwise
 246 no vertex of $I_{y_1 x_3}$ dominates C_2 . Hence, $\{x_1, y_1, x_3, y_3\} \cap N_G(z_{x_1 y_3}) = \emptyset$. If
 247 $I_{x_1 y_3} = \{x_1, z_{x_1 y_3}\}$, then no vertex of $I_{x_1 y_3}$ dominates x_3 , a contradiction. Hence,
 248 $I_{x_1 y_3} = \{y_3, z_{x_1 y_3}\}$. But then no vertex of $I_{x_1 y_3}$ dominates y_1 , again a contradic-
 249 tion. This proves (1).

250

251 (2) By (1), we may assume that $x \in V(C_1)$. Choose $y \in V(C_2)$. Consider
 252 $G + xy$. Let $\{z\} = I_{xy} - \{x, y\}$. By Lemma 4(2), $z \notin S$. Then $z \in V(C_3)$. We
 253 first suppose that $I_{xy} = \{x, z\}$. Then $V(C_2) = \{y\}$. Now consider $G + xz$. Then,
 254 by Lemma 5, $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. If $I_{xz} = \{x, y\}$, then $|V(C_3)| = 1$.
 255 If $I_{xz} = \{z, y\}$, then $|V(C_1)| = 1$. Hence, G contains at least two singleton
 256 components as required.

257 We now suppose that $I_{xy} = \{y, z\}$. By similar arguments, G contains at least
 258 two singleton components. This proves (2).

259

260 (3) By (2), we may assume without loss of generality that $|V(C_1)| = |V(C_2)| =$
 261 1. Put $\{x\} = V(C_1)$ and $\{y\} = V(C_2)$. Since S is a minimum cutset, $x \succ S$
 262 and $y \succ S$. Suppose to the contrary that C_3 is not complete. Then there exist

263 $a, b \in V(C_3)$ where $ab \notin E(G)$. Consider $G + xa$. Let $\{z\} = I_{xa} - \{x, a\}$. Because
 264 no vertex of $\{x, a\}$ is adjacent to a vertex of $\{y, b\}$, $z \succ \{y, b\}$. Then $z \in S$ since
 265 $y \in V(C_2)$ and $b \in V(C_3)$. But this contradicts Lemma 4(2) since $x \succ S$. This
 266 proves (3) and completes the proof of our lemma. ■

267 **Theorem 8.** *Let G be a connected 3- i -critical graph and S a minimum cutset of*
 268 *size 2. If $G - S$ contains exactly three components, then $G \in \mathcal{H}_1$.*

269 **Proof.** Let C_1, C_2 and C_3 be the components of $G - S$. By Lemma 7(2) we may
 270 assume that $|V(C_1)| = 1$ and $|V(C_2)| = 1$. Put $\{a\} = V(C_1)$, $\{b\} = V(C_2)$ and
 271 $S = \{u, v\}$. Since S is a minimum cutset, $N_G(a) = N_G(b) = \{u, v\}$. Further,
 272 if $|V(C_3)| = 1$, then it is easy to see that $i(G) \leq 2$, a contradiction. Hence,
 273 $|V(C_3)| \geq 2$. By Lemma 7(3), C_3 is complete. Note that $N_{C_3}(u) \neq \emptyset$ and
 274 $N_{C_3}(v) \neq \emptyset$ since S is a minimum cutset. If there is a vertex $y \in N_{C_3}(u) - N_{C_3}(v)$,
 275 then $\{y, v\} \succ_i G$, a contradiction. Similarly, if there is a vertex $y \in N_{C_3}(v) -$
 276 $N_{C_3}(u)$, then $\{y, u\} \succ_i G$, again a contradiction. Hence, $N_{C_3}(u) \cup N_{C_3}(v) =$
 277 $N_{C_3}(u) \cap N_{C_3}(v)$. It is easy to see that $|N_{C_3}(u) \cup N_{C_3}(v)| = |N_{C_3}(u) \cap N_{C_3}(v)| \geq 2$
 278 since S is a minimum cutset of size 2. Because $i(G) = 3$, $|V(C_3) - (N_{C_3}(u) \cup$
 279 $N_{C_3}(v))| \geq 1$. Further, $uv \notin E(G)$. Therefore, $G \in \mathcal{H}_1$ as required. This
 280 completes the proof of our theorem. ■

281 **Theorem 9.** *Let G be a connected 3- i -critical graph and S a minimum cutset of*
 282 *size 3. If $G - S$ contains exactly three components, then $G \in \mathcal{H}_i$ for $2 \leq i \leq 6$.*

283 **Proof.** Let C_1, C_2 and C_3 be the components of $G - S$. By Lemma 7(2) we may
 284 assume that $|V(C_1)| = 1$ and $|V(C_2)| = 1$. Put $\{a\} = V(C_1)$, $\{b\} = V(C_2)$ and
 285 $S = \{u_1, u_2, u_3\}$. Since S is a minimum cutset, $N_G(a) = N_G(b) = \{u_1, u_2, u_3\}$.
 286 Further, if $|V(C_3)| = 1$, then S must be independent since $i(G) = 3$ and thus
 287 $G \in \mathcal{H}_4$. So we may suppose that $|V(C_3)| \geq 2$. Note that C_3 is complete by
 288 Lemma 7(3). Further, $N_{C_3}(u_i) \neq \emptyset$ for $1 \leq i \leq 3$ since S is a minimum cutset.

289 **Claim 1:** $G[S]$ contains at most one edge.

290 Suppose to the contrary that $G[S]$ contains at least two edges. Then there ex-
 291 ists $i \in \{1, 2, 3\}$ such that $u_i u_j \in E(G)$ for $1 \leq i \neq j \leq 3$. Clearly, $u_i \succ S \cup \{a, b\}$.
 292 It is easy to see that $i(G) \leq 2$ since C_3 is complete. But this contradicts the fact
 293 that $i(G) = 3$. This settles our claim.

294 We now distinguish two cases according to the number of edges in $G[S]$.

295 **Case 1:** $G[S]$ contains exactly one edge.

296

297

298

299

300 We may assume without loss of generality that $u_1u_2 \in E(G)$. Then $u_1u_3 \notin$
 301 $E(G)$ and $u_2u_3 \notin E(G)$.

302
 303 **Claim 2:** $N_{C_3}(u_3) = N_{C_3}(u_1) \cap N_{C_3}(u_2)$.

304 It is easy to see that if there exists $x \in N_{C_3}(u_3)$ but $x \notin N_{C_3}(u_1) \cap N_{C_3}(u_2)$,
 305 then $\{x, u_j\} \succ_i G$ for $1 \leq j \leq 2$, a contradiction. Hence, $N_{C_3}(u_3) \subseteq N_{C_3}(u_1) \cap$
 306 $N_{C_3}(u_2)$. On the other hand, if $x \in N_{C_3}(u_1) \cap N_{C_3}(u_2)$ but $x \notin N_{C_3}(u_3)$, then
 307 $\{x, u_3\} \succ_i G$, again a contradiction. Hence, $N_{C_3}(u_1) \cap N_{C_3}(u_2) \subseteq N_{C_3}(u_3)$. This
 308 settles our claim.

309
 310 Now put $Y = V(C_3) - N_{C_3}(u_3)$. It is easy to see that $Y \neq \emptyset$ since $i(G) = 3$.

311
 312 **Claim 3:** Each vertex of Y is adjacent to at most one vertex of $\{u_1, u_2\}$.

313 Clearly, if there exists $y \in Y$ such that $yu_1 \in E(G)$ and $yu_2 \in E(G)$, then
 314 $\{y, u_3\} \succ_i G$, a contradiction. This settles our claim.

315
 316 **Case 1.1:** $|Y| = 1$.

317 Put $\{v\} = Y$. If $vu_j \in E(G)$ for some $1 \leq j \leq 2$, then $\{u_j, u_3\} \succ_i G$, a contra-
 318 diction. Hence, $vu_j \notin E(G)$ for $1 \leq j \leq 2$. Consequently, $N_G(v) = V(C_3) - \{v\}$.
 319 Because S is a minimum cutset, $|V(C_3) - \{v\}| \geq 3$. Therefore, $G \in \mathcal{H}_2$.

320
 321 **Case 1.2:** $|Y| \geq 2$.

322
 323 **Claim 4:** Each vertex of Y is adjacent to exactly one vertex of $\{u_1, u_2\}$.
 324 Further, $|N_Y(u_1)| \geq 1$ and $|N_Y(u_2)| \geq 1$.

325 We first suppose to the contrary that there is a vertex $y \in Y$ such that
 326 $yu_1 \notin E(G)$ and $yu_2 \notin E(G)$. Thus $N_S(y) = \emptyset$. Since $|Y| \geq 2$, there is $y_1 \in$
 327 $Y - \{y\}$. Consider $G + u_3y_1$. Let $\{z_1\} = I_{u_3y_1} - \{u_3, y_1\}$. By Lemma 4(2),
 328 $z_1 \notin V(C_3) \cup \{a, b\}$ since C_3 is complete and $\{a, b\} \subseteq N_G(u_3)$. Then $z_1 \in \{u_1, u_2\}$.
 329 If $I_{u_3y_1} = \{u_3, z_1\}$, then no vertex of $I_{u_3y_1}$ is adjacent to y , a contradiction. Hence,
 330 $I_{u_3y_1} = \{y_1, z_1\}$. We may assume that $z_1 = u_1$. Then $u_1y_1 \notin E(G)$. Now consider
 331 $G + u_1y$. Let $\{z_2\} = I_{u_1y} - \{u_1, y\}$. By Lemma 4(2), $z_2 \notin V(C_3) \cup \{a, b, u_2\}$ since
 332 C_3 is complete and $\{a, b, u_2\} \subseteq N_G(u_1)$. Thus $z_2 = u_3$. If $I_{u_1y} = \{u_1, u_3\}$, then
 333 no vertex of I_{u_1y} is adjacent to y_1 , a contradiction. Hence, $I_{u_1y} = \{y, u_3\}$. But
 334 then no vertex of I_{u_1y} is adjacent to u_2 , again a contradiction. This proves that
 335 each vertex of Y is adjacent to at least one vertex of $\{u_1, u_2\}$. It then follows by
 336 Claim 3 that each vertex of Y is adjacent to exactly one vertex of $\{u_1, u_2\}$.

337 We next show that $N_Y(u_1) \neq \emptyset$. Suppose this is not the case. Then u_1 is not
 338 adjacent to any vertex of Y . Then each vertex of Y is adjacent to u_2 by above
 339 argument. Consequently, $\{u_2, u_3\} \succ_i G$, a contradiction. Hence, $N_Y(u_1) \neq \emptyset$.
 340 Similarly, $N_Y(u_2) \neq \emptyset$. This settles our claim.

341

342 Put $Y_1 = N_Y(u_1)$ and $Y_2 = N_Y(u_2)$. It follows by Claim 4 that $Y_1 \cap Y_2 = \emptyset$
 343 and hence $G \in \mathcal{H}_3$.

344

345 **Case 2:** $G[S]$ is independent.

346

347 **Claim 5:** For each $x \in V(C_3)$, either x is adjacent to every vertex of
 348 $\{u_1, u_2, u_3\}$ or x is adjacent to exactly one vertex of $\{u_1, u_2, u_3\}$.

349 We first suppose to the contrary that there is a vertex $x_1 \in V(C_3)$ such
 350 that x_1 is not adjacent to any vertex of $\{u_1, u_2, u_3\}$. Consider $G + x_1u_1$. Let
 351 $\{z\} = I_{x_1u_1} - \{x_1, u_1\}$. By Lemma 4(2) and the fact that C_3 is complete and
 352 $\{a, b\} \subseteq N_G(u_1)$, it follows that $z \in \{u_2, u_3\}$. Consequently, $u_2u_3 \in E(G)$, a
 353 contradiction. Hence, each vertex of $V(C_3)$ is adjacent to at least one vertex of
 354 $\{u_1, u_2, u_3\}$.

355 We now suppose that there is a vertex $x_2 \in V(C_3)$ such that x_2 is adjacent
 356 to u_i and u_j where $1 \leq i \neq j \leq 3$. If $x_2u_k \notin E(G)$ where $k \in \{1, 2, 3\} - \{i, j\}$,
 357 then $\{x_2, u_k\} \succ_i G$, a contradiction. This settles our claim.

358

359 For $1 \leq i \leq 3$, let $Y_i = \{y \in V(C_3) | yu_i \in E(G) \text{ but } yu_j \notin E(G) \text{ for } 1 \leq i \neq j \leq 3\}$.
 360 By Claim 5, $Y_i \cap Y_j = \emptyset$ for $1 \leq i \neq j \leq 3$. Put $X = V(C_3) - (Y_1 \cup Y_2 \cup Y_3)$.
 361 Then each vertex of X is adjacent to every vertex of $\{u_1, u_2, u_3\}$.

362

363 **Claim 6:** For $1 \leq i \leq 3$, if $Y_i \neq \emptyset$, then either $Y_j \neq \emptyset$ for all $1 \leq i \neq j \leq 3$
 364 or $Y_j = \emptyset$ for all $1 \leq i \neq j \leq 3$.

365 Assume that $Y_i \neq \emptyset$ for some $1 \leq i \leq 3$. Suppose to the contrary that $Y_j \neq \emptyset$
 366 but $Y_k = \emptyset$ where $\{j, k\} = \{1, 2, 3\} - \{i\}$. Then $V(C_3) = Y_i \cup Y_j \cup X$. Consider
 367 $G + u_iu_j$. Let $\{z\} = I_{u_iu_j} - \{u_i, u_j\}$. By Lemma 4(2), $z \notin V(C_3) \cup \{a, b\}$. Then
 368 $z = u_k$. If $I_{u_iu_j} = \{u_i, u_k\}$, then no vertex of $I_{u_iu_j}$ is adjacent to a vertex of Y_j ,
 369 a contradiction. Hence, $I_{u_iu_j} = \{u_j, u_k\}$. But then no vertex of $I_{u_iu_j}$ is adjacent
 370 to a vertex of Y_i , again a contradiction. This settles our claim.

371

372 We now distinguish two subcases.

373

374 **Subcase 2.1:** For $1 \leq i \leq 3$, $Y_i = \emptyset$.

375 It is easy to see that $V(C_3) = X$ and thus $G \in \mathcal{H}_4$.

376

377 **Subcase 2.2:** $Y_i \neq \emptyset$ for some $1 \leq i \leq 3$.

378 We first suppose that $Y_j = \emptyset$ for all $1 \leq i \neq j \leq 3$. If $X = \emptyset$, then u_i becomes
 379 a cutvertex of G , contradicting the fact that $S = \{u_1, u_2, u_3\}$ is a minimum cutset.
 380 Hence, $X \neq \emptyset$. Note that $\{u_i\} \cup X$ is a vertex cutset of G . Since the minimum
 381 cardinality of a vertex cutset in G is 3, $|X| \geq 2$. It is easy to see that $G \in \mathcal{H}_5$.

382 We now suppose that $Y_j \neq \emptyset$ for all $1 \leq i \neq j \leq 3$. Then $G \in \mathcal{H}_6$. This
383 completes the proof of our theorem. ■

384 Our last result follows immediately from Theorems 8 and 9.

385 **Corollary 10.** *Let G be a connected 3- i -critical graph and S a minimum cutset*
386 *where $2 \leq |S| \leq 3$. If $\omega(G - S) = 3$, then the minimum degree of G is $|S|$.*

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A characterization of connected 3- i -critical graphs of connectivity two

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Abstract

A subset S of $V(G)$ is an independent dominating set for G if S is independent and each vertex of G is either in S or adjacent to some vertex of S . Let $i(G)$ denote the minimum cardinality of an independent dominating set for G . Graph G is k - i -critical if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G . In this paper, we characterize connected 3- i -critical graphs with minimum cutset of size 2. In fact, we show that if G is a connected 3- i -critical graph with minimum cutset S of size 2 and the number of components of $G - S$ is exactly two, then G is isomorphic to a graph in one of nine classes of connected 3- i -critical graphs.

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1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by \overline{G} . For graphs G_1 and G_2 , we say that G_1 and G_2 are disjoint if $V(G_1) \cap V(G_2) = \emptyset$. A complete graph of order n is denoted by K_n and a complete bipartite graph with bipartitioning sets of order n and m is denoted by $K_{n,m}$. $K_{1,m}$ is also called a star. For vertices u and v of G , $d(u, v)$ denotes the distance between u and v . The diameter of G is the maximum distance between a pair of vertices of G .

For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. S is independent if no two vertices of S are adjacent. The number of components of G is denoted by $\omega(G)$. A vertex subset S of $V(G)$ is a cutset of G if $\omega(G - S) > \omega(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G , denoted by $N_G(v)$, is the set of all vertices of $V(G)$ which are adjacent to v . $N_G[v]$ denotes the closed neighborhood of v , i.e., $N_G[v] = N_G(v) \cup \{v\}$. The non-neighborhood of v in G denoted by $\overline{N}_G(v)$ is $V(G) - N_G[v]$. For $S \subseteq V(G)$, $N_G(v) \cap S$, $N_G[v] \cap S$, $\overline{N}_G(v) \cap S$ and $\overline{N}_G[v] \cap S$ are denoted by $N_S(v)$, $N_S[v]$, $\overline{N}_S(v)$ and $\overline{N}_S[v]$, respectively. Further, $N_G(S)$ denotes $\bigcup_{x \in S} N_G(x)$ and $N_G[S]$ denotes $\bigcup_{x \in S} N_G[x]$. For simplicity, if S is a subgraph of G , then we also denote $N_G(v) \cap V(S)$, $N_G[v] \cap V(S)$, $\overline{N}_G(v) \cap S$ and $\overline{N}_G[v] \cap S$ by $N_S(v)$, $N_S[v]$, $\overline{N}_S(v)$ and $\overline{N}_S[v]$, respectively.

For subsets S and T of $V(G)$, we say that S dominates T , denoted by $S \succ T$, if $T \subseteq N_G[S]$. If $\{s\} \succ T$ and $S \succ V(H)$ where H is a subgraph of G , then we shall write, for simplicity, $s \succ T$ and $S \succ H$, respectively. We say that S is a *dominating set* for G if $S \succ G$. Thus S is a dominating set for G if each vertex of $V(G)$ is either in S or adjacent to some vertex of S . The minimum cardinality of a dominating set for G is called the domination number of G and denoted by $\gamma(G)$.

For a subgraph H of G , if $S \succ H$ and S is independent, then we say that S is an *independent dominating set* for H and denoted by $S \succ_i H$. Thus S is an independent dominating set for G if $S \succ_i G$. The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and denoted by $i(G)$. Observe that for any graph G , $\gamma(G) \leq i(G)$ and if $\gamma(G) = 1$, then $i(G) = 1$.

In 1994, Ao [3] introduced the concept of “independent domination critical”. A graph G is *k-i-critical* if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G . In [1], Ananchuen and Ananchuen proved that if G is a connected *k-i-critical* graph, for $k \geq 3$ with a cutvertex u , then $\omega(G - u) \leq k - 1$. Further, a characterization of such graph G with $\omega(G - u) = k - 1$ was given. Later, in [2], they established that if G is a connected *3-i-critical* graph and S is a vertex cutset of G , then $\omega(G - S) \leq |S| - 2$ if $|S| \geq 6$ and $\omega(G - S) \leq 3$ if $|S| \leq 5$. Further, they also gave a characterization of connected *3-i-critical* graph G containing a minimum cutset S of size 2 with $\omega(G - S) = 3$. The problem that arises is that of characterizing such graph with $\omega(G - S) = 2$. Our results in Section 4 resolve this problem and then provide a complete characterization of connected *3-i-critical* graph G containing a minimum cutset S of size 2.

2 Preliminary results

We begin this section with some terminology. For a pair of non-adjacent vertices u and v of G , I_{uv} denotes a minimum independent dominating set for $G + uv$. Our first two results follow immediately from the definition of k - i -critical graphs.

Lemma 2.1. *Let G be a connected k - i -critical graph and let u and v be non-adjacent vertices of G . Then $|I_{uv}| = k - 1$ and $|I_{uv} \cap \{u, v\}| = 1$. \square*

Lemma 2.2. *Let G be a connected 3- i -critical graph and let u and v be non-adjacent vertices of G . Then*

1. $I_{uv} = \{u, w\}$ or $I_{uv} = \{v, w\}$ for some $w \in V(G) - \{u, v\}$.
2. If $\{w\} = I_{uv} - \{u, v\}$, then $\{u, v, w\}$ is independent. \square

Our next result is a useful tool for establishing our main results.

Lemma 2.3. [2] *Let G be a connected 3- i -critical graph and let u and v be non-adjacent vertices of G . If $I_{uv} = \{u, w\}$ for some $w \in V(G) - \{u, v\}$, then $I_{uw} - \{u, w\} = \{v\}$, i.e., $I_{uw} = \{u, v\}$ or $I_{uw} = \{w, v\}$. \square*

Our last two results in this section were proved by Ao [3].

Lemma 2.4. [3] *The diameter of a connected 3- i -critical graph is at most 3. \square*

Lemma 2.5. [3] *A graph G is 2- i -critical if and only if $\overline{G} \cong \bigcup_{i=1}^n K_{1,r_i}$ for some positive integers r_i and n . \square*

3 Classes of connected 3- i -critical graphs with a minimum cutset of size 2

In this section, we provide classes of connected 3- i -critical graphs with a minimum cutset of size 2 where if G is a graph in such classes and S is a minimum cutset of size 2, then $G - S$ contains exactly two components.

I. The class \mathcal{A}_1

For $1 \leq i \leq 4$, let n_i be a positive integer where if $n_1 \geq 2$, then $n_2 = n_3 = 1$. Define a graph $G \in \mathcal{A}_1$ of order $n_1 + n_2 + n_3 + 2n_4 + 2$ as follows. Set $V(G) = \{u, v\} \cup \bigcup_{i=1}^4 X_i$ where $|X_i| = n_i$ for $1 \leq i \leq 3$ and $|X_4| = 2n_4$. Then, set $G[X_1] = K_{n_1}$, $G[X_2 \cup X_3] = K_{n_2+n_3}$ and $G[X_4]$ is a complete graph without a perfect matching. Further, join u to every vertex of

$\{v\} \cup X_2 \cup X_4$ and join v to every vertex of $X_3 \cup X_4$. Finally, join each vertex of X_1 to every vertex of X_4 . Figure 1 illustrates our construction. Note that in our diagrams a “double line” denotes the join and for simplicity, a “double dash line” denotes the adjacency of $\{u, v\}$ and vertices in our diagram where the star vertices are joined to both u and v , the triangle vertices are only joined to u , the square vertices are only joined to v and the diamond vertices are neither joined to u nor v .

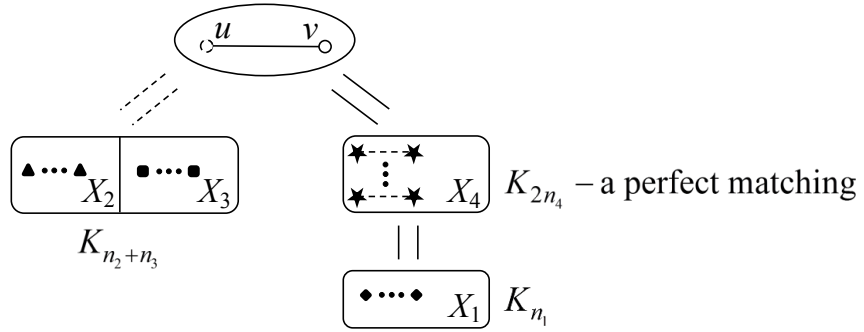


Figure 1: A structure of a graph in \mathcal{A}_1

II. The graph H_1 and the class \mathcal{A}_2

For $1 \leq i \leq 6$, let n_i be a positive integer where $n_6 \geq 2$, define the graph H_1 of order $\sum_{i=1}^5 2n_i + n_6 + 3$ as follows. Set $V(H_1) = \{u, v, z\} \cup \bigcup_{i=1}^6 X_i$ where for $1 \leq i \leq 5$, $|X_i| = 2n_i$ and $|X_6| = n_6 \geq 2$. Put $X_5 = X'_5 \cup X''_5$ where $H_1[X'_5] = H_1[X''_5] = K_{n_5}$. Now for $1 \leq i \leq 5$, set $H_1[X_i]$ to be a complete graph without a perfect matching and $H_1[X_6] = K_{n_6}$. Then for each $x \in X'_5$, there is a unique vertex $\bar{x} \in X''_5$ such that $x\bar{x} \notin E(H_1)$. Further, join u to every vertex of $\{v\} \cup X_2 \cup X_3 \cup X'_5 \cup X_6$, join v to every vertex of $X_2 \cup X_4 \cup X''_5 \cup X_6$ and join z to every vertex of X_6 . Finally, join each vertex of X_i to every vertex of X_j where $1 \leq i \neq j \leq 5$.

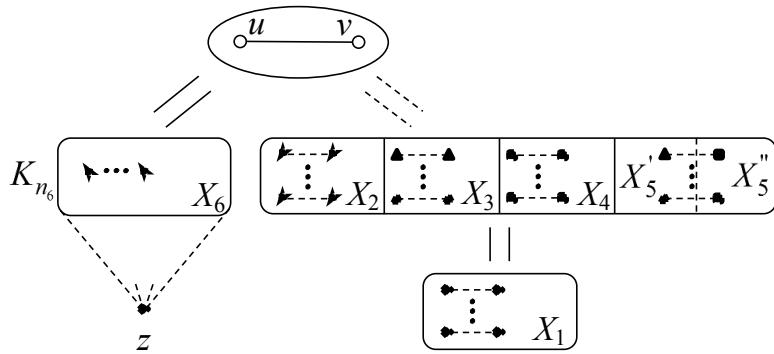


Figure 2: A structure of the graph H_1

We now define a new class of connected 3- i -critical graphs obtained from an induced

subgraph of H_1 . The class \mathcal{A}_2 consists of G_1, G_2, \dots, G_7 where, for $1 \leq i \leq 7$, G_i is obtained from an induced subgraph of H_1 as follows.

