A GENERALIZATION OF COHEN'S THEOREM

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ABSTRACT

In this thesis, the researcher introduced the classes of strongly prime and one-sided strongly prime submodules and used these classes to characterize Noetherian modules. The researcher gave a new characterization of Noetherian modules, following which a finitely generated right R-module M is Noetherian if and only if every one-sided strongly prime submodule is finitely generated. This result can be considered as a generalization of Cohen's Theorem.

KEY WORDS : COHEN'S THEOREM/STRONGLY PRIME SUBMODULES
ONE-SIDED STRONGLY PRIME SUBMODULES
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CONTENTS

CHAPTER I
INTRODUCTION AND LITERATURE REVIEW

An ideal P of a commutative ring R is a *prime ideal* if for any $x, y \in R$ such that $xy \in P$, then either $x \in P$ or $y \in P$. In 1950, I. S. Cohen [15] showed that for a commutative ring R with identity, if every prime ideal is finitely generated, then R is a Noetherian ring. This shows that, to check whether every ideal in a commutative ring is finitely generated, we can only check the class of prime ideals in this commutative ring. However, this theorem does not true for noncommutative rings. Therefore, many authors have been modified Cohen theorem to noncommutative rings. In 1971, K. Koh [43] introduced the class of one-sided prime ideals, following that, a right ideal I of R is *one-sided prime* if $AB \subset I$ and $AI \subset I$, then either $A \subset I$ or $B \subset I$ for any right ideals A, B of the ring R. Using this definition, K. Koh proved that a ring R is right Noetherian if and only if every one-sided prime ideal is finitely generated. Similarly, in 1971, G. O. Michler [56] also studied prime one-sided ideals. He defined that a right ideal I of R to be *one-sided prime* if $aRb \subset I$, then either $a \in I$ or $b \in I$. He also gave a version of Cohen theorem as follows: a ring is right Noetherian if and only if every one-sided prime right ideal is finitely generated. In 1975, V.R. Chandran [11] proved that Cohen Theorem is also true for the class of duo rings. In addition, in 1996, B. V. Zabavskii introduced the definition of almost prime right ideals. A right ideal P of a ring R is called an *almost prime right ideal* if the condition $ab \in P$, where b is a duo-element of R, always implies that $a \in P$ or $b \in P$. From this definition, he proved that a ring is right Noetherian if and only if every almost prime right ideal is finitely generated. In 2011, M. L. Reyes [63] introduced the notion of completely prime right ideals. A right ideal $P \subsetneq R$ is a *completely prime right ideal* if for any $a, b \in R$ such that $aP \subset P$ and $ab \in P$, then either $a \in P$ or $b \in P$. By this definition, M. L. Reyes proved that a ring R is right Noetherian if and only if every completely prime right ideal is finitely generated. Recently, S. I. Bilavska and B. V. Zabavsky [2011] gave a new notion of dr-prime left (right) ideals. This new definition allows them to extend Cohen's theorem for noncommutative rings.

This thesis is arranged as follows. Basic concepts are reviewed in Chapter II. Our main results will be presented in Chapter III. In this part, we study the classes of strongly prime and one-sided strongly prime submodules of a given module. Some properties of strongly prime and one-sided strongly prime submodules are investigated. We use these classes to characterize Noetherian modules. The two nice results are Theorem 3.1.12 and Theorem 3.2.1. Finally, Chapter IV for conclusion.

CHAPTER II
BASIC KNOWLEDGE

Throughout this thesis, all rings are associative with identity and all modules are unitary. For a right R-module M, we denote $S = End_R(M)$, the ring of all R-endomorphisms of M. Let M and N be two right R-modules. The set of all R-homomorphisms from M to N is denoted by $Hom_R(M, N)$. We write R_R $({}_RR)$ to indicate that R_R $({}_RR)$ is a right (left) R-module. The letters $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ will denote the sets of natural, integer, rational and real numbers, respectively.

2.1 Prime ideals in rings

Prime ideals had many applications in Algebra. For examples, prime ideals are used in the localization of commutative rings. Associated prime ideals are an important part in the theory of primary decomposition in commutative algebra. Prime ideals are used not only in topology space but also in number theory, algebraic geometry. In this subsection, we will not introduce all properties of prime ideals, but rather we will introduce some properties of prime ideals without proofs. The results in this part can be found in [25] and [45]. A *prime ideal* in the ring R is any proper ideal P of R such that whenever I and J are ideals of R with $IJ \subsetneq P$, either $I \subsetneq P$ or $J \subsetneq P$. If 0 is a prime ideal of a ring, the ring is called a *prime ring*. Next, we will provide some properties of prime ideals by the following theorem.

Theorem 2.1.1. *[25, Proposition 2.1] For a proper ideal P in a ring R, the following conditions are equivalent:*

(i) P is a prime ideal.

(ii) If I and J are any ideals of R properly containing P, then $IJ \nsubseteq P$.

(iii) R/P is a prime ring.

(iv) If I and J are any left ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

(v) If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$.

From part (v) in theorem above, we can see that the definition of prime ideals coincides with the usual definition of prime ideals in the commutative case.

Definition 2.1.2. An ideal I is a *maximal ideal* of a ring R if $I \neq R$ and no proper ideal of R properly contains I.

Proposition 2.1.3. *[25, Proposition 3.2] Every maximal ideal I of a ring R is a prime ideal.*

Definition 2.1.4. A *minimal prime ideal* in a ring R is any prime ideal of R that does not properly contain any other prime ideals.

Proposition 2.1.5. *[25, Proposition 3.3] Any prime ideal P in a ring R contains a minimal prime ideal.*

2.2 Prime and semiprime rings

Let R be any ring. A nonempty set $S \subseteq R$ is called an *m-system* if for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$. In commutative algebra, a subset S of a commutative ring R is a *multiplicative set* if for any $x, y \in S$, then $xy \in S$. It is easy to check that a multiplicatively closed set S is an m-system. However, the converse is not true. For example, for any $a \in R, \{a, a^2, a^4, a^8, \cdots\}$ is an m-system but not multiplicatively closed in general. It is well-known that an ideal P of a commutative ring R is prime if and only if $R \setminus P$ is a multiplicative set. For arbitrary rings, we have the following result.

Lemma 2.2.1. *[45] An ideal $P \subseteq R$ is prime if and only if $R \setminus P$ is an m-system.*

Proposition 2.2.2. *[45] Let $S \subseteq R$ be an m-system, and let P be an ideal maximal with respect to the property that P is disjoint from S. Then P is a prime ideal.*

Definition 2.2.3. For an ideal I in a ring R, let
$\sqrt{I} := \{s \in R : \text{every } m-\text{system containing } s \text{ meets } I\}$.

Theorem 2.2.4. *[45] For any ring R and any ideal $I \subseteq R, \sqrt{I}$ equals the intersection of all prime ideals containing I. In particular, $\sqrt{I}$ is an ideal in R.*

Definition 2.2.5. An ideal J in a ring R is said to be *semiprime ideal* if for any ideal I of $R, I^2 \subseteq J$ implies that $I \subseteq J$.

Proposition 2.2.6. *[45] For any ideal J, the following statements are equivalent:*

(i) J is semiprime.

(ii) For $a \in R, (a)^2 \subseteq J$ implies that $a \in J$.

(iii) For $a \in R, aRa \subseteq J$ implies that $a \in J$.

(iv) For any left ideal I in $R, I^2 \subseteq J$ implies that $I \subseteq J$.

(v) For any right ideal I in $R, I^2 \subseteq J$ implies that $I \subseteq J$.

Definition 2.2.7. A ring R is called a *prime (resp, semiprime) ring* if 0 is a prime (resp, semiprime) ideal.

Example 2.2.8. (1) Any domain is a prime ring.

