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L-open sets and L^*-open sets

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Abstract

As new classes of generalized open sets and generalized continuous functions, the notions of L-open sets, L^*-open sets, L-continuous functions, L^*-continuous functions and weakly L^*-continuous functions are introduced and investigated. It is proved that L-open sets form a topology and decomposition of continuity is obtained.

Keywords: Lindelof, maximal Lindelof, LC space, generalized open set, continuous functions

1. Introduction

Let (X, τ) be a topological space and let $A \subseteq X$. Denote the set of all countable (resp. Lindelof, compact) subsets of X by $\omega(X, \tau)$ (resp. $L(X, \tau), C(X, \tau)$). A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω-closed sets and ω-open sets as follows: A is called ω-closed (Hdeib, 1982) if it contains all its condensation points. The complement of an ω-closed set is called ω-open. Denote the set of all ω-open sets in (X, τ) by τ_ω. Al-Zoubi & Al-Nashef (2003), proved that (X, τ_ω) is a topological space, $\tau \subseteq \tau_\omega$ and $\{U - C: U \in \tau \text{ and } C \in \omega(X, \tau)\}$ forms a base for τ_ω. A is called co-compact open (notation: coc-open set) (Al Ghour & Samarah, 2012) if for every $x \in A$, there exists an open set $U \subseteq X$ and $K \in C(X, \tau)$ such that $x \in U - K \subseteq A$. Denote the set of all coc-open sets in (X, τ) by τ_K. Al Ghour & Samarah (2012) investigated coc-open sets, and proved that (X, τ_K) is a topological space, $\tau \subseteq \tau_K$ and $\{U - K: U \in \tau \text{ and } K \in C(X, \tau)\}$ forms a base for τ_K and gave a decomposition theorem of continuity. It is not difficult to give an example to show that ω-open sets are not coc-open sets and vice versa, in general. The first main goal of this research is to introduce and investigate L-open sets as a new class of sets that is strictly containing both ω-open sets and coc-open sets. The second main goal of this research is to introduce and study L^*-open sets as a new class of sets that is containing strictly the open sets and give with the L-open sets a decomposition of open sets. The third main goal of this paper is to define three new classes of continuous functions via L-open and L^*-open sets.

Throughout this paper, $\mathbb{R}$, and $\mathbb{Q}$ denote the set of real numbers and the set of rational numbers, respectively. For a subset A of a topological space (X, τ), $Cl_\tau(A)$ and $Int_\tau(A)$ will denote the closure of A and the interior of A, respectively. Also, we write τ_A to denote the relative topology on A when A is nonempty. For a nonempty set X, we will denote the discrete topology on X and the indiscrete topology on X by τ_{disc} and τ_{ind}, respectively. Finally, τ_u will denote the usual topology on $\mathbb{R}$.

2. L-Open Sets

Definition 2.1. Let (X, τ) be a topological space and $A \subseteq X$. A point x in X is in L-closure of A ($x \in Cl_L(A)$) if $(U - H) \cap A \neq \emptyset$ for any $U \in \tau$ and $H \in L(X, \tau)$ with $x \in U - H$. A is called L-closed if $Cl_L(A) = A$. The complement of an L-closed set is called an L-open set. Denote the family of all L-open subsets of (X, τ) by τ_L.

The next result follows directly from Definition 2.1:

Proposition 2.2. A subset A of a topological space (X, τ) is L-open if and only if for every $x \in A$, there exists an open set $U \subseteq X$ and $H \in L(X, \tau)$ such that $x \in U - H \subseteq A$.

For a topological space (X, τ) denote the family of L-open sets $\{U - H: U \in \tau \text{ and } H \in L(X, \tau)\}$ by $\mathcal{B}_L(\tau)$.

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Theorem 2.3. Let (X, τ) be a topological space. Then the collection τ_L forms a topology on X.

Proof. By the definition one has directly that $\emptyset \in \tau_L$. To see that $X \in \tau_L$, let $x \in X$, take $U = X$ and $H = \emptyset$. Then $x \in U - H \subseteq X$.

Let $U_1, U_2 \in \tau_L$ and let $x \in U_1 \cap U_2$. For each $i = 1,2$, we find an open set V_i and $H_i \in L(X, \tau)$ such that $x \in V_i - H_i \subseteq U_i$. Take $V = V_1 \cap V_2$ and $H = H_1 \cup H_2$. Then V is open, $H \in L(X, \tau)$, and $x \in V - H \subseteq U_1 \cap U_2$. It follows that $U_1 \cap U_2$ is L-open.