Put $G_1 = H_1[\{u, v\} \cup X_3 \cup X_4 \cup X_6]$, $G_2 = H_1[\{u, v\} \cup X_2 \cup X_3 \cup X_4 \cup X_6]$, $G_3 = H_1[\{u, v, z\} \cup X_2 \cup X_5 \cup X_6]$ if $|X_5| = 2$, $G_4 = H_1[\{u, v\} \cup X_1 \cup X_2 \cup X_6]$, $G_5 = H_1[\{u, v\} \cup X_1 \cup X_2 \cup X_3 \cup X_6]$, $G_6 = H_1[\{u, v\} \cup X_1 \cup X_3 \cup X_4 \cup X_6]$ and $G_7 = H_1[\{u, v\} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_6]$.

III. The class \mathcal{A}_3

For $1 \leq i \leq 3$, let n_i be a positive integer where $n_2 \geq 2$, define a graph $G \in \mathcal{A}_3$ as follows. Set $V(G) = \{u, v\} \cup X_1 \cup X_2 \cup X_3$ where for $1 \leq i \leq 2$, $|X_i| = n_i$ and $|X_3| = 2n_3$. Now set $G[X_1] = K_{n_1}$, $G[X_2] = K_{n_2}$ and $G[X_3]$ to be a complete graph without a perfect matching. Finally, join u and v to every vertex of $X_2 \cup X_3$ and join each vertex of X_1 to every of vertex X_2 . This defines a graph $G \in \mathcal{A}_3$. Figure 3 illustrates our construction.

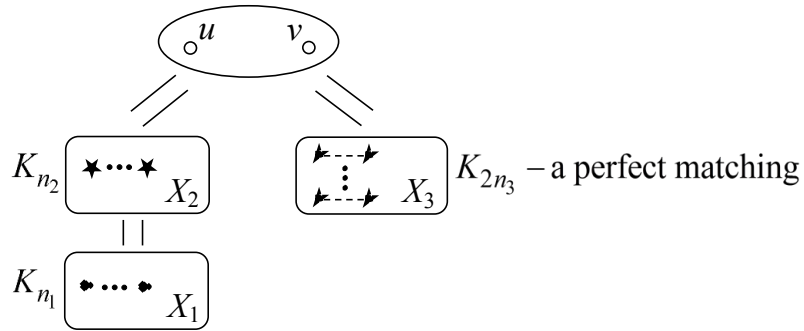


Figure 3: A structure of a graph in \mathcal{A}_3

IV. The graph H_2 and the class \mathcal{A}_4

For $1 \leq i \leq 6$, let n_i be a positive integer where $n_1 = n_5$ and $n_6 \geq 2$, define the graph H_2 of order $n_1 + n_5 + n_6 + 2(n_2 + n_3 + n_4) + 2$ as follows. Set $V(H_2) = \{u, v\} \cup \bigcup_{i=1}^6 X_i$ where $|X_i| = n_i$ for $i \in \{1, 5, 6\}$ and $|X_i| = 2n_i$ for $i \in \{2, 3, 4\}$. Now, for $2 \leq i \leq 4$, set $H_2[X_i]$ and $H_2[X_1 \cup X_5]$ to be complete graphs without a perfect matching where $H_2[X_1] = H_2[X_5] = K_{n_1} = K_{n_5}$ and $H_2[X_6] = K_{n_6}$. Further, join u to every vertex of $X_2 \cup X_3 \cup X_5 \cup X_6$, join v to every vertex of $X_2 \cup X_4 \cup X_5 \cup X_6$. Finally, join each vertex of X_i to every vertex of X_j for $i \in \{2, 3, 4\}$ and $j \in \{1, 2, 3, 4, 5\}$ where $i \neq j$. Figure 4 illustrates our construction.

We now define a new class of connected 3- i -critical graphs obtained from the graph H_2 and an induced subgraph H_2 . The class \mathcal{A}_4 consists of $H_2, H_2[\{u, v\} \cup X_1 \cup X_5 \cup X_6]$ if $|X_5| = |X_1| \geq 2$, $H_2[\{u, v\} \cup X_1 \cup X_i \cup X_5 \cup X_6]$ where $i \in \{2, 3, 4\}$, $H_2[\{u, v\} \cup X_1 \cup X_i \cup X_j \cup X_5 \cup X_6]$ where $i, j \in \{2, 3, 4\}$ and $i \neq j$.

Before we define new classes, we need to set up some terminology. For a graph H and a positive integer n , let X_i a subgraph of H where $1 \leq i \leq n$, we denote the induced

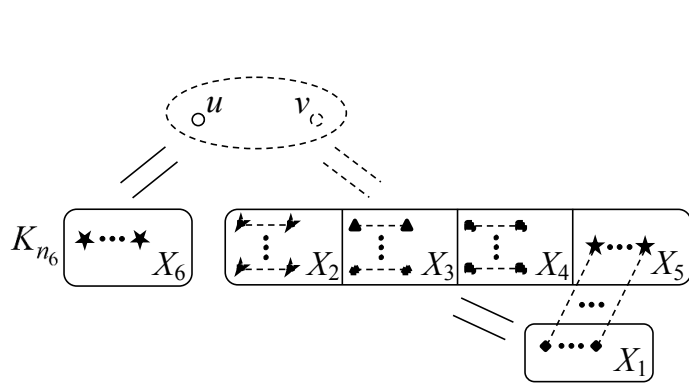


Figure 4: A structure of the graph H_2

subgraph of H by $H_{X_1 \dots X_n}$ where $V(H_{X_1 \dots X_n}) = \bigcup_{i=1}^n V(X_i)$ and $V(X_i) \cap V(X_j) = \emptyset$ for $1 \leq i \neq j \leq n$. Further, if X and Y are subsets of vertices of H where $X \cap Y = \emptyset$, then $E(X, Y)$ denotes set of all edges xy where $x \in X$ and $y \in Y$.

For positive integers m, n and r , let a, b and c be a center vertex of $K_{1,m}$, $K_{1,n}$ and $K_{1,r}$, respectively. A *double star* $DS(m, n)$ is a graph obtained from $K_{1,m}$ and $K_{1,n}$ by adding the edge ab and a *triple star* $TS(m, n, r)$ is a graph obtained from $K_{1,m}$, $K_{1,n}$ and $K_{1,r}$ by adding the edges ab and bc . In what follows, a and b in $DS(m, n)$ are called the *center vertices* while the vertices b and a, c in $TS(m, n, r)$ are called the *middle center vertex* and *end center vertices*, respectively. Figure 5 illustrates $DS(m, n)$ and $TS(m, n, r)$.

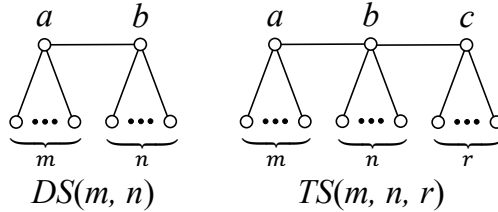


Figure 5: Graphs $DS(m, n)$ and $TS(m, n, r)$

For $1 \leq i \leq 3$, let $P_i \cong K_{2p_i}$ - a perfect matching and $D_i = K_{d'_i} - \bigcup_{j=1}^{d'_i} DS(m_j^i, n_j^i)$ where p_i, m_j^i, n_j^i are positive integers and $d'_i = \sum_{j=1}^{d'_i} (m_j^i + n_j^i + 2)$. Further, let $S_i = K_{s'_i} - \bigcup_{j=1}^{s'_i} K_{1, q_j^i}$ where q_j^i, s_i are positive integers and $s'_i = \sum_{j=1}^{s'_i} (q_j^i + 1)$. Finally, let $T = K_{t'} - \bigcup_{j=1}^t TS(m_j, n_j, r_j)$ where $t' = \sum_{j=1}^t (m_j + n_j + r_j + 3)$ and t is a positive integer.

V. The graph H_3 and the class \mathcal{A}_5

Let H_3 be a graph obtained from $P_1, P_2, D_1, D_2, S_1, S_2, S_3$ where each pair of such graphs is disjoint and $V(H_3) = \bigcup_{i=1}^2 V(P_i) \cup \bigcup_{i=1}^2 V(D_i) \cup \bigcup_{i=1}^3 V(S_i)$, $E(H_3) = \bigcup_{i=1}^2 E(P_i) \cup \bigcup_{i=1}^2 E(D_i) \cup \bigcup_{i=1}^3 E(S_i) \cup E(V(X), V(Y))$ where $X \neq Y$ and $\{X, Y\} \subseteq \{P_1, P_2, D_1, D_2, S_1, S_2, S_3\}$. For $1 \leq j \leq s_2$, let $\emptyset \neq R_j$ be a subset of $\{x | x \text{ is an end vertex of } K_{1, r_j^2} \text{ in } \bar{S}_2\}$. Put $R = \bigcup_{j=1}^{s_2} R_j$. Clearly, $|R| \geq s_2$. We now define a graph $G \in \mathcal{A}_5$ where G is obtained from H_3 by adding three new vertices, say u, v, w , and $E(H_3) \subseteq E(G)$ as

follows. Set $V(G) = V(H_3) \cup \{u, v, w\}$ and $E(G) = E(H_3) \cup \{uw, vw\} \cup E(\{u, v\}, V(S_1) \cup (V(S_2) - R) \cup \{x|x \text{ is an end vertex of a double star in } \overline{D_1} \cup \overline{D_2} \text{ or an end vertex of a star in } \overline{S_3}\}) \cup E(\{u\}, V(P_1) \cup \{x|x \text{ is a center vertex of a double star in } \overline{D_1}\}) \cup E(\{v\}, V(P_2) \cup \{x|x \text{ is a center vertex of a double star in } \overline{D_2}\})$. Figure 6 illustrates our construction. Recall that, in our diagrams, a “double dash line ” denotes the adjacency of $\{u, v\}$ and vertices in our diagram where the star vertices are joined to both u and v , the triangle vertices are only joined to u , the square vertices are only joined to v and the diamond vertices are neither joined to u nor v .

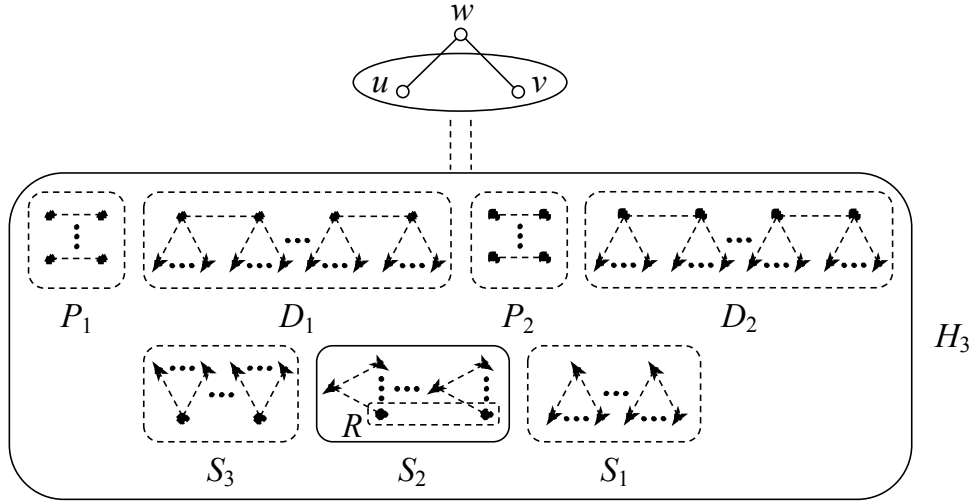


Figure 6: The structure of a graph in the class \mathcal{A}_5

VI. The class \mathcal{A}_6

For $G \in \mathcal{A}_5$, let $G' = G[\{u, v, w\} \cup V(S_2) \cup V(S_3)]$, $G'' = G[\{u, v, w\} \cup V(S_2)]$ if $\overline{S_2}$ contains exactly one star and there are at least two end vertices of the star which are not in R and $G''' = G[\{u, v, w\} \cup V(S_3)]$ if $\overline{S_3}$ contains at least two stars. For $1 \leq i \leq n \leq 5$, let X_i be a graph in $\{P_1, P_2, D_1, D_2, S_1\}$. We now let $G_{X_1 \dots X_n}^1$ be an induced subgraph of G where $V(G_{X_1 \dots X_n}^1) = \{u, v, w\} \cup V(S_2) \cup V(S_3) \cup \bigcup_{i=1}^n V(X_i)$ and for $2 \leq j \leq 3$, let $G_{X_1 \dots X_n}^j$ be an induced subgraph of G where $V(G_{X_1 \dots X_n}^j) = \{u, v, w\} \cup V(S_j) \cup \bigcup_{i=1}^n V(X_i)$. Figure 7 shows a structure of G'' , G''' , $G_{P_1}^3$ and $G_{P_1 D_2}^2$. For $G \in \mathcal{A}_5$, let $\mathcal{A}_6 = \{G', G'', G''', G_{X_1 \dots X_n}^j | 1 \leq j \leq 3 \text{ and } 1 \leq n \leq 5\}$.

VII. The graph H_4 and the class \mathcal{A}_7

Let H_4 be a graph obtained from $P_1, P_2, P_3, D_1, D_2, D_3, S_1$ and T where each pair of such graphs is disjoint and $V(H_4) = \bigcup_{i=1}^3 V(P_i) \cup \bigcup_{i=1}^3 V(D_i) \cup V(S_1) \cup V(T)$, $E(H_4) = \bigcup_{i=1}^3 E(P_i) \cup \bigcup_{i=1}^3 E(D_i) \cup E(S_1) \cup E(T) \cup E(V(X), V(Y))$ where $X \neq Y$ and $\{X, Y\} \subseteq \{P_1, P_2, P_3, D_1, D_2, D_3, S_1, T\}$. We now define a graph $G \in \mathcal{A}_7$ where G is obtained from H_4 by adding three new vertices, say u, v, w and $E(H_4) \subseteq E(G)$ as follows. Set $V(G) = V(H_4) \cup \{u, v, w\}$ and $E(G) = E(H_4) \cup \{uw, uv, vw\} \cup E(\{u, v\}, V(S_1) \cup \{x|x \text{ is$

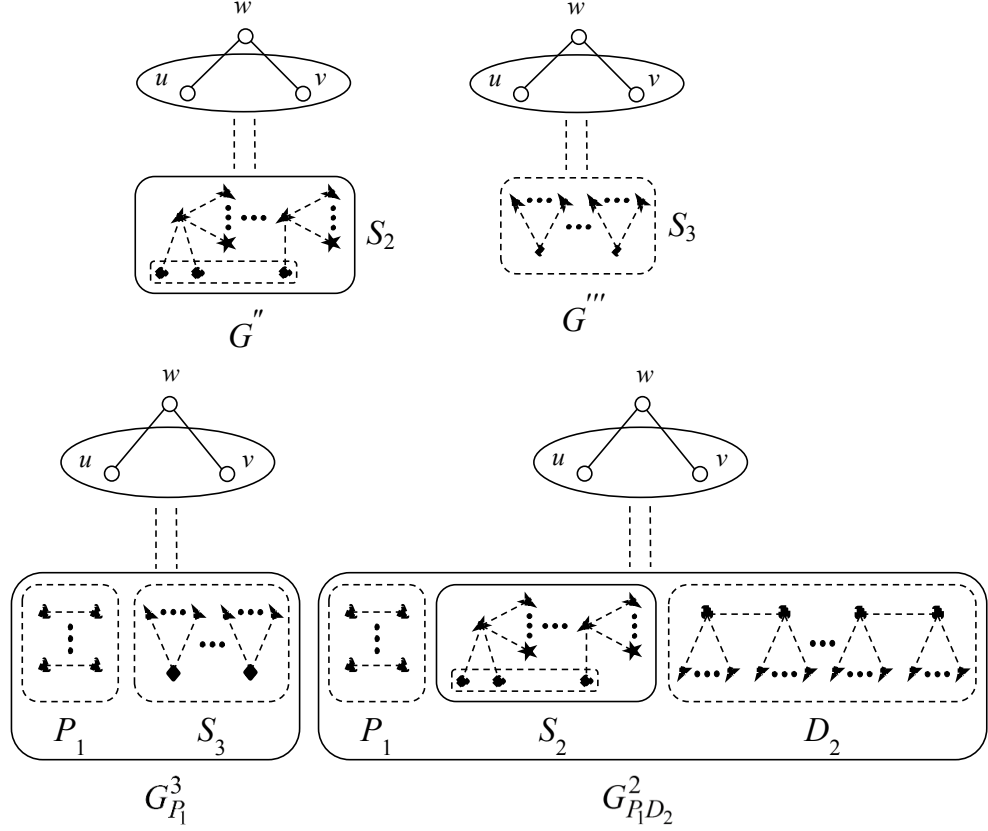


Figure 7: Some graphs in the class \mathcal{A}_6

an end vertex of a double star in $\bigcup_{i=1}^3 \overline{D}_i$ or a triple star in \overline{T}) $\cup E(\{u\}, V(P_1) \cup \{x|x \text{ is a center vertex of a double star in } \overline{D}_1\}) \cup \{x|x \text{ is one of end center vertices of a triple star in } \overline{T}\} \cup E(\{v\}, V(P_2) \cup \{x|x \text{ is a center of a double star in } \overline{D}_2\}) \cup \{x|x \text{ is the end center vertex of a triple star in } \overline{T} \text{ that is not adjacent to } u\}$. Figure 8 illustrates our construction.

VIII. The class \mathcal{A}_8

For $G \in \mathcal{A}_7$, let $R_1 = G[\{u, v, w\} \cup V(P_1) \cup V(P_2)]$, $R_2 = G[\{u, v, w\} \cup V(P_1) \cup V(D_2)]$, $R_3 = G[\{u, v, w\} \cup V(D_1) \cup V(D_2)]$. For $1 \leq i \leq n \leq 4$, let X_i be a graph in $\{R_1, R_2, R_3, S_1\}$ and denote the induced subgraph of $G \in \mathcal{A}_7$ by $G_{X_1 \dots X_n}$. We now define $\mathcal{A}_8 = \{G_{R_1}, G_{R_2}, G_{R_3}, G_{R_1 D_1}, G_{R_1 S_1}, G_{R_2 D_1}, G_{R_2 S_1}, G_{R_3 S_1}, G_{R_1 D_1 D_2}, G_{R_1 D_1 S_1}, G_{R_2 D_1 S_1}, G_{R_1 D_1 D_2 S_1}\}$. Figure 9 illustrates our construction of $G_{R_1 D_1 D_2 S_1}$.

IX. The class \mathcal{A}_9

For $G \in \mathcal{A}_7$, let $W_1 = G[\{u, v, w\} \cup V(P_3)]$, $W_2 = G[\{u, v, w\} \cup V(D_3)]$, $W_3 = G[\{u, v, w\} \cup V(T)]$, $W_4 = G[\{u, v, w\} \cup V(P_3) \cup V(D_3)]$, $W_5 = G[\{u, v, w\} \cup V(P_3) \cup V(T)]$, $W_6 = G[\{u, v, w\} \cup V(D_3) \cup V(T)]$ and $W_7 = G[\{u, v, w\} \cup V(P_3) \cup V(D_3) \cup V(T)]$. Further, let $Q_1 = G[\{u, v, w\} \cup V(D_3)]$ if \overline{D}_3 contains either at least two double stars or exactly one double star $DS(m, n)$ where $m \geq 2, n \geq 2$. For $1 \leq i \leq 5$, let F_i be a graph in

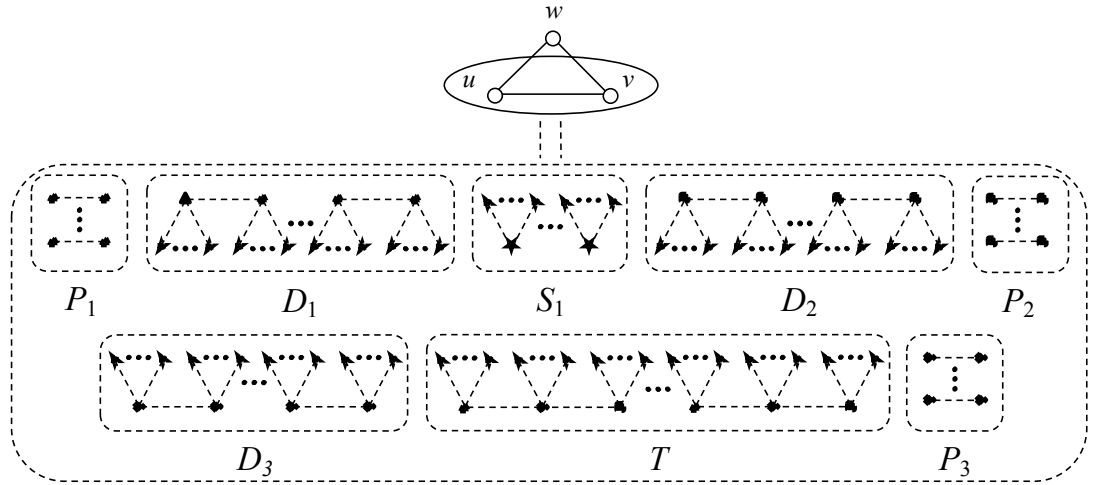


Figure 8: The structure of a graph in the class \mathcal{A}_7

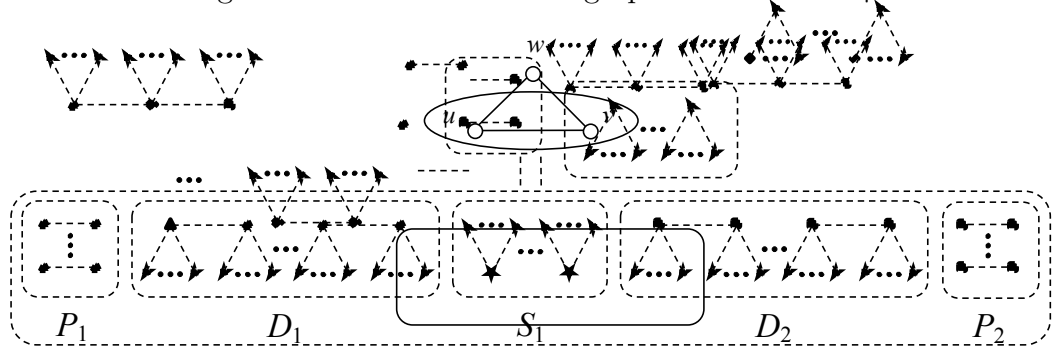


Figure 9: The structure of $G_{R_1D_1D_2S_1}$ in the class \mathcal{A}_8

$\{P_1, P_2, D_1, D_2, S_1\}$ and assume that if $1 \leq i \neq j \leq 5, F_i \neq F_j$. For $3 \leq k \leq 7, 2 \leq k' \leq 7, 1 \leq n \leq 5$ and $l \geq 2$, let $\mathcal{A}_9 = \{G_{Q_1}, G_{W_1D_1}, G_{W_1S_1}, \dots, G_{W_1F_1}, F_1, G_{W_k}, G_{W_k}, F_1, \dots, F_n\}$. Note that $G_{W_7P_1P_2D_1D_2S_1}$ is a graph in \mathcal{A}_7 .

It is not difficult to show that if $G \in \mathcal{A}_9$, for $1 \leq i \leq 9$, then G is a connected 3- i -critical graph where $\{u, v\}$ is a minimum cutset of G and $\omega(G - \{u, v\}) = 2$.

4 The main results

In this section, we establish a characterization of connected 3- i -critical graph G having a minimum cutset S of size 2 with $\omega(G - S) = 2$. Recall that for a pair of non-adjacent vertices x and y of G , I_{xy} denotes a minimum independent dominating set for $G + xy$. We begin with the following lemma.

Lemma 4.1. *Let G be a connected 3- i -critical graph and S a minimum cutset of size 2 where $S = \{u, v\}$ and $\omega(G - S) = 2$. Further, let C_1 and C_2 be the components of $G - S$.*

Then

1. Exactly one of C_1 or C_2 is complete.
2. Suppose C_1 is complete. Then no vertex of $N_{C_2}(u) \cup N_{C_2}(v)$ dominates C_2 . Further, $i(C_2) \leq 2$.

Proof. Since S is a minimum cutset, for $1 \leq i \leq 2$, $N_{C_i}(u) \neq \emptyset$ and $N_{C_i}(v) \neq \emptyset$.

Claim 1: If $x \in V(C_1)$ and $y \in V(C_2)$ where x does not dominate C_1 and y does not dominate C_2 , then $I_{xy} \cap S \neq \emptyset$.

It is easy to see that if $I_{xy} \cap S = \emptyset$, then either $x \succ C_1$ or $y \succ C_2$ since $|I_{xy}| = 2$ and $|I_{xy} \cap \{x, y\}| = 1$.

(1) Suppose to the contrary that neither C_1 nor C_2 is complete. Then there exist $x_1, y_1 \in V(C_1)$ and $x_2, y_2 \in V(C_2)$ such that $x_1 y_1 \notin E(G)$ and $x_2 y_2 \notin E(G)$. By Claim 1, we may suppose without loss of generality that $I_{x_1 x_2} = \{x_1, u\}$. Then $u y_1, u y_2 \in E(G)$. Thus $u \notin I_{y_1 y_2}$ and $u \notin I_{x_1 y_2}$ by Lemma 2.2(2). It follows by Claim 1 that $v \in I_{y_1 y_2}$ and $v \in I_{x_1 y_2}$. Since $v \in I_{y_1 y_2}$, $v x_1 \in E(G)$ and since $v \in I_{x_1 y_2}$, $v x_1 \notin E(G)$ by Lemma 2.2(2), a contradiction. Hence, C_1 or C_2 is complete. Since S is a minimum cutset, $N_{C_1}(u) \neq \emptyset$ and $N_{C_2}(v) \neq \emptyset$. If C_1 and C_2 are complete, then $\{u_1, v_1\} \succ_i G$ where $u_1 \in N_{C_1}(u)$ and $v_1 \in N_{C_2}(v)$, a contradiction. Hence, exactly one of C_1 or C_2 is complete. This proves (1).

(2) By (1), C_2 is not complete.

Claim 2: No vertex of $N_{C_2}(u) \cup N_{C_2}(v)$ dominates C_2 .

