รายงานวิจัยฉบับสมบูรณ์

โครงการ อิทธิพลของพารามิเตอร์เชิงเรขาคณิต และพารามิเตอร์เชิงกลที่มีต่อสมบัติ สมบัติประสิทธิผล (effective properties) ของเมทริกซ์ที่เสริมกำลังด้วยเส้นใย (fibers)

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สัญญาเลขที่ MRG5080422

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สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษาและสำนักงานกองทุนสนับสนุนการวิจัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกอ. และ สกว. ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

รหัสโครงการ:	MRG5080422
ชื่อโครงการ:	อิทธิพลของพารามิเตอร์เชิงเรขาคณิต และพารามิเตอร์เชิงกลที่มีต่อสมบัติประสิทธิผล (effective properties) ของเมทริกซ์ที่เสริมกำลังด้วยเส้นใย (fibers)
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โครงงานวิจัยนี้เสนออีกบทพิสูจน์ของการศึกษาโฮโมจีไนเซชัน (homogenization) จากปัญหาขอบแบบอิลลิบติกกึ่งเชิงเส้นของ Bellieud และ Bouchitte และปัญหาขอบแบบอิลลิบติกเชิงเส้นของ Bellieud และ Gruais ปัญหาทั้งสองมีโดเมนทรงกระบอก มีโครงสร้างแบบเส้นใย โดยที่พารามิเตอร์เชิงกลบนเส้นใยมีค่าสูง เราใช้เทคนิคการลู่เข้าแบบแปรผัน ของฟังก์ชันนัลพลังงาน เราคาดว่าเทคนิคนี้จะสามารถนำไปใช้อธิบายปัญหาเดียวกันในกรณีรูปทรงเรขาคณิตทั่วไปมากขึ้น เนื่องจากฟังก์ชันนัลพลังงานที่ใช้ต้องคำนวณค่าบนปริภูมิ Sobolev ที่เหมาะสมและสัมพันธ์กับทฤษฎีดิสทริบิวชัน ในกรอบของทฤษฎีดิสทริบิวชัน ท้ายสุดเราคำนวณดิสทริบิวชันประกอบของฟังก์ชันนัลเดลต้า โดยใช้แคลคูลัสของนิวทริกซ์

คำหลัก: โฮโมจีไนเซชัน, การลู่เข้าแบบแปรผัน, ดิสทริบิวชัน, นิวทริกซ์

Abstract

Project Code: MRG5080422

Project Title: Influence of Various Geometrical and Mechanical Parameters on Effective Properties of a matrix Reinforced by Fibers

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Project Period: July 2, 2007 – July 1, 2009

We propose another proof of the homogenization study of a quasilinear elliptic boundary value problem by Bellieud and Bouchitte, and the homogenization study of a linear elliptic boundary valued problem by Bellieud and Gruais. Both problems are setting in a cylindrical domain with periodic structure of fiber in which the data takes high value. By studying the variational convergence of the energy functional in both two problems, we expect that this technique can be treated in a more general geometrical and physical cases. The energy functional is applied in a suitable Sobolev spaces which is related to distribution theory. In distribution frame work, We finally calculate a particular composition of distributions of delta functional using neutrix calculus.

Keywords: homogenization, variational convergence, distribution, neutrix

สารบัญ

บทที่ 1

Executive Summary

1.1 Rationale

Looking at new matials such as smart materials and nano materials being used in current technology, we find that sometimes it is difficult to explained such materials under the classical Cauchy continuum frame work. Therefore, there is an attempt to interpret them with an introduction of generalized continuum medium. However, theories to support a new medium is still in the research level. In our work, we are interested in a composite material with Cacuhy continuum medium but with a structure leading to a generalized continuum under homogenization. We investigate the work of Bellieud and Bouchitté in quasilinear elliptic boundary value problem (scalar case), and the work of Bellieud and Gruais in linear elliptic boundary valued problem (vector case). We reprove their results using a variational convergence technique. This technique is used to investigate a sequence of functional in a suitable Sobolev space which is supported by a distribution theory. We finally compute a certain distribution composition of delta functional by neutrix calculus.

1.2 Objective

The objectives of this project are as follows:

1. To use a mathematical technique to prove an existence of generalized continuum.

2. To propose a different proof of Bellieud and Bouchitté using a technique of variational convergence.

3. To propose a different proof of Bellieud and Gruais using a technique of variational convergence.

4. To calculate a certain distribution composition of delta functional.

1.3 Result

1. We propose another proof of results using a variational convergence. (See chapter 2.1).

2. We propose another proof of results using a variational convergence. (See chapter 2.2).

3. We investigate a certain composition of distribution on delta functional. If we let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta functional. We prove that the neutrix composition $\delta^{(s)}[\ln^r(1 + x_+^{1/r})]$ exists and

$$\delta^{(s)}[\ln^r(1+x_+^{1/r})] = \sum_{k=0}^{s} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{(r+1)k+r+s+i-1} s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \delta^{(k)}(x)$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$. Further results are also proved (See chapter 2.3).

1.4 Discussion and Conclusion

1. The analysis in chapter 2.1 can be easily extended to the case when ϕ_p is replaced by any strictly convex function which satisfies

$$\exists\, M > 0,\ \exists\, r \in (1, p)\ ; \qquad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \qquad \forall\, \xi \in \mathbb{R}^3, \tag{1.4.1}$$

the density function associated with $\Phi(u, v)$ becomes

$$W(\nabla u) + 2\pi\gamma|v - u|^p + W\left(\frac{\partial v}{\partial x_3}\right).$$

Indeed, $(1.4.1)$ and Hölder inequality imply

$$\left|\int_{B_\varepsilon} W(\nabla u_\varepsilon)\,dx - \int_{B_\varepsilon} \phi_p(\nabla u_\varepsilon)\,dx\right| \leq M|B_\varepsilon|^{1-\frac{r}{p}} \int_\Omega |\nabla u_\varepsilon|^p\,dx,$$

while our arguments and those of [7] to derive the upper bound and lower bound respectively are valid when ϕ_p is replaced by any convex function satisfying a growth condition like

$$\exists\,\alpha,\beta>0\ ; \qquad \alpha(|\xi|^p-1)\leq W(\xi)\leq \beta(1+|\xi|^p) \qquad \forall\,\xi\in\mathbb{R}^3,$$

which is an obvious consequence of $(1.4.1)$.

Eventually, the key arguments of our analysis are the identification of γ, θ_ε in terms of the solution of capacitary problems and the use of the p-positive homogeneity and convexity of ϕ_p and of the fact that $\phi_p(\xi)\geq\phi_p(\hat{\xi})$. Thus, it is easy to guess what could be $\Phi(u,v)$, when ϕ_p is replaced by any strictly convex function and when the cross sections of the fibers are smooth star-shaped domains of $\mathbb{R}^2$. We hope that our proposed strategy will be able to reduce and overcome the involved technical difficulties.

2. Here was presented another proof of a result of [4] concerning the homogenization of a cylindrical fibered structure. Instead of passing to the limit on a formulation of the problem in terms of variational equality through appropriate sequence of oscillating test fields, we study the variational convergence of the energy functional. Hence, the ingredients in the construction of the appropriate oscillating test fields are clearly justified as providing the “best" upper bound. Thus, it seems possible to consider a more general cross section for the fibers (say $r_\varepsilon\triangle$ with $\partial\triangle$ smooth enough) and a more general quadratic bulk energy density W_M for the matrix in the extent where the solutions $\theta_\varepsilon^\alpha$ of the involved capacitary problems

$$\min\left\{\int_{(-\varepsilon,\varepsilon)^2} W_M(e(\varphi))\,d\hat{x}\ \middle|\ \begin{array}{l}\varphi\in H^1((-\varepsilon,\varepsilon)^2,\mathbb{R}^3),\\ \varphi(\hat{x})=e^\alpha \text{ on } r_\varepsilon\triangle,\\ \varphi(\hat{x})=0 \text{ on } (-\varepsilon,\varepsilon)^2\setminus D(0,R_\varepsilon).\end{array}\right\}$$

are such that

i) $\exists w_M^{\text{cap}}\in\mathbb{S}^3$ such that $(w_M^{\text{cap}})_{\alpha\beta}=\lim_{\varepsilon\to0}\varepsilon^{-2}\int_{(-\varepsilon,\varepsilon)^2} w_M(e(\theta_\varepsilon^\alpha),e(\theta_\varepsilon^\beta))\,d\hat{x}$,

ii) $\lim_{\varepsilon\to0}\int_{\varepsilon i+r_\varepsilon\partial\triangle} W_M'(e(\theta_\varepsilon^\alpha))\cdot(u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl=\lim_{\varepsilon\to0}\int_{\partial D^i_{R_\varepsilon}} W_M'(e(\theta_\varepsilon^\alpha))\cdot(u_\varepsilon-\bar{u}_\varepsilon)\,dl=0.$

3. In the theory of distributions, many arguments show that no meaning can be generally given to expressions of the form $F(f(x))$, where F is a distribution and f is a locally summable function. Using the concepts of a neutrix and neutrix limit due to van der Corput [8], Fisher gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions. This has been exploited in the context of distributions, particularly in connection with the composition of distributions, see [9, 10]. With Fisher's definition, we are able to obtain one particular composition of distribution involving delta functional which is of crucial in applications.

บทที่ 2

Research Content

2.1 Homogenization of Quasilinear Elliptic Problem (Scalar Case)

Let $p \in (1, +\infty)$, we consider the homogenization of the elliptic problem

$$\begin{aligned} - \operatorname{div} \sigma_\varepsilon = f, \;\; \sigma_\varepsilon = a_\varepsilon \left|\nabla u_\varepsilon\right|^{p-2} \nabla u_\varepsilon \quad & \text{on } \Omega \\ u_\varepsilon = u_0 \quad & \text{on } \Gamma_0 \\ \sigma_\varepsilon \cdot n = g \quad & \text{on } \Gamma_1 \end{aligned} \tag{2.1.1}$$

where $\Omega := \omega \times (0, L)$ with $L > 0$ and ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary and containing the origin of coordinates. The homogenization study of $(2.1.1)$ consists in examining the behavior of the sequence of the solution (u_ε) as ε tends to zero. The conductivity coefficient a_ε is ε-periodic and satisfies a uniform lower bound, Γ_0 is an open subset of $\partial\Omega$ with Hausdorff measure $H^2(\Gamma_0)$ strictly positive, $\Gamma_1 = \partial\Omega \backslash \Gamma_0$, and n is the unit exterior normal on $\partial\Omega$. The boundary data u_0 is Lipschitz, while $(f, g) \in L^{p'}(\Omega) \times L^{p'}(\Gamma_1)$, $p' = p/(p-1)$.

The problem $(2.1.1)$ is related to the minimization problem

$$(\mathcal{P}_\varepsilon) \qquad \min \left\{ F_\varepsilon(w) - L(w) : w \in W^{1,p}_{\Gamma_0}(\Omega) \right\},$$

where

$$\begin{aligned} W^{1,p}_{\Gamma_0}(\Omega) &:= \left\{ w \in W^{1,p}(\Omega) : w = u_0 \text{ on } \Gamma_0 \right\}, \\ F_\varepsilon(w) &:= \int_\Omega a_\varepsilon \phi_p(\nabla w)\, dx, \\ \phi_p(\xi) &:= \frac{1}{p}|\xi|^p,\ \forall\, \xi \in \mathbb{R}^n, n = 1,2,3, \\ L(w) &:= \int_\Omega f w\, dx + \int_{\Gamma_1} g w\, dH^2. \end{aligned} \tag{2.1.2}$$

We are interested in the asymptotic behavior of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to 0$. We present another proof of a study of Bellieud and Bouchitté [7] that we expect to be more suitable to treat more general geometrical and physical cases.

The bases of the cylindrical domain Ω are denoted by $\omega_0 = \omega \times \{0\}$ and $\omega_L = \omega \times \{L\}$. For each ε, we consider a periodic distribution of cells $(Y^i_\varepsilon)_{i \in I_\varepsilon}$ such that $Y^i_\varepsilon := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$, and $I_\varepsilon := \left\{ i \in \mathbb{Z}^2 \ :\ Y^i_\varepsilon \subset \omega \right\}$. Let $(D^i_{r_\varepsilon})_{i \in I_\varepsilon}$ be the family of disks of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon := (\varepsilon i_1, \varepsilon i_2)$ of radius $r_\varepsilon \ll \varepsilon$, $T^i_\varepsilon := D^i_{r_\varepsilon} \times (0, L)$ and $T_\varepsilon := \cup_{i \in I_\varepsilon} T^i_\varepsilon$. The set of thin parallel cylinders T_ε represents the fibers (see Figure 2.1 and Figure 2). The conductivity coefficient a_ε is

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \backslash T_\varepsilon, \\ \lambda_\varepsilon & \text{otherwise.} \end{cases}$$

We make the assumptions

$$r_\varepsilon \to 0, \qquad \frac{r_\varepsilon}{\varepsilon} \to 0, \qquad \lambda_\varepsilon \to +\infty, \qquad k_\varepsilon := \lambda_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2} \to k,\ k \geq 0 \qquad \text{as } \varepsilon \to 0. \tag{2.1.3}$$

In [7], it was shown that the asymptotic limit of $(\mathcal{P}_\varepsilon)$ is

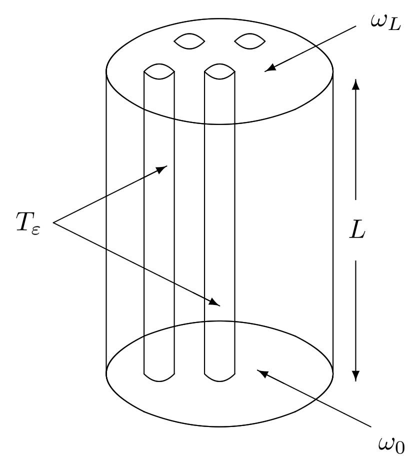
$$\min \left\{ \Phi(u,v) - L(u) : (u,v) \in (L^p(\Omega))^2 \right\},$$

where

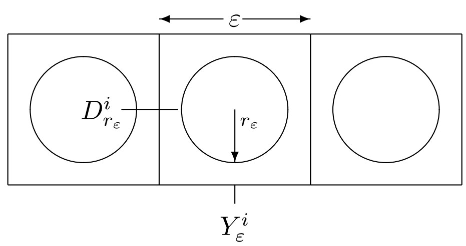
$$\Phi(u,v) = \begin{cases} \int_\Omega \phi_p(\nabla u)\, dx + \frac{k\pi}{p} \int_\Omega \left| \frac{\partial v}{\partial x_3} \right|^p dx + \frac{2\pi\gamma}{p} \int_\Omega |v-u|^p\, dx, \\ \qquad\qquad \text{if } \begin{cases} (u,v) \in W^{1,p}_{\Gamma_0}(\Omega) \times L^p(\omega, W^{1,p}(0,L)), \\ v = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L), \end{cases} \\ +\infty \qquad \text{otherwise,} \end{cases} \tag{2.1.4}$$

and

$$[0, +\infty] \ni \gamma = \begin{cases} \lim\limits_{\varepsilon \to 0} \varepsilon^{-2} |\log r_\varepsilon|^{-1} & \text{if } p = 2, \\ \lim\limits_{\varepsilon \to 0} \left| \frac{2-p}{p-1} \right|^{p-1} r_\varepsilon^{2-p} \varepsilon^{-2} & \text{if } p \neq 2. \end{cases} \tag{2.1.5}$$



รูปที่ 2.1: the domain $\Omega = \omega \times (0, L)$ occupied by a composite material



รูปที่ 2.2: the circular cross section of the fiber, $Y^i_\varepsilon \subset \omega$

Here, the boundary data u_0 is assumed to be Lipschitz in order to ensure that the infimum value of problem $(\mathcal{P}_\varepsilon)$ remains finite as $\varepsilon \to 0$. In case $k = +\infty$, we add further assumption

$$k_\varepsilon r_\varepsilon \to 0, \qquad \text{as } \varepsilon \to 0. \tag{2.1.6}$$

The conditions

$$k > 0 \quad \text{and} \quad \{\, \gamma > 0 \quad \text{or} \quad \omega_0 \subset \Gamma_0 \quad \text{or} \quad \omega_L \subset \Gamma_0 \,\} \tag{2.1.7}$$

guarantee that the functional Φ is coercive in $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L)$.