(2) Any reduced ring is a semiprime ring.

(3) Any simple ring R is a prime ring.

(4) Any direct product of semiprime rings is semiprime.

Proposition 2.2.9. *[45] A ring R is prime (resp, semiprime) if and only if $\mathbb{M}_n(R)$ is prime (resp, semiprime).*

Theorem 2.2.10. *[45] For any ring R, the following three statements are equivalent:*

(i) R is semisimple.

(ii) R is semiprime and left Artinian.

(iii) R is semiprime and satisfies DCC on principal left ideals.

Theorem 2.2.11. *[45] For any ideal $J \subsetneq R$, the following are equivalent:*

(i) J is a semiprime ideal.

(ii) J is an intersection of prime ideals.

(iii) $J := \sqrt{J}$.

Corollary 2.2.12. *[45] For any ideal $I \subseteq R$, $\sqrt{I}$ is the smallest semiprime ideal in R which contains I.*

2.3 Finitely generated and finitely cogenerated modules

A right R-module M is *finitely generated* if there are $m_1, m_2, \ldots, m_k \in M$ such that $M = \sum_{i=1}^{k} m_i R$. This is equivalent to say that there is an epimorphism $R^k \longrightarrow M$, for some $k \in \mathbb{Z}^+$. A characterization of a finitely generated right R-module is given in [36], following that, a right R-module M is finitely generated if and only if for any family $\{A_i, i \in I\}$ of submodules of M such that $\sum_{i\in I} A_i = M$, we can find a finite subset I_0 of I such that $\sum_{i\in I_0} A_i = M$. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of modules. If M is finitely generated then so is N; and if both L and N are finitely generated then so is M.

A right R-module M is *finitely cogenerated* if for any family $\{A_i, i \in I\}$ of submodules of M such that $\bigcap_{i\in I} A_i = 0$, we can find a finite subset I_0 of I such that $\bigcap_{i\in I_0} A_i = 0$. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of modules. If M is finitely cogenerated then so is L; and if both L and N are finitely cogenerated then so is M. We can see that every submodule of a finitely cogenerated module is a finitely cogenerated submodule. In a duality, every factor module of a finitely generated module is again a finitely generated module.

2.4 Noetherian and Artinian modules

Definition 2.4.1. (i) A chain of submodules of M_R :

$$\cdots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \ldots$$

(finite or infinite) is called *stationary* if it contains a finite number of distinct A_i.

(ii) A collection $\mathcal{A}$ of subsets of a set A satisfies the *ascending chain* *condition* (briefly, ACC) if there does not exist a strictly ascending infinite chain $A_1 \subset A_2 \subset \dots$ of subsets from $\mathcal{A}$.

Proposition 2.4.2. *[25, Proposition 1] For a module M, the following conditions are equivalent:*

(i) M has the ACC on submodules.

(ii) Every nonempty family of submodules of M has a maximal element.

(iii) Every submodule of M is finitely generated.

Definition 2.4.3. A module M is *Noetherian* if and only if the equivalent conditions of Proposition 2.4.2 are satisfied. A ring R is called *right (resp. left) Noetherian* if the module R_R (resp. ${}_RR$) is Noetherian.

Theorem 2.4.4. *[36, Theorem 6.1.2] Let M be a right R-module and A, a submodule of M. Then the following conditions are equivalent:*

(i) M is Noetherian.

(ii) A and M/A are Noetherian.

(iii) Every ascending chain $A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n \subset \dots$ of submodules of M is stationary.

(iv) Every submodule of M is finitely generated;

(v) For every family $\{A_i | i \in I\} \neq \emptyset$ of submodules of M, there exists a finite subfamily $\{A_i | i \in I_0\}$ (i.e, $I_0 \subset I$ and finite) such that $\sum_{i \in I} A_i = \sum_{i \in I_0} A_i$.

Corollary 2.4.5. *[36, Corollary 6.1.3] Let M be a right R-module. Then the following statements hold:*

(i) If M is a finite sum of Noetherian submodules, then M is Noetherian.

(ii) If the ring R is a right Noetherian ring and $M = M_R$ is finitely generated, then M is Noetherian.

(iii) Every factor ring of a right Noetherian ring is again right Noetherian.

Theorem 2.4.6. *[25, Theorem 3.4] In a right or left noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetitions allowed) that equals zero.*

Definition 2.4.7. (i) A chain of submodules of M_R :

$$\cdots \supset A_{i-1} \supset A_i \supset A_{i+1} \supset \dots$$

(finite or infinite) is called *stationary* if it contains a finite number of distinct A_i.

(ii) A non-empty family $\mathcal{M}$ of submodules of a right R-module satisfies the *desending chain condition* (briefly, DCC) if every desending chain $M_1 \supset M_2 \supset M_3 \cdots \supset M_n \supset \dots$ of elements of $\mathcal{M}$ is stationary.

(iii) A right R-module M is called *Artinian* if the set of all submodules of M has DCC, or every non-empty set of its submodules has a minimal element by inclusion.

(iv) A ring R is called *right (resp. left) Artinian* if the module R_R (resp. ${}_RR$) is Artinian. The ring R is called *Artinian* if it is both right and left Artinian.

By Theorem 6.1.2 in [25], we have some characterizations of an Artinian module M as follows:

(i) M and M/A are Artinian, where A is a submodule of M.

(ii) Every desending chain $A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_{n-1} \supset A_n \supset \dots$ of submodules of M is stationary.

(iii) Every factor module of M is finitely cogenerated.

(iv) In every family $\{A_i | i \in I\} \neq \emptyset$ of submodules of M, there exists a finite subfamily $\{A_i | i \in I_0\}$ (i.e, $I_0 \subset I$ and I_0 is finite) such that $\bigcap_{i \in I} A_i = \bigcap_{i \in I_0} A_i$.

Theorem 2.4.8. *[36, Theorem 6.1.2] The following properties are equivalent:*

(i) M is Artinian and Noetherian.

(ii) M is a module of finite length.

Corollary 2.4.9. *[36, Corollary 6.1.3] (i) If M is a finite sum of Artinian submodules, then it is Artinian.*

(ii) If the ring R is right Artinian, then every finitely generated right R-module M_R is Artinian.

Corollary 2.4.10. *Let S be a subring of a ring R. If S is right Noetherian and R is finitely generated as a right S-module, then R is right Noetherian.*

Let M_R be a right R-module and φ an endomorphism of M. Then

$\varphi^n (n \in \mathbb{N})$ is also an endomorphism of M. We have:

$$Ker(\varphi) \subset Ker(\varphi^2) \subset Ker(\varphi^3) \subset \dots,$$

$$Im(\varphi) \supset Im(\varphi^2) \supset Im(\varphi^3) \supset \dots.$$

For Noetherian (resp. Artinian) module, the first (resp. the second) chain is stationary. It follows the interesting results:

Theorem 2.4.11. *Let φ be an endomorphism of the module M. Then*

(i) M is Artinian $\Rightarrow \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 : M = Im(\varphi^n) + Ker(\varphi^n)$.

(ii) M is Artinian and φ is an monomorphism $\Rightarrow$ φ is an automorphism.

(iii) M is Noetherian $\Rightarrow \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 : 0 = Im(\varphi^n) \cap Ker(\varphi^n)$.

(iv) M is Noetherian and φ is an epimorphism $\Rightarrow$ φ is an automorphism.

In the next part, we will provide some examples.

(1) Any finite dimensional vector space is a module of finite length. So any finite dimensional vector space is Noetherian and Artinian.

(2) Infinite dimensional vector space V_K is neither Artinian nor Noetherian.

(3) Module $\mathbb{Z}_\mathbb{Z}$ is Noetherian but not Artinian. Note that the ring $\mathbb{Z}$ is right and left Noetherian but it is not Artinian. Conversely, every right Artinian with identity is right Noetherian.