Suppose to the contrary that there is a vertex $x \in N_{C_2}(u)$ such that $x \succ C_2$. Let $y \in N_{C_1}(v)$. Then $\{x, y\} \succ_i G$ since C_1 is complete, a contradiction. This settles our claim.

By Claim 2, it is easy to see that $|V(C_2)| \geq 2$. Let $x \in N_{C_1}(u)$ and $y \in N_{C_2}(v)$. Now consider $G + xy$. By Lemma 2.2(2). $I_{xy} \cap \{u, v\} = \emptyset$. Let $\{z\} = I_{xy} - \{x, y\}$. Then either $I_{xy} = \{x, z\}$ or $I_{xy} = \{y, z\}$. We first suppose that $I_{xy} = \{x, z\}$. Then $z \in V(C_2)$ since $|V(C_2)| \geq 2$. Clearly, $z \succ V(C_2) - \{y\}$ and thus $\{z, y\} \succ_i C_2$. Hence, $i(C_2) \leq 2$. We now suppose that $I_{xy} = \{y, z\}$. By Lemma 2.2(2) and the fact that C_1 is complete, $z \notin V(C_1)$. Thus $z \in V(C_2)$. Hence, $\{y, z\} \succ_i C_2$. This proves that $i(C_2) \leq 2$ and completes the proof of our lemma. \square

Theorem 4.2. Let G and S be defined as in Lemma 4.1. Further, let C_1 and C_2 be the components of $G - S$ where C_1 is complete. If $i(C_2) = 1$, then $G \in \mathcal{A}_1$.

Proof. Let $A = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. Since $i(C_2) = 1$, there exists $a \in V(C_2)$ where $a \succ C_2$. By Lemma 4.1(2), $a \in A$.

Claim 1: $N_{C_1}(u) \cap N_{C_1}(v) = \emptyset$. Further, $V(C_1) = N_{C_1}(u) \cup N_{C_1}(v)$.

If there is a vertex $x \in N_{C_1}(u) \cap N_{C_1}(v)$, then $\{x, a\} \succ_i G$, a contradiction. Hence, $N_{C_1}(u) \cap N_{C_1}(v) = \emptyset$. If there is a vertex $y \in V(C_1) - (N_{C_1}(u) \cup N_{C_1}(v))$, then $d(y, a) > 3$. But this contradicts Lemma 2.4. This settles our claim.

Since S is a minimum cutset, $N_{C_1}(u) \neq \emptyset$ and $N_{C_1}(v) \neq \emptyset$. Thus, by Claim 1, $|V(C_1)| = |N_{C_1}(u)| + |N_{C_1}(v)| \geq 2$.

Claim 2: $uv \in E(G)$.

Suppose this is not the case. Consider $G + uv$. We may assume without loss of generality that $I_{uv} = \{u, z\}$ for some $z \in V(G) - \{u, v\}$. Thus $z \in V(C_1) \cup V(C_2)$ and $z \succ N_{C_1}(v) \cup (V(C_2) - N_G(u))$. But this is not possible. Hence, $uv \in E(G)$ as required.

Claim 3: $N_{C_2}(u) \cap N_{C_2}(v) = N_{C_2}(u) \cup N_{C_2}(v)$.

Suppose there is a vertex $y \in N_{C_2}(u) - N_{C_2}(v)$. Consider $G + yv$. By Lemma 2.2(2) and the fact that $a \succ C_2$, $\{u, a\} \cap I_{yv} = \emptyset$. Put $\{z\} = I_{yv} - \{y, v\}$. Clearly, $z \in V(C_1) \cup V(C_2)$. If $I_{yv} = \{v, z\}$, then $z \succ N_{C_1}(u) \cup A$ which is not possible. Hence, $I_{yv} = \{y, z\}$. Since $y \in V(C_2)$, $z \in V(C_1)$. Then $y \succ C_2$. But this contradicts Lemma 4.1(2) since $y \in N_{C_2}(u)$. Hence, $N_{C_2}(u) - N_{C_2}(v) = \emptyset$. Similarly, $N_{C_2}(v) - N_{C_2}(u) = \emptyset$. This settles our claim.

Claim 4: $G' = G[N_{C_2}(u) \cup N_{C_2}(v)] \cong K_{2m} - a \text{ perfect matching}$.

Let $x \in N_{C_2}(u) \cup N_{C_2}(v)$. Choose $w \in N_{C_1}(u)$. Consider $G + xw$. By Claim 3 and Lemma 2.2(2), $I_{xw} \cap \{u, v\} = \emptyset$. Because C_1 is complete and $|V(C_1)| \geq 2$, it follows that $w \in I_{xw}$. Put $\{z\} = I_{xw} - \{w\}$. Then $z \in V(C_2)$ and thus $z \succ V(C_2) - \{x\}$. Since $wv \notin E(G)$ by Claim 1, $z \in N_{C_2}(v)$. Clearly, $\{x, z\} \succ_i G'$. Hence, $i(G') = 2$ by Lemma 4.1(2). It also follows that $z \succ G' + xz$. Hence, G' is 2- i -critical. By Lemma 2.5, $G' \cong K_m - \bigcup_{i=1}^s K_{1,r_i}$ for some positive integers m, r_i and s .

We next show that $r_i = 1$ for $1 \leq i \leq s$. Suppose this is not the case. Then $\overline{G'}$ contains K_{1,r_j} as an induced subgraph where $r_j \geq 2$ for some $1 \leq j \leq s$. Let x and y_1 and y_2 be the center and the end vertices of K_{1,r_j} in $\overline{G'}$, respectively. Choose $w_1 \in N_{C_1}(u)$. By Claim 1, $w_1v \notin E(G)$. Consider $G + w_1y_1$. Put $\{z_1\} = I_{w_1y_1} - \{w_1, y_1\}$. Lemma 2.2(2) together with Claim 3 and the fact that C_1 is complete implies that $z_1 \notin \{u, v\} \cup V(C_1)$. If $I_{w_1y_1} = \{w_1, z_1\}$, then $z_1 = x$ in order to dominate v . But then no vertex of $I_{w_1y_1}$ is adjacent to y_2 , a contradiction. Hence, $I_{w_1y_1} = \{y_1, z_1\}$. Then $z_1 \in V(C_2)$ and thus no vertex of $I_{w_1y_1}$ is adjacent to a vertex of $N_{C_1}(v)$, again a contradiction. This proves that $r_j = 1$ for $1 \leq j \leq s$.

Claim 5: For each $x \in A$, $x \succ C_2$.

Suppose to the contrary that there are $b \in A$ and $c \in V(C_2) - \{b\}$ such that $bc \notin E(G)$. Clearly, $b \neq a$ since $a \succ C_2$. Consider $G + ub$. Let $\{z\} = I_{ub} - \{u, b\}$. Then $z \notin \{a, v\}$ by Lemma 2.2(2), Claim 2 and the fact that $a \succ C_2$. If $I_{ub} = \{u, z\}$, then $z \succ N_{C_1}(v) \cup \{a\}$ which is not possible since $a \in A = V(C_2) - (N_G(u) \cup N_G(v))$. Hence, $I_{ub} = \{b, z\}$. Then

$z \succ \{c\} \cup V(C_1)$ and thus $z = v$ since $c \in V(C_2)$. But this contradicts the fact that $z \notin \{a, v\}$. This settles our claim.

We now distinguish into two cases.

Case 1: $|A| = 1$.

Clearly, $A = \{a\}$. By Claims 1-5, $G \in \mathcal{A}_1$.

Case 2: $|A| \geq 2$.

By Claims 1-5, we need only to show that $|N_{C_1}(u)| = |N_{C_1}(v)| = 1$. Let $x \in A$ and $y \in N_{C_1}(u)$. Consider $G + xy$. By Lemma 2.2(2), $(I_{xy} - \{x, y\}) \cap (V(C_1) \cup V(C_2) \cup \{u\}) = \emptyset$ since $x \succ V(C_2)$ and $y \succ V(C_1) \cup \{u\}$. Then $I_{xy} - \{x, y\} = \{v\}$. If $I_{xy} = \{y, v\}$, then no vertex of I_{xy} is adjacent to vertices of $A - \{x\}$, a contradiction. Hence, $I_{xy} = \{x, v\}$. It follows by Claim 1 that $N_{C_1}(u) = \{y\}$. Thus $|N_{C_1}(u)| = 1$. By similar arguments, $|N_{C_1}(v)| = 1$. Hence, $G \in \mathcal{A}_1$ where $n_1 \geq 2$ and $n_2 = n_3 = 1$. This completes the proof of our theorem. \square

Theorem 4.3. *Let G, S, C_1 and C_2 be defined as in Theorem 4.2. If $i(C_2) = 2$ and $|V(C_1)| \geq 2$, then $G \in \bigcup_{i=2}^4 \mathcal{A}_i$.*

Proof. We begin with the following claims.

Claim 1: If $N_{C_1}(u) - N_{C_1}(v) \neq \emptyset$, then, for each $y \in N_{C_2}(v)$, there exists a unique vertex $y' \in N_{C_2}(v) - \{y\}$ such that $y \succ C_2 - y'$ and $y' \succ C_2 - y$.

Let $x \in N_{C_1}(u) - N_{C_1}(v)$. Choose $y \in N_{C_2}(v)$. Consider $G + xy$. By Lemma 2.2(2) and the fact that C_1 is complete, $(I_{xy} - \{x, y\}) \cap (\{u, v\} \cup V(C_1)) = \emptyset$. Put $\{z\} = I_{xy} - \{x, y\}$. Then $z \in V(C_2)$. If $I_{xy} = \{y, z\}$, then no vertex of I_{xy} is adjacent to vertices of $V(C_1) - \{x\}$, a contradiction. Hence, $I_{xy} = \{x, z\}$. Thus $z \succ C_2 - \{y\}$ and $zv \in E(G)$ since $xv \notin E(G)$. Now consider $G + xz$. By Lemma 2.3, either $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. By similar arguments as above, $I_{xz} = \{x, y\}$ and thus $y \succ C_2 - z$ and $yv \in E(G)$. Continuing in this fashion, if there is $w \in V(C_2) - \{y, z\}$, then there exists $w' \in V(C_2) - \{y, z, w\}$ such that $w' \succ C_2 - w$, $w'v \in E(G)$ and $w \succ C_2 - w'$, $wv \in E(G)$. This settles our claim.

By similars arguments as in the proof of Claim 1, Claim 2 follows.

Claim 2: If $N_{C_1}(v) - N_{C_1}(u) \neq \emptyset$, then for each $y \in N_{C_2}(u)$, there exists a unique vertex $y' \in N_{C_2}(u) - \{y\}$ such that $y \succ C_2 - y'$ and $y' \succ C_2 - y$.

Claim 3: $N_{C_1}(u) \cup N_{C_1}(v) = N_{C_1}(u) \cap N_{C_1}(v)$.

Suppose to the contrary that there exists $x \in N_{C_1}(u) - N_{C_1}(v)$. If $v \succ C_2$, then $\{x, v\} \succ_i G$ since C_1 is complete, a contradiction. Choose $y \in V(C_2) - N_{C_2}(v)$. Consider $G + xy$. Let $\{z\} = I_{xy} - \{x, y\}$. By Lemma 2.2(2), $z \notin V(C_1) \cup \{u\}$. Further, $z \succ \{v\}$

since x and y are not adjacent to v . We first suppose that $I_{xy} = \{x, z\}$. If $z \in V(C_2) - \{y\}$, then $z \in N_{C_2}(v)$ and thus $zy \in E(G)$ by Claim 1. But this contradicts Lemma 2.2(2). Hence, $z = v$ and thus $V(C_2) - N_{C_2}(v) = \{y\}$. It then follows by Claim 1 that $y \succ C_2$. But this contradicts our hypothesis that $i(C_2) = 2$. Hence, $I_{xy} = \{y, z\}$. Then $z = v$ since $|V(C_1)| \geq 2$ and $z \notin V(C_1) \cup \{u\}$. Thus $y \succ V(C_2) - N_{C_2}(v)$. By Claim 1, $y \succ C_2$, again a contradiction. Hence, $N_{C_1}(u) - N_{C_1}(v) = \emptyset$. By Claim 2 and similar arguments, $N_{C_1}(v) - N_{C_1}(u) = \emptyset$. This settles our claim.

Claim 4: $G[C_2] \cong K_{2m}$ – a perfect matching for some positive integer m .

Let $x \in N_{C_1}(u)$ and $y \in V(C_2)$. Thus $x \in N_{C_1}(v)$ by Claim 1. Consider $G + xy$. By Lemma 2.2(2) together with the fact that C_1 is complete and $|V(C_1)| \geq 2$, $(I_{xy} - \{x, y\}) \cap (\{u, v\} \cup V(C_1)) = \emptyset$. By applying similar arguments as in the proof of Claim 1, our claim follows.

Now put $A = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. We now distinguish two cases.

Case 1: $A = \emptyset$.

Then each vertex of $V(C_2)$ is adjacent to u or v .

Claim 5: Suppose x and y are vertices of C_2 where $xy \notin E(G)$. If $x \in N_{C_2}(u) \cap N_{C_2}(v)$, then $y \in N_{C_2}(u) \cap N_{C_2}(v)$.

Note that, by Claim 4, $N_G[x] = (V(C_2) - \{y\}) \cup \{u, v\}$ and $(V(C_2) - \{x\}) \subseteq N_G[y]$. Suppose to the contrary that $yu \notin E(G)$. Since $A = \emptyset$, $N_G[y] = (V(C_2) - \{x\}) \cup \{v\}$. Consider $G + uy$. Put $\{z\} = I_{uy} - \{u, y\}$. Then, by Lemma 2.2(2), $z \in V(C_1) - N_{C_1}(u)$. Thus $zu \notin E(G)$. By Claim 3, $zv \notin E(G)$. Now consider $G + zx$. Lemma 2.2(2) together with the fact that C_1 is complete implies that $I_{zx} - \{z, x\} = \{y\}$. If $I_{zx} = \{z, y\}$, then no vertex of I_{zx} is adjacent to u , contradiction. Hence, $I_{zx} = \{x, y\}$. But then no vertex of I_{zx} dominates $C_1 - z$, contradicting the fact that $|V(C_1)| \geq 2$. Hence, $yu \in E(G)$. By similar arguments, $yv \in E(G)$. This settles our claim.

By Claim 4, it is easy to see that for each $x \in V(C_2)$ there exists only one vertex of C_2 , say y , such that $xy \notin E(G)$. Let

$X = \{x \in N_{C_2}(u) \cap N_{C_2}(v) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(u) \cap N_{C_2}(v)\}$,

$Y = \{x \in N_{C_2}(u) - N_{C_2}(v) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(u) - N_{C_2}(v)\}$,

$Z = \{x \in N_{C_2}(v) - N_{C_2}(u) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } (N_{C_2}(v) - N_{C_2}(u))\}$

and

$W = \{x \in N_{C_2}(u) - N_{C_2}(v) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(v) - N_{C_2}(u)\} \cup \{x \in N_{C_2}(v) - N_{C_2}(u) \mid \text{the only non-neighbour of } x \text{ in } C_2 \text{ belongs to } N_{C_2}(u) - N_{C_2}(v)\}$.

It follows by Claim 5 that X, Y, Z and W form a partition set of $N_{C_2}(u) \cup N_{C_2}(v)$. Further, $G[X], G[Y], G[Z]$ and $G[W]$ are complete without a perfect matching.

Claim 6: If $Y \neq \emptyset$ ($Z \neq \emptyset$), then $N_{C_1}(u) \cap N_{C_1}(v) = V(C_1)$, $uv \in E(G)$ and $Z \neq \emptyset$ ($Y \neq \emptyset$).

Let $x_1, x_2 \in Y$ where $x_1x_2 \notin E(G)$. Then $vx_1, vx_2 \notin E(G)$. Consider $G + vx_1$. Put $\{z\} = I_{vx_1} - \{v, x_1\}$. If $I_{vx_1} = \{x_1, z\}$, then $z \succ \{x_2\} \cup V(C_1)$ and thus $z = u$. But this contradicts Lemma 2.2(2). Hence, $I_{vx_1} = \{v, z\}$. Since x_2 is the only non-neighbour of x_1 in C_2 and $vx_2 \notin E(G)$, it follows that $z = x_2$. Thus $v \succ C_1$. By Claim 3, $u \succ C_1$. Hence, $N_{C_1}(u) \cap N_{C_1}(v) = V(C_1)$ as required.

We now suppose to the contrary that $Z = \emptyset$. It then follows that $uv \in E(G)$ otherwise $\{u, v\} \succ_i G$ since $A = \emptyset$. If $W = \emptyset$, then $u \succ G$, a contradiction. Hence, $W \neq \emptyset$. Let $w_1, w_2 \in W$ where $w_1w_2 \notin E(G)$, $w_1 \in N_{C_2}(u) - N_{C_2}(v)$ and $w_2 \in N_{C_2}(v) - N_{C_2}(u)$. But then $\{u, w_2\} \succ_i G$, again a contradiction. Hence, $Z \neq \emptyset$. This settles our claim.

It follows by Claim 6 that $Y = \emptyset$ if and only if $Z = \emptyset$.

Claim 7: If $W \neq \emptyset$, then $Y = \emptyset$, $|W| = 2$, $|V(C_1) - (N_{C_2}(u) \cap N_{C_2}(v))| = 1$ and $uv \in E(G)$.

Let $w_1, w_2 \in W$ where $w_1w_2 \notin E(G)$, $w_1 \in N_{C_2}(u) - N_{C_2}(v)$ and $w_2 \in N_{C_2}(v) - N_{C_2}(u)$. Consider $G + vw_1$. By similar arguments as in the proof of Claim 6, $I_{vw_1} = \{v, z\}$ for some $z \in V(G) - \{v, w_1\}$. Since w_2 is the only non-neighbour of w_1 in C_2 and $vw_2 \in E(G)$, it follows that $z \in V(C_1) - N_{C_1}(v)$. Clearly, $v \succ C_2 - w_1$. Thus $Y = \emptyset$ and $|W| = 2$, as required.

By Claim 3, $z \in V(C_1) - (N_{C_1}(u) \cap N_{C_1}(v))$. Consider $G + vz$. By Lemma 2.3, either $I_{vz} = \{v, w_1\}$ or $I_{vz} = \{z, w_1\}$. If $I_{vz} = \{z, w_1\}$, then no vertex of I_{vz} is adjacent to w_2 , a contradiction. Thus $I_{vz} = \{v, w_1\}$. Hence, $V(C_1) - (N_{C_1}(u) \cap N_{C_1}(v)) = \{z\}$ as required.

We now show that $uv \in E(G)$. Suppose this is not the case. Consider $G + uv$. Let $\{z_1\} = I_{uv} - \{u, v\}$ for some $z_1 \in V(G) - \{u, v\}$. If $I_{uv} = \{u, z_1\}$, then $z_1 \succ \{z, w_2\}$. But this is not possible since $z \in V(C_1) - (N_{C_1}(u) \cap N_{C_1}(v))$ and $w_2 \in N_{C_2}(v) - N_{C_2}(u)$. Hence, $I_{uv} = \{v, z_1\}$. But then $z_1 \succ \{z, w_1\}$ which is not possible either. Therefore, $uv \in E(G)$. This settles our claim.

We now distinguish three subcases.

Subcase 1.1: $Y \neq \emptyset$.

By Claim 7, $W = \emptyset$ and by Claim 6, $N_{C_1}(u) \cap N_{C_1}(v) = V(C_1)$, $uv \in E(G)$ and $Z \neq \emptyset$. Then $N_{C_2}(u) \cup N_{C_2}(v) = Y \cup X \cup Z$. Then G is isomorphic to the graph $G_1 \in \mathcal{A}_2$ if $X = \emptyset$ and G is isomorphic to the graph $G_2 \in \mathcal{A}_2$ if $X \neq \emptyset$.

Subcase 1.2: $W \neq \emptyset$.

By Claim 7, $Y = \emptyset$, $|W| = 2$, $|V(C_1) - (N_{C_1}(u) \cup N_{C_1}(v))| = 1$ and $uv \in E(G)$. Then

$Z = \emptyset$ by Claim 6 and thus $V(C_2) = W \cup X$. It is easy to see that $X \neq \emptyset$ otherwise u or v becomes a cutvertex contradicting our hypothesis that $\{u, v\}$ is a minimum cutset. Further, $|N_{C_1}(u) \cap N_{C_1}(v)| \geq 2$. Hence, G is isomorphic to the graph $G_3 \in \mathcal{A}_2$.

Subcase 1.3: $Y = \emptyset$ and $W = \emptyset$.

By Claim 6, $Z = \emptyset$ and thus $N_{C_2}(u) \cup N_{C_2}(v) = X$. Then $u \succ C_2$ and $v \succ C_2$. It is easy to see that if $uv \in E(G)$, $i(G) \leq 2$ since C_1 is complete. Hence, $uv \notin E(G)$. Further, $N_{C_1}(u) \cup N_{C_1}(v) \neq V(C_1)$. Since $\{u, v\}$ is a minimum cutset, $|N_{C_1}(u) \cup N_{C_1}(v)| \geq 2$. It is easy to see that G is isomorphic to a graph in \mathcal{A}_3 .

Case 2: $A \neq \emptyset$.

By Lemma 2.4 and Claim 3, $V(C_1) = N_{C_1}(u) \cup N_{C_1}(v) = N_{C_1}(u) \cap N_{C_1}(v)$.

Claim 8: Suppose there exists a pair of non-adjacent vertices w_1, w_2 of C_2 where $\{w_1, w_2\} \subseteq A$. Then $uv \in E(G)$. Further, for a pair of non-adjacent vertices x and y of $V(C_2) - \{w_1, w_2\}$, if $x \in A$, then $y \in A$.

Consider $G + uw_1$. Let $\{z\} = I_{uw_1} - \{u, w_1\}$. By Lemma 2.2(2) and Claim 4, $z \in \{v, w_2\}$. If $I_{uw_1} = \{w_1, z\}$, then $z \succ \{w_2\} \cup V(C_1)$ which is not possible since $d(w_2, c) > 2$ for each $c \in V(C_1)$. Hence, $I_{uw_1} = \{u, z\}$. It follows that $z = w_2$ since $z \succ \{w_2\}$. Hence, $uv \in E(G)$ as required.

We now suppose to the contrary that $y \notin A$. Assume that $y \in N_G(u)$. Consider $G + ux$. Since $u \succ V(C_1) \cup \{v, y\}$ and $x \succ V(C_2) - \{y\}$ by Claim 4, it follows that $I_{ux} - \{u, x\} \subseteq N_G(u) \cup N_G(x)$. But this contradicts Lemma 2.2(2). Hence, $y \in A$. This settles our claim.

Claim 9: Let x and y be a pair of non-adjacent vertices of C_2 . If $x \in N_{C_2}(u) \cap N_{C_2}(v)$, then $y \in N_{C_2}(u) \cap N_{C_2}(v)$ or $y \in A$.

Suppose this is not the case. Then we may assume that $y \in N_{C_2}(u) - N_{C_2}(v)$. Consider $G + vy$. Since $v \succ V(C_1) \cup \{x\}$ and $y \succ (V(C_2) - \{x\}) \cup \{u\}$, it then follows that $I_{vy} - \{v, y\} \subseteq N_G(v) \cup N_G(y)$. But this contradicts Lemma 2.2(2). This settles our claim.

Claim 10: Let x and y be a pair of non-adjacent vertices of C_2 . If $x \in N_{C_2}(u) - N_{C_2}(v)$ ($N_{C_2}(v) - N_{C_2}(u)$), then $y \in N_{C_2}(u) - N_{C_2}(v)$ ($N_{C_2}(v) - N_{C_2}(u)$).

Clearly, $y \notin N_{C_2}(u) \cap N_{C_2}(v)$ by Claim 9 since $x \in N_{C_2}(u) - N_{C_2}(v)$. Suppose to the contrary that $y \in A$. Consider $G + vx$. Since $v \succ V(C_1)$ and $x \succ (V(C_2) - \{y\}) \cup \{u\}$ by Claim 4, it follows by Lemma 2.2(2) that either $I_{vx} = \{v, y\}$ or $I_{vx} = \{x, y\}$. If $I_{vx} = \{x, y\}$, then no vertex of I_{vx} dominates C_1 , a contradiction. Hence, $I_{vx} = \{v, y\}$. Thus $vu \in E(G)$ since $yu \notin E(G)$. Now consider $G + uy$. Because $u \succ V(C_1) \cup \{x, v\}$ and $y \succ V(C_2) - \{x\}$, $I_{uy} - \{u, y\} \subseteq N_G(u) \cup N_G(y)$. But this contradicts Lemma 2.2(2). Hence, $y \notin A$. Then $y \in N_{C_2}(u) - N_{C_2}(v)$ or $y \in N_{C_2}(v) - N_{C_2}(u)$.

Suppose $y \notin N_{C_2}(u) - N_{C_2}(v)$. Then $y \in N_{C_2}(v) - N_{C_2}(u)$. Consider $G + xv$. Since $x \succ (V(C_2) - \{y\}) \cup \{u\}$ and $v \succ V(C_1) \cup \{y\}$, it follows that $I_{xv} \subseteq N_G(x) \cup N_G(v)$. But this contradicts Lemma 2.2(2). Hence, $y \in N_{C_2}(u) - N_{C_2}(v)$. This settles our claim.

Claim 11: Let x and y be a pair of non-adjacent vertices of C_2 . If $x \in A$ and $y \notin A$, then $y \in N_{C_2}(u) \cap N_{C_2}(v)$ and $uv \notin E(G)$.

It follows by Claim 8 that $y \notin (N_{C_2}(u) - N_{C_2}(v)) \cup (N_{C_2}(v) - N_{C_2}(u))$. Thus $y \in N_{C_2}(u) \cap N_{C_2}(v)$. Consider $G + ux$. Since $u \succ V(C_1) \cup \{y\}$ and $x \succ V(C_2) - \{y\}$, it follows by Lemma 2.2(2) that $I_{ux} - \{u, x\} = \{v\}$. Thus $uv \notin E(G)$ by Lemma 2.2(2) as required.

Recall that $V(C_1) = N_{C_1}(u) \cap N_{C_1}(v)$ and $A \neq \emptyset$. Further, $N_{C_2}(u) \neq \emptyset$ and $N_{C_2}(v) \neq \emptyset$ since $\{u, v\}$ is a minimum cutset. We now distinguish two subcases.

