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Original Article

Performance analysis of an M/G/1 retrial *G*-queue with feedback under working breakdown services

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Abstract

In this paper, we investigate a new type of retrial queueing model with feedback and working breakdown services. The regular busy server may become defective by disasters (negative customers) at any point of time. Negative customers arrive only at the service time of a positive customer and remove the positive customer from the service. At the instant of failure, the main server is sent for repair and the repair period begins immediately. During the repair period, the server gives service at a lower speed (called working breakdown period). The steady state probability generating function for system size and orbit size are obtained using the method of supplementary variable. We also obtain some analytic expressions for various performance measures such as system state probabilities, mean orbit size, and mean system size of this model and some important special cases are discussed. Finally, some numerical examples are presented to study the impact of the system parameters.

Keywords: retrial queue, *G* queue, feedback, working breakdown services

1. Introduction

The topic of the retrial queues in queueing theory has been an interesting research topic during the last two decades. The concept of retrial queues has been a subject of great effort and interest by many researchers (Artalejo, 2010; Artalejo & Gomez-Corral, 2008). Such queueing models are sure to bring applications in the performance analysis of a wide range of systems in data distributed networks, tele-communications, traffic management on high-speed networks, and production engineering.

The concept of negative customers (called *G*-queues) was first developed by Gelenbe (1989) in computers, neural networks, and communication networks. The name *G*-queue (negative customers) was adopted for queues with negative customers in the acknowledgment of Gelenbe. Negative customers (disasters) arrive only at the regular service time of positive customers (ordinary customers). Negative customers cannot accumulate in a queue and do not receive service, and will remove the positive customers already in service from the system. These types of negative customers cause server breakdown and the service channel will fail for a short interval of time. At the instant of failure, the main server is sent for repair and the repair period begins immediately. The repaired server is assumed to be as good as a new server. Tan Van Do (2011) presented a survey on queueing systems with *G*-networks, negative customers, and applications. Further, such models are motivated by recent advanced applications in computer systems and data communication networks. Recently, Kim and Lee (2014) have discussed queueing models with breakdowns and repairs.

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Queueing models with different service rates were studied by various authors in the past. The initiative of these models almost made the change of the service rate dependent on the situation of the system, such as queues in random environment, queues with breakdown, and working breakdown or models with vacations and working vacations. Servi and Finn (2002) introduced an M/M/1 queueing system with working vacations. Wu and Takagi (2006) extended the M/M/1/WV queue to an M/G/1/WV queue. Authors like Arivudainambi *et al.* (2014), Gao *et al.* (2014), Rajadurai *et al.* (2016), Zhang and Hou (2012), Zhang and Liu (2015) and Rajadurai (2018a, 2018b) analyzed queueing systems with working vacations.

The concept of working breakdowns was first introduced by Kalidass and Ramanath (2012). In other words, if the system becomes defective by disasters at any point of time when a regular busy server is in operation, the system should be ready with a substitute (standby) server in preparation for possible main server failures. The substitute server renders services to the customers while the main server is repaired. The service rate of the substitute server is different from (lower than) the main server. At the instant of the repair completion, the main server returns to the system and becomes available. Additionally, the working breakdown service can decrease complaints from the customers who should wait for the main server to be repaired and reduces the cost of waiting customers. Therefore, a more reasonable repair policy is the working breakdown service for unreliable queueing systems. Recently, Kim and Lee (2014) discussed a model M/G/1 queueing system with disasters and working breakdown services.

Motivated by this factor, this work introduces a new class of M/G/1 retrial queue with negative customers, feedback under working breakdown services, and working vacation services. During the period of working vacation and working breakdown, the server works in different rates of services. The analytical results of this model are very useful and helpful for decision makers for the design of a management policy. This model has potential applications in medical service systems for telephone consultation, stochastic production, and inventory systems with a multipurpose production facility and machine replacement problems. The rest of this work is organized as follows. The mathematical model description of this work is described in section 2. The steady state governing equations and the number of customers in the orbit for different states are obtained in section 3. In section 4, some important system performance measures are given. In section 5, we analyze some special cases of our model which are consistent with the existing literature. Numerical examples are presented for various parameters on the system performance and cost optimization is analysed in section 6. Finally, conclusions of the work are given in section 7.

2. Basic Description of the Model

We investigate an M/G/1 retrial *G*-queue with feedback under working vacations and working breakdowns (M/G/1/WB).

Arrival process: There are two types of customers arriving into the system: ordinary customers (positive customers) and disasters (negative customers). Assume that both types of customers arrive from outside the system according to independent Poisson processes with rates λ and δ, respectively.

Retrial process: If an arriving positive customer finds that the server is free, the customer begins his service immediately. Otherwise, when arriving customers find the server busy or lower speed service, the arrivals join the pool of blocked customers called an orbit in accordance with FCFS discipline, which means that only one customer at the head of the orbit queue is allowed access to the server. Inter-retrial times have an arbitrary distribution $R(x)$ with corresponding Laplace Stieltjes Transform (LST) $R^*(\vartheta)$.

Regular service process: Whenever a new positive customer or retry positive customer arrives at the server idle state, then the server immediately starts normal service for the arrivals. The service time has a general distribution which is denoted by the random variable S with distribution function (d.f) $S(x)$ having LST $S^*(\vartheta)$.

Feedback rule: After completion of service for each customer, the unsatisfied customers may rejoin the orbit as feedback customers to receive another service with probability p $(0 \leq p \leq 1)$ or may leave the system with complement probability $q = (1- p)$.