Let $\{U_\alpha : \alpha \in \Delta\}$ be a collection of L-open subsets of (X, τ) and $x \in \bigcup_{\alpha\in\Delta} U_\alpha$. Then there exists $\alpha_0 \in \Delta$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is L-open, then there exists an open set V and $H \in L(X, \tau)$, such that $x \in V - H \subseteq U_{\alpha_0}$. Therefore, we have $x \in V - H \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$. Hence, $\bigcup_{\alpha\in\Delta} U_\alpha$ is L-open.

Remark 2.4. Let (X, τ) be a topological space. Then

(a) The collection $\mathcal{B}_L(\tau)$ forms a base for τ_L.

(b) The collection $\tau \cup \{X - H : H \in L(X, \tau)\}$ forms a subbase for τ_L.

(c) $\tau_\omega \cup \tau_K \subseteq \tau_L$.

Remark 2.5. The inclusion in Remark 2.4 (c) is not equality in general, to see this, let $X = \mathbb{R}$ and $\tau = \tau_u$. Then $\{1\} \in \tau_L - (\tau_\omega \cup \tau_K)$ and so $\tau_\omega \cup \tau_K \neq \tau_L$.

The following is an example of a topological space (X, τ) such that $\tau_\omega = \tau_L$ and $\tau_K \neq \tau_L$:

Example 2.6. Let $X = \mathbb{R}$ and $\tau = \{X\} \cup \{U \subseteq X : 1 \notin U\}$. Then $C(X, \tau) = \{K \subseteq X : 1 \in K\} \cup \{K \subseteq X : 1 \notin K \text{ and } K \text{ is finite}\}$ and

$$L(X, \tau) = \{H \subseteq X : 1 \in K\} \cup \{H \subseteq X : 1 \notin H \text{ and } H \text{ is countable}\}.$$

Thus, $\tau_K = \tau \cup \{U \subseteq X : 1 \in U \text{ and } X - U \text{ is finite}\}$ and

$\tau_L = \tau \cup \{U \subseteq X : 1 \in U \text{ and } X - U \text{ is countable}\}$.

Clearly, $\tau_\omega = \tau_L$ and $\tau_K \neq \tau_L$.

Definition 2.7. (Mukherji & Sarkar, 1979) A topological space (X, τ) is called an LC space if each Lindelof subset is closed.

Theorem 2.8. Let (X, τ) be a topological space. Then the following are equivalent:

(a) (X, τ) is LC.

(b) $\tau = \mathcal{B}_L(\tau)$.

(c) $\tau = \tau_\omega = \tau_L$.

Proof. (a) $\Rightarrow$ (b) $\tau \subseteq \mathcal{B}_L(\tau)$ is obvious. Let $U - H \in \mathcal{B}_L(\tau)$ where $U \in \tau$ and $H \in L(X, \tau)$. Since (X, τ) is LC, then H is closed and hence $U - H \in \tau$. It follows that $\mathcal{B}_L(\tau) \subseteq \tau$.

(b) $\Rightarrow$ (c) By Remark 2.4 (c), it is sufficient to see that $\tau_L \subseteq \tau$. By Remark 2.4 (a), $\mathcal{B}_L(\tau)$ is a base for τ_L. Thus by (b), $\tau_L \subseteq \tau$.

(c) $\Rightarrow$ (a) If $H \in L(X, \tau)$, then $X - H \in \tau_L$ and by (c), $X - H \in \tau$. Therefore, H is closed in X.

Proposition 2.9. For any topological space (X, τ), (X, τ_L) is LC.

Proof. Let $H \in L(X, \tau_L)$. Since $\tau \subseteq \tau_L$, then $L(X, \tau_L) \subseteq L(X, \tau)$ and so $H \in L(X, \tau)$. Thus, we have $X - H \in \tau_L$, and hence H is closed in (X, τ_L).

Corollary 2.10. For any topological space (X, τ), $(\tau_L)_L = \tau_L$.

Proof. This is an immediate consequence of Theorem 2.8 and Proposition 2.9.

As defined in Cameron (1971), a topological space (X, τ) is maximal Lindelof if (X, τ) is Lindelof and there exists no strictly finer Lindelof topology on X.

Theorem 2.11. (Cameron, 1971) A Lindelof topological space is maximal Lindelof if and only if it is LC.

Corollary 2.12. Let (X, τ) be a topological space. Then (X, τ_L) is maximal Lindelof if and only if (X, τ_L) is Lindelof.

Proof. This is an immediate consequence of Proposition 2.9 and Theorem 2.11.

Corollary 2.13. A Lindelof topological space (X, τ) is maximal Lindelof if and only if there is a topology σ on X such that $\sigma_L = \tau$.