We are concerned with the extension of this result to more general cross sections of the fibers and more general energy density than ϕ_p. The aim of this paper is therefore to provide another proof that we expect to be more suitable to treat such general cases. The steps of the proof in [7] are to successively establish:

(i) a compactness property of the sequence (u_ε) such that $F_\varepsilon(u_\varepsilon) < C$,

(ii) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$,

(iii) an upper bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$.

Here we replace the steps (ii) and (iii) by

(ii′) an upper equality of the sequence $(F_\varepsilon(u_\varepsilon))$,

(iii′) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$ which essentially uses a subdiffenrential inequality.

2.1.1 An Alternative Strategy

It consists, under $(2.1.3)$, $(2.1.5)$, $(2.1.6)$ and $(2.1.7)$, in proving the following three propositions. In the sequel, the symbols $\to$, $\rightharpoonup$ and $\stackrel{*}{\rightharpoonup}$ stand for the strong convergence, the weak convergence and the weak star convergence, respectively. As usual, the letter C denotes various constants and for all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2).

Proposition 2.1.1 (compactness property)**.** *Let (u_ε) be a sequence such that $\sup F_\varepsilon(u_\varepsilon)$ is finite. Then (u_ε) is strongly relatively compact in $L^p(\Omega)$ and (v_ε), given by $v_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon$, is bounded in $L^1(\Omega)$ and, up to a subsequence, (v_ε) weakly* converges in the space of bounded measures $\mathcal{M}_b(\Omega)$ to an element v of $L^p(\Omega)$.*

Proposition 2.1.2 (upper bound equality)**.** *For all (u, v) in $(L^p(\Omega))^2$, such that $\Phi(u, v) < +\infty$, there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \stackrel{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$ and*

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \Phi(u, v).$$

Proposition 2.1.3 (lower bound inequality)**.** *For all u in $L^p(\Omega)$ and for all sequences (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \stackrel{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$, one has:*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

The proofs of these propositions are presented in the following sections.

Proof of Proposition 2.1.1

Compactness property was already proved in [7].

Proof of Proposition 2.1.2

Our sole contribution is to prove that we can replace inequality by equality, for that we use the same approximation u'_ε of u as in [7]

$$u'_\varepsilon = (1-\theta_\varepsilon)u + \theta_\varepsilon w_\varepsilon.$$

The function θ_ε is first defined on the closure of $\omega_\varepsilon := \cup_{i\in I_\varepsilon} Y^i_\varepsilon$ as a $(-\varepsilon/2,\varepsilon/2)^2$-periodic continuous function which satisfies $0 \le \theta_\varepsilon \le 1,\ \theta_\varepsilon = 1$ on $D_\varepsilon := \cup_{i\in I_\varepsilon} D^i_{r_\varepsilon},\ \theta_\varepsilon = 0$ on $\overline{\omega}_\varepsilon \setminus \cup_{i\in I_\varepsilon} D^i_{R_\varepsilon}$, where $D^i_{R_\varepsilon}$ is the disk of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon$ of radius R_ε such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. Next θ_ε is assumed to vanish on $\overline{\omega}\setminus\omega_\varepsilon$ and

$$w_\varepsilon(\hat{x},x_3) = \sum_{i\in I_\varepsilon}\left(\frac{1}{|D^i_{r_\varepsilon}|}\int_{D^i_{r_\varepsilon}} v(\hat{y},x_3)\,d\hat{y}\right) 1_{Y^i_\varepsilon}(\hat{x}).$$

The approximation u'_ε does not satisfy the boundary condition on $\Gamma_0\cap(\omega_0\cup\omega_L)$ so that, as in [7], we introduce a sharper approximation

$$u^\#_\varepsilon := u\varphi_\varepsilon + u'_\varepsilon(1-\varphi_\varepsilon).$$

Here φ_ε is a $C^\infty(\overline{\Omega})$ function which satisfies $\varphi_\varepsilon = 1$ on Γ_0, $\varphi_\varepsilon = 0$ on $\overline{\Omega}\setminus\Sigma_\varepsilon$, $|\nabla\varphi_\varepsilon| \le C/r_\varepsilon$ on $\overline{\Omega}$ where $\Sigma_\varepsilon := \{x\in\Omega : \mathrm{dist}(x,\Gamma_0) < r_\varepsilon\}$. We assume that u and v are Lipschitz on $\overline{\Omega}$ and there exists $L>0$ such that

$$\left|\frac{\partial v}{\partial x_3}(\hat{x}',x_3) - \frac{\partial v}{\partial x_3}(\hat{x}'',x_3)\right| < L|\hat{x}'-\hat{x}''| \qquad \forall\,(\hat{x}',x_3),(\hat{x}'',x_3)\in\overline{\Omega}. \tag{2.1.8}$$

Letting Ψ be any continuous function on $\overline{\Omega}$ such that $0\le\Psi\le1$, we introduce $F^\Psi_\varepsilon,\ \Phi^\Psi$ defined by similar formulae as the ones of F_ε and Φ but with $\Psi\,dx$ in place of dx. We will prove the lemma:

Lemma 2.1.4.

$$\lim_{\varepsilon\to0} F^\Psi_\varepsilon(u'_\varepsilon) = \Phi^\Psi(u,v).$$

If Lemma 2.1.4 is proved, then, by a classical approximation process, we can deduce

$$\lim_{\varepsilon\to0} F_\varepsilon(u'_\varepsilon) = \Phi(u,v). \tag{2.1.9}$$

Finally, we complete the proof of $(2.1.9)$ for any (u,v) such that $\Phi(u,v) < +\infty$ by approximation and diagonalization arguments.

Proof. *We split* $F_\varepsilon^\Psi(u'_\varepsilon)$ *in three parts*

$$F_\varepsilon^\Psi(u'_\varepsilon) = F_\varepsilon^{\Psi 1}(u'_\varepsilon) + F_\varepsilon^{\Psi 2}(u'_\varepsilon) + F_\varepsilon^{\Psi 3}(u'_\varepsilon). \tag{2.1.10}$$

First, we consider

$$F_\varepsilon^{\Psi 1}(u'_\varepsilon) := \int_{\Omega\setminus B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u'_\varepsilon)\Psi\,dx = \int_{\Omega\setminus B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u)\Psi\,dx,$$

where $B_\varepsilon := \cup_{i\in\varepsilon I_\varepsilon} D^i_{R_\varepsilon}\setminus\overline{D^i_{r_\varepsilon}} \times (0,L)$. *Hence, the assumption* $R_\varepsilon \ll \varepsilon$ *yields* $\lim_{\varepsilon\to 0}|B_\varepsilon \cup T_\varepsilon| = 0$ *and, consequently,*

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 1}(u'_\varepsilon) = \int_\Omega \phi_p(\nabla u)\Psi\,dx.$$

Next, we pay attention to

$$F_\varepsilon^{\Psi 2}(u'_\varepsilon) := \int_{B_\varepsilon} \phi_p(\nabla u'_\varepsilon)\Psi\,dx.$$

Writing

$$z_\varepsilon := (v-u)\widehat{\nabla}\theta_\varepsilon, \tag{2.1.11}$$

we obtain

$$\nabla u'_\varepsilon = z_\varepsilon + (w_\varepsilon - v)\nabla\theta_\varepsilon + (1-\theta_\varepsilon)\nabla u + \theta_\varepsilon\nabla w_\varepsilon.$$

Let us show

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi\,dx = 0. \tag{2.1.12}$$

The function ϕ_p, *being convex and positively homogeneous of degree* p, *satisfies*

$$\forall\,\xi,\eta\in\mathbb{R}^n,\ \ n=1,2,3,\quad |\phi_p(\xi)-\phi_p(\eta)| \le C|\xi-\eta|(|\xi|^{p-1}+|\eta|^{p-1}). \tag{2.1.13}$$

Therefore, Hölder inequality yields

$$\left|\int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi\,dx\right| \le C\left(\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_\varepsilon}|\nabla u'_\varepsilon|^p\,dx + \int_{B_\varepsilon}|z_\varepsilon|^p\,dx\right)^{\frac{1}{p'}}.$$

The smoothness of (u,v) *implies*

$$\left.\begin{array}{l} u'_\varepsilon = u \text{ on } \Omega\setminus(B_\varepsilon\cup T_\varepsilon),\quad |u'_\varepsilon|\le C \text{ on } \Omega,\quad |\nabla w_\varepsilon| \le C \text{ on } B_\varepsilon, \\ u'_\varepsilon = w_\varepsilon \text{ on } T_\varepsilon,\quad |w_\varepsilon - v| \le CR_\varepsilon \text{ on } B_\varepsilon, \end{array}\right\} \tag{2.1.14}$$

so that

$$\int_{B_\varepsilon}|\nabla u'_\varepsilon|^p\,dx + \int_{B_\varepsilon}|z_\varepsilon|^p\,dx \le C\varepsilon^{-2}\int_{D(r_\varepsilon,R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\,d\hat{x},$$

$$\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p \, dx \le C R_\varepsilon^p \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x},$$

where $D(r_\varepsilon, R_\varepsilon) = \left\{\hat{x} \in \mathbb{R}^2 : r_\varepsilon < |\hat{x}| < R_\varepsilon\right\}$. *Hence, if we choose* θ_ε *such that*

$$\exists\, M > 0 \;; \qquad \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \;\le M \qquad \forall\, \varepsilon > 0, \tag{2.1.15}$$

then (2.1.12) is true. We finally have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) \\
=\;& \lim_{\varepsilon\to 0} \int_{B_\varepsilon} \phi_p(z_\varepsilon)\Psi\, dx \\
=\;& \lim_{\varepsilon\to 0} \int_{B_\varepsilon} |v-u|^p \phi_p(\widehat{\nabla}\theta_\varepsilon)\Psi\, dx \\
=\;& \lim_{\varepsilon\to 0} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \int_0^L \sum_{i\in I_\varepsilon} |v-u|^p(\hat{y}^i_\varepsilon, x_3)\Psi(\hat{y}^i_\varepsilon, x_3)\, dx_3 \quad (\textit{with } \hat{y}^i_\varepsilon \in Y^i_\varepsilon) \\
=\;& \lim_{\varepsilon\to 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \int_0^L \sum_{i\in I_\varepsilon} |Y^i_\varepsilon||v-u|^p(\hat{y}^i_\varepsilon, x_3)\Psi(\hat{y}^i_\varepsilon, x_3)\, dx_3.
\end{aligned}$$

Observe that $\lim_{\varepsilon\to 0}\int_0^L \sum_{i\in I_\varepsilon} |Y^i_\varepsilon||v-u|^p(\hat{y}^i_\varepsilon, x_3)\Psi(\hat{y}^i_\varepsilon, x_3)\, dx_3 = \int_\Omega |v-u|^p\Psi\, dx$. *To get the lowest upper bound in Proposition 2.1.2, it is clear that* θ_ε *has to be the solution of the capacitary problem*

$$(\mathcal{P}_\varepsilon^{cap}) \quad \min\left\{ \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\varphi)\, d\hat{x} : \begin{array}{l} \varphi \in W^{1,p}(D(r_\varepsilon, R_\varepsilon)), \\ \varphi(\hat{x}) = 1 \textit{ on } |\hat{x}| = r_\varepsilon, \\ \varphi(\hat{x}) = 0 \textit{ on } |\hat{x}| = R_\varepsilon. \end{array} \right\}$$

As observed in [7], we have

$$\theta_\varepsilon = \begin{cases} \frac{\log R_\varepsilon - \log|\hat{x}|}{\log R_\varepsilon - \log r_\varepsilon} & \textit{if } p = 2, \\ \frac{R_\varepsilon^s - |\hat{x}|^s}{R_\varepsilon^s - r_\varepsilon^s} & \textit{if } p \ne 2 \quad (s = \frac{p-2}{p-1}) \end{cases}$$

and

$$\int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} = \frac{2\pi}{p}\Gamma_p(r_\varepsilon, R_\varepsilon),$$

where $\Gamma_p(r_\varepsilon, R_\varepsilon) := \begin{cases} \frac{1}{\log R_\varepsilon - \log r_\varepsilon} & \textit{if } p = 2, \\ (\frac{s}{R_\varepsilon^s - r_\varepsilon^s})^{p-1} & \textit{if } p \ne 2 \quad (s = \frac{p-2}{p-1}). \end{cases}$

Note that

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\Gamma_p(r_\varepsilon, R_\varepsilon) = \gamma.$$

If $\gamma < +\infty$, *then* (2.1.15) *is satisfied and*

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = \frac{2\pi\gamma}{p} \int_\Omega |v-u|^p\Psi\, dx.$$

When $\gamma = +\infty$, *it suffices to prove that* $\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = 0$. *Due to* (2.1.14), *the result is a consequence of* $F_\varepsilon^{\Psi 2}(u'_\varepsilon) \le C R_\varepsilon^p \varepsilon^{-2}\Gamma_p(r_\varepsilon, R_\varepsilon)$, *which tends to zero.*

Now, we consider the remaining part

$$F_\varepsilon^{\Psi 3}(u'_\varepsilon) := \int_{T_\varepsilon} \lambda_\varepsilon \phi_p(\nabla w_\varepsilon)\Psi \, dx.$$

Recalling the assumption (2.1.8) *on* v *and using the local Lipschitz property* (2.1.13), *we deduce*

$$\begin{aligned} \lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 3}(u'_\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} \lambda_\varepsilon \phi_p\left(\frac{\partial v}{\partial x_3}\right)\Psi \, dx \\ &= \frac{k\pi}{p}\int_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p \Psi \, dx, \end{aligned}$$

as shown in [7].

Now, we will prove the upper bound equality by using the sharper approximation $(u_\varepsilon^{\#})$. We start with

$$F_\varepsilon(u_\varepsilon^{\#}) = \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^{\#})\, dx + \int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u'_\varepsilon)\, dx. \tag{2.1.16}$$

Conditions (2.1.14) imply $|u'_\varepsilon - u| \le C(r_\varepsilon 1_{T_\varepsilon} + R_\varepsilon 1_{B_\varepsilon})$. Hence

$$\int_{\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u_\varepsilon^{\#})\, dx \le C\left(|\Sigma_\varepsilon| + \int_{\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u'_\varepsilon|^p\, dx + \lambda_\varepsilon |T_\varepsilon \cap \Sigma_\varepsilon| + \left(\frac{R_\varepsilon}{r_\varepsilon}\right)^p |\Sigma_\varepsilon|\right).$$

Lemma 2.1.4 implies that for every $\Psi \in C^0(\overline{\Omega}, [0,1])$, such that $\Psi = 1$ on a small neighborhood of $\Gamma_0 \cap (\omega_0 \cup \omega_L)$,

$$\begin{aligned} \limsup_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u_\varepsilon^{\#})\, dx &\le \limsup_{\varepsilon\to 0}\int_\Omega a_\varepsilon\phi_p(\nabla u'_\varepsilon)\Psi\, dx \\ &= \int_\Omega \left(\phi_p(\nabla u) + \frac{k\pi}{p}\left|\frac{\partial v}{\partial x_3}\right|^p + \frac{2\pi\gamma}{p}|v-u|^p\right)\Psi\, dx. \end{aligned}$$

Thus, by letting Ψ tend to zero, we deduce

$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u_\varepsilon^{\#})\, dx = \lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u'_\varepsilon)\, dx = 0,$$

and

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u'_\varepsilon)\, dx &= \lim_{\varepsilon\to 0}\left(\int_\Omega a_\varepsilon\phi_p(\nabla u'_\varepsilon)\, dx - \int_{\Sigma_\varepsilon} a_\varepsilon\phi_p(\nabla u'_\varepsilon)\, dx\right) \\ &= \int_\Omega \phi_p(\nabla u) + \frac{k\pi}{p}\left|\frac{\partial v}{\partial x_3}\right|^p + \frac{2\pi\gamma}{p}|v-u|^p\, dx, \end{aligned}$$

which proves the result for (u,v) smooth. We complete the proof by a standard approximation of (u,v) and a diagonalization argument [6].