Removal rule and the working breakdown process: Negative customers (disasters) arrive only at the regular service time of the positive customers. Negative customers cannot accumulate in a queue and do not receive service but will remove the positive customers in service from the system. These types of negative customers cause server breakdown and the service channel will fail for a short interval of time. At the instant of failure, the main server is sent for repair and the repair period begins immediately. The repair time follows an exponential distribution with the rate of η. The repaired server is assumed to be as good as a new server. However, when disaster occurs in a regular busy server, the server goes for a working breakdown. During the working breakdown period, the substitute server works at a lower service rate for the arriving customers ($\mu_w < \mu$). When repair ends, if there are customers in the orbit, the server switches to the normal working level and will start a new busy period. Otherwise, it is idle and ready for serving new arrivals. During the working breakdown periods (lower speed services), the service time follows a general random variable S_w with distribution function $S_w(t)$ and LST $S_w^*(\vartheta)$.

Multiple working vacations process: The server begins a working vacation each time the orbit becomes empty and the vacation time follows an exponential distribution with parameter θ. If any customer arrives during a vacation period, the server gives service at a lower speed service rate ($\mu_w < \mu$). If any customers in the orbit at a lower speed service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy period which means vacation interruption happens. Otherwise, it continues the vacation. When a vacation ends, if there are customers in the orbit, the server switches to the normal working level. Otherwise, the server begins another vacation. During the working vacation periods (lower speed services), the service time follows a general random variable S_w with distribution function $S_w(t)$ and LST $S_w^*(\vartheta)$.

Lower speed service process: We consider the working vacation period and working breakdown period as the lower speed service period, and we assume that all random variables (inter-arrival times, retrial times, regular service times and lower speed service times) defined above are independent of each other.

2.1 Practical justifications of the suggested model

The suggested model has practical real life application in medical service systems for telephone consultation. Nowadays, doctors have initiated telephone consultation services for patients who are called positive customers. Here, we consider a telephone consultation service system staffed with a chief physician (main server) and a physician assistant (substitute server or working breakdown server). The physician assistant only provides service to the patients when the chief physician is on vacation (working vacation) and the service rate of the physician assistant is usually slower than the chief physician. In generally, there is a phone operator who is responsible to establish communications between doctors and patients or notes down the order of the calls, corresponding to the 'orbit'. If the line is busy when a patient makes a call, he cannot queue but tries again sometime later (retrial), otherwise he is served immediately by the chief physician or the physician assistant. During the patients' consultation time, the telephone signal status is very low or no network coverage (negative customer), and the patient's call has lost service. Once the signal strength is full (repaired), then the system is again treated as good as new to serve.

When the chief physician finds no patient has called, he will need to rest from his work, i.e. go on a vacation. During the vacation period of the chief physician, the physician assistant will serve the patients, if any, and after his service completion, if there are patients in the system, the chief physician will come back from his vacation whether his vacation has ended or not, i.e. vacation interruption happens. Meanwhile, if there is no patient when a vacation ends, the chief physician begins another work vacation (multiple working vacations), otherwise, the chief physician takes over as the physician assistant. To understand the patient's condition, the chief physician will restart his service no matter how long the physician assistant has served the patient. On the other hand, to minimize the idle time of the chief physician, immediately on a service completion, the phone operator will call (or search for) the customers who are in orbit under FCFS and the search time is assumed to be generally distributed, which is corresponding to the general retrial time policy.

2.2. Notations and probabilities

In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S(0)=0$, $S(\infty)=1$, $S_w(0)=0$, $S_w(\infty)=1$ are continuous at $x = 0$. The following notations and probabilities are used in sequent sections:

$r(x) \equiv$ the hazard rate (conditional completion rate) for retrial of $R(x)$; i.e., $\theta(x)dx = \dfrac{dR(x)}{1-R(x)}$.

$\mu(x) \equiv$ the hazard rate for service of $S(x)$; i.e., $\mu(x)dx = \dfrac{dS(x)}{1-S(x)}$.

$\mu_w(x) \equiv$ the hazard rate for lower rate service of $S_w(x)$; i.e., $\mu_w(x)dx = \dfrac{dS_w(x)}{1-S_w(x)}$.

$N(t) \equiv$ the number of customers in the orbit at time t.

$C(t) \equiv$ the state of the server at time t.

$R^0(t) \equiv$ the elapsed retrial time.

$S^0(t) \equiv$ the elapsed service time on i^{th} phase.

$S_w^0(t) \equiv$ the elapsed lower rate service time.

$P_0(t) \equiv$ the probability that the system is empty at time t.

$W_0(t) \equiv$ the probability that the system is empty at time t and the server is in working vacation and breakdown (lower speed service).

$R_n(x,t) \equiv$ the probability that at time t there are exactly n customers in the orbit with the elapsed retrial time of the test customer undergoing retrial lying in between x and $x+dx$.

$\Pi_n(x,t) \equiv$ the probability that at time t there are exactly n customers in the orbit with the elapsed normal service time of the test customer undergoing service lying in between x and $x+dx$.

$W_n(x,t) \equiv$ the probability that at time t there are exactly n customers in the orbit with the elapsed lower rate service time of the test customer undergoing service lying in between x and $x+dx$.