Proposition 2.14. If (X, τ) is hereditarily Lindelof, then $\tau_L = \tau_{disc}$.

Proof. For every $x \in X$, $X - \{x\}$ is Lindelof and so $\{x\} = X - (X - \{x\}) \in \mathcal{B}_L(\tau) \subseteq \tau_L$. It follows that $\tau_L = \tau_{disc}$.

As a conclusion of Proposition 2.14, for any second countable topological space (X, τ), $\tau_L = \tau_{disc}$.

The following is an example a nondiscrete topological space which shows that the converse of Proposition 2.14 is not true in general:

Example 2.15. Let $X = \mathbb{R}$ and $\tau = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : \mathbb{Q} \subseteq U\}$. It is not difficult to see that $L(X, \tau) = \{H \subseteq X : H - \mathbb{Q} \text{ is countable}\}$. For every $x \in X$, take $H = \mathbb{Q}$ and $U = \mathbb{Q} \cup \{x\}$. Then $H \in C(X, \tau), U \in \tau$ and $\{x\} = U - H$. This shows that $\tau_L = \tau_{disc}$. On the other hand, it is clear that (X, τ) is not hereditarily Lindelof.

Theorem 2.16. Let (X, τ) be a topological space and A be a nonempty subset of X. Then $(\tau_{|A})_L \subseteq \tau_{L_{|A}}$.

Proof. Let $B \in (\tau_{|A})_L$ and $x \in B$. Then there exists $V \in \tau_{|A}$ and a Lindelof set $H \subseteq A$ such that $x \in V - H \subseteq B$. Choose $U \in \tau$ such that $V = U \cap A$. Now $U - H \in \tau_L$ implies that $(U - H) \cap A \in \tau_{L_{|A}}$. Hence, $B \in \tau_{L_{|A}}$.

Question 2.17. Let (X, τ) be a topological space and A be a nonempty subset of X. Is it true that $(\tau_{|A})_L = \tau_{L_{|A}}$?

The following result is a partial answer for Question 2.17:

Theorem 2.18. Let (X, τ) be a topological space and A be a nonempty closed set in (X, τ). Then $(\tau_{|A})_L = \tau_{L_{|A}}$.

Proof. By Theorem 2.16, $(\tau_{|A})_L \subseteq \tau_{L_{|A}}$. Conversely, let $B \in \tau_{L_{|A}}$ and $x \in B$. Choose $C \in \tau_L$ such that $B = C \cap A$. Choose $U \in \tau$ and $H \in L(X,\tau)$ such that $x \in U - H \subseteq C$. Thus, we have $x \in (U \cap A) - (H \cap A) \subseteq B, U \cap A \in \tau_{|A}$ and $H \cap A \in L(A, \tau_{|A})$. It follows that $B \in (\tau_{|A})_L$.

Theorem 2.19. If $f: (X,\tau) \to (Y,\sigma)$ is injective, open, and continuous, then $f: (X,\tau_L) \to (Y,\sigma_L)$ is open.

Proof. Let $G = U - H$ where $U \in \tau$ and $H \in L(X,\tau)$ be a basic element for τ_L. Since f is injective, $f(G) = f(U) - f(H)$. Since $f: (X,\tau) \to (Y,\sigma)$ is open, $f(U) \in \sigma$. And since $f: (X,\tau) \to (Y,\sigma)$ is continuous, $f(H) \in L(Y,\sigma)$. This ends the proof.

In Theorem 2.19, the condition 'continuous' cannot be dropped as the following example shows:

Example 2.20. Consider $f: (\mathbb{R}, \tau_{ind}) \to (\mathbb{R}, \tau)$, where τ as in Example 2.6 and f is the identity function. Then f is bijective and open. On the other hand, since $(\mathbb{R}, \tau_{ind})$ is hereditarily Lindelof, we have $(\tau_{ind})_L = \tau_{disc}$. Since by Example 2.6 $\tau_L \neq \tau_{disc}$, $f: (\mathbb{R}, (\tau_{ind})_L) \to (\mathbb{R}, \tau_L)$ is not open.

3. L^*-Open Sets

For any topological space (X,τ) and $A \subseteq X$, it is clear that $Cl_L(A) = Cl_{\tau_L}(A)$. For simplicity, from now on we will use only $Cl_{\tau_L}(A)$. Also, it is clear that $Cl_{\tau_L}(A) \subseteq Cl_\tau(A)$. The following example shows that $Cl_\tau(A) \neq Cl_{\tau_L}(A)$ in general.

Example 3.1. Consider $(X,\tau) = (R, \tau_u)$. By Proposition 2.14, $\tau_L = \tau_{disc}$. Therefore, $Cl_{\tau_L}(A)(\mathbb{Q}) = \mathbb{Q}$ but $Cl_\tau(\mathbb{R}) = \mathbb{R}$.