2.5 Generators and cogenerators

Generators and cogenerators are notions in categories. They play an important role in module theory. In this subsection, we review the definitions and introduce some properties of generators and cogenerators. We refer the readers to [36] for more details.

Definition 2.5.1. (a) A module U_R is called a *generator* for Mod-R, if

$$\forall M \in Mod - R, M = \sum_{\varphi \in Hom_R(U,M)} Im\varphi.$$

(b) A module V_R is called a *cogenerator* for Mod-R, if

$$\forall M \in Mod - R, 0 = \bigcap_{\varphi \in Hom_R(M,V)} Ker\varphi.$$

(c) An R-module M is called a *self-generator* (*self-cogenerator*) if it generates all its submodules (cogenerates all its factor modules).

For arbitrary modules B and M, we denote

$$Im(B, M) = \sum_{\varphi \in Hom_R(B,M)} Im\varphi.$$

Then B is a generator for $Mod - R$ if for any right R-module $M, Im(B, M)$ is as large as possible for every M and so equals M.
Similarly, for arbitrary modules C and M, we denote

$$Ker(M, C) = \bigcap_{\varphi \in Hom_R(M,C)} Ker\varphi.$$

Then C_R is a cogenerator for $Mod - R$ that $Ker(M, C)$ is as small as possible for every M and so equals 0.

Proposition 2.5.2. *(i) If B is a generator and A is a module such that $Im(A, B) = B$, then A is also a generator;*

(ii) Every module M such that there is an epimorphism from M to R_R is also a generator;

(iii) If C is a cogenerator and D is a module such that $Ker(C, D) = 0$ then D is also a cogenerator.

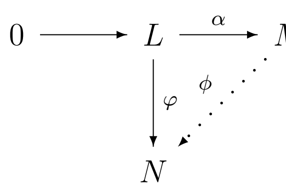
Theorem 2.5.3. *(i) B is a generator if and only if for any* $f \in Hom_R(M,N), f \neq 0$, *there is* $g \in Hom_R(B,M)$ *such that* $f.g \neq 0$.

(ii) C is a cogenerator if and only if for any $h \in Hom_R(L,M), h \neq 0$, *there is* $t \in Hom_R(M,C)$ *such that* $t.h \neq 0$.

2.6 Injective and Projective modules

Injective and projective modules form an important part of the study in Algebra, especially in Ring and Module Theory and in Homological Algebra. In this part, we will introduce the definition of injective and projective modules. Some characterizations of injective and projective modules are also given.

Definition 2.6.1. (1) Let M and N be two right R-modules. A right R-module N is said to be M-injective if for any R-monomorphism $\alpha : L \longrightarrow M$ and R-homomorphism $\varphi : L \longrightarrow N$, there exists a homomorphism $\phi : M \longrightarrow N$ such that $\phi\alpha = \varphi$.



(2) A right R-module K is *injective* if it is M-injective, for all right R-module M.

(3) A right R-module M is called *quasi-injective* if it is M-injective.

Theorem 2.6.2. *[36, Theorem 5.3.1] Let M be a right R-module. Then the following conditions are equivalent:*

(i) M is injective;

(ii) Every monomorphism $\varphi : M \longrightarrow N$ *splits (i.e,* $Im\varphi$ *is a direct summand in* N*);*

(iii) For every monomorphism $\alpha : L \longrightarrow N$ *of right R-modules and any*

homomorphism $\varphi : L \longrightarrow M$, *we can find a homomorphism* $\bar{\varphi} : N \longrightarrow M$ *such that* $\bar{\varphi}\alpha = \varphi$;

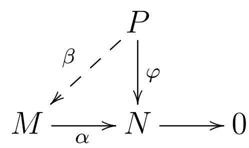
(iv) For every monomorphism $\alpha : L \longrightarrow N$
$Hom(\alpha, 1_M) : Hom_R(N, M) \longrightarrow Hom_R(L, M)$ *is an epimorphism.*

The following Baer's Criterion proves the equivalence between injectivity and R-injectivity.

Theorem 2.6.3. *[36, Theorem 5.7.1] (Baer's Criterion) A module* Q_R *is injective if and only if to every right ideal* I *of* R_R *and to every homomorphism* $\alpha : I \longrightarrow Q$, *there exists a homomorphism* $\beta : R_R \longrightarrow Q$ *with* $\alpha = \beta\gamma$, *where* γ *is the inclusion map of* I *into* R.

Note that the definition of projective modules is dual to the definition of injective modules. Therefore, basic properties of projective modules are also dual to those of injective modules.

Definition 2.6.4. A right R-module P is called M-projective if for every epimorphism $\alpha : M \longrightarrow N$ and every homomorphism $\varphi : P \longrightarrow N$, there exists a homomorphism $\beta : P \longrightarrow M$ such that $\alpha\beta = \varphi$.



Similarly the case of injective modules, we will introduce some characterizations of projective modules by the following theorem.

Theorem 2.6.5. *[36, Theorem 5.3.1] The following properties of a right* R*-module* P *are equivalent:*

(i) P *is projective;*

(ii) Every epimorphism $\varphi : M \longrightarrow P$ *splits (i.e,* $Ker\varphi$ *is a direct summand in* M*);*

(iii) For every epimorphism $\beta : B \longrightarrow C$ *of right* R*-modules and any homomorphism* $\varphi : P \longrightarrow C$, *there is a homomorphism* $\bar{\varphi} : P \longrightarrow B$ *such that*

$\beta\bar{\varphi} = \varphi$;

(iv) For every epimorphism $\alpha : B \longrightarrow C$ $Hom(1_P, \beta) : Hom_R(P; B) \longrightarrow Hom_R(P; C)$ *is an epimorphism.*

Theorem 2.6.6. *[36, Theorem 5.4.1] A module is projective if and only if it is isomorphic to a direct summand of a free module.*

Proposition 2.6.7. *[2, Proposition 16.10] Let* M *be a right* R*-module and* $(U_\alpha)_{\alpha \in A}$ *be a family of right* R*-modules indexed by* A*. Then*

(i) The direct sum $\bigoplus_A U_\alpha$ *is* M*-projective if and only if every* U_α *is* M*-projective,* $\alpha \in A$*.*

(ii) The direct product $\prod_A U_\alpha$ *is* M*-injective if and only if every* U_α *is* M*-injective,* $\alpha \in A$*.*

2.7 Prime submodules

Many authors want to transfer the notion of prime ideals in rings to modules. Therefore, in module theory, prime submodules have been appeared in many contexts and papers. For examples, Andrunakievich [3], Beachy and Blair [6], Dauns [16], Bican et.al. [9], Wisbauer [74], C. P. Lu [51], Behboodi and Koohy [8] gave some definitions of prime submodules. However, from these definitions, we could not find any properties which are similar to that of prime ideals. In 2008, N. V. Sanh [65] proposed a new definition of prime submodules. By this definition, our group could prove many theorems similar to the properties of prime ideals.

Definition 2.7.1. A submodule X of M is called a *fully invariant submodule* of M if for any $f \in S$, we have $f(X) \subset X$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R.

Definition 2.7.2. A fully invariant submodule X of M is called a *prime submodule* of M if for any ideal I of $S = End_R(M)$, and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$.

Example 2.7.3. (1) Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the additive group of intergers modulo 4. Then $X = \langle 2 \rangle$ is a prime submodule of $\mathbb{Z}$-module $\mathbb{Z}_4$.

(2) Let M be a semisimple module. Then $M = \bigoplus_{i \in I} C_i$ with each C_i is a homogeneous component of M. Put $M_j = \bigoplus_{i \in I, i \neq j} C_i$. Then each M_j is a prime submodule of M.