3. Steady State Analysis

For an M/G/1 retrial G-queue with feedback under working vacations and working breakdowns (M/G/1/WVB), we developed the steady state difference-differential equations based on a supplementary variable method. For further development of this retrial queueing model, let us define the random variable where

$$C(t)=\begin{cases}0, & \text{if the server is free and in working vacation and working breakdown period,}\\ 1, & \text{if the server is free and in regular service period,}\\ 2, & \text{if the server is busy and in regular service period on both phases at time } t,\\ 3, & \text{if the server is busy and in lower speed service period period at time } t.\end{cases}$$

Thus the supplementary variables are introduced in order to obtain a bivariate Markov process $\{C(t), N(t);\ t \geq 0\}$. If $C(t)$ = 1 and $N(t)$ > 0, then $R^0(t)$ represents the elapsed retrial time. If $C(t)$ = 2 and $N(t) \geq 0$ then $S^0(t)$ corresponds to the elapsed time of the customer being served in a normal busy period. If $C(t)$ = 3 and $N(t) \geq 0$ then $S_w^0(t)$ corresponds to the elapsed time of the customer being served in a lower rate service period.

Let $\{t_n;\ n = 1,2,...\}$ be the sequence of epochs at which either a normal service or lower service period completion occurs. The sequence of random vectors $Z_n = \{C(t_n +), N(t_n +)\}$ forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix A that $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda),$ for our system to be stable, where $\rho = p + \frac{\lambda}{\delta}\left(1 - S^*(\delta)\right).$

For the process $\{N(t),\ t \geq 0\}$, we define the probabilities $P_0(t) = P\{C(t) = 0,\ N(t) = 0\}$ and $W_0(t) = P\{C(t) = 0,\ N(t) = 0\}$ the probability densities

$$R_n(x,t)dx = P\left\{C(t) = 1, N(t) = n,\ x \leq R^0(t) < x + dx\right\},\ \text{for } t \geq 0,\ x \geq 0 \text{ and } n \geq 1.$$
$$\Pi_n(x,t)dx = P\left\{C(t) = 2, N(t) = n,\ x \leq S_b^0(t) < x + dx\right\},\ \text{for } t \geq 0,\ x \geq 0,\ n \geq 0.$$
$$W_n(x,t)dx = P\left\{C(t) = 4, N(t) = n,\ x \leq S_w^0(t) < x + dx\right\},\ \text{for } t \geq 0,\ x \geq 0, \text{ and } n \geq 0.$$

We assume that the stability condition is fulfilled in the sequel and so that we can set $P_0 = \lim_{t\to\infty} P_0(t)$ and $W_0 = \lim_{t\to\infty} W_0(t)$ limiting densities for $x > 0$ and $n \geq 0$

$$R_n(x) = \lim_{t\to\infty} R_n(x,t)\ ,\ \Pi_n(x) = \lim_{t\to\infty} \Pi_n(x,t) \text{ and } W_n(x) = \lim_{t\to\infty} W_n(x,t).$$

3.1 Steady state equations

The system of governing equations of server states as follows:

$$\lambda P_0 = \eta W_0. \tag{3.1}$$
$$\left(\lambda + \theta + \eta\right) W_0 = \theta Q_0 + q\int_0^\infty \Pi_0(x)\mu(x)dx + q\int_0^\infty W_0(x)\mu_w(x)dx + \delta\int_0^\infty \Pi_n(x)dx,\ n \geq 0. \tag{3.2}$$
$$\frac{dR_n(x)}{dx} + \left(\lambda + r(x)\right)R_n(x) = 0,\ n \geq 1. \tag{3.3}$$
$$\frac{d\Pi_0(x)}{dx} + \left(\lambda + \delta + \mu(x)\right)\Pi_0(x) = 0,\ n = 0. \tag{3.4}$$
$$\frac{d\Pi_n(x)}{dx} + \left(\lambda + \delta + \mu(x)\right)\Pi_n(x) = \lambda\Pi_{n-1}(x),\ n \geq 1. \tag{3.5}$$
$$\frac{dW_0(x)}{dx} + \left(\lambda + \theta + \eta + \mu_w(x)\right)W_0(x) = 0,\ n = 0. \tag{3.6}$$
$$\frac{dW_n(x)}{dx} + \left(\lambda + \theta + \eta + \mu_w(x)\right)W_n(x) = \lambda W_{n-1}(x),\ n \geq 1. \tag{3.7}$$

The steady state boundary conditions at $x = 0$ are

$$R_n(0) = p\int_0^\infty \Pi_n(x)\mu(x)dx + q\int_0^\infty \Pi_{n-1}(x)\mu(x)dx + p\int_0^\infty W_n(x)\mu_w(x)dx + q\int_0^\infty W_{n-1}(x)\mu_w(x)dx,\ n \geq 1. \tag{3.8}$$

$$\Pi_0(0)=\int_0^\infty R_1(x)r(x)dx+(\theta+\eta)\int_0^\infty W_0(x)dx+\lambda P_0,\ n=0. \tag{3.9}$$

$$\Pi_n(0)=\int_0^\infty R_{n+1}(x)r(x)dx+\lambda\int_0^\infty R_n(x)dx+(\theta+\eta)\int_0^\infty W_n(x)dx,\ n\geq 1. \tag{3.10}$$

$$W_n(0)=\begin{cases}\lambda W_0, & n=0\\ 0, & n\geq 1\end{cases} \tag{3.11}$$

The normalizing condition is

$$P_0+W_0+\sum_{n=1}^{\infty}\int_0^\infty R_n(x)dx+\sum_{n=0}^{\infty}\left(\int_0^\infty \Pi_n(x)dx+\int_0^\infty W_n(x)dx\right)=1 \tag{3.12}$$