The following definition is reasonable:

Definition 3.2. Let (X,τ) be a topological space and $A \subseteq X$.

(a) A is called L^*-closed if $Cl_\tau(A) = Cl_{\tau_L}(A)$.

(b) A is called L^*-open if $X - A$ is L^*-closed.

Theorem 3.3. For a topological space (X,τ), we have the following

(a) $\emptyset$ and X are L^*-closed sets in (X,τ).

(b) Every closed set in (X,τ) is L^*-closed set.

(c) Every finite union of L^*-closed sets in (X,τ) is L^*-closed set in (X,τ).

Proof. (a) It is obvious.

(b) Let A be closed in (X,τ). Then $Cl_{\tau_L}(A) \subseteq Cl_\tau(A) = A$. It follows that $Cl_{\tau_L}(A) = Cl_\tau(A) = A$ and hence A is an L^*-closed set in (X,τ).

(c) Suppose $\{A_i \subseteq X: 1 \leq i \leq n\}$ is a finite collection of L^*-closed sets. Then for every $1 \leq i \leq n$, $Cl_\tau(A_i) = Cl_{\tau_L}(A_i)$. Thus,

$$Cl_\tau(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} Cl_\tau(A_i)$$

$$= \bigcup_{i=1}^{n} Cl_{\tau_L}(A_i)$$

$$= Cl_{\tau_L}(\bigcup_{i=1}^{n} A_i).$$

It follows that $\bigcup_{i=1}^{n} A_i$ is L^*-closed.

Corollary 3.4. For a topological space (X,τ), we have the following

(a) $\emptyset, X$ are L^*-open sets in (X,τ).

(b) Every open set in (X,τ) is L^*-open set.

(c) Every finite intersection of L^*-open sets in (X,τ) is L^*-open in (X,τ).

The following theorem characterizes L^*-open sets:

Proposition 3.5. A subset A of a topological space (X,τ) is L^*-open if and only if $Int_\tau(A) = Int_{\tau_L}(A)$.

Proof. A is L^*-open if and only if $X - A$ is L^*-closed if and only if $Cl_\tau(X - A) = Cl_{\tau_L}(X - A)$ if and only if $X - Cl_\tau(X - A) = X - Cl_{\tau_L}(X - A)$ if and only if $Ext_\tau(X - A) = Ext_{\tau_L}(X - A)$ if and only if $Int_\tau(A) = Int_{\tau_L}(A)$.

The union of two L^*-open sets one of them is open need not to be L^*-open in general, as the following example shows:

Example 3.6. Consider the topological space as in Example 2.6. Let $A = (0, \infty)$ and $B = (-\infty, 0)$. Then A is open, also since $Int_\tau(B) = Int_{\tau_L}(B) = B - \{1\}$, then B is L^*-open. On the other hand, $Int_\tau(A \cup B) = \mathbb{R} - \{0,1\}$ but $Int_{\tau_L}(A \cup B) = \mathbb{R} - \{0\}$ which implies that $A \cup B$ is not L^*-open

Example 3.6 shows also that the converse of Corollary 3.4 (b) is not true in general.

Let (X,τ) be a topological space. Denote the family of all L^*-open sets in (X,τ) by $\mathcal{B}_{\tau_{L^*}}$. By Corollary 3.4 (b), $\tau \subseteq \mathcal{B}_{\tau_{L^*}}$. According to Example 3.6, $\mathcal{B}_{\tau_{L^*}}$ does not form a topology on X in general. By Corollary 3.4, $\mathcal{B}_{\tau_{L^*}}$ forms a base for some topology on X. Denote the topology on X which has $\mathcal{B}_{\tau_{L^*}}$ as a base by τ_{L^*}.

Definition 3.7. (Henriksen & Woods, 1988) A topological space (X,τ) is called anti-locally Lindelof if any Lindelof subset of X has empty interior.

If S is the Sorgenfrey line and (X,τ) is the Cartesian product topological space $S \times S$, then (X,τ) is anti-locally Lindelof (see Corollary 2.4 of (Henriksen & Woods, 1988)).

Theorem 3.8. Open sets in an anti-locally Lindelof topological space are L^*-closed sets.