(3) If M is a semisimple module having only one homogeneous component, then 0 is a prime submodule, especially, if M is simple, then 0 is a prime submodule.

The following Theorem of Sanh gives some characterizations of prime submodules.

Theorem 2.7.4. *[65, Theorem 1.2] Let X be a proper fully invariant submodule of M. Then the following conditions are equivalent:*

(i) X is a prime submodule of M;

(ii) For any right ideal I of S, any submodule U of M, if $I(U) \subset X$, then either $I(U) \subset X$ or $U \subset X$;

(iii) For any $\varphi \in S$ and fully invariant submodule U of M, if $\varphi(U) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$;

(iv) For any left ideal I of S and subset A of M, if $IS(A) \subset X$, then either $I(M) \subset X$ or $A \subset X$;

(v) For any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$, then either $\varphi(M) \subset X$ or $m \in X$.

Moreover, if M is a quasi-projective, then the above conditions are equivalent to:

(vi) M/X is a prime module.

In addition, if M is quasi-projective and a self-generator, then the above conditions are equivalent to:

(vii) If I is an ideal of S and U, a fully invariant submodule of M such that $I(M)$ and U properly contain P, then $I(U) \not\subset P$.

The above theorem shows that the structure of prime submodules is very nice and when $M = R_R$, we return to the well-known characterizations of prime ideals.

Corollary 2.7.5. *For a proper ideal P in a ring R, the following conditions are equivalent:*

(i) P is a prime ideal.

(ii) If I and J are any ideals of R properly containing P, then $IJ \nsubseteq P$.

(iii) R/P is a prime ring.

(iv) If I and J are any left ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

(v) If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$.

Sanh [65] introduced the notion of the right ideal I_X of S corresponding to a submodule X of M. For a submodule X of M, I_X is a right ideal of S, especially, we get the following Lemma.

Lemma 2.7.6. *[65, Lemma 1.9] Let M be a right R-module and $S = End(M_R)$. Suppose that X is a fully invariant submodule of M. Then the set $I_X = \{f \in S | f(M) \subset X\}$ is a two-sided ideal of S.*

Theorem 2.7.7. *[65, Theorem 1.10] Let M be a right R- module, $S = End(M_R)$ and X a fully invariant submodule of M. If X is a prime submodule of M, then I_X is a prime ideal of S. Conversely, if M is a self-generator and if I_X is a prime ideal of S, then X is a prime submodule of M.*

Lemma 2.7.8. *[67, Lemma 3] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then the following statements hold:*

(i) If X is a maximal submodule of M, then I_X is a maximal right ideal of S;

(ii) If P is a maximal right ideal of S, then $X := P(M)$ is a maximal submodule of M and $P = I_X$.

Definition 2.7.9. A prime submodule P of a right R-module M is called a *minimal prime submodule* if it is minimal in the class of prime submodules of M.

The following proposition gives us a property similar to that of rings.

Lemma 2.7.10. *[67] If P is a prime submodule of a right R-module M, then P contains a minimal prime submodule of M.*

Definition 2.7.11. A fully invariant submodule X of a right R-module M is called a *semiprime submodule* if it is an intersection of prime submodules of M.

A right R-module M is called a *semiprime module* if 0 is a semiprime submodule of M. Consequently, the ring R is a semiprime ring if R_R is a semiprime module. By symmetry, the ring R is semiprime if ${}_RR$ is a semiprime left R-module.

Example 2.7.12. (1) Every semisimple module is semiprime.

(2) As a $\mathbb{Z}$-module, the module $\mathbb{Z}_4$ is not semiprime.

For a right R-module M, let $P(M)$ be the intersection of all prime submodules of M. By our definition, M is a semiprime module if $P(M) = 0$. We want to get some properties similar to that of prime radical of rings.

Theorem 2.7.13. *[67] Let M be a quasi-projective module. Then $M/P(M)$ is a semiprime module, that is, $P(M/P(M)) = 0$.*

Theorem 2.7.14. *[67] If M is a semiprime module, then S is a semiprime ring.*

For the converse part, we need M to be a self-generator and finitely generated module.

Theorem 2.7.15. *[67] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If S is a semiprime ring, then M is a semiprime module.*

Theorem 2.7.16. *(i) If M is a prime module, then so is M^n for any $n \in \mathbb{N}$.*

(ii) If M is a semiprime module, then so is M^n for any $n \in \mathbb{N}$.

Theorem 2.7.17. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a fully invariant submodule of M. Then the following conditions are equivalent:*

(i) X is a semiprime submodule of M;

(ii) If J is any ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(iii) If J is any ideal of S properly containing X, then $J^2(M) \not\subset X$;

(iv) If J is any right ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(v) If J is any left ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

The following corollary is a direct consequence.

Corollary 2.7.18. *[66] Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a semiprime submodule of M. If J is a right or left ideal of S such that $J^n(M) \subset X$ for some positive integer n, then $J(M) \subset X$.*

2.8 Duo modules

Duo rings were first studied in 1958 by H. Feller [21]. Using the definition of H. Feller, in 1959, G. Thierrin [70] gave some properties of duo rings. In 2006, A. C. Ozcan, A. Harmanci and P. F. Smith [60] studied duo modules. Many results of duo modules are investigated in [60]. In this subsection, we will introduce some properties of duo modules.

Definition 2.8.1. A right R- module M is called a *duo module* if every submodule of M is a fully invariant submodule of M. A ring is called a right duo ring if every right ideal is a two-sided ideal. Moreover, a ring R is said to be a *duo ring* iff R is a left and right duo ring.

It follows immediately from the above definition that if R is a duo ring, then $Rx = xR$ for all $x \in R$. This means that there exists an element $z \in R$ such that $xy = zx$ for all $x, y \in R$. We will introduce some examples of duo modules:

(i) Uniserial Artinian modules; [60]

(ii) Self- injective self- cogenerator modules with commutative endomorphism rings; [75]

(iii) Multiplication modules;

(iv) Comultiplication modules.

Proposition 2.8.2. *[60, Proposition 1.3] Any direct summand of a duo module is also a duo module.*

Proposition 2.8.3. *[60, Proposition 1.4] Let M be a duo module.*

(i) If M is quasi-injective, then every submodule of M is a duo module.

(ii) If M is quasi-projective, then every homomorphic image of M is duo.

Proposition 2.8.4. *[60, Proposition 1.5] Let M be a module such that every countably generated submodule is a duo module. Then M is a duo module.*

Theorem 2.8.5. *[60, Theorem 2.10] Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$. Then M is a duo module if and only if*

(i) M_i is a duo module for all $i \in I$, and

(ii) $N = \bigoplus_{i \in I} (N \bigcap M_i)$ for every submodule N of M.

Corollary 2.8.6. *[60, Corollary 2.11] Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$. Then M is a duo module if and only if $M_i \bigoplus M_j$ is a duo module for all distinct i, j in I.*

2.9 Bounded, fully bounded rings and modules

Let R be any ring. A right R-module M is called a *bounded module* if every essential submodule contains a fully invariant submodule which is essential as a submodule. A ring R is a *right bounded ring* if every essential right ideal of R contains an ideal which is essential as a right ideal. A right R-module M_R is said to be *fully bounded* if for every prime submodule X of M, the prime factor module M/X is a bounded module. A ring R is *right fully bounded* if for every prime ideal I of R, the prime factor ring R/I is a right bounded ring.

Example 2.9.1. (1) A simple Artinian ring has no proper essential right ideals. Therefore, it is right bounded. This implies that any right Artinian ring is right fully bounded.

(2) A simple ring is right bounded if and only if it is Artinian.