3.2. Computation of the steady state solution

In the following, the probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z|\leq 1$, as follows:

$$R(x,z)=\sum_{n=1}^{\infty}R_n(x)z^n;\ \mathrm{R}(0,z)=\sum_{n=1}^{\infty}R_n(0)z^n;\ \Pi(x,z)=\sum_{n=0}^{\infty}\Pi_n(x)z^n;\ \ \Pi(0,z)=\sum_{n=0}^{\infty}\Pi_n(0)z^n;\ W(x,z)=\sum_{n=0}^{\infty}W_n(x)z^n$$

and $W(0,z)=\sum_{n=0}^{\infty}W_n(0)z^n$;

Multiplying the steady state equation and steady state boundary condition (3.2) - (3.10) by z^n and summing over n, (n = 0,1,2...) and solving the partial differential equations, it follows that

$$\frac{\partial R(x,z)}{\partial x}+\left(\lambda+r(x)\right)R(x,z)=0 \tag{3.13}$$

$$\frac{\partial \Pi(x,z)}{\partial x}+\left(\lambda(1-z)+\delta+\mu(x)\right)\Pi(x,z)=0 \tag{3.14}$$

$$\frac{\partial W(x,z)}{\partial x}+\left(\lambda(1-z)+\theta+\eta+\mu_w(x)\right)W(x,z)=0 \tag{3.15}$$

$$R(0,z)=\ (pz+q)\int_0^\infty \Pi(x,z)\mu(x)dx+(pz+q)\int_0^\infty W(x,z)\mu_w(x)dx-\delta\int_0^\infty \Pi(x,z)dx-(\lambda+\eta)W_0 \tag{3.16}$$

$$\Pi(0,z)=\frac{1}{z}\int_0^\infty R(x,z)r(x)dx+\lambda\int_0^\infty R(x,z)dx+(\theta+\eta)\int_0^\infty W(x,z)dx+\lambda P_0 \tag{3.17}$$

$$W(0,z)=\lambda W_0 \tag{3.18}$$

Solving the partial differential equations 3.13–3.15, it follows that

$$R(x,z)=R(0,z)[1-R(x)]e^{-\lambda x} \tag{3.19}$$

$$\Pi(x,z)=\Pi(0,z)[1-S(x)]e^{-A(z)x}. \tag{3.20}$$

$$W(x,z)=W(0,z)[1-S_w(x)]e^{-A_w(z)x}. \tag{3.21}$$

where $A(z)=\left(\delta+\lambda(1-z)\right)$ and $A_w(z)=\left(\theta+\eta+\lambda(1-z)\right)$.

Inserting equations 3.19–3.21 and 3.16 and make some calculations, finally we get,

$$\Pi(0,z)=\frac{R(0,z)}{z}\left[R^*(\lambda)+z\left(1-R^*(\lambda)\right)\right]+\lambda P_0+\lambda W_0V(z). \tag{3.22}$$

where $V(z)=\dfrac{(\theta+\eta)\left[1-S_w^*\left(A_w(z)\right)\right]}{(\theta+\eta)+\lambda(1-z)}$ and $S(z)=\dfrac{\delta\left[1-S^*\left(A(z)\right)\right]}{\delta+\lambda(1-z)}$

Using equations 3.19–3.21 and 3.22 in 3.16, we get

$$R(0,z)=(pz+q)\ \Pi(0,z)\left(S^*\left(A(z)\right)+S(z)\right)+(pz+q)W(0,z)S_w^*\left(A_w(z)\right)-(\lambda+\eta)W_0 \tag{3.23}$$

Using equations 3.18 and 3.22 in equation 3.23, we get

$$\begin{aligned}&\left(z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)\right)R(0,z)\\&\qquad=zW_0\left(\left(S^*\left(A(z)\right)+S(z)\right)\left(\lambda V(z)+\eta\right)+\lambda\left((pz+q)S_w^*\left(A_w(z)\right)-1\right)-\eta\right)\end{aligned} \tag{3.24}$$

From the above equation, we know that the key element for obtaining $P(0,z)$ is to find the zeros of $f(z)=z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)=0$ in the range $0<z<1$ for the equation $f(z)=0.$ (from Gao *et al.* [2014]). From this, we give the lemma in Appendix B.

From equation 3.24, we get

$$R(0,z)=\frac{zW_0\left(\left(S^*\left(A(z)\right)+S(z)\right)\left(\lambda V(z)+\eta\right)+\lambda\left((pz+q)S_w^*\left(A_w(z)\right)-1\right)-\eta\right)}{\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)\right\}} \tag{3.25}$$

Using the equation 3.25 in equation 3.22, we get

$$\Pi(0,z)=\frac{W_0\left\{z\left(\lambda V(z)+\eta\right)+\left(\lambda\left((pz+q)S_w^*\left(A_w(z)\right)-1\right)-\eta\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\right\}}{\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)\right\}} \tag{3.26}$$

Using equations 3.18 and 3.25–3.26 in equations 3.19–3.21, then the limiting probability generating functions (PGFs) are $R(x,z),\ \Pi(x,z)$ and $W(x,z).$