Subcase 2.1: There exists a pair of non-adjacent vertices w_1, w_2 of C_2 where $\{w_1, w_2\} \subseteq A$.

By Claim 8, $uv \in E(G)$ and for each pair of non-adjacent vertices x and y of $V(C_2) - \{w_1, w_2\}$, if $x \in A$, then $y \in A$. It then follows by Claims 3, 8, 9 and 10 that G is isomorphic to a graph $G_i \in \mathcal{A}_2$ where $i \in \{4, 5, 6, 7\}$.

Subcase 2.2: For each pair of non-adjacent vertices of C_2 , say x and y , $x \notin A$ or $y \notin A$.

Since $A \neq \emptyset$, there exists $t_1 \in A$. Let t_2 be the non-neighbour of t_1 in C_2 . By our hypothesis and Claim 11, $t_2 \in N_{C_2}(u) \cap N_{C_2}(v)$. Further, $uv \notin E(G)$. It is easy to see that if $|V(C_2) - A| = 1$, then $|A| = 1$ and $V(C_2) - A = N_{C_2}(u) \cap N_{C_2}(v)$ and thus G is disconnected. Consequently, $|V(C_2) - A| \geq 2$. It follows by Claims 9 and 10 that G is isomorphic to a graph in \mathcal{A}_4 . This completes the proof of Case 2 and our theorem. \square

In what follows, for subsets W and L of $V(G)$ where $W \cap L = \emptyset$, $W(\sim L)$ denotes the set of all vertices of W which are not adjacent to some vertex of L .

Theorem 4.4. Let G, S, C_1 and C_2 be defined as in Theorem 4.2. If $i(C_2) = 2$ and $|V(C_1)| = 1$, then $G \in \bigcup_{i=5}^9 \mathcal{A}_i$.

Proof. Let $\{w\} = V(C_1)$. Then $N_G(w) = \{u, v\}$ by the fact that S is a minimum cutset. Put $B = N_{C_2}(u)$, $C = N_{C_2}(v)$ and $A = V(C_2) - (B \cup C)$. By our hypothesis that $i(C_2) = 2$, we have following observation.

Observation 1: No vertex of $V(C_2)$ dominates C_2 .

Claim 1: Suppose $xy \notin E(G)$ where either $x \in B - C$ and $y \in C - B$ or x or y belongs to $B \cap C$. Then $I_{xy} - \{x, y\} = \{w\}$ and thus either $x \succ C_2 - y$ or $y \succ C_2 - x$. Further, if $x \in B \cap C$ and $y \in (B - C) \cup (C - B)$, then $I_{xy} = \{x, w\}$ and thus $x \succ C_2 - y$.

By Lemma 2.2(2), $I_{xy} \cap \{u, v\} = \emptyset$. Since $N_G(w) = \{u, v\}$, $I_{xy} - \{x, y\} = \{w\}$. If $I_{xy} = \{x, w\}$, then $x \succ C_2 - y$. Similarly, if $I_{xy} = \{y, w\}$, then $y \succ C_2 - x$. We now suppose that $x \in B \cap C$ and $y \in (B - C) \cup (C - B)$. If $I_{xy} = \{y, w\}$, then $y \succ C_2 - x$ and thus either $\{v, y\} \succ_i G$ or $\{u, y\} \succ_i G$, a contradiction. This settles our claim.

By similar arguments as in the proof of Claim 1, Claim 2 follows.

Claim 2: If $x \in B \cap C$ and $y \in A \cup (B \cap C)$ where $xy \notin E(G)$, then either $x \succ C_2 - y$ or $y \succ C_2 - x$ and thus $\{x, y\} \succ_i G - w$.

Claim 3: For $x \in B - C$, there exists a unique vertex $\bar{x} \in A \cup (B - C)$ where $x \succ (A \cup (B - C)) - \{\bar{x}\}$ and $\bar{x} \succ (A \cup (B - C)) - \{x\}$. Further, if $uv \notin E(G)$, then $\bar{x} \in B - C$, each vertex of $B - C$ dominates A and $G[B - C] \cong K_{2n}$ - a perfect matching for some positive integer n .

Consider $G + xv$. By Lemma 2.2(2), $I_{xv} \cap (\{u, w\} \cup (B \cap C) \cup (C - B)) = \emptyset$ since $xu, xv \in E(G)$ and $C \subseteq N_G(v)$. Because $N_G(w) = \{u, v\}$, $v \in I_{xv}$. Thus $I_{xv} = \{v, \bar{x}\}$ for some $\bar{x} \in (A \cup (B - C)) - \{x\}$. Clearly, $\bar{x} \succ (A \cup (B - C)) - \{x\}$. Now consider $G + \bar{x}v$. By similar arguments, $I_{\bar{x}v} = \{v, x\}$ since $\bar{x} \succ (A \cup (B - C)) - \{x\}$. Thus $x \succ (A \cup (B - C)) - \{\bar{x}\}$ as required.

We now suppose that $uv \notin E(G)$. Since $I_{xv} = \{v, \bar{x}\}$, \bar{x} dominates u and thus $\bar{x} \in (B - C) - \{x\}$. By above argument, $x \succ (A \cup (B - C)) - \{\bar{x}\}$ and $\bar{x} \succ (A \cup (B - C)) - \{x\}$. If $|B - C| = 2$, we are done. So suppose there exists $x_1 \in (B - C) - \{x, \bar{x}\}$. Consider $G + x_1v$. By the above arguments, there exists $\bar{x}_1 \in (B - C) - \{x_1\}$ such that $x_1 \succ (A \cup (B - C)) - \{\bar{x}_1\}$ and $\bar{x}_1 \succ (A \cup (B - C)) - \{x_1\}$. By Lemma 2.2(2), $\{x_1, \bar{x}_1\} \cap \{x, \bar{x}\} = \emptyset$. Continuing in this fashion, $G[B - C]$ is isomorphic to a complete graph without a perfect matching. Further, each vertex of $B - C$ dominates A . This settles our claim.

By the same arguments, Claim 4 follows.

Claim 4: For $x \in C - B$, there exists a unique vertex $\bar{x} \in A \cup (C - B)$ where $x \succ (A \cup (C - B)) - \{\bar{x}\}$ and $\bar{x} \succ (A \cup (C - B)) - \{x\}$. Further, if $uv \notin E(G)$, then $\bar{x} \in C - B$, each vertex of $C - B$ dominates A and $G[C - B] \cong K_{2m}$ - a perfect matching for some positive integer m .

For the rest of our proof, for $x \in B - C(C - B)$, we shall denote the only non-neighbour of x in $A \cup (B - C)(A \cup (C - B))$ by \bar{x} .

Claim 5: If $x \in B - C$ and $y \in C - B$, then $xy \in E(G)$. Thus each vertex of $B - C(C - B)$ dominates $C - B(B - C)$.

By Claims 3 and 4, there exist $\bar{x} \in A \cup (B - C)$ and $\bar{y} \in A \cup (C - B)$ such that $x\bar{x}, y\bar{y} \notin E(G)$. Suppose to the contrary that $xy \notin E(G)$. Consider $G + xy$. By Claim 1, either $x \succ C_2 - y$ or $y \succ C_2 - x$. But this is not possible since $x\bar{x}, y\bar{y} \notin E(G)$. Hence, $xy \in E(G)$. This settles our claim.

Now put $A_1 = \{x \in A | x \succ A\}$ and $A_2 = A - A_1$

Claim 6: $G[A_2]$ is isomorphic to a complete graph without a perfect matching and for each pair of non-adjacent vertices x_1 and x_2 of A_2 , $x_i \succ (B - C) \cup (C - B) \cup (A - \{x_j\})$ where $1 \leq i \neq j \leq 2$. Further, if $uv \notin E(G)$, then $A_2 = \emptyset$ and thus $G[A]$ is complete.

Let $x \in A_2$. Then there exists $y \in A_2$ such that $xy \notin E(G)$. Consider $G + xy$. Put $\{z\} = I_{xy} - \{x, y\}$. Then $z \in \{u, v, w\}$ since $N_G(w) = \{u, v\}$. We may assume without loss of generality that $I_{xy} = \{x, z\}$. Clearly, $x \succ A - \{y\}$ since z is not adjacent to any vertex of A . Now consider $G + xz$. By Lemma 2.3, $I_{xz} = \{x, y\}$ or $I_{xz} = \{z, y\}$. If $I_{xz} = \{x, y\}$, then no vertex of I_{xz} is adjacent to vertices of $\{u, v, w\} - \{z\}$, a contradiction. Hence, $I_{xz} = \{z, y\}$ and thus $y \succ A - \{x\}$. If $|A_2| = 2$, then we are done. So suppose that $|A_2| \geq 3$. By applying similar arguments as above and as in the proof of Claim 3, $G[A_2]$ is isomorphic to a complete graph without a perfect matching.

Now let $x_1, x_2 \in A_2$ where $x_1x_2 \notin E(G)$. It follows by Claims 3 and 4 that, $x_i \succ (B - C) \cup (C - B)$ for $1 \leq i \leq 2$. Since $G[A_2]$ is a complete graph without a perfect matching, $x_i \succ (B - C) \cup (C - B) \cup (A - \{x_j\})$, where $1 \leq i \neq j \leq 2$ as required.

We now suppose that $uv \notin E(G)$ and $A_2 \neq \emptyset$. Since $G[A_2]$ is a complete graph without a perfect matching, there exist $y_1, y_2 \in A_2$ where $y_1y_2 \notin E(G)$. Consider $G + uy_1$. Put $\{z_1\} = I_{uy_1} - \{u, y_1\}$. We first suppose that $I_{uy_1} = \{u, z_1\}$. Since $uv, uy_2 \notin E(G)$, z_1 dominates $\{v, y_2\}$. It follows that $z_1 \in C - B$ by Lemma 2.2(2) and $z_1 \succ C - B$. But this contradicts Claim 3. Hence, $I_{uy_1} = \{y_1, z_1\}$. Since $y_1y_2, y_1w \notin E(G)$, z_1 dominates $\{y_2, w\}$. But this is not possible since $d(y_2, w) = 3$. Hence, $A_2 = \emptyset$ as required. This settles our claim.

Let

$$\begin{aligned} X_1 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in B - C\} \\ X_2 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in C - B\} \\ X_3 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in B \cap C\} \\ X_4 &= \{x \in B \cap C \mid xy \notin E(G) \text{ for some } y \in A\}. \end{aligned}$$

By Observation 1, $B \cap C = X_1 \cup X_2 \cup X_3 \cup X_4$. The following claim follows immediately by Claims 1, 3 and 4.

Claim 7: If $x \in X_1(X_2)$, then there exists a unique vertex $y \in B - C(C - B)$ such that $N_G[x] = V(G) - \{y, w\}$.

Claim 8: For $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$, $X_i \cap X_j = \emptyset$. Further, if $uv \in E(G)$, $X_3 \cap X_4 = \emptyset$.

It follows by Claim 7 that $X_i \cap X_j = \emptyset$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$. We now assume that $uv \in E(G)$. Suppose to the contrary that there exists $x \in X_3 \cap X_4$. Let $y_1 \in B \cap C$ and $y_2 \in A$ where $xy_1, xy_2 \notin E(G)$. By Claim 1, either $x \succ C_2 - y_2$ or $y_2 \succ C_2 - x$. Since $xy_1 \notin E(G)$, $y_2 \succ C_2 - x$. But then $\{y_2, u\} \succ_i G$, a contradiction. Hence, $X_3 \cap X_4 = \emptyset$ as required.

By the definition of X_3 and X_4 together with Claims 7 and 8, we have the following

claim.

Claim 9: If $x \in X_3 \cup X_4$, then $x \succ (B - C) \cup (C - B) \cup X_1 \cup X_2$. Further, suppose $uv \in E(G)$. Then if $x \in X_3$, then $x \succ A \cup (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup X_4$ and if $x \in X_4$, then $x \succ (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup X_3$.

For $x \in B - C(C - B)$, by Claim 3 (Claim 4) there is a unique vertex \bar{x} of $A \cup (B - C)(A \cup (C - B))$ which is not adjacent to x . Let

$$\begin{aligned} Y_1 &= \{x \in (B - C)(\sim X_1) | \bar{x} \in B - C\} \\ Y_2 &= \{x \in (B - C)(\sim X_1) | \bar{x} \in A\} \\ Z_1 &= \{x \in (C - B)(\sim X_2) | \bar{x} \in C - B\} \\ Z_2 &= \{x \in (C - B)(\sim X_2) | \bar{x} \in A\}. \end{aligned}$$

Observation 2: $X_1 \neq \emptyset$ if and only if $Y_1 \cup Y_2 \neq \emptyset$ and $X_2 \neq \emptyset$ if and only if $Z_1 \cup Z_2 \neq \emptyset$.

Observation 3: If $x \in Y_2 \cup Z_2$, then $\bar{x} \in A_1$ and $uv \in E(G)$.

Claim 10: If $y \in Y_1$, then $\bar{y} \in Y_1$.

Since $y \in Y_1$, there is a vertex $x \in X_1$ such that $xy \notin E(G)$. Suppose to the contrary that $\bar{y} \succ X_1$. By Claims 7 and 9, $\bar{y} \succ B \cap C$ and thus $\bar{y} \succ C_2 - y$ by Claims 3 and 5. Consider $G + w\bar{y}$. By Lemma 2.2(2) and the fact that $N_G(w) = \{u, v\}$, it follows that $I_{w\bar{y}} - \{w, \bar{y}\} = \{y\}$. If $I_{w\bar{y}} = \{w, y\}$, then no vertex of $I_{w\bar{y}}$ dominates x , a contradiction. Hence, $I_{w\bar{y}} = \{\bar{y}, y\}$. But then no vertex of $I_{w\bar{y}}$ dominates v , again a contradiction. Hence, there exists a vertex $x_1 \in X_1$ such that $\bar{y}x_1 \notin E(G)$. This settles our claim.

By similar arguments as in the proof of Claim 10, Claim 11 follows.

Claim 11: If $z \in Z_1$, then $\bar{z} \in Z_1$.

Claim 12: If $Y_1 \neq \emptyset$, then each component of $\overline{G}[Y_1 \cup (X_1(\sim Y_1))]$ is a double star $DS(m, n)$, in \overline{C}_2 , for some positive integers m and n where the center and the end vertices of $DS(m, n)$ are in Y_1 and $X_1(\sim Y_1)$, respectively.

Let $y \in Y_1$. Then $yx_1 \notin E(G)$ for some $x_1 \in X_1$. By Claim 10, $\bar{y} \in Y_1$ and thus there exists $x_2 \in X_1$ such that $\bar{y}x_2 \notin E(G)$. By Claim 7, $x_1 \neq x_2$ and $N_G[x_1] = V(G) - \{y, w\}$ and $N_G[x_2] = V(G) - \{\bar{y}, w\}$. Further, by Claims 3, 5, 7 and 9, $V(C_2) - (\{y\} \cup X_1) \subseteq N_G[y]$ and $V(C_2) - (\{\bar{y}\} \cup X_1) \subseteq N_G[\bar{y}]$. It then follows by Claim 7 that $\overline{G}[\{y, \bar{y}, x_1, x_2\} \cup \overline{N}_{X_1}(y) \cup \overline{N}_{X_1}(\bar{y})]$ is a component in \overline{C}_2 and forms a double star $DS(m, n)$ where $m = |\overline{N}_{X_1}(y)| \geq 1$ and $n = |\overline{N}_{X_1}(\bar{y})| \geq 1$. Further, y and \bar{y} are center vertices and vertices of $\overline{N}_{X_1}(y) \cup \overline{N}_{X_1}(\bar{y})$ are end vertices of $DS(m, n)$ in \overline{C}_2 . This settles our claim.

By similar arguments as in the proof of Claim 12, Claim 13 follows.

Claim 13: If $Z_1 \neq \emptyset$, then each component of $\overline{G}[Z_1 \cup X_2(\sim Z_1)]$ is a double star

$DS(m, n)$, in \overline{C}_2 , for some positive integers m and n where the center and the end vertices of $DS(m, n)$ are in Z_1 and $X_2(\sim Z_1)$, respectively.

We now distinguish two cases according to uv .

Case 1: $uv \notin E(G)$.

Then $A \neq \emptyset$ otherwise $i(G) = 2$ and thus $G[A]$ is complete by Claim 6. Further, by Claims 3 and 4, $G[B - C]$ and $G[C - B]$ are complete without a perfect matching and thus $Y_2 = Z_2 = \emptyset$. It follows that $X_1(\sim Y_1) = X_1$ and $X_2(\sim Z_1) = X_2$. Since a vertex of A dominates $A \cup (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup (X_3 - X_4)$ by the definitions of X_3 and X_4 and Claims 3, 4, 6 and 7, $X_4 \neq \emptyset$ otherwise $i(G) = 2$. Consequently, $i(G[A \cup X_3 \cup X_4]) \geq 2$.

Claim 14: Each component of $\overline{G}[A \cup X_3 \cup X_4]$, in \overline{C}_2 , is a star $K_{1,r}$ for some positive integer r . Further, for each pair of non-adjacent vertices x and y , $|\{x, y\} \cap A| \leq 1$, $\{x, y\} \succ_i G - w$ and if x is the center and y is an end vertex of $K_{1,r}$ in $\overline{G}[A \cup X_3 \cup X_4]$, then $\{y, w\} \succ_i G - x$.

Claims 1, 3, 4, 7, 8 and 9 together with the definitions of X_3 and X_4 imply that if $z \in X_3 \cup X_4$, then $\overline{N}_{C_2}(z) \subseteq X_3 \cup X_4 \cup A$. Further, if $z \in A$, then $\overline{N}_{C_2}(z) \subseteq X_4$. Now let $x, y \in A \cup X_3 \cup X_4$ where $xy \notin E(G)$. By Claim 6, $|\{x, y\} \cap A| \leq 1$. We may suppose without loss of generality that $x \notin A$. Then $x \in X_3 \cup X_4 \subseteq B \cap C$. Consider $G + xy$. By Claim 1, either $x \succ C_2 - y$ or $y \succ C_2 - x$. Hence, $i(G[A \cup X_3 \cup X_4] + xy) = 1$. Therefore, $i(G[A \cup X_3 \cup X_4]) = 2$ and $G[A \cup X_3 \cup X_4]$ is 2- i -critical. Thus, by Lemma 2.5, each component of $\overline{G}[A \cup X_3 \cup X_4]$ is a star $K_{1,r}$, in \overline{C}_2 .

Since $x \in B \cap C$ and either $x \succ C_2 - y$ or $y \succ C_2 - x$, it follows that $\{x, y\} \succ_i G - w$ and thus $\{x, w\} \succ_i G - y$ or $\{y, w\} \succ_i G - x$ by Lemma 2.3. It is easy to see that if x is a center and y is an end vertex of $K_{1,r}$, in \overline{C}_2 , then $\{y, w\} \succ_i G - x$. This settles our claim.

Now let $S_1 = \{K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4] \mid V(K_{1,r}) \cap A = \emptyset\}$, $S_2 = \{K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4] \mid \text{some end vertex of } K_{1,r} \text{ belongs to } A\}$ and $S_3 = \{K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4] \mid \text{the center of } K_{1,r} \text{ belongs to } A\}$. Clearly, $\overline{G}[A \cup X_3 \cup X_4] = S_1 \cup S_2 \cup S_3$ and $X_3 \cap X_4 = \emptyset$ if and only if all end vertices of stars in S_2 belong to A .

Claim 15: For $1 \leq i \neq j \leq 3$, $S_i \cap S_j = \emptyset$

Let xy be an edge of $K_{1,r} \in \overline{G}[A \cup X_3 \cup X_4]$ where x is the center and y is an end vertex of $K_{1,r}$. By Claim 6, $|\{x, y\} \cap A| \leq 1$. It is easy to see that if $|\{x, y\} \cap A| = 0$, then $\{x, y\} \subseteq X_3$. If there exists an end $y_1 \neq y$ of $K_{1,r}$ where $y_1 \in A$, then $K_{1,r} \in S_2$ otherwise $K_{1,r} \in S_1$. We now suppose that $|\{x, y\} \cap A| = 1$. If $\{x, y\} \cap A = \{x\}$, then $K_{1,r} \in S_3$ and if $\{x, y\} \cap A = \{y\}$, $K_{1,r} \in S_2$. This settles our claim.

Claim 16: Suppose $\overline{G}[A \cup X_3 \cup X_4] = K_{1,r}$ for some positive integer r . Then $K_{1,r} \in S_2$ or $K_{1,r} \in S_3$. Further, if $K_{1,r} \in S_3$, then $|B - C| \geq 2$ or $|C - B| \geq 2$ and if $K_{1,r} \in S_2$, then either at least two end vertices of $K_{1,r}$ belong to X_3 or $|B - C| \geq 2$ or $|C - B| \geq 2$.

Since $A \neq \emptyset$, and $X_4 \neq \emptyset$, it follows that $K_{1,r} \notin S_1$. Thus $K_{1,r} \in S_2$ or $K_{1,r} \in S_3$.

Let x, y_1, y_2, \dots, y_r be the center and end vertices of $K_{1,r}$, respectively. We first suppose that $K_{1,r} \in S_3$. Then $x \in A$ and thus $\{y_1, y_2, \dots, y_r\} = X_4$. Because G is connected and $\{u, v\}$ is a minimum cutset of G , it follows that there exist at least two vertices of $V(C_2) - (A \cup X_4)$, joining to x . Thus $B - C$ or $C - B$ is not empty by Observation 2 and the fact that $G[B - C]$ and $G[C - B]$ are complete graphs without a perfect matching. Clearly, $|B - C| \geq 2$ or $|C - B| \geq 2$.

We now suppose that $K_{1,r} \in S_2$. Assume that $y_1 \in A$. Thus $x \in X_4$. By similar arguments as above, if $|\{y_1, y_2, \dots, y_r\} - A| \leq 1$, then $B - C$ or $C - B$ is not empty. So we suppose that $|\{y_1, y_2, \dots, y_r\} - A| \geq 2$. Thus $x \in X_3 \cup X_4$ and thus $\{y_1, y_2, \dots, y_r\} - A \subseteq X_3$ as required. This settles our claim.

We are now ready to characterize graphs in this case. Recall that $Y_2 = Z_2 = \emptyset$. Let F be a component of \overline{C}_2 and $x \in V(F)$. We first suppose that $x \in (B - C) - Y_1 ((C - B) - Z_1)$. By Claims 3(4), 5, 7 and 9, there exists a unique vertex $\bar{x} \in (B - C) - Y_1 ((C - B) - Z_1)$ where $N_{C_2}[x] = V(C_2) - \{\bar{x}\}$ and $N_{C_2}[\bar{x}] = V(C_2) - \{x\}$. Thus $V(F) = \{x, \bar{x}\}$ and F is a K_2 , in \overline{C}_2 , by Claim 3(4).

We next suppose that $x \in Y_1 \cup X_1 (\sim Y_1) = Y_1 \cup X_1$. Clearly, $V(F) \cap X_1 \neq \emptyset$ and $V(F) \cap Y_1 \neq \emptyset$. We may assume that $x \in V(F) \cap X_1$. Let $y \in V(F) \cap Y_1$. By Claim 7, $N_G[x] = V(G) - \{y, w\}$ and thus $N_{C_2}[x] = V(C_2) - \{y\}$. By Claim 10, there exists a unique vertex $\bar{y} \in Y_1$ where $V(C_2) - (\{\bar{y}\} \cup X_1) \subseteq N_{C_2}[y]$ and $V(C_2) - (\{y\} \cup X_1) \subseteq N_{C_2}[\bar{y}]$ by Claim 3, 5, 7 and 9. It then follows that $V(F) \subseteq Y_1 \cup X_1$. By Claim 12, F is a double star, in \overline{C}_2 , where y and \bar{y} are center vertices and vertices of $\overline{N}_{X_1}(y) \cup \overline{N}_{X_1}(\bar{y})$ are end vertices.

By similar arguments, if $x \in Z_1 \cup (X_2 (\sim Z_1)) = Z_1 \cup X_2$, then $V(F) \subseteq Z_1 \cup X_2$ and F is a double star in \overline{C}_2 .

Finally, we may suppose that $x \in A \cup X_3 \cup X_4$. It is easy to see that if $V(F) \cap A = \emptyset$, then $V(F) \subseteq X_3$ by Claim 9 and thus F is a star by Claim 14. So we may now suppose that $V(F) \cap A \neq \emptyset$ and $x \in V(F) \cap A$. Then there exists $y \in V(F) \cap X_4$ such that $xy \notin E(G)$. By Claim 9, $V(F) \subseteq A \cup X_3 \cup X_4$ and thus F is a star by Claim 14.

We may conclude here that a component F of \overline{C}_2 is a K_2 or a star or a double star.

Recall that $A \neq \emptyset$ and $X_4 \neq \emptyset$. We first suppose that $B - C = C - B = \emptyset$. By Observation 2, $X_1 = X_2 = \emptyset$ and thus $V(G) = \{u, v, w\} \cup A \cup X_3 \cup X_4$. It follows by Claim 16 that $\overline{G}[A \cup X_3 \cup X_4]$ contains either at least two stars or exactly one star with the center of the star belongs to X_4 , A is a subset of the set of end vertices of the star and at least two end vertices of the star are in X_3 . Therefore, G is isomorphic to a graph in $\{G', G'', G''', G_{S_1}^1, G_{S_1}^2, G_{S_1}^3\} \subseteq \mathcal{A}_6$.