Proof. Let (X,τ) be a topological space and $A \in \tau$. Suppose on the contrary that there is $x \in Cl_\tau(A) - Cl_{\tau_L}(A)$. Since $x \notin Cl_{\tau_L}(A)$, there exists $B \in \tau_L$ such that $x \in B$ and $B \cap A = \emptyset$. Choose $U \in \tau$ and $H \in L(X,\tau)$ such that $x \in U - H \subseteq B$. Thus we have $U \cap A \subseteq H$. Since $x \in Cl_\tau(A)$, then $U \cap A \neq \emptyset$

and hence $Int_\tau(H) \neq \emptyset$. This contradicts the assumption that (X,τ) is anti-locally Lindelof.

Recall that a topological space (X,τ) is locally indiscrete if every open set in (X,τ) is closed.

Corollary 3.9. If (X,τ) is anti-locally Lindelof such that $\mathcal{B}_{\tau_{L^*}} = \tau$, then (X,τ) is locally indiscrete.

Proof. Let A be an open set in (X,τ). Then by Theorem 3.8, A is L^*-closed. So $X - A$ is L^*-open and by assumption, $X - A$ is open. Thus, A is closed.

In Theorem 3.8 the condition 'anti-locally Lindelof' cannot be dropped as we can see by the following example:

Example 3.10. Consider $(X,\tau) = (\mathbb{R},\tau_u)$ as in Example 3.1. Take $A = (0,1)$, then $Cl_{\tau_L}(A) = A$ but $Cl_\tau(A) = [0,1]$.

The following example shows that the converse of Corollary 3.4 (b) is not true in general, even if we add the condition 'anti-locally Lindelof' on the topological space:

Example 3.11. Let (X,τ) be the Cartesian product topological space $S \times S$ where S is the Sorgenfrey line, then as we mentioned above (X,τ) is anti-locally Lindelof. If every L^*-open set in (X,τ) is open, then by Corollary 3.9, (X,τ) is locally indiscrete. We are going to show that (X,τ) is not locally indiscrete. Let $A = (0,\infty) \times \mathbb{R}$. Then $A \in \tau$, on the other hand, if A is closed, then $X - A = (-\infty,0] \times \mathbb{R} \in \tau$ and so there exist $a,b,c,d \in \mathbb{R}$ such that $(0,0) \in [a,b) \times [c,d) \subseteq (-\infty,0] \times \mathbb{R}$ which implies that $b > 0$ and $((b/2),0) \in (-\infty,0] \times \mathbb{R}$, which is not true. It follows that (X,τ) is not locally indiscrete.

The following is a decomposition theorem of openness in terms of L-openness and L^*-openness:

Theorem 3.12. For any topological space (X,τ), $\tau = \tau_L \cap \mathcal{B}_{\tau_{L^*}}$.

Proof. By Proposition 2.4 (c), $\tau \subseteq \tau_\omega \subseteq \tau_L$, and by Corollary 3.4 (b), $\tau \subseteq \mathcal{B}_{\tau_{L^*}}$. It follows that $\tau \subseteq \tau_L \cap \mathcal{B}_{\tau_{L^*}}$. Conversely, let $A \in \tau_L \cap \mathcal{B}_{\tau_{L^*}}$. Since A is L^*-open, then $Int_\tau(A) = Int_{\tau_L}(A)$. Also, since A is L-open then $Int_{\tau_L}(A) = A$. It follows that $Int_\tau(A) = A$ and hence $A \in \tau$. It follows that $\tau_L \cap \mathcal{B}_{\tau_{L^*}} \subseteq \tau$.

Corollary 3.13. For any topological space $(X,\tau), \tau = \tau_L \cap \tau_{L^*}$.

Corollary 3.14. If (X,τ) is a hereditarily Lindelof topological space, then $\mathcal{B}_{\tau_{L^*}} = \tau$.

Proof. By Proposition 2.16, $\tau_L = \tau_{disc}$. Theorem 3.12 ends the proof.

Corollary 3.15. If (X,τ) is a second countable topological space, then $\mathcal{B}_{\tau_{L^*}} = \tau$.

Proposition 3.16. If (X,τ) is an LC topological space, then $\mathcal{B}_{\tau_{L^*}} = \tau_{disc}$.

Proof. Let $A \subseteq X$. By Theorem 2.8, we have $Int_\tau(A) = Int_{\tau_L}(A)$ and hence A is L^*-open.

Corollary 3.17. For any topological space (X,τ), $(\tau_L)_{L^*} = \tau_{disc}$.

Proof. By Proposition 2.9, (X,τ_L) is LC. So by Proposition 3.16, we must have $(\tau_L)_{L^*} = \tau_{disc}$.