2.1.2 Proof of Proposition 2.1.3

It is enough to consider $\liminf_{\varepsilon} F_{\varepsilon}(u_{\varepsilon}) < +\infty$. Due to the compactness property, (u, v) is in $(L^p(\Omega))^2$. We first consider the term $F_{\varepsilon}^2(u_{\varepsilon})$. Let (u_{η}, v_{η}) be Lipschitz on $\overline{\Omega}$ such that $\lim_{\eta\to 0} \|u_{\eta} - u\|_{L^p(\Omega)} + \|v_{\eta} - v\|_{L^p(\Omega)} = 0$. We define $(v_{\eta} - u_{\eta})_{\varepsilon} := \sum_{i\in I_{\varepsilon}} (v_{\eta} - u_{\eta})(\hat{x}_{\varepsilon}^i, x_3) 1_{Y_{\varepsilon}^i}$ and $z_{\eta\varepsilon} := (v_{\eta} - u_{\eta})_{\varepsilon} \widehat{\nabla}\theta_{\varepsilon}$. Because of the local Lipschitz property (2.1.13) of ϕ_p and $(u, v) \in (L^p(\Omega))^2$, Hölder inequality implies

$$\lim_{\varepsilon\to 0} \int_{B_{\varepsilon}} \phi_p(z_{\eta\varepsilon}) - \phi_p(z_{\varepsilon})\, dx = 0.$$

The proof of the upper bound equality allows us to write

$$\lim_{\varepsilon\to 0} \phi_p(z_{\eta\varepsilon}) = \frac{2\pi\gamma}{p} \int_{\Omega} |v_{\eta} - u_{\eta}|^p\, dx.$$

The convexity of ϕ_p and the fact that $\phi_p(\nabla u_{\varepsilon}) \geq \phi_p(\widehat{\nabla} u_{\varepsilon})$ yield

$$\begin{aligned} \liminf_{\varepsilon\to 0} F_{\varepsilon}^2(u_{\varepsilon}) &\geq \liminf_{\varepsilon\to 0} \int_{B_{\varepsilon}} \phi_p(\widehat{\nabla} u_{\varepsilon})\, dx \\ &\geq \liminf_{\varepsilon\to 0} \int_{B_{\varepsilon}} \phi_p(z_{\eta\varepsilon})\, dx \\ &\quad + \liminf_{\varepsilon\to 0} \int_{B_{\varepsilon}} \phi_p'(z_{\eta\varepsilon}) \cdot (\widehat{\nabla} u_{\varepsilon} - z_{\eta\varepsilon})\, dx. \end{aligned} \tag{2.1.17}$$

The very definition of ϕ_p implies

$$\begin{aligned} \phi_p'(\xi) &= |\xi|^{p-2}\xi & &\forall\, \xi \in \mathbb{R}^n,\ n = 1, 2, 3, \\ \phi_p'(t\xi) &= \phi_p'(t)\phi_p'(\xi) & &\forall\, (t, \xi) \in \mathbb{R} \times \mathbb{R}^n, n = 1, 2, 3, \\ \phi_p'(\xi) \cdot \xi &= p\phi_p(\xi) & &\forall\, \xi \in \mathbb{R}^n,\ n = 1, 2, 3. \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0} \int_{B_{\varepsilon}} \phi_p'(z_{\eta\varepsilon}) \cdot z_{\eta\varepsilon}\, dx = 2\pi\gamma \int_{\Omega} |v_{\eta} - u_{\eta}|^p\, dx. \tag{2.1.18}$$

For the other term of (2.1.17), we have

$$\int_{B_{\varepsilon}} \phi_p'(z_{\eta\varepsilon}) \cdot \widehat{\nabla} u_{\varepsilon}\, dx = \sum_{i\in I_{\varepsilon}} \int_0^L \phi_p'(v_{\eta} - u_{\eta})(\hat{x}_{\varepsilon}^i, x_3) \int_{D^i(r_{\varepsilon}, R_{\varepsilon})} \phi_p'(\widehat{\nabla}\theta_{\varepsilon}) \cdot \widehat{\nabla} u_{\varepsilon}\, d\hat{x}\, dx_3,$$

where $D^i(r_\varepsilon, R_\varepsilon) = D^i_{R_\varepsilon} \setminus \overline{D^i_{r_\varepsilon}}$. Let ν be the outer normal on $\partial D^i(r_\varepsilon, R_\varepsilon)$, the very definition of θ_ε as a solution of $(\mathcal{P}^{\text{cap}}_\varepsilon)$ yields

$$\begin{aligned}\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}u_\varepsilon\,d\hat{x} &= \int_{\partial D^i(r_\varepsilon,R_\varepsilon)} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl \\ &= \int_{\partial D^i_{R_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl + \int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl \\ &= -\tilde{u}^i_\varepsilon \int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl + \tilde{v}^i_\varepsilon \int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl\end{aligned}$$

where $\tilde{u}^i_\varepsilon := \frac{\int_{\partial D^i_{R_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl}{\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl} = \frac{1}{2\pi R_\varepsilon}\int_{\partial D^i_{R_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl$, $\tilde{u}_\varepsilon := \sum_{i\in I_\varepsilon} \tilde{u}^i_\varepsilon 1_{Y^i_\varepsilon}$, $\tilde{v}^i_\varepsilon := \frac{\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)v_\varepsilon\,dl}{\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl} = \frac{1}{2\pi r_\varepsilon}\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)v_\varepsilon\,dl$, $\tilde{v}_\varepsilon := \sum_{i\in I_\varepsilon} \tilde{v}^i_\varepsilon 1_{Y^i_\varepsilon}$. Thus,

$$\begin{aligned}\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}u_\varepsilon\,d\hat{x} &= (\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon)\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl \\ &= (\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon)\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}\theta_\varepsilon\,d\hat{x} \\ &= 2\pi\Gamma_p(r_\varepsilon,R_\varepsilon)(\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon),\end{aligned}$$

and

$$\int_{B_\varepsilon} \phi_p'(z_{\eta\varepsilon})\cdot\widehat{\nabla}u_\varepsilon\,dx = 2\pi\Gamma_p(r_\varepsilon,R_\varepsilon)\int_\Omega \phi_p'((v_\eta-u_\eta)_\varepsilon)(\tilde{v}_\varepsilon-\tilde{u}_\varepsilon)\,dx.$$

It was shown in [7] that $(\tilde{v}_\varepsilon-\tilde{u}_\varepsilon)\rightharpoonup(v-u)$ in $L^p(\Omega)$. On the other hand, $(v_\eta-u_\eta)$ being smooth and ϕ_p' being continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$, we have $\phi_p'((v_\eta-u_\eta)_\varepsilon)\to\phi_p'(v_\eta-u_\eta)$ in $L^{p'}(\Omega)$. Hence,

$$\lim_{\varepsilon\to0}\int_{B_\varepsilon} \phi_p'(z_{\eta\varepsilon})\cdot\widehat{\nabla}u_\varepsilon\,dx = 2\pi\gamma\int_\Omega \phi_p'(v_\eta-u_\eta)(v-u)\,dx. \tag{2.1.19}$$

Therefore, $(2.1.17)$, $(2.1.18)$ and $(2.1.19)$ imply

$$\begin{aligned}\liminf_{\varepsilon\to0} F^2_\varepsilon(u_\varepsilon) \quad &\geq \quad \frac{2\pi\gamma}{p}\int_\Omega |v_\eta-u_\eta|^p\,dx \\ &\quad +2\pi\gamma\left[\int_\Omega |v_\eta-u_\eta|^p\,dx - \int_\Omega \phi_p'(v_\eta-u_\eta)(v-u)\,dx\right].\end{aligned}$$

The expected lower bound for $F^2_\varepsilon(u_\varepsilon)$ is obtained by letting η tend to zero.

To complete the proof it suffices to use the arguments of [7] concerning the lower bounds for $F^1_\varepsilon(u_\varepsilon)$, $F^3_\varepsilon(u_\varepsilon)$ and the fact that v belongs to $L^p(\omega, W^{1,p}(0,L))$.

2.1.3 The Final Result

The following theorem, a convergence result for the minimizer of $(\mathcal{P}_\varepsilon)$, is a standard consequence of the previous three propositions.

Theorem 2.1.5. *Let the assumptions* (2.1.3) *and* (2.1.5) *hold with* $(k,\gamma) \in (0,+\infty)$. *Then the unique solution* $\overline{u}_\varepsilon$ *of* $(\mathcal{P}_\varepsilon)$ *converges weakly in* $W^{1,p}(\Omega)$ *to the unique solution* $\overline{u}$ *of the problem*

$$\min\left\{\min\left\{\Phi(u,v) - L(u) : v \in L^p(\Omega)\right\} : u \in L^p(\Omega)\right\},$$

where Φ *and* L *are defined by* (2.1.4) *and* (2.1.2) *respectively.*

Proof. *The proof of this theorem is the same as that in [7].*

2.2 Homogenization of Linear Elliptic Problem (Vector Case)

We intend to study the macroscopic behavior of a cylindrical micro-fibered structure made of a linearly isotropic elastic matrix surrounding a periodic distribution of very thin linearly isotropic elastic fibers of very high stiffness. As usual, we make no difference between the real physical space and $\mathbb{R}^3$ whose an orthonormal basis is denoted by $\{e^\alpha\}$ and, for all $\xi = (\xi_1, \xi_2, \xi_3)$ of $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2). Let ω a bounded domain of $\mathbb{R}^2$, containing the origin, with a Lipschitz continuous boundary $\partial\omega$ and L a positive number so that $\Omega := \omega \times (0, L)$ is a reference configuration of the fibered structure.

For each $\varepsilon > 0$, $(Y_\varepsilon^i) \in I_\varepsilon$, where $Y_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$ and $I_\varepsilon := \left\{i \in \mathbb{Z}^2 \mid Y_\varepsilon^i \subset \omega\right\}$, denotes a periodic distribution of cells. Let $(D_{r_\varepsilon}^i)_{i \in I_\varepsilon}$ the family of disk of $\mathbb{R}^2$ centered at $\hat{x}_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2)$ of radius $r_\varepsilon \ll \varepsilon$, and $T_\varepsilon^i := D_{r_\varepsilon}^i \times (0, L)$. The set $T_\varepsilon := \cup_{i \in I_\varepsilon} T_\varepsilon^i$ of thin parallel cylinders is the domain occupied by the fibers.

The Lamé coefficients λ_ε and μ_ε of the structure are such that

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_0 > 0, & \text{if } x \in \Omega \setminus T_\varepsilon \\ \lambda_{\varepsilon 1}, & \text{if } x \in T_\varepsilon \end{cases}, \qquad \mu_\varepsilon(x) = \begin{cases} \mu_0 > 0, & \text{if } x \in \Omega \setminus T_\varepsilon \\ \mu_{\varepsilon 1}, & \text{if } x \in T_\varepsilon. \end{cases}$$

The structure is clamped on the part $\Gamma_0 := \omega \times \{0, L\}$ of the boundary $\partial\Omega$ of Ω, subjected to body forces of density f and to surface forces of density g on $\Gamma_1 := \partial\Omega \setminus \Gamma_0$.

The problem of finding the equilibrium configuration of the structure reads as

$$\begin{cases} -\operatorname{div}\sigma_\varepsilon = f & \text{in } \Omega, \\ \sigma_\varepsilon := \lambda_\varepsilon \operatorname{tr} e(u_\varepsilon) I + 2\mu_\varepsilon e(u_\varepsilon), & e(u_\varepsilon) = (\nabla u)_s := \frac{1}{2}(\nabla u_\varepsilon + \nabla u_\varepsilon^T), \\ u_\varepsilon = 0 & \text{on } \Gamma_0, \\ \sigma_\varepsilon n = g & \text{on } \Gamma_1. \end{cases} \tag{2.2.1}$$

where u_ε, σ_ε denote the displacement and stress fields and n is the unit outward normal. It is well-known that if $\lambda_{\varepsilon 1},\ \mu_{\varepsilon 1} > 0,\ f \in L^2(\Omega, \mathbb{R}^3),\ g \in L^2(\Gamma_1, \mathbb{R}^3)$, then the problem, which can also be written

$$(\mathcal{P}_\varepsilon) \qquad \min\left\{F_\varepsilon(w) - L(w) \mid w \in H^1_{\Gamma_0}(\Omega, \mathbb{R}^3)\right\}, \tag{2.2.2}$$

where

$$F_\varepsilon(w) := \int_\Omega W_\varepsilon(e(w))\, dx, \qquad L(w) := \int_\Omega f \cdot w\, dx - \int_{\Gamma_1} g \cdot w\, ds,$$

and

$$W_\varepsilon(e) := \frac{1}{2}\lambda_\varepsilon \operatorname{tr}^2 e + \mu_\varepsilon |e|^2,\ \forall e \in \mathbb{S}^3 \text{ the space of symmetric } 3 \times 3 \text{ matrices},$$

$$H^1_{\Gamma_0}(\Omega, \mathbb{R}^3) = \{\, v \in H^1(\Omega, \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_0 \,\}$$

has a unique solution $\bar{u}_\varepsilon$.

To determine the macroscopic (or efficient) behavior of the micro-fibered structure, we aim to study the asymptotic behavior of $\bar{u}_\varepsilon$ when ε goes to zero. Let

$$k_\varepsilon := \mu_{\varepsilon 1} \frac{|T_\varepsilon|}{|\Omega|}, \qquad l_\varepsilon := \frac{\lambda_{\varepsilon 1}}{\mu_{\varepsilon 1}},$$

and assume that, as $\varepsilon \to 0$,

$$\begin{gathered} r_\varepsilon \to 0, \qquad \frac{r_\varepsilon}{\varepsilon} \to 0, \qquad \lambda_{\varepsilon 1} \to +\infty, \qquad \mu_{\varepsilon 1} \to +\infty, \\ k_\varepsilon \to k \in [0, +\infty], \qquad r_\varepsilon^2 k_\varepsilon \to \kappa \in [0, +\infty], \qquad l_\varepsilon \to l \in [0, +\infty), \\ (\varepsilon^2 |\ln r_\varepsilon|)^{-1} \to \gamma \in [0, +\infty). \end{gathered} \tag{2.2.3}$$

Let 1_{T_ε} the characteristic function of T_ε and $\mathcal{M}_b(\Omega, \mathbb{R}^3)$ the space of bounded $\mathbb{R}^3$-valued measures in Ω, it was proven in [4] that, as ε tends to zero, $\bar{u}_\varepsilon$ weakly converges in $H^1(\Omega, \mathbb{R}^3)$ toward $\bar{u}$ and $\bar{v}_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} \bar{u}_\varepsilon 1_{T_\varepsilon}$ weakly* converges in $\mathcal{M}_b(\Omega, \mathbb{R}^3)$ toward an element $\bar{v}$ of $L^2(\Omega, \mathbb{R}^3)$ solving

$$(\mathcal{P}^{\text{hom}}) \qquad \min\left\{\Phi(u, v) - L(u) \mid (u, v) \in L^2(\Omega, \mathbb{R}^3)^2\right\},$$

with

$$\Phi(u,v) = \begin{cases} \displaystyle\int_\Omega W_0(e(u))\,dx \\ \displaystyle + \mu_0 \pi\gamma \int_\Omega (v-u)^T \begin{pmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (v-u)\,dx & \text{if } (u,v) \in \mathcal{D}, \\ \displaystyle + \frac{1}{2}\frac{3l+2}{2(l+1)} k \int_\Omega \left|\frac{\partial v_3}{\partial x_3}\right|^2 dx \\ \displaystyle + \frac{1}{2}\frac{3l+2}{2(l+1)}\frac{\kappa}{4} \int_\Omega \left|\frac{\partial^2 v_1}{\partial x_3^2}\right|^2 + \left|\frac{\partial^2 v_2}{\partial x_3^2}\right|^2 dx, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$W_0(e(u)) := \frac{1}{2}\lambda_0 \operatorname{tr}^2 e(u) + \mu_0 |e(u)|^2, \qquad \chi := \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0},$$
$$\mathcal{D} := H^1_{\Gamma_0}(\Omega, \mathbb{R}^3) \times \left\{ v \in L^2(\omega, H^2_0(0,L;\mathbb{R}^3)) \;\middle|\; \frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = 0 \text{ on } \Gamma_0 \right\}.$$

Thus, the macroscopic behavior of the micro-fibered structure is the one of a so-called generalized elastic continuum medium involving an additional state variable and its first two derivatives. This additional state variable accounts for the microstructure in the extent where it describes the asymptotic behavior of a suitable scaling of the displacement field in the fibers. Our main concern is to understand this result more deeply and in a more general setting e.g., a different cross-section of the fibers, a more general behavior of the matrix or the fibers. Nevertheless, here, we confine to give another proof of the result of [4] by directly studying the variational convergence (as in the scalar case [5]) of F_ε and shall divide our proof into three steps:

1. a compactness property for each sequence (u_ε) such that $F_\varepsilon(u_\varepsilon)$ is bounded,
2. an upper bound equality for the sequence $(F_\varepsilon(u_\varepsilon))$,
3. a lower bound inequality for the sequence $(F_\varepsilon(u_\varepsilon))$.