The Jacobson radical of R, denoted by $J(R)$, is defined to be the intersection of all maximal ideals of R. The equality $\bigcap_{k=1}^{\infty} J^k(R) = 0$ is well-known as

Jacobson conjecture, which appeared in [32, pp 200]. This had been proved for commutative rings by W. Krull. In 1965 I. N. Herstein showed that it is false if the ring is right but not left Noetherian. The Jacobson conjecture is not true by a result of Jategaonkar in 1974. Jategaonkar proved that there is a right Noetherian, right serial ring with $\bigcap_{k=1}^{\infty} J^k(R) \neq 0$. We introduce some results about Jacobson conjecture. These results are given in [12] and [71].

Theorem 2.9.2. *[12, Theorem 7.5] Let R be a left and right Noetherian right fully bounded ring with Jacobson radical $J(R)$. Then* $\bigcap_{k=1}^{\infty} J^k(R) = 0$.

Theorem 2.9.3. *[12, Theorem 7.11] Let $J(R)$ be the Jacobson radical of a left and right Noetherian right fully bounded ring. Then* $\bigcap_{k=1}^{\infty} J^k(R) = 0$.

Theorem 2.9.4. *[71, Theorem 1] Let R be a ring. If R is a left Noetherian, right distributive, then* $\bigcap_{k=1}^{\infty} J^k(R) = 0$.

Theorem 2.9.5. *If R is a right Noetherian, left distribution, then* $\bigcap_{k=1}^{\infty} J^k(R) = 0$.

Theorem 2.9.6. *[12, Theorem 6.7] Let R be a left and right Noetherian right serial ring with Jacobson radical $J(R)$. Then* $\bigcap_{k=1}^{\infty} J^k(R) = 0$.

Proposition 2.9.7. *[12, Proposition 7.12] Let R be a right Noetherian right fully bounded ring, then R is right bounded.*

Next part, we will introduce some properties of fully bounded modules. For more details, we refer the readers to [69].

Lemma 2.9.8. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If I is an essential right ideal of S, then $I(M)$ is an essential submodule of M.*

Theorem 2.9.9. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then M is a bounded module if and only if its endomorphism ring $S = End(M_R)$ is a right bounded ring.*

Theorem 2.9.10. *[69] If M is a bounded module, then so is M^n, for any $n \in \mathbb{N}$.*

The following corollary is a direct consequence.

Corollary 2.9.11. *[69] If the ring R is right bounded, then the right R-module R^n is bounded for any $n \in \mathbb{N}$, and therefore the matrix ring $M_n(R)$ of all square matrices of order n with coefficients in R will be right bounded.*

Theorem 2.9.12. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a prime module, then M is a bounded module if and only if every essential submodule of M contains a non-zero fully inraviant submodule of M.*

Theorem 2.9.13. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then M is a fully bounded module if and only if S is a right fully bounded ring.*

Theorem 2.9.14. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M_R is a fully bounded Noetherian module, then M_R is a bounded module.*

Theorem 2.9.15. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a fully bounded Noetherian module and X is a fully invariant submodule of M, then M/X is a bounded module.*

Corollary 2.9.16. *[69] Let M be a quasi-projective, finitely generated right R-module which is a self-generator and $f : M \longrightarrow N$ be an epimorphism. Suppose that $Ker f$ is a fully invariant submodule of M. If M is a fully bounded Noetherian module, then N is a bounded module.*

2.10 IFP rings and modules

Definition 2.10.1. A submodule X of a right R-module M is said to have *"insertion factor property"* (briefly, an IFP-submodule) if for any endomorphism φ of M and any element $m \in M$, if $\varphi(m) \in X$, then $\varphi Sm \subset X$. A right ideal I of R is an IFP-right ideal if it is an IFP submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right R-module M is called an IFP-module if 0 is an

IFP-submodule of M. A ring is IFP if 0 is an IFP ideal.

Some properties of IFP-modules can be found in [68].

Proposition 2.10.2. *[68] Let X be a submodule of a right R-module M. If X is an-IFP submodule and M is quasi-projective,then M/X is an IFP-module. Conversely, if M/X is IFP and X is fully invariant, then X is an IFP-submodule of M.*

Let X be a submodule of M. Define $I_X = \{f \in S | f(M) \subset X\}$. Then we can see that I_X is a right ideal of S. Moreover, if X is fully invariant in M, then I_X is a two-sided ideal of S. The following simple proposition is useful.

Proposition 2.10.3. *[68] If X is an IFP submodule of M, then I_X is is an IFP right ideal of S. The converse is true if M is a self-generator.*

Proposition 2.10.4. *[68] If X is a prime submodule of a right R-module M and if M is a self-generator, then X is an IFP-submodule of M. Especially, every prime ideal of a ring R is IFP.*

Proposition 2.10.5. *[68] Let M be a right R-module and $S = End(M)$. The following conditions are equivalent:*

(i) M is an IFP-module;

(ii) For any $m \in M$, $\mathbf{l}_S(m)$ is an ideal of S;

(iii) For any $\varphi \in S$, $ker(\varphi)$ is a fully invariant submodule of M;

If M is quasi-projective, then the above conditions are equivalent to:

(iv) For any $\varphi \in S$, $ker(\varphi)$ is an IFP-ideal of S;

(v) $M/ker(I)$ is an IFP-module for any subset I of S;

If M is a self-generator, then the above conditions (i), (ii) and (iii) are equivalent to:

(vi) For any $m \in M$, $\mathbf{l}_S(m)$ is an IFP-ideal of S;

(vii) $S/\mathbf{l}_S(A)$ is an IFP-ring for any subset $A \subset M$.

The following Corollary is a direct consequence of the above Proposition.

Corollary 2.10.6. *For a ring R the following conditions are equivalent:*

(i) R is an IFP-ring;

(ii) For any $a \in R$, $\mathbf{l}_R(a)$ is an ideal of R;

(iii) For any $a \in R$, $r_R(a)$ is an ideal of$R$;

(iv) For any $a \in R$, $\mathbf{l}_R(a)$ is an IFP-ideal of R;

(v) For any $a \in R$, $r_R(a)$ is an ideal of$R$;

(vi) For any $a \in R$, $R/r_R(a)$ is an IFP-ring;

(vii) For any $a \in R$, $R/\mathbf{l}_R(a)$ is an IFP ring.

Definition 2.10.7. A module M is called *fully IFP* if M/U is IFP for every proper fully invariant submodule U of M. A ring is called *fully IFP* if R/I is IFP for every proper ideal I of R.

Due to [28], fully IFP rings are called *homomorphically IFP rings.* Next, we give the relationship between a fully IFP module and its endomorphism ring.

Proposition 2.10.8. *[68] Let M be a right R-module which is a self-generator. If M is a fully IFP module, then S is a fully IFP ring. Conversely, if S is a fully IFP ring, then M is a fully IFP module.*

Proposition 2.10.9. *[68] Let M be a right R-module which is a self-generator. If M is a fully IFP module, then M is an IFP module.*

Recall from that a module N is called *M-generated* if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called *finitely M-generated.* We can see that if M is a quasi-projective and X is a finitely M-generated submodule, then $I_X = \{f \in S \mid f(M) \subset X\}$ is a finitely generated right ideal of S. We now introduce a result about Anderson's Theorem for modules as a generalization of Anderson's Theorem for noncommutative rings. To do that, we have a following proposition.

Proposition 2.10.10. *[67, Proposition 1.1] Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then we have the following:*

(i) If X is a minimal prime submodule of M, then I_X is a minimal

prime ideal of S.

(ii) If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.

For following Proposition, we refer to Huh et al [28].

Proposition 2.10.11. *[28, Theorem 3] Let R be a homomorphically IFP ring and I be a proper ideal of R. If every minimal prime ideal over I is finitely generated, then there are only finitely many minimal prime ideals over I.*

Motivated this result we can prove the following Proposition as a generalization of Anderson's theorem for modules.