3.3. Steady state results

If the system is in steady state condition $\rho<R^*(\lambda),$ the PGFs are as follows:
(i) the number of customers in the orbit when the server is idle;

$$R(z)=\int_0^\infty R(x,z)dx=\frac{zW_0\left(1-R^*(\lambda)\right)\left(\left(S^*\left(A(z)\right)+S(z)\right)\left(\lambda V(z)+\eta\right)+\lambda\left((pz+q)S_w^*\left(A_w(z)\right)-1\right)-\eta\right)}{\lambda\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)\right\}} \tag{3.27}$$

(ii) the number of customers in the orbit when the server is regularly busy;

$$\Pi(z)=\int_0^\infty \Pi(x,z)dx=\frac{W_0\left(1-S^*\left(A(z)\right)\right)\left\{z\left(\lambda V(z)+\eta\right)+\left(\lambda\left((pz+q)S_w^*\left(A_w(z)\right)-1\right)-\eta\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\right\}}{A(z)\times\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)\right\}} \tag{3.28}$$

(iii) the number of customers in the orbit when the server is at a lower speed service;

$$W(z)=\int_0^\infty W(x,z)dx=\left\{\frac{\lambda W_0 V(z)}{(\theta+\eta)}\right\}. \tag{3.29}$$

Using the normalizing condition, we can determine P_0 and W_0, by setting z = 1 in equations 3.26–3.28 and applying the L-Hospitals rule whenever necessary and then we get $P_0+W_0+R(1)+\Pi(1)+W(1)=1.$
The probability that the server is idle at a lower speed service is equation 3.30,

$$W_0=\frac{R^*(\lambda)-p-\frac{\lambda}{\delta}\left(1-S^*(\delta)\right).}{\left\{\left(q+\frac{\eta}{\lambda}\right)R^*(\lambda)+\frac{\lambda}{(\theta+\eta)}\left(1-S_w^*(\theta+\eta)\right)\left(q+1-S_w^*(\theta+\eta)\left(1-R^*(\lambda)\right)\right)-\frac{\lambda}{\delta}S_w^*(\theta+\eta)\left(1-S^*(\delta)\right)\right\}} \tag{3.30}$$

The probability that the server is idle in regular service is equation 3.31,

$$P_0=\frac{R^*(\lambda)-p-\frac{\lambda}{\delta}\left(1-S^*(\delta)\right)}{\frac{\lambda}{\delta}\left\{\left(q+\frac{\eta}{\lambda}\right)R^*(\lambda)+\frac{\lambda}{(\theta+\eta)}\left(1-S_w^*(\theta+\eta)\right)\left(q+1-S_w^*(\theta+\eta)\left(1-R^*(\lambda)\right)\right)-\frac{\lambda}{\delta}S_w^*(\theta+\eta)\left(1-S^*(\delta)\right)\right\}} \tag{3.31}$$

Corollary 3.1. If the system satisfies the steady state condition, The PGF of the number of customers in the system ($K_s(z)$) is obtained using

$$K_s(z)=P_0+W_0+R(z)+z\left(\Pi(z)+W(z)\right) \tag{3.32}$$

The PGF of the number of customers in the orbit ($K_o(z)$) is obtained using

$$K_o(z)=P_0+W_0+R(z)+\Pi(z)+W(z). \tag{3.33}$$

4. System performance measures

Our analysis is based on the following system characteristics of the retrial queueing system.

4.1. System state probabilities

(i) Let R be the steady state probability that the server is idle during the retrial,

$$R=R(1)=\frac{W_0\left(1-R^*(\lambda)\right)\left\{p+\left(1-S_w^*(\theta+\eta)\right)\left[\frac{\lambda}{\delta}\left(1-S^*(\delta)\right)+\frac{\lambda}{(\theta+\eta)}\left(1-S_w^*(\theta+\eta)\right)\right]+\frac{\eta}{\delta}\left(1-S^*(\delta)\right)\right\}}{\left(R^*(\lambda)-\rho\right).}$$

(ii) Let Π be the steady-state probability that the server is busy,

$$\Pi=\Pi(1)=W_0\frac{\left(1-S^*(\delta)\right)}{\delta}\frac{\left\{pS_w^*(\theta+\eta)+\lambda\left(1-S_w^*(\theta+\eta)\right)\left(\frac{\lambda}{(\theta+\eta)}+R^*(\lambda)\right)+\eta R^*(\lambda)\right\}}{\left(R^*(\lambda)-\rho\right)}$$

(iii) Let W be the steady state probability that the server is at lower speed service,

$$W=W(1)=\frac{\lambda W_0\left(1-S_w^*(\theta+\eta)\right)}{(\theta+\eta)}.$$

(iv) Let W_{wb} be the steady state probability that the server is on WVB,

$$W_{wb}=W+W_0=\frac{W_0\left((\theta+\eta)+\lambda\left(1-S_w^*(\theta+\eta)\right)\right)}{(\theta+\eta)}.$$

(v) Let F_f be the steady state probability of server failure,

$$F_f = \delta \times \Pi(1) = \frac{W_0\left(1-S^*(\delta)\right)\left\{pS_w^*(\theta+\eta)+\lambda\left(1-S_w^*(\theta+\eta)\right)\left(\frac{\lambda}{(\theta+\eta)}+R^*(\lambda)\right)+\eta R^*(\lambda)\right\}}{\left(R^*(\lambda)-\rho\right)}.$$

4.2. Mean system size and orbit size

(i) The expected number of customers in the orbit (L_q) is obtained by differentiating equation 3.32 with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z)$$

(ii) The expected number of customers in the system (L_s) is obtained by differentiating equation 3.31 with respect to z and evaluating at z = 1