2.2.1 An Alternative Strategy

Actually, the result of [4] is a standard consequence [3] of the following three propositions:

Proposition 2.2.1. ***(Compactness property)*** *Let (u_ε) be a sequence such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon)$ is finite. Then (u_ε) is strongly relatively compact in $L^2(\Omega,\mathbb{R}^3)$ and (v_ε) is bounded in $L^1(\Omega,\mathbb{R}^3)$ and, up to a subsequence, (v_ε) weakly* converges in $\mathcal{M}_b(\Omega,\mathbb{R}^3)$ to an element v of $L^2(\Omega,\mathbb{R}^3)$.*

Proposition 2.2.2. ***(Upper bound equality)*** *For all (u,v) in $L^2(\Omega,\mathbb{R}^3)^2$ with $\Phi(u,v)<+\infty$, there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^2(\Omega,\mathbb{R}^3)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega,\mathbb{R}^3)$ and*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \Phi(u,v).$$

Proposition 2.2.3. ***(Lower bound inequality)*** *For all u in $L^2(\Omega,\mathbb{R}^3)$ and for all sequence (u_ε) such that $u_\varepsilon \to u$ in $L^2(\Omega,\mathbb{R}^3)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega,\mathbb{R}^3)$, one has:*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u,v).$$

The convergence symbols $\to$, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ are used for the strong convergence, the weak convergence and the weak* convergence, respectively. The proof of these propositions are presented in the following subsections, where, as a common practice, C denote various constants which may vary from line to line.

Proof of Proposition 2.2.1

A proof of this proposition can be found in [4].

Proof of Proposition 2.2.2

We split F_ε into three parts:

$$F_\varepsilon^1(w) := \int_{\Omega\setminus(B_\varepsilon\cup T_\varepsilon)} W_\varepsilon(e(w))\,dx,$$
$$F_\varepsilon^2(w) := \int_{B_\varepsilon} W_\varepsilon(e(w))\,dx, \qquad F_\varepsilon^3(w) := \int_{T_\varepsilon} W_\varepsilon(e(w))\,dx,$$

where $B_\varepsilon := (D_{R_\varepsilon}\setminus \overline{D}_{r_\varepsilon})\times(0,L)$, $D_{r_\varepsilon} := \cup_{i\in I_\varepsilon} D^i_{r_\varepsilon}$, $D_{R_\varepsilon} := \cup_{i\in I_\varepsilon} D^i_{R_\varepsilon}$, $D^i_{R_\varepsilon}$ is the disk of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon$ of radius R_ε such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$.. We point out that our proof is in the same spirit as that of [5], where the main ingredient stems from [4] and essentially confine to the convergence of F_ε^2. We first assume u and v to be smooth on $\overline{\Omega}$ and construct an L^2-approximation u_ε of u by:

$$u_\varepsilon = \sum_{\alpha=1}^{3}\big(u_\alpha(e^\alpha-\theta^\alpha_\varepsilon)+\bar{w}_{\varepsilon\alpha}\theta^\alpha_\varepsilon\big). \tag{2.2.4}$$

Here, for each $\alpha \in \{1,2,3\}$, the vector field $\theta_\varepsilon^\alpha$ is first defined on the closure of $\omega_\varepsilon := \cup_{i\in I_\varepsilon} Y_\varepsilon^i$ as a $(-\varepsilon/2, \varepsilon/2)^2$-periodic element of $H^1(\Omega, \mathbb{R}^3)$ which does not depend on x_3 and satisfies $\theta_\varepsilon^\alpha = e^\alpha$ on D_{r_ε}, $\theta_\varepsilon^\alpha = 0$ on $\overline{\omega}_\varepsilon \setminus D_{R_\varepsilon}$. Next $\theta_\varepsilon^\alpha$ is assumed to vanish on $\overline{\omega} \setminus \omega_\varepsilon$ and

$$\bar{w}_\varepsilon(x) = w_\varepsilon(x) + V_\varepsilon(x) \tag{2.2.5}$$

where

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i\in I_\varepsilon} \left(\frac{1}{|D_{r_\varepsilon}^i|} \int_{D_{r_\varepsilon}^i} v(\hat{y}, x_3)\, d\hat{y} \right) 1_{Y_\varepsilon^i}(\hat{x}),$$

and V_ε stems from w_ε in such a way that $F_\varepsilon^3(u_\varepsilon)$ converges. The true expressions of V_ε can be found in [4] (formula (5.16) and (5.52) with ψ and φ in place of v and u).
As $R_\varepsilon \ll \varepsilon$ implies $\lim_{\varepsilon\to 0} |B_\varepsilon \cup T_\varepsilon| = 0$, we have

$$\lim_{\varepsilon\to 0} F_\varepsilon^1(u_\varepsilon) = \int_\Omega W_0(e(u))\, dx.$$

To find the limit of $F_\varepsilon^2(u_\varepsilon)$, we introduce

$$z_\varepsilon := \sum_{\alpha=1}^{3} (v-u)_\alpha\, e(\theta_\varepsilon^\alpha), \tag{2.2.6}$$

and compute the linearized strain of u_ε from (2.2.4):

$$e(u_\varepsilon) = z_\varepsilon + e(u) + \sum_{\alpha=1}^{3} \left[(\bar{w}_{\varepsilon\alpha} - v_\alpha)\, e(\theta_\varepsilon^\alpha) + (\theta_\varepsilon^\alpha \otimes \nabla(\bar{w}_\varepsilon - u)_\alpha)_s \right].$$

We claim that a good choice of $\theta_\varepsilon^\alpha$ yields

$$\lim_{\varepsilon\to 0} \left(\int_{B_\varepsilon} W_0(e(u_\varepsilon))\, dx - \int_{B_\varepsilon} W_0(z_\varepsilon)\, dx \right) = 0.$$

Note that W_0, being convex and positively homogeneous of degree 2, satisfies (see [1]):

$$\forall\, \xi, \eta \in \mathbb{S}^3, \qquad |W_0(\xi) - W_0(\eta)| \le C|\xi - \eta|(|\xi| + |\eta|) \tag{2.2.7}$$

so that Cauchy-Schwarz inequality implies

$$\begin{aligned} &\left| \int_{B_\varepsilon} W_0(e(u_\varepsilon)) - W_0(z_\varepsilon)\, dx \right| \\ &\le C \left(\int_{B_\varepsilon} |e(u_\varepsilon) - z_\varepsilon|^2\, dx \right)^{1/2} \left(\int_{B_\varepsilon} |e(u_\varepsilon)|^2\, dx + \int_{B_\varepsilon} |z_\varepsilon|^2\, dx \right)^{1/2}. \end{aligned}$$

Because u and v are smooth, we have

$$|\nabla \bar{w}_\varepsilon| \leq C \text{ on } B_\varepsilon, \qquad |\bar{w}_\varepsilon - v| \leq CR_\varepsilon \text{ on } B_\varepsilon,$$

consequently,

$$\int_{B_\varepsilon} |z_\varepsilon|^2 \, dx \leq C\varepsilon^{-2} \sum_{\alpha=1}^{3} \int_{D(r_\varepsilon, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 \, d\hat{x}$$

with $D(r_\varepsilon, R_\varepsilon) = D(0, R_\varepsilon) \setminus \overline{D(0, r_\varepsilon)}$, where for all $R > 0$, $D(0, R) := \{\hat{x} \in \mathbb{R}^2 \mid |\hat{x}| < R\}$ and

$$\begin{aligned}
&\int_{B_\varepsilon} |e(u_\varepsilon) - z_\varepsilon|^2 \, dx \\
&\qquad \leq CR_\varepsilon^2 \varepsilon^{-2} \Big(1 + \sum_{\alpha=1}^{3} R_\varepsilon^2 \int_{D(0,R_\varepsilon)} |\theta_\varepsilon^\alpha|^2 \, d\hat{x} + \int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 \, d\hat{x}\Big) \\
&\qquad \leq CR_\varepsilon^2 \varepsilon^{-2} \Big(1 + \sum_{\alpha=1}^{3} \int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha|^2 \, d\hat{x}\Big)
\end{aligned}$$

by due account of the Korn inequality in $D(0, R_\varepsilon)$. Therefore,

$$\begin{aligned}
&\Big|\int_{B_\varepsilon} W_0(e(u_\varepsilon)) \, dx - \int_{B_\varepsilon} W_0(z_\varepsilon) \, dx\Big| \\
&\quad \leq CR_\varepsilon \varepsilon^{-1} \Big(1 + \sum_{\alpha=1}^{3} \int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 \, d\hat{x}\Big)^{1/2} \cdot \varepsilon^{-1} \Big(\sum_{\alpha=1}^{3} \int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha|^2 \, d\hat{x}\Big)^{1/2},
\end{aligned}$$

thus, assuming that $\theta_\varepsilon^\alpha$ satisfies

$$\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 \, d\hat{x} \leq \frac{C}{|\ln r_\varepsilon|}, \qquad \forall \alpha = \{1, 2, 3\}, \tag{2.2.8}$$

it suffices to study the asymptotic behavior of $\int_{B_\varepsilon} W_0(z_\varepsilon) \, dx$. Let us denote the bilinear form associated with the quadratic form W_0 by w_0:

$$w_0(e, e') = \frac{1}{2} \lambda_0 (\operatorname{tr} e)(\operatorname{tr} e') + \mu_0 e \cdot e', \qquad \forall e, e' \in \mathbb{S}^3.$$

Note that

$$\begin{aligned}
\int_{B_\varepsilon} W_0(z_\varepsilon) \, dx &= \sum_{\alpha,\beta=1}^{3} \int_{B_\varepsilon} (v - u)_\alpha (v - u)_\beta w_0\big(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)\big) \, dx \\
&= \varepsilon^{-2} \sum_{\alpha,\beta=1}^{3} \Big(\int_{D(r_\varepsilon, R_\varepsilon)} w_0\big((e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)\big) \, d\hat{x} \\
&\qquad \int_0^L \sum_{i \in I_\varepsilon} |Y_\varepsilon^i| (v - u)_\alpha (\hat{x}_\varepsilon^i, x_3)(v - u)_\beta (\hat{x}_\varepsilon^i, x_3) \, dx_3\Big) + O(\varepsilon).
\end{aligned}$$

Here, it clearly appears that in order to get the lowest upper bound for F^2_ε, $\theta^\alpha_\varepsilon$ has to be the unique solution of the capacitary problem

$$(\mathcal{P}^{\text{cap},\,\alpha}_\varepsilon) \qquad \min\left\{ \int_{D(r_\varepsilon,R_\varepsilon)} W_\varepsilon(e(\varphi))\,d\hat{x} \;\middle|\; \begin{array}{l} \varphi \in H^1((-\varepsilon,\varepsilon)^2,\mathbb{R}^3), \\ \varphi(\hat{x}) = e^\alpha \text{ on } D(0,r_\varepsilon) = \{\,|\hat{x}| < r_\varepsilon\,\}, \\ \varphi(\hat{x}) = 0 \text{ on } (-\varepsilon,\varepsilon)^2 \setminus D(0,R_\varepsilon). \end{array} \right\}$$

It is shown in [4] (see Appendix) that

$$\left.\begin{array}{l} \text{i) inequality (2.2.8) is true,} \\ \text{ii) } \exists w_0^{\text{cap}} \in \mathbb{S}^3 \text{ such that} \\ \qquad \lim_{\varepsilon\to0} \varepsilon^{-2} \int_{D(r_\varepsilon,R_\varepsilon)} w_0\big((e(\theta^\alpha_\varepsilon), e(\theta^\beta_\varepsilon)\big)\,d\hat{x} = (w_0^{\text{cap}})_{\alpha\beta} \qquad \forall \alpha,\beta \in \{1,2,3\}. \end{array}\right\} \tag{2.2.9}$$

Therefore,

$$\lim_{\varepsilon\to0} \int_{B_\varepsilon} W_0(z_\varepsilon)\,dx = \int_\Omega w_0^{\text{cap}}(v-u)\cdot(v-u)\,dx,$$

with (see Appendix)

$$w_0^{\text{cap}} = \pi\gamma\mu_0 \begin{pmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We complete the proof of the convergence of $F^1_\varepsilon(u_\varepsilon)$ and $F^2_\varepsilon(u_\varepsilon)$ for any (u,v) such that $\Phi(u,v) < \infty$ by approximation and diagonalization arguments. Eventually, as mentioned earlier, V_ε is chosen in such a way that a tedious computation shows that $F^3_\varepsilon(u_\varepsilon)$ has the expected limit.

2.2.2 Proof of Proposition 2.2.3

We assume here that $\liminf_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Compactness property yields that (u,v) belongs to $L^2(\Omega,\mathbb{R}^3)^2$.

We begin with the term $F^2_\varepsilon(u_\varepsilon)$. Let (u_η,v_η) be Lipschitz on $\overline{\Omega}$ with the property $\lim_{\eta\to0}\|u_\eta - u\|_{L^2(\Omega,\mathbb{R}^3)} + \|v_\eta - v\|_{L^2(\Omega,\mathbb{R}^3)} = 0$. Next we define an approximation $(v_\eta - u_\eta)_\varepsilon := \sum_{i\in I_\varepsilon}(v_\eta - u_\eta)(\hat{x}^i_\varepsilon, x_3)1_{Y^i_\varepsilon}$ of $(v_\eta - u_\eta)$, and associate $z_{\eta\varepsilon}$ to (u_η,v_η) by (2.2.6). Let $\tilde{z}_{\eta\varepsilon} := \sum_{\alpha=1}^3 (v_\eta - u_\eta)_{\varepsilon\alpha} e(\theta^\alpha_\varepsilon)$. Because of local Lipschitz property (2.2.7) of W_0 and $(u,v) \in L^2(\Omega,\mathbb{R}^3)^2$, Cauchy-Schwarz inequality implies

$$\lim_{\varepsilon\to0}\Big(\int_{B_\varepsilon} W_0(\tilde{z}_{\eta\varepsilon})\,dx - \int_{B_\varepsilon} W_0(z_{\eta\varepsilon})\,dx\Big) = 0.$$

The proof of upper bound equality shows

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(z_{\eta\varepsilon})\,dx = \int_\Omega w_0^{\mathsf{cap}}(v_\eta - u_\eta)\cdot(v_\eta - u_\eta)\,dx.$$

Therefore, W_0, being convex and 2-positively homogeneous, the subdifferential inequality gives:

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(e(u_\varepsilon))\,dx\\
&\qquad\geq \liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(\tilde z_{\eta\varepsilon})\,dx + \liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0'(\tilde z_{\eta\varepsilon})\cdot(e(u_\varepsilon)-\tilde z_{\eta\varepsilon})\,dx\\
&\qquad= -\int_\Omega w_0^{\mathsf{cap}}(v_\eta - u_\eta)\cdot(v_\eta - u_\eta)\,dx + \liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0'(\tilde z_{\eta\varepsilon})\cdot e(u_\varepsilon)\,dx
\end{aligned} \tag{2.2.10}$$

Letting $D^i(r_\varepsilon, R_\varepsilon) := D^i_{R_\varepsilon}\setminus\overline{D^i_{r_\varepsilon}}$, we have:

$$\int_{B_\varepsilon} W_0'(\tilde z_{\eta\varepsilon})\cdot e(u_\varepsilon)\,dx = \sum_{i\in I_\varepsilon}\sum_{\alpha=1}^{3}\int_0^L (v_\eta - u_\eta)_\alpha(\hat x^i_\varepsilon, x_3)\left(\int_{D^i(r_\varepsilon,R_\varepsilon)} W_0'(e(\theta^\alpha_\varepsilon))\cdot e(u_\varepsilon)\,d\hat x\right)dx_3.$$