Proposition 2.10.12. *[68] Let M be a quasi-projective, finitely generated, fully IFP right R-module which is a self-generator. Assume that U is a proper fully invariant submodule of M. If every minimal prime submodule over U is finitely generated, then there are only finitely many minimal prime submodules over U.*

The following Corollary is a direct consequence of the above Proposition.

Corollary 2.10.13. *[68] Let M be a quasi-projective, finitely generated, fully IFP right R-module which is a self-generator. If every minimal prime submodule of M is finitely generated, then there are only finitely many minimal prime submodules of M.*

CHAPTER III
CHARACTERIZATIONS OF NOETHERIAN MODULES

3.1 Strongly prime and one-sided strongly prime submodules

In this part, we study the classes of strongly prime and one-sided strongly prime submodules. Some properties of strongly prime and one-sided strongly prime submodules are investigated. First, we will introduce the definition of strongly prime and one-sided strongly prime submodules.

Definition 3.1.1. A fully invariant proper submodule U of M is called *strongly prime* if for any $f \in S$, any $m \in M$, $f(m) \in U$, then either $f(M) \subset U$ or $m \in U$. Especially, a proper ideal I of a ring R is *strongly prime* if for any $a, b \in R, ab \in I$, then either $a \in I$ or $b \in I$.

Definition 3.1.2. A proper submodule U of M is called *one-sided strongly prime* if for any $f \in S$ and $m \in M$ such that $f(U) \subset U$ and $f(m) \in U$, then either $f(M) \subset U$ or $m \in U$. In particular, a right ideal $P \subsetneq R$ is an *one-sided strongly prime ideal* if for any $a, b \in R$ such that $aP \subset P, ab \in P$ then either $a \in P$ or $b \in P$.

Proposition 3.1.3. *Every strongly prime submodule is prime.*

Proof. Let X be a strongly prime submodule of M and $\varphi(U) \subset X$ with $\varphi \in S$ and U, a fully invariant submodule of M. If $\varphi(M) \not\subset X$, then $u \in X$, for all $u \in U$, that is $U \subset X$, proving that X is prime. □

The following proposition can be considered as an example to show that the class of one-sided strongly prime submodules is larger than that of strongly prime submodules.

Proposition 3.1.4. *Every maximal submodule is an one-sided strongly prime submodule. In particular, every maximal right ideal of a ring R is an one-sided strongly prime right ideal.*

Proof. Let U be a maximal submodule of M and $\varphi \in S, m \in M$ such that $\varphi(U) \subset U$ and $\varphi(m) \in U$. Suppose that $m \notin U$. Then $U + mR = M$ and hence $\varphi(M) = \varphi(U) + \varphi(m)R \subset U$, proving that U is an one-sided strongly prime submodule. □

Example 3.1.5. We will present some examples of strongly prime and one-sided strongly prime submodules:

1) Every prime ideal in a duo ring is a strongly prime ideal. Indeed, suppose that P is a prime ideal and $ab \in P$. Put $C = \{c \in R \mid ac \in P\}$. We can verify that C is a right ideal. Since R is a duo ring, C is a two-sided ideal. Note that from $ab \in P$, we see that $b \in C$. Since C is a two-sided ideal of R, we can see that $Rb \subset C$. This shows that $aRb \subset P$, proving that P is a strongly prime ideal.

2) Every prime submodule in a duo module is a strongly prime submodule. In fact, suppose that U is a prime submodule and M is a duo module. Let $\varphi(m) \subset U$. Then we have $U \supset \varphi(m)R = \varphi(mR)$. Since M is a duo module, we see that mR is a fully invariant submodule of M. This implies that $S(mR) = mR$. Hence $\varphi(mR) = \varphi S(mR) \subset U$. By the primeness of U, either $\varphi(M) \subset U$ or $m \in U$, showing that U is a strongly prime submodule of M.

3) Let $\mathbb{M}_3(k)$ be a matrix ring and k be a division ring. Let R be the following subring of $\mathbb{M}_3(k)$:

$$R := \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} .$$

Let $P \subset R$ be the right ideal of R of the form $P := \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$.

It is easy to verify that if $xP \subset P$, then $x_{12} = 0; x_{13} = 0$. Suppose that $xy \in P$.

Then $x_{11}y_{11} = 0; x_{11}y_{12} = 0; x_{11}y_{13} = 0$. From $x_{12} = 0, x_{13} = 0$, we see that either $x \in P$ or $y \in P$, proving that P is an one-sided strongly prime ideal of R.

Definition 3.1.6. An R-module M is called *strongly prime* if 0 is a strongly prime submodule of M. A ring R is called a *strongly prime ring* if 0 is a strongly prime ideal of R.

Proposition 3.1.7. *Let M be a right R-module which is a quasi-projective module. Then the following are equivalent:*

(1) X is a strongly prime submodule of M,

(2) M/X is a strongly prime module.

Proof. (1) $\Rightarrow$ (2). Suppose that $\bar{\varphi}(\bar{m}) = \bar{0}$, where $\bar{\varphi} \in End(M/X)$. This implies that $\bar{\varphi}\nu(m) = \bar{0}$. Since M is a quasi-projective module, we can find $f \in S$ such that $\nu f = \overline{\varphi}\nu$, where ν is the natural epimorphism from M to $\bar{M} = M/X$. By (1), either $f(M) \subset X$ or $m \in X$. If $f(M) \subset X$, then $\overline{\varphi}(M/X) = \overline{\varphi}\nu(M) = \bar{0}$. If $m \in X$, then we have $\nu(m) = \bar{m} = \bar{0}$. Hence $\bar{0}$ is a strongly prime submodule of M/X, showing that M/X is a strongly prime module.

(2) $\Rightarrow$ (1). Let $\varphi(m) \in X$, for some $\varphi \in S$ and $m \in M$. Then $\nu\varphi(m) = \bar{0}$. Since X is a fully invariant submodule of M, we can find an endomorphism $f \in \bar{S} = End(M/X)$ such that $\nu\varphi = f\nu$. It follows that $f(\bar{m}) = \bar{0}$, which is a strongly prime submodule. Hence either $f(\bar{M}) = \bar{0}$ or $\bar{m} = \bar{0}$. If $f(\bar{M}) = \bar{0}$, then $f\nu(M) = \bar{0}$. This shows that $\nu\varphi(M) = \bar{0}$. Hence $\varphi(M) \subset X$. If $\bar{m} = \bar{0}$, then $m \in X$. This proves that X is a strongly prime submodule. □

Note that in the proof (2) $\Rightarrow$ (1) we do not need the quasi-projectivity of M. The following corollary is a direct consequence of proposition above.

Corollary 3.1.8. *Let I be an ideal of the ring R. Then I is a strongly prime if and only if R/I is a strongly prime ring.*

Lemma 3.1.9. *Let M, N be right R-modules and $f : M \longrightarrow N$ be an epimorphism. Suppose that $Kerf$ is a fully invariant submodule of M. Then,*

(1) For any $\varphi \in S$, there exists $\phi \in \bar{S} = End(N)$ such that $\phi f = f\varphi$.

(2) If V is a fully invariant submodule of N, then $U = f^{-1}(V)$ is a fully invariant submodule of M.

Proof. (1) Let $y \in N$. Then $y = f(m)$ for some $m \in M$. Put $\psi(y) = f\varphi(m)$. If $y = f(m) = f(m')$, then $m - m' \in Kerf$. Since $Kerf$ is a fully invariant submodule of $M, \varphi(m - m') \in Kerf$. Thus $f\varphi(m - m') = 0$, proving that ψ is well-defined and moreover it is an R-homomorphism with $f\varphi = \psi f$.