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z)$$

(iii) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are found using Little's formula

$$W_s = \frac{L_s}{\lambda} \quad \text{and} \quad W_q = \frac{L_q}{\lambda}.$$

4.3 Mean busy period and mean busy cycle

Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions. The results follow directly by applying the argument of an alternating renewal process which leads to

$$P_0 = \frac{E(T_0)}{E(T_b)+E(T_0)};\ E(T_b) = \frac{1}{\lambda}\left(\frac{1}{P_0}-1\right) \text{ and } E(T_c) = \frac{1}{\lambda P_0} = E(T_0)+E(T_b). \tag{4.1}$$

where T_0 is the time length that the system is in empty state. Since the inter-arrival time between two customers follows exponential distribution with parameter λ, we have $E(T_0) = \left(1/\lambda\right)$. Inserting equation 3.31 into equation 4.1 and using the above results, we can get

$$E(T_b) = \frac{1}{\lambda}\frac{\left\{\frac{\delta}{\lambda}\left(q+\frac{\delta}{\lambda}\right)R^*(\lambda)+\frac{\delta}{(\theta+\delta)}\left(1-S_w^*(\theta+\delta)\right)\left(q+1-S_w^*(\theta+\delta)\left(1-R^*(\lambda)\right)\right)-\frac{\delta}{\alpha}S_w^*(\theta+\delta)\left(1-S_b^*(\delta)\right)\right\}}{R^*(\lambda)-p-\frac{\lambda}{\delta}\left(1-S_b^*(\delta)\right)}. \tag{4.2}$$

$$E(T_c) = \frac{\left\{\left(q+\frac{\delta}{\lambda}\right)R^*(\lambda)+\frac{\lambda}{(\theta+\delta)}\left(1-S_w^*(\theta+\delta)\right)\left(q+1-S_w^*(\theta+\delta)\left(1-R^*(\lambda)\right)\right)-\frac{\lambda}{\alpha}S_w^*(\theta+\delta)\left(1-S_b^*(\delta)\right)\right\}}{\delta\left(R^*(\lambda)-p-\frac{\lambda}{\alpha}\left(1-S_b^*(\delta)\right)\right)}. \tag{4.3}$$

5. Special Cases

We present three special cases of our model.

Case (i): No negative arrival, No feedback, No repair, and No working breakdown

Let $\alpha = \delta = c = 0$; our model can be reduced to an M/G/1 retrial queue with working vacations. The results coincide with the results of Gao *et al.* (2014).

Case (ii): No negative arrival, No feedback, No repair, and No working breakdown

Let $(\alpha, \delta, \theta, p) \to (0, 0, 0, 0)$; our model can be reduced to $M/G/1$ retrial queue with single working vacation. This model results coincide with Arivudainambi *et al.* (2014).

Case (iii): No negative arrival, No feedback, No repair, No working breakdown and vacation

Let $(\alpha, \delta, \theta, p) \to (0, 0, 0, 0)$; our model can be reduced to M/G/1 retrial queue with general retrial times. The following result coincides with the results of Gomes Corral (1999).

6. Numerical Examples

In this section, based on the results obtained, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. Without loss of generality, we assume that the retrial times, service times, vacation times, and repair times are exponential, 2-stage Erlang, and 2-stage hyper-exponential distributed with the parameters α, p, and θ. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition.

The tables give the computed values of various characteristics of our model, i.e. probability that the server is idle (P_0), the mean orbit size (L_q), probability that server is idle during retrial time (R), busy (Π), and working breakdown (W). The exponential distribution is $f(x)=\phi e^{-\phi x}, x>0$, Erlang distribution of order 2 is $f(x)=\phi^2 xe^{-\phi x}, x>0$, and the hyper-exponential distribution of order 2 is $f(x)=c\phi e^{-\phi x}+(1-c)\phi^2 e^{-\phi^2 x}, x>0$ and $(0<c<1)$.

In Table 1, we show the effect of failure rate (α) on P_0 and L_q. As the system failure rate increases, the probability of no patients in the buffer increases and the number of patients waiting in the buffer decreases. That is, if the negative arrival rate increases, the probability that the server is idle (P_0) increases, the coefficient of variation (ρ) decreases, the mean orbit size (L_q) increases and probability that the server is busy (Π) also increases for the values of $\lambda=2,\ a=3,\ \mu_b=4,\ \mu_w=2,\ p=0.5,\ \theta=1,\ \delta=4$;

Table 1. Effect of failure rate (α) on P_0 and L_q.

Failure rate (α)	*Exp*			*Erlang*			*Hyp-Exp*		
	P_0	L_q	$\Pi(1)$	P_0	L_q	$\Pi(1)$	P_0	L_q	$\Pi(1)$
0.50	0.4655	0.4753	0.0468	0.0671	4.1174	0.0911	0.5113	0.4118	0.0600
0.60	0.4666	0.8583	0.0555	0.0695	7.2554	0.1072	0.5122	0.7498	0.0709
0.70	0.4678	1.4096	0.0639	0.0721	11.6741	0.1227	0.5132	1.2382	0.0814
0.80	0.4690	2.1658	0.0721	0.0748	17.5734	0.1376	0.5142	1.9101	0.0916
0.90	0.4702	3.1663	0.0801	0.0777	25.1399	0.1519	0.5152	2.8011	0.1015

In Table 2 with the increase of feedback probability (p), then the probability that the server is idle (P_0) decreases, the coefficient of variation (ρ) increases, and the mean orbit size (L_q) increases. In other words, as the number of patients increases for retransmission, the probability of no patients in the waiting line decreases and the number of packets in the line increases for the values of $\lambda=2,\ a=3,\ \mu_b=4,\ \mu_w=2,\ \theta=1,\ p=0.5,\ \alpha=0.3,\ \delta=4$;

Table 2. Effect of feedback probability (p) on P_0 and L_q.