If ν denotes the outer normal along to both $\partial D^i_{r_\varepsilon}$ and $\partial D^i_{R_\varepsilon}$. The very definition of $\theta^\alpha_\varepsilon$ as a solution of $(\mathcal{P}^{\mathsf{cap},\alpha}_\varepsilon)$ and Green formula imply:

$$\begin{aligned}
&\int_{D^i(r_\varepsilon,R_\varepsilon)} W_0'(e(\theta^\alpha_\varepsilon))\cdot e(u_\varepsilon)\,d\hat x\\
&\qquad= -\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot u_\varepsilon\,dl + \int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot u_\varepsilon\,dl\\
&\qquad= -\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot(u_\varepsilon - \bar{\bar u}_\varepsilon)\,dl + \int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot(u_\varepsilon - \bar u_\varepsilon)\,dl\\
&\qquad\quad+\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot(\bar{\bar u}_\varepsilon - \bar u_\varepsilon)\,dl\\
&\qquad= -\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot(u_\varepsilon - \bar{\bar u}_\varepsilon)\,dl + \int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu\cdot(u_\varepsilon - \bar u_\varepsilon)\,dl\\
&\qquad\quad+2\sum_{\beta=1}^{3}(\bar{\bar u}_\varepsilon - \bar u_\varepsilon)^i_\beta\int_{D(r_\varepsilon,R_\varepsilon)} w_0(e(\theta^\alpha_\varepsilon), e(\theta^\beta_\varepsilon))\,d\hat x,
\end{aligned}$$

where

$$(\bar{\bar u}_\varepsilon)^i(x_3) = \frac{1}{|\partial D^i_{r_\varepsilon}|}\int_{\partial D^i_{r_\varepsilon}} u_\varepsilon(\hat x, x_3)\,dl, \qquad \bar{\bar u}_\varepsilon(x) = \sum_{i\in I_\varepsilon}(\bar{\bar u}_\varepsilon)^i(x_3)1_{Y^i_\varepsilon}(\hat x),$$

$$(\bar u_\varepsilon)^i(x_3) = \frac{1}{|\partial D^i_{R_\varepsilon}|}\int_{\partial D^i_{R_\varepsilon}} u_\varepsilon(\hat x, x_3)\,dl, \qquad \bar u_\varepsilon(x) = \sum_{i\in I_\varepsilon}(\bar u_\varepsilon)^i(x_3)1_{Y^i_\varepsilon}(\hat x),$$

$u_\varepsilon(\cdot, x_3)$ being, by Fubini's theorem, well defined in $H^1(\omega, \mathbb{R}^3)$ for a.e. $x_3 \in (0, L)$.
Actually, the standard estimates

$$\int_0^L \int_{\partial D^i_{r_\varepsilon}} |u_\varepsilon - \bar{\bar{u}}^i_\varepsilon| \, dl \, dx_3 \leq r_\varepsilon \left\{ \int_{T^i_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2},$$

$$\int_0^L \int_{\partial D^i_{R_\varepsilon}} |u_\varepsilon - \bar{u}^i_\varepsilon| \, dl \, dx_3 \leq R_\varepsilon \left\{ \int_{D^i_{R_\varepsilon} \times (0,L)} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2}$$

and the estimates (see Appendix)

$$|W_0'(e(\theta^\alpha_\varepsilon))\nu|_{L^\infty(\partial D(0, r_\varepsilon))} \leq \frac{C}{r_\varepsilon |\ln r_\varepsilon|}, \qquad |W_0'(e(\theta^\alpha_\varepsilon))\nu|_{L^\infty(\partial D(0, R_\varepsilon))} \leq \frac{C}{R_\varepsilon |\ln r_\varepsilon|}$$

for $\alpha = 1, 2, 3$ imply that

$$\left| \sum_{i \in I_\varepsilon} \sum_{\alpha=1}^3 \int_0^L (v_\eta - u_\eta)_\alpha (\hat{x}^i_\varepsilon, x_3) \left(\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu \cdot (u_\varepsilon - \bar{\bar{u}}_\varepsilon) \, dl \right) dx_3 \right|$$
$$\leq \frac{C}{|\ln r_\varepsilon|} \sum_{i \in I_\varepsilon} \left\{ \int_{T^i_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2} \leq \frac{C\varepsilon}{\varepsilon^2 |\ln r_\varepsilon|} \left\{ \int_\Omega |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2} \leq C\varepsilon$$

and

$$\left| \sum_{i \in I_\varepsilon} \sum_{\alpha=1}^3 \int_0^L (v_\eta - u_\eta)_\alpha (\hat{x}^i_\varepsilon, x_3) \left(\int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta^\alpha_\varepsilon))\nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) \, dl \right) dx_3 \right|$$
$$\leq \frac{C}{|\ln r_\varepsilon|} \sum_{i \in I_\varepsilon} \left\{ \int_{T^i_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2} \leq \frac{C\varepsilon}{\varepsilon^2 |\ln r_\varepsilon|} \left\{ \int_\Omega |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2} \leq C\varepsilon.$$

Thus,

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon}) \cdot e(u_\varepsilon) \, dx$$
$$= 2 \sum_{i \in I_\varepsilon} \sum_{\alpha, \beta = 1}^3 \int_0^L (v_\eta - u_\eta)_\alpha (\hat{x}^i_\varepsilon, x_3) \left(\int_{D(r_\varepsilon, R_\varepsilon)} w_0(e(\theta^\alpha_\varepsilon), e(\theta^\beta_\varepsilon)) \, d\hat{x} \right) (\bar{\bar{u}}_\varepsilon - \bar{u}_\varepsilon)^i_\beta (x_3) \, dx_3$$
$$= 2 \int_\Omega w_0^{\text{cap}} (v_\eta - u_\eta) \cdot (v - u) \, dx,$$

by due account of (2.2.9) and of the weak convergence in $L^2(\Omega, \mathbb{R}^3)$ of $(\bar{\bar{u}}, \bar{u})$ toward (v, u) (see [4] P.68). Hence, letting η tends to zero, yields

$$\liminf_{\varepsilon \to 0} F^2_\varepsilon (u_\varepsilon) \geq \int_\Omega w_0^{\text{cap}} (v - u) \cdot (v - u) \, dx.$$

Because $|B_\varepsilon \cup T_\varepsilon|$ tends to zero, a classical semi-continuity argument taking into account the convexity of W_0 yields

$$\liminf_{\varepsilon\to 0} F_\varepsilon^1(u_\varepsilon) \geq \int_\Omega W_0(e(u))\,dx.$$

For the third term $F_\varepsilon^3(u_\varepsilon)$, we may extend the strategy of [2] to all cases of relative behaviors of the parameters by due account of the function V_ε introduced by [4].

2.3 Neutrix Composition on Delta Functional

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$ and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots,$ it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in $\mathcal{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t)\rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

If f is an infinitely differentiable function having a single simple zero at the point $x = x_0$, then the distribution $\delta^{(r)}(f(x))$ is defined by

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x - x_0) \tag{2.3.1}$$

for $r = 0, 1, 2, \ldots,$ see [14].

Fisher generalized equation (2.3.1) in [9] as follows:

Definition 2.3.1. *Let f be an infinitely differentiable function. We say that the neutrix composition $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a,b),* *with* $-\infty < a < b < \infty$*, if*

$$\underset{n\to\infty}{\text{N}-\lim}\int_{-\infty}^{\infty}\delta_n^{(r)}(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ *in* $\mathcal{D}[a,b]$*, where* N *is the neutrix, see [8], having domain* N' *the positive and range* N'' *the real numbers, with negligible functions which are finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \qquad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the usual sense as n *tends to infinity.*

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

Definition 2.3.1 was later generalized with the following definition in [10] and was originally called the neutrix composition of distributions.

Definition 2.3.2. *Let* F *be a distribution in* $\mathcal{D}'$ *and let* f *be a locally summable function. We say that the neutrix composition* $F(f(x))$ *exists and is equal to* h *on the open interval* (a,b)*, with* $-\infty < a < b < \infty$*, if*

$$\underset{n\to\infty}{\text{N}-\lim}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ *in* $\mathcal{D}[a,b]$*, where* $F_n(x) = F(x) * \delta_n(x)$ *for* $n = 1, 2, \dots$*. In particular, we say that the composition* $F(f(x))$ *exists and is equal to* h *on the open interval* (a,b) *if*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ *in* $\mathcal{D}[a,b]$.

We now prove the following theorem.

Theorem 2.3.3. *The neutrix composition* $\delta^{(s)}[\ln^r(1+x_+^{1/r})]$ *exists and*

$$\delta^{(s)}[\ln^r(1+x_+^{1/r})] = \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{(r+1)k+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\delta^{(k)}(x) \tag{2.3.2}$$

for $s = 0, 1, 2, \dots$ *and* $r = 1, 2, \dots$*.*

Proof. *To prove equation (2.3.2), we will first of all evaluate*

$$\underset{n\to\infty}{\text{N}-\lim}\langle\delta_n^{(s)}[\ln^r(1+x_+^{1/r})], \varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[a,1]$, where $a<0$.

By Taylor's Theorem, we have

$$\varphi(x)=\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}x^k+\frac{x^{s+1}}{(s+1)!}\varphi^{(s+1)}(\xi x),$$

where $0<\xi<1$. Then if $\varphi(x)$ in $\mathcal{D}[a,1]$, we have

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(s)}[\ln^r(1+x_+^{1/r})],\varphi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}\int_a^1\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^k\,dx\\ &+\operatorname*{N-lim}_{n\to\infty}\frac{1}{(s+1)!}\int_a^1\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^{s+1}\varphi^{(s+1)}(\xi x)\,dx.\end{aligned} \tag{2.3.3}$$

For large enough n, we have

$$\begin{aligned}\int_a^1\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^k\,dx &= n^{s+1}\int_a^1\rho^{(s)}[n\ln^r(1+x_+^{1/r})]x^k\,dx\\ &= n^{s+1}\int_a^0\rho^{(s)}[n\ln^r(1+x_+^{1/r})x^k\,dx+n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1+x_+^{1/r})]x^k\,dx\\ &= n^{s+1}\rho^{(s)}(0)\int_a^0x^k\,dx+n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx\\ &= -\frac{n^{s+1}a^{k+1}\rho^{(s)}(0)}{k+1}+n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx\\ &= E_1+E_2.\end{aligned} \tag{2.3.4}$$

It follows immediately that

$$\operatorname*{N-lim}_{n\to\infty}E_1 = 0. \tag{2.3.5}$$

Making the substitution $t=n\ln^r(1+x^{1/r})$, we have

$$\begin{aligned}E_2 &= n^{s+1}\int_0^1\rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx\\ &= n^{s+1-1/r}\int_0^1t^{1/r-1}\{\exp[(t/n)^{1/r}]-1\}^{kr+r-1}\exp[(t/n)^{1/r}]\rho^{(s)}(t)\,dt\\ &= n^{s+1-1/r}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}(-1)^{kr+r+i-1}\\ &\qquad\times\int_0^1t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt,\end{aligned} \tag{2.3.6}$$

where

$$n^{s+1-1/r}\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt$$
$$= \sum_{j=0}^{sr+r-2}\int_0^1 \frac{(i+1)^j t^{(j+1)/r-1}}{j!n^{(j+1)/r-s-1}}\rho^{(s)}(t)\,dt$$
$$+\frac{1}{(sr+r-1)!}\int_0^1 (i+1)^{sr+r-1}t^s\rho^{(s)}(t)+O(n^{-1/r}).$$

It follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty} n^{s+1-1/r}\int_0^1 t^{1/r-1}\exp[(i+1)(t/n)^{1/r}]\rho^{(s)}(t)\,dt = \frac{(-1)^s s!(i+1)^{sr+r-1}}{2(sr+r-1)!} \tag{2.3.7}$$

for $i = 0,1,2,\dots,kr+r-1$ *and it now follows from equations* (2.3.6) *and* (2.3.7) *that*

$$\mathrm{N}\!-\!\lim_{n\to\infty} E_2 = \sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{kr+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!} \tag{2.3.8}$$

Then using equations(2.3.4)*,* (2.3.5) *and* (2.3.8)*, we see that*

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_a^1 \delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^k\,dx$$
$$= \sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{kr+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!} \tag{2.3.9}$$

for $k = 0,1,2,\dots,s$.

When $k = s+1$*, we have*

$$\int_0^1 \left|\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^{s+1}\right|dx$$
$$\le n^{s+1-1/r}\int_0^1 t^{1/r-1}\{\exp[(t/n)^{1/r}]-1\}^{sr+2r-1}\exp[(t/n)^{1/r}]|\rho^{(s)}(t)|\,dt$$
$$= n^{s+1-1/r}\int_0^1 t^{1/r-1}[(t/n)^{1/r}+O(n^{-2/r})]^{sr+2r-1}[1+O(n^{-1/r})]|\rho^{(s)}(t)|\,dt$$
$$= n^{s+1-1/r}\int_0^1 t^{1/r-1}[(t/n)^{s+2-1/r}+O(n^{-(s+2)})]|\rho^{(s)}(t)|\,dt$$
$$= O(n^{-1}) \tag{2.3.10}$$

and so if ψ *is an arbitrary function in* $\mathcal{D}[a,1]$*, we have*

$$\lim_{n\to\infty}\int_0^1 \left|\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]x^{s+1}\psi(x)\right|dx = 0. \tag{2.3.11}$$

It then follows from equations (2.3.3), (2.3.9) and (2.3.11) *that*

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty}\langle \delta_n^{(s)}[\ln^r(1+x_+^{1/r})],\varphi(x)\rangle \\
&\quad = \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{kr+r+s+i-1}s!(i+1)^{sr+r-1}\varphi^{(k)}(0)}{2(sr+r-1)!k!}+0 \\
&\quad = \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{(r+1)k+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\langle \delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

proving equation (2.3.2) on the interval $[a,1]$.

Since $\delta_n^{(s)}[\ln^r(1+x_+^{1/r})]=0$ *for* $x>0$, *it follows that equation (2.3.2) holds for* $x>a$ *and since* $a<0$ *is arbitrary, it follows that equation (2.3.2) holds on the real line, completing the proof of the theorem.*

Theorem 2.3.4. *The neutrix composition* $\delta^{(s)}[\ln^r(1+|x|^{1/r})]$ *exists and*

$$\begin{aligned}
&\delta^{(s)}[\ln^r(1+|x|^{1/r})] \\
&= \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+k+i-1}[1+(-1)^k]s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\delta^{(k)}(x),
\end{aligned} \tag{2.3.12}$$

for $s=0,1,2,\ldots$ *and* $r=1,2,\ldots$.

Proof. *To prove equation* (2.3.12), *we now have to evaluate*

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle \delta_n^{(s)}[\ln^r(1+|x|^{1/r})],\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ *in* $\mathcal{D}[-1,1]$. *By Taylor's Theorem, we have*

$$\varphi(x)=\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}x^k+\frac{x^{s+1}}{(s+1)!}\varphi^{(s+1)}(\xi x),$$

where $0<\xi<1$. *Then if* φ *is in* $\mathcal{D}[-1,1]$, *we have*

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty}\langle \delta_n^{(s)}[\ln^r(1+|x|^{1/r})],\varphi(x)\rangle \quad &= \quad \mathrm{N}\!-\!\lim_{n\to\infty}\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+x^{1/r})]x^k\,dx \\
&+\mathrm{N}\!-\!\lim_{n\to\infty}\frac{1}{(s+1)!}\int_{-1}^{1}\delta_n^{(s)}[\ln^r(1+|x|^{1/r}]x^{s+1}\varphi^{(s+1)}(\xi x)\,dx.
\end{aligned} \tag{2.3.13}$$

Since

$$\int_{-1}^{0} \delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx \tag{2.3.14}$$
$$= (-1)^k \int_0^1 \delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx$$
$$= (-1)^k n^{s+1} \int_0^1 \rho^{(s)}[n\ln^r(1+x^{1/r})]x^k\,dx, \tag{2.3.15}$$

it follows from equations(2.3.6) *and* (2.3.8) *that*

$$\underset{n\to\infty}{\mathrm{N-lim}} \int_{-1}^{1} \delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx$$
$$= \frac{[1+(-1)^k]}{2} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+i-1} s!(i+1)^{sr+r-1}}{(sr+r-1)!} \tag{2.3.16}$$

for $k = 0, 1, 2, \ldots, s$.