4. L-Continuity and L^*-Continuity

Definition 4.1. A function $f:(X,\tau) \to (Y,\sigma)$ is called L-continuous (resp. L^*-continuous, weakly L^*-continuous) at a point $x \in X$, if for every $V \in \sigma$ with $f(x) \in V$ there is $U \in \tau_L$ (resp. $U \in \mathcal{B}_{\tau_{L^*}}, U \in \tau_{L^*}$) such that $x \in U$ and $f(U) \subseteq V$. If f is L-continuous (resp. L^*-continuous, weakly L^*-continuous) at each point of X, then f is said to be L-continuous (resp. L^*-continuous, weakly L^*-continuous).

The following result follows immediately and its proof is left to the reader:

Proposition 4.2. A function $f:(X,\tau) \to (Y,\sigma)$ is

(a) L-continuous if and only if $f:(X,\tau_L) \to (Y,\sigma)$ is continuous.

(b) L^*-continuous if and only if for each $V \in \sigma, f^{-1}(V)$ is L^*-open.

(c) Weakly L^*-continuous if and only if $f:(X,\tau_{L^*}) \to (Y,\sigma)$ is continuous.

Corollary 4.3. Every L^*-continuous function is weakly L^*-continuous

Proof. The proof follows directly from parts (b) and (c) of Proposition 4.2.

The following example shows that Corollary 4.3 is not reversible in general. It is also an example of an L-continuous function that is not L^*-continuous.

Example 4.4. Let $(X,\tau), A$ and B be the topological space and the sets as in Example 3.6. Let $Y = \{a,b\}$ and $\sigma = \{\emptyset, Y, \{b\}\}$. Define $f:(X,\tau) \to (Y,\sigma)$ by $f(0) = a$ and $f(x) = b$ otherwise. As it can be seen from Example 3.6, $A \cup B \in \tau_{L^*} - \mathcal{B}_{\tau_{L^*}}$. Since $f^{-1}(\{b\}) = A \cup B \in \tau_{L^*} - \mathcal{B}_{\tau_{L^*}}$, then f is weakly L^*-continuous that is not L^*-continuous. Also, since $f^{-1}(\{b\}) = A \cup B \in \tau_L$, then f is L-continuous.

The following example shows that L-continuity does not imply even weakly L^*-continuity in general:

Example 4.5. Let $(X,\tau) = (\mathbb{R},\tau_u)$ and $(Y,\sigma) = (\mathbb{R},\tau_{disc})$. Since $(\mathbb{R},\tau_u)$ is second countable, then by Proposition 2.14 $(X,\tau_L) = (\mathbb{R},\tau_{disc})$ and by Corollary 3.15, $\mathcal{B}_{\tau_{L^*}} = \tau$. So the identity function $I:(X,\tau) \to (Y,\sigma)$ is an L-continuous function that is not weakly L^*-continuous.

The following is an example of an L^*-continuous function that is not L-continuous:

Example 4.6. Let (X,τ) be the topological space as in Example 2.6, $Y = \{a,b\}$ and $\sigma = \{\emptyset, Y, \{a\}\}$. Define $f:(X,\tau) \to (Y,\sigma)$ by $f(1) = a$ and $f(x) = b$ otherwise. Since $Int_\tau(\{1\}) = Int_{\tau_L}(\{1\}) = \emptyset$, then $f^{-1}(\{a\}) = \{1\} \in \mathcal{B}_{\tau_{L^*}}$ and hence f is L^*-continuous. On the other hand, since $f^{-1}(\{a\}) = \{1\} \notin \tau_L$, then f is not L-continuous.

In the following result we list two decomposition theorems for continuity:

Proposition 4.7. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following are equivalent:

(a) f is continuous.

(b) f is both L-continuous and L^*-continuous.

(c) f is both L-continuous and weakly L^*-continuous.

Proof. (a) ⇔ (b) The Proof follows immediately from Theorem 3.12 and Proposition 4.2.
(a) ⇔ (c) The Proof follows directly from Corollary 3.13 and Proposition 4.2.

As defined in (Hdeib, 1989), a function $f:(X,\tau)\to(Y,\sigma)$ is ω-continuous if the inverse image of each open set is ω-open.

Any ω-continuous function is L-continuous, however, the function I in Example 4.5 is L-continuous but not ω-continuous.

Remark 4.8. Let $f:(X,\tau)\to(Y,\sigma)$ be a function for which (X,τ) is LC. Then the following are equivalent:

(a) f is continuous.

(b) f is ω-continuous.

(c) f is L-continuous.

Proof. This is an immediate consequence of Theorem 2.8.

Remark 4.9. If $f:(X,\tau)\to(Y,\sigma)$ is any function for which (X,τ) is LC, then f is L^*-continuous.

Proof. This is an immediate consequence of Proposition 3.16.

Remark 4.10. Let $f:(X,\tau)\to(Y,\sigma)$ be a function with (X,τ) is hereditarily Lindelof (in particular, (X,τ) is second countable). Then the following are equivalent:

(a) f is continuous.