(2) Suppose that V is a fully invariant submodule of N and $U := f^{-1}(V)$. Then by homomorphism theorem, for each $\varphi \in S$, there exists $\alpha \in S$ such that $f\varphi = \alpha f$. Since $Kerf$ is fully invariant, $f\varphi(U) = \alpha f(U) = \alpha(V) \subset V$. This shows that $\varphi(U) \subset f^{-1}(V) = U$, i.e., U is a fully invariant submodule of M. □

Lemma 3.1.10. *Let M be a quasi-projective module and X, a strongly prime submodule of M. If $A \subset P$ be a fully invariant submodule of M, then P/A is a strongly prime submodule of M/A.*

Proof. Let $\bar{S} = End_R(M/A), \varphi \in \bar{S}$ and $m + A \in M/A$ with $\varphi(m + A) \subset P/A$. By the quasi-projectivity of M, we can find an endomorphism $f \in S$ such that $\varphi\nu = \nu f$ where $\nu : M \longrightarrow M/A$ is the natural epimorphism. From $f(m) + A = \nu f(m) = \varphi\nu(m) = \varphi(m + A) \in P/A$, we see that $f(m) \in P$. By hypothesis, either $m \in P$ or $f(M) \subset P$. This implies that either $m + A \in P/A$ or $\varphi(M/A) = (f(M) + A)/A \subset P/A$, showing that P/A is strongly prime. □

Proposition 3.1.11. *Let M be a quasi-projective module and $f : M \longrightarrow N$ be an epimorphism such that $Kerf$ is a fully invariant submodule of M. Then,*

(1) If Y is a strongly prime submodule of N, then $X = f^{-1}(Y)$ is a strongly prime submodule of M.

(2) If X is a strongly prime submodule of M, then $f(X)$ is a strongly prime submodule of N.

Proof. (1) By Lemma 3.1.9, we have $X = f^{-1}(Y)$ is a fully invariant submodule of M. It is easy to see that X is different M. Suppose that $\varphi \in S$ and $\varphi(m) \in X$. We will show that either $\varphi(M) \subset X$ or $m \in X$. From Lemma 3.1.9 again, there exists $\gamma \in S' = End(N)$ such that $\gamma f = f\varphi$. From $\varphi(m) \in X$, we can see that

$f\varphi(m) \in f(X) = Y$. Since $\gamma f = f\varphi$, we have $\gamma f(m) \in Y$. By assumption, we must have either $f(m) \in Y$ or $\gamma(N) \subset Y$. If $\gamma(N) \subset Y$, then $\gamma f(M) \subset Y$. It follows that $f\varphi(M) \subset Y$. Hence $\varphi(M) \subset f^{-1}(Y) = X$. If $f(m) \in Y$, then $m \in f^{-1}(Y) = X$. Therefore X is a strongly prime submodule.

(2) Note that $f(X)$ is a fully invariant submodule of N. Suppose that $f(X) = N = f(M)$. Then we have $M \subset X + Kerf = X$, a contradiction. This implies that $f(X)$ is different N. Let $\gamma(n) \in f(X)$, where $\gamma \in S' = End(N)$. We will show that $\gamma(N) \subset f(X)$ or $n \in f(X)$. Since M is a quasi-projective module, there is $\varphi \in S$ such that $\gamma f = f\varphi$. From this, we can see that $\gamma(n) = \gamma(f(f^{-1})(n)) = f\varphi(f^{-1}(n)) \subset f(X)$. It follows that $\varphi(f^{-1}(n)) \subset X + Kerf = X$. If X is a strongly prime submodule, then we have either $\varphi(M) \subset X$ or $f^{-1}(n) \in X$. If $\varphi(M) \subset X$, then $f\varphi(M) \subset f(X)$. Thus $\gamma f(M) \subset f(X)$ and hence $\gamma(N) \subset f(X)$. If $f^{-1}(n) \in X$, then $n \in f(X)$. This shows that $f(X)$ is a strongly prime submodule. □

Next, we give the relationship between a strongly prime and prime submodule by the following theorem.

Theorem 3.1.12. *Let M be an R-module. A submodule X of M is a strongly prime submodule if and only if it is prime and IFP.*

Proof. Suppose that X is a strongly prime submodule of M. For any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$, then $\varphi(m) \in X$. Since X is a strongly prime submodule, we have either $\varphi(M) \subset X$ or $m \in X$. This implies that X is a prime submodule. We assume that $\varphi(m) \in X$. We need to prove that $\varphi S(m) \subset X$. Since $\varphi(m) \in X$, we can see that either $\varphi(M) \subset X$ or $m \in X$. If $m \in X$, then we have $g(m) \in g(X) \subset X$, for all $g \in S$. This means that $S(m) \subset X$. Therefore $\varphi S(m) \subset X$. Suppose that $\varphi(M) \subset X$. We can see that $\varphi S(M) = \varphi(M) \subset X$. This follows that $\varphi S(m) \subset X$, as desired.

Suppose that X is a prime submodule and has IFP. If $\varphi(m) \in X$, then we want to show that either $\varphi(M) \subset X$ or $m \in X$. Since X has IFP, we have $\varphi S(m) \subset X$. By primeness of X, we can see that either $\varphi(M) \subset X$ or $m \in X$. This shows that X is a strongly prime submodule, as required. □

The following result is a direct consequence.

Corollary 3.1.13. *An ideal I of a ring R is a strongly prime ideal if and only if it is prime and IFP.*

Proposition 3.1.14. *Let M be a right R-module. If X is a strongly prime submodule of M, then I_X is a strongly prime ideal of S. Conversely, if M is a self-generator and I_X is a strongly prime ideal of S, then X is a strongly prime submodule.*

Proof. Suppose that X is a strongly prime submodule. From Theorem 3.1.12, we see that X is prime and IFP. By Theorem 2.7.7, I_X is a prime ideal of S. It is well-known from [68, Lemma 2] that if X has IFP, then I_X is an IFP-right ideal of S. Hence I_X is a strongly prime ideal of S, by Corollary 3.1.13.

Conversely, suppose that M is a self-generator and I_X is a strongly prime ideal of S. Then I_X is prime and IFP. By Theorem 2.7.7, we see that X is prime. Similarly, from [68, Lemma 2] again, X has IFP. Applying Theorem 3.1.12, X is a strongly prime submodule, as desired. □

Proposition 3.1.15. *Let M be a right R-module which is a self-generator. If X is an one-sided strongly prime submodule of M, then I_X is an one-sided strongly prime ideal of S. Conversely, if I_X is an one-sided strongly prime right ideal of S, then X is an one-sided strongly prime submodule of M.*

Proof. Suppose that X is an one-sided strongly prime submodule and $\varphi, \alpha \in S$ such that $\varphi I_X \subset I_X$ and $\varphi\alpha \in I_X$. Then $\varphi\alpha(m) \in X$ for all $m \in M$. Since M is a self-generator, we have $X = \sum_{f \in I_X} f(M)$. Hence $\varphi(X) \subset X$. We assume that $\varphi \notin I_X$. Since X is an one-sided strongly prime submodule, we must have $\alpha(m) \in X$, for all $m \in M$. This shows that $\alpha \in I_X$. Hence I_X is an one-sided strongly prime right ideal of S.