When $k = s+1$, *we have as in the proof of equation* (2.3.11),

$$\lim_{n\to\infty} \int_{-1}^{1} \left|\delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^{s+1}\psi(x)\right| dx = 0, \tag{2.3.17}$$

for an arbitrary continuous function ψ. *It then follows from equations*(2.3.13), (2.3.16) *and* (2.3.17) *that*

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle \delta_n^{(s)}[\ln^r(1+x^{1/r})], \varphi(x)\rangle = \underset{n\to\infty}{\mathrm{N-lim}} \sum_{k=0}^{s} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} \delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^k\,dx$$
$$+ \lim_{n\to\infty} \frac{1}{(s+1)!} \int_{-1}^{1} \delta_n^{(s)}[\ln^r(1+|x|^{1/r})]x^{s+1}\varphi^{(s+1)}(\xi x)\,dx$$
$$= \sum_{k=0}^{s} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+i-1}[1+(-1)^k]s!(i+1)^{sr+r-1}\varphi^{(k)}(0)}{2(sr+r-1)!k!}$$
$$= \sum_{k=0}^{s} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+k+i-1}[1+(-1)^k]s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \langle \delta^{(k)}(x), \varphi(x)\rangle,$$

proving equation(2.3.12) *on the interval* $[-1,1]$. *However, it is clear that* $\delta_n^{(s)}[\ln^r(1+|x|^{1/r})] = 0$ *outside this interval and so equation* (2.3.12) *is proved on the real line.*

Theorem 2.3.5. *The neutrix composition $\delta^{(r^s-1)}[\ln^{1/r}(1+|x|)]$ exists and*

$$\delta^{(r^s-1)}[\ln^{1/r}(1+|x|)] = \sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}[1+(-1)^k]r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!k!}\delta^{(k)}(x) \tag{2.3.18}$$

for $s = 1, 2, \ldots$ and $r = 2, 3, \ldots$.

Proof. *This time we must evaluate*

$$\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)], \varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{r^{s-1}}}{(r^{s-1})!}\varphi^{(r^{s-1})}(\xi x),$$

where $0<\xi<1$. Then if $\varphi(x)$ in $\mathcal{D}[-1,1]$, we have

$$\begin{aligned}&\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)], \varphi(x)\rangle\\ &= \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx\\ &+ \operatorname*{N-lim}_{n\to\infty}\frac{1}{(r^{s-1})!}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^{s-1}}\varphi^{(r^{s-1})}(\xi x)\,dx.\end{aligned} \tag{2.3.19}$$

For large enough n, we have

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx &= n^{r^s}\int_{-1}^{1}\rho^{(r^s-1)}[n\ln^{1/r}(1+|x|)]x^k\,dx\\ &= n^{r^s}[1+(-1)^k]\int_0^1\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx.\end{aligned} \tag{2.3.20}$$

Making the substitution $t = n\ln^{1/r}(1+x)$, we have

$$\begin{aligned}&n^{r^s}\int_0^1\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx\\ &\qquad= rn^{r^s-r}\int_0^1 t^{r-1}\{\exp[(t/n)^r]-1\}^k\exp[(t/n)^r]\rho^{(r^s-1)}(t)\,dt\\ &\qquad= rn^{r^s-r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt,\end{aligned}$$

where

$$rn^{r^s-r}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt$$
$$=\sum_{j=0}^{\infty}\int_0^1 \frac{r(i+1)^j t^{r(j+1)-1}}{j!n^{r(j+1)-r^s}}\rho^{(r^s-1)}(t)\,dt.$$

It follows that

$$\mathrm{N-}\lim_{n\to\infty} rn^{r^s-r}\int_0^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt$$
$$=\int_0^1 \frac{r(i+1)^{r^{s-1}-1}t^{r^s-1}}{(r^{s-1}-1)!}\rho^{(r^s-1)}(t)\,dt$$
$$=\frac{(-1)^{r^s-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!} \tag{2.3.21}$$

for $i=0,1,2,\dots,k$ *and so*

$$\mathrm{N-}\lim_{n\to\infty} n^{r^s}\int_0^1 \rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx$$
$$=\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}. \tag{2.3.22}$$

It now follows from equations(2.3.20) *and*(2.3.22) *that*

$$\mathrm{N-}\lim_{n\to\infty}\int_{-1}^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k\,dx$$
$$=[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!}, \tag{2.3.23}$$

for $k=0,1,2,\dots,r^{s-1}-1$.

When $k=r^{s-1}$, *we have*

$$\int_0^1\left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+x))]x^{r^{s-1}}\right|dx$$
$$\le rn^{r^s-r}\int_0^1 t^{r-1}\{\exp[(t/n)^r]-1\}^{r^{s-1}}\exp[(t/n)^r]|\rho^{(r^s-1)}(t)|\,dt$$
$$=rn^{r^s-r}\int_0^1 t^{r-1}[(t/n)^r+O(n^{-2r})]^{r^{s-1}}[1+O(n^{-r})]|\rho^{(r^s-1)}(t)|\,dt$$
$$=rn^{r^s-r}\int_0^1 t^{r-1}[(t/n)^{r^s}+O(n^{-(r^s+r)})]|\rho^{(r^s-1)}(t)|\,dt$$
$$=O(n^{-r})$$

and so if ψ is an arbitrary function in $\mathcal{D}[a,1]$, we have

$$\lim_{n\to\infty}\int_0^1\left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^{r^{s-1}}\psi(x)\right|dx = 0.$$

Then if φ is an arbitrary function in $\mathcal{D}[-1,1]$, we have

$$\lim_{n\to\infty}\int_{-1}^1\left|\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^{r^{s-1}}\varphi^{r^{s-1}}(\xi x)\right|dx = 0 \tag{2.3.24}$$

and it follows from equations(2.3.19), (2.3.23) *and* (2.3.24) *that*

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]x^k,\varphi(x)\rangle \\
&= \sum_{k=0}^{r^{s-1}-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s+k-i-1}r(r^s-1)!(i+1)^{r^{s-1}-1}\varphi^{(k)}(0)}{2(r^{s-1}-1)!k!} \\
&= \sum_{k=0}^{r^{s-1}-1}[1+(-1)^k]\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{r^s-i-1}(r^s-1)!(i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

proving equation(2.3.18) *on the interval $[-1,1]$. However, it is clear that $\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)]=0$ outside this interval and so equation* (2.3.18) *is proved.*

Finally we have

Theorem 2.3.6. *The neutrix composition $\delta^{(r^s-1)}(\ln^{1/r}|1+x|)$ exists and*

$$\begin{aligned}
&\delta^{(r^s-1)}(\ln^{1/r}|1+x|) \\
&\qquad = \sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i r(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!k!}\delta^{(k)}(x)
\end{aligned} \tag{2.3.25}$$

for $s=1,2,\dots$ and $r=1,3,5,\dots$.

Proof. *This time we must evaluate*

$$\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(r^s-1)}(\ln^{1/r}|1+x|),\varphi(x)\rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1,1]$.

For large enough n, we have on making the substitution $t=n\ln^{1/r}(1+x)$,

$$\begin{aligned}
\int_{-1}^1\delta_n^{(r^s-1)}(\ln^{1/r}|1+x|)x^k\,dx &= \int_{-1}^1\delta_n^{(r^s-1)}[\ln^{1/r}(1+x)]x^k\,dx \\
&= rn^{r^s-r}\int_{-1}^1 t^{r-1}\{\exp[(t/n)^r]-1\}^k\exp[(t/n)^r]\rho^{(r^s-1)}(t)\,dt \\
&= rn^{r^s-r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\int_{-1}^1 t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt,
\end{aligned}$$

where

$$rn^{r^s-r}\int_{-1}^{1} t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt$$
$$= \sum_{j=0}^{\infty}\int_{-1}^{1}\frac{r(i+1)^j t^{r(j+1)-1}}{j!n^{r(j+1)-r^s}}\rho^{(r^s-1)}(t)\,dt.$$

Noting that r^s-1 *is an even integer when* r *is an odd integer, and using equation* (2.3.21), *it follows that*

$$\underset{n\to\infty}{\mathrm{N-lim}}\, rn^{r^s-r}\int_{-1}^{1} t^{r-1}\exp[(i+1)(t/n)^r]\rho^{(r^s-1)}(t)\,dt$$
$$= \int_{-1}^{1}\frac{r(i+1)^{r^{s-1}-1}t^{r^s-1}}{(r^{s-1}-1)!}\rho^{(r^s-1)}(t)\,dt$$
$$= \frac{r(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!}$$

for $i=0,1,2,\ldots,k$ *and so*

$$\underset{n\to\infty}{\mathrm{N-lim}}\, n^{r^s-r}\int_{-1}^{1}\rho^{(r^s-1)}[n\ln^{1/r}(1+x)]x^k\,dx$$
$$= \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!}.$$

Thus

$$\underset{n\to\infty}{\mathrm{N-lim}}\int_{-1}^{1}\delta_n^{(r^s-1)}[\ln^{1/r}|1+x|]x^k\,dx$$
$$= \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!}, \tag{2.3.26}$$

for $k=0,1,2,\ldots,r^{s-1}-1$.

When $k=r^{s-1}$, *it follows as above that*

$$\int_{-1}^{1}\left|\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^{r^{s-1}}\right| = O(n^{-r})$$

and so if φ *is an arbitrary function in* $\mathcal{D}[-1,1]$, *then*

$$\lim_{n\to\infty}\int_{-1}^{1}\left|\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)]x^{r^{s-1}}\varphi^{(r^{s-1})}(x)\right|dx = 0. \tag{2.3.27}$$

Now let φ *be an arbitrary function in* $\mathcal{D}[-1,1]$. *By Taylor's Theorem, we have*

$$\varphi(x) = \sum_{k=0}^{r^{s-1}-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^{r^{s-1}}}{(r^{s-1})!}\varphi^{(r^{s-1})}(\xi x),$$

where $0 < \xi < 1$. *Then if* φ *is in* $\mathcal{D}[-1,1]$ *and using equations* (2.3.26) *and* (2.3.27), *we have*

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(r^{s-1}-1)}(\ln^{1/r}|1+x|), \varphi(x)\rangle \\
&= \operatorname*{N-lim}_{n\to\infty} \sum_{k=0}^{r^{s-1}-1} \frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1} \delta_n^{(r^s-1)}[\ln^{1/r}|1+x|]x^k\,dx \\
&+ \operatorname*{N-lim}_{n\to\infty} \frac{1}{(r^{s-1})!}\int_{-1}^{1} \delta_n^{(r^s-1)}(\ln^{1/r}(|1+x|)x^{r^s}\varphi^{(r^{s-1})}(\xi x)\,dx \\
&= \sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}r(r^s-1)!(i+1)^{r^{s-1}-1}\varphi^{(k)}(0)}{(r^{s-1}-1)!k!} \\
&= \sum_{k=0}^{r^{s-1}-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i}(r^s-1)!(i+1)^{r^{s-1}-1}}{(r^{s-1}-1)!k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

proving equation (2.3.25) *on the interval* $[-1,1]$. *However, it is clear that* $\delta_n^{(r^s-1)}[\ln^{1/r}(|1+x|)] = 0$ *outside this interval and so equation* (2.3.25) *is proved on the real line.*

บทที่ 3

Output of the Project

- S. Orankitjaroen, N. Sontichai, C. Licht, A. Kananthai, *Mathematical modeling of fiber reinforced structures by homogenization*, Thai Journal of Mathematic, Special Issue Annual Meeting in Mathematics, (2008), 103–115.
- S. Orankitjaroen, N. Sontichai, C. Licht, A. Kananthai, *Remark on the Homogenization of a Microfibered Linearly Elastic Material*, In preparation for an international conference ICMA-MU 2009.
- T. Kraiweeradechachai, S. Orankitjaroen[1], B. Fisher and E. Özçağ, *Further Results on the Neutrix Composition of the Delta Function*, (Accepted for East-West Journal of Mathematics in volume 11, no 2 (2009) that will be appeared in December 2009).

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บทที่ 4

Appendix

The Vector Capacitary Problem

Taking advantage of the cylindrical geometry, Bellieud and Gruais [4] showed that $\theta_\varepsilon^\alpha$ and $\sigma_\varepsilon^\alpha := W_0'(e(\theta_\varepsilon^\alpha))$ are such that

$$\theta_\varepsilon^1(x_1, x_2) = \theta_\varepsilon^2(x_2, x_1), \qquad \theta_{\varepsilon 3}^1 = \theta_{\varepsilon 3}^2 = 0,$$
$$\theta_\varepsilon^3 = \frac{\ln(R_\varepsilon/r)}{\ln(R_\varepsilon/r_\varepsilon)} e^3 \text{ in } D(r_\varepsilon, R_\varepsilon), \qquad r = |\hat{x}|,$$

$$(\theta_{\varepsilon 1}^1 + i\theta_{\varepsilon 2}^1)(\hat{x}) = \frac{A}{2\mu_0}\left(\chi(\ln z + \overline{\ln z}) + \frac{z^2}{r_\varepsilon^2 + R_\varepsilon^2} - \frac{z}{\bar{z}} - \frac{2z\bar{z}}{\chi(r_\varepsilon^2 + R_\varepsilon^2)} + \frac{2\chi(r_\varepsilon^2 \ln R_\varepsilon - R_\varepsilon^2 \ln r_\varepsilon)}{R_\varepsilon^2 - r_\varepsilon^2} + \frac{r_\varepsilon^2 R_\varepsilon^2}{(r_\varepsilon^2 + R_\varepsilon^2)\bar{z}^2}\right) + \frac{R_\varepsilon^2}{R_\varepsilon^2 - r_\varepsilon^2},$$

with

$$\chi := \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0}, \qquad A := \frac{\mu_0}{\frac{R_\varepsilon^2 - r_\varepsilon^2}{\chi(r_\varepsilon^2 + R_\varepsilon^2)} - \chi \ln \frac{R_\varepsilon}{r_\varepsilon}}, \text{ } z \text{ the complex number } x_1 + ix_2,$$

and

$$\sigma_\varepsilon^1 \nu = \frac{\mu_0(\chi + 1)(1 + o(1))}{\chi r_\varepsilon |\ln r_\varepsilon|} e^1 \qquad \text{on } \partial D(0, r_\varepsilon),$$
$$\sigma_\varepsilon^1 \nu = \frac{\mu_0(1 + o(1))}{\chi R_\varepsilon |\ln r_\varepsilon|}\left[4(1 + \frac{1}{\chi}\cos^2\theta - (1 + \chi + \frac{2}{\chi}))\right] e^1 + 2(1 + \frac{1}{\chi})\sin 2\theta e^2 \qquad \text{on } \partial D(0, R_\varepsilon),$$
$$\sigma_\varepsilon^3 \nu = \frac{-\mu}{r \ln R_\varepsilon / r_\varepsilon} e^3 \qquad \text{on the circle of radius } r.$$

Thus, for each α, $\beta = 1, 2, 3$, i)

$$|\sigma_\varepsilon^\alpha \nu|_{L^\infty(\partial D(0,r_\varepsilon))} \leq \frac{C}{r_\varepsilon |\ln r_\varepsilon|}, \qquad |\sigma_\varepsilon^\alpha \nu|_{L^\infty(\partial D(0,R_\varepsilon))} \leq \frac{C}{R_\varepsilon |\ln r_\varepsilon|},$$

ii)

$$\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 \, d\hat{x} \leq C \int_{D(0,R_\varepsilon)} W_0(e(\theta_\varepsilon^\alpha)) \, d\hat{x} \leq \frac{C}{2} \int_{\partial D(0,R_\varepsilon)} (\sigma_\varepsilon^\alpha \nu)_\alpha \, dl \leq \frac{C}{|\ln r_\varepsilon|},$$

iii)

$$\begin{aligned}(w_0^{\mathsf{cap}})_{\alpha\beta} &:= \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} w_0\big((e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)\big) \, d\hat{x} \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon^{-2}}{2} \int_{\partial D(0,r_\varepsilon)} (\sigma_\varepsilon^\alpha \nu)_\beta \, dl,\end{aligned}$$

satisfies

$$w_0^{\mathsf{cap}} = \pi \gamma \mu_0 \begin{pmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Paper Published

Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2008): 103–115

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Mathematical Modeling of Fiber Reinforced Structures by Homogenization

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Abstract : We present another proof of a study of Bellieud and Bouchitté that we expect to be more suitable to treat more general geometrical and physical cases. We consider the homogenization of the quasi-linear elliptic problem

$$\begin{aligned} -\operatorname{div}\sigma_\varepsilon = f,\ \ \sigma_\varepsilon = a_\varepsilon \left|\nabla u_\varepsilon\right|^{p-2} \nabla u_\varepsilon \quad & \text{on } \Omega \\ u_\varepsilon = u_0 \quad & \text{on } \Gamma_0 \\ \sigma_\varepsilon \cdot n = g \quad & \text{on } \Gamma_1 \end{aligned}$$

where Ω is a bounded cylindrical open subset of $\mathbb{R}^3$ and $1 < p < +\infty$. The fibers occupy a set of thin parallel cylinders periodically distributed in Ω. The conductivity coefficient a_ε is ε-periodic and takes very high values in the fibers.