(b) f is L^*-continuous.

Proof. This follows from Corollary 3.14.

Theorem 4.11. (a) If $f:(X,\tau)\to(Y,\sigma)$ is L-continuous and $g:(Y,\sigma)\to(Z,\mu)$ is continuous, then $g\circ f:(X,\tau)\to(Z,\mu)$ is L-continuous.

(b) If $f:(X,\tau)\to(Y,\sigma)$ is L^*-continuous and $g:(Y,\sigma)\to(Z,\mu)$ is continuous, then $g\circ f:(X,\tau)\to(Z,\mu)$ is L^*-continuous.

(c) If $f:(X,\tau)\to(Y,\sigma)$ is weakly L^*-continuous and $g:(Y,\sigma)\to(Z,\mu)$ is continuous, then $g\circ f:(X,\tau)\to(Z,\mu)$ is weakly L^*-continuous.

Proof. (a) Since $f:(X,\tau)\to(Y,\sigma)$ is L-continuous, then by Proposition 4.2 (a), $f:(X,\tau_L)\to(Y,\sigma)$ is continuous. Therefore, $g\circ f:(X,\tau_L)\to(Z,\mu)$ is continuous. Again by Proposition 4.2 (a), it follows that $g\circ f:(X,\tau)\to(Z,\mu)$ is L-continuous.

(b) Let $W\in\mu$. Since $g:(Y,\sigma)\to(Z,\mu)$ is continuous, then $g^{-1}(W)\in\sigma$. Since $f:(X,\tau)\to(Y,\sigma)$ is L^*-continuous, then $f^{-1}(g^{-1}(W))\in\mathcal{B}_{\tau_{L^*}}$. Therefore, $(g\circ f)^{-1}(W)=f^{-1}(g^{-1}(W))\in\mathcal{B}_{\tau_{L^*}}$.

(c)Similar to that used in (a)

The following Example shows in Theorem 4.11 (a) that the continuity condition on g cannot be replaced by L-continuity:

Example 4.12. Let $X=\mathbb{R}, Y=\{a,b,c\}, Z=\{m,n\}, \tau$ be as in Example 2.6, $\sigma=\{\emptyset,Y,\{b\},\{b,c\}\}$, and $\mu=\{\emptyset,Z,\{m\}\}$. Define the function $f:(X,\tau)\to(Y,\sigma)$ by $f(x)=a$ if $x\in\{0,1\}$ and $f(x)=c$ otherwise, and define the function $g:(Y,\sigma)\to(Z,\mu)$ by $g(a)=g(b)=m$ and $g(c)=n$. Then f and g are L-continuous functions, but $g\circ f$ is not L-continuous since $(g\circ f)^{-1}(\{m\})=\{0,1\}\notin\tau_L$.

Questions 4.13. Is it true that the composition of two L^*-continuous (resp. weakly L^*-continuous) functions is L^*-continuous (resp. weakly L^*-continuous)?

Theorem 4.14. Let $\{f_\alpha:(X,\tau)\to(Y_\alpha,\sigma_\alpha):\alpha\in\Delta\}$ be a family of functions. Then the function $f:(X,\tau)\to(\prod_{\alpha\in\Delta}Y_\alpha,\tau_{prod})$ defined by $f(x)=(f_\alpha(x))_{\alpha\in\Delta}$ is L-continuous (resp. L^*-continuous, weakly L^*-continuous) if and only if for each $\alpha\in\Delta$, f_α is L-continuous (resp. L^*-continuous, weakly L^*-continuous).

Proof. Only we prove it for L^*-continuous, the others are similar. Suppose that f is L^*-continuous and let $\beta\in\Delta$. Then $f_\beta=\pi_\beta\circ f$ where $\pi_\beta:(\prod_{\alpha\in\Delta}Y_\alpha,\tau_{prod})\to(Y_\beta,\sigma_\beta)$ is the projection function on Y_β. Since π_β is continuous, then by Theorem 4.11 (b), f_β is L^*-continuous. Conversely, suppose for each $\alpha\in\Delta$, f_α is L^*-continuous. Let A be any subbasic open set of $(\prod_{\alpha\in\Delta}Y_\alpha,\tau_{prod})$, say $A=\pi_\beta^{-1}(U)$ for some $\beta\in\Delta$ and $U\in\sigma_\beta$. Then $f^{-1}(A)=f^{-1}\left(\pi_\beta^{-1}(U)\right)=\left(\pi_\beta\circ f\right)^{-1}(U)=f_\beta^{-1}(U)$. Since by assumption f_β is L^*-continuous, then we have $f_\beta^{-1}(U)\in\mathcal{B}_{\tau_{L^*}}$. By Proposition 4.2 (b), it follows that f is L^*-continuous.