Feedback probability (p)	*Exp*			*Erlang*			*Hyp-Exp*		
	P_0	L_q	$R(1)$	P_0	L_q	$R(1)$	P_0	L_q	$R(1)$
0.20	0.8249	9.6753	0.0842	0.6608	14.1653	0.2025	0.7922	10.1492	0.1098
0.30	0.7887	10.3047	0.1135	0.6168	15.2601	0.2406	0.7530	10.8233	0.1427
0.40	0.7465	11.0918	0.1476	0.5689	16.6287	0.2821	0.7083	11.6602	0.1801
0.50	0.6967	12.1088	0.1879	0.5165	18.3939	0.3276	0.6570	12.7318	0.2231
0.60	0.6371	13.4826	0.2361	0.4590	20.7668	0.3774	0.5973	14.1619	0.2731

Table 3 shows that when the vacation rate (θ) increases, then the probability that the server is idle (P_0) increases, the coefficient of variation (ρ) decreases, the mean orbit size (L_q) decreases and the probability that the server is busy in working vacation (W) also decreases for the values of $\lambda=2,\ a=3,\ \mu_b=4,\ \mu_w=2,\ p=0.5,\ \alpha=0.3,\ \delta=4$;

Table 3. Effect of vacation rate (θ) on P_0 and L_q.

Vacation rate (θ)	*Exp*			*Erlang*			*Hyp-Exp*		
	P_0	L_q	$W(1)$	P_0	L_q	$W(1)$	P_0	L_q	$W(1)$
4.00	0.6311	2.5172	0.0631	0.1988	9.1100	0.0957	0.5595	2.7898	0.0452
5.00	0.6468	2.4839	0.0505	0.2287	8.0546	0.0765	0.5724	2.7526	0.0355
6.00	0.6574	2.4633	0.0421	0.2486	7.4984	0.0638	0.5808	2.7301	0.0292
7.00	0.6649	2.4494	0.0361	0.2629	7.1557	0.0547	0.5867	2.7150	0.0248
8.00	0.6705	2.4393	0.0316	0.2736	6.9236	0.0478	0.5910	2.7042	0.0215

For the effects of the parameters $\lambda,\ r,\ \theta,\ \eta,\ \mu,\ \mu_w$ *and* δ on the system performance measures, three dimensional graphs are illustrated in Figures 1–4. In Figure 1, we see that the behavior of the mean orbit size (L_q) decreases as the values of

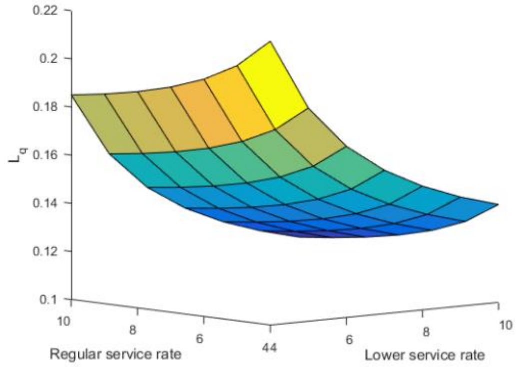

Figure 1. Lq versus μ and μw.

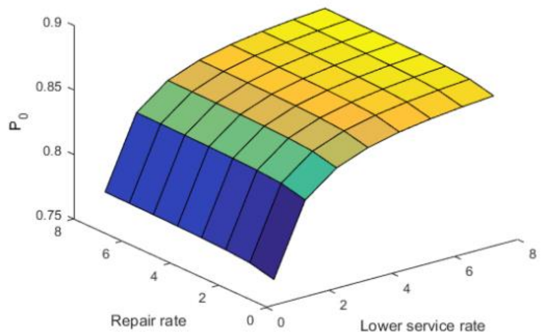

Figure 2. P_0 versus μ_w and η.

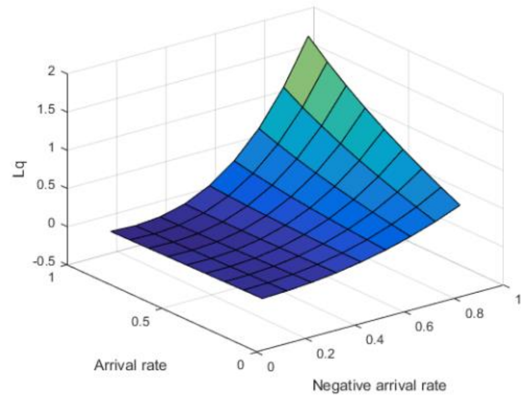

Figure 3. L_q versus λ and α.