We next suppose that $B - C \neq \emptyset$ but $C - B = \emptyset$. Then, by Observation 2, $X_2 = \emptyset$ and thus $V(G) = \{u, v, w\} \cup (B - C) \cup A \cup X_1 \cup X_3 \cup X_4$. Then G is isomorphic to a graph in $\{G_{P_1}^j, G_{D_1}^j, G_{P_1 D_1}^j, G_{P_1 S_1}^j, G_{D_1 S_1}^j, G_{P_1 D_1 S_1}^j\} \subseteq \mathcal{A}_6$ where $1 \leq j \leq 3$. By similar arguments, if $C - B \neq \emptyset$ but $B - C = \emptyset$, then G is isomorphic to a graph in $\{G_{P_2}^j, G_{D_2}^j, G_{P_2 D_2}^j, G_{P_2 S_1}^j, G_{D_1 S_1}^j, G_{P_1 D_1 S_1}^j\} \subseteq \mathcal{A}_6$ where $1 \leq j \leq 3$. Finally, we suppose that $B - C \neq \emptyset$ and $C - B \neq \emptyset$. Then G is isomorphic to a graph in $\{G_{P_1 P_2}^j, G_{P_1 D_2}^j, G_{P_1 P_2 D_2}^j, G_{P_1 P_2 S_1}^j, G_{P_1 D_2 S_1}^j, G_{P_1 P_2 D_2 S_1}^j, G_{D_1 P_2}^j, G_{D_1 D_2}^j, G_{D_1 P_2 D_2}^j, G_{D_1 P_2 S_1}^j, G_{D_1 D_2 S_1}^j\}$

$G_{D_1P_1P_2S_1}^j, G_{P_1D_1P_2}^j, G_{P_1D_1D_2}^j, G_{P_1D_1P_2D_2}^j, G_{D_1D_2P_2S_1}^j, G_{P_1D_1D_2S_1}^j, G_{P_1D_1P_2D_2S_1}^j\} \subseteq \mathcal{A}_6$ where $1 \leq j \leq 3$. This proves Case 1.

Case 2: $wv \in E(G)$.

Before we characterize this case, we need to establish some more claims. By Claim 8, $X_3 \cap X_4 = \emptyset$. By the definition of X_3 , $i(G[X_3]) \geq 2$. By applying similar arguments as in the proof of Claim 14 together with the fact that for each $x \in X_3$, $x \succ A \cup (B - C) \cup (C - B) \cup X_1 \cup X_2 \cup X_4$ by Claim 9, Claim 17 follows.

Claim 17: Each component of $\overline{G}[X_3]$ is a star $K_{1,r}$ for some positive integer r . Further, if $x, y \in X_3$ where $xy \notin E(G)$, then $\{x, y\} \succ_i G - \{w\}$.

Claim 18: Suppose $x, y \in A_2$ where $xy \notin E(G)$. If x is not adjacent to some vertex $x_1 \in X_4$, then y is not adjacent to some vertex $y_1 \in X_4 - \{x_1\}$.

By applying similar arguments as in the proof of Claim 10 together with the definition of X_3 and Claims 1, 6, 7 and 9, our claim follows.

Claim 19: If $A_2(\sim X_4) \neq \emptyset$, then each component of $\overline{G}[(X_4(\sim A_2)) \cup (A_2(\sim X_4))]$ is a double star $DS(m, n)$ for some positive integers m and n where the center and the end vertices of $D(m, n)$ are in $A_2(\sim X_4)$ and $X_4(\sim A_2)$, respectively.

By applying similar arguments as in the proof of Claim 12 together with Claims 2, 6, 7 and 18, our claim follows.

Claim 20: For $x \in A_1$, there exist $y_1 \in B - C$ and $y_2 \in C - B$ such that $xy_1, xy_2 \notin E(G)$ and $x \succ (A \cup (B - C) \cup (C - B)) - \{y_1, y_2\}$ and $y_i \succ (A \cup (B - C) \cup (C - B)) - \{x\}$ for $1 \leq i \leq 2$. Further, there exist three distinct vertices of $B \cap C$, say a_1, a_2, b such that $y_1a_1, y_2a_2, xb \notin E(G)$.

It is easy to see that if $x \succ B - C$, then $\{x, v\} \succ_i G$ since $wv \in E(G)$, a contradiction. Hence, there exists $y_1 \in B - C$ such that $xy_1 \notin E(G)$. Similarly, there exists $y_2 \in C - B$ such that $xy_2 \notin E(G)$. By the definition of A_1 together with Claims 3 and 4, $x \succ (A \cup (B - C) \cup (C - B)) - \{y_1, y_2\}$. Further, by Claims 3, 4 and 5, $y_i \succ (A \cup (B - C) \cup (C - B)) - \{x\}$ for $1 \leq i \leq 2$.

Suppose to the contrary that $y_1 \succ B \cap C$. Then $y_1 \succ C_2 - x$. Consider $G + wy_1$. Put $\{z\} = I_{wy_1} - \{w, y_1\}$. Then $z = x$ by Lemma 2.2(2) since $y_1 \succ C_2 - x$ and $N_G(w) = \{u, v\}$. If $I_{wy_1} = \{w, x\}$, then no vertex of I_{wy_1} is adjacent to y_2 , a contradiction. Hence, $I_{wy_1} = \{y_1, x\}$. But then no vertex of I_{wy_1} is adjacent to v , again a contradiction. Hence, there exists $a_1 \in B \cap C$ such that $y_1a_1 \notin E(G)$. Similarly, there exists $a_2 \in B \cap C$ such that $y_2a_2 \notin E(G)$.

We next show that x does not dominate $B \cap C$. Suppose this is not the case. Then $x \succ B \cap C$ and thus $x \succ C_2 - \{y_1, y_2\}$. Consider $G + wx$. Put $\{z_1\} = I_{wx} - \{w, x\}$. By Lemma 2.2(2), $z_1 \in \{y_1, y_2\}$ since $x \succ C_2 - \{y_1, y_2\}$ and $N_G(w) = \{u, v\}$. If $I_{wx} = \{w, z_1\}$, then no vertex of I_{wx} is adjacent to either a_1 or a_2 , a contradiction. Hence, $I_{wx} = \{x, z_1\}$. Then no vertex of I_{wx} is adjacent to either v or u , again a contradiction. Hence, there

exists $b \in B \cap C$ such that $xb \notin E(G)$. By Claims 1 and 2, a_1, a_2 and b are all distinct. This settles our claim.

Claim 21: Suppose $A_1 \neq \emptyset$. Then each component of $\overline{G}[A_1 \cup Y_2 \cup Z_2 \cup (X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup (X_4(\sim A_1))]$ is a triple star $TD(m, n, r)$ for some positive integers m, n and r where the middle center vertex is in A_1 one of the end center vertices is in Y_2 and the another end center vertices is in Z_2 while all the end vertices are in $(X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup (X_4(\sim A_1))$.

Let $x \in A_1$. By Claim 20, there exist $y_1 \in B - C$ and $y_2 \in C - B$ such that $xy_1, xy_2 \notin E(G)$ and $x \succ (A \cup (B - C) \cup (C - B)) - \{y_1, y_2\}$ and for $1 \leq i \leq 2$, $y_i \succ (A \cup (B - C) \cup (C - B)) - \{x\}$. Further, there exist $a_1, a_2, b \in B \cap C$ where $y_1a_1, y_2a_2, xb \notin E(G)$. Clearly, $a_1 \in X_1(\sim Y_2)$, $a_2 \in X_2(\sim Z_2)$ and $b \in X_4(\sim A_1)$. By Claims 1, 2, 5, 7, 9 and 20, $\overline{N}_{C_2}(x) \subseteq \{y_1, y_2\} \cup (X_4(\sim A_1))$, $\overline{N}_{C_2}(y_1) \subseteq \{x\} \cup (X_1(\sim Y_2))$ and $\overline{N}_{C_2}(y_2) \subseteq \{x\} \cup (X_2(\sim Z_2))$. Further, for $c \in \{y_1, y_2, x\}$, if $c_1 \in \overline{N}_{C_2}(c) - \{y_1, y_2, x\}$, then $c_1 \succ C_2 - c$. Hence, $\overline{G}[\{x, y_1, y_2\} \cup \overline{N}_{C_2}(x) \cup \overline{N}_{C_2}(y_1) \cup \overline{N}_{C_2}(y_2)]$ is a component in \overline{C}_2 and forms a triple star $TS(m, n, r)$ where $m = |\overline{N}_{C_2}(y_1)|$, $n = |\overline{N}_{C_2}(x)|$ and $r = |\overline{N}_{C_2}(y_2)|$. Clearly, m, n and r are positive integers. Further, x is the middle center vertex, y_1 and y_2 are the end center vertices and each vertex of $\overline{N}_{C_2}(x) \cup \overline{N}_{C_2}(y_1) \cup \overline{N}_{C_2}(y_2)$ is an end vertex of $TS(m, n, r)$ as required. This settles our claim.

Let F be a component of \overline{C}_2 and $x \in V(F)$. By the same arguments as in the proof of Case 1, if $x \in (B - C) - (Y_1 \cup Y_2)((C - B) - (Z_1 \cup Z_2))$, F is K_2 . If $x \in Y_1 \cup (X_1(\sim Y_1))(Z_1 \cup (X_2(\sim Z_1)))$, F is a double star. We now suppose that $x \in X_3$. By Claim 9, $V(F) \subseteq X_3$ and thus F is a star by Claim 17.

We next suppose that $x \in X_4(\sim A_1)$. Then there exists $x_1 \in A_1$ such that $xx_1 \notin E(G)$. By Claim 20, there exist $a_1, a_2, b \in B \cap C, y_1 \in B - C, y_2 \in C - B$ where $x_1y_1, x_1y_2, y_1a_1, y_2a_2, x_1b \notin E(G)$. (Note that b and x might not be different). Then $\{x, x_1, y_1, y_2, a_1, a_2, b\} \subseteq V(F)$. By similar arguments as in the proof of Claim 21, $V(F) \subseteq (X_4(\sim A_1)) \cup (X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup \{x_1, y_1, y_2\}$. By Claim 21, F is a triple star.

Finally, we assume that $x \in X_4(\sim A_2)$. Then there exists $w_1 \in A_2$ such that $xw_1 \notin E(G)$. By Claim 6, there exists a unique vertex $w_2 \in A_2 - \{w_1\}$ such that $w_i \succ (B - C) \cup (C - B) \cup (A - \{w_j\})$ for $1 \leq i \neq j \leq 2$. By Claim 18, there exists $c \in B \cap C$ such that $w_2c \notin E(G)$. By Claim 2, $x \succ C_2 - w_1$ and $c \succ C_2 - w_2$. Further, by Claims 7 and 9, $w_i \succ X_1 \cup X_2 \cup X_3$ for $1 \leq i \leq 2$. Thus $V(F) \subseteq (X_4(\sim A_2)) \cup \{w_1, w_2\}$. Hence, F is a double star by Claim 19.

We are now ready to characterize this case by distinguishing three cases.

Case 2.1: $A_1 = A_2 = \emptyset$.

Since $uv \in E(G)$, $u \succ V(G) - (C - B)$ and $v \succ V(G) - (B - C)$. Then $B - C \neq \emptyset$ and $C - B \neq \emptyset$ otherwise $u \succ G$ or $v \succ G$. Note that $X_4 = \emptyset$ and X_1, X_2, X_3 are partition sets of $B \cap C$ by Claim 8. It then follows that $Y_2 = Z_2 = \emptyset$ and thus $X_1(\sim Y_1) = X_1$ and $X_2(\sim Z_1) = X_2$. If $X_1 = X_2 = X_3 = \emptyset$, then G is isomorphic to $G_{R_1} \in \mathcal{A}_8$. We now sup-

pose that $X_3 = \emptyset$. If exactly one of $\{X_1, X_2\}$ is not empty, then G is isomorphic to a graph in $\{G_{R_1D_1}, G_{R_2}\} \subseteq \mathcal{A}_8$ and if both of X_1 and X_2 are not empty, then G is isomorphic to a graph in $\{G_{R_3}, G_{R_2D_1}, G_{R_1D_1D_2}\} \subseteq \mathcal{A}_8$. We next suppose that $X_3 \neq \emptyset$. If $X_1 = X_2 = \emptyset$, then G is isomorphic to $G_{R_1S_1} \in \mathcal{A}_8$ and if at least one of $\{X_1, X_2\}$ is not empty, then G is isomorphic to a graph in $\{G_{R_1D_1S_1}, G_{R_2S_1}, G_{R_3S_1}, G_{R_2D_1S_1}, G_{R_1D_1D_2S_1}\} \subseteq \mathcal{A}_8$. This proves Case 2.1.

Case 2.2: $A_1 = \emptyset$ but $A_2 \neq \emptyset$.

Then, by Claim 6, $G[A] = G[A_2]$ is a complete graph without a perfect matching and each vertex of $A_2 = A$ dominates $(B - C) \cup (C - B)$. It then follows by Claims 3 and 4 that $G[B - C]$ and $G[C - B]$ are complete graphs without a perfect matching. Further, $Y_2 = Z_2 = \emptyset$. By Claim 5, each vertex of $B - C$ ($C - B$) dominates $C - B(B - C)$. We now distinguish three subcases.

Case 2.2.1: $B \cap C = \emptyset$.

Then $B - C \neq \emptyset$ and $C - B \neq \emptyset$ otherwise v or u becomes a cutvertex, contradicting the fact that $\{u, v\}$ is a minimum cutset. It is easy to see that $G \cong G_{W_1P_1P_2} \in \mathcal{A}_9$.

Case 2.2.2: $X_4 = \emptyset$ but $X_1 \cup X_2 \cup X_3 \neq \emptyset$.

Then $A_2(\sim X_4) = \emptyset$. By Claims 12, 13 and 17, G is isomorphic to a graph in $\{G_{W_1D_1}, G_{W_1D_1P_1}, G_{W_1D_1P_1P_2}, G_{W_1S_1}, G_{W_1S_1P_1}, G_{W_1S_1P_1P_2}\} \subseteq \mathcal{A}_9$ if exactly one of X_1, X_2 and X_3 is not empty. If exactly two of X_1, X_2 and X_3 are not empty, then G is isomorphic to a graph in $\{G_{W_1D_1D_2}, G_{W_1D_1D_2P_1}, G_{W_1D_1D_2P_1P_2}, G_{W_1D_1S_1}, G_{W_1D_1S_1P_1}, G_{W_1D_1S_1P_1P_2}\} \subseteq \mathcal{A}_9$. Finally, if none of X_1, X_2 and X_3 is empty, then G is isomorphic to a graph in $\{G_{W_1D_1S_1D_2}, G_{W_1D_1S_1D_2P_1}, G_{W_1D_1S_1P_1P_2}\} \subseteq \mathcal{A}_9$.

Case 2.2.3: $X_4 \neq \emptyset$.

By Claim 19, $\overline{G}[(X_4(\sim A_2)) \cup (A_2(\sim X_4))] \cong \bigcup_{j=1}^d DS(m_j, n_j)$. Note that if $d = 1$, then either $m \geq 2$ and $n \geq 2$, or $A_2 - (A_2(\sim X_4)) \neq \emptyset$ or $(B - C) \cup (C - B) \neq \emptyset$ or $X_3 \neq \emptyset$ otherwise an end vertex of the double star becomes a cutvertex, contradicting the fact that $\{u, v\}$ is a minimum cutset of G . For this case, G is isomorphic to a graph in $\{G_{Q_1}, G_{w_4}, G_{W_2F_1, \dots, F_n}, G_{W_4F_1 \dots F_n}\} \subseteq \mathcal{A}_9$ where $F_i \in \{P_1, D_1, P_2, D_2, S_1\}$ and $F_i \neq F_j$ for $1 \leq i \neq j \leq n \leq 5$.

Case 2.3: $A_1 \neq \emptyset$ (A_2 may be empty).

By Claim 21, each component of $\overline{G}[A_1 \cup Y_2 \cup Z_2 \cup (X_1(\sim Y_2)) \cup (X_2(\sim Z_2)) \cup (X_4(\sim A_1))]$ is a triple star $TD(m, n, r)$ for some positive integers m, n and r where the middle center vertex is in A_1 , one of the end center vertices is in Y_2 and the another end center is in Z_2 while all the end vertices are in $X_1(\sim Y_2) \cup X_2(\sim Z_2) \cup X_4(\sim A_1)$. By Claims 3, 4, 5, 6, 12, 13 and 17, G is isomorphic to a graph in $\{G_{W_i}, G_{W_iF_1 \dots F_n}\} \subseteq \mathcal{A}_9$ where $i \in \{3, 5, 6, 7\}$, $F_i \in \{P_1, D_1, P_2, D_2, S\}$ and $F_i \neq F_j$ for $1 \leq i \neq j \leq n \leq 5$. This completes the proof of Case 2 and our theorem. \square

As a consequence of Lemma 4.1 and Theorems 4.2 - 4.4, we have:

Theorem 4.5. *Let G be a connected 3- i -critical graph and S a minimum cutset of size 2 where $\omega(G - S) = 2$. Then $G \in \bigcup_{i=1}^9 \mathcal{A}_i$. Further, $\delta(G) = 2$ or G contains $K_{1,3}$ as an induced subgraph \square*

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On independent domination critical graphs and k -factor critical

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Abstract

A subset S of $V(G)$ is an independent dominating set for G if S is independent and each vertex of G is either in S or adjacent to some vertex of S . Let $i(G)$ denote the minimum cardinality of an independent dominating set for G . For a positive integer t , a graph G is t - i -critical if $i(G) = t$, but $i(G + uv) < t$ for any pair of non-adjacent vertices u and v of G . Further, for a positive integer k , a graph G is k -factor-critical if for every $S \subseteq V(G)$ with $|S| = k$, $G - S$ has a perfect matching. In this paper, we provide sufficient conditions for connected 3- i -critical graphs to be k -factor-critical in terms of connectivity and minimum degree.

Keywords: *independent domination, critical, k-factor-critical*

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1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by \overline{G} . For $S \subseteq V(G)$, S is independent if no two vertices of S are adjacent. The number of components of G and the number of odd components of G are denoted by $\omega(G)$ and $\omega_0(G)$, respectively. For a vertex v of G , the neighbour of v in G , denoted by $N_G(v)$, is $\{u \in V(G) | u \text{ is adjacent to } v\}$ and $\deg_G(v) = |N_G(v)|$. The minimum degree of G , denoted by $\delta(G)$, is $\min \{\deg_G(v) | v \in V(G)\}$. For $S \subseteq V(G)$, $N_G(v) \cap S$ is denoted by $N_S(v)$. For a positive integer k , a graph G is said to be k -connected if $|V(G)| > k$ and $G - X$ is connected for every $X \subseteq V(G)$ if $|X| < k$. Further, a graph G is k -factor-critical if for every $S \subseteq V(G)$ with $|S| = k$, $G - S$ has a perfect matching. 1-factor-critical and 2-factor-critical graphs are also called *factor-critical* and *bicritical*, respectively.

For $S \subseteq V(G)$, S is a dominating set for G if each vertex of G either belongs to S or is adjacent to a vertex of S . A dominating set which is also independent is called *independent dominating set*. The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and denoted by $i(G)$.

In 1994, Ao [7] introduced the concept of “independent domination critical”. For a positive integer t , a graph G is t - i -critical if $i(G) = t$, but $i(G + uv) < t$ for any pair of non-adjacent vertices u and v of G . It is easy to see that the only 1- i -critical graphs are K_n for some positive integer n . Ao [7] proved that G is 2- i -critical if and only if $\overline{G} \cong \bigcup_{i=1}^n K_{1,r_i}$ for some positive integers r_i and n . For $t \geq 3$, there are very few known results concerning connected t - i -critical graphs. Some properties of connected 3- i -critical graphs were established by Ao [7], Ananchuen and Ananchuen [1, 2] and Ananchuen etc.[3]. However, none of them concerns k -factor-critical property. In this paper, we provide sufficient conditions for connected 3- i -critical graphs to be k -factor-critical, for a positive integer k , in terms of the connectivity and minimum degree. In fact, we prove the following theorem in Section 3.

Theorem 1.1. *For a positive integer r , let G be a t -connected 3- i -critical graph of order n where $n \equiv r \pmod{2}$ and*

$$t = \begin{cases} r + 1, & \text{for } 1 \leq r \leq 12, \\ \lfloor \frac{7r+1}{6} \rfloor, & \text{for } r \geq 13. \end{cases}$$

Further, for $1 \leq r \leq 12$, $\delta(G) \geq r + 2$. Then G is r -factor-critical. □

We conclude this section by pointing out that sufficient conditions for critical graphs with respect to ordinary domination (see [4] and [5]) and connected domination (see [6]) to be k -factor-critical were investigated. All of these results concern only $1 \leq k \leq 3$.

2 Preliminary results

In this section, we state some results that we make use of in establishing our main results. We begin with some terminology. For a pair of non-adjacent vertices u and v of G , I_{uv}

denotes a minimum independent dominating set for $G + uv$. Our first two results follow immediately from the definition of k - i -critical graphs.

Lemma 2.1. *Let G be a connected k - i -critical graph and let u and v be non-adjacent vertices of G . Then $|I_{uv}| = k - 1$ and $|I_{uv} \cap \{u, v\}| = 1$. \square*

Lemma 2.2. *Let G be a connected 3- i -critical graph and let u and v be non-adjacent vertices of G . Then*

1. $I_{uv} = \{u, w\}$ or $I_{uv} = \{v, w\}$ for some $w \in V(G) - \{u, v\}$.
2. If $\{w\} = I_{uv} - \{u, v\}$, then $\{u, v, w\}$ is independent. \square

The next two results provide necessary and sufficient condition for a graph to contain a perfect matching and to be k -factor-critical.

Theorem 2.3. *(Tutte's Theorem)(see Page 76 in [8]) A nontrivial graph G has a perfect matching if and only if, for every proper subset S of $V(G)$, $\omega_0(G - S) \leq |S|$. \square*

Theorem 2.4. *[9] A graph G is k -factor-critical if and only if $\omega_0(G - S) \leq |S| - k$, for every $S \subseteq V(G)$ and $|S| \geq k$. \square*

Our last three results concern the number of components in connected 3- i -critical graphs in terms of the size of their cutsets.

Lemma 2.5. *[1] For a positive integer $k \geq 3$, let G be a connected k - i -critical graph containing u as a cutvertex. Then $\omega(G - u) \leq k - 1$. \square*

Theorem 2.6. *[2] Let G be a connected 3- i -critical graph and S a vertex cutset. Then*

$$\omega(G - S) \leq \begin{cases} |S| - 2, & \text{for } |S| \geq 6, \\ 3, & \text{for } |S| \leq 5. \end{cases} \quad \square$$

Lemma 2.7. *[2] Let G be a connected 3- i -critical graph and S a minimum cutset where $2 \leq |S| \leq 3$. If $\omega(G - S) = 3$, then $\delta(G) = |S|$. \square*

3 The main results

In this section, we establish the proof of Theorem 1.1. We begin with some lemmas. Our first result follows immediately from Lemma 2.5, Theorems 2.3 and 2.6.

Lemma 3.1. *If G is a connected 3- i -critical graph of even order, then G has a perfect matching. \square*

Lemma 3.2. *Let G be a 3-connected 3-edge- i -critical graph of even order. If $\delta(G) \geq 4$, then G is bicritical.*

Proof. Suppose to the contrary that G is not bicritical. By Theorem 2.4, there exists $S \subseteq V(G)$ where $|S| \geq 2$ and $\omega_0(G - S) > |S| - 2$. Theorem 2.3 and Lemma 3.1 together with the fact that G is of even order implies that $\omega_0(G - S) = |S|$. Consequently, $|S| \leq 3$ by Theorem 2.6. Then, by the connectivity of G , $|S| = 3$ and thus $\omega(G - S) = \omega_0(G - S) = |S| = 3$. By Lemma 2.7, $\delta(G) = 3$, contradicting the minimum degree of G . This completes the proof of our lemma. \square

Lemma 3.3. *Let G be a connected 3- i -critical graph and S a cutset of G . Let x_1, x_2, y_1, y_2 be four distinct vertices of $G - S$ where $x_i y_i \notin E(G)$ for $1 \leq i \leq 2$ and these four vertices belong to at least two different components of $G - S$. Further, let $\{z_i\} = I_{x_i y_i} - \{x_i, y_i\}$, $1 \leq i \leq 2$. If either $\omega(G - S) = 3$ but no components of $G - S$ is singleton or $\omega(G - S) \geq 4$, then $\{z_1, z_2\} \subseteq S$ and $z_1 \neq z_2$.*

Proof. By our hypothesis, it is easy to see that, for $1 \leq i \leq 2$, there are at least two components of $G - (S \cup \{x_i, y_i\})$ which are dominated by z_i . Then $z_i \in S$ and thus $\{z_1, z_2\} \subseteq S$ as required. Let C be a component of $G - S$ containing x_1 . We first suppose that $y_1 \in V(C) - \{x_1\}$. Then $x_2 \notin V(C)$ or $y_2 \notin V(C)$ and thus $z_1 x_2 \in E(G)$ or $z_1 y_2 \in E(G)$ since $\{z_1\} = I_{x_1 y_1} - \{x_1, y_1\}$. Hence, $z_1 \neq z_2$ because $\{x_2, y_2, z_2\}$ is independent by Lemma 2.2(2).

We now assume that $y_1 \notin V(C)$. Since $\{x_2, y_2\} \subseteq V(G) - S$, it follows that $|N_G(x_2) \cap \{x_1, y_1\}| \leq 1$ and $|N_G(y_2) \cap \{x_1, y_1\}| \leq 1$. Thus $x_1 \in N_G(z_2)$ or $y_1 \in N_G(z_2)$ since $\{z_2\} = I_{x_2 y_2} - \{x_2, y_2\}$. Consequently, $z_1 \neq z_2$ because $\{x_1, y_1, z_1\}$ is independent by Lemma 2.2(2). This completes the proof of our lemma. \square

Lemma 3.4. *Let G be a connected 3- i -critical graph and S a cutset of G . Let x_1, x_2 and x_3 be three distinct vertices of $G - S$ and no any two vertices of $\{x_1, x_2, x_3\}$ belong to the same component of $G - S$. Further, let $\{z_1\} = I_{x_1 x_2} - \{x_1, x_2\}$ and $\{z_2\} = I_{x_1 x_3} - \{x_1, x_3\}$. If either $\omega(G - S) = 3$ but no components of $G - S$ is singleton or $\omega(G - S) \geq 4$, then $\{z_1, z_2\} \subseteq S$ and $z_1 \neq z_2$.*

Proof. By similar arguments as in the proof of Lemma 3.3, $\{z_1, z_2\} \subseteq S$. By Lemma 2.2(2), $\{x_1, x_2, z_1\}$ and $\{x_1, x_3, z_2\}$ are independent. Since x_3 does not belong to the same component containing x_1 or x_2 , $z_1 x_3 \in E(G)$ because $\{z_1\} = I_{x_1 x_2} - \{x_1, x_2\}$. By Lemma 2.2(2), $z_1 \neq z_2$. This completes the proof of our lemma. \square

Theorem 3.5. *For a positive integer k , let G be a t -connected 3- i -critical graph of even order where*

$$t = \begin{cases} 2k + 1, & \text{for } 1 \leq k \leq 6, \\ \lfloor \frac{7k}{3} \rfloor, & \text{for } k \geq 7. \end{cases}$$

Further, for $1 \leq k \leq 6$, $\delta(G) \geq 2k + 2$. Then G is $2k$ -factor-critical.