Keywords : Variational convergence, Homogenization

1 Introduction

Let $p \in (1, +\infty)$, we consider the homogenization of the elliptic problem

$$\begin{aligned} -\operatorname{div}\sigma_\varepsilon = f,\ \ \sigma_\varepsilon = a_\varepsilon \left|\nabla u_\varepsilon\right|^{p-2} \nabla u_\varepsilon \quad & \text{on } \Omega \\ u_\varepsilon = u_0 \quad & \text{on } \Gamma_0 \\ \sigma_\varepsilon \cdot n = g \quad & \text{on } \Gamma_1 \end{aligned} \tag{1.1}$$

where $\Omega := \omega \times (0, L)$ with $L > 0$ and ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary and containing the origin of coordinates. The homogenization study of (1.1) consists in examining the behavior of the sequence of the solution (u_ε) as ε tends to zero. The conductivity coefficient a_ε is ε-periodic and satisfies a uniform lower bound, Γ_0 is an open subset of $\partial\Omega$ with Hausdorff measure $H^2(\Gamma_0)$ strictly positive, $\Gamma_1 = \partial\Omega\backslash\Gamma_0$, and n is the unit exterior normal on $\partial\Omega$. The boundary data u_0 is Lipschitz, while $(f, g) \in L^{p'}(\Omega) \times L^{p'}(\Gamma_1)$, $p' = p/(p-1)$.

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The problem (1.1) is related to the minimization problem

$$(\mathcal{P}_\varepsilon) \quad \min\left\{F_\varepsilon(w) - L(w) : w \in W^{1,p}_{\Gamma_0}(\Omega)\right\},$$

where

$$\begin{aligned} W^{1,p}_{\Gamma_0}(\Omega) &:= \left\{ w \in W^{1,p}(\Omega) : w = u_0 \text{ on } \Gamma_0 \right\}, \\ F_\varepsilon(w) &:= \int_\Omega a_\varepsilon \phi_p(\nabla w)\, dx, \\ \phi_p(\xi) &:= \frac{1}{p}|\xi|^p, \ \forall\, \xi \in \mathbb{R}^n, n = 1, 2, 3, \\ L(w) &:= \int_\Omega f w\, dx + \int_{\Gamma_1} g w\, dH^2. \end{aligned} \tag{1.2}$$

We are interested in the asymptotic behavior of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to 0$. We present another proof of a study of Bellieud and Bouchitté [2] that we expect to be more suitable to treat more general geometrical and physical cases.

The bases of the cylindrical domain Ω are denoted by $\omega_0 = \omega \times \{0\}$ and $\omega_L = \omega \times \{L\}$. For each ε, we consider a periodic distribution of cells $(Y^i_\varepsilon)_{i \in I_\varepsilon}$ such that $Y^i_\varepsilon := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$, and $I_\varepsilon := \left\{i \in \mathbb{Z}^2 \ : \ Y^i_\varepsilon \subset \omega\right\}$. Let $(D^i_{r_\varepsilon})_{i \in I_\varepsilon}$ be the family of disks of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon := (\varepsilon i_1, \varepsilon i_2)$ of radius $r_\varepsilon \ll \varepsilon$, $T^i_\varepsilon := D^i_{r_\varepsilon} \times (0, L)$ and $T_\varepsilon := \cup_{i \in I_\varepsilon} T^i_\varepsilon$. The set of thin parallel cylinders T_ε represents the fibers (see Figure 1 and Figure 2). The conductivity coefficient a_ε is

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \backslash T_\varepsilon, \\ \lambda_\varepsilon & \text{otherwise.} \end{cases}$$

We make the assumptions

$$r_\varepsilon \to 0, \quad \frac{r_\varepsilon}{\varepsilon} \to 0, \quad \lambda_\varepsilon \to +\infty, \quad k_\varepsilon := \lambda_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2} \to k, \ k \geq 0 \quad \text{as } \varepsilon \to 0. \tag{1.3}$$

In [2], it was shown that the asymptotic limit of $(\mathcal{P}_\varepsilon)$ is

$$\min\left\{\Phi(u, v) - L(u) : (u, v) \in (L^p(\Omega))^2\right\},$$

where

$$\Phi(u, v) = \begin{cases} \int_\Omega \phi_p(\nabla u)\, dx + \frac{k\pi}{p} \int_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p dx + \frac{2\pi\gamma}{p} \int_\Omega |v - u|^p\, dx, \\ \qquad \text{if } \begin{cases} (u, v) \in W^{1,p}_{\Gamma_0}(\Omega) \times L^p(\omega, W^{1,p}(0, L)), \\ v = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L), \end{cases} \\ +\infty \quad \text{otherwise,} \end{cases} \tag{1.4}$$

and

$$[0, +\infty] \ni \gamma = \begin{cases} \lim\limits_{\varepsilon \to 0} \varepsilon^{-2} |\log r_\varepsilon|^{-1} & \text{if } p = 2, \\ \lim\limits_{\varepsilon \to 0} \left|\frac{2-p}{p-1}\right|^{p-1} r_\varepsilon^{2-p} \varepsilon^{-2} & \text{if } p \neq 2. \end{cases} \tag{1.5}$$

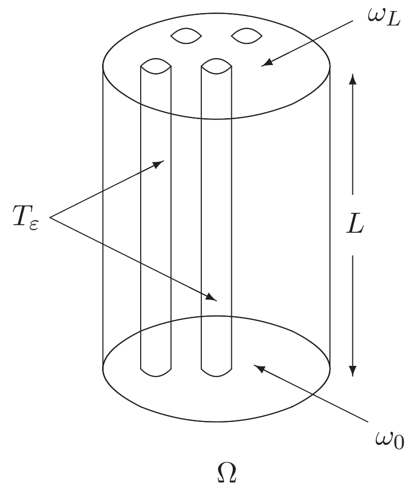

Figure 1: the domain $\Omega = \omega \times (0, L)$ occupied by a composite material

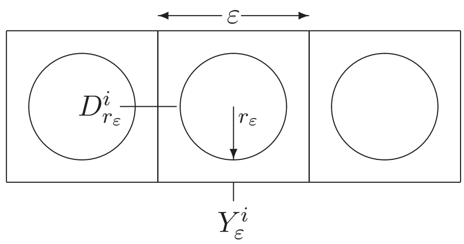

Figure 2: the circular cross section of the fiber, $Y^i_\varepsilon \subset \omega$

Here, the boundary data u_0 is assumed to be Lipschitz in order to ensure that the infimum value of problem $(\mathcal{P}_\varepsilon)$ remains finite as $\varepsilon \to 0$. In case $k = +\infty$, we add further assumption

$$k_\varepsilon r_\varepsilon \to 0, \quad \text{as } \varepsilon \to 0. \tag{1.6}$$

The conditions

$$k > 0 \quad \text{and} \quad \{\, \gamma > 0 \quad \text{or} \quad \omega_0 \subset \Gamma_0 \quad \text{or} \quad \omega_L \subset \Gamma_0 \,\} \tag{1.7}$$

guarantee that the functional Φ is coercive in $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L)$.

We are concerned with the extension of this result to more general cross sections of the fibers and more general energy density than ϕ_p. The aim of this paper is therefore to provide another proof that we expect to be more suitable to treat such general cases. The steps of the proof in [2] are to successively establish:

(i) a compactness property of the sequence (u_ε) such that $F_\varepsilon(u_\varepsilon) < C$,

(ii) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$,

(iii) an upper bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$.

Here we replace the steps (ii) and (iii) by

(ii′) an upper equality of the sequence $(F_\varepsilon(u_\varepsilon))$,

(iii′) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$ which essentially uses a subdiffenrential inequality.

2 An Alternative Strategy

It consists, under (1.3), (1.5), (1.6) and (1.7), in proving the following three propositions. In the sequel, the symbols $\to$, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ stand for the strong convergence, the weak convergence and the weak star convergence, respectively. As usual, the letter C denotes various constants and for all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2).

Proposition 2.1 (compactness property). *Let (u_ε) be a sequence such that* $\sup F_\varepsilon(u_\varepsilon)$ *is finite. Then (u_ε) is strongly relatively compact in $L^p(\Omega)$ and (v_ε), given by $v_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon$, is bounded in $L^1(\Omega)$ and, up to a subsequence, (v_ε) weakly* converges in the space of bounded measures $\mathcal{M}_b(\Omega)$ to an element v of $L^p(\Omega)$.*

Proposition 2.2 (upper bound equality). *For all (u, v) in $(L^p(\Omega))^2$, such that $\Phi(u, v) < +\infty$, there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$ and*

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \Phi(u, v).$$

Proposition 2.3 (lower bound inequality). *For all u in $L^p(\Omega)$ and for all sequences (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$, one has:*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

The proofs of these propositions are presented in the following sections.

2.1 Proof of Proposition 2.1

Compactness property was already proved in [2].

2.2 Proof of Proposition 2.2

Our sole contribution is to prove that we can replace inequality by equality, for that we use the same approximation u'_ε of u as in [2]

$$u'_\varepsilon = (1 - \theta_\varepsilon) u + \theta_\varepsilon w_\varepsilon.$$

The function θ_ε is first defined on the closure of $\omega_\varepsilon := \cup_{i \in I_\varepsilon} Y^i_\varepsilon$ as a $(-\varepsilon/2, \varepsilon/2)^2$-periodic continuous function which satisfies $0 \leq \theta_\varepsilon \leq 1$, $\theta_\varepsilon = 1$ on $D_\varepsilon := \cup_{i \in I_\varepsilon} D^i_{r_\varepsilon}$, $\theta_\varepsilon = 0$ on $\overline{\omega}_\varepsilon \setminus \cup_{i \in I_\varepsilon} D^i_{R_\varepsilon}$, where $D^i_{R_\varepsilon}$ is the disk of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon$ of radius R_ε such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. Next θ_ε is assumed to vanish on $\overline{\omega} \setminus \omega_\varepsilon$ and

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i \in I_\varepsilon} \left(\frac{1}{|D^i_{r_\varepsilon}|} \int_{D^i_{r_\varepsilon}} v(\hat{y}, x_3)\, d\hat{y} \right) 1_{Y^i_\varepsilon}(\hat{x}).$$

The approximation u'_ε does not satisfy the boundary condition on $\Gamma_0 \cap (\omega_0 \cup \omega_L)$ so that, as in [2], we introduce a sharper approximation

$$u^{\#}_\varepsilon := u \varphi_\varepsilon + u'_\varepsilon (1 - \varphi_\varepsilon).$$

Here φ_ε is a $C^\infty(\overline{\Omega})$ function which satisfies $\varphi_\varepsilon = 1$ on Γ_0, $\varphi_\varepsilon = 0$ on $\overline{\Omega}\backslash\Sigma_\varepsilon$, $|\nabla\varphi_\varepsilon| \leq C/r_\varepsilon$ on $\overline{\Omega}$ where $\Sigma_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \Gamma_0) < r_\varepsilon\}$. We assume that u and v are Lipschitz on $\overline{\Omega}$ and there exists $L > 0$ such that

$$\left|\frac{\partial v}{\partial x_3}(\hat{x}', x_3) - \frac{\partial v}{\partial x_3}(\hat{x}'', x_3)\right| < L|\hat{x}' - \hat{x}''| \quad \forall\,(\hat{x}', x_3), (\hat{x}'', x_3) \in \overline{\Omega}. \tag{2.1}$$

Letting Ψ be any continuous function on $\overline{\Omega}$ such that $0 \leq \Psi \leq 1$, we introduce F_ε^Ψ, Φ^Ψ defined by similar formulae as the ones of F_ε and Φ but with $\Psi\,dx$ in place of dx. We will prove the lemma:

Lemma 2.4.

$$\lim_{\varepsilon\to 0} F_\varepsilon^\Psi(u'_\varepsilon) = \Phi^\Psi(u, v).$$

If Lemma 2.4 is proved, then, by a classical approximation process, we can deduce

$$\lim_{\varepsilon\to 0} F_\varepsilon(u'_\varepsilon) = \Phi(u, v). \tag{2.2}$$

Finally, we complete the proof of (2.2) for any (u, v) such that $\Phi(u, v) < +\infty$ by approximation and diagonalization arguments.

Proof of Lemma 2.4. We split $F_\varepsilon^\Psi(u'_\varepsilon)$ in three parts

$$F_\varepsilon^\Psi(u'_\varepsilon) = F_\varepsilon^{\Psi 1}(u'_\varepsilon) + F_\varepsilon^{\Psi 2}(u'_\varepsilon) + F_\varepsilon^{\Psi 3}(u'_\varepsilon). \tag{2.3}$$

First, we consider

$$F_\varepsilon^{\Psi 1}(u'_\varepsilon) := \int_{\Omega\backslash B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u'_\varepsilon)\Psi\,dx = \int_{\Omega\backslash B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u)\Psi\,dx,$$

where $B_\varepsilon := \cup_{i\in\varepsilon I_\varepsilon} D^i_{R_\varepsilon}\backslash\overline{D^i_{r_\varepsilon}} \times (0, L)$. Hence, the assumption $R_\varepsilon \ll \varepsilon$ yields $\lim_{\varepsilon\to 0}|B_\varepsilon \cup T_\varepsilon| = 0$ and, consequently,

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 1}(u'_\varepsilon) = \int_\Omega \phi_p(\nabla u)\Psi\,dx.$$

Next, we pay attention to

$$F_\varepsilon^{\Psi 2}(u'_\varepsilon) := \int_{B_\varepsilon} \phi_p(\nabla u'_\varepsilon)\Psi\,dx.$$

Writing

$$z_\varepsilon := (v - u)\widehat{\nabla}\theta_\varepsilon, \tag{2.4}$$

we obtain

$$\nabla u'_\varepsilon = z_\varepsilon + (w_\varepsilon - v)\nabla\theta_\varepsilon + (1 - \theta_\varepsilon)\nabla u + \theta_\varepsilon\nabla w_\varepsilon.$$

Let us show

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi\,dx = 0. \tag{2.5}$$

The function ϕ_p, being convex and positively homogeneous of degree p, satisfies

$$\forall\, \xi, \eta \in \mathbb{R}^n,\ n = 1, 2, 3, \quad |\phi_p(\xi) - \phi_p(\eta)| \le C|\xi - \eta|(|\xi|^{p-1} + |\eta|^{p-1}). \tag{2.6}$$

Therefore, Hölder inequality yields

$$\left|\int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi\, dx\right| \le C\left(\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p\, dx\right)^{\frac{1}{p}} \left(\int_{B_\varepsilon} |\nabla u'_\varepsilon|^p\, dx + \int_{B_\varepsilon} |z_\varepsilon|^p\, dx\right)^{\frac{1}{p'}}.$$

The smoothness of (u, v) implies

$$\left.\begin{array}{l} u'_\varepsilon = u \text{ on } \Omega\backslash(B_\varepsilon \cup T_\varepsilon), \quad |u'_\varepsilon| \le C \text{ on } \Omega, \quad |\nabla w_\varepsilon| \le C \text{ on } B_\varepsilon, \\ u'_\varepsilon = w_\varepsilon \text{ on } T_\varepsilon, \quad |w_\varepsilon - v| \le CR_\varepsilon \text{ on } B_\varepsilon, \end{array}\right\} \tag{2.7}$$

so that

$$\int_{B_\varepsilon} |\nabla u'_\varepsilon|^p\, dx + \int_{B_\varepsilon} |z_\varepsilon|^p\, dx \le C\varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x},$$

$$\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p\, dx \le CR_\varepsilon^p \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x},$$

where $D(r_\varepsilon, R_\varepsilon) = \left\{\hat{x} \in \mathbb{R}^2 : r_\varepsilon < |\hat{x}| < R_\varepsilon\right\}$. Hence, if we choose θ_ε such that

$$\exists\, M > 0\ ; \qquad \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x}\ \le M \qquad \forall\, \varepsilon > 0, \tag{2.8}$$

then (2.5) is true. We finally have

$$\begin{aligned} & \lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) \\ = \ & \lim_{\varepsilon\to 0} \int_{B_\varepsilon} \phi_p(z_\varepsilon)\Psi\, dx \\ = \ & \lim_{\varepsilon\to 0} \int_{B_\varepsilon} |v - u|^p \phi_p(\widehat{\nabla}\theta_\varepsilon)\Psi\, dx \\ = \ & \lim_{\varepsilon\to 0} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \int_0^L \sum_{i\in I_\varepsilon} |v - u|^p(\hat{y}_\varepsilon^i, x_3)\Psi(\hat{y}_\varepsilon^i, x_3)\, dx_3 \quad (\text{with } \hat{y}_\varepsilon^i \in Y_\varepsilon^i) \\ = \ & \lim_{\varepsilon\to 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} \int_0^L \sum_{i\in I_\varepsilon} |Y_\varepsilon^i||v - u|^p(\hat{y}_\varepsilon^i, x_3)\Psi(\hat{y}_\varepsilon^i, x_3)\, dx_3. \end{aligned}$$