Conversely, suppose that I_X is an one-sided strongly prime right ideal of S. Since M is a self-generator, we have $I_X(M) = X$. Assume that $\varphi(X) \subset X, \varphi(m) \in X$ and $m \notin X$. We wish to prove that $\varphi(M) \subset X$. From our assumption, we can see that $\varphi I_X \subset I_X$. Put $mR = \sum_{\psi \in A} \psi(M)$, for some subset A of S. Then

$X \supset \varphi(m)R = \varphi(mR) = \varphi(\sum_{\psi \in A} \psi(M)) = \sum_{\psi \in A} \varphi\psi(M)$. This implies that $\varphi\psi(M) \subset X$ for all $\psi \in A$. Since I_X is an one-sided strongly prime right ideal and $m \notin X$, we have $\varphi \in I_X$. This shows that X is an one-sided strongly prime submodule of M, as required. □

3.2 Characterizations of Noetherian modules

It is well-known that a ring R is right Noetherian if and only if every direct sum of injective modules is again injective. Cohen [1950] gave another characterization for commutative rings by the property that every prime ideal is finitely generated. From commutative rings to non-commutative one, the way is so far. Many authors tried to generalize Cohen's Theorem as we will present in the corollaries 3.2.2, 3.2.3, and so on. From rings to modules, this is a very long way. The main result in my thesis is a new characterization of Noetherian modules over any associative ring with identity. By introducing the class of one-sided strongly prime submodules, we now can prove directly the following beautiful Theorem.

Theorem 3.2.1. *Let M be a finitely generated right R-module. Then M is a Noetherian right R-module if and only if every one-sided strongly prime submodule of M is finitely generated.*

Proof. One way is clear. Suppose on the contrarily that there is a submodule A of M which is not finitely generated. By Zorn's Lemma, the set $\mathcal{F} = \{X \subset M | A \subset X$ and X is not finitely generated $\}$ has a maximal element, A_0 says. Since M is finitely generated, A_0 is a proper submodule of M. We now prove that A_0 is one-sided strongly prime. Suppose that there are $\varphi \in S, m \in M$ such that $\varphi(m) \in A_0$ with $\varphi(A_0) \subset A_0$ but $\varphi(M) \not\subset A_0$ and $m \notin A_0$. Then $A_0 + \varphi(M)$ contains properly A_0, and hence it is finitely generated, that is $A_0 + \varphi(M) = x_1R + x_2R + \cdots + x_nR$ for some $x_1, x_2, \ldots, x_n \in M$. Let $K = \{a \in M | \varphi(a) \in A_0\}$. By assumption, $A_0 \subset K$ and $m \in K$. Since $m \notin A_0, K$ contains properly $A_0 + mR$ and hence it is finitely generated. Since $x_i \in A_0 + \varphi(M)$, we can write $x_i = b_i + \varphi(m_i)$ where $b_i \in A_0$ and $m_i \in M$. By definition, $\varphi(K) \subset A_0$. It follows that $b_1R + b_2R + \cdots + b_nR \subset A_0$. We now prove that $A_0 \subset b_1R + b_2R + \cdots + b_nR + \varphi(K)$. For any $w \in A_0$, we have $w \in A_0 + \varphi(M)$. We can write $w = \sum_{i=1}^{n} x_ir_i = \sum_{i=1}^{n}(b_i + \varphi(m_i))r_i = \sum_{i=1}^{n} b_ir_i + \sum_{i=1}^{n} \varphi(m_ir_i) + \varphi(\sum_{i=1}^{n} m_ir_i)$. Since $w \in A_0$ and $\sum_{i=1}^{n} b_ir_i \in A_0$, we have $\varphi(\sum_{i=1}^{n} m_ir_i) \in A_0$ and hence $\sum_{i=1}^{n} m_ir_i \in K$. This implies that $w \in b_1R+b_2R+\cdots+b_nR+\varphi(K)$. Therefore $b_1R+b_2R+\cdots+b_nR+\varphi(K) \subset A_0$. This proves that $A_0 = b_1R+b_2R+\cdots+b_nR+\varphi(K)$. Since K is finitely generated, we can see that $\varphi(K)$ is finitely generated and hence A_0 is finitely generated, which is a contradiction. Therefore, every submodule of M is finitely generated, proving that M is Noetherian. □

Note that one-sided strongly prime right ideals are called *completely prime right ideals* by M. L. Reyes in [63]. The following Corollary can be considered as an immediately consequence of our theorem.

Corollary 3.2.2. *(Reyes, [63, Theorem 3.8]) A ring R is right Noetherian if and only if every one-sided strongly prime right ideal is finitely generated.*

Recall that a right R- module M is called a *duo module* if every submodule of M is a fully invariant submodule of M. A ring is called a right duo ring if every right ideal is a two-sided ideal. It is easy to see that a fully invariant one-sided strongly prime submodule of M is a strongly prime submodule of M. Thus, if M is a duo module, then every one-sided strongly prime submodule of M is also a strongly prime submodule of M. This leads to another corollary.

Corollary 3.2.3. *A finitely generated, duo right R- module is Noetherian if and only if every strongly prime submodule of M is finitely generated.*

From this corollary, putting $M = R_R$, we get:

Corollary 3.2.4. *(Chandran, [11, Theorem 2]) If R is a left (resp. right) duo ring and suppose that every prime ideal in R is finitely generated, then R is left (resp. right) Noetherian.*

Note that the definition of strongly prime ideals coincides with the usual definition of prime ideals in the commutative case. Therefore, the following Corollary is a direct consequence of Theorem 3.2.1.

Corollary 3.2.5. *(Cohen1950, [15, Theorem 2]) A commutative ring R with identity is Noetherian if and only if every prime ideal of R is finitely generated.*

Before proceeding to prove Cohen's Theorem for the class of fully bounded modules, we need to have the following Proposition.

Proposition 3.2.6. *Let M be a right R-module which is a self-generator. If S is a right Noetherian ring, then M is a Noetherian module.*

Proof. Suppose that we have an ascending chain of submodules of M, $M_1 \subset M_2 \subset M_3 \subset ... \subset M_n ...$, says. This shows that $I_{M_1} \subset I_{M_2} \subset I_{M_3} \subset ... \subset I_{M_n} ...$ is an escending chain of right ideals of S. Since S is a right Noetherian, the chain $I_{M_1} \subset I_{M_2} \subset I_{M_3} \subset ... \subset I_{M_n} ...$ is stationary. Thus there is an integer n such that $I_{M_n} = I_{M_k}$, for all $k > n$. From hypothesis, we have $M_n = I_{M_n}(M) = I_{M_k}(M) = M_k$, for all $k > n$. Hence $M_1 \subset M_2 \subset M_3 \subset ... \subset M_n ...$, is stationary. Therefore, M is a Noetherian module, completing the proof of our theorem.

The following result is given in [44, p.95]. This Proposition can be considered as Cohen theorem for the class of fully bounded rings.

Proposition 3.2.7. *[44] A right fully bounded ring is right noetherian iff all of its prime ideals are finitely generated as right ideals.*

Motivated this result, we have the following result for the class of fully bounded modules.

Theorem 3.2.8. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then a fully bounded module M is a right Noetherian module if and only if every prime submodule is finitely generated.*

Proof. ($\Rightarrow$) This is an immediate consequence of the definition of a Noetherian module.

($\Leftarrow$) Suppose that every prime submodule of M is finitely generated. Then we can prove that every prime right ideal of S is finitely generated. Applying Theorem 2.9.13, S is a right fully bounded ring. Since Proposition 3.2.7, we see that S is a right Noetherian ring. Proposition 3.2.6 implies that M is a Noetherian module. The proof of our is now complete.

□

CHAPTER IV
CONCLUSION

In the thesis, we study the classes of strongly prime and one-sided strongly prime submodules and use these classes to characterize Noetherian modules. Some properties of strongly prime and one-sided strongly prime submodules are investigated. We can see that every strongly prime submodule is prime. It is natural to ask a question that when a prime submodule is strongly prime. We answered it by Theorem 3.1.12 in the thesis. We gave a characterization of Noetherian modules by the class of one-sided strongly prime submodules (Theorem 3.2.1). This can be considered as a generalization of Cohen theorem in 1950 in commutative rings. This theorem is very interesting and covers some well-known results.

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