Proof. We shall prove by mathematical induction. Clearly, our result holds for $k = 1$ by Lemma 3.2. We now suppose that our result holds for $k - 1$ where $k \geq 2$. Suppose to the contrary that G satisfies our hypothesis but G is not $2k$ -factor-critical. By Theorem 2.4, there is a vertex cutset S where $|S| \geq 2k$ and $\omega_0(G - S) > |S| - 2k$. By our induction hypothesis, G is $2(k - 1)$ -factor-critical and thus, by Theorem 2.4, $\omega_0(G - S) \leq$

$|S| - 2(k-1)$. Since G is of even order, $\omega_0(G-S) = |S| - 2k + 2 \geq 2$. Then $|S| \geq t \geq 2k+1$ and thus $\omega_0(G-S) \geq 3$. Because $\delta(G) \geq 2k+2$, if $\omega_0(G-S) = 3$ (that is $|S| = 2k+1$), then no components of $G-S$ is singleton.

For $1 \leq i \leq |S| - 2k + 2$, let C_i be an odd component of $G-S$. Choose $x_i \in V(C_i)$. Then, for $1 \leq i \neq j \leq |S| - 2k + 2$, the only vertex of $I_{x_i x_j} - \{x_i, x_j\}$ must be in S by Lemmas 3.3 and 3.4. Thus $|S| \geq \binom{|S|-2k+2}{2}$. It also follows by Lemma 2.2(2) that x_i is not adjacent to the only vertex of $I_{x_i x_j} - \{x_i, x_j\}$ which is in S . Then $|N_S(x_i)| \leq |S| - (|S| - 2k + 1) = 2k - 1$. Since $\delta(G) \geq 2k + 2$, $2k + 2 \leq \deg_G(x_i) \leq (|V(C_i)| - 1) + |N_S(x_i)| \leq |V(C_i)| + 2k - 2$. Hence, for $1 \leq i \leq |S| - 2k + 2$, $|V(C_i)| \geq 5$ because C_i is odd.

For $1 \leq m \neq n \leq |S| - 2k + 2$, let $x_{m_1}, \dots, x_{m_5} \in V(C_m)$ and $x_{n_1}, \dots, x_{n_5} \in V(C_n)$ where $x_{m_i} \neq x_{m_j}$ and $x_{n_i} \neq x_{n_j}$ for $1 \leq i \neq j \leq 5$. Consider $G + x_{m_i} x_{n_i}$ for $1 \leq i \leq 5$. Put $\{z_{x_{m_i} x_{n_i}}\} = I_{x_{m_i} x_{n_i}} - \{x_{m_i}, x_{n_i}\}$. Then, by Lemmas 3.3 and 3.4, $T_{mn} = \{z_{x_{m_i} x_{n_i}} \mid 1 \leq i \leq 5\} \subseteq S$, $|T_{mn}| = 5$ and for $1 \leq m \neq n, m' \neq n' \leq |S| - 2k + 2$, if $\{m, n\} \neq \{m', n'\}$, then $T_{mn} \cap T_{m'n'} = \emptyset$. Consequently, $|S| \geq 5 \binom{|S|-2k+2}{2} = \frac{5}{2}(|S| - 2k + 2)(|S| - 2k + 1)$. Hence, $5|S|^2 + (13 - 20k)|S| + 5(4k^2 - 6k + 2) \leq 0$. Consequently, $|S| \leq \frac{20k-13+\sqrt{80k-31}}{10}$.

We next show that $\frac{20k-13+\sqrt{80k-31}}{10} < 2k+1$ for $1 \leq k \leq 6$ and $\frac{20k-13+\sqrt{80k-31}}{10} < \frac{7k}{3} - 1$ for $k \geq 7$. We first assume that $1 \leq k \leq 6$. Suppose to the contrary that $\frac{20k-13+\sqrt{80k-31}}{10} \geq 2k+1$. Then $\sqrt{80k-31} \geq 23$ and thus $k \geq 7$, a contradiction. Hence, $\frac{20k-13+\sqrt{80k-31}}{10} < 2k+1$ for $1 \leq k \leq 6$ as required. We now assume that $k \geq 7$. Again, suppose to the contrary that $\frac{20k-13+\sqrt{80k-31}}{10} \geq \frac{7k}{3} - 1$. Then $3\sqrt{80k-31} \geq 10k+9$ and thus $0 \geq 100k^2 - 540k + 360 = (10k - 27)^2 - 369 \geq 1480$ since $k \geq 7$, a contradiction. Hence, $\frac{20k-13+\sqrt{80k-31}}{10} < \frac{7k}{3} - 1$ for $k \geq 7$. Therefore, $|S| < t$, contradicting the connectivity of G . Hence, G is $2k$ -factor-critical as required. This completes the proof of our theorem. \square

We now turn our attention to connected 3- i -critical graphs of odd order.

Lemma 3.6. *Let G be a 2-connected 3- i -critical graph of odd order. If $\delta(G) \geq 3$, then G is factor-critical.*

Proof. Suppose to the contrary that G is not factor-critical. By Theorem 2.4, there exists $S \subseteq V(G)$ where $|S| \geq 1$ and $\omega_0(G-S) > |S| - 1$. Since G is of odd order, $\omega_0(G-S) \geq |S| + 1$. This implies by Theorem 2.6 that $1 \leq |S| \leq 2$. Then, by the hypothesis that G is 2-connected, $|S| = 2$ and thus $\omega(G-S) = \omega_0(G-S) = 3$. By Lemma 2.7, $\delta(G) = 2$, contradicting the minimum degree requirement of G . This completes the proof of our lemma. \square

By applying similar arguments as in the proof of Theorem 3.5, we have the following results.

Theorem 3.7. *For a positive integer k , let G be a t -connected 3- i -critical graph of odd order where*

$$t = \begin{cases} 2k, & \text{for } 1 \leq k \leq 6, \\ \lfloor \frac{7k-3}{3} \rfloor, & \text{for } k \geq 7. \end{cases}$$

Further, for $1 \leq k \leq 6$, $\delta(G) \geq 2k+1$. Then G is $(2k-1)$ -factor-critical. \square

As a consequence of Theorems 3.5 and 3.7, Theorem 1.1 follows.

We conclude our paper by pointing out that the bound on minimum degree hypothesis in Theorem 1.1 is best possible. The graph G in Figure 1 is 2-connected 3- i -critical graph with minimum degree 2. Clearly, G is not factor-critical. For positive integers $r \geq 2$ and $n \geq 2r + 2$, let G be $(2K_1 \cup K_{r-1}) \vee (2K_1 \cup K_{n-r-3})$. It is not difficult to see that G is $(r + 1)$ -connected 3- i -critical with minimum degree $r + 1$. But G is not r -factor-critical.

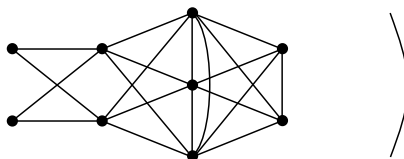


Figure 1: A 2-connected 3- i -critical graph with minimum degree 2

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Extendability of the complementary prism of 2-regular graphs

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Abstract

Let G be a simple graph. The complementary prism of G , denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and \overline{G} , the complement of G , by adding the edges of a perfect matching between the corresponding vertices of G and \overline{G} . A connected graph G of order at least $2k + 2$ is k -extendable if for every matching M of size k in G , there is a perfect matching in G containing all edges of M . The problem that arises is that of investigating the extendability of $G\overline{G}$. In this paper, we investigate the extendability of $G\overline{G}$ where G contains G_1, \dots, G_l as its components and the extendability of $G_i\overline{G}_i$ is known for $1 \leq i \leq l$. We then apply this result to establish the extendability of $G\overline{G}$ when G is 2-regular.

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Keywords: extendable, complementary prism, regular graph

1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by \overline{G} . For a vertex v of G , $deg_G(v)$ and $N_G(v)$ denote the degree and the neighbour set of v . Further, the closed neighbour set of v , denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For disjoint graphs G_1 and G_2 , the join of G_1 and G_2 is denoted by $G_1 \vee G_2$. For positive integers m and $n \geq 3$, K_m and C_n denote a complete graph of order m and a cycle of order n , respectively. For $S \subseteq V(G)$, the induced subgraph of S in G is denoted by $G[S]$. A graph G is said to be H -free if G does not contain H as an induced subgraph. A subset M of $E(G)$ is called a matching in G if no two edges of M have a common end vertex. M is a maximum matching in G if there is no matching M'

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in G such that $|M'| > |M|$. A vertex v of G is said to be M -saturated if v is an end vertex of some edge in a matching M ; otherwise, v is M -unsaturated. If each vertex of G is M -saturated, then M is called a perfect matching. Note that if M is a perfect matching of G , then $|M| = \frac{|V(G)|}{2}$.

In 1980, Plummer [5] introduced a concept of k -**extendable**. For a positive integer k , a connected graph G of order at least $2k+2$ is said to be k -extendable if for every matching M of size k in G , there is a perfect matching in G containing all edges of M . It is easy to see that K_{2n} is k -extendable for $1 \leq k \leq n-1$ and a cycle of even order is 1-extendable but not 2-extendable. Since 1980 the concept of k -extendable graphs was investigated by several researchers. For excellence surveys in this topic, a reader is directed to ([7],[8] and [9]). A closely concept to k -extendable graphs is k -**factor-critical** graphs introduced by Favaron [2]. A graph G is said to be k -factor-critical if for every subset $S \subseteq V(G)$ with $|S| = k$, $G - S$ has a perfect matching. Favaron also pointed out some relationship between extendable graphs and factor-critical graphs as we shall see in Theorem 2.5.

Haynes et al. [4] introduced the concept of **complementary prism** of a graph. For a simple graph G , the complementary prism of G , denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and \overline{G} by adding the edges of a perfect matching between the corresponding vertices of G and \overline{G} . Examples of the complementary prism of graphs are shown in Figures 1 and 2. Note that the graph $C_5\overline{C}_5$ in Figure 1 is isomorphic to the Petersen graph. One might ask what property that a graph G should have so that $G\overline{G}$ is k -extendable for some k . A problem that arises is that of investigating the extendability of $G\overline{G}$. In this paper, we first consider the extendability of $G\overline{G}$ where G contains G_1, \dots, G_l as its components and the extendability of $G_i\overline{G}_i$ is known for $1 \leq i \leq l$. In fact, we prove the following theorem:

Theorem 1.1. *For positive integers i and l where $1 \leq i \leq l$, let G_1, \dots, G_l be components of G . If $G_i\overline{G}_i$ is k -extendable of order $p_i \geq 2k+2$ for some positive integer k , then $G\overline{G}$ is k -extendable. \square*

We then apply Theorem 1.1 to establish the extendability of 2-regular graphs. We show that:

Theorem 1.2. *Let G be a 2-regular H -free graph where $H \in \{C_3, C_4, C_5\}$. Then $G\overline{G}$ is 2-extendable. \square*

The condition of H -free and the extendability of $G\overline{G}$ stated in Theorem 1.2 are all best possible. For positive integers $n \geq 8$ and $3 \leq i \leq 5$, let $H_i = C_i \cup C_{n-i}$. Then the graph $H_i\overline{H}_i$, shown in Figure 2, is not 2-extendable since there is no perfect matching containing the edge x_1x_2 and y_1y_2 . Note that ‘‘a double line’’ in our diagram denotes the join between corresponding graphs. Hence, the hypothesis H -free where $H \in \{C_3, C_4, C_5\}$

in Theorem 1.2 cannot be dropped. Finally, the extendability of $G\overline{G}$ in Theorem 1.2 is best possible by Theorem 2.2(2) and the fact that the minimum degree of $G\overline{G}$ is 3.

The proof of Theorems 1.1 and 1.2 are in Sections 3 and 4, respectively.

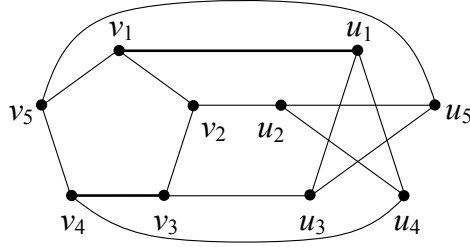


Figure 1: The graph $C_5\overline{C}_5$

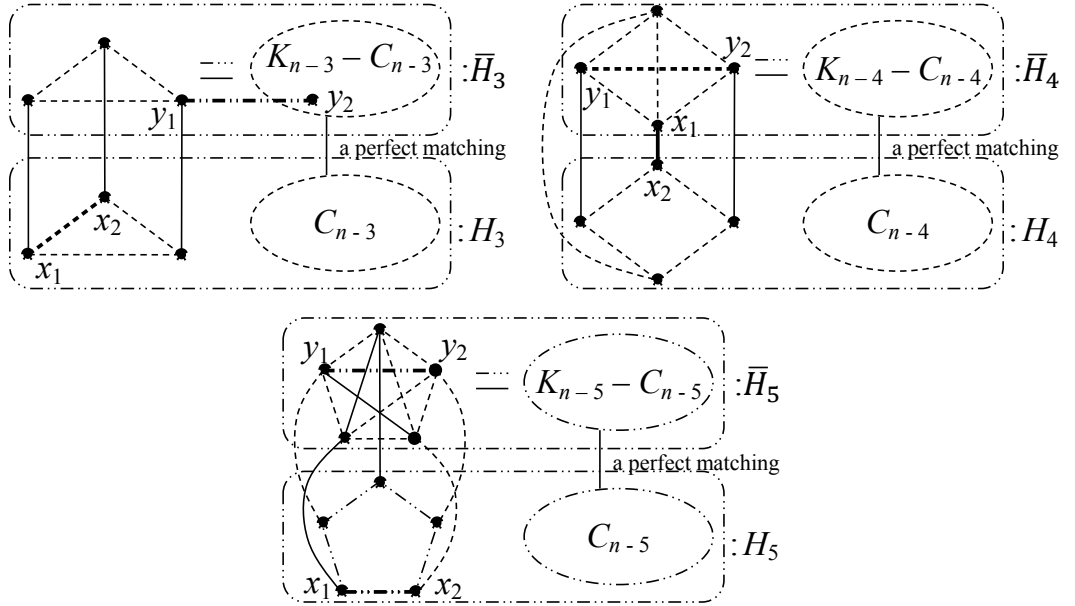


Figure 2: The graph $H_i\overline{H}_i$, $i \in \{3, 4, 5\}$

2 Preliminary results

In this section, we provide results that we make use of in establishing our results in the next two sections. We begin with a result on an existence of a perfect matching in a graph.

Theorem 2.1. (see [1] on page 76) *A graph G has a perfect matching if and only if for a subset S of $V(G)$, the number of odd components of $G - S$ is at most $|S|$. \square*

The next three theorems proved by Plummer concern some properties of extendable graphs.

Theorem 2.2. [5] *For positive integers k and p , let G be a graph of order $p \geq 2k + 2$. If G is k -extendable, then*

1. G is $(k - 1)$ -extendable, and
2. G is $(k + 1)$ -connected. □

Theorem 2.3. [5] *If G is 2-extendable non-bipartite graph of order $p \geq 6$, then G is 2-factor-critical.* □

Theorem 2.4. [6] *Let $k \geq 1$ be an integer and let G be a $(2k+1)$ -connected $K_{1,3}$ -free graph with an even number of vertices. Then G is k -extendable.* □

Our next result provides a relationship between extendable graphs and factor-critical graphs proved by Favaron.

Theorem 2.5. [2] *If G is a $2k$ -extendable non-bipartite graph for $2k \geq 2$, then G is a $2k$ -factor-critical graph.* □

We conclude this section with our results proved in [3].

Lemma 2.6. [3] *Let G be a k -extendable non-bipartite graph and M a matching of G with $|M| \leq k - 1$. Then $G - V(M)$ is a $(k - |M|)$ -extendable non-bipartite graph. Further, if $k - |M|$ is even, then $G - V(M)$ is $(k - |M|)$ -factor critical.* □

Lemma 2.7. [3] *Let G be a k -extendable graph for some integer $k \geq 2$ and let $S \subseteq V(G)$ be a cutset of G . If $G[S]$ contains $t \leq k - 1$ independent edges, then $|S| \geq k + t + 1$.* □

3 Fundamental results

In this section, we provide the proof of Theorem 1.1. We first establish a useful lemma. For a matching M , we simply denote the set of end vertices of edges in M by $V(M)$.

Lemma 3.1. *Let $G\bar{G}$ be a k -extendable graph for some positive integer k . Suppose M is a matching in $G\bar{G}$ and $S \subseteq V(\bar{G})$ where $V(M) \cap S = \emptyset$ and $|M| + |S| \leq k$. Then*

1. *If $|S|$ is even, then there is a perfect matching in $G\bar{G} - (V(M) \cup S)$.*

2. If $|S|$ is odd, then there is a vertex $\bar{y} \in V(\bar{G}) - (V(M) \cup S)$ such that $G\bar{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching.

Proof. Observe that $G\bar{G}$ is non-bipartite.

(1) It is easy to see that if $S = \emptyset$, then, by Theorem 2.2(1), $G\bar{G} - (V(M) \cup S) = G\bar{G} - V(M)$ contains a perfect matching since $G\bar{G}$ is k -extendable. So we may now assume that $S \neq \emptyset$. Then $2 \leq |S| \leq |M| + |S| \leq k$. Thus $|M| \leq k - 2$. By Lemma 2.6 and the fact that $G\bar{G}$ is non-bipartite, $G\bar{G} - V(M)$ is $(k - |M|)$ -extendable non-bipartite. Since $|S| \leq k - |M|$, $G\bar{G} - V(M)$ is $|S|$ -extendable non-bipartite by Theorem 2.2(1). Hence, $G\bar{G} - V(M)$ is $|S|$ -factor-critical by Theorem 2.5 and the fact that $|S|$ is even. Therefore, $G\bar{G} - (V(M) \cup S)$ contains a perfect matching. This proves (1).

(2) Since $|S|$ is odd, $|S| \geq 1$ and thus $|M| \leq k - |S| \leq k - 1$. We first show that there are a vertex $\bar{u} \in S$ and a vertex $\bar{v} \in V(\bar{G}) - (V(M) \cup S)$ such that $\bar{u}\bar{v} \in E(\bar{G})$. Suppose this is not the case. Let $\bar{u}_0 \in S$. Then $N_{G\bar{G}}[\bar{u}_0] \subseteq S \cup V(M) \cup \{u_0\}$ where u_0 is the only vertex in G which is adjacent to \bar{u}_0 . Put $S' = (S - \{\bar{u}_0\}) \cup \{u_0\}$. Clearly, \bar{u}_0 becomes an isolated vertex in $G\bar{G} - (V(M) \cup S')$ and $|V(M) \cup S'| = 2|M| + |S'| = 2|M| + |S| \leq k + |M|$. So $V(M) \cup S'$ is a cutset of $G\bar{G}$. But this contradicts Lemma 2.7 since $G\bar{G}[V(M) \cup S']$ contains a matching of size at least $|M|$ and at most $|M| + \frac{1}{2}|S'| < |M| + |S| \leq k$ and $|V(M) \cup S'| \leq k + |M|$. Hence, there are a vertex $\bar{u} \in S$ and a vertex $\bar{v} \in V(\bar{G}) - (V(M) \cup S)$ such that $\bar{u}\bar{v} \in E(\bar{G})$ as required.

Now let $\bar{x} \in S$ and a vertex $\bar{y} \in V(\bar{G}) - (V(M) \cup S)$ such that $\bar{x}\bar{y} \in E(\bar{G})$. Consider $M \cup \{\bar{x}\bar{y}\}$. Clearly, $|M \cup \{\bar{x}\bar{y}\}| \leq k$. We first suppose that $|M \cup \{\bar{x}\bar{y}\}| = k$. Because $|M| \leq k - |S|$, $|S| = 1$ and thus $S = \{\bar{x}\}$. Since $G\bar{G}$ is k -extendable, $G\bar{G} - (V(M) \cup \{\bar{x}\bar{y}\}) = G\bar{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching as required. So we now suppose that $|M \cup \{\bar{x}\bar{y}\}| \leq k - 1$. By Lemma 2.6 and the fact that $G\bar{G}$ is non-bipartite, $G\bar{G} - (V(M) \cup \{\bar{x}\bar{y}\})$ is $(k - (|M| + 1))$ -extendable non-bipartite. Since $k - |M| - 1 \geq |S| - 1$ and $|S| - 1$ is even, it then follows by Theorems 2.2(1) and 2.5 that $G\bar{G} - (V(M) \cup \{\bar{x}\bar{y}\})$ is $(|S| - 1)$ -factor-critical. Hence, $G\bar{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching as required. This proves (2) and completes the proof of our lemma. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1

Proof. Clearly, our result holds for $l = 1$. So we now suppose $l \geq 2$. For simplicity, the induced subgraphs $G\bar{G}[V(G_i)]$, $G\bar{G}[V(\bar{G}_i)]$ and $G\bar{G}[V(G_i\bar{G}_i)]$ are denoted by G_i , \bar{G}_i and $G_i\bar{G}_i$, respectively.

Let M be a matching of size k in $G\bar{G}$. For $1 \leq i \leq l$, let $M_i = M \cap E(G_i\bar{G}_i)$ and $S_i = \{x \in V(G_i\bar{G}_i) | xy \in M \text{ and } y \notin V(G_i\bar{G}_i)\}$. Observe that $S_i \subseteq V(\bar{G}_i)$ and $E(G\bar{G}[\bigcup_{i=1}^l S_i]) = M - \bigcup_{i=1}^l M_i$. We first suppose that $|S_i|$ is even for $1 \leq i \leq l$. Then, by Lemma 3.1(1), there is a perfect matching F_i in $G_i\bar{G}_i - (V(M_i) \cup S_i)$ for $1 \leq i \leq l$. Hence, $(\bigcup_{i=1}^l F_i) \cup M$ is a perfect matching in $G\bar{G}$ containing M as required.

We now suppose that $|S_i|$ is odd for some i . Let l_o be the number of components G_i of G in which $|S_i|$ is odd. We may now renumber the components of G in such a way that for the first l_o components of G , $|S_i|$ is odd for $1 \leq i \leq l_o$ and for the last $l - l_o$ components of G , $|S_i|$ is even. Since $\sum_{i=1}^{l_o} |S_i| = 2(|M - \bigcup_{i=1}^l M_i|) - \sum_{i>l_o} |S_i|$ is even, l_o is even. By Lemma 3.1(2), there is $\bar{y}_i \in V(\bar{G}_i) - (V(M_i) \cup S_i)$ such that $G_i \bar{G}_i - (V(M_i) \cup S_i \cup \{\bar{y}_i\})$ contains a perfect matching, say F'_i , for $1 \leq i \leq l_o$. Clearly, $G\bar{G}[\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_o}\}]$ is a complete graph of even order. So there is a perfect matching in $G\bar{G}[\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_o}\}]$, say F' . By Lemma 3.1(1), if $l_o < l$, then there is a perfect matching F'_i in $G_i \bar{G}_i - (V(M_i) \cup S_i)$ for $l_o + 1 \leq i \leq l$. Therefore, $\bigcup_{i=1}^l F'_i \cup F' \cup M$ is a perfect matching in $G\bar{G}$ containing M as required. Hence, $G\bar{G}$ is k -extendable. This completes the proof of our theorem. \square

Our next result follows immediately from Theorems 1.1 and 2.2(1).

Corollary 3.2. *For positive integers i and l where $1 \leq i \leq l$, let G_1, \dots, G_l be components of G . If $G_i \bar{G}_i$ is k_i -extendable of order $p_i \geq 2k_i + 2$ for some positive integer k_i , then $G\bar{G}$ is k_0 -extendable where $k_0 = \min\{k_1, k_2, \dots, k_l\}$. \square*

4 The extendability of 2-regular graphs

According to Theorem 1.1, it is enough to establish the following theorem in order to prove Theorem 1.2.

Theorem 4.1. *Let G be a connected 2-regular graph of order $n \geq 6$. Then $G\bar{G}$ is 2-extendable. \square*

In order to establish Theorem 4.1, we need to set up some lemmas. For simplicity, we now assume that G is a connected 2-regular graph and put $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_{i+1} | 1 \leq i \leq n\}$ where the subscript is read modulo n . Observe that $G \cong C_n$, $\bar{G} \cong K_n - \{v_i v_{i+1} | 1 \leq i \leq n\}$ and thus \bar{G} is $(n - 3)$ -regular of order n . In what follows, the symbols G , \bar{G} , $V(G)$, $E(G)$, n and v_k for $1 \leq k \leq n$ are referred to these set up. Further, all subscripts are read modulo n . Our first lemma follows immediately from the fact that if x is a vertex of C_p where $p \geq 3$, then $C_p - x$ is a path of order $p - 1$.