Observe that $\lim_{\varepsilon\to 0} \int_0^L \sum_{i\in I_\varepsilon} |Y_\varepsilon^i||v - u|^p(\hat{y}_\varepsilon^i, x_3)\Psi(\hat{y}_\varepsilon^i, x_3)\, dx_3 = \int_\Omega |v - u|^p \Psi\, dx$. To get the lowest upper bound in Proposition 2.2, it is clear that θ_ε has to be the solution of the capacitary problem

$$(\mathcal{P}_\varepsilon^{\text{cap}}) \quad \min\left\{\int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\varphi)\, d\hat{x} : \begin{array}{l} \varphi \in W^{1,p}(D(r_\varepsilon, R_\varepsilon)), \\ \varphi(\hat{x}) = 1 \text{ on } |\hat{x}| = r_\varepsilon, \\ \varphi(\hat{x}) = 0 \text{ on } |\hat{x}| = R_\varepsilon. \end{array}\right\}$$

As observed in [2], we have

$$\theta_\varepsilon = \begin{cases} \frac{\log R_\varepsilon - \log|\hat{x}|}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } p = 2, \\ \frac{R_\varepsilon^s - |\hat{x}|^s}{R_\varepsilon^s - r_\varepsilon^s} & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}) \end{cases}$$

and

$$\int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon)\, d\hat{x} = \frac{2\pi}{p}\Gamma_p(r_\varepsilon, R_\varepsilon),$$

where $\Gamma_p(r_\varepsilon, R_\varepsilon) := \begin{cases} \frac{1}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } p = 2, \\ (\frac{s}{R_\varepsilon^s - r_\varepsilon^s})^{p-1} & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}). \end{cases}$

Note that

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\Gamma_p(r_\varepsilon, R_\varepsilon) = \gamma.$$

If $\gamma < +\infty$, then (2.8) is satisfied and

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = \frac{2\pi\gamma}{p}\int_\Omega |v-u|^p \Psi\, dx.$$

When $\gamma = +\infty$, it suffices to prove that $\lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = 0$. Due to (2.7), the result is a consequence of $F_\varepsilon^{\Psi 2}(u'_\varepsilon) \leq C R_\varepsilon^p \varepsilon^{-2}\Gamma_p(r_\varepsilon, R_\varepsilon)$, which tends to zero.

Now, we consider the remaining part

$$F_\varepsilon^{\Psi 3}(u'_\varepsilon) := \int_{T_\varepsilon} \lambda_\varepsilon \phi_p(\nabla w_\varepsilon)\Psi\, dx.$$

Recalling the assumption (2.1) on v and using the local Lipschitz property (2.6), we deduce

$$\begin{aligned} \lim_{\varepsilon\to 0} F_\varepsilon^{\Psi 3}(u'_\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} \lambda_\varepsilon \phi_p\left(\frac{\partial v}{\partial x_3}\right)\Psi\, dx \\ &= \frac{k\pi}{p}\int_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p \Psi\, dx, \end{aligned}$$

as shown in [2]. ■

Now, we will prove the upper bound equality by using the sharper approximation $(u_\varepsilon^\#)$. We start with

$$F_\varepsilon(u_\varepsilon^\#) = \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#)\, dx + \int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\, dx. \tag{2.9}$$

Conditions (2.7) imply $|u'_\varepsilon - u| \leq C(r_\varepsilon 1_{T_\varepsilon} + R_\varepsilon 1_{B_\varepsilon})$. Hence

$$\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#)\, dx \leq C\left(|\Sigma_\varepsilon| + \int_{\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u'_\varepsilon|^p\, dx + \lambda_\varepsilon |T_\varepsilon \cap \Sigma_\varepsilon| + \left(\frac{R_\varepsilon}{r_\varepsilon}\right)^p |\Sigma_\varepsilon|\right).$$

Lemma 2.4 implies that for every $\Psi \in C^0(\overline{\Omega}, [0,1])$, such that $\Psi = 1$ on a small neighborhood of $\Gamma_0 \cap (\omega_0 \cup \omega_L)$,

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^{\#})\, dx &\leq \limsup_{\varepsilon\to 0} \int_\Omega a_\varepsilon \phi_p(\nabla u'_\varepsilon)\Psi\, dx \\ &= \int_\Omega \left(\phi_p(\nabla u) + \frac{k\pi}{p}\left|\frac{\partial v}{\partial x_3}\right|^p + \frac{2\pi\gamma}{p}|v-u|^p\right)\Psi\, dx.\end{aligned}$$

Thus, by letting Ψ tend to zero, we deduce

$$\lim_{\varepsilon\to 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^{\#})\, dx = \lim_{\varepsilon\to 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\, dx = 0,$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\, dx &= \lim_{\varepsilon\to 0}\left(\int_\Omega a_\varepsilon \phi_p(\nabla u'_\varepsilon)\, dx - \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon)\, dx\right) \\ &= \int_\Omega \phi_p(\nabla u) + \frac{k\pi}{p}\left|\frac{\partial v}{\partial x_3}\right|^p + \frac{2\pi\gamma}{p}|v-u|^p\, dx,\end{aligned}$$

which proves the result for (u, v) smooth. We complete the proof by a standard approximation of (u, v) and a diagonalization argument [1].

2.3 Proof of Proposition 2.3

It is enough to consider $\liminf_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Due to the compactness property, (u, v) is in $(L^p(\Omega))^2$. We first consider the term $F^2_\varepsilon(u_\varepsilon)$. Let (u_η, v_η) be Lipschitz on $\overline{\Omega}$ such that $\lim_{\eta\to 0} \|u_\eta - u\|_{L^p(\Omega)} + \|v_\eta - v\|_{L^p(\Omega)} = 0$. We define $(v_\eta - u_\eta)_\varepsilon := \sum_{i\in I_\varepsilon}(v_\eta - u_\eta)(\hat{x}^i_\varepsilon, x_3)1_{Y^i_\varepsilon}$ and $z_{\eta\varepsilon} := (v_\eta - u_\eta)_\varepsilon \widehat{\nabla}\theta_\varepsilon$. Because of the local Lipschitz property (2.6) of ϕ_p and $(u, v) \in (L^p(\Omega))^2$, Hölder inequality implies

$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon} \phi_p(z_{\eta\varepsilon}) - \phi_p(z_\varepsilon)\, dx = 0.$$

The proof of the upper bound equality allows us to write

$$\lim_{\varepsilon\to 0} \phi_p(z_{\eta\varepsilon}) = \frac{2\pi\gamma}{p}\int_\Omega |v_\eta - u_\eta|^p\, dx.$$

The convexity of ϕ_p and the fact that $\phi_p(\nabla u_\varepsilon) \geq \phi_p(\widehat{\nabla} u_\varepsilon)$ yield

$$\begin{aligned}\liminf_{\varepsilon\to 0} F^2_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \int_{B_\varepsilon} \phi_p(\widehat{\nabla} u_\varepsilon)\, dx \\ &\geq \liminf_{\varepsilon\to 0} \int_{B_\varepsilon} \phi_p(z_{\eta\varepsilon})\, dx \\ &\quad + \liminf_{\varepsilon\to 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon})\cdot(\widehat{\nabla} u_\varepsilon - z_{\eta\varepsilon})\, dx. \qquad (2.10)\end{aligned}$$

The very definition of ϕ_p implies

$$\begin{aligned} \phi_p'(\xi) &= |\xi|^{p-2}\xi && \forall\, \xi \in \mathbb{R}^n,\ n = 1,2,3, \\ \phi_p'(t\xi) &= \phi_p'(t)\phi_p'(\xi) && \forall\, (t,\xi) \in \mathbb{R}\times\mathbb{R}^n, n = 1,2,3, \\ \phi_p'(\xi)\cdot\xi &= p\phi_p(\xi) && \forall\, \xi \in \mathbb{R}^n,\ n = 1,2,3. \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \phi_p'(z_{\eta\varepsilon})\cdot z_{\eta\varepsilon}\,dx = 2\pi\gamma\int_\Omega |v_\eta - u_\eta|^p\,dx. \tag{2.11}$$

For the other term of (2.10), we have

$$\int_{B_\varepsilon} \phi_p'(z_{\eta\varepsilon})\cdot\widehat{\nabla}u_\varepsilon\,dx = \sum_{i\in I_\varepsilon}\int_0^L \phi_p'(v_\eta - u_\eta)(\hat{x}_\varepsilon^i, x_3)\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}u_\varepsilon\,d\hat{x}\,dx_3,$$

where $D^i(r_\varepsilon, R_\varepsilon) = D^i_{R_\varepsilon}\setminus\overline{D^i_{r_\varepsilon}}$. Let ν be the outer normal on $\partial D^i(r_\varepsilon,R_\varepsilon)$, the very definition of θ_ε as a solution of $(\mathcal{P}^{\mathrm{cap}}_\varepsilon)$ yields

$$\begin{aligned} \int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}u_\varepsilon\,d\hat{x} &= \int_{\partial D^i(r_\varepsilon,R_\varepsilon)} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl \\ &= \int_{\partial D^i_{R_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl + \int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl \\ &= -\tilde{u}^i_\varepsilon\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl + \tilde{v}^i_\varepsilon\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl \end{aligned}$$

where $\tilde{u}^i_\varepsilon := \frac{\int_{\partial D^i_{R_\varepsilon}}(\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl}{\int_{\partial D^i_{r_\varepsilon}}(\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl} = \frac{1}{2\pi R_\varepsilon}\int_{\partial D^i_{R_\varepsilon}}(\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)u_\varepsilon\,dl$, $\tilde{u}_\varepsilon := \sum_{i\in I_\varepsilon}\tilde{u}^i_\varepsilon 1_{Y^i_\varepsilon}$, $\tilde{v}^i_\varepsilon := \frac{\int_{\partial D^i_{r_\varepsilon}}(\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)v_\varepsilon\,dl}{\int_{\partial D^i_{r_\varepsilon}}(\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl} = \frac{1}{2\pi r_\varepsilon}\int_{\partial D^i_{r_\varepsilon}}(\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)v_\varepsilon\,dl$, $\tilde{v}_\varepsilon := \sum_{i\in I_\varepsilon}\tilde{v}^i_\varepsilon 1_{Y^i_\varepsilon}$. Thus,

$$\begin{aligned} \int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}u_\varepsilon\,d\hat{x} &= (\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon)\int_{\partial D^i_{r_\varepsilon}} (\phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\nu)\,dl \\ &= (\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon)\int_{D^i(r_\varepsilon,R_\varepsilon)} \phi_p'(\widehat{\nabla}\theta_\varepsilon)\cdot\widehat{\nabla}\theta_\varepsilon\,d\hat{x} \\ &= 2\pi\Gamma_p(r_\varepsilon,R_\varepsilon)(\tilde{v}^i_\varepsilon - \tilde{u}^i_\varepsilon), \end{aligned}$$

and

$$\int_{B_\varepsilon} \phi_p'(z_{\eta\varepsilon})\cdot\widehat{\nabla}u_\varepsilon\,dx = 2\pi\Gamma_p(r_\varepsilon,R_\varepsilon)\int_\Omega \phi_p'((v_\eta-u_\eta)_\varepsilon)(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon)\,dx.$$

It was shown in [2] that $(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon) \rightharpoonup (v-u)$ in $L^p(\Omega)$. On the other hand, $(v_\eta - u_\eta)$ being smooth and ϕ_p' being continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$, we have $\phi_p'((v_\eta-u_\eta)_\varepsilon) \to \phi_p'(v_\eta - u_\eta)$ in $L^{p'}(\Omega)$. Hence,

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} \phi_p'(z_{\eta\varepsilon})\cdot\widehat{\nabla}u_\varepsilon\,dx = 2\pi\gamma\int_\Omega \phi_p'(v_\eta - u_\eta)(v-u)\,dx. \tag{2.12}$$

Therefore, (2.10), (2.11) and (2.12) imply

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon^2(u_\varepsilon) \quad &\geq \quad \frac{2\pi\gamma}{p}\int_\Omega |v_\eta - u_\eta|^p\,dx \\ &\quad +2\pi\gamma\left[\int_\Omega |v_\eta - u_\eta|^p\,dx - \int_\Omega \phi_p'(v_\eta - u_\eta)(v-u)\,dx\right].\end{aligned}$$

The expected lower bound for $F_\varepsilon^2(u_\varepsilon)$ is obtained by letting η tend to zero.

To complete the proof it suffices to use the arguments of [2] concerning the lower bounds for $F_\varepsilon^1(u_\varepsilon)$, $F_\varepsilon^3(u_\varepsilon)$ and the fact that v belongs to $L^p(\omega, W^{1,p}(0,L))$.

2.4 The Final Result

The following theorem, a convergence result for the minimizer of $(\mathcal{P}_\varepsilon)$, is a standard consequence of the previous three propositions.

Theorem 2.5. *Let the assumptions* (1.3) *and* (1.5) *hold with* $(k,\gamma) \in (0,+\infty)$. *Then the unique solution* $\overline{u}_\varepsilon$ *of* $(\mathcal{P}_\varepsilon)$ *converges weakly in* $W^{1,p}(\Omega)$ *to the unique solution* $\overline{u}$ *of the problem*

$$\min\left\{\min\left\{\Phi(u,v) - L(u) : v \in L^p(\Omega)\right\} : u \in L^p(\Omega)\right\},$$

where Φ *and* L *are defined by* (1.4) *and* (1.2) *respectively.*

Proof. The proof of this theorem is the same as that in [2]. ■

3 Conclusions and Remarks

The previous analysis can be easily extended to the case when ϕ_p is replaced by any strictly convex function which satisfies

$$\exists\, M > 0,\ \exists\, r \in (1, p)\ ; \qquad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \qquad \forall\, \xi \in \mathbb{R}^3, \tag{3.1}$$

the density function associated with $\Phi(u, v)$ becomes

$$W(\nabla u) + 2\pi\gamma|v - u|^p + W\left(\frac{\partial v}{\partial x_3}\right).$$

Indeed, (3.1) and Hölder inequality imply

$$\left|\int_{B_\varepsilon} W(\nabla u_\varepsilon)\, dx - \int_{B_\varepsilon} \phi_p(\nabla u_\varepsilon)\, dx\right| \leq M|B_\varepsilon|^{1-\frac{r}{p}} \int_\Omega |\nabla u_\varepsilon|^p\, dx,$$

while our arguments and those of [2] to derive the upper bound and lower bound respectively are valid when ϕ_p is replaced by any convex function satisfying a growth condition like

$$\exists\, \alpha, \beta > 0\ ; \qquad \alpha(|\xi|^p - 1) \leq W(\xi) \leq \beta(1 + |\xi|^p) \qquad \forall\, \xi \in \mathbb{R}^3,$$

which is an obvious consequence of (3.1).

Eventually, the key arguments of our analysis are the identification of γ, θ_ε in terms of the solution of capacitary problems and the use of the p-positive homogeneity and convexity of ϕ_p and of the fact that $\phi_p(\xi) \geq \phi_p(\hat{\xi})$. Thus, it is easy to guess what could be $\Phi(u, v)$, when ϕ_p is replaced by any strictly convex function and when the cross sections of the fibers are smooth star-shaped domains of $\mathbb{R}^2$. We hope that our proposed strategy will be able to reduce and overcome the involved technical difficulties.

4 Acknowlagement

Somsak was supported by a research fund from Commission on Higher Education and the Thailand Research Fund (MRG5080422). Nuttawat was supported by a scholarship from The Ministry Staff Development Project of the Ministry of University Affairs.

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(Received 13 May 2008)

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