Corollary 4.15. A function $f:(X,\tau)\to(Y,\sigma)$ is L-continuous (resp. L^*-continuous, weakly L^*-continuous) if and only if the graph function $h:(X,\tau)\to(X\times Y,\tau_{prod})$, given by $h(x)=(x,f(x))$ for every $x\in X$ is L-continuous (resp. L^*-continuous, weakly L^*-continuous).

Theorem 4.16. The restriction of an L-continuous function on a nonempty closed set is L-continuous.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be L-continuous and A be a nonempty closed set. Let $U\in\sigma$. Then $f^{-1}(U)\in\tau_L$. So by Theorem 2.18, we have $(f_{|A})^{-1}(U)=f^{-1}(U)\cap A\in(\tau_{|A})_L$. It follows that the restriction function $f_{|A}:(A,\tau_{|A})\to(Y,\sigma)$ is L-continuous.

Question 4.17. Is it true that the restriction of an L^*-continuous (resp. weakly L^*-continuous) function on a nonempty closed set is L^*-continuous (resp. weakly L^*-continuous)?

Proposition 4.18. Let (X,τ) be a topological space and let A and B be two L-closed sets in (X,τ) with $X=A\cup B$. If $f:(X,\tau)\to(Y,\sigma)$ is a function such that $f_{|A}:(A,\tau_{|A})\to$

(Y,σ) and $f_{|B} : (B,\tau_{|B}) \to (Y,\sigma)$ are L-continuous functions, then f is L-continuous.

Proof. It is similar to that used in Theorem 4.8 of (Al Ghour & Samarah, 2012) and left to the reader.

Corollary 4.19. Let (X,τ) be a topological space, A and B be two closed sets in (X,τ) with $X = A \cup B$ and $f: (X,\tau) \to (Y,\sigma)$ be a function. Then $f: (X,\tau) \to (Y,\sigma)$ is L-continuous iff $f_{|A} : (\mathrm{A},\tau_{|A}) \to (Y,\sigma)$ and $f_{|B} : (\mathrm{A},\tau_{|B}) \to (Y,\sigma)$ are L-continuous

Proof. This is an immediate consequence of Theorem 4.16 and Proposition 4.18.

Theorem 4.20. Let $f: (X,\tau) \to (Y,\sigma)$ be a function. If there is $A \in \tau_L$ containing $x \in X$ such that the restriction of f to A, $f_{|A} : (\mathrm{A},\tau_{|A}) \to (Y,\sigma)$ is L-continuous at x, then f is L-continuous at x.

Proof. Let $V \in \sigma$ with $f(x) \in V$. Since $f_{|A}$ is L-continuous at x, there is $U \in (\tau_{|A})_L$ such that $x \in U$ and $(f_{|A})(U) = f(U) \subseteq V$. By Theorem 2.16, $(\tau_{|A})_L \subseteq \tau_{L_{|A}}$ and so $U \in \tau_{L_{|A}}$. Since $A \in \tau_L$, then $U \in \tau_L$. This ends the proof.

Corollary 4.21. Let $f: (X,\tau) \to (Y,\sigma)$ be a function. If $\mathcal{B}$ is a cover of X which consists of nonempty L-open sets such that for each $A \in \mathcal{B}, f_{|A} : (\mathrm{A},\tau_{|A}) \to (Y,\sigma)$ is L-continuous, then f is L-continuous.

Proof. Let $x \in X$. Choose $A_x \in \mathcal{B}$ such that $x \in A_x$. Then by Theorem 4.20, f is L-continuous at x.

References

Al Ghour, S., & Samarah S. (2012). Cocompact open sets and continuity. *Abstract and Applied Analysis*, Article ID 548612.

Al-Zoubi, K., & Al-Nashef, B. (2003). The topology of ω-open subsets. *Al-Manarah Journal, 9*(2), 169-179.

Cameron, D. (1971). Maximal and minimal topologies. *Transactions of the American Mathematical Society, 160*, 229-248.

Hdeib, H., (1982). ω-Closed mappings. *Revista Colombiana de Mathematicas, 16*, 65-78.

Hdeib, H., (1989). ω-continuous functions. *Dirasat Journal, 16*, 136-153.

Henriksen, M., & Woods, R. (1988). Weak P-spaces and L-Closed spaces. *Questions Answers General Topology, 6*, 201-207.

Mukherji, T. & Sarkar, M. (1979). On a class of almost discrete spaces. *Matematički Vesnik, 31*, 459-474.