Lemma 4.2. *1. If n is even and e is an edge of G , then there is a perfect matching in G containing the edge e .*

2. If n is odd, then, for each $1 \leq k \leq n$, $G - v_k$ contains a maximum matching of size $\frac{n-1}{2}$. In fact, a maximum matching of size $\frac{n-1}{2}$ is $\{v_{k+1}v_{k+2}, v_{k+3}v_{k+4}, \dots, v_{k+n-2}v_{k+n-1}\}$ which is also a perfect matching in $G - v_k$. \square

Lemma 4.3. *For an integer $n \geq 5$, \bar{G} is $(n - 3)$ -connected.*

Proof. Let S be a minimum cutset of \overline{G} . For a positive integer $k \geq 2$, let H_1, \dots, H_k be components of $\overline{G} - S$. Since \overline{G} is $(n-3)$ -regular, $|V(H_i)| \geq n-2-|S|$. Then $n = |V(\overline{G})| = \sum_{i=1}^k |V(H_i)| + |S| \geq 2(n-2-|S|) + |S| = 2n-4-|S|$ and thus $|S| \geq n-4$. Suppose $|S| = n-4$. It is easy to see that $|V(H_i)| = 2$ and $k = 2$. Thus $n \geq 7$ since \overline{G} is $(n-3)$ -regular. It follows that $\overline{G} \cong 2K_2 \vee H$ where H is $(n-7)$ -regular of order $n-4$. Thus G contains C_4 as an induced subgraph. But this contradicts the fact that $G \cong C_n$ where $n \geq 5$. Hence, $|S| \geq n-3$ and then \overline{G} is $(n-3)$ -connected. This completes the proof of our lemma. \square

Lemma 4.4. *For an integer $n \geq 6$, if n is even, then \overline{G} is $(\frac{n-4}{2})$ -extendable and if n is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is $(\frac{n-5}{2})$ -extendable.*

Proof. Observe that \overline{G} is $K_{1,3}$ -free otherwise G contains C_3 as an induced subgraph which contradicts the fact that $G \cong C_n$ and $n \geq 6$. By Theorem 2.4 and Lemma 4.3, \overline{G} is $(\frac{n-4}{2})$ -extendable if n is even. We now suppose that n is odd. Then $n \geq 7$ and $\overline{G} - v_k$ is $(n-4)$ -connected by Lemma 4.3. Hence, by Theorem 2.4, $\overline{G} - v_k$ is $(\frac{n-5}{2})$ -extendable. This proves our lemma. \square

As a consequence of Theorem 2.2(1) and Lemma 4.4, we have the following corollaries.

Corollary 4.5. *For an integer n , if $n \geq 8$ is even, then \overline{G} is 2-extendable and if $n \geq 9$ is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is 2-extendable.* \square

Corollary 4.6. *For an integer n , if $n \geq 6$ is even, then \overline{G} is 1-extendable and if $n \geq 7$ is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is 1-extendable.* \square

Corollary 4.7. *For an integer $n \geq 6$, let v_i, v_j, v_k be three distinct vertices of \overline{G} where $1 \leq i, j, k \leq n$, then $\overline{G} - \{v_i, v_j\}$ has a perfect matching if n is even and $\overline{G} - \{v_i, v_j, v_k\}$ has a perfect matching if n is odd.*

Proof. Our result follows from Theorems 2.2(1) and 2.3 together with Lemma 4.4 if $n \geq 8$. For $6 \leq n \leq 7$, our result follows from Theorem 2.1, Lemma 4.3 and the fact that \overline{G} is $K_{1,3}$ -free. \square

We are now ready to prove Theorem 4.1. For simplicity, put $V(\overline{G}) = \{u_1, \dots, u_n\}$ where $u_i \in V(\overline{G})$ corresponds to $v_i \in V(G)$. Then $V(G\overline{G}) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v_i u_i | 1 \leq i \leq n\}$.

Proof of Theorem 4.1

Proof. Let $T = \{e_1, e_2\}$ be a matching of size 2 in $G\overline{G}$. It is easy to see that if $\{e_1, e_2\} \subseteq \{v_i u_i | 1 \leq i \leq n\}$, then $\{v_i u_i | 1 \leq i \leq n\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . So we may now assume without loss of generality that $e_1 \notin$

$\{v_i u_i | 1 \leq i \leq n\}$. For simplicity, the set of end vertices of the edge e_i is denoted by $V(e_i)$ for $1 \leq i \leq 2$. To show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 , we distinguish five cases according to the edges e_1 and e_2 .

Case 1: $\{e_1, e_2\} \subseteq E(\overline{G})$.

By Corollary 4.5 and the fact that $G \cong C_n$, it is easy to see that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 if $n \geq 8$ is even. For $n = 6$, it is not difficult to show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 as well.

So we now suppose that $n \geq 9$ is odd. Choose a vertex $u_j \in V(\overline{G}) - (V(e_1) \cup V(e_2))$. Then, by Corollary 4.5, there is a perfect matching \overline{M}_1 , in $\overline{G} - u_j$, containing the edges e_1 and e_2 . By Lemma 4.2(2), there is a perfect matching M_1 in $G - v_j$. Hence, $M_1 \cup \overline{M}_1 \cup \{v_j u_j\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now consider $n = 7$. Observe that $V(\overline{G}) - (V(e_1) \cup V(e_2))$ contains an edge, say e_3 , otherwise G contains C_3 as an induced subgraph. Put $\{u_{j'}\} = V(\overline{G}) - \bigcup_{i=1}^3 V(e_i)$. By Lemma 4.2(2), there is a perfect matching M_2 in $G - v_{j'}$. Thus $M_2 \cup \{e_1, e_2, e_3, v_{j'} u_{j'}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 1.

Case 2: $e_1 \in E(G), e_2 \in E(\overline{G})$.

Suppose $e_1 = v_j v_{j+1}$ and $e_2 = u_k u_{k'}$ where $1 \leq j, k, k' \leq n$ and $k \neq k'$. By Lemma 4.2(1) and Corollary 4.6, it is easy to see that there is a perfect matching containing the edges e_1 and e_2 if n is even. So we now suppose that n is odd.

We first suppose that $j + 2 \notin \{k, k'\}$. Then a maximum matching M_1 , in $G - v_{j+2}$, containing the edge $e_1 = v_j v_{j+1}$ is a matching of size $\frac{n-1}{2}$. Thus M_1 is a perfect matching in $G - v_{j+2}$ by Lemma 4.2(2). By Corollary 4.6, $\overline{G} - u_{j+2}$ has a perfect matching \overline{M}_1 containing the edge e_2 . Then $M_1 \cup \overline{M}_1 \cup \{v_{j+2} u_{j+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

By similar arguments, if $j - 1 \notin \{k, k'\}$, then there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . We may now assume that $\{j - 1, j + 2\} = \{k, k'\}$. Then $e_2 = u_k u_{k'} = u_{j-1} u_{j+2}$. Now consider $G - v_{j+4}$. Since $n \geq 7$, $j + 4 \notin \{j - 1, j + 2\}$. Then a maximum matching M_2 , in $G - v_{j+4}$, of size $\frac{n-1}{2}$ must contain the edge $e_1 = v_j v_{j+1}$. By Lemma 4.2(2), M_2 is a perfect matching in $G - v_{j+4}$. By Corollary 4.6, $\overline{G} - u_{j+4}$ has a perfect matching \overline{M}_2 containing the edge e_2 . Then $M_2 \cup \overline{M}_2 \cup \{v_{j+4} u_{j+4}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

Case 3: $e_1 \in E(G), e_2 \in \{v_i u_i | 1 \leq i \leq n\}$.

Let $e_2 = v_k u_k$ for some $1 \leq k \leq n$. Consider $G - v_k$. Observe that $G - v_k$ is a path of order $n - 1$. Let M_1 and M_2 be matchings in $G - v_k$ where $E(G - v_k) = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. We may assume that $|M_1| \geq |M_2|$. We first suppose that n is odd. Then $|M_1| = \frac{n-1}{2}$ and $|M_2| = \frac{n-3}{2}$. Further, v_{k-1} and v_{k+1} are M_2 -unsaturated. By Lemma 4.2(2), M_1 is a perfect matching in $G - v_k$. If $e_1 \in M_1$, then $M_1 \cup \overline{M}_1 \cup \{v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_1 is a perfect matching, in $\overline{G} - u_k$. Note that \overline{M}_1 exists by Corollary 4.6. We now suppose that $e_1 \in M_2$. By

Corollary 4.7, there is a perfect matching \overline{M}_2 , in $\overline{G} - \{u_{k-1}, u_k, u_{k+1}\}$. Hence, $M_2 \cup \overline{M}_2 \cup \{v_{k-1}u_{k-1}, v_k u_k, v_{k+1}u_{k+1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now suppose that n is even. Then $|M_1| = |M_2| = \frac{n-2}{2}$. Then either v_{k-1} or v_{k+1} is M' -unsaturated where $M' \in \{M_1, M_2\}$. Suppose without loss of generality that $e_1 \in M_1$ and v_{k-1} is M_1 -unsaturated. By Corollary 4.7, there is a perfect matching \overline{M}_3 , in $\overline{G} - \{u_{k-1}, u_k\}$. Hence, $M_1 \cup \overline{M}_3 \cup \{v_{k-1}u_{k-1}, v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 3.

Case 4: $e_1 \in E(\overline{G}), e_2 \in \{v_i u_i | 1 \leq i \leq n\}$.

Let $e_1 = u_j u_{j'}$ and $e_2 = v_k u_k$ for some $1 \leq j, j', k \leq n$. Clearly, $k \notin \{j, j'\}$. We first suppose that n is odd. By Lemma 4.2(2), $G - v_k$ contains M_1 as a perfect matching. By Corollary 4.6, $\overline{G} - u_k$ has a perfect matching containing the edge e_1 , say \overline{M}_1 . Thus $M_1 \cup \overline{M}_1 \cup \{v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now suppose that $n \geq 8$ is even. Let M_2 and M_3 be perfect matchings in G containing the $v_k v_{k+1}$ and $v_{k-1} v_k$, respectively. Observe that if $S \subseteq V(\overline{G})$ with $|S| = 4$, then $\overline{G}[S]$ contains a matching of size two since \overline{G} is $(n-3)$ -regular and G does not contain C_3 as an induced subgraph. We first suppose that $k+1 \notin \{j, j'\}$. By Corollary 4.5, $\overline{G} - \{u_j, u_{j'}, u_k, u_{k+1}\}$ contains \overline{M}_2 as a perfect matching. Then $(M_2 - \{v_k v_{k+1}\}) \cup \overline{M}_2 \cup \{u_j u_{j'}, v_k u_k, v_{k+1} u_{k+1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . By similar arguments, if $k-1 \notin \{j, j'\}$, then $(M_3 - \{v_{k-1} v_k\}) \cup \overline{M}_3 \cup \{u_j u_{j'}, v_{k-1} u_{k-1}, v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_3 is a perfect matching in $\overline{G} - \{u_j, u_{j'}, u_{k-1}, u_k\}$. Finally, we suppose that $\{j, j'\} = \{k-1, k+1\}$. By Corollary 4.5 and the observation that $\overline{G}[S]$ contains a matching of size two if $S \subseteq V(\overline{G})$ with $|S| = 4$, $\overline{G} - \{u_{k-1}, u_k, u_{k+1}, u_{k+3}\}$ contains \overline{M}_4 as a perfect matching. Then $(M_2 - \{v_k v_{k+1}, v_{k+2} v_{k+3}\}) \cup \overline{M}_4 \cup \{u_j u_{j'}, v_k u_k, v_{k+1} v_{k+2}, v_{k+3} u_{k+3}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . For $n = 6$, it is routine to show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 4.

Case 5: $\{e_1, e_2\} \subseteq E(G)$.

Let M be a maximum matching in G containing the edge e_1 . Clearly, M is a perfect matching if n is even and if n is odd, then there is exactly one M -unsaturated vertex, say v_j , for some $1 \leq j \leq n$. We first suppose that $e_2 \in M$. Then there is a perfect matching F containing the edges e_1 and e_2 where $F = M \cup \overline{M}$ if n is even and $F = M \cup \overline{M}_1 \cup \{v_j u_j\}$ if n is odd where \overline{M} and \overline{M}_1 are perfect matchings in \overline{G} and $\overline{G} - u_j$, respectively. Such \overline{M} and \overline{M}_1 exist by Lemma 4.4.

We now suppose that $e_2 \notin M$. Put $e_2 = v_k v_{k+1}$ where $1 \leq k \leq n$. We first assume that n is even. Then $\{v_{k-1} v_k, v_{k+1} v_{k+2}\} \subseteq M - \{e_1\}$ since $\{e_1, e_2\}$ is a matching, M is a perfect matching and $G \cong C_n$. Clearly, $\{v_{k-1}, v_{k+2}\} \cap V(e_1) = \emptyset$. By Corollary 4.7, there exists a perfect matching in \overline{M}_2 in $\overline{G} - \{u_{k-1}, u_{k+2}\}$. Then $(M - \{v_{k-1} v_k, v_{k+1} v_{k+2}\}) \cup \overline{M}_2 \cup \{v_k v_{k+1}, v_{k-1} u_{k-1}, v_{k+2} u_{k+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 .

We now suppose that n is odd. Recall that v_j is the only M -unsaturated of G . If $\{v_k, v_{k+1}\} \cap \{v_j\} = \{v_k\}$, then $\{v_{k+1} v_{k+2}\} \subseteq M - \{e_1\}$ and thus $(M - \{v_{k+1} v_{k+2}\}) \cup \overline{M}_3$

$\cup \{v_k v_{k+1}, v_{k+2} u_{k+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_3 is a perfect matching in $\overline{G} - u_{k+2}$. Note that \overline{M}_3 exists by Corollary 4.6. Similarly, if $\{v_k, v_{k+1}\} \cap \{v_j\} = \{v_{k+1}\}$, then $M - \{v_{k-1} v_k\} \cup \overline{M}_4 \cup \{v_k v_{k+1}, v_{k-1} u_{k-1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where \overline{M}_4 is a perfect matching in $\overline{G} - u_{k-1}$. We now consider the case that $\{v_k, v_{k+1}\} \cap \{v_j\} = \emptyset$. Observe that $j \notin \{k-1, k+2\}$ since $e_2 \notin M$ and v_j is M -unsaturated. Then $\{v_{k-1} v_k, v_{k+1} v_{k+2}\} \subseteq M - \{e_1\}$. By Corollary 4.7, there exists a perfect matching \overline{M}_5 in $\overline{G} - \{u_j, u_{k-1}, u_{k+2}\}$. Then $(M - \{v_{k-1} v_k, v_{k+1} v_{k+2}\}) \cup \overline{M}_5 \cup \{v_k v_{k+1}, v_{k-1} u_{k-1}, v_{k+2} u_{k+2}, v_j u_j\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 . This proves Case 5 and completes the proof of our theorem \square

Note that the bound on n in Theorem 4.1 is sharp since the graph $C_5 \overline{C}_5$ in Figure 1 is not 2-extendable because there is no perfect matching containing the edges $v_1 u_1$ and $v_3 v_4$.

Corollary 4.8. *Let G be a connected 2-regular graph of order $n \geq 4$. Then $G\overline{G}$ is 1-extendable.*

Proof. Our result follows from Theorems 2.2(1) and 4.1 if $n \geq 6$. It is not difficult to show that the result is true for $4 \leq n \leq 5$. \square

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Matching extension in complementary prism of regular graphs

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Abstract : Let \overline{G} denote the complement of a simple graph G . The complementary prism of G denoted by $G\overline{G}$ can be obtained by taking a copy of G and a copy of \overline{G} and then joining corresponding vertices by an edge. A connected graph G of order at least $2k + 2$ is k -extendable if for every matching M of size k in G , there is a perfect matching in G containing all edges of M . In this paper, we establish some sufficient conditions for the complementary prism of regular graphs to be 2-extendable.

Keywords : matching; extendable; complementary prism; regular graph.

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1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by \overline{G} . For $S \subseteq V(G)$, $G[S]$ denotes the induced subgraph of G by S . A neighbor set of a vertex v in G is denoted by $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. For $v \in V(G)$ and $T \subseteq V(G)$, a neighbor set of a vertex v in T is denoted by $N_T(v) = \{u \in T | uv \in E(G)\}$ and if $X \subseteq V(G)$, $N_G(X)$ denotes $\bigcup_{v \in X} N_G(v)$. The number of components of G , the number of odd and even components of G are denoted by $c(G)$, $c_o(G)$ and $c_e(G)$, respectively. A complete graph of order r is denoted by K_r . For graphs H and G , G is called H -free if G does not contain

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H as an induced subgraph. A subgraph H is called a clique if $H \cong K_r$, for some r . A set $M \subseteq E(G)$ is called a matching if no two edges of M have common end vertex. A vertex u is saturated by M if there is an edge in M incident with u . For simplicity, the set of all vertices saturated by M is denoted by $V(M)$. M is called a maximum matching in G if there is no matching N in G of size greater than $|M|$. A perfect matching in G is a matching that saturates all vertices of G . A connected graph G of order at least $2k + 2$ is k -extendable if for every matching M of size k in G , there is a perfect matching in G containing all edges of M . A graph G is k -factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For $k = 1$ and $k = 2$, k -factor-critical graph is also called factor-critical and bicritical, respectively.

The concept of k -extendable graphs was introduced, in 1980, by Plummer [9]. He gave a sufficient condition for a graph to be k -extendable in terms of minimum degree. A fundamental theorem (see Theorem 2.2) that mainly used in studying matching extension was established. He also proved that 2-extendable non-bipartite graphs are bicritical. Some sufficient conditions for special classes of graphs to be k -extendable were established (see [8, 10, 14]). For survey of this topic, a reader is directed to Plummer [11, 12, 13].

The concept of k -factor-critical graphs was introduced, in 1996, by Favaron [6]. She gave a necessary and sufficient condition for a graph to be k -factor-critical and also provided a relationship between n -extendable graphs and k -factor - critical graphs.

A *complementary prism* of G , denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of its complement \overline{G} and then joining corresponding vertices by an edge. A complementary prism is a specific case of complementary product of graphs introduced by Haynes et al.[3] in 2007. Haynes et al. ([3, 4, 5]) studied some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number and the domination number.

According to the definition of the complementary prism of G , it is easy to see that $G\overline{G}$ contains a perfect matching. A problem that arises is that of investigating properties of G so that $G\overline{G}$ is k -extendable for some k . In [7], Janseana etc. established that if G is a 2-regular H -free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable. In this paper, we concentrate on connected r -regular graphs for $r \geq 3$. In fact, we prove that for a connected graph G of order p , if G is either 3-regular F -free where $p \geq 8$ or r_0 -regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable where the graph F is shown in Figure 1. We further extend this result to disconnected graphs. We show that if each component G_i of G is 3-regular F -free of order at least 8 or r_0 -regular of order at least $2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable. These results are in Section 3.

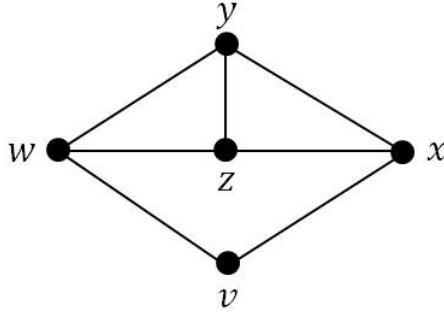


Figure 1: the graph F

2 Preliminary results

In this section, we state some results which are used in establishing our results in Section 3. Our first result is a well known theorem for studying an existence of a perfect matching in graphs established by Tutte.

Theorem 2.1. [2] (Tutte's Theorem) *A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G - S) \leq |S|$.* \square

In 1980, Plummer [9] established a fundamental theorem on k -extendable graphs as following.

Theorem 2.2. [9] *Let G be a graph of order $p \geq 2k + 2$ and $k \geq 1$. If G is k -extendable, then*

- (a) G is $(k - 1)$ -extendable, and
- (b) G is $(k + 1)$ -connected. \square

Ananchuen and Caccetta [1] gave a necessary condition for a neighbor set of a vertex having minimum degree in extendable graphs. They showed that:

Theorem 2.3. [1] *If G is a k -extendable graph on $p \geq 2k + 2$ vertices with $\delta(G) = k + t$, $1 \leq t \leq k \leq p$. If $d_G(u) = \delta(G)$, then the induced subgraph $G[N_G(u)]$ has at most $t - 1$ independent edges.* \square

A necessary and sufficient condition for a graph to be k -extendable and to be k -factor-critical were provided by Yu [15] and Favaron [6], respectively.

Theorem 2.4. [15] *A graph G is k -extendable ($k \geq 1$) if and only if for any $S \subseteq V(G)$,*

- (a) $c_o(G - S) \leq |S|$ and
- (b) $c_o(G - S) = |S| - 2t$, ($0 \leq t \leq k - 1$) implies that $F(S) \leq t$, where $F(S)$ is the size of a maximum matching in $G[S]$. \square

Theorem 2.5. [6] *A graph G is k -factor-critical if and only if $|V(G)| \equiv k \pmod{2}$ and for any $S \subseteq V(G)$ with $|S| \geq k$, $c_o(G - S) \leq |S| - k$. \square*

We now turn our attention to some results concerning complementary prism of graphs.

Theorem 2.6. [7] *For positive integers l and i where $1 \leq i \leq l$, let G_1, \dots, G_l be components of G . If $G_i \overline{G}_i$ is k -extendable of order $p_i \geq 2k + 2$ for some positive integer k , then $G\overline{G}$ is k -extendable. \square*

Theorem 2.7. [7] *Let G be a 2-regular H -free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable. \square*

3 Main results

We begin this section by establishing some lemmas concerning complementary prism of graphs and of regular graphs. These results are essential for establishing Theorem 3.10, the main result of our paper. To simplify our discussion of complementary prisms, G and \overline{G} are referred to subgraph copies of G and \overline{G} , respectively, in $G\overline{G}$. For a vertex v of G , there is exactly one vertex of \overline{G} which is adjacent to v in $G\overline{G}$. This vertex is denoted by \overline{v} . That is $\{\overline{v}\} = N_{\overline{G}}(v)$. Conversely, v is the only vertex of G which is adjacent to \overline{v} . Similarly, for $\phi \neq X = \{x_1, x_2, \dots, x_k\} \subseteq V(G)$, $\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_k\} \subseteq V(\overline{G})$ is denoted by \overline{X} and vice versa. Clearly, $|X| = |\overline{X}|$.

Lemma 3.1. *Let G be a graph. Then $G\overline{G}$ is even and connected.*

Proof. Clearly, $G\overline{G}$ is even. Let $u, v \in V(G\overline{G})$. It is easy to see that if $u, v \in V(G)(V(\overline{G}))$, then either $uv \in E(G)$ or $u\overline{u}\overline{v}v$ is a $u - v$ path. We may now assume that $u \in V(G)$ and $v \in V(\overline{G})$. Clearly, $uv \in E(G\overline{G})$ if $v = \overline{u}$. So suppose that $v = \overline{w}$ for some $w \in V(G) - \{u\}$. Then either $u\overline{u}\overline{w}$ or $uw\overline{w}$ is a $u - v$ path. This proves that $G\overline{G}$ is connected and completes the proof of our lemma. \square

For a graph G , it is easy to see that $G\overline{G}$ has a perfect matching. It then follows by Theorem 2.1 that for a cutset $S \subseteq V(G\overline{G})$, $c_o(G\overline{G} - S) \leq |S|$. The next lemma provides a relationship of a cutset and the number of odd components in a complementary prism.

Lemma 3.2. *Let $G\overline{G}$ be a complementary prism and let $S = A \cup \overline{B}$ be a cutset of $G\overline{G}$, where $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Then*

- a) $c_o(G\overline{G} - S) = |S| - 2t$, for some $t \geq 0$.
- b) $c_o(G\overline{G} - S) = |A| + |B| - 2t \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |A| + |B| - 2|A \cap B|$. Consequently, $|A \cap B| \leq t$.
- c) If $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |A| + |B| - 2|A \cap B|$, then each component of $G[B - A]$ and $\overline{G}[\overline{A} - \overline{B}]$ is singleton and hence $G[A - B]$ is a clique.

Proof. a) Since $G\bar{G}$ contains a perfect matching and is of even order, it follows by Theorem 2.1 that there is a non-negative integer t such that $c_o(G\bar{G} - S) = |S| - 2t$, for any cutset $S \subseteq V(G\bar{G})$.

We first observe that $|B - A| + |\bar{A} - \bar{B}| = |B - A| + |A - B| = |A| + |B| - 2|A \cap B|$ since $|A| = |A - B| + |A \cap B|$ and $|B| = |B - A| + |A \cap B|$.

b) Let $C = V(G) - (A \cup B)$. By Lemma 3.1, if $C \neq \phi$, then $G\bar{G}[C \cup \bar{C}]$ is even and connected. Thus $c_o(G\bar{G} - S) \leq c_o(G\bar{G} - (S \cup C \cup \bar{C})) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) \leq |B - A| + |\bar{A} - \bar{B}| = |A| + |B| - 2|A \cap B|$ as required.

c) follows by the fact that $|B - A| + |A - B| = |A| + |B| - 2|A \cap B|$. \square

For an induced subgraph H of G , Com_H denotes the set of all components in H . If $X \subseteq V(G)$, then we use Com_X for $Com_{G[X]}$. For a cutset S of $G\bar{G}$, put $A = S \cap V(G)$, $\bar{B} = S \cap V(\bar{G})$ and $C = V(G) - (A \cup B)$. Thus $S = A \cup \bar{B}$. Further, let $T_{B-A} = \{F | F \text{ is an odd component of } G[B - A] \text{ and } N_G(u) - V(F) \subseteq A \text{ for all } u \in V(F)\}$. $T_{\bar{A}-\bar{B}} = \{F | F \text{ is an odd component of } \bar{G}[\bar{A} - \bar{B}] \text{ and } N_{\bar{G}}(\bar{u}) - V(F) \subseteq \bar{B} \text{ for all } \bar{u} \in V(F)\}$. Finally, let $L = L_G \cup L_{\bar{G}}$, where $L_G = \{F | F \text{ is an odd component in } G[B - A] \text{ and } N_{G\bar{G}}(V(F)) \cap C \neq \phi\}$ and $L_{\bar{G}} = \{F | F \text{ is an odd component in } \bar{G}[\bar{A} - \bar{B}] \text{ and } N_{G\bar{G}}(V(F)) \cap \bar{C} \neq \phi\}$. Note that if $C = \phi$, then $L = \phi$. Clearly, $T_{B-A} \cap L_G = \phi$ and $T_{\bar{A}-\bar{B}} \cap L_{\bar{G}} = \phi$. It is easy to see that, if G is connected and $G[B - A]$ contains only odd components, then $Com_{B-A} = T_{B-A} \cup L_G$. Similarly, if \bar{G} is connected and $\bar{G}[\bar{A} - \bar{B}]$ contains only odd components, then $Com_{\bar{A}-\bar{B}} = T_{\bar{A}-\bar{B}} \cup L_{\bar{G}}$. In what follows, the symbols Com_H , S , A , \bar{B} , C , T_{B-A} , $T_{\bar{A}-\bar{B}}$, L , L_G and $L_{\bar{G}}$ are referred to these set up.