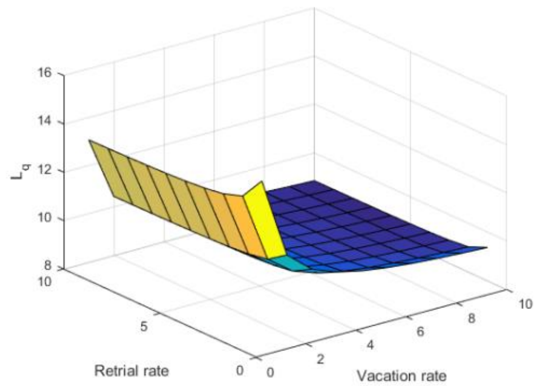

Figure 4. L_q versus r and θ.

the lower service rate (μ_w) and regular service rate (μ) increase. The surface displays an upward trend as expected for increasing the value of the lower speed service rate (μw) and repair rate (η) against the idle probability P_0 in Figure 2. From Figure 3, the surface displays a downward trend as expected to increase the value of arrival rate (λ) and negative arrival rate (δ) against the mean orbit size L_q in Figure 7. In Figure 4, we examine the behaviour of the mean orbit size (L_q) decreases for increasing the value of vacation rate (θ) and retrial rate (r).

From the above numerical examples, we can find the influence of parameters on the performance measures in the system.

7. Conclusions

We studied an M/G/1 retrial *G*-queue with feedback under working vacations and working breakdowns (M/G/1/WVB). By applying the PGF approach and the supplementary variable technique, the PGFs for the numbers of customers in the system and its orbit when it is free, regular busy, and on lower speed service are derived. Various system performance measures and some important special cases were discussed. The explicit expressions for the average queue length of orbit and system were obtained. Finally, some numerical examples were presented to study the impact of the system parameters. The novelty of this investigation is the introduction of working breakdown queueing models in the presence of retrial queues with multiple working vacations. This proposed model has potential practical real life applications in a production ordering system to enhance the performance of the production facility and to stop a production facility from becoming overloaded, in computer processing systems and in medical service systems for telephone consultation.

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Appendix A

The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$, where $\rho = p + \frac{\lambda}{\delta}\left(1 - S^*(\delta)\right)$.

Proof. To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (Pakes, 1969), which states that the chain $\{Z_n;\ n \in N\}$ is an irreducible and aperiodic a Markov chain is ergodic if there exists a non-negative function *f(j)*, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E\left[f(z_{n+1}) - f(z_n) / z_n = j\right]$ is finite for all $j \in N$ and $\psi_j \le -\varepsilon$ for all $j \in N$, except perhaps for a finite number *j*'s. In our case, we consider the function $f(j) = j$. then we have

$$\psi_j = \begin{cases} \rho - 1, & \text{if } j = 0, \\ \rho - R^*(\lambda), & \text{if } j = 1, 2... \end{cases}$$

Clearly the inequality $\rho < R^*(\lambda)$ is a sufficient condition for ergodicity.

To prove the necessary condition, as noted in Sennott *et al.* (1983), Markov chain $\{Z_n;\ n \ge 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \ge 0$ and there exits $j_0 \in$ N such that $\psi_j \ge 0$ for $j \ge j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n;\ n \in N\}$. Then $\rho \ge R^*(\lambda)$ implies non-ergodicity of the Markov chain.

Appendix B

Lemma 3.1. If $\rho < R^*(\lambda)$, the equation $z - (pz + q)\left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(S^*\left(A(z)\right) + S(z)\right)$ has no roots in the range $0 < z < 1$ and has the minimal nonnegative root $z = 1$.

Proof. We only need to prove that

$$u(z) \stackrel{\Delta}{=} (pz + q)\left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(S^*\left(A(z)\right) + S(z)\right)$$

is a probability generating function of the number of customers that arrive in the system. Denote by U the time period from the epoch a service completion occurs, leaving the orbit non-empty, to the next service completion epoch, by N_U the number of primary customers that arrive during U and define

$$u_j(t)dt = P\left(t < U \le t + dt,\ N(U) = j\right).$$

Then, $u_j(t) = e^{-\lambda t}\alpha(t) * a_j(t) + (1-\delta_{j,0})\lambda e^{-\lambda t}(1-R(t)) * a_{j-1}(t),\ j = 0,1,2...$ where * means convolution, $\alpha(t)$ is the p.d.f. of inter-retrial times, $b(t)$ is the p.d.f. of normal service times and $a_j(t)dt\ = e^{-\lambda t}\dfrac{(\lambda t)^j}{j!}b(t).$ Denote by $N_U(z)$ the probability generating function of N_U, we have that

$$
\begin{aligned}
N_U(z) &= \sum_{j=0}^{\infty} z^j \int_0^{\infty} u_j(t)dt \\
&= \sum_{j=0}^{\infty} z^j \int_0^{\infty} \left(e^{-\lambda t}\alpha(t) * a_j(t) + (1-\delta_{j,0})\lambda e^{-\lambda t}(1-R(t)) * a_{j-1}(t)\right)dt \\
&= \ (pz+q)\left(R^*(\lambda) + z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right) + S(z)\right) \\
&= \ u(z),
\end{aligned}
$$

which proves the expected result that $u(z) \overset{\Delta}{=} (pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)$ is exactly a probability generating function. From assumption $\rho < R^*(\lambda),$ we have $E[N_u] = \dfrac{d}{dz}u(z)|_{z=1} = 1-\left(R^*(\lambda)-\rho\right) < 1.$ and the convex function $u(z)$ is a monotonically increasing function of z for $0 \le z \le 1,$ and $u(0)\ = P\left(N_U = 0\right) < 1,\ u(1) = 1.$ So we can easily prove the expected result of Lemma 3.1.

Then for $\rho < R^*(\lambda),\ \ z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(S^*\left(A(z)\right)+S(z)\right)$ never vanishes in the range $0 < z < 1.$