The next lemma follows from our set up.

Lemma 3.3. *Let G be an r -regular connected graph of order $p \geq 2r + 1$ and $G\bar{G}$ a complementary prism. If $|A| < r$, then T_{B-A} contains no singleton components. Similarly, if $|\bar{B}| < p - r - 1$, then $T_{\bar{A}-\bar{B}}$ contains no singleton components. \square*

Lemma 3.4. *For $r \geq 3$, let G be a connected r -regular graph of order $p \geq 2r + 1$. Let $A, B, T_{B-A}, T_{\bar{A}-\bar{B}}$ be defined as above. Then*

- a) *If $G[A] = K_r$, then each component of T_{B-A} is of order at least 3.*
- b) *If $|A \cap B| = 1$ and $G[A - B] \cong K_r$, then the number of singleton components in T_{B-A} is at most 1.*
- c) *If $|A \cap B| = 1$ and $G[A - B] \cong K_{r-1}$, then the number of singleton components in T_{B-A} is at most 2.*

Proof. a) It follows by the fact that G is connected r -regular of order $p \geq 2r + 1$.

b) Suppose to the contrary that T_{B-A} contains two singleton components, say F_1 and F_2 where $V(F_1) = \{y_1\}$ and $V(F_2) = \{y_2\}$. Because $|A \cap B| = 1$, y_1 and y_2 are adjacent to at least $r - 1$ vertices of $A - B$. Since $G[A - B] = K_r$ and $r \geq 3$, it follows that there exists a vertex of $A - B$, say y_3 , such that $\{y_1, y_2\} \cup (A - B) \subseteq N_G(y_3)$. Thus $d_G(y_3) \geq r + 1$, a contradiction

c) By applying similar arguments as in the proof of (b), (c) follows. \square

Let x be a real number, $\lfloor x \rfloor_e$ is denoted a greatest even integer less than or equal to x , that is, $\lfloor x \rfloor_e = 2\lfloor x/2 \rfloor$. Note that if x is an integer and $\lfloor x \rfloor_e = k$ then $x = k$ or $x = k + 1$.

Lemma 3.5. *Let $G\bar{G}$ be a complementary prism and $L = L_G \cup L_{\bar{G}}$ be defined as above. Then $c_o(G\bar{G} - S) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) - \lfloor |L| \rfloor_e$. Consequently $c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) - c_o(G\bar{G} - S) \leq |L| \leq c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) - c_o(G\bar{G} - S) + 1$.*

Proof. If $C = \phi$, then $|L| = 0$ and thus $c_o(G\bar{G} - S) = c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}])$ as required. We now suppose that $C \neq \phi$. By Lemmas 3.2(a) and (b), $c_o(G\bar{G} - S) \leq c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}])$. By Lemma 3.1, $G\bar{G}[C \cup \bar{C}]$ is even and connected. So it must be contained in some component of $G\bar{G} - S$, say F . If $x \in V(F) - (C \cup \bar{C})$, then x is in some component of $G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]$, say M . So $V(M) \subseteq V(F)$. If M is odd, then $M \in L$. Note that each odd component of L is a subgraph of F . Hence, $|V(F)|$ has the same parity with $|L|$ and $c_o(G\bar{G} - S) = c_o(G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]) - |L| + \epsilon$, where $\epsilon = 1$ if $|L|$ is odd and $\epsilon = 0$ if $|L|$ is even. So $c_o(G\bar{G} - S) = c_o(G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]) - \lfloor |L| \rfloor_e$. Thus $\lfloor |L| \rfloor_e = c_o(G[B - A] \cup \bar{G}[\bar{A} - \bar{B}]) - c_o(G\bar{G} - S)$. By properties of $\lfloor x \rfloor_e$, our result follows. This proves our lemma. \square

Lemma 3.6. *If G is an r -regular graph of order $p \geq 2r + 1$, then \bar{G} is connected.*

Proof. Note that \bar{G} is $(p - r - 1)$ -regular graph of order p . Suppose \bar{G} is disconnected. Then each component must have order at least $p - r$. So $p \geq 2(p - r)$ and thus $p \leq 2r$, a contradiction. This proves our lemma. \square

Lemma 3.7. *Let G be a connected r -regular graph of order $p \geq 2r + 1$. Let S be a cutset of $G\bar{G}$. Then $S \cap V(G) \neq \phi$ and $S \cap V(\bar{G}) \neq \phi$.*

Proof. By Lemma 3.6, \bar{G} is connected. Hence, G and \bar{G} are connected. Suppose without loss of generality that $S \cap V(G) = \phi$. So $S \subseteq V(\bar{G})$. Since $G = G\bar{G} - V(\bar{G})$ is connected and each vertex \bar{u} of $V(\bar{G}) - S$ is adjacent to a vertex u in G , it follows that $G\bar{G} - S$ is connected, a contradiction. Hence, $S \cap V(G) \neq \phi$. By similar arguments, $S \cap V(\bar{G}) \neq \phi$. This proves our lemma. \square

Theorem 3.8. *Let G be a connected r -regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. Then $G\bar{G}$ is bicritical. Consequently, $G\bar{G}$ is 1-extendable.*

Proof. Suppose $G\bar{G}$ is not bicritical. By Theorem 2.5, there is a cutset $S \subseteq V(G\bar{G})$, where $|S| \geq 2$ such that $c_o(G\bar{G} - S) > |S| - 2$. It follows by Lemmas 3.2(a) that $c_o(G\bar{G} - S) = |S|$ for $|S| \geq 2$. Note that, by Lemma 3.7, $A = S \cap V(G)$ and $\bar{B} = S \cap V(\bar{G})$ are not empty. Thus \bar{A} and B are not empty. By Lemma 3.2 (b), $A \cap B = \phi$ and thus $c_o(G[B - A]) + c_o(\bar{G}[\bar{A} - \bar{B}]) = c_o(G[B]) + c_o(\bar{G}[\bar{A}]) = c_o(G\bar{G} - S) = |S| = |B| + |\bar{A}|$. By Lemma 3.2(c), each component of $G[B]$ and $\bar{G}[\bar{A}]$ is singleton. Hence, $G[A] \cong K_{|A|}$. Since G is r -regular of order $p \geq 2r + 1$,

$|A| \leq r + 1$. If $|A| = r + 1$, then $G[A] \cong K_{r+1}$ is a disconnected component in G , a contradiction. So $1 \leq |A| \leq r$. By Lemmas 3.3 and 3.4(a), no singleton component in $G[B]$ belongs to T_{B-A} . Since each component of $G[B]$ is singleton, $T_{B-A} = \phi$. Because $c_o(G\overline{G} - S) = c_o(\overline{G}[\overline{A}]) + c_o(G[B])$, it follows by Lemma 3.5 that $0 \leq |L| \leq 1$. Since $B \neq \phi$ and $G[B]$ contains only singleton components, it follows that $1 \leq |B| = |T_{B-A}| + |L_G| \leq 1$. Hence, $|B| = |L_G| = 1$. Therefore, $|\overline{B}| = 1 < r \leq p - r - 1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Hence, $T_{\overline{A}-\overline{B}} = \phi$. Since each component of $\overline{G}[\overline{A}]$ is singleton, it is contained in $L_{\overline{G}}$. So $|L_{\overline{G}}| = |\overline{A}| = |A| \geq 1$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| \geq 2$, a contradiction. Hence, $G\overline{G}$ is bicritical. It then follows that $G\overline{G}$ is 1-extendable. This proves our theorem. \square

The next lemma follows by Theorem 2.3.

Lemma 3.9. *Let G be a connected r -regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. If G contains a triangle, then $G\overline{G}$ is not r -extendable. \square*

By Lemma 3.9, if G is a 3-regular graph of order $p \geq 8$ containing a triangle, then $G\overline{G}$ is not 3-extendable. The next theorem provides a sufficient condition for a connected r -regular graph G which $G\overline{G}$ is 2-extendable, for $r \geq 4$. In case $r = 3$, if G contains the graph F in Figure 1 as an induced subgraph, then $\{yz, \overline{w}\overline{x}\}$ cannot be extended to a perfect matching in $G\overline{G}$. Hence, $G\overline{G}$ is not 2-extendable. We next show that the complementary prism of connected 3-regular F -free graphs and connected r -regular graphs for $r \geq 4$ are 2-extendable.

Theorem 3.10. *Suppose G is a connected graph of order p . If G is either 3-regular F -free where $p \geq 8$ and F is the graph in Figure 1 or r_0 -regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

Proof. Observe that \overline{G} is $(p - r - 1)$ -regular where $r \in \{3, r_0\}$ and $p - r - 1 \geq 4$. By Theorem 3.8, $G\overline{G}$ is bicritical. Suppose to the contrary that $G\overline{G}$ is not 2-extendable. Then there is a matching $M \subseteq E(G\overline{G})$ of size two such that $G\overline{G} - V(M)$ contains no perfect matching. By Theorem 2.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. Let $S = T \cup V(M)$. Clearly, $|S| \geq 4$. Thus $c_o(G\overline{G} - S) > |S| - 4$. Because $G\overline{G}$ is bicritical, by Theorem 2.5, $c_o(G\overline{G} - S) \leq |S| - 2$. It follows by parity that $c_o(G\overline{G} - S) = |S| - 2$ and $G\overline{G}[S]$ contains a matching of size at least two. Let $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$. By Lemma 3.2 (b), $|A \cap \overline{B}| \leq 1$. Further, by Lemma 3.7, $A \neq \phi$ and $\overline{B} \neq \phi$. So $\overline{A} \neq \phi$ and $B \neq \phi$. We distinguish 2 cases according to $|A \cap \overline{B}|$.

Case 1: $|A \cap \overline{B}| = 1$. Put $\{u\} = A \cap \overline{B}$. By Lemma 3.2(b) $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |S| - 2$. By Lemma 3.5, $|L| \leq 1$. Further, by Lemma 3.2(c), each component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is singleton. Thus, $G[A - B]$ is a clique, $|Com_{\overline{A}-\overline{B}}| = |\overline{A} - \overline{B}|$ and $|Com_{B-A}| = |B - A|$. Since G is connected, it is easy to see that if $|A - B| \geq r + 1$, then $G[A - B] \cong K_{|A-B|}$ contains a vertex of

degree greater than r or $G \cong K_{r+1}$ is a graph of order less than p , a contradiction. Hence, $|A - B| \leq r$.

We first show that $|T_{B-A}| \geq 2$. Suppose to the contrary that $|T_{B-A}| \leq 1$. Since $G[B - A]$ contains only singleton components and $|L_G| \leq |L| \leq 1$, it follows that $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 2$. Thus $|\overline{B}| = |B| = |B - A| + |B \cap A| \leq 3 < 4 \leq p - r - 1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Consequently, $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = L_{\overline{G}}$. Therefore, $|\overline{A} - \overline{B}| = |L_{\overline{G}}| \leq 1$ since $\overline{G}[\overline{A} - \overline{B}]$ contains only singleton components. So $|A| = |\overline{A}| = |\overline{A} - \overline{B}| + |\overline{A} \cap \overline{B}| \leq 2 < r$. By Lemma 3.3, T_{B-A} contains no singleton components. So $T_{B-A} = \phi$. Since $T_{\overline{A}-\overline{B}} = \phi$ and $T_{B-A} = \phi$, it follows that every odd component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is in L . Because $|L| \leq 1$ and $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ contains only singleton components, it follows that $|\overline{A} - \overline{B}| + |B - A| \leq 1$. Hence, $|S| = |A - B| + |A \cap B| + |\overline{A} \cap \overline{B}| + |\overline{B} - \overline{A}| = |A - B| + 2|A \cap B| + |\overline{B} - \overline{A}| \leq 3 < 4$, contradicting the fact that $|S| \geq 4$. Therefore, $|T_{B-A}| \geq 2$.

Let $D_1, D_2 \in T_{B-A}$. Since $G[B - A]$ contains only singleton components, $D_i \cong K_1$, for $1 \leq i \leq 2$. Put $\{v_i\} = V(D_i)$. By Lemma 3.3, $|A| \geq r$. Consequently, $|A - B| \geq r - 1$. Because $|A - B| \leq r$, $r - 1 \leq |A - B| \leq r$. Since $G[A - B]$ is clique, $|A \cap B| = 1$ and $|T_{B-A}| \geq 2$, it follows by Lemmas 3.4 (b) and (c) that $|A - B| = r - 1$ and $|T_{B-A}| = 2$. Thus $|A| = |A - B| + |A \cap B| = r$. Because $r - 1 = |A - B| = |\overline{A} - \overline{B}| = |Com_{\overline{A}-\overline{B}}| = |T_{\overline{A}-\overline{B}}| + |L_{\overline{G}}| \leq |T_{\overline{A}-\overline{B}}| + 1$, it follows that $|T_{\overline{A}-\overline{B}}| \geq r - 2 \geq 1$. Thus $T_{\overline{A}-\overline{B}}$ contains a singleton component. By Lemma 3.3, $|\overline{B}| \geq p - r - 1 \geq 4$. Therefore, $|\overline{B} - \overline{A}| = |\overline{B}| - |\overline{B} \cap \overline{A}| \geq p - r - 2 \geq 3$. On the other hand, $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 3$. Then $|B - A| = |\overline{B} - \overline{A}| = 3$. Thus $3 = |T_{B-A}| + |L_G| = 2 + |L_G|$. It follows that $L = L_G = \{K_1\}$ and consequently $L_{\overline{G}} = \phi$. Since $|A| = r$, $deg_G v_1 = deg_G v_2 = r$ and $N_G(v_1) = N_G(v_2) \subseteq A$, it follows that $N_G(v_1) = N_G(v_2) = A$.

We now put $\{\overline{w}\} = V(K_1)$ where $K_1 \in T_{\overline{A}-\overline{B}}$. Clearly, $N_{\overline{G}}(\overline{w}) \subseteq \overline{B} - \{\overline{v}_1, \overline{v}_2\}$ since v_1 and v_2 are adjacent to every vertex in A . Because $|\overline{B}| = |\overline{B} - \overline{A}| + |\overline{A} \cap \overline{B}| = 3 + 1 = 4$, $|N_{\overline{G}}(\overline{w})| \leq |\overline{B}| - |\{\overline{v}_1, \overline{v}_2\}| = 2$ thus \overline{G} is t -regular where $t \leq 2$. This contradicts the fact that \overline{G} is $(p - r - 1)$ -regular where $p - r - 1 \geq 4$. Therefore, Case 1 cannot occur.

Case 2: $|A \cap B| = 0$. By Lemmas 3.2(a) and (b), $|S| - 2 = c_o(G\overline{G} - S) \leq c_o(\overline{G}[\overline{A}]) + c_o(G[B]) \leq |\overline{A}| + |B| = |S|$. By parity, $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S|$ or $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2$. We distinguish 2 cases.

Case 2.1 : $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| = |\overline{A}| + |B|$. Clearly, each component of $\overline{G}[\overline{A}] \cup G[B]$ is singleton. So $G[A] \cong K_{|A|}$. It is easy to see that if $|A| \geq r + 1$, then $G[A]$ contains a vertex of degree greater than r or $G[A]$ is a disconnected component in G , a contradiction. Hence, $|A| \leq r$. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. Therefore, $T_{B-A} = \phi$. Thus $|L_G| = |B|$. Because $c_o(G[B]) + c_o(\overline{G}[\overline{A}]) - c_o(G\overline{G} - S) = |S| - (|S| - 2) = 2$, by Lemma 3.5, $2 \leq |L| \leq 3$. Since $B \neq \phi$ and $|B| = |L_G| \leq |L|$, it follows that $1 \leq |B| \leq 3$. Because $|\overline{B}| = |B| \leq 3 < 4 \leq p - r - 1$, by Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no

singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Hence, $|L_{\overline{G}}| = |\overline{A}| = |A|$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| = |B| + |\overline{A}| = |S|$ and thus $2 \leq |S| \leq 3$ since $2 \leq |L| \leq 3$, contradicting the fact that $|S| \geq 4$. Hence, Case 2.1 cannot occur.

Case 2.2 : $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2 = |\overline{A}| + |B| - 2$. Put $s = |S|$. It is easy to see that $\overline{G}[\overline{A}] \cup G[B]$ contains all singleton components except exactly one non-singleton component which is of order 2 or 3. Hence, $\overline{G}[\overline{A}] \cup G[B]$ is isomorphic to a graph in $\{(s-2)K_1 \cup K_2, (s-3)K_1 \cup P_3, (s-3)K_1 \cup K_3\}$. If $|\overline{A}| \geq r+2 \geq 5$, then $\overline{G}[\overline{A}]$ must contain a singleton component, say F , where $V(F) = \{\overline{u}\}$. It follows that $\deg_{\overline{G}} \overline{u} \geq r+1$, a contradiction. Hence, $|A| = |\overline{A}| \leq r+1$. Since $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) - c_o(G\overline{G} - S) = (|S| - 2) - (|S| - 2) = 0$, by Lemma 3.5, $|L| \leq 1$. We distinguish 2 subcases according to the non-singleton component.

Subcase 2.2.1 : The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $G[B]$. So $\overline{G}[\overline{A}] \cong |\overline{A}|K_1$ and $G[A] \cong K_{|\overline{A}|} \cong K_{|A|}$. Clearly, $|A| \leq r$ otherwise $G[A]$ is a disconnected component in G . By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. So every singleton component in $G[B]$ is contained in L_G . Since $|L_G| \leq |L| \leq 1$, $G[B]$ contains at most 1 singleton component. We first show that $T_{\overline{A}-\overline{B}} = \phi$. Suppose this is not the case. Then there is $K_1 \in T_{\overline{A}-\overline{B}}$ since $\overline{G}[\overline{A}]$ contains only singleton components. By Lemma 3.3, $|B| = |\overline{B}| \geq p-r-1 \geq 4$. Because $G[B]$ contains a non-singleton component of order either 2 or 3 and at most 1 singleton component, it follows that $G[B]$ is isomorphic to a graph in $\{K_1 \cup P_3, K_1 \cup K_3\}$. Thus $|B| = 4$ and either $T_{B-A} = \{P_3\}$ or $T_{B-A} = \{K_3\}$, and $L_G = \{K_1\}$. Thus $L_{\overline{G}} = \phi$. So $Com_{\overline{A}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = T_{\overline{A}-\overline{B}}$. Therefore, each vertex of \overline{A} is adjacent to every vertex of \overline{B} since \overline{G} is $(p-r-1)$ -regular and $p-r-1 \geq 4$. It follows that there is no edge joining vertices of A and B . But this contradicts the fact that $T_{B-A} \neq \phi$. Hence, $T_{\overline{A}-\overline{B}} = \phi$ as required.

Therefore, $Com_{\overline{A}} = L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$ and $|\overline{A}| = |A| \neq 0$, it follows that $|Com_{\overline{A}}| = |L_{\overline{G}}| = 1$. Further, $L_G = \phi$ and $\overline{G}[\overline{A}] = K_1$. Thus $Com_B = T_{B-A}$. Because $|A| = |\overline{A}| = 1 < r \leq 3$, by Lemma 3.3, T_{B-A} contains no singleton components. So $G[B]$ contains no singleton components and $G[B]$ is isomorphic to a graph in $\{P_3, K_3\}$ since $|B| = |S| - |A| \geq 3$. Then $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size less than two, contradicting the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, Subcase 2.2.1 cannot occur.

Subcase 2.2.2 : The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $\overline{G}[\overline{A}]$. So $G[B] \cong |B|K_1$. We first show that $T_{B-A} \neq \phi$. Suppose this is not the case. Then $T_{B-A} = \phi$ and thus $Com_B = T_{B-A} \cup L_G = L_G$. Since $B \neq \phi$ and $|L_G| + |L_{\overline{G}}| = |L| \leq 1$, it follows that $|L_G| = 1$ and $|L_{\overline{G}}| = 0$. Consequently, $|B| = 1$ since $G[B] \cong |B|K_1$. Because $|\overline{B}| = |B| = 1 < r$, $T_{\overline{A}-\overline{B}}$ contains no singleton components by Lemma 3.3. Hence, $\overline{G}[\overline{A}]$ contains exactly one non-singleton component of order 2 or 3. Thus $|A| = |\overline{A}| \leq 3$. It is easy to see that $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size at most one since $|\overline{B}| = 1$. This contradicts the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, $T_{B-A} \neq \phi$. Further, $|T_{B-A}| \geq |B| - 1$ since $|L_G| \leq |L| \leq 1$ and $|T_{B-A}| + |L_G| = |B|$.

Because $G[B] \cong |B|K_1$, there exists $K_1 \in T_{B-A}$. By Lemma 3.3, $|A| \geq r$. So $r \leq |A| \leq r+1$. We first suppose that $|A| = r+1$. Let F_t be the non-singleton component of order t in $\overline{G}[A]$ and let $\overline{A}_1 = V(F_t)$. Then $2 \leq t \leq 3$ and $\overline{G}[A] \cong (r+1-t)K_1 \cup F_t$. It is easy to see that $G[A]$ contains $r+1-t$ vertices of degree r and each vertex of $A_1 = \overline{A}_1$ has degree, in $G[A]$, at least $r+1-t$ and at most $r-1$. Let $\{w\} = V(K_1)$ where $K_1 \in T_{B-A}$, then $N_G(w) \subseteq A_1$ and thus $3 \leq r = \deg_G(w) \leq t \leq 3$. It then follows that $N_G(w) = A_1$ and $t = r = 3$. Thus \bar{w} is not adjacent to any vertex of \overline{A}_1 and $\overline{G}[A] \cong K_1 \cup F_3$. Further, each vertex of A_1 has degree at least $|T_{B-A}| + 1 = |B| - |L_G| + 1 \geq |B|$ since $|L_G| \leq 1$. Thus $|B| \leq 3$ since G is now 3-regular. Because \overline{G} is $(p-r-1)$ -regular where $p-r-1 \geq 4$ and each vertex of $V(F_3) = \overline{A}_1$ has degree at most 3 in $\overline{G}[\overline{A} \cup \overline{B}]$ since it must be adjacent to at most one vertex in \overline{B} , it follows that $F_3 \in L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$, the only singleton component, K_1 , of $\overline{G}[A]$ must be in $T_{\overline{A}-\overline{B}}$. By Lemma 3.3, $|\overline{B}| \geq p-r-1 \geq 4$. But this contradicts the fact that $|\overline{B}| = |B| \leq 3$. Therefore, $|A| = r$.

Consequently, for each $w \in V(K_1)$ where $K_1 \in T_{B-A}$, $N_G(w) = A$. Now let $\bar{v} \in \overline{A}$. Then $\deg_{\overline{G}}(\bar{v}) \leq |\overline{B}| - |T_{B-A}| = |B| - |T_{B-A}| = |L_G| \leq 1$. Further, $\deg_{\overline{A}}(\bar{v}) \leq 2$ since each component of $\overline{G}[A]$ has order at most 3. Because \overline{G} is $(p-r-1)$ -regular where $p-r-1 \geq 4$, \bar{v} is adjacent to some vertex of \overline{C} . Consequently, each odd component of $\overline{G}[A]$ is contained in $L_{\overline{G}}$. Because $|\overline{A}| = |A| = r \geq 3$, $\overline{G}[A]$ contains a non-singleton component of order either 2 or 3 and $|L_{\overline{G}}| \leq |L| \leq 1$, it follows that $c_o(\overline{G}[A]) = 1$. Therefore, $\overline{G}[A]$ is isomorphic to a graph in $\{K_1 \cup K_2, P_3, K_3\}$. Hence, $r = |A| = 3$, $|L| = |L_{\overline{G}}| = 1$, $Com_B = T_{B-A} = \{|B|K_1\}$. Further, for $x \in B, y \in A, N_G(x) = A$ and $\deg_G(y) = r = 3 \geq |B| = |\overline{B}|$.

We first suppose that $\overline{G}[A] \cong K_3$. Then $G[A]$ is independent and thus $\overline{G}[\overline{B}]$ must contain a matching of size at least two since $G\overline{G}[S]$ contains a matching of size at least two. So $|B| = |\overline{B}| \geq 4$. But this contradicts the fact that $|B| = |\overline{B}| \leq 3$. Hence, $\overline{G}[A] \neq K_3$. Therefore, $\overline{G}[A]$ is isomorphic to a graph in $\{P_3, K_1 \cup K_2\}$. In either case, $G[A]$ contains a maximum matching of size one. Then $2 \leq |\overline{B}| \leq 3$ since $G\overline{G}[A \cup \overline{B}]$ contains a matching of size at least two.

We now suppose that $\overline{G}[A] \cong K_1 \cup K_2$. Then $G[A] \cong P_3$ and then the vertex of degree two in P_3 has degree, in G , greater than $r = 3$, again a contradiction. Hence, $\overline{G}[A] \neq K_1 \cup K_2$. Consequently, $\overline{G}[A] \cong P_3$ and then $G[A] \cong K_1 \cup K_2$. Clearly, $|B| \neq 3$ otherwise $G[A]$ contains a vertex of degree greater than $r = 3$. So $|B| = 2$ and thus $G[A \cup B]$ contains the graph F in Figure 1 as an induced subgraph. But this contradicts our hypothesis that G is 3-regular F -free graph. This completes the proof of our theorem. \square

It is clear that a connected 3-regular graph containing F , in Figure 1, as an induced subgraph contains v as a cut vertex. So 2-connected 3-regular graphs are F -free. The next corollary follows by this fact and Theorem 3.10.

Corollary 3.11. *If G is a 2-connected r -regular graph of order $p \geq 2r + 1$, for $r \geq 3$, then $G\overline{G}$ is 2-extendable.* \square

According to Theorems 2.6 and 3.10, we have the following theorem.

Theorem 3.12. *If each component G_i of G is 3-regular F -free of order at least 8 where F is the graph in Figure 1 or r_0 -regular of order at least $2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

We conclude our paper by posing following problem.

Problem : Establish sufficient condition for a complementary prism of r -regular graphs to be k -extendable for $r \geq k \geq 3$.

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